INTRODUCTION TO SHIMURA VARIETIES

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ABSTRACT. These are lecture notes for a course on Shimura varieties I am currently teaching at Zhejiang University. Comments are highly welcome and much appreciated.

CONTENTS

1. INTRODUCTION

In this first lecture, we will learn, very roughly, what Shimura varieties are and why they are interesting. Everything brought up today will be covered in much more detail later in the course, and it will be perfectly normal that many terms will be new during a first reading. Our goal today is only to get an overview.

1.1. **Why study Shimura varieties?** Shimura varieties combine two interesting properties:

- They are varieties defined over number fields which makes them interesting from a number theory perspective. Most importantly, their étale cohomology groups are representations of Galois groups of number fields.

- Their definition is in terms of connected reductive algebraic groups $G/\mathbb{Q}$. They come equipped with an action of the adelic points $G(\mathbb{A}_f)$, which implies that their étale cohomology groups are also $G(\mathbb{A}_f)$-representations.

Hence, the étale cohomology groups of Shimura varieties are both Galois and $G(\mathbb{A}_f)$-representations. Conjecturally, this two-fold structure is described by the global Langlands correspondence. Conversely, one can use the cohomology of Shimura varieties to prove important cases of this correspondence. This is the main motivation for our course, and our overall aim is to learn about several important ideas in this context.

Let us mention that Shimura varieties are also interesting for other reasons. For example, the study of heights on the Siegel variety plays an important role in Faltings's proof of the Mordell Conjecture [4]. Another example is the Gross–Zagier formula [6], which states an identity between height pairings of complex multiplication points on the modular curve and derivatives of L-functions. It plays a major role in the proof of cases of the Birch–Swinnerton-Dyer Conjecture. Its higher-dimensional generalizations, the arithmetic Gan–Gross–Prasad Conjectures [5,16], are an important topic in current arithmetic geometry research. In a related direction, the Kudla program [8] seeks to establish connections

between cycles on Shimura varieties and modular forms or Eisenstein series. The proof of the averaged Colmez conjecture [1,15] has been an application of such ideas.

1.2. This course.
The first part of our course will be an introduction to Shimura varieties. We will learn how to define them in terms of moduli spaces of abelian varieties and how to relate this definition to the group-theoretic one of Deligne. One of our goals is to obtain familiarity with the adelic formalism which will become important later.

In the second part of the course, we will study the cohomology of Shimura varieties. We will first get to know Matsushima's formula, which expresses the Betti cohomology of compact Shimura varieties in terms of automorphic representations. We will then learn about point counting in characteristic p (Langlands–Kottwitz method). The aim here is to give an orbital integral expression for the number of $\mathbb{F}_{p^n}$-points of the reduction mod p of the Shimura variety.

1.3. References.
The following two are our main background references.

- The introductory lecture notes by Milne [11]. They focus on the group-theoretic definition of Shimura varieties and the definition of canonical models.

- The first few articles in the lecture notes volume [7]. They provide an introduction to PEL type Shimura varieties. The article of Yihang Zhu [17] is directly related to the material of the second part of the course.

1.4. Prerequisites.
We will assume as little as possible. The only necessary background is some familiarity with varieties and algebraic number theory.

In the rest of this introduction, we sketch the definition of Shimura varieties and give an outline of the course contents.

1.5. Shimura data.
Shimura varieties are attached to Shimura data. The formalism starts with a connected reductive group G over $\mathbb{Q}$. For example, G might be one of the following.

- $G = \mathrm{GL}_2$

- $G = \mathrm{GSp}_{2g}$, the general symplectic group in $2g$ variables. Let $J = \begin{pmatrix} & 1_g \\ -1_g & \end{pmatrix}$ be the matrix defining the standard symplectic form on $\mathbb{Q}^{2g}$. Then GSp is defined by

$$\mathrm{GSp}_{2g}(\mathbb{Q}) = \left\{ g \in GL_{2g}(\mathbb{Q}) \mid {}^t g \cdot J \cdot g = c \cdot J \text{ for some } c \in \mathbb{Q}^{\times} \right\}. \tag{1.1}$$

  It is related to the usual symplectic group Sp_{2g} by the exact sequence

$$1 \longrightarrow \mathrm{Sp}_{2g} \longrightarrow \mathrm{GSp}_{2g} \stackrel{c}{\longrightarrow} \mathrm{GL}_1 \longrightarrow 1.$$

  The map c is called the *similitude factor*. Note that $\mathrm{GSp}_2 = \mathrm{GL}_2$ and $\mathrm{Sp}_2 = \mathrm{SL}_2$, recovering the previous example.

- $G = \mathrm{U}(V)$, a unitary group. Let $K/\mathbb{Q}$ be an imaginary quadratic extension. (This means that $\mathbb{R} \otimes_{\mathbb{Q}} K \cong \mathbb{C}$.) Let V be an n-dimensional hermitian K-vector space. If V is not positive or negative definite then $\mathrm{U}(V)$ can occur as part of a Shimura datum.

Next, the formalism requires the datum of a homomorphism of real algebraic groups

$$h : \mathbb{C}^{\times} \longrightarrow G(\mathbb{R}) \tag{1.2}$$

which satisfies certain axioms introduced by Deligne [2]. Such an h is called a *Deligne homomorphism*. If $g \in G(\mathbb{R})$ is a real point of G, then we may conjugate h to define a new Deligne homomorphism,

$$(ghg^{-1})(z) := gh(z)g^{-1}.$$

Let $S_h \subset G(\mathbb{R})$ denote the centralizer of h, meaning the subgroup of elements g with $ghg^{-1} = h$. The quotient $X = G(\mathbb{R})/S_h$ is precisely the set of Deligne homomorphisms that are conjugate to h. An important consequence of Deligne's axioms is that X is a finite union of hermitian symmetric domains for $G(\mathbb{R})$. In particular, it is a complex manifold. The pair (G, X) is called a *Shimura datum*.

Example 1.1. Consider $G = \mathrm{GL}_2$. We can embedd $\mathbb{C}$ into $\mathrm{M}_2(\mathbb{R})$ as $\mathbb{R}$-algebra by

$$h(a + bi) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

If we restrict this embedding to unit groups, then we obtain a Deligne homomorphism $h : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$. Its centralizer is precisely $h(\mathbb{C}^\times)$ and the quotient X is the set of complex structures on $\mathbb{R}^2$. Since $\mathbb{C}^\times$ is connected and since $\mathrm{GL}_2(\mathbb{R})$ has two connected components, X has two connected components. We want to give a more explicit description of X.

Recall that $\mathbb{P}^1(\mathbb{C})$ is the space of complex lines in $\mathbb{C}^2$. Clearly, the Lie group $\mathrm{GL}_2(\mathbb{C})$ acts on it by its natural action on $\mathbb{C}^2$. The subgroup $\mathrm{GL}_2(\mathbb{R})$ preserves the real projective line $\mathbb{P}^1(\mathbb{R})$ and hence acts on the complement,

$$\mathrm{GL}_2(\mathbb{R}) \ \circlearrowright \ \mathbb{C}\backslash\mathbb{R}, \quad g \cdot \tau = \frac{a\tau + b}{c\tau + d}. \tag{1.3}$$

The complement $\mathbb{C}\backslash\mathbb{R}$ is the union of the upper and lower half plane which we often denote by $\mathbb{H}^\pm$. As an open subset of $\mathbb{C}$, it is naturally a complex manifold. Let us compute the stabilizer of i:

$$i = \frac{ai + b}{ci + d} \quad \Longleftrightarrow \quad -c + di = ai + b \tag{1.4}$$
$$\Longleftrightarrow \quad a = d, \ \ c = -b.$$

That is, the stabilizer of i is precisely $h(\mathbb{C}^\times)$. Moreover, it is clear that $\mathrm{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}^\pm$ because

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix} \cdot i = ai + b.$$

Hence, we see that

$$X \xrightarrow{\sim} \mathbb{H}^\pm, \quad ghg^{-1} \longmapsto g \cdot i \tag{1.5}$$

as smooth manifolds in a $\mathrm{GL}_2(\mathbb{R})$-equivariant way. We have not defined the complex structure on X, but it is, in fact, given by the complex structure on $\mathbb{H}^\pm$ under (1.5).

Remark 1.2. Some groups, such as GL_n with $n \geq 3$, cannot occur as part of a Shimura datum. For example, the dimension of the symmetric space for $\mathrm{GL}_3(\mathbb{R})$ is

$$\dim \mathrm{SL}_3(\mathbb{R}) - \dim \mathrm{SO}(3) = 8 - 3$$

which is odd and hence cannot be a complex manifold.

1.6. **Shimura varieties over $\mathbb{C}$.** Given a Shimura datum (G, X), one next defines a complex variety in the following way. Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$, and let $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ be its factorization into finite and archimedean part. (We will review these definitions later in the course.) Given an open compact subgroup $K \subset G(\mathbb{A}_f)$, the quotient $G(\mathbb{A}_f)/K$ is a discrete countably infinite set with transitive $G(\mathbb{A}_f)$-action. Hence, the product $X \times G(\mathbb{A}_f)/K$ is a countable union of copies of X. We consider the diagonal action

$$G(\mathbb{Q}) \ \circlearrowright \ X \times G(\mathbb{A}_f)/K.$$

If K is small enough then the $G(\mathbb{Q})$-action is free. (The technical term is "neat" and we will get to know it later in the course.) It is also properly discontinuous, so we can form the quotient complex manifold

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) := G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_f)/K\big). \tag{1.6}$$

At this point, we have defined the complex points of the *Shimura variety for Shimura datum (G, X) and level K* as a complex manifold. The theorem of Baily–Borel states that there is a unique way to endow it with an algebraic structure.

Theorem 1.3 (Baily–Borel, see [11, Corollary 3.16]). *There exists a quasi-projective complex variety $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ such that there exists an isomorphism of complex manifolds $\mathrm{Sh}_K(G, X)_{\mathbb{C}}(\mathbb{C}) \xrightarrow{\sim} \mathrm{Sh}_K(G, X)(\mathbb{C})$. This variety is unique up to isomorphism.*

Remark 1.4. Simple examples of non-unique algebraic structures on complex manifolds can be found in [13].

Example 1.5. Let us again consider the case $G = \mathrm{GL}_2$ and let us give an example of a connected component of (1.6). Let $\widehat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$ be the subring of integral elements of $\mathbb{A}_f$. For $n \geq 1$, consider the kernel

$$K(n) = \ker\big(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})\big)$$

which is an open compact subgroup of $G(\mathbb{A}_f)$. It is small enough (in the above sense) if $n \geq 3$. The intersection

$$\Gamma(n) := \mathrm{GL}_2(\mathbb{Q}) \cap K(n)$$

is the classical congruence subgroup

$$\Gamma(n) = \left\{\gamma \in \mathrm{GL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \bmod n\right\}.$$

The quotients $\Gamma(n)\backslash\mathbb{H}^+$ and $\Gamma(n)\backslash\mathbb{H}^-$ will be two of the connected components of the complex manifold $\mathrm{Sh}_{K(n)}(\mathrm{GL}_2, \mathbb{H}^{\pm})$.

1.7. **Shimura varieties over number fields.** Finally, one descends $\mathrm{Sh}_K(G, X)$ to a number field. Starting from a Shimura datum (G, X), Deligne defines a number field $E \subset \mathbb{C}$ called the *reflex field*. In a suitable sense, it is the smallest field over which the conjugacy class X is defined.

Example 1.6. Consider the three examples from §1.5.

- If $G = \mathrm{GL}_2$ or more generally $G = \mathrm{GSp}_{2g}$, then the reflex field is $\mathbb{Q}$.
- If $G = U(V)$ is a non-definite unitary group for an imaginary-quadratic field $K/\mathbb{Q}$, then the reflex field is the subfield $E \subset \mathbb{C}$ that is isomorphic to K.

Deligne [2] gave a definition of *canonical model* of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ over E. It is a variety $\mathrm{Sh}_K(G, X)$ over $\mathrm{Spec}(E)$ together with an isomorphism

$$\mathbb{C} \otimes_E \mathrm{Sh}_K(G, X) \xrightarrow{\sim} \mathrm{Sh}_K(G, X)_{\mathbb{C}}$$

that satisfies a certain reciprocity law for complex multiplication points. Deligne proves that the canonical model $\mathrm{Sh}_K(G, X)$ is unique up to isomorphism if it exists.

Theorem 1.7 (Borovoi, Milne [9]). *For every Shimura datum, the canonical model exists.*

Definition 1.8. Let (G, X) be a Shimura datum with reflex field E and let $K \subset G(\mathbb{A}_f)$ be a sufficiently small level subgroup. The Shimura variety of level K attached to (G, X) is the canonical model $\mathrm{Sh}_K(G, X)$ from Theorem 1.7.

Remark 1.9. Historically, the study of Shimura varieties started with Shimura in the 1960s. He first considered moduli spaces of abelian varieties with **P**olarization, **E**ndomorphisms, and **L**evel structure (PEL). These are the Shimura varieties defined by *PEL type* Shimura data.

Shimura also studied several non-PEL cases and defined the corresponding Shimura varieties as varieties over number fields. Deligne [2] gave a group-theoretic framework for Shimura's work. His definition in terms of a reciprocity law for complex multiplication points is extrapolated from the Shimura–Taniyama reciprocity law for abelian varieties with complex multiplication. Deligne also constructed the canonical model for abelian type Shimura varieties. The proof of existence in the general case was completed by Milne based on ideas of Borovoi. See here for a short summary of the history by Milne [10, §6].

Example 1.10. Consider the two cases from Example 1.6. The unitary group $U(V)$ has no PEL type Shimura data. For the group GSp_{2g}, there exists a PEL type Shimura datum (GSp_{2g}, X). Consider a principal congruence level subgroup

$$K(n) = \ker\left(\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) \longrightarrow \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})\right)$$

with $n \geq 3$. Then the canonical model $\mathrm{Sh}_{K(n)}(\mathrm{GSp}_{2g}, X)$ can be described as a moduli space of principally polarized abelian varieties with level-n-structure. For example, if we look at $\mathbb{C}$-points and specialize to GL_2, then we obtain

$$\mathrm{Sh}_{K(n)}(\mathrm{GL}_2, X)(\mathbb{C}) \xrightarrow{\sim} \{(E, \eta)/\mathbb{C}\}/\sim \tag{1.7}$$

where the right hand side denotes the set of isomorphism classes of pairs (E, η) with

- E an elliptic curve over $\mathbb{C}$,
- $\eta : (\mathbb{Z}/n\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[n]$ a choice of basis for the n-torsion.

The datum η is called a *level structure* for E. Proving (1.7) will be one of our first goals.

1.8. **Further topics.** We will say more about this when the time comes. For now, let us start looking at Shimura varieties in detail.

Part 1. **The Shimura variety of** GL_2

2. The upper half plane

In Example 1.1, we have introduced the action of $\mathrm{GL}_2(\mathbb{Q})$ on the union of upper and lower half plane $\mathbb{H}^{\pm} = \mathbb{C}\backslash\mathbb{R}$. Recall that it is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

In Example 1.5, we have seen that we are especially interested in actions by subgroups such as $\mathrm{GL}_2(\mathbb{Z})$ and $\Gamma(n)$. Our aim in this section is to give a definition of such *arithmetic subgroups* and to prove properties about their action on $\mathbb{H}^{\pm}$.

Note that elements of $\mathrm{GL}_2(\mathbb{Z})$ have determinant 1 or -1, and that the elements of determinant -1 interchange upper and lower half plane. So we will focus on the action of $\mathrm{SL}_2(\mathbb{Q})$ on the upper half plane $\mathbb{H} \subset \mathbb{H}^{\pm}$.

2.1. The fundamental domain. Let $\mathcal{F}$ be the area defined by

$$\mathcal{F} = \left\{ \tau \in \mathbb{H} \;\middle|\; |\tau| \geq 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2} \right\}. \tag{2.1}$$

Its interior $\mathcal{F}^{\circ}$ is the open subset where $|\tau| > 1$ and $-1/2 \leq \mathrm{Re}(\tau) \leq 1/2$.
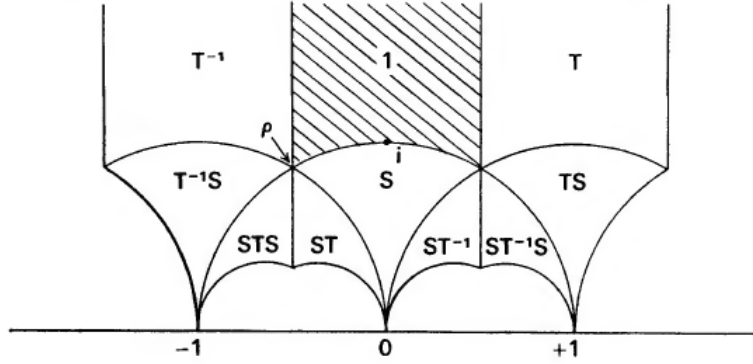

FIGURE 1. The area $\mathcal{F}$ is depicted in grey. The remaining areas show translates of $\mathcal{F}$ under the action of the elements S and T defined in (2.3). By Proposition 2.1 and Remark 2.2, these translates cover all of $\mathbb{H}$. The picture is taken from [14, §VII].

Proposition 2.1. *The set $\mathcal{F}$ is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ on $\mathbb{H}$. That is, it has the following two properties.*

(1) For every $\tau \in \mathbb{H}$, there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\tau \in \mathcal{F}$.

(2) $\mathcal{F}^{\circ} \cap \gamma\mathcal{F}^{\circ} = \emptyset$ whenever $\gamma \notin \{\pm 1\}$.

Proof. Fix $\tau \in \mathbb{H}$ and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z})$ be any element. By direct computation, we see that

$$\mathrm{Im}(\gamma\tau) \;=\; \mathrm{Im}\left(\frac{(a\tau + b)(c\tau - d)}{|c\tau + d|^2} \right) \;=\; \frac{(ad - bc)\mathrm{Im}(\tau)}{|c\tau + d|^2} \;=\; \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}. \tag{2.2}$$

The denominator $|c\tau + d|^2$ defines a positive definite quadratic form in $(c, d) \in \mathbb{Z}^2$. It hence takes a minimum on the set of (c, d) that occur as the bottom row of an element of $\mathrm{SL}_2(\mathbb{Z})$. (These are precisely the (c, d) with $\gcd(c, d) = 1$.) So we see that $\{\mathrm{Im}(\gamma\tau) \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$ has a maximum.

Let γ be such that $\mathrm{Im}(\gamma\tau)$ is maximal. Consider the two matrices

$$S = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \tag{2.3}$$

and observe that they act as the very simple transformations

$$S\tau = -\frac{1}{\tau}, \qquad T\tau = \tau + 1. \tag{2.4}$$

In particular, acting with a suitable power T^m, $m \in \mathbb{Z}$, we can translate $\gamma\tau$ to assume it lies in the strip $-1/2 \leq \mathrm{Re}(z) \leq 1/2$. Then also $|\gamma\tau| \geq 1$ because otherwise $\mathrm{Im}(S\gamma\tau) > \mathrm{Im}(\gamma\tau)$ would contradict the maximality of $\mathrm{Im}(\gamma\tau)$. This proves statement (1) of the proposition.

We now prove statement (2). Assume that τ and $\gamma\tau$ both lie in $\mathcal{F}^{\circ}$, our aim being to show that $\gamma \in \{\pm 1\}$. After possibly replacing the pair (γ, τ) by $(\gamma^{-1}, \gamma\tau)$, we can assume that $\mathrm{Im}(\gamma\tau) \geq \mathrm{Im}(\tau)$. Considering again (2.2), this means that $|c\tau + d|^2 \leq 1$.

Clearly, we now have $c = 0$ because $|c\tau + d| > 1$ for every $c \neq 0$ (use $\tau \in \mathcal{F}^{\circ}$). This means that γ is of the form

$$\gamma = \pm \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$$

for some $m \in \mathbb{Z}$. Since both τ and $\gamma\tau$ have real part in $(-1/2, 1/2)$, the only possibility is $m = 0$. This finishes the proof. $\qquad\square$

Remark 2.2. One can show that the matrices S and T from (2.3) generate $\mathrm{SL}_2(\mathbb{Z})$. That is, every element of $\mathrm{SL}_2(\mathbb{Z})$ can be written as a product of the three elements S, T and T^{-1}. The proof is not difficult and can be found in [14, §VII.1, Theorem 2].

2.2. **Arithmetic subgroups of** $\mathrm{SL}_2(\mathbb{Q})$. We now define arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Q})$.

Definition 2.3. (1) For $n \geq 1$, we define the *principal congruence subgroup* $\Gamma(n)$ by

$$\Gamma(n) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ | \ \gamma \equiv 1 \bmod n\}.$$

(2) We call a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$ *arithmetic* if it contains a principal congruence group $\Gamma(n)$ with finite index.

The group SL_2 has a very interesting property which will come up again later. Namely, for each $n \geq 1$, the projection map

$$\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \tag{2.5}$$

is surjective. This is not hard to show directly, but also follows from Theorem 3.15 (2) below.

Example 2.4. By the surjectivity we just stated for SL_2, the image of the projection map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ is the set of matrices with determinant ± 1. In particular, this projection is not surjective when $n = 5$ or $n \geq 7$.

In the context of Definition 2.3, the surjectivity of (2.5) implies that $\Gamma(n) \trianglelefteq \mathrm{SL}_2(\mathbb{Z})$ is a normal subgroup of index equal to $|\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})|$. In particular, if a group Γ contains $\Gamma(n)$ with finite index, then it also contains all $\Gamma(mn)$ with finite index.

Proposition 2.5. *Let Γ be an arithmetic subgroup.*

(1) There exists a lattice $\Lambda \subset \mathbb{Q}^2$ such that $\Gamma \subseteq \mathrm{SL}(\Lambda)$.

(2) More precisely, there exist an integer n and an element $g \in \mathrm{GL}_2(\mathbb{Q})$, $\det(g) > 0$, such that

$$\Gamma(m) \ \subseteq \ g\Gamma g^{-1} \ \subseteq \ \mathrm{SL}_2(\mathbb{Z}).$$

Proof. The two statements are proved by very simple and universal arguments. First, by assumption on Γ, there exists an integer n such that $\Gamma(n) \subseteq \Gamma$ with finite index. Let $\gamma_1, \ldots, \gamma_r$ be representatives for the cosets $\Gamma/\Gamma(n)$. Then Γ stabilizes the lattice

$$\Lambda := \sum_{i=1}^{r} \gamma_i \cdot \mathbb{Z}^2.$$

Indeed, since $\gamma\mathbb{Z}^2 = \mathbb{Z}^2$ for every $\gamma \in \Gamma(n)$, we can also write Λ as

$$\Lambda = \sum_{\gamma \in \Gamma} \gamma \cdot \mathbb{Z}^2,$$

and from this second expression the Γ-stability is clear. This means that $\Gamma \subseteq \mathrm{SL}(\Lambda)$ which proves statement (1).

Let $\lambda_1, \lambda_2 \in \Lambda$ be a basis as $\mathbb{Z}$-module. Viewing λ_1 and λ_2 as column vectors, the base change matrix $g = (\lambda_1 \ \lambda_2)$ lies in $\mathrm{GL}_2(\mathbb{Q})$ and has the property $g\mathbb{Z}^2 = \Lambda$. Changing λ_1 to $-\lambda_1$ if necessary, we may assume $\det(g) > 0$. Then $\mathrm{SL}_2(\mathbb{Z}) = g^{-1}\mathrm{SL}(\Lambda)g$ and hence $g\Gamma g^{-1} \subseteq \mathrm{SL}_2(\mathbb{Z})$.

We still need to show that $g\Gamma g^{-1}$ contains a principal congruence subgroup. This is the content of the next lemma which completes the proof. $\square$

Lemma 2.6. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$ be an arithmetic subgroup and $g \in \mathrm{GL}_2(\mathbb{Q})$. Then $g\Gamma g^{-1}$ is again an arithmetic subgroup.*

Proof. Let d be the least common multiple of all the denominators of all the entries of g and g^{-1}. Then, if $A \in d^2 m\mathrm{M}_2(\mathbb{Z})$ is an integer matrix divisible by $d^2 m$, we find $g^{-1}Ag \in m\mathrm{M}_2(\mathbb{Z})$. This shows that $g^{-1}\Gamma(d^2 m)g \subseteq \Gamma(m)$ which is equivalent to

$$\Gamma(d^2 m) \ \subseteq \ g\Gamma(m)g^{-1}. \tag{2.6}$$

Now, for the given Γ, choose n with $\Gamma(n) \subseteq \Gamma$. Conjugating this relation by g and using (2.6), we find $\Gamma(d^2 n) \subseteq g\Gamma g^{-1}$ which proves that $g\Gamma g^{-1}$ is again arithmetic. $\square$

In other words, Proposition 2.5 shows that the arithmetic subgroups in $\mathrm{SL}_2(\mathbb{Q})$ are precisely the $\mathrm{GL}_2(\mathbb{Q})$-conjugates of groups between $\mathrm{SL}_2(\mathbb{Z})$ and some $\Gamma(n)$.

2.3. Stabilizers.

Definition 2.7. We say that an arithmetic subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$ is *neat* if it is torsion free.

Proposition 2.8. *Let Γ be a neat arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Q})$. Then Γ acts with trivial stabilizers on $\mathbb{H}$. That is, if $\gamma\tau = \tau$ for some $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$, then $\gamma = 1$.*

Proof. We have seen in (1.4) that the stabilizer of $i \in \mathbb{H}$ in $\mathrm{GL}_2(\mathbb{R})$ is a copy of $\mathbb{C}^\times$. The unit circle $\mathbb{C}^1 \subset \mathbb{C}^\times$ is compact and equals the intersection $\mathbb{C}^\times \cap \mathrm{SL}_2(\mathbb{R})$. For a general point $\tau \in \mathbb{H}$, we can write $\tau = g \cdot i$ for some $g \in \mathrm{SL}_2(\mathbb{R})$:

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a^{1/2} & \\ & a^{-1/2} \end{pmatrix} \cdot i = ai + b.$$

The stabilizers S_i and S_τ of τ and i in $\mathrm{SL}_2(\mathbb{R})$ are then related by $S_\tau = gS_ig^{-1}$. In this way, we see that for every $\tau \in \mathbb{H}$, the stabilizer $S_\tau \subset \mathrm{SL}_2(\mathbb{R})$ is isomorphic to $\mathbb{C}^1$, in particular compact.

Assume that $\gamma\tau = \tau$, where $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$. This is equivalent to $\gamma \in \Gamma \cap S_\tau$. Since $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup, the intersection $\Gamma \cap S_\tau$ is a discrete subgroup of S_τ. Since the discrete subgroups of $\mathbb{C}^1$ are all finite cyclic (generated by a root of unity), and since Γ is torsion-free by assumption, we see that $\Gamma \cap S_\tau = \{1\}$. Hence $\gamma = 1$, and the proof is complete. $\square$

The next proposition provides a simple criterion for detecting neatness.

Proposition 2.9. *For all $n \geq 3$, the principal congruence subgroup $\Gamma(n)$ is neat. In particular, if $\Gamma \subseteq \Gamma(n)$ is an arithmetic subgroup, then Γ is neat.*

Proof. The minimal polynomial $\Phi_d(T)$ of a primitive d-th root of unity has degree $\varphi(d)$ (Euler φ-function). Recall that $\Phi_d(T)$ is called the d-th cyclotomic polynomial and that

$$T^m - 1 = \prod_{d \mid m} \Phi_d(T)$$

because the roots of $T^m - 1$ are precisely the m-th roots of unity, and each such root of unity is a primitve d-th root of unity for a unique divisor $d \mid m$.

The only values for d such that $\varphi(d) \leq 2$ are 1, 2, 3, 4, and 6. These are precisely the values for d such that $\mathbb{Q}(\zeta_d)$ has degree ≤ 2 over $\mathbb{Q}$.

Let $n \geq 1$ and let $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ be a torsion element, say $\gamma^m = 1$. Then the minimal polynomial of γ divides $T^m - 1$. We know that the minimal polynomial and the characteristic polynomial of a matrix have the same irreducible factors. So the characteristic polynomial $P(T)$ of γ is a product of $\Phi_d(T)$ with $d \mid m$. The only possibilities for $P(T)$ are hence[1]

$$(T-1)^2, \ (T+1)^2, \ (T-1)(T+1), \ T^2+1, \ T^2+T+1, \text{ and } T^2-T+1. \qquad (2.7)$$

If $n \geq 3$ and if γ is integral with $\gamma \equiv 1 \bmod n$, then also $P(T) \equiv (T-1)^2 \bmod n$, leaving $P(T) = (T-1)^2$ as the only possibility. This means that γ is either equal to 1 or $\mathrm{GL}_2(\mathbb{Q})$-conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix} \right)$ (Jordan normal form). But γ is also a torsion element by assumption, so $\gamma = 1$ is the only possibility. $\qquad \square$

Exercise 2.10. Extend the argument of the previous proof to GL_n. That is, given $n \geq 1$, find an integer $m \geq 1$ such that for $\gamma \in \mathrm{GL}_n(\mathbb{Z})$,

$$\gamma \equiv 1 \bmod m \quad \implies \quad \gamma \text{ non-torsion.}$$

Conclusion 2.11. In this lecture, we saw the definition of neat arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Q})$. We have seen in Proposition 2.8 that such groups act freely on $\mathbb{H}$. So the quotient $\Gamma\backslash\mathbb{H}$ will be a Riemann surface and the quotient map

$$\mathbb{H} \longrightarrow \Gamma\backslash\mathbb{H} \qquad (2.8)$$

a holomorphic covering map in the sense of topology. We have seen in Proposition 2.5 that, in order to study $\Gamma\backslash\mathbb{H}$, we may always assume $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then we can think of $\Gamma\backslash\mathbb{H}$ as being glued from finitely many $\mathrm{SL}_2(\mathbb{Z})$-translates of the fundamental domain $\mathcal{F}$ as in Figure 2.1 along their edges.

---

[1] The product $(T-1)(T+1)$ cannot actually occur, of course, because $\det(\gamma) = 1$ for $\gamma \in \mathrm{SL}_2(\mathbb{Q})$. This does not affect the argument, though.

3. Adelic double quotients

In this lecture, we study the adelic double quotients $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K)$ and relate them to the quotients $\Gamma\backslash\mathbb{H}$ from the previous lecture. We will first revisit the definition of the adeles and explain the definition of $\mathrm{GL}_2(\mathbb{A}_f)$ as a topological group in more detail. In fact, we will use this opportunity to also study groups of the form $G(\mathbb{A}_f)$ more generally.

3.1. **The adeles.** We begin by defining the ring of *integral adeles*. It is the profinite ring given by[2] $\widehat{\mathbb{Z}} := \lim \mathbb{Z}/n\mathbb{Z}$. The transition maps here are given by the projections $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, whenever $m \mid n$. Concretely, we have

$$\widehat{\mathbb{Z}} = \Big\{(x_1, x_2, \ldots) \in \prod_{n\geq 1} \mathbb{Z}/n\mathbb{Z} \;\Big|\; x_{dn} \equiv x_n \bmod n \text{ for all } d, n \geq 1\Big\}.$$

Recall that the Chinese remainder theorem identifies $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \prod_p \mathbb{Z}/p^{v_p(n)}\mathbb{Z}$. If we apply this identification to each term of the limit, then we obtain an isomorphism

$$\widehat{\mathbb{Z}} \xrightarrow{\sim} \prod_p \mathbb{Z}_p, \qquad (x_1, x_2, \ldots) \longmapsto \big((x_1, x_p, x_{p^2}, \ldots)\big)_p. \tag{3.1}$$

We endow each $\mathbb{Z}_p$ with the usual p-adic topology and their product with the product topology. Then (3.1) is an isomorphism of topological rings.

Definition 3.1. The *ring of finite adeles* is defined by $\mathbb{A}_f := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Since $\widehat{\mathbb{Z}}$ is torsion-free, we can view it as a subring $\widehat{\mathbb{Z}} \subset \mathbb{A}_f$. We endow $\mathbb{A}_f$ with the topology such that $\widehat{\mathbb{Z}}$ is an open subring.

Let us unravel this definition. First, on the level of rings, $\mathbb{A}_f$ is the ring of fractions x/m with $x \in \widehat{\mathbb{Z}}$ and $m \geq 1$, where the usual rules of arithmetic apply. Using (3.1), we can more explicitly describe it as the subring

$$\mathbb{A}_f = \Big\{(x_p) \in \prod_p \mathbb{Q}_p \;\Big|\; x_p \in \mathbb{Z}_p \text{ for almost all } p\Big\}.$$

Now we describe the topology. In $\widehat{\mathbb{Z}}$, a neighborhood basis of 0 is given by all the kernels of the projections $\widehat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$. These are precisely the ideals $n\widehat{\mathbb{Z}}$. Under the isomorphism (3.1), they are the subsets of the form

$$\prod_{p\in S} p^{m_p}\mathbb{Z}_p \times \prod_{p\notin S} \mathbb{Z}_p$$

where S is a finite set of primes and $(m_p)_{p\in S}$ a tuple of non-negative integers. Such sets forming a neighborhood basis of 0 means that the sets

$$\big\{x + n\widehat{\mathbb{Z}} \;\big|\; x \in \widehat{\mathbb{Z}},\, n \geq 1\big\} \tag{3.2}$$

give a basis of the topology on $\widehat{\mathbb{Z}}$. Declaring $\widehat{\mathbb{Z}} \subset \mathbb{A}_f$ an open subring then simply means that the sets $n\widehat{\mathbb{Z}}$ also form a neighborhood basis of 0 in $\mathbb{A}_f$. Equivalently, the sets

$$\big\{x + n\widehat{\mathbb{Z}} \;\big|\; x \in \mathbb{A}_f,\, n \geq 1\big\} \tag{3.3}$$

provide a basis for the topology on $\mathbb{A}_f$.

Definition 3.2. The ring of adeles is defined as the product $\mathbb{A} := \mathbb{A}_f \times \mathbb{R}$ endowed with the product topology.

Proposition 3.3. *The subring $\mathbb{Q} \subset \mathbb{A}$ is discrete.*

---

[2]We use lim and colim to denote the limit and the colimit. In other references, these might be called $\varprojlim$ and $\varinjlim$.

Proof. By definitions, the product $U = \widehat{\mathbb{Z}} \times (-1, 1)$ is an open subset of $\mathbb{A}$. The intersection $U \cap \mathbb{Q}$ consists of those rational numbers that lie in $\mathbb{Z} = \widehat{\mathbb{Z}} \cap \mathbb{Q}$ and in the interval $(-1, 1)$. In other words, $U \cap \mathbb{Q} = \{0\}$. Thus, $\{0\} \subset \mathbb{Q}$ is an open subset for the subspace topology. By additive translation invariance of the topology ($\mathbb{A}$ is a topological ring), the same argument applies for all rational numbers. This shows that the subspace topology on $\mathbb{Q}$ is the discrete topology as claimed. $\qquad\square$

Let $F/\mathbb{Q}$ be a finite extension. The adeles of F can be defined in the same way as for $\mathbb{Q}$. First, we define the integral adeles with profinite topology

$$\widehat{O}_F := \varprojlim_{\mathfrak{a} \subseteq O_F} O_F/\mathfrak{a} \xrightarrow{\sim} \prod_{\mathfrak{p}} O_{F,\mathfrak{p}}. \tag{3.4}$$

The we tensor by $\mathbb{Q}$ over $\mathbb{Z}$, or equivalently by F over O_F, to define the finite adeles:

$$\begin{aligned} \mathbb{A}_{F,f} &:= \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{O}_F \\ &\xrightarrow{\sim} \Big\{ (x_{\mathfrak{p}}) \in \prod_{\mathfrak{p}} F_{\mathfrak{p}} \,\Big|\, x_{\mathfrak{p}} \in O_{F,\mathfrak{p}} \text{ for almost all } \mathfrak{p} \Big\}. \end{aligned} \tag{3.5}$$

Again, the topology on $\mathbb{A}_{F,f}$ is defined by declaring $\widehat{O}_F$ to be an open subring. Finally, we define the adeles as the product

$$\mathbb{A}_F := \mathbb{A}_{F,f} \times (\mathbb{R} \otimes_{\mathbb{Q}} F) \xrightarrow{\sim} \mathbb{A}_{F,f} \times \prod_{\sigma: F \to \mathbb{R}} \mathbb{R} \times \prod_{\{\sigma, \overline{\sigma}\}: F \to \mathbb{C}} \mathbb{C}. \tag{3.6}$$

Here, the real factors have their real vector space topology, and the last two products are over the real (resp. complex) places of F.

Recall that O_F is a free abelian group of rank equal to $d = [F : \mathbb{Q}]$. Let $\alpha_1, \dots, \alpha_d$ be a $\mathbb{Z}$-module basis of O_F. Such a choice provides isomorphisms of $\widehat{\mathbb{Z}}$-, $\mathbb{A}_f$-, resp. $\mathbb{A}$-modules

$$\widehat{\mathbb{Z}}^n \xrightarrow{\sim} O_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \quad \mathbb{A}_f^n \xrightarrow{\sim} F \otimes_{\mathbb{Q}} \mathbb{A}_f, \quad \mathbb{A}^n \xrightarrow{\sim} F \otimes_{\mathbb{Q}} \mathbb{A}. \tag{3.7}$$

We endow $\widehat{\mathbb{Z}}^n$, $\mathbb{A}_f^n$ and $\mathbb{A}^n$ with the product topology and use the isomorphisms in (3.7) to define from this the topology on the three tensor products. This topology is independent of the choice of $\alpha_1, \dots, \alpha_d$.

Remark 3.4. The previous definition is a general principle. Let R be a topological ring and let M be a finite free R-module. Any choice of R-basis $\alpha_1, \dots, \alpha_d$ defines an isomorphism $R^d \xrightarrow{\sim} M$ and, in this way, endows M with a topology.

Any two such isomorphisms differ by an element of $\mathrm{GL}_d(R)$. Since the action of every $g \in \mathrm{GL}_d(R)$ on R^d is continuous, the topology is independent of the chosen basis.

Proposition 3.5. *Multiplication defines isomorphisms of topological rings*

$$O_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \xrightarrow{\sim} \widehat{O}_F, \quad F \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} \mathbb{A}_{F,f}, \quad F \otimes_{\mathbb{Q}} \mathbb{A} \xrightarrow{\sim} \mathbb{A}_F.$$

Proof. Every ideal $\mathfrak{a} \subseteq O_F$ contains an ideal nO_F with $n \in \mathbb{Z}_{\geq 1}$. So we can rewrite (3.4) as $\widehat{O}_F = \varprojlim O_F/nO_F$. Having chosen $\alpha_1, \dots, \alpha_d$, we obtain

$$\begin{aligned} \widehat{O}_F &= \varprojlim \Big(\bigoplus_{i=1}^{d} \mathbb{Z}/n\mathbb{Z} \cdot \alpha_i \Big) \\ &\xrightarrow{\sim} \bigoplus_{i=1}^{d} \big(\varprojlim \mathbb{Z}/n\mathbb{Z} \big) \cdot \alpha_i \\ &\xrightarrow{\sim} \bigoplus_{i=1}^{d} \widehat{\mathbb{Z}} \cdot \alpha_i. \end{aligned}$$

This shows that $O_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \xrightarrow{\sim} \widehat{O}_F$ as topological rings. The statements for $\mathbb{A}_{F,f}$ and $\mathbb{A}_F$ follow from this. $\qquad\square$

Corollary 3.6. *Let $F/\mathbb{Q}$ be a finite extension. Then $F \subset \mathbb{A}_F$ is discrete.*

Proof. Since $\mathbb{Q}$ is discrete in $\mathbb{A}$, we have that $\mathbb{Q}^n$ is discrete in $\mathbb{A}^n$. Choosing a $\mathbb{Q}$-basis $\alpha = (\alpha_1, \dots, \alpha_d)$ for F, we obtain a commutative square of the form

$$
\begin{array}{ccc}
\mathbb{Q}^n & \hookrightarrow & \mathbb{A}^n \\
\alpha \| & & \| \alpha \\
F & \hookrightarrow & \mathbb{A}_F.
\end{array}
$$

By Proposition 3.5, the right vertical identification is a homeomorphism. Hence we obtain that F is discrete in $\mathbb{A}_F$. $\qquad\square$

3.2. **Groups of the form** $G(\mathbb{A}_f)$. Let us formulate the problem more generally.

Question 3.7. Let X be an affine variety[3] over $\mathbb{Q}$ and let R be a topological $\mathbb{Q}$-algebra. We assume that points of R are closed. For example, R could be $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, $\mathbb{A}_f$ or $\mathbb{A}$. How to define the topological space $X(R)$ in a natural way?

The answer is very simple. Let us write $\mathcal{A}^N = \operatorname{Spec} \mathbb{Q}[t_1, \dots, t_N]$ for affine N-space over $\mathbb{Q}$ to avoid confusion with the adele notation. We endow $\mathcal{A}^N(R) = R^N$ with the product topology.

Let $f_1, \dots, f_m \in \mathbb{Q}[t_1, \dots, t_N]$ be polynomials and let $X = V(f_1, \dots, f_m) \subseteq \mathcal{A}^N$ be their vanishing locus. Then $X(R) \subseteq R^N$ is a closed subset because it equals the intersection $\cap_{i=1}^m f_i^{-1}(0)$, and we endow it with the subspace topology.

Definition 3.8. Let X be an affine $\mathbb{Q}$-variety. Choose a presentation $\varphi : X \xrightarrow{\sim} V(f_1, \dots, f_m)$ as above. The topology on $X(R)$ is defined as the subspace topology with respect to $\varphi(R) : X(R) \hookrightarrow R^N$.

Lemma 3.9. *This topology on $X(R)$ is independent of the choices of N, $(f_1, \dots, f_m)$ and φ.*

Proof. Assume that we are given two affine varieties $V(f_1, \dots, f_{m_1}) \subseteq \mathcal{A}^{N_1}$ as well as $V(g_1, \dots, g_{m_2}) \subseteq \mathcal{A}^{N_2}$. Assume that

$$
\varphi : V(f_1, \dots, f_{m_1}) \xrightarrow{\sim} V(g_1, \dots, g_{m_2})
$$

is an isomorphism of $\mathbb{Q}$-varieties. Then φ and $\psi = \varphi^{-1}$ lift to morphisms $\Phi : \mathcal{A}^{N_1} \to \mathcal{A}^{N_2}$ and $\Psi : \mathcal{A}^{N_2} \to \mathcal{A}^{N_1}$. The induced maps

$$
R^{N_1} \overset{\Phi}{\underset{\Psi}{\rightleftarrows}} R^{N_2}
$$

are continuous because they are given by polynomials. Hence their restrictions φ and ψ are continuous as well. Since $\psi = \varphi^{-1}$, this shows that φ is a homeomorphism. $\qquad\square$

Example 3.10. Consider the group variety GL_n. One possible presentation as a closed subset of an affine space is given by

$$
\operatorname{GL}_n \xrightarrow{\sim} V\big(1 - t \cdot \det\big((t_{ij})_{i,j=1}^n\big)\big) \subset \mathcal{A} \times_{\operatorname{Spec}(\mathbb{Q})} \mathcal{A}^{n^2}
$$

$$
g = (t_{ij})_{i,j=1}^n \longmapsto (\det(g)^{-1}, g).
$$

For example, if $n = 1$, we recover the closed immersion[4]

$$
\mathbb{G}_m \hookrightarrow \mathcal{A}^2, \quad t \longmapsto (t^{-1}, t).
$$

---

[3]More generally, an affine finite type $\mathbb{Q}$-scheme.

[4]$\mathbb{G}_m$ is just another notation for GL_1. The notation symbolizes *multiplicative group*.

According to Definition 3.8, the topology on $\mathrm{GL}_n(\mathbb{A}_f)$ is then given as the subspace topology with respect to
$$\mathrm{GL}_n(\mathbb{A}_f) \hookrightarrow \mathbb{A}_f \times \mathrm{M}_n(\mathbb{A}_f), \quad g \longmapsto (\det(g)^{-1}, g).$$
The product $\widehat{\mathbb{Z}} \times \mathrm{M}_n(\widehat{\mathbb{Z}})$ is an open subset on the right hand side. So the intersection
$$\mathrm{GL}_n(\widehat{\mathbb{Z}}) = \mathrm{GL}_n(\mathbb{A}_f) \cap (\widehat{\mathbb{Z}} \times \mathrm{M}_n(\widehat{\mathbb{Z}}))$$
is an open subset of $\mathrm{GL}_n(\mathbb{A}_f)$. (The elements of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ are precisely those elements of $\mathrm{GL}_n(\mathbb{A}_f) \cap \mathrm{M}_n(\widehat{\mathbb{Z}})$ whose inverse determinant again lies in $\widehat{\mathbb{Z}}$.) As a closed subset of the profinite set $\widehat{\mathbb{Z}} \times \mathrm{M}_n(\widehat{\mathbb{Z}})$, $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ is again profinite. In fact, we have
$$\mathrm{GL}_n(\widehat{\mathbb{Z}}) \xrightarrow{\sim} \lim_{m \geq 1} \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$$
as topological group. The principal congruence subgroups
$$K(m) := \ker\left(\mathrm{GL}_n(\widehat{\mathbb{Z}}) \longrightarrow \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})\right)$$
form a neighborhood basis of 1 in $\mathrm{GL}_n(\mathbb{A}_f)$.

Example 3.11. We always view $\mathbb{A}_f^\times$ with the topology coming from $\mathbb{A}_f^\times = \mathbb{G}_m(\mathbb{A}_f)$. Then the inclusion map $\mathbb{A}_f^\times \to \mathbb{A}_f$ is continuous because it is induced from the morphism of varieties $\mathbb{G}_m \to \mathcal{A}$, $t \mapsto t$. But it is not an open immersion. For example, $\widehat{\mathbb{Z}}^\times$ is open in $\mathbb{A}_f^\times$, but not in $\mathbb{A}_f$.

Exercise 3.12. Prove the claim in the previous example. That is, show that none of the open subsets $1 + n\widehat{\mathbb{Z}} \subseteq \widehat{\mathbb{Z}}$, which form a neighborhood basis of $1 \in \widehat{\mathbb{Z}}$, is contained in $\widehat{\mathbb{Z}}^\times$.

Example 3.13. Let G be a general linear algebraic group over $\mathbb{Q}$. There always exist some $N \geq 1$ and a closed immersion $G \hookrightarrow \mathrm{GL}_N$. Then $G(\mathbb{A}_f) \subseteq \mathrm{GL}_N(\mathbb{A}_f)$ has the subspace topology. In particular, the intersections $G(\mathbb{A}_f) \cap K(m)$ with all congruence subgroups form a neighborhood basis of $1 \in G(\mathbb{A}_f)$.

This applies, for example, to the standard representations
$$\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2, \quad \mathrm{Sp}_{2g} \hookrightarrow \mathrm{GL}_{2g}, \quad \mathrm{GSp}_{2g} \hookrightarrow \mathrm{GL}_{2g}.$$
Let V be a quadratic $\mathbb{Q}$-vector space. Then it applies to the closed immersions
$$SO(V) \hookrightarrow \mathrm{GL}(V), \quad O(V) \hookrightarrow \mathrm{GL}(V).$$

Remark 3.14. For local fields k, such as $k \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}_p\}$, the situation is more straightforward in the following sense. If $X \hookrightarrow Y$ is an open immersion of k-varieties, then $X(k) \to Y(k)$ is an open immersion with respect to the topologies from Definition 3.8. In particular, the topology on $X(k)$ from Definition 3.8 agrees with the subspace topology in $Y(k)$.

This remark applies, for example, to
$$\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R}) \quad \text{and} \quad \mathrm{GL}_n(\mathbb{Q}_p) \subset \mathrm{M}_n(\mathbb{Q}_p).$$

3.3. **General adelic double quotients.** Let us begin with a general theorem which we will not prove.

Theorem 3.15 ([11, Theorem 4.16]). *(1) Let $G/\mathbb{Q}$ be a connected reductive algebraic group. Then, for every compact open subgroup $K \subset G(\mathbb{A}_f)$, the double quotient $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite.*

(2, Strong approximation) Let $G/\mathbb{Q}$ be a connected, simply connected semi-simple group of non-compact type. Then $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$. In particular, for every compact open subgroup $K \subset G(\mathbb{A}_f)$,
$$G(\mathbb{A}_f) = \{\gamma \cdot k \mid \gamma \in G(\mathbb{Q}), \ k \in K\}.$$

As our first application, we obtain a more concrete description of the adelic double quotients that make up the complex points of a Shimura variety (1.6). Let (G, X) be a Shimura datum and let $K \subset G(\mathbb{A}_f)$ be a level subgroup. In particular, G is a connected reductive group over $\mathbb{Q}$, so Theorem 3.15 (1) applies. So we find finitely many double coset representatives $g_1, \ldots, g_r \in G(\mathbb{A}_f)$,

$$G(\mathbb{A}_f) = \bigsqcup_{i=1}^{r} G(\mathbb{Q}) g_i K. \tag{3.8}$$

Each of the sets on the right hand side of (3.8) is $G(\mathbb{Q})$-stable. Moreover, $G(\mathbb{Q})$ acts transitively on the cosets $G(\mathbb{Q})g_i K/K$, and the stabilizer of the coset $g_i K \in G(\mathbb{Q})g_i K/K$ is the subgroup

$$\Gamma_i := G(\mathbb{Q}) \cap g_i K g_i^{-1}.$$

So we obtain

$$
\begin{aligned}
G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K) &= \bigsqcup_{i=1}^{r} G(\mathbb{Q}) \backslash (X \times G(\mathbb{Q})g_i K/K) \\
&\xrightarrow{\sim} \bigsqcup_{i=1}^{r} \Gamma_i \backslash X \times \{g_i K\}.
\end{aligned}
\tag{3.9}
$$

If K is small enough, which we will make precise for GL_2 in a minute, then each Γ_i is torsion-free and acts without stabilizers on X. Each quotient $\Gamma_i \backslash X$ is then a complex manifold in the same way as we saw before in Conclusion 2.11.

Exercise 3.16. Work out (3.9) for yourself. For example, first prove the following variant. Let H be a group acting on sets X and Y. Let $Y = \bigsqcup_{i \in I} G \cdot y_i$ be the decomposition of Y into orbits and let Γ_i be the stabilizer of y_i in H. Then

$$H \backslash (X \times Y) \xrightarrow{\sim} \bigsqcup_{i \in I} \Gamma_i \backslash X.$$

Specialize to the situation $H = G(\mathbb{Q})$ and $Y = G(\mathbb{A}_f)/K$.

Exercise 3.17. The group SL_n is connected, simply connected, semi-simple and of non-compact type, so $\mathrm{SL}_n(\mathbb{Q}) \subset \mathrm{SL}_n(\mathbb{A}_f)$ is dense (Strong approximation, see Theorem 3.15 (2)). Using this property, show that

$$\mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})$$

is surjective for all $m \geq 1$. In particular, this shows the surjectivity of (2.5).

3.4. **Back to GL_2.** The description in (3.9) is still quite abstract. We now want to make it completely explicit for congruence subgroups of GL_2. Let us begin by studying $\mathbb{G}_m$.

Proposition 3.18. Let $K(m) = \ker\left(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/m\mathbb{Z})^\times\right)$ be the m-th congruence subgroup of $\mathbb{A}_f^\times$. Then there is an isomorphism

$$\mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K(m) \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^\times. \tag{3.10}$$

Proof. Let $x = (x_p)_p \in \mathbb{A}_f^\times$ be an element. Here, the component x_p lies in $\mathbb{Q}_p^\times$, and almost all components x_p even lie in $\mathbb{Z}_p^\times$. For each prime p, let $v_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ denote the valuation normalized by $v_p(p) = 1$. Take the vector of valuations of all the entries of x:

$$(e_p)_p, \quad e_p = v_p(x_p).$$

Only finitely many of the e_p are non-zero. There is a rational number in $\mathbb{Q}_{>0}$ with the same valuations, namely $t = \prod_p p^{e_p}$. So $t^{-1}x$ lies in $\widehat{\mathbb{Z}}^\times$ which shows that every double coset in (3.10) has a representative in $\widehat{\mathbb{Z}}^\times$. Purely formally, we now obtain

$$\mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K(m) \xrightarrow{\sim} \big(\mathbb{Q}_{>0}^\times \cap \widehat{\mathbb{Z}}^\times\big) \backslash \widehat{\mathbb{Z}}^\times / K(m). \tag{3.11}$$

The rational number t is, in fact, uniquely determined which reflects that $\mathbb{Q}_{>0}^\times \cap \widehat{\mathbb{Z}}^\times = \{1\}$. So (3.11) simplifies to $\widehat{\mathbb{Z}}^\times / K(m)$, which is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^\times$ as claimed. $\square$

We write $\mathrm{GL}_n(\mathbb{Q})_{>0}$ for the subgroup of elements of $\mathrm{GL}_n(\mathbb{Q})$ with positive determinant.

Proposition 3.19. *Let $K \subset \mathrm{GL}_n(\mathbb{A}_f)$ be an open compact subgroup. The determinant map $\det : \mathrm{GL}_n(\mathbb{A}_f) \to \mathbb{A}_f^\times$ induces a bijection*

$$\det : \mathrm{GL}_n(\mathbb{Q})_{>0} \backslash \mathrm{GL}_n(\mathbb{A}_f) / K \xrightarrow{\sim} \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / \det(K). \tag{3.12}$$

Proof. The group SL_n is connected, simply connected, semi-simple and of non-compact type, so $\mathrm{SL}_n(\mathbb{Q}) \subset \mathrm{SL}_n(\mathbb{A}_f)$ is dense (Strong approximation, see Theorem 3.15 (2)). We will use this property freely.

Consider the determinant map in (3.12). It is clearly surjective because already the map $\det : \mathrm{GL}_n(\mathbb{A}_f) \to \mathbb{A}_f^\times$ is surjective. So our task is to prove that (3.12) is injective.

The source in (3.12) is only a set, so we cannot argue with kernels. Instead, we consider two elements $g_1, g_2 \in \mathrm{GL}_n(\mathbb{A}_f)$ with the same image, meaning that

$$\det(g_1) \in \mathbb{Q}_{>0}^\times \det(g_2) \det(K). \tag{3.13}$$

Our task is to show that $g_1 \in \mathrm{GL}_n(\mathbb{Q})g_2 K$.

First, observe that $\det : \mathrm{GL}_n(\mathbb{Q})_{>0} \to \mathbb{Q}_{>0}^\times$ is surjective. So we find elements $h \in \mathrm{GL}_n(\mathbb{Q})_{>0}$ and $k \in K$ such that $\det(g_1) = \det(hg_2k)$. So after replacing g_2 by hg_2k, we may assume $\det(g_1) = \det(g_2)$.

Next, we consider the conjugate group $g_2 K g_2^{-1}$. Strong approximation for SL_n implies that

$$\mathrm{SL}_n(\mathbb{A}_f) = \mathrm{SL}_n(\mathbb{Q}) \cdot (g_2 K g_2^{-1} \cap \mathrm{SL}_n(\mathbb{A}_f)).$$

Hence, there are $h' \in \mathrm{SL}_n(\mathbb{Q})$ and $k' \in K \cap \mathrm{SL}_n(\mathbb{A}_f)$ with

$$g_1 g_2^{-1} = h' g_2 k' g_2^{-1}.$$

This is equivalent to $g_1 = h' g_2 k'$, showing that the double cosets of g_1 and g_2 are equal as claimed. $\square$

Corollary 3.20. *Let $K(m) \subset \mathrm{GL}_2(\mathbb{A}_f)$ be the m-th congruence subgroup. There is a bijection of connected components*

$$\pi_0\big(\mathrm{GL}_2(\mathbb{Q}) \backslash (\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f) / K(m))\big) \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^\times. \tag{3.14}$$

Moreover, the connected components are all of the form $\Gamma \backslash \mathbb{H}$ with $\Gamma = \mathrm{GL}_2(\mathbb{Q})_{>0} \cap gK(m)g^{-1}$ for some element $g \in \mathrm{GL}_2(\mathbb{A}_f)$.

Proof. The two connected components of $\mathbb{H}^\pm$ are interchanged by the elements of negative determinant in $\mathrm{GL}_2(\mathbb{Q})$. Hence, we obtain

$$\pi_0\big(\mathrm{GL}_2(\mathbb{Q}) \backslash (\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f) / K(m))\big) \xrightarrow{\sim} \pi_0\big(\mathrm{GL}_2(\mathbb{Q})_{>0} \backslash (\mathbb{H} \times \mathrm{GL}_2(\mathbb{A}_f) / K(m))\big)$$

$$\xrightarrow{\sim} \quad \mathrm{GL}_2(\mathbb{Q})_{>0} \backslash \mathrm{GL}_2(\mathbb{A}_f) / K(m).$$
$$\tag{3.15}$$

Here, the second isomorphism simply used that $\mathbb{H}$ is connected. Next, observe that

$$L := \det(K(m)) = \ker\big(\widehat{\mathbb{Z}}^\times \longrightarrow (\mathbb{Z}/m\mathbb{Z})^\times\big)$$

is the m-th congruence subgroup in $\mathbb{A}_f^\times$. So, by Proposition 3.19, the determinant allows to rewrite (3.15) as

$$\det : \mathrm{GL}_2(\mathbb{Q})_{>0} \backslash \mathrm{GL}_2(\mathbb{A}_f)/K(m) \xrightarrow{\sim} \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times /L.$$

By Proposition 3.18, the last expression can be identified with $(\mathbb{Z}/m\mathbb{Z})^\times$ as claimed.

The final statement (each connected component being isomorphic to some $\Gamma \backslash \mathbb{H}$ with Γ of the form $\mathrm{GL}_2(\mathbb{Q})_{>0} \cap gK(m)g^{-1}$) is a special case of the decomposition in (3.9), except that we have already replaced $(\mathbb{H}^\pm, \mathrm{GL}_2(\mathbb{Q}))$ by $(\mathbb{H}, \mathrm{GL}_2(\mathbb{Q})_{>0})$. $\qquad\square$

Let us go further and prove a criterion that ensures that all the occurring Γ are torsion free. The arguments will be similar to the ones we saw in §2.3.

Proposition 3.21. *For any $m \geq 3$ and $g \in \mathrm{GL}_2(\mathbb{A}_f)$, the intersection $\Gamma = \mathrm{GL}_2(\mathbb{Q}) \cap gK(m)g^{-1}$ is torsion free.*

Proof. Let γ be an element of $K(m)$. Then, since $K(m) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$, the characteristic polynomial $P_\gamma(T)$ lies in $\widehat{\mathbb{Z}}[T]$. Since $\gamma \equiv 1 \bmod m$, we even know $P_\gamma(T) \equiv (T-1)^2 \bmod m$. In general, for every $n \geq 1$ and any ring R, the characteristic polynomial of an element from $\mathrm{GL}_n(R)$ is invariant under conjugation. So, in our setting, the same properties hold for $P_\gamma(T)$ for $\gamma \in gK(m)g^{-1}$.

Assume that $\gamma \in \Gamma = \mathrm{GL}_2(\mathbb{Q}) \cap gK(m)g^{-1}$. Then, the characteristic polynomial of γ has rational coefficients, and hence lies in the intersection

$$\mathbb{Q}[T] \cap \big((T-1)^2 + n\widehat{\mathbb{Z}}[T]\big).$$

This means that $P_\gamma(T) \in \mathbb{Z}[T]$ and $P_\gamma(T) \equiv (T-1)^2 \bmod m$.

If γ is a torsion element, then we have already seen during the proof of Proposition 2.9 that $P_\gamma(T)$ comes from the list (2.7). By the congruence condition we just established, the only possibility is $P_\gamma(T) = (T-1)^2$. The matrix $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ is not torsion, so cannot be the Jordan normal form of γ. We conclude that $\gamma = 1$, showing that Γ is torsion-free as claimed. $\qquad\square$

Conclusion 3.22. Let us come back to the situation of Corollary 3.20. Assume that $m \geq 3$. Then the connected components of

$$\mathrm{GL}_2(\mathbb{Q}) \backslash \big(\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)\big)$$

are in natural bijection with $(\mathbb{Z}/m\mathbb{Z})^\times$. Each connected component is of the form $\Gamma \backslash \mathbb{H}$ for a torsion free arithmetic subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Q})$.

4. Group schemes

Our current aim is to endow the complex manifolds $\mathrm{Sh}_K(\mathrm{GL}_2, \mathbb{H}^{\pm})(\mathbb{C})$ with an algebraic structure and to even define them over $\mathbb{Q}$ (see 1.7). This relies on their description as moduli spaces of elliptic curves:

Definition 4.1. Let k be a field. An *elliptic curve* over k is a proper, smooth, connected and 1-dimensional k-group scheme.

Later in the course, we will also consider other Shimura varieties and describe them as moduli spaces of abelian varieties:

Definition 4.2. An *abelian variety* over k is a proper, smooth and connected k-group scheme.

In this lecture, we will first discuss some background on group schemes. This will also be useful for talking about groups like GL_n, GSp_{2g} etc. which we have secretly already considered as group schemes over $\mathrm{Spec}\,\mathbb{Z}$ or $\mathrm{Spec}\,\mathbb{Q}$ in previous lectures. In general, group schemes are also an interesting topic in itself and come up in all areas of algebra.

Recommended reading closely related to our course: My lecture notes on moduli spaces of elliptic curves [3]. Parts of our discussion here are taken from [3, §2].

General reference on algebraic groups: Milne's book [12], especially §1 about basic definitions.

4.1. **Basic definitions.** We give the definition over a general base S, but the case to keep in mind is $S = \mathrm{Spec}(k)$ for a field k.

Definition 4.3. Let S be a scheme. A *group scheme* over S is a pair (G, m) that consists of an S-scheme G and an S-scheme morphism (called multiplication morphism)

$$m : G \times_S G \longrightarrow G$$

such that for every S-scheme T, the resulting map on T-valued points

$$m(T) : G(T) \times G(T) \longrightarrow G(T)$$

makes $G(T)$ into a group. We call G *commutative* if $G(T)$ is a commutative group for every T.

Observe that for every morphism $u : T' \to T$ of S-schemes, the diagram

$$\begin{array}{ccc}
G(T) \times G(T) & \xrightarrow{\ m(T)\ } & G(T) \\
{\scriptstyle u^* \times u^*}\big\downarrow & & \big\downarrow{\scriptstyle u^*} \\
G(T') \times G(T') & \xrightarrow{\ m(T')\ } & G(T')
\end{array} \tag{4.1}$$

commutes which means that $u^* : G(T) \to G(T')$ is a group homomorphism. Furthermore, if (G, m) is a group scheme over S, then the Yoneda Lemma implies the existence of two additional S-scheme morphisms:

$$\begin{aligned}
e &: S \longrightarrow G, &&\text{(neutral element section)} \\
i &: G \longrightarrow G, &&\text{(inversion morphism).}
\end{aligned} \tag{4.2}$$

The first one is simply the neutral element $e \in G(S)$ of the group $G(S)$. Given $u : T \to S$, the pullback $u^*(e) = e \circ u \in G(T)$ is the neutral element of $G(T)$. The second one is characterized as the unique morphism that provides the inverse in all the groups $\{G(T)\}_{T \to S}$:

$$i(T) : G(T) \longrightarrow G(T), \quad g \longmapsto g^{-1}.$$

The datum (G, m, e, i) satisfies the group axioms in a scheme sense, meaning that the three diagrams

$$
\begin{array}{ccc}
S \times_S G & \xrightarrow{\ e \times \mathrm{id}\ } & G \times_S G \\
& \searrow & \downarrow{\scriptstyle m} \\
& & G,
\end{array}
\tag{4.3}
$$

$$
\begin{array}{ccc}
G \times_S G \xrightarrow{\ \mathrm{id} \times i\ } G \times_S G & \qquad & G \times_S G \times_S G \xrightarrow{\ m \times \mathrm{id}\ } G \times_S G \\
\downarrow \qquad\qquad \downarrow{\scriptstyle m} & & {\scriptstyle \mathrm{id} \times m}\downarrow \qquad\qquad \downarrow{\scriptstyle m} \\
S \xrightarrow{\ e\ } G, & & G \times_S G \xrightarrow{\ m\ } G,
\end{array}
\tag{4.4}
$$

all commute. In fact, one may also reverse the above logic and obtains the more classical definition of a group scheme over S: It is the same as an S-scheme G together with a morphism $m : G \times_S G \to G$ such that there exist morphisms $e : S \to G$ and $i : G \to G$ such that the diagrams in (4.3) and (4.4) commute. The group scheme (G, m) is commutative if and only if multiplication interchanges with switching the factors in the sense that also the following diagram commutes:

$$
\begin{array}{ccc}
G \times_S G & \xrightarrow{\ (g,h) \,\mapsto\, (h,g)\ } & G \times_S G \\
{\scriptstyle m}\searrow & & \swarrow{\scriptstyle m} \\
& G. &
\end{array}
\tag{4.5}
$$

Definition 4.4. Let (G_1, m_1) and (G_2, m_2) be group schemes over S. A group scheme morphism from G_1 to G_2 is a morphism of S-schemes $f : G_1 \to G_2$ such that $m_2 \circ (f \times f) = f \circ m_1$. Equivalently, it is an S-morphism f such that for all $T \to S$, the induced map

$$
f(T) : G_1(T) \longrightarrow G_2(T)
$$

is a group homomorphism.

If (G, m) is a *commutative* S-group scheme, then $\mathrm{End}_{S-\mathrm{Grp.Sch.}}(G, m)$ forms a (possibly non-commutative) ring because endomorphisms can be "added" (meaning multiplied in G) and multiplied (meaning composed). Concretely, sum and product of two elements $f, g \in \mathrm{End}(G)$ are given by

$$
f + g := m \circ (f, g), \qquad fg := f \circ g.
$$

In particular, we can add the identity n times to itself and obtain the n-th power endomorphism $[n] : G \to G$. On each of the groups $G(T)$, it is given by $[n](g) = g^n$. This is even defined for $n \in \mathbb{Z}$ and $[n] \circ i = [-n]$. In total, these give the ring map

$$
[\cdot] : \mathbb{Z} \to \mathrm{End}(G).
\tag{4.6}
$$

Coming back to general group schemes, we next define kernels. This is straightforward because fiber products exist in the category of S-schemes. (Defining quotients, on the other hand, is tricky. We refer to [3, §13] for some cases.)

Definition 4.5. Let $f : G_1 \to G_2$ be a homomorphism of S-group schemes. Let $e_2 : S \to G_2$ be the neutral element section of G_2. The kernel of f is defined as the fiber product

$$
\begin{array}{ccc}
\ker(f) & \longrightarrow & S \\
\downarrow & & \downarrow{\scriptstyle e_2} \\
G_1 & \xrightarrow{\ f\ } & G_2.
\end{array}
\tag{4.7}
$$

It is clear from its definition that $\ker(f)$ has the property

$$\ker(f)(T) = \ker\left(f(T) : G_1(T) \longrightarrow G_2(T)\right). \tag{4.8}$$

The multiplication morphism of G_1 restricts to a multiplication on $\ker(f)$ which makes $\ker(f)$ into a group scheme:

$$
\begin{array}{ccc}
\ker(f) \times_S \ker(f) & \dashrightarrow & \ker(f) \\
\downarrow & & \downarrow \\
G \times_S G & \xrightarrow{\quad m \quad} & G.
\end{array} \tag{4.9}
$$

Remark 4.6. Recall that if $X \to S$ is a separated morphism, then every section $\sigma : S \to X$ is a closed immersion. Thus, if $G \to S$ is a separated group scheme (e.g. affine or proper), then the neutral element e is a closed immersion. It follows that if in (4.7) $G_2 \to S$ is separated, then $\ker(f) \to G_1$ is a closed immersion.

4.2. **A commutative example: The multiplicative group.** Assume that $S = \operatorname{Spec} R$ is affine. Define $\mathbb{G}_{m,S} = \operatorname{Spec} R[t, t^{-1}]$ which we would like to make into a group scheme over S. Recall that $\operatorname{Spec}(-)$ is an anti-equivalence from R-algebras to affine S-schemes. We define the multiplication map $m : \mathbb{G}_{m,S} \times_S \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$ as $\operatorname{Spec}(m^*)$ where m^* is

$$
\begin{aligned}
m^* : R[t, t^{-1}] &\longrightarrow R[t, t^{-1}] \otimes_R R[t, t^{-1}] \\
t &\longmapsto \qquad t \otimes t.
\end{aligned} \tag{4.10}
$$

We next verify that this makes $\mathbb{G}_{m,S}$ into an S-group scheme. For every S-scheme T, we identify

$$
\begin{aligned}
\mathbb{G}_{m,S}(T) &\xrightarrow{\ \sim\ } \mathcal{O}_T(T)^\times \\
[g : T \to \mathbb{G}_{m,S}] &\longmapsto g^*(t).
\end{aligned} \tag{4.11}
$$

Note that this map is obviously defined; the fact that it is an isomorphism is the adjunction $\operatorname{Mor}_S(T, \operatorname{Spec}(A)) \xrightarrow{\sim} \operatorname{Hom}_R(A, \mathcal{O}_T(T))$. Given two morphisms $g_1, g_2 : T \to \mathbb{G}_{m,S}$, we have

$$
\begin{array}{ccccc}
R[t, t^{-1}] & \xrightarrow{\ m^* \ } & R[t, t^{-1}] \otimes_R R[t, t^{-1}] & \xrightarrow{\ g_1^* \otimes g_2^* \ } & \mathcal{O}_T(T) \\
t & \longmapsto & t \otimes t & \longmapsto & g_1^*(t) g_2^*(t).
\end{array}
$$

Thus we see that the operation $m(T)$ on $\mathbb{G}_{m,S}(T)$ translates to the usual multiplication under (4.11). In particular, $m(T)$ is a group structure for every T, and hence $(\mathbb{G}_{m,S}, m)$ a group scheme.

We can next calculate the neutral element e and the inversion map i from (4.2). Under (4.11), the unit element $1 \in R^\times$ corresponds to

$$e^* : R[t, t^{-1}] \longrightarrow R, \quad t \longmapsto 1.$$

Taking $e = \operatorname{Spec}(e^*)$ gives the neutral element section. The inversion map $i = \operatorname{Spec}(i^*)$ is given by

$$i^* : R[t, t^{-1}] \longrightarrow R[t, t^{-1}], \quad t \longmapsto t^{-1}. \tag{4.12}$$

The n-th power maps are given as $[n] = \operatorname{Spec}([n]^*)$ with

$$[n]^* : R[t, t^{-1}] \longrightarrow R[t, t^{-1}], \quad t \longmapsto t^n. \tag{4.13}$$

Note that (4.12) and (4.13) are compatible in the sense that $i = [-1]$, which is always the case for a commutative group scheme. The next proposition, on the other hand, is very specific to $\mathbb{G}_m$.

Proposition 4.7. *Let S be a connected scheme. Then* $\operatorname{End}(\mathbb{G}_{m,S}) = \mathbb{Z}$.

Proof. We only consider the case $S = \mathrm{Spec}(k)$. The extension to general S can be found in [3, Proposition 2.12].

By definition, a group scheme endomorphism f of $\mathbb{G}_{m,k}$ is the same as $f = \mathrm{Spec}(f^*)$ for a unique k-algebra morphism $f^* : k[t, t^{-1}] \longrightarrow k[t, t^{-1}]$ such that

$$(f^* \otimes f^*) \circ m^* = m^* \circ f^* \tag{4.14}$$

where $m^*(t) = t \otimes t$ is as in (4.10). Giving a k-algebra morphism f^* is equivalent to specifying its image $f^*(t) \in k[t, t^{-1}]^\times$. These units are

$$k[t, t^{-1}]^\times = \{\lambda t^n \mid \lambda \in k^\times, n \in \mathbb{Z}\}.$$

If $f^*(t) = \lambda t^n$, then (4.14) evaluated at t becomes

$$\lambda t^n \otimes \lambda t^n \overset{?}{=} \lambda (t \otimes t)^n \tag{4.15}$$

which holds if and only if $\lambda^2 = \lambda$, meaning $\lambda = 1$. Note that $f^*(t) = t^n$ precisely defines the multiplication-by-n morphism $[n]$ (meaning taking n-th power in this context) and thus $\mathrm{End}(\mathbb{G}_{m,k}) = \mathbb{Z}$ is proved. $\qquad\square$

We next determine the kernel $\mu_{n,S} := \ker([n])$. By definition, see (4.7), we need to compute the fiber product

$$
\begin{array}{ccc}
\mu_{n,S} & \longrightarrow & S \\
\downarrow & & \downarrow \\
\mathbb{G}_{m,S} & \overset{[n]}{\longrightarrow} & \mathbb{G}_{m,S}.
\end{array}
$$

Fiber products of affine schemes are computed by tensor products of rings, so we get

$$
\begin{aligned}
\mu_{n,S} &= \mathrm{Spec}\left(R \otimes_{1 \leftarrow t, \ R[t,t^{-1}], \ t \mapsto t^n} R[t, t^{-1}]\right) \\
&= \mathrm{Spec}\left(R[t]/(t^n - 1)\right).
\end{aligned} \tag{4.16}
$$

In terms of (4.8) and (4.11), we see

$$\mu_{n,S}(T) = \{\zeta \in \mathcal{O}_T(T)^\times \mid \zeta^n = 1\}. \tag{4.17}$$

That is, $\mu_{n,S}$ is the *group scheme of n-th roots of unity*. Let us assume that $S = \mathrm{Spec}(k)$. We observe the following interesting phenomenon:

Assume that n is prime to $\mathrm{char}(k)$. Then $t^n - 1 \in k[t]$ is a separable polynomial. Hence, $\mu_{n,k} = \mathrm{Spec}\, k[t]/(t^n - 1)$ is an étale k-scheme. On the other hand, if $p = \mathrm{char}(k) \mid n$, then $t^n - 1$ is not separable and $k[t]/(t^n - 1)$ is not reduced. For example,

$$
\begin{aligned}
\mu_{p,k} &= \mathrm{Spec}\, k[t]/(t^p - 1) \\
&= \mathrm{Spec}\, k[t]/(t - 1)^p \\
&\overset{\sim}{\longrightarrow} \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^p)
\end{aligned}
$$

is completely infinitesimal. We have the following general results in this direction.

Theorem 4.8 (Cartier, [12, Corollary 8.38]). *Let k be a field of characteristic 0 and let G/k be a finite type group scheme. Then G is smooth.*

A morphism $f : X \to S$ is said to be *finite locally free of rank n* if it is finite and if $f_*(\mathcal{O}_X)$ is locally free of rank n as $\mathcal{O}_S$-module.

Theorem 4.9. *Let G be a commutative S-group scheme which is finite locally free of rank n. Assume that $n \in \mathcal{O}_S(S)^\times$. Then G is étale.*

4.3. **A non-commutative example:** GL_n. Let $S = \mathrm{Spec}(R)$ be affine as before. The (underlying scheme of the) general linear group in n variables over S is defined as

$$\mathrm{GL}_{n,S} = \mathrm{Spec}\, R\left[t_{ij},\ 1 \le i,j \le n;\ \det((t_{ij})_{i,j})^{-1}\right].$$

For every S-scheme T, we can (exercise) identify $\mathrm{GL}_{n,S}(T)$ with $\mathrm{GL}_n(\mathcal{O}_T(T))$ by

$$\Phi : [g : T \to \mathrm{GL}_{n,S}] \longmapsto (g^*(t_{ij}))_{i,j}. \tag{4.18}$$

We have the usual matrix multiplication on $\mathrm{GL}_n(\mathcal{O}_T(T))$. In terms of (4.18), it comes from the multiplication morphism $m : \mathrm{GL}_{n,S} \times_S \mathrm{GL}_{n,S} \to \mathrm{GL}_{n,S}$ which is given in coordinates by

$$m^*(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}.$$

The pair $(\mathrm{GL}_{n,S}, m)$ is then an S-group scheme. The identity map $e = \mathrm{Spec}(e^*)$ is given by

$$e^* : R[t_{ij}, \det((t_{ij})_{ij})^{-1}] \longrightarrow R, \quad e^*(t_{ij}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The inverse of the matrix $(t_{ij})_{ij} \in \mathrm{GL}_n(R[t_{ij}, \det((t_{ij})_{ij})^{-1}])$ has an expression of the form $\det((t_{ij})_{ij})^{-1} \cdot (s_{ij})_{ij}$ where the s_{ij} are polynomials in the variables t_{ij}. (In fact, the s_{ij} are the entries of the adjunct, or adjugate matrix.) Then the inverse morphism $i : \mathrm{GL}_{n,S} \to \mathrm{GL}_{n,S}$ is given in coordinates by

$$i^*(t_{k\ell}) = \det((t_{ij})_{ij})^{-1} s_{k\ell}.$$

Clearly, $\mathrm{GL}_{1,S}$ is the same as the multiplicative group $\mathbb{G}_{m,S}$. For every S-scheme T, we have a determinant morphism $\mathrm{GL}_n(\mathcal{O}_T(T)) \to \mathcal{O}_T(T)^\times$. With respect to our identifications (4.11) and (4.18), these come from the group scheme homomorphism

$$\det : \mathrm{GL}_{n,S} \longrightarrow \mathbb{G}_{m,S}, \quad \det^*(t) = \det((t_{ij})_{ij}). \tag{4.19}$$

Its kernel $\ker(\det)$ is the group subscheme $\mathrm{SL}_{n,S} \subset \mathrm{GL}_{n,S}$. Being a closed subscheme of an affine scheme, it is again affine. It can be described explicitly by

$$\mathrm{SL}_{n,S} = \mathrm{Spec}\left(R[t_{ij},\ 1 \le i,j \le n]/(\det((t_{ij})_{ij}) - 1)\right).$$

4.4. **Linear algebraic groups.** We now specialize to the case of finite type group schemes over a field k. A general classification theorem essentially reduces their study to the affine and the proper case.

Theorem 4.10 (see [12, §8a]). *Let G/k be a connected finite type k-group scheme. Then there exists a unique maximal normal, connected, affine closed group sub-scheme $N \subseteq G$. The quotient G/N is an abelian variety.*

Affine finite type k-group schemes are also called *linear algebraic groups*. The reason for this name is that they can always be realized as a group of linear automorphisms of some vector space. That is, they always embed into some GL_N.

Theorem 4.11 (see [12, Corollary 4.10]). *Let G be an affine finite type k-group scheme. Then there exist an integer n and a closed immersion group scheme morphism $G \to \mathrm{GL}_n$.*

4.5. **Abelian varieties.** We have already defined abelian varieties in Definition 4.2. The main point of this definition is that abelian varieties are proper. This implies that they are necessarily commutative which also explains their name.

Theorem 4.12 ([3, Corollary 3.7]). *Let (A, m) be an abelian variety over k. Then (A, m) is a commutative group scheme.*

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