Automata & Language Theory (PGU-14022)
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Homework#1

1. Show that for all sets S and T, $\bar{S} - T = \bar{S} \cap \bar{T}$.
2. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.
3. Give conditions on S_1 and S_2 necessary and sufficient to ensure that

$$S_1 = (S_1 \cup S_2) - S_2.$$

4. Let $G = (V, E)$ be any graph. Prove the following claim: If there is any walk between $v_i \in V$ and $v_j \in V$, then there must be a simple path of length no larger than $|V| - 1$ between these two vertices.
5. Show that

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

6. Show that $2 - \sqrt{2}$ is irrational.
7. Show that every positive integer can be expressed as the product of prime numbers.
8. Prove that the set of all prime numbers is infinite.
9. Use induction on n to show that $|u^n| = n|u|$ for all strings u and all n.
10. Show that $(L^*)^* = L^*$ for all languages.
11. Find grammars for $\Sigma = \{a, b\}$ that generate the sets of
    a. all strings with exactly two a's.
    b. all strings with at least two a's.
    c. all strings with no more than three a's.
    d. all strings with at least three a's.
    e. all strings that start with a and end with b.
    f. all strings with an even number of b's.

    In each case, give convincing arguments that the grammar you give does indeed generate the indicated language.
12. Show that the grammars

$$S \to aSb|ab|\lambda$$

and

$$S \to aaSbb|aSb|ab|\lambda$$

are equivalent.

13. For $\Sigma = \{a, b\}$, construct dfa's that accept the sets consisting of

    (a) all strings of even length.
    (b) all strings of length greater than 5.
    (c) all strings with an even number of a's.
    (d) all strings with an even number of a's and an odd number of b's.

14. With $\Sigma = \{a, b\}$, give a dfa for $L = \{w_1 a w_2 : |w_1| \geq 3, |w_2| \leq 4\}$.

15. Consider the set of strings on {0, 1} defined by the requirements below. For each, construct an accepting dfa.

    a. Every 00 is followed immediately by a 1. For example, the strings 101, 0010, 0010011001 are in the language, but 0001 and 00100 are not.
    b. All strings that contain the substring 000, but not 0000.
    c. The leftmost symbol differs from the rightmost one.
    d. Every substring of four symbols has, at most, two 0's. For example, 001110 and 011001 are in the language, but 10010 is not because one of its substrings, 0010, contains three zeros.
    e. All strings of length five or more in which the third symbol from the right end is different from the leftmost symbol.
    f. All strings in which the leftmost two symbols and the rightmost two symbols are identical.
    g. All strings of length four or greater in which the leftmost two symbols are the same, but different from the rightmost symbol.

16. Let GM be the transition graph for some dfa M. Prove the following:

    a. If L (M) is infinite, then GM must have at least one cycle for which there is a path from the initial vertex to some vertex in the cycle and a path from some vertex in the cycle to some final vertex.
    b. If L (M) is finite, then no such cycle exists.

Good Luck