CS–E4850 Computer Vision
Week 09

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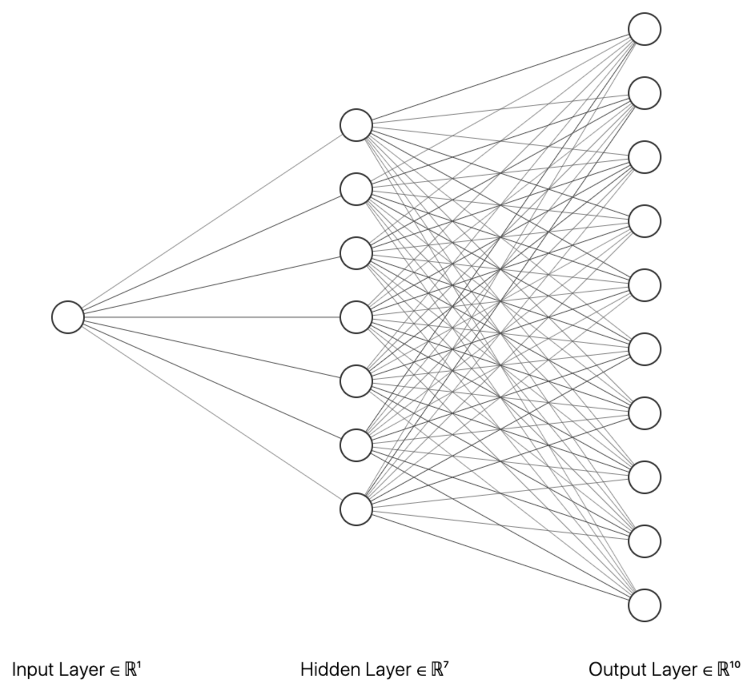
1 Problem 1

Picture of Fully Connected Neural Network with 1 input[1]



Part 1

Given m=1:

$$\begin{aligned}
E &= \frac{1}{m}\sum_{j=1}^{m} -t_j \cdot log(y_j) \\
E &= -t \cdot log(\mathbf{y}) = -t \cdot log(\sigma(\mathbf{W}x)) = -t \cdot log\frac{1}{1+e^{-\mathbf{W}x}}
\end{aligned} \tag{1}$$

Now we need to computer partial derivates of E with respect to t and x:

$$\begin{aligned}
&\frac{\partial E}{\partial t} = -log\frac{1}{1+e^{-\mathbf{W}x}} \\
&\frac{\partial E}{\partial x} = -\frac{t}{\sigma(\mathbf{W}x)} \cdot \frac{\partial \sigma(\mathbf{W}x)}{\partial x} \quad = -t(1+e^{-\mathbf{W}x}) \cdot \mathbf{W}\frac{1}{1+e^{-\mathbf{W}x}}(1-\frac{1}{1+e^{-\mathbf{W}x}}) \\
&\quad = -t\mathbf{W}\frac{e^{-\mathbf{W}x}}{1+e^{-\mathbf{W}x}}
\end{aligned} \tag{2}$$

Part 2

$$\begin{aligned}
\frac{\partial E}{\partial z_i^{(2)}} &= \sum_{j=1}^{n}\frac{\partial E_j}{\partial y_j^{(2)}}\frac{\partial y_j^{(2)}}{\partial z_i^{(2)}} \\
\frac{\partial E_j}{\partial y_j^{(2)}} &= \frac{\partial(-t_j \cdot log y_j)}{\partial z_i^{(2)}} = -\frac{t_j}{y_i}
\end{aligned} \tag{3}$$

Case $(i \neq j)$

$$\frac{\partial y_j^{(2)}}{\partial z_i^{(2)}} = \frac{\partial \sigma(z_j^{(2)})}{\partial z_i^{(2)}} = -y_i y_j \tag{4}$$

Case $(i = j)$

$$\frac{\partial y_j^{(2)}}{\partial z_i^{(2)}} = \frac{\partial \sigma(z_i^{(2)})}{\partial z_i^{(2)}} = y_i(1-y_i) \tag{5}$$

$$
\begin{aligned}
\frac{\partial E}{\partial z_i^{(2)}} &= \sum_{j=1}^{n} \frac{\partial E_j}{\partial y_j^{(2)}} \frac{\partial y_j^{(2)}}{\partial z_i^{(2)}} = \sum_{i \neq j} \frac{\partial E_j}{\partial y_j^{(2)}} \frac{\partial y_j^{(2)}}{\partial z_i^{(2)}} + \sum_{i=j} \frac{\partial E_j}{\partial y_j^{(2)}} \frac{\partial y_j^{(2)}}{\partial z_i^{(2)}} \\
&= \sum_{i \neq j} [-\frac{t_j}{y_i}(-y_i y_j)] - \frac{t_j}{y_i} y_i(1-y_i) = \sum_{i \neq j} [(t_j y_i)] + t_i y_i - t_i \\
&= y_i - t_i
\end{aligned} \tag{6}
$$

Therefore we have:

$$
\frac{\partial E}{\partial \mathbf{z}^{(2)}} = (\mathbf{y}^{(2)} - \mathbf{t})^\top \tag{7}
$$

Part 3

$$
\begin{aligned}
\frac{\partial E}{\partial \mathbf{z}^{(2)}} &= (\mathbf{y}^{(2)} - \mathbf{t})^\top \\
\frac{\partial E}{\partial \mathbf{y}^{(1)}} &= \frac{\partial E}{\partial z(2)} \cdot \frac{\partial z(2)}{\partial y(1)} = (\mathbf{y}^{(2)} - \mathbf{t})^\top \cdot \mathbf{W}^{(2)}
\end{aligned} \tag{8}
$$

Part 4

$$
\begin{aligned}
\frac{\partial E}{\partial W_{uv}^{(2)}} &= \frac{\partial E}{\partial \mathbf{z}^{(2)}} \frac{\partial \mathbf{z}^{(2)}}{\partial W_{uv}^{(2)}} \\
&= (\frac{\partial E}{\partial \mathbf{z}^{(2)}})_u y_v^{(1)} \\
&= (y_u^{(2)} - t_u) y_v^{(1)} \\
\rightarrow \frac{\partial E}{\partial \mathbf{W}^{(2)}} &= [(\mathbf{y}^{(2)} - \mathbf{t})\mathbf{y}^{(1)}]^\top
\end{aligned} \tag{9}
$$

Part 5

$$
\begin{aligned}
\frac{\partial \mathbf{y}^{(1)}}{\partial \mathbf{z}^{(1)}} &= \mathbf{y}^{(1)}(1 - \mathbf{y}^{(1)}) = diag(\mathbf{y}^{(1)} * (1 - \mathbf{y}^{(1)})) \\
\frac{\partial \sigma(z)}{\partial z} &= \frac{-(-1)e^{-z}}{(1+e^{-z})^2} = \frac{1}{1+e^{-z}} \cdot \frac{e^{-z}}{1+e^{-z}} = \sigma(z)(1 - \sigma(z))
\end{aligned} \tag{10}
$$

Part 6

$$\frac{\partial E}{\partial \mathbf{z}^{(1)}} = \frac{\partial E}{\partial \mathbf{y}^{(1)}}\frac{\partial \mathbf{y}^{(1)}}{\partial \mathbf{z}^{(1)}} = (\mathbf{y}^{(2)} - t)^\top \mathbf{W}^{(2)} diag(\mathbf{y}^{(1)}) * (1 - \mathbf{y}^{(1)}) \tag{11}$$

Part 7

$$\begin{aligned}
\frac{\partial E}{\partial W_{uv}^{(1)}} &= \frac{\partial E}{\partial \mathbf{z}^{(1)}}\frac{\partial \mathbf{z}^{(1)}}{\partial W_{uv}^{(1)}} \\
&= \frac{\partial E}{\partial z_u^{(1)}}\frac{\partial \mathbf{z}_u}{\partial w_{uv}^{(1)}} \\
&= \frac{\partial E}{\partial z_u^{(1)}}\frac{\partial (\sum_i w_{ui} x_i)}{\partial w_{uv}} \\
&= \frac{\partial E}{\partial z_u^{(1)}} x_v \\
\rightarrow \frac{\partial E}{\partial \mathbf{W}^{(1)}} &= (\frac{\partial E}{\partial \mathbf{z}^{(1)}})^\top \mathbf{x}^\top
\end{aligned} \tag{12}$$

Part 8

If we have $m > 1$, we compute above for each batch of data.

Part 9

In case of L_2 normalization, we normalize the partial derivatives also, by multiplying with λ and value of the $W \cdot (\lambda \times W)$