

# Probability

---

Md. Aminul Islam Shazid

# Outline

- 1 Introduction
- 2 Key Concept and Terms
- 3 Definition of Probability
- 4 Some Laws of Probability

# Introduction

---

# What is Probability?

Probability deals with uncertainty and quantifies how likely an event is to occur.

- Many real-life situations involve uncertainty rather than certainty
- Probability helps us make informed decisions under uncertainty
- It provides a numerical measure of chance, between 0 and 1
- A probability close to 0 indicates a rare event
- A probability close to 1 indicates a highly likely event

# Examples

Probability concepts appear naturally in daily activities.

- Weather forecasting: chance of rain tomorrow
- Medical testing: likelihood that a test result is correct
- Games and sports: chances of winning or losing
- Traffic planning: probability of congestion at a given time
- Finance: risk assessment and expected returns

## Example: Tossing a Coin

- The experiment consists of tossing a fair coin once
- Possible outcomes are Head (H) and Tail (T)
- Each outcome has an equal chance of occurring
- Probability of Head = 0.5
- Probability of Tail = 0.5

## **Key Concept and Terms**

---

# Basic Principal of Counting

- If an event can occur in  $m$  possible ways and for each of the  $m$  possible ways that the first event can occur, there are  $n$  possible ways that a second event can occur, then there are in total  $m \times n$  possible ways that the two events can occur together
- For example, if a person can go from place A to place B in three possible ways, and B to C in two ways, then there are a total of six ways to go from A to C

# Generalized Basic Principle of Counting

- If an event can occur in  $m_1$  possible ways and for each of the possible ways that the first event can occur, there are  $m_2$  possible ways that a second event can occur, and again for each of the  $m_1 \times m_2$  possible ways that the first two events can occur, there are  $m_3$  possible ways that a third event can occur, and so on, then there are in total  $m_1 \times m_2 \times m_3 \dots$  possible ways that all these events can occur together

# Permutation

A permutation is an arrangement of objects where the order matters.

- Number of permutations of  $r$  objects chosen from  $n$  distinct objects:

$${}^n P_r = \frac{n!}{(n-r)!}$$

- Used when positions or order are important
- Example:
  - Number of ways to arrange 3 students out of 5 in a row:

$${}^5 P_3 = \frac{5!}{2!} = 60$$

# Combination

A combination is a selection of objects where the order does not matter.

- Number of combinations of  $r$  objects chosen from  $n$  distinct objects:

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

- Used when only selection matters, not arrangement
- Example:
  - Number of ways to choose 3 students from 5:

$${}^5C_3 = \frac{5!}{3!2!} = 10$$

# Experiment

- An experiment is any process that can be repeated under certain conditions and that produces an observable result
- The result of an experiment is called an outcome
- Example:
  - Tossing a coin or a dice
  - Measuring daily rainfall
  - Conducting chemical reactions

# Outcome

- An outcome is a single possible result of an experiment
- There can be one or more *potential* outcomes
- Each experiment produces exactly one outcome
- Outcomes may be numerical or categorical
- Example:
  - Getting a head when tossing a coin
  - Getting a 4 when throwing a dice

# Types of Experiment

Experiments can be categorized into two types based on the nature of their outcome(s):

- Deterministic: outcome is known or can be predicted with certainty
- Random: outcome is unknown and cannot be predicted with certainty

# Random Experiment

A random experiment is an experiment whose outcome cannot be predicted with certainty.

- The same experiment may produce different outcomes on repetition
- Potential outcomes are known, but which one will occur is uncertain
- Examples:
  - Tossing a coin
  - Rolling a dice
  - Drawing a card from a shuffled deck

# Deterministic Experiment

A deterministic experiment is an experiment whose outcome can be predicted with certainty.

- Repeating the experiment under identical conditions gives the same result
- No randomness is involved
- Examples:
  - Calculating the sum of two fixed numbers
  - Measuring the boiling point of pure water at standard pressure

# Iteration (Trial or Repetition)

An iteration refers to repeating an experiment under identical conditions.

- Each repetition is called a trial
- Iterations help study long-run behavior of outcomes
- Examples:
  - Tossing a coin 100 times
  - Rolling a die repeatedly and recording outcomes

# Sample Space

The sample space is the set of all possible outcomes of a random experiment.

- Denoted by  $S$ .
- Each outcome is called a sample point
- Example:
  - Tossing a coin once:  $S = \{H, T\}$
  - Rolling a die:  $S = \{1, 2, 3, 4, 5, 6\}$
  - Tossing a coin twice:  $S = \{HH, HT, TH, TT\}$

# Event

An event is any *subset* of the sample space.

- An event may contain one or more outcomes
- A *simple (elementary) event* contains exactly one outcome
- Example (dice roll):
  - Event:
    - ▶ Getting an even number: {2, 4, 6}
    - ▶ Getting four or higher: {4, 5, 6}
  - Simple event: getting a 4: {4}

# Mutually Exclusive Events

Two or more events are mutually exclusive if they cannot occur simultaneously.

- They have no common outcomes
- For events  $A$  and  $B$ :

$$A \cap B = \emptyset$$

- Example (dice roll):
  - $A$ : getting an even number
  - $B$ : getting an odd number

# Collectively Exhaustive Events

Events are collectively exhaustive if their union covers the entire sample space.

- At least one of the events must occur
- For events  $A_1, A_2, \dots, A_n$ :

$$A_1 \cup A_2 \cup \dots \cup A_n = S$$

- Example:
  - Tossing a coin:  $A_1 = \{H\}$  and  $A_2 = \{T\}$
  - Throwing a dice:  $A_1$  = getting an even number,  $A_2$  = getting an odd number

# Impossible and Sure Events

- An impossible event is an event that cannot occur
  - Probability is 0
  - Example: getting a 7 on a fair dice
- A sure (certain) event is an event that always occurs
  - Probability is 1
  - Example: getting a number less than 7 on a fair die

# Equally Likely Events

Events are equally likely if each has the same chance of occurring.

- Common in experiments with symmetry
- Example:
  - Tossing a fair coin:  $P(H) = P(T) = 0.5$
  - Rolling a fair die: each outcome has probability  $1/6$

# Definition of Probability

---

# Classical Definition of Probability

The classical definition applies when outcomes are equally likely.

- If  $A$  is an event:

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

- Example:
  - Probability of getting an even number on a dice:

$$P(\text{getting an even number}) = \frac{3}{6} = \frac{1}{2}$$

## Example

In a community of 400 people, 20 people has a particular disease. If a person is selected randomly from that community, what is the probability that he/she do not have the disease?

**Solution:** Probability of having the disease,  $P(D) = 20/400$

Therefore, probability of not having the disease,  
 $P(D^c) = 1 - P(D) = 1 - 20/400 = 0.95$

# Frequency (Empirical) Definition of Probability

Probability is defined as the long-run relative frequency of an event.

- Based on repeated experiments
- If an event  $A$  occurs  $f$  times in  $n$  trials:

$$P(A) \approx \lim_{n \rightarrow \infty} \frac{f}{n}$$

- Becomes more accurate as  $n$  increases
- For example, if a coin is tossed 1,000 times and 520 heads are seen, then probability of getting a head is  $520/1000 = 0.52$

# Axiomatic Definition of Probability

Probability is defined using a set of axioms.

- Proposed by Kolmogorov
- For any event  $A$ :
  - $0 \leq P(A) \leq 1$  (*probability is a number between 0 and 1*)
  - $P(S) = 1$  (*probability of sample space is 1*)
  - For a sequence of disjoint (mutually exclusive) events  $A_1, A_2, \dots, A_k$ :

$$P(A_1, A_2, \dots, A_k) = P(A_1) + P(A_2) + \dots + P(A_k)$$

- Forms the foundation of modern probability theory

## Some Laws of Probability

---

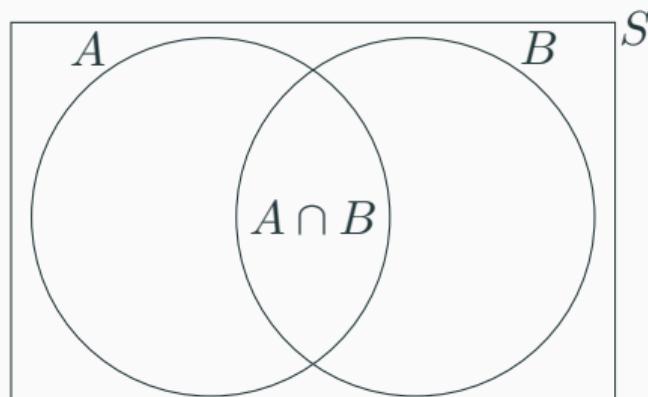
# Addition Law of Probability (in Case of Two Joint Events)

The addition law gives the probability that at least one of two events occurs (either event  $A$ , or  $B$  or both).

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Example:** Let  $P(A) = 0.5$ ,  $P(B) = 0.4$ , and  $P(A \cap B) = 0.2$ .

$$P(A \cup B) = 0.5 + 0.4 - 0.2 = 0.7$$



## Addition Law of Probability (Three Joint Events)

When there are three events:

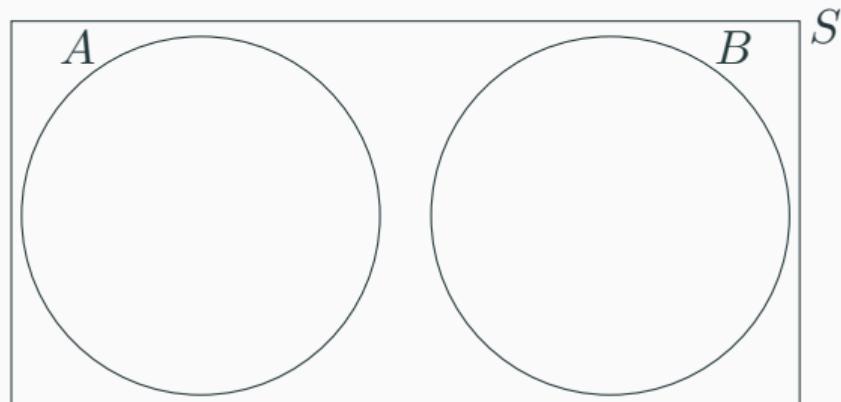
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

The above is the probability that at least one or two or all three of the events occur.

## Addition Law of Probability (Disjoint Case)

The addition law gives the probability that at least one of two events occurs.

$$P(A \cup B) = P(A) + P(B)$$



Here,  $P(A \cap B) = 0$  because  $A \cap B = \emptyset$

## Addition Law of Probability (Disjoint Case, More than Two Events)

For a sequence of disjoint (mutually exclusive) events  $A_1, A_2, \dots, A_k$ :

$$P(A_1, A_2, \dots, A_k) = P(A_1) + P(A_2) + \dots + P(A_k)$$

## Example: Addition Law

In a company, 60% of the employees have motorcycle, 40% have private car and 20% have both.

If an employee is selected randomly from that company, then

- ① What is the probability that the employee has *at least* one type of vehicle?
- ② What is the probability that the employee has *exactly* one type of vehicle?
- ③ What is the probability that the employee has neither motorcycle nor private car?

## Solution

Let,

$M$  = event that the employee has a motorcycle  
 $C$  = event that the employee has a car

Then,  $M \cap C$  = event that the employee has both

Here,  $P(M) = 0.6$ ,  $P(C) = 0.4$  and  $P(M \cap C) = 0.2$

## Solution (cont.)

- ① Probability of having *at least* one type of vehicle:

$$P(M \cup C) = P(M) + P(C) - P(M \cap C) = 0.6 + 0.4 - 0.2 = 0.8$$

- ② Probability of having *exactly* one type of vehicle:

$$P(M \cup C) - P(M \cap C) = 0.8 - 0.2 = 0.6$$

- ③ Probability of having *neither* type of vehicle:

$$1 - P(M \cup C) = 1 - 0.8 = 0.2$$

# Conditional Probability

Conditional probability measures the likelihood of an event given that another event has occurred.

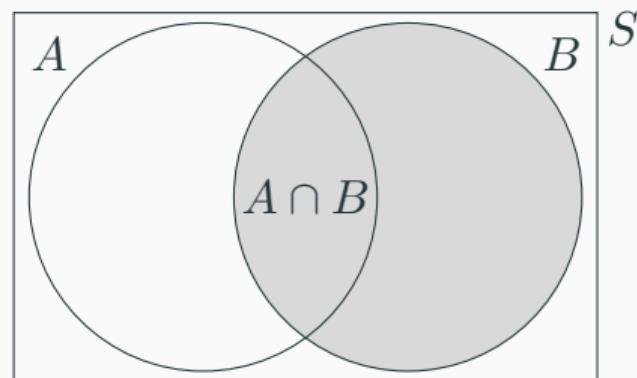
Probability that the event  $A$  will occur given that the  $B$  has already occurred:

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Since  $B$  has already occurred:

- if the event  $A$  is to occur, then the set of favourable outcomes is given by  $A \cap B$
- it also shrinks the sample space to  $B$  only

# Conditional Probability (cont.)



The information that event  $B$  has already occurred, shrinks the samples space to the set denoted by  $B$ .

## Conditional Probability: Example 1

The sample space of a dice throw is,  $S = \{1, 2, 3, 4, 5, 6\}$ , then the probability of getting a 2 is:  $P(2) = 1/6$ .

Suppose it is known that an even number occurred (the specific number that occurred, is not yet disclosed).

Then the sample space shrinks, it becomes:  $S^* = \{2, 4, 6\}$ . Then  $P(2|\text{even}) = 1/3$ .

It can also be written as:  $P(2|\text{even}) = \frac{P(\text{2 and even})}{P(\text{even})} = \frac{1/6}{3/6} = \frac{1}{3}$ .

## Conditional Probability: Example 2

In a company, 60% of the employees have motorcycle, 40% have private car and 20% have both.

If an employee is selected randomly from that company. What is the probability that:

- ① the employee has a car given that he/she has a motorcycle?
- ② the employee has a motorcycle given that he/she has a car?

# Conditional Probability: Example 2 Solution

- ① Probability that the employee has a car given that he/she owns a motorcycle:

$$P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{0.2}{0.6} = 1/3$$

- ② Probability that the employee has a motorcycle given that he/she owns a car:

$$P(M|C) = \frac{P(C \cap M)}{P(C)} = \frac{0.2}{0.4} = 1/2$$

# Multiplication Law of Probability

- For two **dependent events**  $A$  and  $B$ , the probability that, both events will occur simultaneously is:

$$P(A \cap B) = P(A) P(B | A) = P(A | B) P(B)$$

- For two **independent events**, the probability of both occurring simultaneously is:

$$P(A \cap B) = P(A) P(B)$$

## Multiplication Law of Probability (cont.)

Suppose there are three events:  $A$ ,  $B$  and  $C$ .  
Then one of the  $3!$  ways to write the multiplication law is:

$$P(A \cap B \cap C) = P(A | B \cap C) P(B \cap C) = P(A | B \cap C) P(B | C) P(C)$$

## Multiplication Law: Example

In rainy season, it rains 70% of the days in Bangladesh. When it rains, 80% times it makes thunderstorms.

What is the probability that, in a particular day of rainy season, it will rain and it will thunderstorm?

Suppose:

$R$  = event that it rains in that day

$T$  = event that thunderstorm occurs

Then,  $P(R) = 0.7$  and  $P(T | R) = 0.8$ .

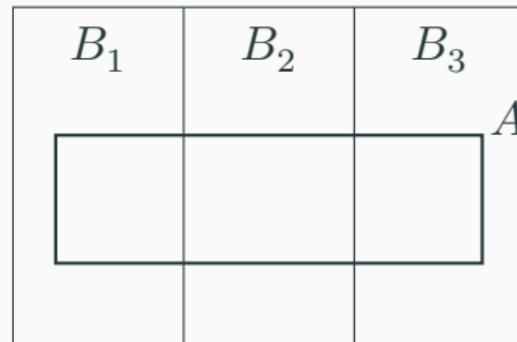
Therefore,  $P(T \cap R) = P(T | R) P(R) = 0.8 \times 0.7 = 0.56$

# Law of Total Probability

Suppose  $B_1, B_2, \dots, B_n$  are mutually exclusive and collectively exhaustive events with  $P(B_i) > 0$ . Then for any event  $A$ :

$$P(A) = \sum_{i=1}^n P(A | B_i)P(B_i)$$

The probability of  $A$  is found by breaking the sample space into disjoint parts and weighting conditional probabilities by their proportions.



## Law of Total Probability: Example

A factory has two machines. Machine  $B_1$  produces 60% of items, machine  $B_2$  produces 40%. 2% of the items produced by  $B_1$  are defective, this rate is 5% for machine  $B_2$ .

Let  $D$  be the event that an item is defective.

$$P(B_1) = 0.6, \quad P(B_2) = 0.4$$

$$P(D | B_1) = 0.02, \quad P(D | B_2) = 0.05$$

$$P(D) = 0.02(0.6) + 0.05(0.4) = 0.012 + 0.02 = 0.032$$

Overall probability of a defective item is 0.032.

## Law of Total Probability (cont.)

For two events  $E$  and  $F$ , the law can also be written as:

$$P(E) = P(E | F) P(F) + P(E | F^c) P(F^c)$$

# Bayes' Theorem

Bayes' theorem reverses conditional probabilities. Assuming  $P(B) > 0$ :

$$\begin{aligned} P(A | B) &= \frac{P(B \cap A)}{P(B)} \\ &= \frac{P(B | A) P(A)}{P(B)} \\ &= \frac{P(B | A) P(A)}{P(B | A) P(A) + P(B | A^c) P(A^c)} \end{aligned}$$

**Thank you.**

---