Probability

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Outline

Introduction

What is Probability?

Probability deals with uncertainty and quantifies how likely an event is to occur.

- Many real-life situations involve uncertainty rather than certainty
- Probability helps us make informed decisions under uncertainty
- It provides a numerical measure of chance, between 0 and 1
- A probability close to 0 indicates a rare event
- A probability close to 1 indicates a highly likely event

Examples

Probability concepts appear naturally in daily activities.

- Weather forecasting: chance of rain tomorrow
- Medical testing: likelihood that a test result is correct
- Games and sports: chances of winning or losing
- Traffic planning: probability of congestion at a given time
- Finance: risk assessment and expected returns

Example: Tossing a Coin

- The experiment consists of tossing a fair coin once
- Possible outcomes are Head (H) and Tail (T)
- Each outcome has an equal chance of occurring
- Probability of Head = 0.5
- Probability of Tail = 0.5

Key Concept and Terms

Basic Principal of Counting

- If an event can occur in m possible ways and for each of the m possible ways that the first event can occur, there are n possible ways that a second event can occur, then there are in total $m \times n$ possible ways that the two events can occur together
- For example, if a person can go from place A to place B in three possible ways, and B to C in two ways, then there are a total of six ways to go from A to C

Generalized Basic Principle of Counting

- If an event can occur in m_1 possible ways and for each of the possible ways that the first event can occur, there are m_2 possible ways that a second event can occur, and again for each of the $m_1 \times m_2$ possible ways that the first two events can occur, there are m_3 possible ways that a third event can occur, and so on, then there are in total $m_1 \times m_2 \times m_3 ...$ possible ways that all these events can occur together

Permutation

A permutation is an arrangement of objects where the order matters.

- Number of permutations of r objects chosen from n distinct objects:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

- Used when positions or order are important
- Example:
  - Number of ways to arrange 3 students out of 5 in a row:

$$^{5}P_{3} = \frac{5!}{2!} = 60$$

Combination

A combination is a selection of objects where the order does not matter.

- Number of combinations of r objects chosen from n distinct objects:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

- Used when only selection matters, not arrangement
- Example:
  - Number of ways to choose 3 students from 5:

$$^{5}C_{3} = \frac{5!}{3!2!} = 10$$

Experiment

- An experiment is any process that can be repeated under certain conditions and that produces an observable result
- The result of an experiment is called an outcome
- Example:
  - Tossing a coin or a dice
  - Measuring daily rainfall
  - Conducting chemical reactions

Outcome

- An outcome is a single possible result of an experiment
- There can be one or more *potential* outcomes
- Each experiment produces exactly one outcome
- Outcomes may be numerical or categorical
- Example:
  - Getting a head when tossing a coin
  - Getting a 4 when throwing a dice

Types of Experiment

Experiments can be categorized in to two types based on the nature of their outcome(s):

- Deterministic: outcome is known or can be predicted with certainty
- Random: outcome is unknown and cannot be predicted with certainty

Random Experiment

A random experiment is an experiment whose outcome cannot be predicted with certainty.

- The same experiment may produce different outcomes on repetition
- Potential outcomes are known, but which one will occur is uncertain
- Examples:
  - Tossing a coin
  - Rolling a dice
  - Drawing a card from a shuffled deck

Deterministic Experiment

A deterministic experiment is an experiment whose outcome can be predicted with certainty.

- Repeating the experiment under identical conditions gives the same result
- No randomness is involved
- Examples:
  - Calculating the sum of two fixed numbers
  - Measuring the boiling point of pure water at standard pressure

Iteration (Trial or Repetition)

An iteration refers to repeating an experiment under identical conditions.

- Each repetition is called a trial
- Iterations help study long-run behavior of outcomes
- Examples:
  - Tossing a coin 100 times
  - Rolling a die repeatedly and recording outcomes

Sample Space

The sample space is the set of all possible outcomes of a random experiment.

- Denoted by S.
- Each outcome is called a sample point
- Example:
  - Tossing a coin once: $S = \{H, T\}$
  - Rolling a die: $S = \{1, 2, 3, 4, 5, 6\}$
  - Tossing a coin twice: $S = \{HH, HT, TH, TT\}$

Event

An event is any *subset* of the sample space.

- An event may contain one or more outcomes
- A *simple (elementary) event* contains exactly one outcome
- Example (dice roll):
  - Event:
    - Getting an even number: $\{2, 4, 6\}$
    - Getting four or higher: $\{4, 5, 6\}$
  - Simple event: getting a 4: $\{4\}$

Mutually Exclusive Events

Two or more events are mutually exclusive if they cannot occur simultaneously.

- They have no common outcomes
- For events A and B:

$$A \cap B = \emptyset$$

- Example (dice roll):
  - A: getting an even number
  - B: getting an odd number

Collectively Exhaustive Events

Events are collectively exhaustive if their union covers the entire sample space.

- At least one of the events must occur
- For events $A_1, A_2, \ldots, A_n$:

$$A_1 \cup A_2 \cup \cdots \cup A_n = S$$

- Example:
  - Tossing a coin: $A_1 = \{H\}$ and $A_2 = \{T\}$
  - Throwing a dice: A_1 = getting an even number, A_2 = getting an odd number

Impossible and Sure Events

- An impossible event is an event that cannot occur
  - Probability is 0
  - Example: getting a 7 on a fair dice
- A sure (certain) event is an event that always occurs
  - Probability is 1
  - Example: getting a number less than 7 on a fair die

Equally Likely Events

Events are equally likely if each has the same chance of occurring.

- Common in experiments with symmetry
- Example:
  - Tossing a fair coin: $P(H) = P(T) = 0.5$
  - Rolling a fair die: each outcome has probability $1/6$

Definition of Probability

Classical Definition of Probability

The classical definition applies when outcomes are equally likely.

- If A is an event:

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

- Example:
    - Probability of getting an even number on a dice:

$$P(\text{getting an even number}) = \frac{3}{6} = \frac{1}{2}$$

Example

In a community of 400 people, 20 people has a particular disease. If a person is selected randomly from that community, what is the probability that he/she do not have the disease?

Solution: Probability of having the disease, $P(D) = 20/400$

Therefore, probability of not having the disease, $P(D^c) = 1 - P(D) = 1 - 20/400 = 0.95$

Frequency (Empirical) Definition of Probability

Probability is defined as the long-run relative frequency of an event.

- Based on repeated experiments
- If an event A occurs f times in n trials:

$$P(A) \approx \lim_{n \to \infty} \frac{f}{n}$$

- Becomes more accurate as n increases
- For example, if a coin is tossed $1{,}000$ times and 520 heads are seen, then probability of getting a head is $520/1000 = 0.52$

Axiomatic Definition of Probability

Probability is defined using a set of axioms.

- Proposed by Kolmogorov
- For any event A:
  - $0 \leq P(A) \leq 1$ *(probability is a number between* 0 *and* 1*)*
  - $P(S) = 1$ *(probability of sample space is* 1*)*
  - For a sequence of disjoint (mutually exclusive) events $A_1, A_2, ..., A_k$:

$$P(A_1, A_2, ..., A_k) = P(A_1) + P(A_2) + ... + P(A_k)$$

- Forms the foundation of modern probability theory

Some Laws of Probability

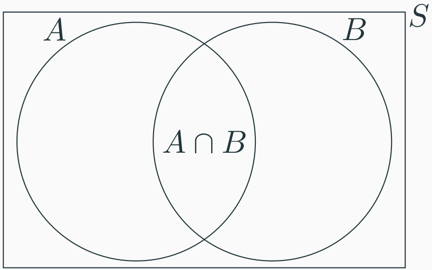
Addition Law of Probability (in Case of Two Joint Events)

The addition law gives the probability that at least one of two events occurs (either event A, or B or both).

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: Let $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cap B) = 0.2$.

$$P(A \cup B) = 0.5 + 0.4 - 0.2 = 0.7$$



Addition Law of Probability (Three Joint Events)

When there are three events:

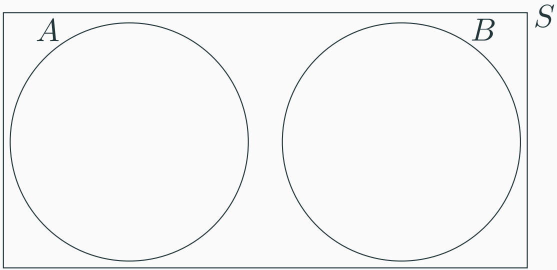
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

The above is the probability that at least one or two or all three of the events occur.

Addition Law of Probability (Disjoint Case)

The addition law gives the probability that at least one of two events occurs.

$$P(A \cup B) = P(A) + P(B)$$



Here, $P(A \cap B) = 0$ because $A \cap B = \emptyset$

Addition Law of Probability (Disjoint Case, More than Two Events)

For a sequence of disjoint (mutually exclusive) events $A_1, A_2, ..., A_k$:

$$P(A_1, A_2, ..., A_k) = P(A_1) + P(A_2) + ... + P(A_k)$$

Example: Addition Law

In a company, 60% of the employees have motorcycle, 40% have private car and 20% have both.

If an employee is selected randomly from that company, then

1. What is the probability that the employee has *at least* one type of vehicle?
2. What is the probability that the employee has *exactly* one type of vehicle?
3. What is the probability that the employee has neither motorcycle nor private car?

Solution

Let,
M = event that the employee has a motorcycle
C = event that the employee has a car

Then, $M \cap C$ = event that the employee has both

Here, $P(M) = 0.6$, $P(C) = 0.4$ and $P(M \cap C) = 0.2$

Solution (cont.)

1. Probability of having *at least* one type of vehicle:
   $P(M \cup C) = P(M) + P(C) - P(M \cap C) = 0.6 + 0.4 - 0.2 = 0.8$
2. Probability of having *exactly* one type of vehicle:
   $P(M \cup C) - P(M \cap C) = 0.8 - 0.2 = 0.6$
3. Probability of having neither type of vehicle:
   $1 - P(M \cup C) = 1 - 0.8 = 0.2$

Conditional Probability

Conditional probability measures the likelihood of an event given that another event has occurred.

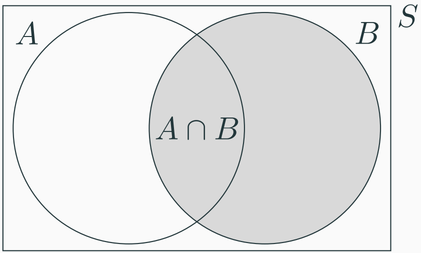
Probability that the event A will occur given that the B has already occurred:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Since B has already occurred:

- if the event A is to occur, then the set of favourable outcomes is given by $A \cap B$
- it also shrinks the sample space to B only

Conditional Probability (cont.)



The information that event B has already occured, shrinks the samples space to the set denoted by B.

Conditional Probability: Example 1

The sample space of a dice throw is, $S = \{1, 2, 3, 4, 5, 6\}$, then the probability of getting a 2 is: $P(2) = 1/6$.

Suppose it is known that an even number occured (the specific number that occured, is not yet disclosed).

Then the sample space shrinks, it becomes: $S^* = \{2, 4, 6\}$. Then $P(2|\text{even}) = 1/3$.

It can also be written as: $P(2|\text{even}) = \frac{P(2\,\text{and even})}{P(\text{even})} = \frac{1/6}{3/6} = \frac{1}{3}$.

Conditional Probability: Example 2

In a company, 60% of the employees have motorcycle, 40% have private car and 20% have both.

If an employee is selected randomly from that company. What is the probability that:

1. the employee has a car given that he/she has a motorcycle?
2. the employee has a motorcycle given that he/she has a car?

Conditional Probability: Example 2 Solution

1. Probability that the employee has a car given that he/she owns a motorcycle:
$$P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{0.2}{0.6} = 1/3$$
2. Probability that the employee has a motorcycle given that he/she owns a car:
$$P(M|C) = \frac{P(C \cap M)}{P(C)} = \frac{0.2}{0.4} = 1/2$$

Multiplication Law of Probability

- **For two *dependent events*** A and B, the probability that, both events will occur simultaneously is:

$$P(A \cap B) = P(A)\,P(B \mid A) = P(A \mid B)\,P(B)$$

- **For two *independent events***, the probability of both occuring simultaneously is:

$$P(A \cap B) = P(A)\,P(B)$$

Multiplication Law of Probability (cont.)

Suppose there are three events: A, B and C.
Then one of the $3!$ ways to write the multiplication law is:

$$P(A \cap B \cap C) = P(A \mid B \cap C)\, P(B \cap C) = P(A \mid B \cap C)\, P(B \mid C)\, P(C)$$

Multiplication Law: Example

In rainy season, it rains 70% of the days in Bangladesh. When it rains, 80% times it makes thunderstorms.

What is the probability that, in a particular day of rainy season, it will rain and it will thunderstorm?

Suppose:
R = event that it rains in that day
T = event that thunderstorm occurs

Then, $P(R) = 0.7$ and $P(T \mid R) = 0.8$.

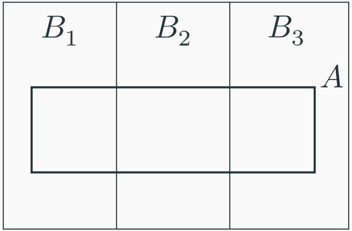
Therefore, $P(T \cap R) = P(T \mid R)\, P(R) = 0.8 \times 0.7 = 0.56$

Law of Total Probability

Suppose $B_1, B_2, \ldots, B_n$ are mutually exclusive and collectively exhaustive events with $P(B_i) > 0$. Then for any event A:

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

The probability of A is found by breaking the sample space into disjoint parts and weighting conditional probabilities by their proportions.



Law of Total Probability: Example

A factory has two machines. Machine B_1 produces 60% of items, machine B_2 produces 40%. 2% of the items produced by B_1 are defective, this rate is 5% for machine B_2.

Let D be the event that an item is defective.

$P(B_1) = 0.6, \quad P(B_2) = 0.4$
$P(D \mid B_1) = 0.02, \quad P(D \mid B_2) = 0.05$

$$P(D) = 0.02(0.6) + 0.05(0.4) = 0.012 + 0.02 = 0.032$$

Overall probability of a defective item is 0.032.

Law of Total Probability (cont.)

For two events E and F, the law can also be written as:

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F^c) \\ &= P(E \mid F)\, P(F) + P(E \mid F^c)\, P(F^c) \end{aligned}$$

Bayes' Theorem

Bayes' theorem reverses conditional probabilities. Assuming $P(B) > 0$:

$$
\begin{aligned}
P(A \mid B) &= \frac{P(B \cap A)}{P(B)} \\
&= \frac{P(B \mid A)\, P(A)}{P(B)} && (1) \\
&= \frac{P(B \mid A)\, P(A)}{P(B \mid A)\, P(A) + P(B \mid A^c)\, P(A^c)} && (2)
\end{aligned}
$$

Equation (1) ans (2) are known as the Bayes' Rule.

Bayes' Theorem (cont.)

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$

This theorem provies a framework to update one's belief when faced with new information.

- $P(A)$ is the *prior* belief regarding the probability of event A
- The event B has occured, is the new information
- $P(A \mid B)$ is the *posterior* or the updated belief after it gets known that B has occured

Bayes' Theorem: Example 1

Let $P(A) = 0.3$, $P(B \mid A) = 0.5$, and $P(B) = 0.4$.

$$P(A \mid B) = \frac{0.5 \times 0.3}{0.4} = 0.375$$

Bayes' Theorem: Example 2

Continuing from the example in page 40, suppose an item is found to be defective, calculate the probability that:

1. it was produced by machine B_1
2. it was produced by machine B_2

Recall the following from page 40:

$P(B_1) = 0.6, \quad P(B_2) = 0.4$

$P(D \mid B_1) = 0.02, \quad P(D \mid B_2) = 0.05$

$P(D) = 0.02(0.6) + 0.05(0.4) = 0.012 + 0.02 = 0.032$

Bayes' Theorem: Example 2 Solution

1. Probability that the item was produced by B_1 given that the item was defective:

$$P(B_1 \mid D) = \frac{P(D \mid B_1)\, P(B_1)}{P(D)} = \frac{0.02 \times 0.6}{0.032} = 0.375$$

2. Probability that the item was produced by B_2 given that the item was defective:

$$P(B_2 \mid D) = \frac{P(D \mid B_2)\, P(B_2)}{P(D)} = \frac{0.05 \times 0.4}{0.032} = 0.625$$

Thank you.