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Computer Vision Ex6

Q1. a) $E = \sum \| x_i' - M x_i - t \|^2 = \sum_{i=1}^{n} \| \begin{pmatrix} x_i' \\ y_i' \end{pmatrix} - \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \|^2$

$$E = \sum_{i=1}^{n} \left[ (x_i' - m_1 x_i - m_2 y_i - t_1)^2 + (y_i' - m_3 x_i - m_4 y_i - t_2)^2 \right]$$

* $\frac{\partial E}{\partial m_1} = -2 \sum_{i=1}^{n} x_i (x_i' - m_1 x_i - m_2 y_i - t_1)$

$\frac{\partial E}{\partial m_2} = -2 \sum_{i=1}^{n} y_i (x_i' - m_1 x_i - m_2 y_i - t_1)$

$\frac{\partial E}{\partial m_3} = -2 \sum_{i=1}^{n} x_i (y_i' - m_3 x_i - m_4 y_i - t_2)$

$\frac{\partial E}{\partial m_4} = -2 \sum_{i=1}^{n} y_i (x_i' - m_3 x_i - m_4 y_i - t_2)$

$\frac{\partial E}{\partial t_1} = -2 \sum_{i=1}^{n} (x_i' - m_1 x_i - m_2 y_i - t_1)$

$\frac{\partial E}{\partial t_2} = -2 \sum_{i=1}^{n} (y_i' - m_3 x_i - m_4 y_i - t_2)$

b) By putting above equations equal to zero, we would get:

$\sum x_i' x_i = \sum m_1 x_i^2 + m_2 y_i x_i + t_1 x_i$

$\sum x_i' y_i = \sum m_1 x_i y_i + m_2 y_i^2 + t_1 y_i$

$\sum y_i' x_i = \sum m_3 x_i^2 + m_4 y_i x_i + t_2 x_i$

$\sum y_i' y_i = \sum m_3 x_i y_i + m_4 y_i^2 + t_2 y_i$

$\sum x_i' = \sum m_1 x_i + m_2 y_i + t_1$

$\sum y_i' = \sum m_3 x_i + m_4 y_i + t_2$

So we can express the system $Sh = u$ as

$$\underbrace{\begin{bmatrix} \sum x_i^2 & \sum x_i y_i & 0 & 0 & \sum x_i & 0 \\ \sum x_i y_i & \sum y_i^2 & 0 & 0 & \sum y_i & 0 \\ 0 & 0 & \sum x_i^2 & \sum x_i y_i & 0 & \sum x_i \\ 0 & 0 & \sum x_i y_i & \sum y_i^2 & 0 & \sum y_i \\ \sum x_i & \sum y_i & 0 & 0 & n & 0 \\ 0 & 0 & \sum x_i & \sum y_i & 0 & n \end{bmatrix}}_{=S} \underbrace{\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix}}_{=h} = \underbrace{\begin{bmatrix} \sum x_i' x_i \\ \sum x_i' y_i \\ \sum y_i' x_i \\ \sum y_i' y_i \\ \sum x_i' \\ \sum y_i' \end{bmatrix}}_{=u}$$

c) $(x_1, y_1) = (0,0) \rightarrow (x_1', y_1') = (1,2)$

$(x_2, y_2) = (1,0) \rightarrow (x_2', y_2') = (3,2)$

$(x_3, y_3) = (0,1) \rightarrow (x_3', y_3') = (1,4)$

$$
S = \begin{bmatrix} \sum x_i^2 & \sum x_i y_i & 0 & 0 & \sum x_i & 0 \\ \sum x_i y_i & \sum y_i^2 & 0 & 0 & \sum y_i & 0 \\ 0 & 0 & \sum x_i^2 & \sum x_i y_i & 0 & \sum x_i \\ 0 & 0 & \sum x_i y_i & \sum y_i^2 & 0 & \sum y_i \\ \sum x_i & \sum y_i & 0 & 0 & n & 0 \\ 0 & 0 & \sum x_i & \sum y_i & 0 & n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{bmatrix}
$$

$$
S^{-1} = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} \sum x_i' x_i \\ \sum x_i' y_i \\ \sum y_i' x_i \\ \sum y_i' y_i \\ \sum x_i' \\ \sum y_i' \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix}
$$

$$
h = S^{-1} u = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}
$$

Q2. a)

$$x_1' = sRx_1 + t \ , \ x_2' = sRx_2 + t \ \rightarrow \ x_2' - x_1' = sR(x_2 - x_1)$$

$$\rightarrow \ v' = sR\,v$$

Now assume $v' = [v_1', v_2']$, $v = [v_1, v_2]$

$$\left.\begin{array}{l} v_1' = s(\cos\theta \cdot v_1 - \sin\theta\, v_2) \\ v_2' = s(\sin\theta \cdot v_1 + \cos\theta\, v_2) \end{array}\right\} \rightarrow \begin{array}{l} v' \cdot v = v_1' v_1 + v_2' v_2 \\ = s(\cos\theta\, v_1^2 - \cancel{\sin\theta\, v_1 v_2} + \cancel{\sin\theta\, v_1 v_2} + \cos\theta\, v_2^2) \\ = s(\cos\theta)(v_1^2 + v_2^2) \end{array}$$

$$|v'||v| = s\sqrt{\cos^2_\theta v_1^2 + \sin^2_\theta v_2^2 - \cancel{2\sin_\theta \cos_\theta v_1 v_2} + \cancel{2\sin_\theta \cos_\theta v_1 v_2} + \cos^2_\theta v_2^2 + \sin^2_\theta v_1^2} \cdot \sqrt{v_1^2 + v_2^2}$$

$$= s\,(\cos^2_\theta + \sin^2_\theta)^{1/2}\,(v_1^2 + v_2^2)$$

$$\rightarrow \cos\theta = \frac{v' \cdot v}{|v'|.|v|} \rightarrow \theta = \arccos\left(\frac{v' \cdot v}{|v'||v|}\right)$$

b) $v_1'^2 + v_2'^2 = s^2\,(\underbrace{\cos^2_\theta + \sin^2_\theta}_{1})(v_1^2 + v_2^2) \rightarrow s^2 = \dfrac{|v'|^2}{|v|^2} \rightarrow s = \dfrac{|v'|}{|v|}$

c) $$x_2' - sRx_2 = t \ \rightarrow \ \underbrace{x_2' - \frac{\|x_2' - x_1'\|}{\|x_2 - x_1\|} \cdot \begin{bmatrix} \frac{v' \cdot v}{|v'|.|v|} & -\sqrt{1 - \left(\frac{v' \cdot v}{|v'|.|v|}\right)^2} \\ \sqrt{1 - \left(\frac{v' \cdot v}{|v'|.|v|}\right)^2} & \frac{v' \cdot v}{|v'|.|v|} \end{bmatrix} x_2}$$

$$= t$$

d)

$$v' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \quad \rightarrow \quad S = \frac{1}{1/2} = 2$$

$$\theta = \arccos\left( \frac{-1/2 + 1/2}{1 \cdot \sqrt{1/2}} \right) = \arccos(0) = \pi/2 \rightarrow R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1+1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = t$$