MS-E1651 - Numerical Matrix Computations
Exercise 2

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1. (P29)

   Solution:

   (a) We are given $A = LL^T$. We want to prove that A is symmetric, i.e., $A = A^T$.

   Let's compute the transpose of A^T:

   $$(A^T)^T = A$$

   Now, let's compute the transpose of LL^T:

   $$(LL^T)^T = (L^T)^T L^T = LL^T$$

   Since $(LL^T)^T$ is equal to LL^T and $(A^T)^T$ is equal to A, we can conclude that A is symmetric. Using the definition of (1.27), show that if a matrix A has a decomposition $A = LL^T$, it must be positive definite.

   To show this, let's use the definition of positive definiteness:

   A matrix A is positive definite if for any nonzero vector x, the expression $x^T Ax$ is greater than zero. Given $A = LL^T$, we want to show that $x^T Ax > 0$ for all nonzero x. Let x be a nonzero vector, then:

   $$x^T Ax = x^T LL^T x = (L^T x)^T (L^T x) = \|L^T x\|^2 \geq 0$$

   Since the squared norm of any vector is non-negative ($\|v\|^2 \geq 0$ for any vector v), we have shown that $x^T Ax \geq 0$ for all nonzero x.

   Therefore, matrix A is positive definite.

   (b) To find the Cholesky decomposition of a matrix A using the recursive definition:

   For $n = 1$, $rchol(A) = \sqrt{A}$.

   For $n > 1$, we split A as follows:

   $$A = \begin{bmatrix} a_{11} & \mathbf{a}_{21}^T \\ \mathbf{a}_{21} & A_{22} \end{bmatrix}$$

   Let $L_2 = rchol\left(A_{22} - \frac{\mathbf{a}_{21}\mathbf{a}_{21}^T}{a_{11}}\right)$.

   By equation (1.31), we have:

   $$rchol(A) = \begin{bmatrix} \sqrt{a_{11}} & \mathbf{0} \\ \frac{1}{\sqrt{a_{11}}}\mathbf{a}_{21} & L_2 \end{bmatrix}$$

Compute the Cholesky decomposition of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 4 \\ 2 & 4 & 15 \end{bmatrix}$$

We start by splitting A into the specified form:

$$A = \begin{bmatrix} 1 & \begin{bmatrix} 2 \\ 2 \end{bmatrix}^T \\ \begin{bmatrix} 2 \\ 2 \end{bmatrix} & \begin{bmatrix} 8 & 4 \\ 4 & 15 \end{bmatrix} \end{bmatrix}$$

Next, we compute L_2 for the bottom-right submatrix using the recursive definition:

$$L_2 = rchol\left(\begin{bmatrix} 8 & 4 \\ 4 & 15 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix}}{1} \right)$$

Now, compute the subtraction and L_2:

$$L_2 = rchol\left(\begin{bmatrix} 8 & 4 \\ 4 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \right)$$

so we have

$$L_2 = rchol\left(\begin{bmatrix} 4 & 0 \\ 0 & 11 \end{bmatrix} \right)$$

Now, recursively compute the Cholesky decomposition of this 2×2 matrix. We start by splitting $\left(\begin{bmatrix} 4 & 0 \\ 0 & 11 \end{bmatrix} \right)$ into the specified form:

$$\begin{bmatrix} 4 & \begin{bmatrix} 0 \end{bmatrix}^T \\ \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} 11 \end{bmatrix} \end{bmatrix}$$

so we have:

$$L_2 = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{11} \end{bmatrix}$$

So, the final result for $rchol(A)$ can be constructed using the recursive formula:

$$rchol(A) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 0 & \sqrt{11} \end{bmatrix}$$

(c) Let F be the matrix:

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

And let A be the matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 4 \\ 2 & 4 & 15 \end{bmatrix}$$

We want to compute $F^T AF$:

1. Calculate F^T, which is the transpose of matrix F:

$$F^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2. Now, compute $F^T AF$ by performing the matrix multiplications:

$$F^T AF = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 4 \\ 2 & 4 & 15 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

3. Calculate the product $F^T AF$ by performing the multiplications:

$$F^T AF = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 4 \\ 2 & 4 & 15 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 8 \end{bmatrix}$$

So, with the updated matrix F, the result of $F^T AF$ is the same as the given matrix A:

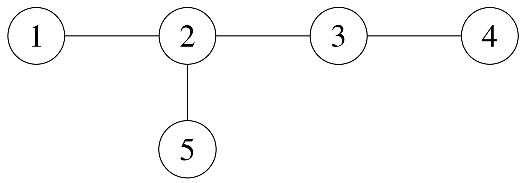
$$F^T AF = \begin{bmatrix} 15 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 8 \end{bmatrix}$$

The matrix F has a trivial null space because it is a permutation matrix. On the other hand, we found Cholesky decomposition for matrix A and based on the part (a), matrix A is positive definite. According to Lemma 1.3, $F^T AF$ is also positive definite.

2. (P30)

Solution:

(a) The graph $G(A)$ can be visualized as follows:



So, this is the graph $G(A)$ corresponding to the matrix A. It consists of five vertices and the edges connecting them as described above.

(b) For each vertex $i \in V(A)$, we will compute the set $\text{reach}(i, \{1, \ldots, i-1\})$, which represents the set of vertices that can be reached from vertex i by following set along the edges of the graph $G(A)$. Let's compute these sets:

1. For vertex 1: $\text{reach}(1, \emptyset) = \{2\}$
2. For vertex 2: $\text{reach}(2, \{1\}) = \{3, 5\}$
3. For vertex 3: $\text{reach}(3, \{1, 2\}) = \{4, 5\}$
4. For vertex 4: $\text{reach}(4, \{1, 2, 3\}) = \{5\}$
5. For vertex 5: $\text{reach}(5, \{1, 2, 3, 4\}) = \emptyset$

(c)

- off-diagonal non-zeros on column 1 are $\text{reach}(1, \emptyset) = \{2\}$.
- off-diagonal non-zeros on column 2 are $\text{reach}(2, \{1\}) = \{3, 5\}$.
- off-diagonal non-zeros on column 3 are $\text{reach}(3, \{1, 2\}) = \{4, 5\}$.
- off-diagonal non-zeros on column 4 are $\text{reach}(4, \{1, 2, 3\}) = \{5\}$
- off-diagonal non-zeros on column 5 are $\text{reach}(5, \{1, 2, 3, 4\}) = \emptyset$.

The non-zeros of the computed factor are

$$\begin{bmatrix} x & 0 & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 \\ 0 & 0 & x & x & 0 \\ 0 & x & x & x & x \end{bmatrix}$$

(d) The code is in p35d.m