PERCOLATION THEORY AND ITS APPLICATION

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CONTENTS

INTRODUCTION

Percolation comes from a Latin verb percolare meaning "to put through a sieve". Percolation as a mathematical theory was first introduced by Broadbent and Hammersley in 1957, as a stochastic way of modelling the flow of a fluid or gas through a porous medium of small channels which may or may not let gas or fluid pass. It is one of the simplest models exhibiting a phase transition, and the occurrence of a critical phenomenon is central to the appeal of percolation. percolation is used to model the fingering and spreading of oil in water, to estimate whether one can build non-defective integrated circuits, to model the spread of infections and forest fires. From a mathematical point of view percolation is attractive because it exhibits relations between probabilistic and algebraic/topological properties of graphs.

The classic motivating problem behind percolation theory examines the flow of water over a porous stone: under what conditions will the water travel successfully through the channels of the stone and reach the bottom? We consider the network of passages within the stone; some passages will be wide enough for the water to pass through, while others will be too narrow. If a series of sufficiently wide passages exists between the top of the stone and the bottom, the water will be able to pass through. We can formalize this problem by viewing the interior of the stone as a graph in which vertices correspond to points within the stone and edges correspond to wide passages between points. Percolation theory seeks to determine if and when a path can form between points of this graph, i.e. when a traversable passage exists through the stone.

The percolation abstraction can be used to study the class of what might be termed “connectivity problems”—problems that hinge on the presence of a path through a random medium. Representative problems include the study of the pas-sage of neurotransmitters between neurons, the calculation of the electrical resistance of a mixture of two metals, and the spread of a pathogen through a population.

What is Percolation?

- In statistical physics, chemistry and materials science, percolation (from Latin *percolare*, "to filter" or "trickle through") refers to the movement and filtering of fluids through porous materials.

  Many natural substances like rocks and soil (e.g. aquifers, petroleum reservoirs), zeolites, biological tissues (e.g. bones, wood, cork), and man-made materials such as cements and ceramics can be considered as porous medium.

- **Some examples of percolation in real life:**

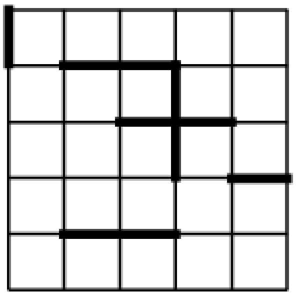
1. Coffee percolation, where the solvent is water, the permeable substance is the coffee grounds, and the soluble constituents are the chemical compounds that give coffee its color, taste, and aroma. In 1880, Hanson Goodrich invented the coffee percolator.
2. Spreading of various epidemic diseases like, POLIO, Spanish flu, Influenza etc.
3. Movement of weathered material down on a slope under the earth's surface.
4. Propagation of fire in a forest can be modelled using percolation theory.
5. Electrical resistance in a mixture of two media.

Classification of Percolation

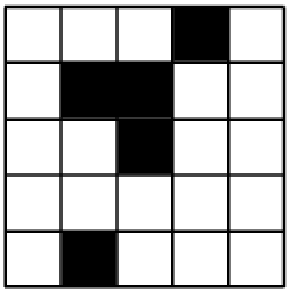
Here we only think of 2-dimensional integer lattice. Mainly there are two types of percolation model in this lattice.

1. Bond Percolation model
2. Site Percolation model

A lattice is a partially ordered set (also called a poset) in which any two elements have a unique supremum (an elements least upper bound; called their join) and an Infrimum (greatest lower bound; called their meet).



bond percolation



site percolation

<u>Bond</u> <u>Percolation:</u>

In discrete percolation theory, bond percolation is a percolation model on a regular L^2 lattice in 2-dimensional Euclidean space which considers the lattice **graph edges** as the relevant entities (left figure). We can say there exists an edge between two vertices of the lattice L if the distance between these two vertices is unity.

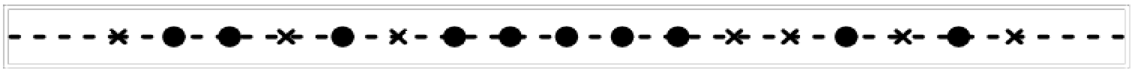
<u>Site</u> <u>Percolation:</u>

Site percolation is a percolation model on a regular L^2 lattice in 2-dimensional Euclidean space which considers the lattice **vertices** as the relevant entities (right figure).

Every bond model may be reformulated as a site model on a different lattice but not vice versa.

PERCOLATION IN 1D:

Imagine a 1d lattice with an infinite number of sites of equal spacing arranged in a line. Each site has a probability p of being occupied, and consequently 1 −p of being empty (not occupied).
These are the only two states possible.

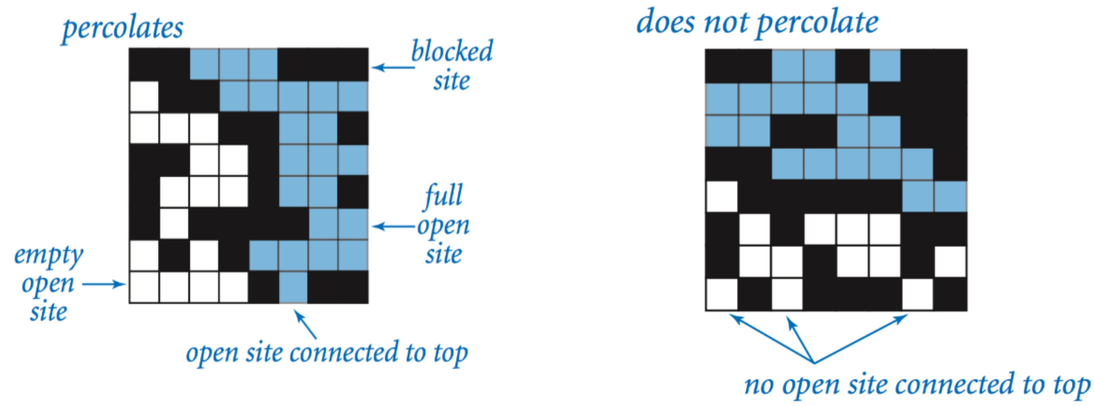


In this figure sites are occupied with probability p. The crosses are empty sites, the solid circles are occupied sites. In the part of the infinite 1d lattice shown above, there is one cluster of size 5, one cluster of size 2, and three clusters of size 1.A percolating cluster in 1d spans from −∞ to +∞. Clearly, in 1d this can only be achieved if all sites are occupied that is, the percolation threshold p_c = 1, as a single empty site would prevent a cluster to percolate.

PERCOLATION IN 2D:

Now we try to model percolation concept in a 2-dimensional lattice(grid).

We think for $\mathbf{Z}^2$ lattice.



We place Bernoulli coins to each of block of the lattice and toss the coins such a way that if head appears in that toss then we say that this particular site is vacant otherwise it is occupied.

In the left side grid, if we pour some water in the top of the grid then water will reach to the bottom of the grid because full open site of the top of the row is connected to the bottom row. So, we can say that this particular system will percolate. While in another grid top full open site are not connected to the bottom so the system will not percolate.

Preliminaries:

Let P(A) denote the probability for an event A and P(A1∩A2) the joint probability for event A1 and A2.

Definition1:

Two events A1 and A2 are independent ⇔P(A1∩A2) =P(A1) P(A2).

Definition2:

More generally, we define $n \geq 3$ events A1, A2, ..., An to be mutually independent if P(A1∩A2∩⋯∩An) =P(A1) P(A2) ⋯P(An) and if any subcollection containing at least two but fewer than n events are mutually independent.

Let each site in a lattice be vacant at random with probability p, that is, each site is vacant (with probability p) or occupied (with probability 1−p) independent of the status (empty or occupied) of any of the other sites in the lattice. We call p as the vacant probability of the concentration.

PERCOLATION PROBABILITY:

One of the principal objects of study is the percolation probability.

Firstly, we define the percolation event as;

$$A= \{\#C\ (0) \leftrightarrow \text{infinite}\}$$

i.e., the number of open sites which is connected to the origin's Cluster is infinite or not?

i.e., Is there exists several occupied sites so that the system will percolate?

Now consider the percolation probability as;

$$\theta(p) = P_p(A)=Pp\ (|C| = \infty)$$

where C = C (0) is, as usual, the open cluster at the origin.

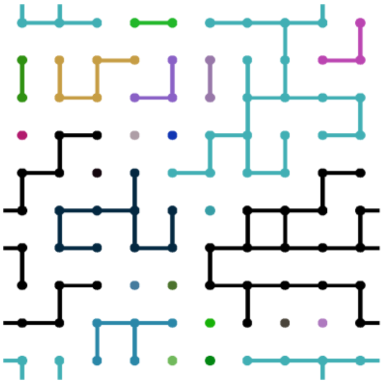
Here the event A is increasing, and therefore θ is non-decreasing.

This probability depends on p which is the parameter of this system.

Here I use p as the vacant probability.

PERCOLATION CLUSTER:

If $k \in Z^2$ is any lattice point on Z^2 then $C(k)$ denotes the cluster of k which is the set of all lattice point connected to k.



Here each of colours represents different cluster in Z^2 lattice. we can say, one lattice point is connected to another lattice point if there exists open path consisting of open edges.

Site Percolation theory deals with the numbers and properties of the clusters formed when sites are occupied with probability p.

PERCOLATION THRESHOLD:

There exists $p_c \in (0, 1)$ such that,

$$\theta p = 0; \text{ if } p > p_c$$

$$> 0; \text{ if } p < p_c$$

That means we find a critical probability (p_c) such that, before that p_c the system will always percolate but after just exceeding p_c the system changes its behaviour from percolating to non-percolating state. This is a radical change of state of percolation system.

Objectives:

1. Here my aim is to find the critical value (p_c). But the problem is we cannot find p_c in mathematical way. In 2d integer lattice, if we increase the size of the grid then we can get an idea about p_c. For this I use monte carlo simulation method in r programming language.
2. How it is applicable in various kinds of reallife phenomenon?

SIMULATION RESULT:

I calculate the percolation probability for different vacant probability to get an idea about the percolation threshold. The following tables shows how percolation probability changes for different vacant probability.

p	Θ_p
0.5	0.056
0.51	0.023
0.52	0.019
0.53	0.012
0.54	0.006
0.55	0.005
0.56	0.002
0.57	0.001
0.58	0.001
0.59	0
0.6	0

TABLE1(N=20, S=1000)

p	Θ_p
0.45	0.061
0.46	0.052
0.47	0.015
0.48	0.007
0.49	0.003
0.5	0.001
0.51	0
0.52	0

TABLE2(N=50, S=1000)

p	Θ_p
0.45	0.007
0.46	0.005
0.47	0.003
0.48	0.002
0.49	0.001
0.5	0.001
0.51	0
0.52	0

TABLE3(N=100, S=1000)

p	Θ_p
0.45	0.0625
0.46	0.0567
0.47	0.0428
0.48	0.0183
0.49	0.0065
0.5	0.0018
0.51	0
0.52	0

TABLE4(N=100, S=10,000)

p	Θ_p
0.45	0.014
0.46	0.009
0.47	0.005
0.48	0.003
0.49	0.002
0.5	0.001
0.51	0
0.52	0

TABLE5(N=200, S=1000)

Here from this simulation result we can see that when we take grid size N=20 with repetition 1000 times then the percolation probability becomes 0 when p>0.58. But when increase grid size as 50,100,200 respectively then the percolation probability becomes 0 after p>0.5, where p is vacant probability of site. So, we get percolation threshold p_c=0.5

APPLICATIONS:

1. **Oil fields**

Percolation theory can be used as a simple idealized model for predicting the distribution of the oil or gas inside porous rocks or oil reservoirs. The network considered in this case is formed from the pores filled with oil in a rock. The pores can be connected, which would correspond to open bonds in the percolation model, or can be isolated. To the probability with which a site is occupied in the percolation problem it corresponds the porosity or the average concentration of oil in the rock. In order to obtain a good oil production from a well it is desirable to position it in an area with high porosity. In order to predict the amount of oil that will be produced one needs to estimate the porosity of the rock in the area where the oil reservoir is assumed to be located. To obtain such an estimation, the porosity of rock samples is determined. The difficulty comes from the fact that the samples are usually rock logs with the diameter in the order of centimeters, so in order to obtain the final result one needs to extrapolate the measurement to the scale of the reservoir, which is at least in the order of kilometers. The fundamental question is whether such an extrapolation is legitimate. Percolation theory predicts that this approach is valid when the probability p in the percolation problem is appreciably higher than the percolation threshold p_c. On the other hand, if the probability is close to the threshold, even though there might exist an extended cluster that could be quite ramified containing a lot of ‘holes’, it is possible that the sample will contain other clusters that unfortunately cannot be reached. In this case the decision to invest would prove to be not profitable.

2. In traffic

percolation theory has been applied to study traffic in a city. The quality of the global traffic in a city at a given time can be characterized by a single parameter, the percolation critical threshold. The critical threshold represents the velocity below which one can travel in a large fraction of the city network. Above this threshold the city network break into clusters of many sizes and one can travel within relatively small neighbourhoods. Critical exponents characterizing the cluster size distribution of good traffic are similar to those of percolation theory. It is also found that during rush hours the traffic network can have several metastable states of different network sizes and the alternate between these states.

3. Propagation of fire in a forest

The propagation of a fire through a forest can be modeled using percolation. In the simplest model, the forest is represented by a lattice, whose squares are either occupied by a tree or not. Each square can be in one of three states: (a) non burnt tree, (b) a burning tree, or (c) no-tree (either already burnt or there was never a tree there). The spread of fire can be simulated as follows: The ignition point can be set or is chosen randomly to model randomness as is the case of fire ignition by a lightning. Once a tree is burning, there is a probability *p* that fire will spread to a neighboring tree. As soon as the fire spreads to another tree, the state of that tree changes to "burning" and that tree can now propagate the fire to another neighboring tree. The spread of fire is simulated by marking certain sites in the lattice as burning and following an iterative procedure that checks which other trees or sites are going to burn. Varying the conditions that are used to decide whether a tree propagates the fire to its neighbors (probability *p*) one can account for static attributes, such as fuel type, elevation and slope, as well as dynamic attributes, such as the direction of the wind or the humidity of the air and the temperature. A cluster in this case is a set of trees corresponding to trees that burnt or are burning. One may be interested in determining the probability that a certain point is reached by the burning cluster.

Here I want to estimate the total number of burnt trees or total burnt area using percolation theory. The percolation theory, tell us that there is a relevant critical exponent called γ with the following expression,

$$S \sim (p - p_c)^{-\gamma}$$

where S is the size of the cluster or cluster mass that in our case is the total number of burned areas. The theoretical value of this exponent is $\gamma = 43/18 \sim 2.38$

CONCLUSIONS

From this project I get the percolation threshold is 0.5

Actually, I try to predict the burned area of forest fires, of Montesinho natural park in the northeast region of Portugal using this model. I collect the data from internet source. The source of the data is given here

https://archive.ics.uci.edu/ml/datasets/forest+fires

Firstly, from this data I estimate the value of p. But after using the model

$$S \sim (p - p_c)^{-\gamma}$$

I cannot determine the value of constant. I also plot the data and fit a straight line to know there exists a linear relationship or not so that I can get the constant. But there is no linear relationship. So, this model cannot predict the total burned area accurately. This portion is left out for future study.

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APPENDIX:

```
EMPTY    = 0
OCCUPIED = 1
FLOW     = 2
S=1000
N=20
p=0.5
x=replicate(S,{
# creating a N*N grid
create.grid <- function(N, p) {
grid = matrix(rbinom(N**2, 1, p), nrow = N)
attributes(grid)$p = p
return(grid)
}
g1 = create.grid(N, p)
# Cheacking for flow
flow <- function(g1, i = NA, j = NA) {
# -> Cycle through cells in top row
if (is.na(i) || is.na(j)) {
for (j in 1:ncol(g1)) {
g1 = flow(g1, 1, j)
}
return(g1)
}
# -> Check specific cell
if (i < 1 || i > nrow(g1) || j < 1 || j > ncol(g1)) return(g1)
```

```
if (g1[i,j] == OCCUPIED || g1[i,j] == FLOW) return(g1)
g1[i,j] = FLOW
g1 = flow(g1, i+1, j)       # down
g1 = flow(g1, i-1, j)       # up
g1 = flow(g1, i, j+1)       # right
g1 = flow(g1, i, j-1)       # left
g1
}
# Check whether flow reaches to bottom of the lattice.
percolates <- function(g1) {
g1 <- flow(g1)
for (j in 1:ncol(g1)) {
if (g1[nrow(g1), j] == FLOW) return(TRUE)
}
return(FALSE)
}
percolates(g1)
})
per=(length(x[x==TRUE]))/S
per
```