Lacunary Eta-quotients Modulo Powers of Primes

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Partitions

Definition

A **partition** of n is a nonincreasing sequence of positive integers $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_m)$ that sum to n, and

$$p(n) := \#\{\text{partitions of } n\}.$$

Example

The partitions of 4 *are given by the following set of size* $p(4) = 5$,

$$\{4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1\}.$$

Divisibility of $p(n)$

Theorem (Ramanujan)

For all $n \geq 0$ the partition function has the following congruences:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Conjecture (Parkin-Shanks)

For $p(n)$, $\frac{1}{2}$ of the values are even and $\frac{1}{3}$ are divisible by 3.

Divisibility of $p(n)$

Definition

For $F(q) = \sum a(n)q^n$ with integer coefficients, define

$$\delta(F, M; X) := \frac{\#\{n \leq X \ : \ a(n) \equiv 0 \pmod{M}\}}{X}.$$

Divisibilty of $p(n)$

Example (Evidence For Parkin-Shanks)

Consider $P(q) := \sum p(n)q^n$.

X	$\delta(P, 2; X)$	$\delta(P, 3; X)$
100,000	0.4980	0.3334
200,000	0.5012	0.3332
300,000	0.5008	0.3335
400,000	0.5000	0.3339
500,000	0.5000	0.3343
$\vdots$	$\vdots$	$\vdots$
∞	1/2	1/3

Lacunarity of Power Series

Definition

For $F(q) = \sum a(n)q^n \in \mathbb{Z}[[q]]$, F is **lacunary** modulo M if

$$\lim_{X \to \infty} \delta(F, M; X) = 1.$$

Example

Consider

$$F(q) = \prod_{n=1}^{\infty}(1 - q^n) = \sum_{k=0}^{\infty}(-1)^k q^{\frac{3k^2+k}{2}}.$$

$F(q)$ is lacunary modulo any positive integer M.

t-regular Partitions

Definition

A t-**regular partition** is a partition with no parts divisible by t,

$$b_t(n) = \#\{t\text{-regular partitions of } n\}.$$

Lemma

The generating function for $b_t(n)$ is

$$G_t(q) := \sum_{n=0}^{\infty} b_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})}{(1-q^n)}.$$

Lacunarity of t-regular Partitions

Theorem (Gordon-Ono)

Let p be a prime such that $p^a \mid t$. If $p^a \geq \sqrt{t}$, then $G_t(q)$ is lacunary modulo p^j for any positive integer j.

Example (10-regular partition)

X	$\delta(G_{10}, 2; X)$	$\delta(G_{10}, 5; X)$
2,000	0.47950	0.58300
4,000	0.48650	0.60300
6,000	0.48900	0.61583
8,000	0.49188	0.62075
10,000	0.49180	0.62560
$\vdots$	$\vdots$	$\vdots$
∞	1/2	1

Extending Theorem of Gordon and Ono

Question

Can we extend this theorem to other generating functions?

Answer

Yes ... We give two theorems extending their results.

Dedekind's Eta-function

Definition

The **Dedekind's eta-function** is

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i \tau}$ and τ is in the upper half plane $\mathcal{H}$.

Remark

The eta-function $\eta(\tau)$ is a weight $k = \frac{1}{2}$ modular form.

Eta-quotients

Definition

An **eta-quotient** is a function, $f(\tau)$, of the form

$$f(\tau) := \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta},$$

where $N \geq 1$ and $r_\delta \in \mathbb{Z}$.

Remark

The generating functions of $p(n)$ and $b_t(n)$ are eta-quotients.

Eta-quotients

Theorem 1 (C-M-S-Z)

Suppose $G(\tau)$ is an integer weight eta-quotient

$$G(\tau) = \frac{\eta(\delta_1\tau)^{r_1}\eta(\delta_2\tau)^{r_2}\cdots\eta(\delta_u\tau)^{r_u}}{\eta(\gamma_1\tau)^{s_1}\eta(\gamma_2\tau)^{s_2}\cdots\eta(\gamma_t\tau)^{s_t}}.$$

If p is a prime such that p^a divides $\gcd(\delta_1,\ldots,\delta_u)$, and

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t}\gamma_i s_i}{\sum_{i=1}^{u}\frac{r_i}{\delta_i}}},$$

then $G(\tau)$ is lacunary modulo p^j for any positive integer j.

Ferrers Diagrams

Definition

The **Ferrers diagram** of a partition $\lambda = \{\lambda_1, \ldots, \lambda_m\}$ is a series of m rows with λ_i boxes in the ith row.

Definition

The **hook length** of a box is the number of boxes to the right or below a given box plus one.

Hook Lengths

Example

The Ferrers diagram of $\lambda = (4, 2, 1)$*, a partition of* 7*, is*

6	4	2	1
3	1		
1			

Definition

Let $\mathcal{H}(\lambda)$ be the multi-set of hook lengths of λ and

$$\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) : h \equiv 0 \pmod{t}\}.$$

Nekrasov-Okounkov and Han

Theorem (Nekrasov-Okounkov)

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right) = \prod_{n\geq 1} (1-q^n)^{z-1}$$

Theorem (Han)

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \prod_{h\in\mathcal{H}_t(\lambda)} \left(y-\frac{tyz}{h^2}\right) = \prod_{n\geq 1} \frac{(1-q^{tn})^t}{(1-(yq^t)^n)^{t-z}(1-q^n)}$$

Han's Equation as Eta-quotients

Definition

Han's (t, y)-extension for $y = 1$ and $y = -1$:

$$G_{1,t,z}(\tau) = q^{\frac{1-tz}{24}} \frac{\eta(t\tau)^z}{\eta(\tau)},$$

$$G_{-1,t,z}(\tau) = q^{\frac{1-tz}{24}} \frac{\eta(t\tau)^{2t-z}\eta(4t\tau)^{t-z}}{\eta(\tau)\eta(2t\tau)^{3(t-z)}}.$$

Han's Equation as Eta-quotients

Corollary (C-M-S-Z)

Suppose z odd, $0 < z < t$, and p prime such that $p^a \mid t$.

1) If

$$p^a \geq \sqrt{\frac{t}{z}}$$

then $G_{1,t,z}(\tau)$ is lacunary modulo p^j for any positive integer j.

2) If

$$p^a \geq 2\sqrt{\frac{t + 6t^3 - 6t^2 z}{9t - 5z}},$$

then $G_{-1,t,z}(\tau)$ is lacunary modulo p^j for any positive integer j.

Remark

If $z = 1$, then we recover the result of Gordon and Ono.

Han's Equation as Eta-quotients

Example

Define

$$G(\tau) := G_{1,18,3}(\tau) = \frac{\eta(18\tau)^3}{\eta(\tau)},$$

then we have:

X	$\delta(G,2;X)$	$\delta(G,3;X)$	$\delta(G,5;X)$
200,000	0.498880	0.687250	0.199315
400,000	0.498670	0.693443	0.199788
600,000	0.499515	0.696803	0.200428
800,000	0.500148	0.699180	0.200126
1,000,000	0.500073	0.701041	0.200324
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1/2	1	1/5

Generalized Eta-Function

Definition

The **generalized eta-function** is

$$\eta_{\delta,g}(\tau) := e^{\pi i P_2(\frac{g}{\delta})\delta\tau} \prod_{\substack{\ell>0 \\ \ell\equiv g \pmod{\delta}}} (1-q^\ell) \prod_{\substack{\ell>0 \\ \ell\equiv -g \pmod{\delta}}} (1-q^\ell)$$

where P_2 is the second Bernoulli polynomial:

$$P_2(n) := (n - \lfloor n \rfloor)^2 - (n - \lfloor n \rfloor) + \frac{1}{6}.$$

Remark

When $g = 0$ or $g = \frac{\delta}{2},$ $\eta_{\delta,0}(\tau) = \eta(\delta\tau)^2$ and $\eta_{\delta,\frac{\delta}{2}}(\tau) = \frac{\eta(\frac{\delta}{2}\tau)^2}{\eta(\delta\tau)^2}$. Otherwise $\eta_{\delta,g}(\tau)$ is a meromorphic modular form of weight $k = 0$.

Rogers-Ramanujan Identities

Theorem (Rogers-Ramanujan)

$$\sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$\sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

Remark

Combining these expressions we obtain

$$\frac{\eta_{5,1}(\tau)}{\eta_{5,2}(\tau)} = q^{\frac{1}{5}} \prod_{n=0}^{\infty} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Generalized Eta-quotients

Definition

Let $H(\tau)$ be a generalized eta-quotient of the form

$$H(\tau) := \frac{\prod_{i=1}^{u} \eta_{\delta_i,g_i}^{r_i}(\tau)}{\prod_{i=1}^{t} \eta_{\gamma_i,h_i}^{s_i}(\tau)} \cdot \frac{\prod_{i=1}^{v} \eta^{r'_i}_{\delta'_i,\frac{\delta'_i}{2}}(\tau)}{\prod_{i=1}^{x} \eta^{s'_i}_{\gamma'_i,\frac{\gamma'_i}{2}}(\tau)} \cdot \frac{\prod_{i=1}^{w} \eta_{\delta''_i,0}^{r''_i}(\tau)}{\prod_{i=1}^{y} \eta_{\gamma''_i,0}^{s''_i}(\tau)},$$

where we assume $H(\tau)$ is modular on $\Gamma_1(N)$. Define

$$\mathcal{D}_H := \gcd\{\delta_i, \delta'_i, \delta''_i, \tfrac{\gamma'_i}{2}\}.$$

Generalized Eta-quotients

Theorem 2 (C-M-S-Z)

Given $H(\tau)$, if p is a prime such that $p^a \mid \mathcal{D}_H$, and

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \gamma_i s_i + \frac{1}{2}\sum_{i=1}^{x} \gamma_i' s_i' + \sum_{i=1}^{y} \gamma_i'' s_i'' + \sum_{i=1}^{v} \delta_i' r_i'}{-\frac{1}{2}\sum_{i=1}^{u} \delta_i r_i + \frac{1}{2}\sum_{i=1}^{v} \frac{r_i'}{\delta_i'} + \sum_{i=1}^{w} \frac{r_i''}{\delta_i''} - \frac{1}{2}\sum_{i=1}^{x} \gamma_i' s_i'}},$$

then $H(\tau)$ is lacunary modulo p^j for any positive integer j.

Congruence Subgroups

Definition

For N a positive integer, the level N **congruence subgroups** are

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0,\ a \equiv d \equiv 1 \pmod{N} \right\}.$$

Modular Forms

Definition

A function f is a **modular form** of weight k for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ if:

- f is analytic on the upper half-plane $\mathcal{H}$,
- for any $A \in \Gamma$, f satisfies the equation

$$f(A\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

- the Fourier expansion of f has the form

$$f(\tau) = \sum_{n=0}^{\infty} c(n) q^n,$$

where $q := e^{2\pi i \tau}$.

Modularity of Eta-quotients

Proposition

Let $f(\tau) = \prod_{\delta | N} \eta(\delta\tau)^{r_\delta}$ *with* $k = \frac{1}{2}\sum_{\delta | N} r_\delta \in \mathbb{Z}$. *If*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24},$$

$$\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(\tau)$ *is modular with weight* k *on* $\Gamma_0(N)$.

Cusps on $\Gamma_0(N)$

Definition

A **cusp** of Γ is an equivalence class of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

Proposition

The set of representatives of the cusps of $\Gamma_0(N)$ is

$$C_0(N) := \left\{ \tfrac{c}{d} : \ d \mid N, \ (c, N) = 1 \right\},$$

where c runs through a complete residue system modulo N.

Order of Vanishing of Eta-quotients

Theorem

Let c, d, and N be positive integers with $d \mid N$ and $(c, d) = 1$. Then the order of vanishing of $f(\tau)$ at the cusp $\frac{c}{d}$ is given by

$$\frac{N}{24d\left(d, \frac{N}{d}\right)} \sum_{\delta \mid N} \frac{(d, \delta)^2 r_\delta}{\delta}.$$

Remark

Since eta-quotients are holomorphic on $\mathcal{H}$, we can check the order of vanishing at every cusp to determine if an eta-quotient is holomorphic.

Modularity of Generalized Eta-quotients

Theorem

If $f(\tau) = \prod_{\delta | N} \eta_{\delta,g}^{r_{\delta,g}}(\tau)$ *is such that*

$$\sum_{\substack{\delta | N \\ g}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} \equiv 0 \pmod{2},$$

$$\sum_{\substack{\delta | N \\ g}} \frac{N}{6\delta} r_{\delta,g} \equiv 0 \pmod{2},$$

then $f(\tau)$ *is modular on* $\Gamma_1(N)$.

Order of Vanishing of Generalized Eta-quotients

Proposition

The set of representatives of the cusps of $\Gamma_1(N)$ *is*

$$C_1(N) := \left\{ \frac{\lambda}{\mu\epsilon} : \epsilon \mid N,\ 1 \leq \lambda, \mu \leq N, \right\},$$

where $(\mu, \lambda) = (\lambda, N) = (\mu, N) = 1.$

Theorem

Given $H(\tau)$, *and a cusp,* $\frac{\lambda}{\mu\epsilon}$ *of* $\Gamma_1(N)$, *the order of vanishing of* $H(\tau)$ *at the cusp is*

$$\frac{N}{2} \sum_{\substack{\delta \mid N \\ g}} \frac{(\delta, \epsilon)^2}{\delta\epsilon} P_2\left(\frac{\lambda g}{(\delta, \epsilon)}\right) r_{\delta,g}.$$

Serre's Theorem

Theorem (Serre)

If $f(\tau)$ is a holomorphic modular form of integer weight with integer coefficients, then $f(\tau)$ is lacunary modulo any positive integer M.

Proof of Theorem 1

Lemma

If $f(x) = 1 + \sum_{n=1}^{\infty} a(n)x^n \in \mathbb{Z}$ *such that* $a(n) \equiv 0 \pmod{p}$ *for all* $n \geq 1$, *then*

$$f^{p^j}(x) \equiv 1 \pmod{p^{j+1}}.$$

Proof of Theorem 1

Definition

Given an arbitrary eta-quotient, $G(\tau)$, prime p, and $a \in \mathbb{Z}^+$, let

$$f_{G,p^a}(\tau) := \prod_{i=1}^{t} \left(\frac{\eta^{p^a}(24\gamma_i\tau)}{\eta(24p^a\gamma_i\tau)} \right)^{s_i}.$$

Remark

By the previous lemma $f_{G,p^a}^{p^j}(\tau) \equiv 1 \pmod{p^{j+1}}$.

Proof of Theorem 1

Definition

Given $G(\tau)$, p^a, and any $j \in \mathbb{Z}^+$, let

$$F_{G,p^a,j}(\tau) := G(24\tau) f_{G,p^a}^{p^j}(\tau)$$
$$= \frac{\prod_{i=1}^{u} \eta^{r_i}(24\delta_i\tau)}{\prod_{i=1}^{t} \eta^{s_i}(24\gamma_i\tau)} \prod_{i=1}^{t} \left(\frac{(\eta^{p^a}(24\gamma_i\tau))}{(\eta(24p^a\gamma_i\tau))} \right)^{s_i p^j}.$$

Remark

Since $f_{G,p^a}^{p^j}(\tau) \equiv 1 \pmod{p^{j+1}}$, $F_{G,p^a,j}(\tau) \equiv G(24\tau) \pmod{p^{j+1}}$. Thus if $F_{G,p^a,j}$ is lacunary modulo p^j then $G(\tau)$ is as well.

Proof of Theorem 1

Proof.

- Using the equation for order of vanishing at a cusp $\frac{c}{d}$, if

$$p^a \mid \gcd(\delta_i)$$

and

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} s_i \gamma_i}{\sum_{i=1}^{u} \frac{r_i}{\delta_i}}},$$

then $F_{G,p^a,j}(\tau)$ is holomorphic.

- By Serre's Theorem, $F_{G,p^a,j}(\tau)$ is lacunary modulo any positive integer.
- $G(\tau)$ is lacunary modulo p^j.

$\square$

Proof of Theorem 2

Recall that an arbitrary generalized eta-quotient $H(\tau)$ is of the following form,

$$H(\tau) := \frac{\prod_{i=1}^{u} \eta_{\delta_i,g_i}^{r_i}(\tau)}{\prod_{i=1}^{t} \eta_{\gamma_i,h_i}^{s_i}(\tau)} \cdot \frac{\prod_{i=1}^{v} \eta_{\delta_i',\frac{\delta_i'}{2}}^{r_i'}(\tau)}{\prod_{i=1}^{x} \eta_{\gamma_i',\frac{\gamma_i'}{2}}^{s_i'}(\tau)} \cdot \frac{\prod_{i=1}^{w} \eta_{\delta_i'',0}^{r_i''}(\tau)}{\prod_{i=1}^{y} \eta_{\gamma_i'',0}^{s_i''}(\tau)}.$$

Proof of Theorem 2

Definition

Given $H(\tau)$, prime p, and $a \in \mathbb{Z}^+$, define $\widetilde{N} = 24L$ where $L = \operatorname{lcm}\{\delta_i, \delta_i', \delta_i'', \gamma_i, \gamma_i', \gamma_i''\}$. Let

$$f_{H,p^a}(\tau) := \prod_{i=1}^{t} \left(\frac{\eta^{p^a}_{\gamma_i,0}(\widetilde{N}\tau)}{\eta_{\gamma_i p^a,0}(\widetilde{N}\tau)} \right)^{s_i} \prod_{i=1}^{v} \left(\frac{\eta^{p^a}_{\delta_i',0}(\widetilde{N}\tau)}{\eta_{\delta_i' p^a,0}(\widetilde{N}\tau)} \right)^{r_i'}$$
$$\prod_{i=1}^{x} \left(\frac{\eta^{p^a}_{\frac{\gamma_i'}{2},0}(\widetilde{N}\tau)}{\eta_{\frac{\gamma_i' p^a}{2},0}(\widetilde{N}\tau)} \right)^{s_i'} \prod_{i=1}^{y} \left(\frac{\eta^{p^a}_{\gamma_i'',0}(\widetilde{N}\tau)}{\eta_{\gamma_i'' p^a,0}(\widetilde{N}\tau)} \right)^{s_i''},$$

Remark

Again, $f^{p^j}_{H,p^a}(\tau) \equiv 1 \pmod{p^{j+1}}$.

Proof of Theorem 2

Definition

Given $H(\tau)$, p^a and $j \in \mathbb{Z}^+$, let

$$F_{H,p^a,j}(\tau) := H(\widetilde{N}\tau) f_{H,p^a}^{p^j}(\tau) \equiv H(\widetilde{N}\tau) \pmod{p^{j+1}}.$$

Proof.

By the order of vanishing formula at a cusp $\frac{\lambda}{\mu\epsilon}$, if $p^a | \gcd(\delta_i, \delta_i', \delta_i'', \frac{\gamma_i}{2})$, and

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \gamma_i s_i + \frac{1}{2}\sum_{i=1}^{x} \gamma_i' s_i' + \sum_{i=1}^{y} \gamma_i'' s_i'' + \sum_{i=1}^{v} \delta_i' r_i'}{-\frac{1}{2}\sum_{i=1}^{u} \delta_i r_i + \frac{1}{2}\sum_{i=1}^{v} \frac{r_i'}{\delta_i'} + \sum_{i=1}^{w} \frac{r_i''}{\delta_i''} - \frac{1}{2}\sum_{i=1}^{x} \gamma_i' s_i'}},$$

then $F_{H,p^a,j}$ is holomorphic. Thus $H(\tau)$ is lacunary modulo p^j. $\square$

Conclusion

Theorem 1 (C-M-S-Z)

Suppose $G(\tau)$ is an arbitrary eta-quotient of the form

$$G(\tau) = \frac{\eta(\delta_1\tau)^{r_1}\eta(\delta_2\tau)^{r_2}\cdots\eta(\delta_u\tau)^{r_u}}{\eta(\gamma_1\tau)^{s_1}\eta(\gamma_2\tau)^{s_2}\cdots\eta(\gamma_t\tau)^{s_t}}$$

with integer weight k and p is a prime such that p^a divides $\gcd(\delta_1, \ldots, \delta_u)$. If

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \gamma_i s_i}{\sum_{i=1}^{u} \frac{r_i}{\delta_i}}},$$

then $G(\tau)$ is lacunary modulo p^j for any positive integer j.

Conclusion

Example

$$G(\tau) := G_{1,18,3}(\tau) = \frac{\eta(18\tau)^3}{\eta(\tau)}$$

X	$\delta(G, 2; X)$	$\delta(G, 3; X)$	$\delta(G, 5; X)$
200,000	0.498880	0.687250	0.199315
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800,000	0.500148	0.699180	0.200126
1,000,000	0.500073	0.701041	0.200324
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1/2	1	1/5

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