Periodic Poisson Solver

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We want to compute the solution to the discretized Poisson equation on the unit square with periodic boundary conditions.

$$(\Delta^{(h)} f)_{\boldsymbol{j}} \equiv \frac{1}{h^2}(-4f_{\boldsymbol{j}} + f_{\boldsymbol{j}+(1,0)} + f_{\boldsymbol{j}+(-1,0)} + f_{\boldsymbol{j}+(0,1)} + f_{\boldsymbol{j}+(0,-1)}) = \rho_{\boldsymbol{j}}$$

$$\boldsymbol{j} = (j_0, j_1) \ , \ 0 \leq j_0, j_1 < N-1 \ , \ h = \frac{1}{N}$$

We do this using the complex FFT in two dimensions.

1. Compute the forward FFT

$$\hat{\rho}_{\boldsymbol{k}} = \mathcal{F}_N(\rho)_{\boldsymbol{k}} = \sum_{0 \leq j_0, j_1 < N-1} \rho_{\boldsymbol{j}} z^{\boldsymbol{j}\cdot\boldsymbol{k}} \ , \ \boldsymbol{k} = (k_0, k_1) \ , \ 0 \leq k_0, k_1 < N-1$$

   Uses `FFTMD::forwardCC`.

2. Divide the Fourier coefficients by the symbol of the operator $\Delta^{(h)}$.

$$\begin{aligned} \hat{f}_{\boldsymbol{k}} =& \frac{\hat{\rho}_{\boldsymbol{k}}}{2cos(2\pi k_0 h) - 2cos(2\pi k_1 h) - 4} \text{ if } \boldsymbol{k} \neq (0,0) \\ =& 0 \text{ if } \boldsymbol{k} = (0,0) \end{aligned}$$

   Done as a `for` loop.

3. Take the inverse Fourier transform, and normalize.

$$f_{\boldsymbol{j}} = \frac{1}{N^2}\mathcal{F}_N^{-1}(\hat{f})_{\boldsymbol{j}} \ , \ 0 \leq j_0, j_1 < N-1$$

   Uses `FFTMD::inverseCC`.