# MS-E1651 - Numerical Matrix Computations
# Exercise 5

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October 15, 2023

1. (P58)

   **Solution:**

   (a)

   Notice that since $||x||_\infty > 0$ you can push the denominator into the max, then into the absolute value, and then apply the distributive property to the sum, i.e.

$$\frac{\max_i ||\Sigma_j a_{ij} x_j|}{||x||_\infty} = \max_i |\Sigma_j a_{ij} \frac{x_j}{||x||_\infty}| \tag{1}$$

   You can then upper bound this value due to the fact that $x_j \leq ||x||_\infty \to \frac{x_j}{||x||_\infty} \leq 1$, hence:

$$\max_i |\Sigma_j a_{ij} \frac{x_j}{||x||_\infty}| \leq \max_i |\Sigma_j a_{ij}| \tag{2}$$

   It is enough to see such $x$ exists, because if we define $x = [1, 1, 1, 1, \cdots, 1, 1]$, we would have $\frac{x_j}{||x||_\infty} = 1$.

   (b)

$$||x||_2 = \sqrt{x_1^2 + \cdots + x_n^2} \leq \sqrt{n \times \max_i |x_i|^2} = \sqrt{n} \max_i |x_i| = \sqrt{n} ||x_i||_\infty \tag{3}$$

$$||x||_\infty = \max_i |x_i| = \sqrt{\max_i |x_i|^2} \leq \sqrt{x_1^2 + \cdots + x_n^2} = ||x||_2 \tag{4}$$

2. (P59)

   **Solution:**

   (a)

   - Start with the equation $x = c + Cx$.
   - Move $Cx$ to the other side by subtracting $Cx$ from both sides:

$$x - Cx = c.$$

   - Factor out $x$ from the left side:

$$x(1 - C) = c.$$

   - Since null space of $(1 - C)$ is only zero, it's invertible (i.e., full rank), so we would have

$$x = c.(1 - C)^{-1}$$

   - So, we have found a solution for $x$, and it is given by $x = c(1 - C)^{-1}$.

(b) Let's use induction:

**Basis**: The statement clearly holds for $n = 0$

**Induction step**: Let's assume the statement holds for $n = k$, and we would prove it also holds for $n = k + 1$.

$$\begin{aligned} x_{k+1} &= c + Cx_k && (5)\\ &= c + C(x - e_k) && (6)\\ &= c + C(x - C^k e_0) && (7)\\ &= c + Cx - C^{k+1} e_0 && (8) \end{aligned}$$

Now let's calculate $e_{k+1}$ as follows

$$\begin{aligned} e_{k+1} = x - x_{k+1} &= x - (c + Cx - C^{k+1} e_0) && (9)\\ &= C^{k+1} e_0 && (10) \end{aligned}$$

Note that in the last step, we know from part (a), that $x = c + Cx$.

Moreover we know $||AB|| \le ||A|| \cdot ||B||$, so we have

$$||e_i|| = ||C^i e_0|| \le ||C^i|| \cdot ||e_0|| \le ||C^{i-1}|| \cdot ||C|| \cdot ||e_0|| \le \cdots \le ||C||^i \cdot ||e_0|| \quad (11)$$

(c) Let's find eigendecomposition of mentioned matrix.

The eigenvalues and eigenvectors of the matrix is:

$$\begin{aligned} \lambda_1 &= \frac{-\sqrt{2}}{x}, v_1 = (1, \sqrt{2}, 1) && (12)\\ \lambda_2 &= \frac{\sqrt{2}}{x}, v_2 = (1, -\sqrt{2}, 1) && (13)\\ \lambda_3 &= 0, v_3 = (-1, 0, 1) && (14) \end{aligned}$$

So $\Lambda = \begin{bmatrix} \frac{-\sqrt{2}}{x} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{x} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$, so we would have

$$CQ = Q\Lambda$$

This is also $Q^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -\sqrt{2} & 1 \\ 1 & \sqrt{2} & 1 \\ -2 & 0 & 2 \end{bmatrix}$, so $C^i = Q\Lambda^i Q^{-1}$, so if $C$ wants to converge to 0, since $Q$, and $Q^{-1}$ are definite and fixed, just $\Lambda$ needs to converge to zero. So $\lim |\frac{\sqrt{2}}{x}|^i = 0$, so $|x| > \sqrt{2}$.

3. (P66)

   **Solution:**

   (a) Note that based on Lemma 2.2, there is an $x$, which $Ax = b$

$$\begin{aligned} p_{i+1} &= A(x - x_{i+1}) && (15)\\ \text{A is s.d.p} \rightarrow p_{i+1}^T &= e_{i+1}^T A && (16)\\ e_{i+1}^T A p_i = p_{i+1}^T p_i &= (p_i - \alpha_i A p_i)^T p_i && (17)\\ &= p_i^T p_i - \alpha_i p_i^T A p_i = p_i^T p_i - p_i^T r_i = 0 && (18) \end{aligned}$$

Furthurmore

$$
\begin{aligned}
x_{i+1} &= x_i + \alpha_i p_i && (19)\\
\text{subtracting both side from x} \rightarrow e_{i+1} &= e_i - \alpha_i p_i && (20)\\
Ae_{i+1} &= Ae_i - \alpha_i Ap_i && (21)\\
||e_{i+1}||_A = e_{i+1}^T Ae_{i+1} &= (e_i - \alpha_i p_i)^T Ae_i - \alpha_i e_{i+1}^T Ap_i = (e_i - \alpha_i p_i)^T Ae_i = e_i^T Ae_i - \alpha_i p_i^T Ae_i && (22)\\
&= e_i^T Ae_i - \alpha_i p_i^T r_i = ||e_i||_A - \alpha_i^2 p_i^T Ap_i && (23)
\end{aligned}
$$

(b)

$$
\text{Condition Number } \kappa(A) = ||A|| \cdot ||A^{-1}|| \tag{24}
$$

Based on P65b and P65c,

$$
\frac{(p_i^T r_i)^2}{(p_i^T Ap_i)^2} p_i^T Ap_i = \frac{(p_i^T r_i)^2}{(p_i^T Ap_i)} = \frac{(e_i^T A^T Ae_i)^2}{(p_i^T Ap_i)} = \frac{1}{(p_i^T Ap_i)} ||Ae_i||^4 \geq \frac{1}{(p_i^T Ap_i)} \frac{1}{||A^{-1}||^2} ||e_i||_A^4 = \frac{1}{||p_i||_A^2} \frac{1}{||A^{-1}||^2} ||e_i||_A^4
$$

$$
\frac{|1}{||p_i||_A^2} \frac{1}{||A^{-1}||^2} ||e_i||_A^4 = \frac{1}{||Ae_i||_A^2} \frac{1}{||A^{-1}||^2} ||e_i||_A^4 \geq \frac{1}{||A||_2^2 ||e_i||_A^2} \frac{1}{||A^{-1}||^2} ||e_i||_A^4 = \frac{1}{||A||_2^2 ||A^{-1}||^2} ||e_i||_A^2 = \frac{1}{(\kappa(A))^2} ||e_i||_A^2
$$

(c)

Let's use induction in this part.

Basis: For $i = 0$, we would have

$$
||e_0||_A^2 \leq (1 - \kappa(A)^{-2})^i ||e_0||_A^2
$$

The term in the paranthesis is equal to 1, so the base of induction is clearly correct.

Induction step:

Assume that inequality holds for $i = k - 1$, let's prove it also holds for $i = k$:

$$
\begin{aligned}
(1 - \kappa(A)^{-2})^k ||e_0||_A^2 = (1 - \kappa(A)^{-2})(1 - \kappa(A)^{-2})^{k-1} ||e_0||_A^2 &\geq (1 - \kappa(A)^{-2}) ||e_{k-1}||_A^2 && (25)\\
= ||e_{k-1}||_A^2 - \frac{1}{(\kappa(A))^2} ||e_{k-1}||_A^2 \geq ||e_{k-1}||_A^2 - \alpha_{k-1} p_{k-1}^T Ap_{k-1} &= ||e_k||_A^2 && (26)
\end{aligned}
$$

So it also proved for $i = k$.

4. (P53)

**Solution:**

(a) As the condition number increases, J deviates further from being flat, and the contours become more elliptical.