MS-E1651 - Numerical Matrix Computations
Exercise 1

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October 2, 2023

1. (P9)

   Solution: If $n = 1$, then we can directly solve for x:

$$x = \frac{b}{L}$$

For $n > 1$, let L be an $n \times n$ lower triangular matrix, and x and b be $n \times 1$ column vectors. We can represent L and x as block matrices as follows:

$$L = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$$

,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

,

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Here, $l_{11},\ x_1,\ b_1$ is a scalar, $L_{21},\ x_2,\ b_2 \in \mathbb{R}^{(n-1)}$ column vector, $L_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ matrix.

We can express the linear system $Lx = b$ in block form as follows:

$$\begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Since L is invertible, $l_{11} \neq 0$, so we have:

$$det(L) = det(L_{22})det([l_{11}])$$

Since $det([l_{11}]) \neq 0 \neq det(L)$, then $det(L_{22}) \neq 0$, and L_{22} is also invertible.

So we can conclude that

$$x_1 = \frac{b_1}{l_{11}}$$

Then we have the subproblem $L_{22}x_2 = (b_2 - L_{21}x_1)$. Since the coefficient matrix $L_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ is invertible and lower triangular, x_2 can be obtained recursivley from $x_2 = trilsolve(L_{22}, b_2 - L_{21}x_1)$.

2. (P21)

   Solution: We will prove that the inverse of any $n \times n$ lower triangular matrix is lower triangular by induction on n.

- Base Case: For $n = 1$, consider a 1×1 lower triangular matrix L given by $L = [l]$, where $l \neq 0$. Its inverse is $L^{-1} = [l^{-1}]$, which is also a 1×1 matrix, and thus it is lower triangular.
- Induction Hypothesis: Assume that the statement holds for positive integer $n = k$.
- Induction Step: We want to prove that the statement also holds for $n = k + 1$. Consider an $(k + 1) \times (k + 1)$ lower triangular matrix L_{k+1} as follows:

$$L_{k+1} = \begin{bmatrix} L_k & 0 \\ \mathbf{v} & l_{k+1} \end{bmatrix}$$

Here, L_k is a $k \times k$ lower triangular matrix, $\mathbf{v}$ is a $k \times 1$ column vector, and l_{k+1} is the $(k + 1)$-th diagonal element. Now to find the inverse of L_{k+1}, denoted as L_{k+1}^{-1}:

$$L_{k+1}^{-1} = \begin{bmatrix} L_k^{-1} & 0 \\ \mathbf{u} & \frac{1}{l_{k+1}} \end{bmatrix}$$

where L_k^{-1} is the inverse of L_k (by the induction hypothesis), and $\mathbf{u}$ is a column vector we need to determine.

To find $\mathbf{u}$, we consider the matrix equation $L_{k+1} \cdot L_{k+1}^{-1} = I_{k+1}$, where I_{k+1} is the $(k + 1) \times (k + 1)$ identity matrix:

$$\begin{bmatrix} L_k & 0 \\ \mathbf{v} & l_{k+1} \end{bmatrix} \begin{bmatrix} L_k^{-1} & 0 \\ \mathbf{u} & \frac{1}{l_{k+1}} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ \mathbf{0} & 1 \end{bmatrix}$$

This equation yields two equations:

1. $L_k \cdot L_k^{-1} = I_k$, which is true since the inverse of a lower triangular matrix is lower triangular (inductive hypothesis).

2. $\mathbf{v} \cdot L_k^{-1} + l_{k+1} \cdot \mathbf{u} = \mathbf{0}$ and $l_{k+1} \cdot \frac{1}{l_{k+1}} = 1$. Since L_k^{-1} is lower triangular, the equation $\mathbf{v} \cdot L_k^{-1} + l_{k+1} \cdot \mathbf{u} = \mathbf{0}$ involves only lower triangular matrices and vectors, implying that $\mathbf{u}$ must also be a lower triangular vector.

Therefore, by induction, we have shown that the inverse of any $n \times n$ lower triangular matrix is lower triangular.

3. (P5) (a) Show that

$$\det \begin{bmatrix} I_{n\times n} & 0 \\ 0 & A_{22}^{m\times m} \end{bmatrix} = \det(A_{22}).$$

Hint: recall the Laplace expansion for computing determinants and use induction with respect to parameter n.

(b) Modify the proof in (a) to show that

$$\det \begin{bmatrix} I_{n\times n} & A_{12}^{n\times m} \\ 0 & A_{22}^{m\times m} \end{bmatrix} = \det(A_{22}).$$

Solution: (a) To prove that $\det \begin{pmatrix} I_{n\times n} & 0 \\ 0 & A_{22}^{m\times m} \end{pmatrix} = \det(A_{22})$, we can use induction on the size of the matrix. We will start with the base case and then proceed with the induction step.

- Base Case: For $n = 1$, you can indeed prove the base case by simply observing that opening the determinant expression from both sides will result in a multiplication of 1 (for the identity matrix) and all other terms being zero, ultimately leaving you with just $\det(A_{22})$
- Induction Step: Assume that the statement holds for some positive integer $n = k$.

  Now, let's consider a matrix of size $(k + 1) \times (k + 1)$:

$$M = \begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & \mathbf{0}_{1\times m} \\ 0 & \mathbf{0}_{m\times 1} & A_{22}^{m\times m} \end{bmatrix}$$

  To compute its determinant, based on the induction hypothesis for $n = k$ and for $n = 1$, respectively, we have:

$$\det(M) = \det(\begin{matrix} I_{1\times 1} & 0 \\ 0 & A_{22}^{m\times m} \end{matrix}) = \det(\begin{matrix} I_{1\times 1} & 0 \\ 0 & A_{22}^{m\times m} \end{matrix}) = \det(A_{22}),$$

(b) To prove that $\det \begin{pmatrix} I_{n\times n} & A_{12}^{n\times m} \\ 0 & A_{22}^{m\times m} \end{pmatrix} = \det(A_{22})$, we can modify the previous proof. We will use induction on the size of the matrix.

- Base Case: For $n = 1$, if we consider the determinant expression of left side of equation, in the terms that we choose an element from $A_{12}^{1\times m}$, there is at least a zero, because we have not choose any element from column 1 yet. So these terms are equal to zero. The terms of left expression is the same as terms of right expression by just multiplying in a one.
- Induction Step: Assume that the statement holds for some positive integer $n = k$.

  Now, let's consider a matrix of size $(k + 1) \times (k + 1)$:

$$M = \begin{bmatrix} I_{k\times k} & c & X_{12}^{k\times m} \\ 0 & 1 & \mathbf{d}_{1\times m} \\ 0 & \mathbf{0}_{m\times 1} & A_{22}^{m\times m} \end{bmatrix}$$

  To compute its determinant, based on the induction hypothesis for $n = k$ and for $n = 1$, respectively, we have:

$$\det(M) = \det \begin{pmatrix} 1 & d^{1\times m} \\ 0 & A_{22}^{m\times m} \end{pmatrix} = \det(A_{22}^{m\times m})$$