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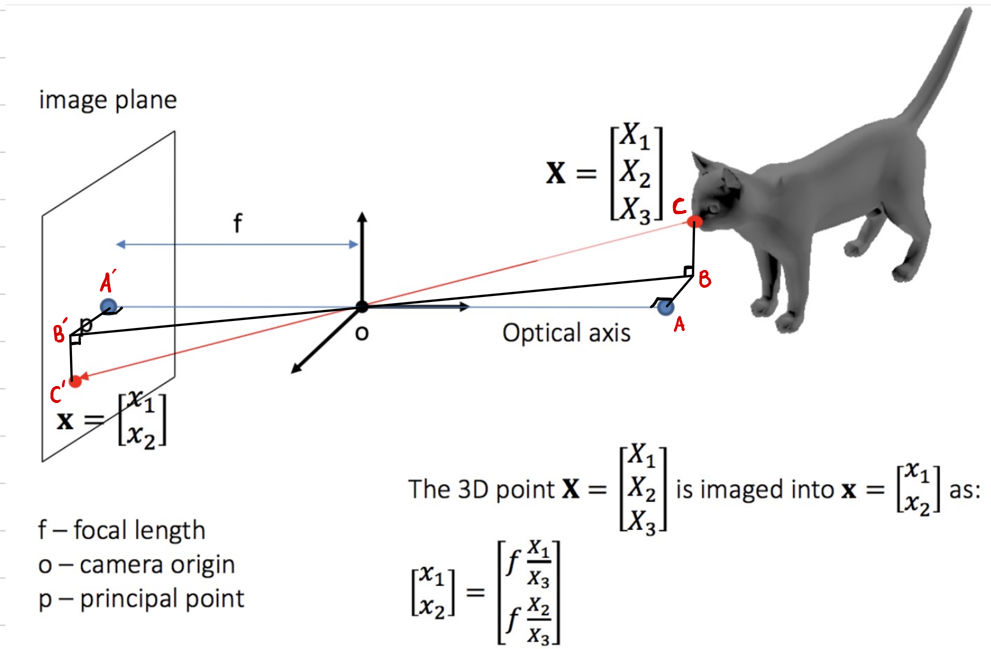
Exercise Round 2

In this solution $\emptyset \to$ zero
$\theta \to$ theta

Ex 1. Pinhole camera.



From point C and C′ we draw lines BC and B′C′ such that they are parallel with vertical side of frame and from points A and A′ we draw lines AB and A′B′ such that they are parallel with horizontal side of frame and obtain points B and B′. Now we are going to show $\triangle ABO \sim \triangle A'B'O$ and $\triangle BCO \sim \triangle B'C'O$.

$$\hat{A} = \hat{A}' = 90$$
$$A\hat{O}B = A'\hat{O}B' \quad \text{vertically opposite angles}$$
$$90 - A\hat{O}B = A\hat{B}O = A'\hat{B}'O = 90 - A'\hat{O}B'$$
$$\left.\right\} \xrightarrow[\text{angles}]{\text{3 equal}} \triangle A\hat{O}B \sim \triangle A'\hat{O}B' \longrightarrow \left\{ \frac{AB}{A'B'} = \frac{AO}{A'O} = \frac{BO}{B'O} \quad (1) \right.$$

$$C\hat{B}O = C'\hat{B}'O = 90$$
$$C\hat{O}B = C'\hat{O}B' \quad \text{vertically opposite angles}$$
$$90 - C\hat{O}B = B\hat{C}O = B'\hat{C}'O = 90 - C'\hat{O}B'$$
$$\left.\right\} \xrightarrow[\text{angles}]{\text{3 equal}} \triangle B\hat{O}C \sim \triangle B'\hat{O}C' \longrightarrow \left\{ \frac{BC}{B'C'} = \frac{BO}{B'O} = \frac{CO}{C'O} \quad (2) \right.$$

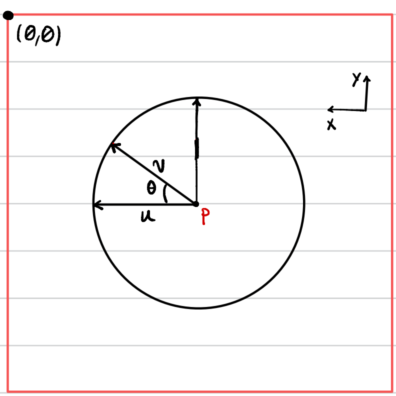
$$(1) \longrightarrow \frac{AB}{A'B'} = \frac{AO}{A'O} \longrightarrow \frac{X_1}{x_1} = \frac{X_3}{f} \longrightarrow \boxed{x_1 = f\frac{X_1}{X_3}}$$

$$(1),(2) \longrightarrow \frac{BC}{B'C'} \overset{(2)}{=} \frac{BO}{B'O} \overset{(1)}{=} \frac{AO}{A'O} \longrightarrow \frac{BC}{B'C'} = \frac{AO}{A'O} \longrightarrow \frac{X_2}{x_2} = \frac{X_3}{f} \longrightarrow \boxed{x_2 = f\frac{X_1}{X_3}}$$

Ex 2. Pixel coordinate frame.

a,b) As long as, part a is weaker than part b. Basically, part b is a generalized version of part a.

Assume the red square is the frame. $\vec{u}$ and $\vec{v}$ are unit vectors and directions.



Now we want to solve the equation: $X_p = r_u\vec{u} + r_v\vec{V}$ s.t. r_u, and r_v are constant.

$$[x_p, y_p] = r_u \vec{u} + r_v \vec{v} = (r_u + r_v \cos_\theta)\vec{u} + r_v \sin_\theta \vec{y}$$

$$\vec{v} = \cos_\theta \vec{u} + \sin_\theta \vec{y}$$

$$\left.\begin{aligned} x_p &= r_u + r_v \cos_\theta \\ y_p &= r_v \sin_\theta \end{aligned}\right\} \rightarrow$$

Now each unit distance in direction $\vec{u}$ has m_u pixels and each unit distance in direction $\vec{v}$ has m_v pixels.

$y_p = r_v \sin_\theta \rightarrow r_v = y_p / \sin_\theta$

So now $X_p = \left(x_p - \frac{y_p}{\sin_\theta}\cos_\theta\right)\vec{u} + \frac{y_p}{\sin_\theta}\vec{v}$

$$P = (u_0, v_0) + \left(\left[x_p - \frac{y_p}{\sin_\theta}\cos_\theta\right] m_u , \frac{y_p}{\sin_\theta} \cdot m_v\right)$$

Note that in all calculation if $\vec{v}$ is parallel with $\vec{y}$, then $\sin_\theta = 1$ and $\cos_\theta = 0$. Thus, the formulation becomes easier.

$$(u_0, v_0) + (x_p m_u , y_p \cdot m_v)$$

Ex 3. Intrinsic camera calibration matrix.

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} \xrightarrow[\text{last question}]{\text{based on}} (u_0, v_0) + (x_p m_u , y_p \cdot m_v)$$

$\downarrow$ homogeneous coordinates

$$\begin{aligned} \lambda x_1 &= f X_1 \\ \lambda x_2 &= f X_2 \\ \lambda &= X_3 \end{aligned} \qquad \lambda \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

we want to obtain pixel coordinate by matrix multiplication of a matrix to $\lambda \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix}$.

$$\lambda \begin{bmatrix} x_p m_u + u_0 \\ y_p m_v + v_0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} m_u & 0 & u_0 \\ 0 & m_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}}_{= A} \cdot \lambda \cdot \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = A \cdot \underbrace{\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= B} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

based on 2(a), we can conclude that there is matrix A in such way.

So $A \cdot B = K_{3 \times 3}$ which is
$$\begin{bmatrix} m_u & 0 & u_0 \\ 0 & m_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} f m_u & 0 & u_0 \\ 0 & f m_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

So this matrix is our K

camera's intrinsic calibration matrix.

Ex 4. Camera projection matrix.

$$\begin{tikzcd} \text{World Coords} & \text{Camera Coords} & \text{Film coords} & \text{Pixel Coords} \\ \begin{matrix} U \\ V \\ W \end{matrix} \arrow[r] \arrow[r, bend right, "\text{Rotation and transformation}"'] & \begin{matrix} X \\ Y \\ Z \end{matrix} \arrow[r] \arrow[rr, bend right, "\text{Matrix } K_{3 \times 3}"'] & \begin{matrix} x \\ y \end{matrix} \arrow[r, dash] & \begin{matrix} u \\ v \end{matrix} \end{tikzcd}$$

Now we want to combine these matrices and use homogeneous coordinates.

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} f m_u & 0 & u_0 & 0 \\ 0 & f m_v & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times \leftarrow \begin{bmatrix} U' \\ V' \\ W' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \\ 1 \end{bmatrix}$$

$\rightarrow$ so our matrix P is equal to

$$P = \left[\begin{array}{c|c} K & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right] \times \left[\begin{array}{c|c} R & t \\ \hline 0\ 0\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} KR & Kt \end{array}\right] = K \left[\begin{array}{c|c} R & t \end{array}\right]$$

Ex5. Rotation matrix.

a) Decompose x first: $\quad x = x_{\parallel} + x_{\perp}$

That $x_{\parallel}$ is the parallel component to u: $x_{\parallel} = (x.u)\,u$

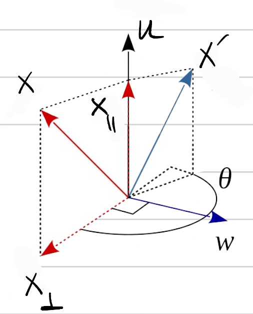
And $x_{\perp}$ is the perpendicular component to u: $x_{\perp} = x - x_{\parallel} = x - (x.u)u$

$$= -u \times (u \times x)$$

Now assume that x' is the rotated vector about u by angle θ.

It is clear that $x'_{\parallel} = x_{\parallel}$, and this parallel component will not change magnitude or direction.

Only perpendicular component of x will change direction, but not its size. $|x'_\perp| = |x_\perp|$



$$x'_\perp = \cos_\theta x_\perp + \sin_\theta \cdot \underbrace{w}_{u \times x_\perp} = \cos_\theta x_\perp + \sin_\theta \cdot \underbrace{u \times x_\perp}$$

$$= u \times (x - x_\parallel)$$

$$= u \times x - \underbrace{\cancel{u \times u_\parallel}}_{0} = u \times x$$

$$x'_\perp = \cos_\theta x_\perp + \sin_\theta \, u \times x$$

$$x' = x_\parallel + \cos_\theta x_\perp + \sin_\theta \, u \times x$$

$$= x_\parallel + \cos_\theta (x - x_\parallel) + \sin_\theta \, u \times x$$

$$= \cos_\theta x + (1 - \cos_\theta) x_\parallel + \sin_\theta \, u \times x$$

$$= \cos_\theta x + (1 - \cos_\theta)(u \cdot x) u + \sin_\theta \, u \times x \quad \ldots \checkmark$$

b) $a \times b = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k)$

$$= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Now we use this matrix form of cross product for $(u \times x)$

$$\begin{bmatrix}(u\times x)_x\\(u\times x)_y\\(u\times x)_z\end{bmatrix}=\begin{bmatrix}u_y x_z-u_z x_y\\u_z x_x-u_x x_z\\u_x x_y-u_y x_x\end{bmatrix}=\underbrace{\begin{bmatrix}0&-u_z&u_y\\u_z&0&-u_x\\-u_y&u_x&0\end{bmatrix}}_{=U}\begin{bmatrix}x_x\\x_y\\x_z\end{bmatrix}$$

Thus we get that $Ux = u\times x$

$$x' = x\cos_\theta + (u\times x)\sin_\theta + \underbrace{u(u.x)}_{(x-x_\perp)=x+u\times(u\times x)}(1-\cos_\theta)$$

$$\rightarrow x' = x\cos_\theta + (u\times x)\sin_\theta + x(1-\cos_\theta) + (1-\cos_\theta)\,u\times(u\times x)$$

$$= x + Ux\sin_\theta + (1-\cos_\theta)\,U(Ux)$$

$$x' = x + \sin_\theta Ux + (1-\cos_\theta)U^2x$$

$$\rightarrow Rx = x' = x + \sin_\theta Ux + (1-\cos\theta)U^2x$$

Factorization by x $\rightarrow$ $R = I_{3\times3} + \sin_\theta U + (1-\cos_\theta)U^2$