Grandi's Infinite Series under the Lens of Dirac Delta Function

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Abstract: We provide simple proofs for new identity corresponding to the limiting value of the partial sum (LVPS) of Grandi's series by using the properties of Dirac Delta function and the completeness relation. The different set of orthogonal functions which have been considered in the completeness relation for giving different proofs for the same identity corresponds to the Eigen functions for a quantum particle moving in a ring, 1 dimensional box of finite length and sphere of finite radius. The LVPS is found to be 1/2, the same value was also obtained incorrectly by Leibnitz, Grandi and others but for the complete sum of the Grandi's series which is shown here to be indeterminate.

Introduction

A series is said to be indeterminant if infinite sequence of the partial sums of the series does not have a limit. One of the first mathematicians to study geometric series was the Greek philospher Aristotle [1-2], who stated in the 300s BC that "the sum of a series of infinitely many addends (potentially considered) can be a finite quantity." Archimedes reconsidered the concept of geometric series in his work [3], "The Quadrature of the Parabola." Most of the results for validity of summable indeteminant series were discussed with great vigor during 17th and 18th century. Many great mathematicians namely Grandi, Leibnitz and others were frequently in doubt about their validity of evaluation procedure. Throughout these periods, the controversies surrounded mainly due to improper way of evaluating the sum. One of the most enduring and controversial discussions regarding the indeteminant series occurred when the infinite series

$$\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1........ \quad (1)$$

was suggested [4-5] by Guido Grandi(1671-1742), Italian philosopher, mathematician and engineer toward the end of the 17th century. Grandi's series, appears in many areas of physics, namely in the viewpoint of quantized fermion fields [6] (for example chiral bag model) and also for bosons such as in the Casimir effect [7-8]. Leibniz studied [9-10] Grandi's series in some letters (1713-1716) to Christian Wolf (1678-1754). He introduced the "probabilistic argument" that impressed Johann and Daniel Bernoulli very much. In order to evaluate the series, Leibniz used the logic that if one could "stop" the series 1−1+1−1+..., at the $n-$th place, it is possible to obtain 0 or 1 with the same "probability" depending on whether n is odd or even. So the most probable value is the average of 0 and 1 that is ½. Grandi had obtained the same value by setting the value of $x=1$ in the following series:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... \quad (2)$$

It is well known that the above expansion is valid for $|x|<1$ therefore the procedure adopted by Grandi for the evaluation of the series was incorrect. It is to be noted that the same result can also be obtained for Grandi's indeterminate geometric series by using the algebraic methods that are used to evaluate convergent geometric series to obtain the value of the series:

$$L = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1....... = L - (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1....) = 1 - L$$

resulting in $L = 1/2$. This procedure cannot be accepted because the result has been obtained based on statement that $\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1.......$ is finite (convergent) which is false as the series is divergent. So the associative law is not applicable for the same.

Our objective here is to prove using the property of Dirac Delta function and the completeness relation [11] that the complete sum of Grandi's series does not exist but the partial sum of the same series is convergent and its limiting value is 1/2.

We assume that the arbitrary function $f(x)$ may be expanded in terms of the complete set of orthogonal functions as

$$f(x) = \sum_{n=n_0}^{\infty} c_n \Psi_n(x)$$

(3)

The uniqueness of our decomposition requires orthonormality of our basis function:

$$\int_a^b dx \Psi_n^*(x) \Psi_m(x) = \delta_{nm} \tag{4}$$

and completeness of our basis implies that

$$\sum_{n=n_0}^{\infty} \Psi_n^*(x') \Psi_n(x) = \delta(x - x') \tag{5}$$

where n_0 is the starting index having a value of $0,1$ or $-\infty$ depending upon the index assigned to the eigen values. Here δ corresponds to Dirac Delta function.

Orthogonal Functions: Quantum Particle in a Ring

We first consider the case of a quantum particle in a one-dimensional ring. We take the potential energy to be zero for a particle at a distance r from the center and to be infinitely high at all other positions. The Schrödinger equation:

$$-\frac{\hbar}{2mr^2} \frac{\partial^2 \psi}{\partial \phi^2} = E\psi \tag{6}$$

Where E and ψ respectively represents eigen values and eigen functions. The eigen functions will be functions of only angular variable ϕ and periodic in ϕ with a period 2π and that they may be normalized leading to the following conditions

$$\int_0^{2\pi} d\phi \Psi_m^*(\phi) \Psi_m(\phi) = 1 \tag{7}$$

and

$$\Psi_m(\phi) = \Psi_m(\phi + 2\pi) \tag{8}$$

Under these conditions, the solution to the Schrödinger equation (Eq.6) is given by

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp[im\phi] \tag{9}$$

where $m = 0, \pm 1, \pm 2 .... \pm \infty$. Since $\psi_m(\phi)$ form a complete orthonormal system of function in Hilbert space, then any arbitrary function in this space can be expanded in terms of orthonormal functions.

To decide the lower and upper limits of the sum in the completeness relation defined in Eq.5 corresponding to the orthonormal functions $\{\psi_m(\phi)\}$, we consider the following n-th partial sum

$$S_m\left(\phi-\phi^{'}\right) = \sum_{k=-m}^{+m} \frac{\exp\left[ik\phi\right]}{\sqrt{2\pi}} \frac{\exp\left[-ik\phi^{'}\right]}{\sqrt{2\pi}} \tag{10}$$

The above series can, in fact, be explicitly evaluated by using the formula for the sum of a geometric series, and the resulting expression can be written as

$$S_m\left(\phi-\phi^{'}\right) = \frac{Sin\left[(m+\frac{1}{2})\left(\phi-\phi^{'}\right)\right]}{2\pi Sin\left[\frac{\left(\phi-\phi^{'}\right)}{2}\right]} \tag{11}$$

Partial sum $S_m\left(\phi-\phi^{'}\right)$ is plotted in Fig.1 for two different values of m. It is found that the plot becomes taller and thinner at the centre, converging to a delta function with increasing the values of m

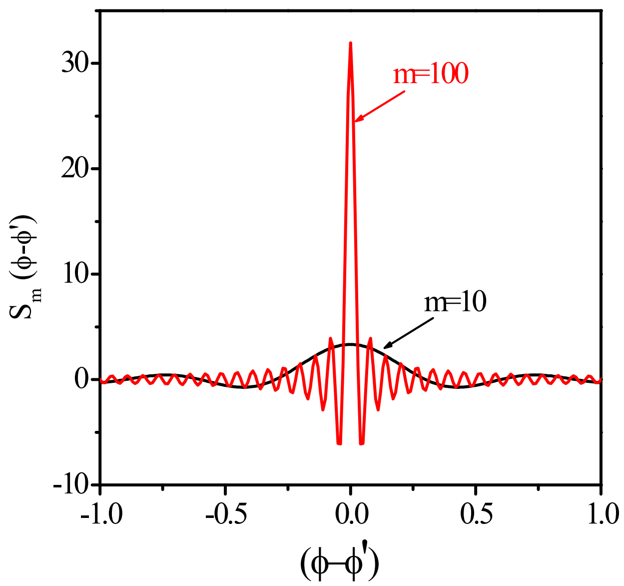

Fig.1 : Plot of $S_m\left(\phi-\phi^{'}\right)$ vs $\left(\phi-\phi^{'}\right)$ for different values of

It can be shown for $m\left(\phi-\phi^{'}\right) << 1$, by using L'Hospital rule, that

$$\lim_{(\phi-\phi')\to 0} S_m\left(\phi-\phi'\right) = \lim_{(\phi-\phi')\to 0} \frac{Sin\left[(m+\frac{1}{2})\left(\phi-\phi'\right)\right]}{2\pi Sin\left[\frac{\left(\phi-\phi'\right)}{2}\right]} = \frac{(m+\frac{1}{2})}{\pi} \to \infty \text{ as } m \to \infty \tag{12}$$

However, away from the centre, the partial sum $S_m\left(\phi-\phi'\right)$ does not go to zero. Rather, it oscillates more and more rapidly. As $m \to \infty$ the oscillations become "extremely fast", with the amplitudes nearly canceling each other out, and the net effect being approaching zero away from the centre. However, at $\left(\phi-\phi'\right) \to 0$, oscillation is minimum(nearly zero). So the partial sum in the limit $m \to \infty$ becomes the delta function, i.e.

$$\delta(\phi-\phi') = \lim m \to \infty \sum_{k=-m}^{+m} \frac{\exp\left[ik\left(\phi-\phi'\right)\right]}{2\pi}. \tag{13}$$

The above equation can also be written as

$$\frac{1}{2} + \delta(\phi-\phi') = \lim_{m\to\infty} \sum_{k=0}^{k=+m} \cos k(\phi-\phi'). \tag{14}$$

By setting the values

$$\phi-\phi' = \pi, \frac{\pi}{2}, \frac{3\pi}{2} ..... \tag{15}$$

into the Eq.12 we arrive at the identity

$$\lim_{m\to\infty} \sum_{k=0}^{k=+m} (-1)^k = \frac{1}{2} \tag{16}$$

The above identity is valid as long as it satisfies the following limit value:

$$\lim_{m\to\infty} \lim_{x\to 0} \frac{Sin\left(mx\right)}{Sin\left(x\right)} = m \tag{17}$$

We find that the limiting value of the partial sum of Grandi's series converges to the finite value 1/2 (Eq.16). In order to prove that the complete sum of Grandi's series is indeterminate, we first combine Eq.9 and Eq.11, then obtain

$$\sum_{k=0}^{m} \cos k(\phi-\phi') = \frac{Sin\left[(m+\frac{1}{2})\left(\phi-\phi'\right)\right]}{4\pi Sin\left[\frac{\left(\phi-\phi'\right)}{2}\right]} + \frac{1}{2} \tag{18}$$

Now setting $m = \infty$ and $\phi - \phi^{'} = \pi$ into the Eq.(18), we get the identity

$$\sum_{k=0}^{\infty}(-1)^k = \frac{Sin\infty}{4\pi} + \frac{1}{2} \tag{19}$$

It is interesting to note that the series which appears in the left hand of Eq.19 corresponds to the complete sum of Grandi's infinite series which is indeterminate due to presence of $Sin\infty$ in the right hand side of Eq.19.

Orthogonal Functions: Quantum particle in a one dimensional box

Our next objective is to give another proof for the Eq.16 by using different complete set of orthogonal functions. To construct different set of orthogonal functions, we consider the wave function of a quantum particle in a one dimensional box of finite length a. The time-independent Schrödinger equation for a particle of mass m moving in one dimension box with energy E is given by

$$-\frac{\hbar}{2m}\frac{d^2\psi}{dx^2} = E\psi \tag{20}$$

where $\hbar$ is Dirac constant, and ψ is the Schrodinger wave function. According to the boundary conditions, the probability of finding the particle at $x=0$ or $x=a$ (length of the box) is zero. Considering the boundary conditions and total probability of finding the particle inside the box is unity, we obtain the normalized wave function defined as :

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), n = 1,2,3...... \qquad 0 < x < a \tag{21}$$

We now consider the partial sum

$$S_m(x,\ x^{'}) = \frac{2}{a}\sum_{k=1}^{m} \sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{k\pi x'}{a}\right),\ 0<\left[x, x^{'}\right] < a \tag{22}$$

which can also be written as

$$S_m(x,\ x^{'}) = \frac{1}{2a}\left[\frac{Sin\left[(m+\frac{1}{2})\frac{\left(x-x^{'}\right)}{a}\right]}{Sin\left[\frac{\left(x-x^{'}\right)}{2a}\right]} - \frac{Sin\left[(m+\frac{1}{2})\frac{\left(x+x^{'}\right)}{a}\right]}{Sin\left[\frac{\left(x+x^{'}\right)}{2a}\right]}\right] \qquad 0<\left[x, x^{'}\right] < a\ . \tag{23}$$

In the limit $m \to \infty$, the partial sum yields

$$\lim_{m\to\infty} S_m(x,\ x^{'}) = \delta(x - x^{'})\ + \delta(x + x^{'})\ 0 < \left[x, x^{'}\right] < a \tag{24}$$

If we introduce the property of Delta function viz.

$$\delta(x) = 0 \quad x \neq 0 \tag{25}$$

into the Eq.24, then we obtain

$$\delta(x-x^{'}) = \lim m \to \infty \frac{2}{a}\sum_{n=1}^{m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right), \quad 0<\left[x, x^{'}\right]<a \tag{26}$$

Now setting the values

$$x = \frac{a}{4}, \ x^{'} = \frac{a}{2} \tag{27}$$

into the Eq.(26), we find that

$$\frac{1}{\sqrt{2}} + \lim m \to \infty \sum_{n=1}^{m} \sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right) = 0 \tag{28}$$

If we introduce the following identity

$$\sin(A+B) = \sin A\cos B + \cos A \sin B \tag{29}$$

into the Eq.(28), then we obtain

$$1+\lim_{m\to\infty} \frac{\sum_{n=1}^{m}(-1)^n\left[\sin\left(\frac{n\pi}{2}\right)+\cos\left(\frac{n\pi}{2}\right)\right]}{2} = \frac{1}{2} \tag{30}$$

The Eq.30 can also be written as

$$\lim_{m\to\infty}\sum_{k=0}^{m}(-1)^k = \frac{1}{2}, \tag{31}$$

which is identical to the result defined in Eq.(14). In order to prove that the Grandi's series is indeterminate, we first combine the Eq.(22), Eq.(23) and Eq.(27), then we obtain

$$\frac{1}{\sqrt{2}} + \sum_{n=1} \sin\frac{(2n+1)\pi}{4}\sin\frac{(2n+1)\pi}{2} = \frac{1}{4}\left[\frac{Sin\left[(m+\frac{1}{2})\frac{\pi}{4}\right]}{Sin\left[\frac{\pi}{8}\right]}\right] \tag{32}$$

Introducing the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$ into the Eq.(32) yields

$$1+\sum_{n=1}^{m}\frac{\left[\sin\left(\frac{n\pi}{2}\right)+\cos\left(\frac{n\pi}{2}\right)\right]\left[(-1)^n\right]}{2}=\frac{1}{2}+\frac{1}{4\sqrt{2}}\left[\frac{Sin\left[(m+\frac{1}{2})\frac{\pi}{4}\right]}{Sin\left[\frac{\pi}{8}\right]}\right] \tag{33}$$

Now setting $m=\infty$ and $\phi-\phi^{'}=\pi$ into the Eq.(33) we get the identity

$$\sum_{n=0}^{\infty}\left[(-1)^n\right]=\frac{1}{2}+\frac{1}{4\sqrt{2}}\left[\frac{Sin\left[(\infty+\frac{1}{2})\frac{\pi}{4}\right]}{Sin\left[\frac{\pi}{8}\right]}\right]. \tag{34}$$

It is to be noted that the series which appears left hand of Eq.34 corresponds to the complete sum of Grandi's infinite series which is indeterminate due to presence of $Sin(\infty+\frac{1}{2})\frac{\pi}{4}$ in the right side of Eq.34.

Orthogonal Functions: Quantum particle in a Sphere

To give another proof for the Eqs.(16) and (31) we first consider the complete set of orthogonal functions [12] corresponding to the Eigen functions of the Schrodinger equation for a quantum particle confined inside a sphere of finite radius a. The completeness relation in three dimensions for these complete set of orthogonal functions is defined as

$$\frac{\delta(r-r^{'})}{r^2\sin(\theta)}\delta(\theta-\theta^{'})\delta(\phi-\phi^{'})=\sum_{i=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\frac{J_{l+\frac{1}{2}}(k_{il}r)}{\sqrt{r}}Y_{lm}\left(\theta,\phi\right)\frac{J_{l+\frac{1}{2}}(k_{il}r)}{\sqrt{r}}Y_{lm}^{*}\left(\theta^{'},\phi^{'}\right)}{\frac{a^2}{2}\left(J_{l+\frac{1}{2}}(k_{il}a)\right)^2} \tag{35}$$

where $J_{l+1/2}(r)$ and $Y_{lm}\left(\theta,\phi\right)$ respectively represent the Bessel function and spherical harmonics and $k_{1l},k_{2l},k_{3l}..$ are the positive roots of the equation $J_{l+1/2}(k_{il}a)=0$, numbered in increasing order. We first integrate both sides of Eq.(35) over the domain of $\left(r,\theta,\phi\right)$, to obtain

$$\int_0^a r^2 dr \int_0^{\pi}\int_0^{2\pi} Sin\theta d\theta d\phi Y_{l'm'}{}^{*}(\theta,\phi)\frac{\delta(r-r')}{r^2\sin(\theta)}\delta(\theta-\theta')\delta(\phi-\phi')$$

$$= \lim L\to\infty \sum_{i=1}^{L}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\int_0^a\int_0^{\pi}\int_0^{2\pi} d\theta d\phi r^2 dr \sin\theta d\phi$$

$$\text{x}\frac{\sqrt{4\pi}\,\dfrac{J_{l+\frac{1}{2}}(k^i{}_{l+\frac{1}{2}}r)}{\sqrt{r}}\displaystyle\int_0^{\pi}\int_0^{2\pi}\sin\theta d\theta d\phi Y_{lm}(\theta,\phi)Y^{*}{}_{0,0}(\theta,\phi)\frac{J_{l+\frac{1}{2}}(k^i{}_{l+\frac{1}{2}}r')}{\sqrt{r}}Y_{lm}{}^{*}(\theta',\phi')}{\dfrac{a^2}{2}\left(J_{l+\frac{3}{2}}(k^i{}_{l+\frac{1}{2}}a)\right)^2} \tag{36}$$

Using the identities

$$\int_0^{\pi} d\theta\sin\theta\int_0^{2\pi} d\varphi Y_{lm}(\theta,\varphi)Y_{l',m'}(\theta,\varphi)^{*} = \delta_{ll'}\delta_{mm'} \tag{37a}$$

$$\int_0^a r^{\nu+1}dr J_\nu(k_0 r) = \frac{a^{\nu+1}}{k_0}J_{\nu+1}(k_0 a), \int_0^a dr r^{\left(l+\frac{1}{2}\right)+1}J_{l+\frac{1}{2}}(k^i{}_{l+\frac{1}{2}}r) = \frac{a^{\left(l+\frac{1}{2}\right)+1}}{k^i{}_{l+\frac{1}{2}}}J_{l+\frac{1}{2}}(k^i{}_{l+\frac{1}{2}}a), \tag{37b}$$

we obtain

$$\frac{2}{\sqrt{a}}\lim L\to\infty\sum_{i=1}^{L}\frac{\dfrac{J_{\frac{1}{2}}(k^i{}_{\frac{1}{2}}r')}{\sqrt{r}'}}{k^i{}_{\frac{1}{2}}\left(J_{\frac{3}{2}}(k^i{}_{\frac{1}{2}}a)\right)} = 1 \tag{38}$$

Now substituting the expression of Bessel functions namely

$$\frac{J_{\frac{1}{2}}(k^i{}_{\frac{1}{2}}r')}{\sqrt{a}\sqrt{r}'} = \frac{\sqrt{\dfrac{2}{n\pi^2}}\sin\left(n\pi\left(\dfrac{r'}{a}\right)\right)}{a\left(\dfrac{r'}{a}\right)} \tag{39}$$

$$k^i{}_{\frac{1}{2}}\left(J_{\frac{3}{2}}(k^i{}_{\frac{1}{2}}a)\right) = \frac{n\pi}{a}\sqrt{\frac{2}{n\pi^2}}(-1)^{n+1} \tag{40}$$

into the Eq.38, we obtain

$$\lim m\to\infty\sum_{n=1}^{m}(-1)^{n+1}\frac{Sin(\frac{n\pi r'}{a})}{n\left(\dfrac{r'}{a}\right)} = \frac{\pi}{2} \tag{41}$$

Now taking the limit $r' \to 0$, we obtain the identical identities defined in Eqs.(16) and (31) viz.

$$\lim m \to \infty \sum_{n=1}^{m} (-1)^{n+1} = \frac{1}{2} \tag{42}$$

Conclusion

The Grandi's series that has caused several disagreements and controversies throughout the history of science have been discussed in details based on the completeness relation. We have shown here using three different completeness relations (Eqs.(14), (26) and (35)) that the sum of complete Grandi's series becomes indeteminant (Eq.s (19) and (34)), but the limiting value of the partial sum converges to the value 1/2 (Eq.s(16), (31) and (42)), the same value was however also obtained incorrectly by Leibnitz, Grandi and others but for the complete sum of the Grandi's series which is shown here to be indeterminate.

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These results are the same as those Euler got by taking the sum of the series to be the value of the function from which the series is derived.

Euler took up the subject of sums of series in a major paper of 1754/55 entitled "On Divergent Series" **[5],** in which he recognized the distinction between convergent and divergent series. Apropos of the former he says that for those series in which by constantly adding terms we approach closer and closer to a fixed number, which happens when the terms continually decrease, the series is said to be convergent and the fixed number is its sum. Series whose terms do not decrease and may even increase are divergent.

On divergent series, Euler says one should not use the term "sum" because this refers to actual addition. Euler then states a general principle which explains what he means by the definite value of a divergent series. He points out that the divergent series comes from finite algebraic expressions and then says that the value of the series is *the value of the algebraic expression from which the series comes.* Euler further states, "Whenever an infinite series is obtained as* the development of some closed expression, it may be used in mathematical operations as the equivalent of that expression, even for values of the variable for which the series diverges." He repeats this principle in his *Institutiones* of 1755:

using Euler further states, "Whenever an infinite series is obtained as* the development of some closed expression, it may be used in mathematical operations as the equivalent of that

expression, even for values of the variable for which the series diverges." He repeats this principle in his *Institutiones* of 1755: In accord with Euler principle, we will show that the algebraic expression corresponds to the completeness relation where the sum of nearly infinite number of product of orthogonal functions is expressed in terms of one-dimensional Dirac delta function and the Grandi's divergent series arises and obtain the limit value of partial sum of Grandi's series. However, the algebraic expression does not holds when one include in the sum the infinite numbers of orthogonal functions. However, where the sum of nearly infinite number of product of orthogonal functions is expressed in