CS-E4850 Computer Vision
Exercise Round 1

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September 4, 2023

1. Homogeneous coordinates.

   (a) The equation of a line in the plane is

   $$ax + by + c = 0 \tag{1}$$

   Show that by using homogeneous coordinates this can be written as

   $$x^T l = 0 \tag{2}$$

   where $l = (a,\ b,\ c)^T$.

   Solution: We are talking about points lies on the line in the plane. As far as we know, a point lies on the line $(a,\ b,\ c)$ if and only if $ax + by + c = 0$. This point $(x,\ y)$ in $\mathbb{R}^2$ can be also represent in $\mathbb{R}^3$ by adding a final coordinate 1. We can also represent the Equation **??** as inner product of two vectors $(x,\ y,\ 1)$ and $(a,\ b,\ c)^T$. To consider the set of vectors $(kx,\ ky,\ k)^T$ for varying values of k to be a representation of the point $(x,\ y)^T$ in $\mathbb{R}^2$, because $(kx,\ ky,\ k)l = 0$ if and only if $(x,\ y,\ 1)l = 0$. Thus, just as with lines, points are represented by homogeneous vectors. An arbitrary homogeneous vector representative of a point is of the form $x = (x_1, x_2, x_3)^T$, representing the point $(\frac{x1}{x3}, \frac{x2}{x3})^T$ in $\mathbb{R}^2$. Points, then, as homogeneous vectors are also elements of $\mathbb{P}^2$.

   (b) Show that the intersection of two lines l and l' is the point $x = l \times l'$.

   Solution: It is clear that for intersection we need a point x that satisfies in both equations $x^T l = 0$ and $x^T l' = 0$. Let $x = l \times l'$. It is clear that due to the hint, $l^T(l \times l') = l'^T(l \times l') = 0$, because two of vectors in the scalar triple product are parallel. Therefore, $x^T l = x^T l' = 0$, so x satisfies both equations.

   (c) Show that the line through two points x and x' is $l = x \times x'$.

   Solution: Again, we need a line l that holds the following equation:

   $$x^T l = x^T l' = 0 \tag{3}$$

   Consider $l = (x \times x')$, due to the scalar triple product hint, $x^T(x \times x') = x'^T(x \times x') = 0$, so $(x \times x')$ can be a line that satisfies the Equation **??**.

   (d) Show that for all $\alpha \in \mathbb{R}$ the point $y = \alpha x + (1 - \alpha)x'$ lies on the line through points x and x'.

   Solution: According to previous part, $l = x \times x'$. As we know inner product is a linear function, so $y^T l = \alpha x^T l + (1 - \alpha)x'^T l = 0 + 0$; so y is also on this line.

2. Transformations in 2D

(a) Use homogeneous coordinates and give the matrix representations of the following transformation groups: translation, Euclidean transformation (rotation+translation), similarity transformation (scaling+rotation+translation), affine transformation, projective transformation.

Solution:

- Translation:

$$\begin{pmatrix} x+t_x \\ y+t_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{4}$$

- Euclidean:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} cos(\theta) & -sin(\theta) & t_x \\ sin(\theta) & cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

$$\Rightarrow \begin{pmatrix} cos(\theta) & -sin(\theta) & t_x \\ sin(\theta) & cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \tag{6}$$

- Similarity:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta) & -sin(\theta) & t_x \\ sin(\theta) & cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a\ cos(\theta) & -a\ sin(\theta) & t_x \\ a\ sin(\theta) & a\ cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$\Rightarrow \begin{pmatrix} a\ cos(\theta) & -a\ sin(\theta) & t_x \\ a\ sin(\theta) & a\ cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \tag{8}$$

- Affine:

$$\begin{pmatrix} a_1 & a_2 & t_x \\ a_3 & a_4 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \tag{9}$$

- Projective:

$$\begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{10}$$

$$x' = \frac{u}{w} \ , \ y' = \frac{v}{w} \tag{11}$$

(b) What is the number of degrees of freedom in these transformations?

Solution:

- Translation: DoF=2
- Euclidean: DoF= 3
- Similarity: DoF=4
- Affine: DoF=6
- Projective: DoF=8

(c) Why is the number of degrees of freedom in a projective transformation less than the number of elements in a 3×3 matrix?

Solution: Let consider the following 3×3 matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \tag{12}$$

Now we apply this matrix on our homogeneous 3-vector.

$$k' \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} k \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{13}$$

k' and k are just homogeneous scaling factors that are pulled outside of the homogeneous vectors. Therefore, we have

$$\frac{k'}{k} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a_{1,1}}{a_{3,3}} & \frac{a_{1,2}}{a_{3,3}} & \frac{a_{1,3}}{a_{3,3}} \\ \frac{a_{2,1}}{a_{3,3}} & \frac{a_{2,2}}{a_{3,3}} & \frac{a_{2,3}}{a_{3,3}} \\ \frac{a_{3,1}}{a_{3,3}} & \frac{a_{3,2}}{a_{3,3}} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{14}$$

Now take $w = \frac{k'}{k}$, and we have:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{a_{1,1}}{a_{3,3}} & \frac{a_{1,2}}{a_{3,3}} & \frac{a_{1,3}}{a_{3,3}} \\ \frac{a_{2,1}}{a_{3,3}} & \frac{a_{2,2}}{a_{3,3}} & \frac{a_{2,3}}{a_{3,3}} \\ \frac{a_{3,1}}{a_{3,3}} & \frac{a_{3,2}}{a_{3,3}} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{15}$$

So it is right that our matrix had nine elements at the begining, but now only eight of them are independent. As such, it follows that a projective transformation has eight degrees of freedom which is less than nine.

3. Planar projective transformation

The equation of a line on a plane, $ax + by + c = 0$, can be written as $l^T x = 0$, where $l = [a\ b\ c]^T$ and x are homogeneous coordinates for lines and points, respectively. Under a planar projective transformation, represented with an invertible 3×3 matrix H, points transform as

$$x' = Hx$$

(a) Given the matrix H for transforming points, as defined above, define the line transformation (i.e. transformation that gives l' which is a transformed version of l).

Solution: It is clear that because H is invertible, there is a matrix H^{-1}. It is clear that $l^T x = x^T l = 0$ gives us the equation $l^T H^{-1} H x = 0$. This means that all of the point $x' = Hx$ lie on the line $l'^T = l^T H^{-1}$. Thus, the transformed version of l is as follows:

$$l'^T = l^T H^{-1}$$

(b) A projective invariant is a quantity which does not change its value in the transformation. Using the transformation rules for points and lines, show that two lines, l_1, l_2, and two points, x_1, x_2, not lying on the lines have the following invariant under projective transformation:

Solution: Consider transformed version of lines and points as below:

$$l_1'^T = l^T{}_1 H^{-1} \tag{16}$$
$$l_2'^T = l^T{}_2 H^{-1} \tag{17}$$
$$x_1' = Hx_1 \tag{18}$$
$$x_2' = Hx_2 \tag{19}$$

Now we check the invariant function after the transformation:

$$I' = \frac{(l_1'^T x_1')(l_2'^T x_2')}{(l_1'^T x_2')(l_2'^T x_1')} = \frac{(l^T{}_1 H^{-1} H x_1)(l^T{}_2 H^{-1} H x_2)}{(l^T{}_1 H^{-1} H x_2)(l^T{}_2 H^{-1} H x_1)} = \frac{(l_1^T x_1)(l_2^T x_2)}{(l_1^T x_2)(l_2^T x_1)} = I \tag{20}$$

So we have the above invariant under projective transformation, and also projective invariants defined via homogeneous coordinates must be invariant also to arbitrary scaling of the homogeneous coordinate vectors with a non-zero scaling factor, so if we use fewer number of lines of number of points, this will not be an invariant, because they might be an extra factor k stands in numerator or denominator.