Impedance Control of Euler-Lagrange Systems with Unknown Dynamics

***Abstract*—**

I. INTRODUCTION

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II. DESIRED DYNAMICS AND PROBLEM FORMULATION

In various robot-environment interaction scenarios we desire the ability to have adaptable interactions with environments that are not completely known or predictable. To achieve this simultaneous force-motion control in task space we often desire to control the way the robot interacts with the environment, or essentially we try to control the dynamics of the system.

The simple second order dynamics, similar to the standard spring-mass-damper, are well studied and understood, and hence a very popular choice of dynamics to be imposed on the system.

A. Conventional UAM Model and Dynamics

We have considered consider an generalized aerial-manipulator system with a n degrees-of-freedom (DoFs) manipulator system as in Fig. **??**. For this system, let the position p and orientation q (in Euler angles) of the quadrotor be defined as $p \triangleq \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3$ and $q \triangleq \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^T \in \mathbb{R}^3$ and we define $\alpha \triangleq \begin{bmatrix} \alpha_1 & \alpha_2 & .. & \alpha_n \end{bmatrix}^T \in \mathbb{R}^n$ for the n-link manipulator system. The Euler-Lagrangian dynamical model for this system can be represented as [**?**]

$$M(\chi(t))\ddot{\chi}(t)+C(\chi(t),\dot{\chi}(t))\dot{\chi}(t)+g(\chi(t))+d(t) = \tau+\tau_{ext} \tag{1}$$

where $\chi = \begin{bmatrix} p^T & q^T & \alpha^T \end{bmatrix}^T \in \mathbb{R}^{6+n}$ is the generalize state vector; $M \in \mathbb{R}^{(6+n)\times(6+n)}$ represents the inertia matrix; $C \in \mathbb{R}^{(6+n)\times(6+n)}$ is the coriolis and centrifugal matrix; $g \in \mathbb{R}^{6+n}$ is the vector of gravity forces; $d \in \mathbb{R}^{6+n}$ is the bounded external disturbance; $\tau \in \mathbb{R}^{6+n}$ is the control input vector and $\tau_{ext} \in \mathbb{R}^{6+n}$ is the torque due to external forces acting on the end-effector. These dynamics terms can be split as:

$$M = \begin{bmatrix} M_{pp} & M_{pq} & M_{p\alpha} \\ M_{pq}^T & M_{qq} & M_{q\alpha} \\ M_{p\alpha}^T & M_{q\alpha}^T & M_{\alpha\alpha} \end{bmatrix}, \quad \begin{matrix} M_{pp}, M_{qq}, M_{pq} \in \mathbb{R}^{3\times3} \\ M_{p\alpha}, M_{q\alpha} \in \mathbb{R}^{3\times n} \\ M_{\alpha\alpha} \in \mathbb{R}^{n\times n} \end{matrix} \tag{2a}$$

$$C = \begin{bmatrix} C_p \\ C_q \\ C_\alpha \end{bmatrix}, \quad \begin{matrix} C_p, C_q \in \mathbb{R}^{3\times(6+n)} \\ C_\alpha \in \mathbb{R}^{n\times(6+n)} \end{matrix} \tag{2b}$$

$$g = \begin{bmatrix} g_p \\ g_q \\ g_\alpha \end{bmatrix}, d = \begin{bmatrix} d_p \\ d_q \\ d_\alpha \end{bmatrix}, \quad \begin{matrix} g_p, g_q, d_p, d_q \in \mathbb{R}^3 \\ g_\alpha, d_\alpha \in \mathbb{R}^n \end{matrix} \tag{2c}$$

$$\tau = \begin{bmatrix} \tau_p \\ \tau_q \\ \tau_\alpha \end{bmatrix}, \quad \begin{matrix} \tau_p, \tau_q \in \mathbb{R}^3 \\ \tau_\alpha \in \mathbb{R}^n \end{matrix}. \tag{2d}$$

$$\tau_{ext} = \begin{bmatrix} 0 \\ 0 \\ J_\alpha^T F_{ext} \end{bmatrix}, \quad \begin{matrix} F_{ext} \in \mathbb{R}^3 \\ J_\alpha \in \mathbb{R}^{n\times3}. \end{matrix} \tag{2e}$$

Where J_α is the analytic jacobian for the manipulator.

Here, $\tau_\alpha \triangleq \begin{bmatrix} \tau_{\alpha_1} & \tau_{\alpha_2} & .. & \tau_{\alpha_n} \end{bmatrix}^T$ is used for the control input for the manipulator; $\tau_q \triangleq \begin{bmatrix} u_2(t) & u_3(t) & u_4(t) \end{bmatrix}^T$ is used as the control inputs for roll, pitch and yaw of the quadrotor; $\tau_p = R_B^W U$ is the generalized control input for quadrotor position in Earth-fixed frame, such that $U(t) \triangleq \begin{bmatrix} 0 & 0 & u_1(t) \end{bmatrix}^T \in \mathbb{R}^3$ being the force vector in body-fixed frame and $R_B^W \in \mathbb{R}^{3\times3}$ being the $Z-Y-X$ Euler angle rotation matrix describing the rotation from the body-fixed coordinate frame to the Earth-fixed frame, given by

$$R_B^W = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & s_\phi c_\theta & c_\theta c_\phi \end{bmatrix}, \tag{3}$$

where $c_{(\cdot)}, s_{(\cdot)}$ and denote $\cos(\cdot), \sin(\cdot)$ respectively.

B. Distributed UAM Dynamics

For ease of control design and analysis and using (2), the UAM dynamics (1) can be re-written as

$$M_{pp}\ddot{p} + M_{pq}\ddot{q} + M_{p\alpha}\ddot{\alpha} + C_p\dot{\chi} + g_p + d_p = \tau_p \tag{4a}$$

$$M_{pq}^T\ddot{p} + M_{qq}\ddot{q} + M_{q\alpha}\ddot{\alpha} + C_q\dot{\chi} + g_q + d_q = \tau_q \tag{4b}$$

$$M_{p\alpha}^T\ddot{p} + M_{q\alpha}^T\ddot{q} + M_{\alpha\alpha}\ddot{\alpha} + C_\alpha\dot{\chi} + g_\alpha + d_\alpha = \tau_\alpha + \tau_{\alpha ext} \tag{4c}$$

$$\tau_p = R_B^W U \tag{4d}$$

$$\tau_{\alpha ext} = J_\alpha^T F_{ext} \tag{4e}$$

Need to format the equation numbers into this.

where (**??**), (**??**) and (**??**) represent the quadrotor position dynamics, quadrotor attitude dynamics and manipulator dynamics along with their interactions, respectively.

The following standard system properties hold via Euler-Lagrange mechanics [**?**]:

Property 1: The matrix $M_(\chi)$ is uniformly positive definite and $\exists \underline{m}, \overline{m} \in \mathbb{R}^{+}$ such that $0 < \underline{m} I \leq M(\chi) \leq \overline{m} I$.

Property 2: $\exists \bar{c}, \bar{g}, \bar{d} \in \mathbb{R}^{+}$ such that $||C(\chi)|| \leq \bar{c}||\dot{\chi}||$, $||g(\chi)|| \leq \bar{g}$ and $||d(t)|| \leq \bar{d}$.

The following assumption highlights the available knowledge of various system parameters for the control design:

Assumption 1 (Uncertainty): The system dynamics terms M, C, g, d and their bounds $\overline{m}, \underline{m}, \bar{c}, \bar{g}, \bar{d}$ defined in Properties 1-2 are unknown for control design.

Assumption 2: The desired trajectories $\chi_d = \begin{bmatrix} p_d^T & q_d^T & \alpha_d^T \end{bmatrix}^T$ and their time-derivatives $\dot{\chi}_d, \ddot{\chi}_d$ are designed to be sufficiently bounded. Furthermore, $\chi, \dot{\chi}, \ddot{\chi}$ are available for feedback.

Control Problem Under Properties 1-2 and Assumptions 1-2, to design a distributed adaptive control framework for the aerial manipulator system (4), with an important objective being to impose desired impedance dynamics on the manipulator for the attenuation of forces arising from environmental interactions.

The following section solves the control problem.

III. Controller Design and Analysis

The proposed control framework consists of designing (i) quadrotor position control, (ii) quadrotor attitude control and (iii) the manipulator control as per the dynamics (4). Note that such design process is simultaneous in nature and not decoupled. Defining tracking error as

$$e \triangleq \chi(t) - \chi_d(t), \;\; \xi(t) \triangleq \begin{bmatrix} e(t) & \dot{e}(t) \end{bmatrix}, \tag{5}$$

we elaborate the proposed control designs in the following subsections.

A. Quadrotor Position Control

Taking the quadrotor position tracking error as $e_p(t) \triangleq p(t) - p_d(t)$, let us define an error variable as

$$s_p = \dot{e}_p + \Phi_p e_p \tag{6}$$

where $\Phi_p \in \mathbb{R}^{3\times 3}$ is a positive definite gain matrix. Multiplying the time derivative of (16) by $\bar{M}_{pp}$ and using (**??**) yields

$$\begin{aligned} \bar{M}_{pp}\dot{s}_p &= \bar{M}_{pp}(\ddot{p} - \ddot{p}_d + \Phi_p \dot{e}_p) \\ &= \tau_p + E_p - \bar{M}_{pp}(\ddot{p}_d - \Phi_p \dot{e}_p), \end{aligned} \tag{7}$$

where $\bar{M}_{pp}$ is a user-defined constant positive definite matrix and $E_p \triangleq -\{(M_{pp} - \bar{M}_{pp})\ddot{p} + M_{pq}\ddot{q} + M_{p\alpha}\ddot{\alpha} + C_p\dot{\chi} + g_p + d_p\}$. The selection of $\bar{M}_{pp}$ would be discussed later (cf. Remark **??**).

The control law is proposed as

$$\tau_p = -\Lambda_p s_p - \Delta\tau_p + \bar{M}_{pp}(\ddot{p}_d - \Phi_p \dot{e}_p), \tag{8a}$$

$$\Delta\tau_p = \begin{cases} \rho_p \frac{s_p}{||s_p||} & \text{if } ||s_p|| \geq \varpi_p \\ \rho_p \frac{s_p}{\varpi_p} & \text{if } ||s_p|| < \varpi_p \end{cases} \tag{8b}$$

where Λ_p is a user-defined positive definite gain matrix and $\varpi_p > 0$ is a scalar used to avoid chattering; ρ_p tackles system uncertainties, whose design will be discussed later.

Using Property 2 and the relation (2) we have $||C_p|| \leq ||C|| \leq \bar{c}||\dot{\chi}||$, $||g_p|| \leq ||g|| \leq \bar{g}$ and $||d_p|| \leq ||d|| \leq \bar{d}$. Using these relations, the inequalities $||\ddot{\chi}|| \geq ||\ddot{p}||$, $||\ddot{\chi}|| \geq ||\ddot{q}||$ and $||\ddot{\chi}|| \geq ||\ddot{\alpha}||$, $||\xi|| \geq ||\dot{e}||$, $||\xi|| \geq ||e||$ and substituting $\dot{\chi} = \dot{e} + \dot{\chi}_d$ into (**??**) yields

$$||E_p|| \leq K_{p0}^* + K_{p1}^*||\xi|| + K_{p2}^*||\xi||^2 + K_{p3}^*||\ddot{\chi}|| \tag{9}$$

$$\begin{aligned} &K_{p0}^* = \bar{g} + \bar{d} + \bar{c}||\dot{\chi}_d||^2, \\ &K_{p1}^* = 2\bar{c}||\dot{\chi}_d||, \;\; K_{p2}^* = \bar{c}, \\ &K_{p3}^* = (||(M_{pp} - \bar{M}_{pp})|| + ||M_{pq}|| + ||M_{p\alpha}||) \end{aligned}$$

are unknown scalars. Based on the upper bound structure in (9), the gain ρ_p in (8b) is designed as

$$\rho_p = \hat{K}_{p0} + \hat{K}_{p1}||\xi|| + \hat{K}_{p2}||\xi||^2 + \hat{K}_{p3}||\ddot{\chi}|| + \zeta_p + \gamma_p \tag{10}$$

where $\hat{K}_{pi}$ are the estimates of K_{pi}^* $i = 0, 1, 2, 3$, and ζ_p, γ_p are auxiliary gain used for closed-loop stabilization (cf. Remark **??**). The gains $\hat{K}_{pi}$ are adapted via the following laws:

$$\dot{\hat{K}}_{pia} = ||s_p||||\xi||^i - \nu_{pia}\hat{K}_{pia}, \dot{\hat{K}}_{pi\bar{a}} = 0, i = 0, 1, 2 \tag{11a}$$

$$\dot{\hat{K}}_{p3a} = ||s_p||||\ddot{\chi}|| - \nu_{p3a}\hat{K}_{p3a}, \dot{\hat{K}}_{p3\bar{a}} = 0 \tag{11b}$$

$$\dot{\zeta}_{pa} = -(1 + \hat{K}_{p3a}||s_a||||\ddot{\chi}||)\zeta_{pa} + \bar{\epsilon}_{pa}, \dot{\zeta}_{p\bar{a}} = 0 \tag{11c}$$

$$\dot{\gamma}_{pa} = 0, \dot{\gamma}_{p\bar{a}} = \left(1 + \frac{\varrho_{p\bar{a}}}{2}\sum_{i=0}^{3}\hat{K}_{pi\bar{a}}^2\right)\gamma_{p\bar{a}} + \epsilon_{p\bar{a}} \tag{11d}$$

$$\text{with } \nu_{pi} > \varrho_p/2, i = 0, 1, 2 \tag{11e}$$

$$\hat{K}_{pi}(t_0) > 0, \;\; \zeta_p(t_0) = \bar{\zeta}_p > \bar{\epsilon}_p, \gamma_p(t_0) = \bar{\gamma}_p > \epsilon_p \tag{11f}$$

where a and $\bar{a} \in \Omega\backslash a$ denote the active and inactive subsystems respectively; $\nu_{pia}, \bar{\epsilon}_{pa}, \epsilon_{p\bar{a}} \in \mathbb{R}^{+}$ are user-defined scalars and $\varrho_p \triangleq \frac{\min\{\lambda_{\min}(\Lambda_p), \lambda_{\min}(\Phi_p)\}}{\max\{\bar{M}_{pp}, 1\}}$ $\forall \in \Omega$ and t_0 is the initial time and from **??** and intial condition **??**, it can be verified that $\exists \underline{\zeta}_p, \underline{\gamma}_p \in \mathbb{R}^{+}$ such that

$$\begin{aligned} &\hat{K}_{pi}(t) \geq 0, \quad 0 < \underline{\zeta}_p \leq \zeta_p(t) < \bar{\zeta}_p, \\ &\text{and} \;\; 0 < \underline{\gamma}_p \leq \gamma_p(t) < \bar{\gamma}_p \;\; \forall t \geq t_0. \end{aligned} \tag{12a}$$

B. Quadrotor Attitude Control

To achieve the attitude control, the tracking error in orientation/attitude is defined as [**?**]

$$e_q = ((R_d)^T R_B^W - (R_B^W)^T R_d)^v \tag{13}$$

$$\dot{e}_q = \dot{q} - R_d^T R_B^W \dot{q}_d \tag{14}$$

where $(.)^v$ represents *vee* map, which converts elements of $SO(3)$ to $\in \mathbb{R}^3$ and R_d is the rotation matrix as in (3) evaluated at $(\phi_d, \theta_d, \psi_d)$.

The quadrotor attitude sub-dynamics (4b) is rearranged as

$$\bar{M}_{qq}\ddot{q} + E_q = \tau_q \tag{15}$$

where $\bar{M}_{qq}$ is a user-defined constant positive definite matrix (cf. Remark **??**) and $E_q \triangleq (M_{qq} - \bar{M}_{qq})\ddot{q} + M_{pq}^T\ddot{p} + M_{q\alpha}\ddot{\alpha} + C_q\dot{\chi} + g_q + d_q$.

For quadrotor attitude tracking error, let us define an error variable as

$$s_q = \dot{e}_q + \Phi_q e_q \tag{16}$$

where $\Phi_q \in \mathbb{R}^{3\times3}$ is a positive definite gain matrix. Multiplying the time derivative of (**??**) by $\bar{M}_{qq}$ and using (**??**) yields

$$\begin{aligned} \bar{M}_{qq}\dot{s}_q &= \bar{M}_{qq}(\ddot{q} - \ddot{q}_d + \Phi_q\dot{e}_q) \\ &= \tau_q - E_q - \bar{M}_{qq}(\ddot{q}_d - \Phi_q\dot{e}_q), \end{aligned} \tag{17}$$

The control law is proposed as

$$\tau_q = -\Lambda_q s_q + \Delta\tau_q + \bar{M}_{qq}(\ddot{q}_d - \Phi_q\dot{e}_q), \tag{18a}$$

$$\Delta\tau_q = \begin{cases} \rho_q \frac{s_q}{||s_q||} & \text{if } ||s_q|| \geq \varpi_q \\ \rho_q \frac{s_q}{\varpi_q} & \text{if } ||s_q|| < \varpi_q \end{cases} \tag{18b}$$

where Λ_q is a user-defined positive definite gain matrix and $\varpi_q > 0$ is a scalar used to avoid chattering; ρ_q tackles system uncertainties, whose design will be discussed later. Using Property 2 and the relation (2) we have $||C_q|| \leq ||C|| \leq \bar{c}||\dot{\chi}||$, $||g_q|| \leq ||g|| \leq \bar{g}$ and $||d_q|| \leq ||d|| \leq \bar{d}$. Using these relations, the inequalities $||\ddot{\chi}|| \geq ||\ddot{p}||$, $||\ddot{\chi}|| \geq ||\ddot{q}||$ and $||\ddot{\chi}|| \geq ||\ddot{\alpha}||$, $||\xi|| \geq ||\dot{e}||$, $||\xi|| \geq ||e||$ and substituting $\dot{\chi} = \dot{e} + \dot{\chi}_d$ into (**??**) yields

$$||E_q|| \leq K_{q0}^* + K_{q1}^*||\xi|| + K_{q2}^*||\xi||^2 + K_{q3}^*||\ddot{\chi}|| \tag{19}$$

where

$$\begin{aligned} &K_{q0}^* = \bar{g} + \bar{d} + \bar{c}||\dot{\chi}_d||^2, \\ &K_{q1}^* = 2\bar{c}||\dot{\chi}_d||, \ K_{q2}^* = \bar{c}, \\ &K_{q3}^* = (||(M_{qq} - \bar{M}_{qq})|| + ||M_{pq}^T|| + ||M_{q\alpha}||) \end{aligned}$$

$K_{p0}^* = K_{q0}^*$
are unknown scalars. Based on the upper bound structure in (19), the gain ρ_q in (18b) is designed as

$$\rho_q = \hat{K}_{q0} + \hat{K}_{q1}||\xi|| + \hat{K}_{q2}||\xi||^2 + \hat{K}_{q3}||\ddot{\chi}|| + \zeta_q + \gamma_q \tag{20}$$

where $\hat{K}_{qi}$ are the estimates of K_{qi}^* $i = 0,1,2,3$, and ζ_q, γ_q are auxiliary gain used for closed-loop stabilization (cf. Remark **??**). The gains $\hat{K}_{qi}$ are adapted via the following laws:

$$\dot{\hat{K}}_{qia} = ||s_q||||\xi||^i - \nu_{qia}\hat{K}_{qia}, \dot{\hat{K}}_{qi\bar{a}} = 0, i = 0,1,2 \tag{21a}$$

$$\dot{\hat{K}}_{q3a} = ||s_q||||\ddot{\chi}|| - \nu_{q3a}\hat{K}_{q3a}, \dot{\hat{K}}_{q3\bar{a}} = 0 \tag{21b}$$

$$\dot{\zeta}_{qa} = -(1 + \hat{K}_{q3a}||s_q||||\ddot{\chi}||)\zeta_{qa} + \bar{\epsilon}_{qa}, \dot{\zeta}_{q\bar{a}} = 0 \tag{21c}$$

$$\dot{\gamma}_{qa} = 0, \dot{\gamma}_{q\bar{a}} = \left(1 + \frac{\varrho_{q\bar{a}}}{2}\sum_{i=0}^{3}\hat{K}_{qi\bar{a}}^2\right)\gamma_{q\bar{a}} + \epsilon_{q\bar{a}} \tag{21d}$$

$$\text{with } \nu_{qi} > \varrho_q/2, i = 0,1,2 \tag{21e}$$

$$\hat{K}_{qi}(t_0) > 0, \ \zeta_q(t_0) = \bar{\zeta}_q > \bar{\epsilon}_q, \gamma_q(t_0) = \bar{\gamma}_q > \epsilon_q \tag{21f}$$

where a and $\bar{a} \in \Omega\backslash a$ denote the active and inactive subsystems respectively; $\nu_{qia}, \bar{\epsilon}_{qa}, \epsilon_{q\bar{a}} \in \mathbb{R}^+$ are user-defined scalars and $\varrho_q \triangleq \frac{\min\{\lambda_{\min}(\Lambda_q), \lambda_{\min}(\Phi_q)\}}{\max\{M_{qq}, 1\}}$ $\forall \in \Omega$ and t_0 is the initial time and from **??** and intial condition **??**, it can be verified that $\exists \underline{\zeta}_q, \underline{\gamma}_q \in \mathbb{R}^+$ such that

$$\begin{aligned} &\hat{K}_{qi}(t) \geq 0, \quad 0 < \underline{\zeta}_q \leq \zeta_q(t) < \bar{\zeta}_q, \\ &\text{and} \quad 0 < \underline{\gamma}_q \leq \gamma_q(t) < \bar{\gamma}_q \quad \forall t \geq t_0. \end{aligned} \tag{22a}$$

C. Manipulator Impedance Control

For control design purpose, the UAM manipulator sub-dynamics (4c) is rearranged as

$$\bar{M}_{\alpha\alpha}\ddot{\alpha} + E_\alpha = \tau_\alpha + \tau_{\alpha ext} \tag{23}$$

where $\bar{M}_{\alpha\alpha}$ is a user-defined constant positive definite matrix and $E_\alpha \triangleq (M_{\alpha\alpha} - \bar{M}_{\alpha\alpha})\ddot{\alpha} + M_{p\alpha}^T\ddot{p} + M_{q\alpha}^T\ddot{q} + C_\alpha\dot{\chi} + g_\alpha + d_\alpha$. The selection of $\bar{M}_{pp}$ would be discussed later (cf. Remark **??**).

Taking the UAM manipulator tracking error as $e_\alpha(t) \triangleq p(t) - p_\alpha(t)$, let us define an error variable as

$$s_\alpha = \dot{e}_\alpha + \Phi_\alpha e_\alpha \tag{24}$$

where $\Phi_\alpha \in \mathbb{R}^{3\times3}$ is a positive definite gain matrix. Multiplying the time derivative of (24) by $\bar{M}_{\alpha\alpha}$ and using (**??**) yields

$$\begin{aligned} \bar{M}_{\alpha\alpha}\dot{s}_\alpha &= \bar{M}_{\alpha\alpha}(\ddot{\alpha} - \ddot{\alpha}_d + \Phi_\alpha\dot{e}_\alpha) \\ &= \tau_\alpha - E_\alpha - \bar{M}_{\alpha\alpha}(\ddot{\alpha}_d - \Phi_\alpha\dot{e}_\alpha), \end{aligned} \tag{25}$$

The control law is proposed as

$$\tau_\alpha = -\Lambda_\alpha s_\alpha + \Delta\tau_\alpha + \bar{M}_{\alpha\alpha}(\ddot{\alpha}_d - \Phi_\alpha\dot{e}_\alpha), \tag{26a}$$

$$\Delta\tau_\alpha = \begin{cases} \rho_\alpha \frac{s_\alpha}{||s_\alpha||} & \text{if } ||s_\alpha|| \geq \varpi_\alpha \\ \rho_\alpha \frac{s_\alpha}{\varpi_\alpha} & \text{if } ||s_\alpha|| < \varpi_\alpha \end{cases} \tag{26b}$$

where Λ_α is a user-defined positive definite gain matrix and $\varpi_\alpha > 0$ is a scalar used to avoid chattering; ρ_α tackles system uncertainties, whose design will be discussed later.

Using Property 2 and the relation (2) we have $||C_\alpha|| \leq ||C|| \leq \bar{c}||\dot{\chi}||$, $||g_\alpha|| \leq ||g|| \leq \bar{g}$ and $||d_\alpha|| \leq ||d|| \leq \bar{d}$. Using these relations, the inequalities $||\ddot{\chi}|| \geq ||\ddot{p}||$, $||\ddot{\chi}|| \geq ||\ddot{q}||$ and $||\ddot{\chi}|| \geq ||\ddot{\alpha}||$, $||\xi|| \geq ||\dot{e}||$, $||\xi|| \geq ||e||$ and substituting $\dot{\chi} = \dot{e} + \dot{\chi}_d$ into (**??**) yields

$$||E_\alpha|| \leq K_{\alpha0}^* + K_{\alpha1}^*||\xi|| + K_{\alpha2}^*||\xi||^2 + K_{\alpha3}^*||\ddot{\chi}|| \tag{27}$$

$$
\begin{aligned}
&K_{\alpha 0}^* = \bar{g} + \bar{d} + \bar{c}||\dot{\chi}_d||^2, \\
&K_{\alpha 1}^* = 2\bar{c}||\dot{\chi}_d||, \ K_{\alpha 2}^* = \bar{c}, \\
&K_{\alpha 3}^* = (||(M_{\alpha\alpha} - \bar{M}_{\alpha\alpha})|| + ||M_{p\alpha}^T|| + ||M_{q\alpha}^T||)
\end{aligned}
$$

are unknown scalars. Based on the upper bound structure in (55), the gain ρ_α in (54b) is designed as

$$\rho_\alpha = \hat{K}_{\alpha 0} + \hat{K}_{\alpha 1}||\xi|| + \hat{K}_{\alpha 2}||\xi||^2 + \hat{K}_{\alpha 3}||\ddot{\chi}|| + \zeta_\alpha + \gamma_\alpha \tag{28}$$

where $\hat{K}_{\alpha i}$ are the estimates of $K_{\alpha i}^*$ $i = 0, 1, 2, 3$, and ζ_α, γ_α are auxiliary gain used for closed-loop stabilization (cf. Remark **??**). The gains $\hat{K}_{\alpha i}$ are adapted via the following laws:

$$\dot{\hat{K}}_{\alpha ia} = ||s_\alpha||||\xi||^i - \nu_{\alpha ia}\hat{K}_{\alpha ia}, \dot{\hat{K}}_{\alpha i\bar{a}} = 0, i = 0, 1, 2 \tag{29a}$$

$$\dot{\hat{K}}_{\alpha 3a} = ||s_\alpha||||\ddot{\chi}|| - \nu_{\alpha 3a}\hat{K}_{\alpha 3a}, \dot{\hat{K}}_{\alpha 3\bar{a}} = 0 \tag{29b}$$

$$\dot{\zeta}_{\alpha a} = -(1 + \hat{K}_{\alpha 3a}||s_\alpha||||\ddot{\chi}||)\zeta_{\alpha a} + \bar{\epsilon}_{\alpha a}, \dot{\zeta}_{\alpha\bar{a}} = 0 \tag{29c}$$

$$\dot{\gamma}_{\alpha a} = 0, \dot{\gamma}_{\alpha\bar{a}} = \left(1 + \frac{\varrho_{\alpha\bar{a}}}{2}\sum_{i=0}^{3}\hat{K}_{\alpha i\bar{a}}^2\right)\gamma_{\alpha\bar{a}} + \epsilon_{\alpha\bar{a}} \tag{29d}$$

$$\text{with } \nu_{\alpha i} > \varrho_\alpha/2, i = 0, 1, 2 \tag{29e}$$

$$\hat{K}_{\alpha i}(t_0) > 0, \ \zeta_\alpha(t_0) = \bar{\zeta}_\alpha > \bar{\epsilon}_\alpha, \gamma_\alpha(t_0) = \bar{\gamma}_\alpha > \epsilon_\alpha \tag{29f}$$

where a and $\bar{a} \in \Omega\backslash a$ denote the active and inactive sub-systems respectively; $\nu_{\alpha ia}, \bar{\epsilon}_{\alpha a}, \epsilon_{\alpha\bar{a}} \in \mathbb{R}^+$ are user-defined scalars and $\varrho_\alpha \triangleq \frac{\min\{\lambda_{\min}(\Lambda_\alpha), \lambda_{\min}(\Phi_\alpha)\}}{\max\{\bar{M}_{\alpha\alpha}1\}}$ $\forall \in \Omega$ and t_0 is the initial time and from **??** and intial condition **??**, it can be verified that $\exists \underline{\zeta}_\alpha, \underline{\gamma}_\alpha \in \mathbb{R}^+$ such that

$$
\begin{aligned}
&\hat{K}_{\alpha i}(t) \geq 0, \quad 0 < \underline{\zeta}_\alpha \leq \zeta_\alpha(t) < \bar{\zeta}_\alpha, \\
&\text{and} \ \ 0 < \underline{\gamma}_\alpha \leq \gamma_\alpha(t) < \bar{\gamma}_\alpha \quad \forall t \geq t_0.
\end{aligned} \tag{30a}
$$

1) Desired Dynamics selection: As mentioned earlier the most popular choice of dynamics to be imposed is the standard second order dynamics similar to a Mass-Spring-Damper system.

$$\bar{M}_{\alpha\alpha}(\ddot{\alpha}(t) - \ddot{\alpha}_d(t)) + B_d(\dot{\alpha}(t) - \dot{\alpha}_d(t)) + K_d(\alpha(t) - \alpha_d(t)) = \tau_{\alpha ext} - \tau_{\alpha d} \tag{31}$$

Where $\bar{M}_{\alpha\alpha}$ represents the desired Inertia matrix to be imposed; B_d represents the desired damping; K_d represents the desired spring constant for the system; r_d represents the desired trajectory;$\tau_{\alpha d} = J(\alpha)^T F_d$ where $J(\alpha)$ is the analytic jacobian matrix and F_d represents the desired contact force.

Let $\alpha(t) - \alpha_d(t) \triangleq e_\alpha(t)$ and $\tau_{\alpha ext} - \tau_{\alpha d} \triangleq e_\tau(t)$, then equation (31) can be rewritten as

$$\bar{M}_{\alpha\alpha}\ddot{e_\alpha}(t) + B_d\dot{e_\alpha}(t) + K_d e_\alpha(t) = e_\tau \tag{32}$$

and subsequently we can write it as

$$\ddot{e_\alpha}(t) + K_1\dot{e_\alpha}(t) + K_2 e_\alpha(t) = K_3 e_\tau \tag{33}$$

where $K_1 = \bar{M}_{\alpha\alpha}^{-1}B_d$; $K_2 = \bar{M}_{\alpha\alpha}^{-1}K_d$; and $K_3 = \bar{M}_{\alpha\alpha}^{-1}$.

2) Control Problem Formulation: Now as the intended dynamics are defined as 33, we define the deviation from the intended Impedance dynamics as ΔI (34)

$$\Delta I \triangleq \ddot{e_\alpha}(t) + K_1\dot{e_\alpha}(t) + K_2 e_\alpha(t) - K_3 e_\tau \tag{34}$$

Our control objective is to reduce the Deviation Error ΔI to zero which will in turn enforce the dynamics (33) and subsequently (31) on the system.

3) Sliding surface selection: Considering the force and position tracking problem we define a sliding variable s as

$$s \triangleq \dot{e_\alpha}(t) + \lambda e_\alpha - g_f(t) \tag{35}$$

where g_f is an unknown function.

Then $\dot{s}$ is given as

$$\dot{s} = \ddot{e_\alpha}(t) + \lambda\dot{e}_\alpha - \dot{g_f}(t) \tag{36}$$

now, from (36) and (34)

$$\Delta I = \dot{s} - \lambda\dot{e_\alpha} + \dot{g_f}(t) + K_1\dot{e_\alpha}(t) + K_2 e_\alpha(t) - K_3 e_\tau \tag{37}$$

$$\Delta I = \dot{s} + (K_1 - \lambda)[\dot{e_\alpha} + \frac{K_2}{(K_1 - \lambda)}e_\alpha(t)] + \dot{g_f}(t) - K_3 e_\tau \tag{38}$$

$$\Delta I = \dot{s} + \alpha[\dot{e_\alpha} + \beta e_\alpha(t)] + \dot{g_f}(t) - K_3 e_\tau \tag{39}$$

where $\alpha \triangleq K_1 - \lambda$ and $\beta \triangleq \frac{K_2}{(K_1 - \lambda)}$

Let $\beta = \lambda$, then λ can be solved in terms of K_1 and K_2 using the following equation

$$\lambda(K_1 - \lambda) = K_2 \tag{40}$$

So, equation (39) becomes

$$\Delta I = \dot{s} + \alpha[\dot{e_\alpha} + \lambda e_\alpha(t)] + \dot{g_f}(t) - K_3 e_\tau \tag{41}$$

using (41) and (35)

$$\Delta I = \dot{s} + \alpha[s + g_f(t)] + \dot{g_f}(t) - K_3 e_\tau \tag{42}$$

rearranging the terms,

$$\Delta I = \dot{s} + \alpha s + \dot{g_f}(t) + \alpha g_f(t) - K_3 e_\tau \tag{43}$$

let g_f satisfy the equation

$$\dot{g_f}(t) + \alpha g_f(t) - K_3 e_\tau = 0 \tag{44}$$

taking the Laplace transform we can see that

$$G_f(\boldsymbol{S}) = \frac{K_3 e_\tau(\boldsymbol{S})}{\boldsymbol{S} + \alpha} \tag{45}$$

where $\boldsymbol{S}$ is the Laplace Variable. This is the standard structure of a Low-Pass filter, and we can define $g_f(t)$ to be a Low-Pass filtered force error signal and consider it as known if e_τ is known.

Hence, the sliding variable s is now fully known.

Now, from (43) and (**??**)

$$\Delta I = \dot{s} + \alpha s \tag{46}$$

Sub-Control Problem: Now, if we can force s and $\dot{s}$ to go to zero using any control strategy then we can ensure that the deviation error ΔI goes to zero, enforcing the desired impedance dynamics (33) on the system.

4) Controller Derivation and Adaptation: For control design purpose, the UAM manipulator sub-dynamics (4c) is rearranged as

$$\bar{M}_{\alpha\alpha}\ddot{\alpha} + E_\alpha = \tau_\alpha + \tau_{\alpha ext} \tag{47}$$

where $\bar{M}_{\alpha\alpha}$ is a user-defined constant positive definite matrix and

$$E_\alpha \triangleq (M_{\alpha\alpha} - \bar{M}_{\alpha\alpha})\ddot{\alpha} + M_{p\alpha}^T\ddot{p} + M_{q\alpha}^T\ddot{q} + C_\alpha\dot{\chi} + g_\alpha + d_\alpha \tag{48}$$

The selection of $\bar{M}_{\alpha\alpha}$ would be discussed later (cf. Remark **??**).

Now, from(36)

$$\dot{s}_\alpha = \ddot{\alpha}(t) - \ddot{\alpha_d}(t) + \lambda\dot{e}_\alpha - \dot{g_f}(t) \tag{49}$$

Multiplying with $\bar{M}_{\alpha\alpha}$ on both we get

$$\bar{M}_{\alpha\alpha}\dot{s}_\alpha = \bar{M}_{\alpha\alpha}\ddot{\alpha}(t) - \bar{M}_{\alpha\alpha}[\ddot{\alpha_d} - \lambda\dot{e_\alpha} + \dot{g_f}] \tag{50}$$

Then from equation (47) we know,

$$\bar{M}_{\alpha\alpha}\ddot{\alpha} = -E_\alpha + \tau_\alpha + \tau_{\alpha ext} \tag{51}$$

Combining this with (50)

$$\bar{M}_{\alpha\alpha}\dot{s}_\alpha = -E_\alpha + \tau_\alpha + \tau_{\alpha ext} - \bar{M}_{\alpha\alpha}[\ddot{\alpha_d} - \lambda\dot{e}_\alpha + \dot{g_f}] \tag{52}$$

Substituting $\dot{g_f}$ from (44) into (50) we get

$$\bar{M}_{\alpha\alpha}\dot{s}_\alpha = -E_\alpha + \tau_\alpha + \tau_{\alpha ext} - \bar{M}_{\alpha\alpha}[\ddot{\alpha_d} - \lambda\dot{e}_\alpha - \alpha g_f(t) + K_3 e_\tau] \tag{53}$$

The control law is proposed as

$$\tau_\alpha = -\Lambda_\alpha s_\alpha + \Delta\tau_\alpha - \tau_{\alpha ext} + \bar{M}_{\alpha\alpha}[\ddot{\alpha_d} - \lambda\dot{e}_\alpha - \alpha g_f(t) + K_3 e_\tau], \tag{54a}$$

$$\Delta\tau_\alpha = \begin{cases} \rho_\alpha \frac{s_\alpha}{||s_\alpha||} & \text{if } ||s_\alpha|| \geq \varpi_\alpha \\ \rho_\alpha \frac{s_\alpha}{\varpi_\alpha} & \text{if } ||s_\alpha|| < \varpi_\alpha \end{cases} \tag{54b}$$

where Λ_α is a user-defined positive definite gain matrix and $\varpi_\alpha > 0$ is a scalar used to avoid chattering; ρ_α tackles system uncertainties, whose design will be discussed later.

Using Property 2 and the relation (2) we have $||C_\alpha|| \leq ||C|| \leq \bar{c}||\dot{\chi}||$, $||g_\alpha|| \leq ||g|| \leq \bar{g}$ and $||d_\alpha|| \leq ||d|| \leq \bar{d}$. Using these relations, the inequalities $||\ddot{\chi}|| \geq ||\ddot{p}||$, $||\ddot{\chi}|| \geq ||\ddot{q}||$ and $||\ddot{\chi}|| \geq ||\ddot{\alpha}||$, $||\xi|| \geq ||\dot{e}||$, $||\xi|| \geq ||e||$ and substituting $\dot{\chi} = \dot{e} + \dot{\chi}_d$ into (48) yields the upper bound structure for E_α as

$$||E_\alpha|| \leq K_{\alpha 0}^* + K_{\alpha 1}^*||\xi|| + K_{\alpha 2}^*||\xi||^2 + K_{\alpha 3}^*||\ddot{\chi}|| \tag{55}$$

$$\begin{aligned} &K_{\alpha 0}^* = \bar{g} + \bar{d} + \bar{c}||\dot{\chi}_d||^2, \\ &K_{\alpha 1}^* = 2\bar{c}||\dot{\chi}_d||, \ K_{\alpha 2}^* = \bar{c}, \\ &K_{\alpha 3}^* = (||(M_{\alpha\alpha} - \bar{M}_{\alpha\alpha})|| + ||M_{p\alpha}^T|| + ||M_{q\alpha}^T||) \end{aligned}$$

are unknown scalars. Based on the upper bound structure in (55), the gain ρ_α in (54b) is designed as

$$\rho_\alpha = \hat{K}_{\alpha 0} + \hat{K}_{\alpha 1}||\xi|| + \hat{K}_{\alpha 2}||\xi||^2 + \hat{K}_{\alpha 3}||\ddot{\chi}|| + \zeta_\alpha + \gamma_\alpha \tag{56}$$

where $\hat{K}_{\alpha i}$ are the estimates of $K_{\alpha i}^*$ $i = 0, 1, 2, 3$, and ζ_α, γ_α are auxiliary gain used for closed-loop stabilization (cf. Remark **??**). The gains $\hat{K}_{\alpha i}$ are adapted via the following laws:

$$\dot{\hat{K}}_{\alpha i a} = ||s_\alpha||||\xi||^i - \nu_{\alpha i a}\hat{K}_{\alpha i a}, \dot{\hat{K}}_{\alpha i \bar{a}} = 0, i = 0, 1, 2 \tag{57a}$$

$$\dot{\hat{K}}_{\alpha 3 a} = ||s_\alpha||||\ddot{\chi}|| - \nu_{\alpha 3 a}\hat{K}_{\alpha 3 a}, \dot{\hat{K}}_{\alpha 3 \bar{a}} = 0 \tag{57b}$$

$$\dot{\zeta}_{\alpha a} = -(1 + \hat{K}_{\alpha 3 a}||s_\alpha||||\ddot{\chi}||)\zeta_{\alpha a} + \bar{\epsilon}_{\alpha a}, \dot{\zeta}_{\alpha \bar{a}} = 0 \tag{57c}$$

$$\dot{\gamma}_{\alpha a} = 0, \dot{\gamma}_{\alpha \bar{a}} = \left(1 + \frac{\varrho_{\alpha \bar{a}}}{2}\sum_{i=0}^{3}\hat{K}_{\alpha i \bar{a}}^2\right)\gamma_{\alpha \bar{a}} + \epsilon_{\alpha \bar{a}} \tag{57d}$$

$$\text{with } \nu_{\alpha i} > \varrho_\alpha/2, i = 0, 1, 2 \tag{57e}$$

$$\hat{K}_{\alpha i}(t_0) > 0, \ \zeta_\alpha(t_0) = \bar{\zeta}_\alpha > \bar{\epsilon}_\alpha, \gamma_\alpha(t_0) = \bar{\gamma}_\alpha > \epsilon_\alpha \tag{57f}$$

where a and $\bar{a} \in \Omega\backslash a$ denote the active and inactive sub-systems respectively; $\nu_{\alpha i a}, \bar{\epsilon}_{\alpha a}, \epsilon_{\alpha \bar{a}} \in \mathbb{R}^+$ are user-defined scalars and $\varrho_\alpha \triangleq \frac{\min\{\lambda_{\min}(\Lambda_\alpha), \lambda_{\min}(\Phi_\alpha)\}}{\max\{M_{\alpha\alpha}1\}}$ $\forall \in \Omega$ and t_0 is the initial time and from **??** and intial condition **??**, it can be verified that $\exists \underline{\zeta}_\alpha, \underline{\gamma}_\alpha \in \mathbb{R}^+$ such that

$$\begin{aligned} &\hat{K}_{\alpha i}(t) \geq 0, \quad 0 < \underline{\zeta}_\alpha \leq \zeta_\alpha(t) < \bar{\zeta}_\alpha, \\ &\text{and} \quad 0 < \underline{\gamma}_\alpha \leq \gamma_\alpha(t) < \bar{\gamma}_\alpha \quad \forall t \geq t_0. \end{aligned} \tag{58a}$$