MS-E1651 - Numerical Matrix Computations
Exercise 4

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October 2, 2023

1. (P48)

   Solution:

$$||A||_2 = \sigma_{max}(A) \tag{1}$$

$$||A||_F = \sqrt{\Sigma_{i,j}|a_{ij}|^2} = \sqrt{trace(A^*A)} = \sqrt{\Sigma_i \sigma_i^2} \tag{2}$$

   Since $\sigma_{max}(A)$ is among the eigenvalues, we can conclude

$$||A||_2 \leq ||A||_F$$

   and if since all of eigenvalues are less than $\sigma_{max}(A)$, we can conclue

$$||A||_F \leq \sqrt{n}||A||_2$$

2. (P49)

   Solution:

   (a) Let's write down the equation of each norm:

$$||\delta A||_F = \sqrt{\Sigma_{i,j}|\delta a_{ij}|^2} \leq \sqrt{\Sigma_{i,j}\epsilon_{ij}^2|a_{ij}|^2} \leq \sqrt{\Sigma_{i,j}\epsilon^2|a_{ij}|^2} = \epsilon\sqrt{\Sigma_{i,j}|a_{ij}|^2} = \epsilon||A||_F \tag{3}$$

   (b) Let's write down the equation of each norm:

$$||\delta A||_1 = \max_j \Sigma_i|\delta a_{ij}| \leq \max_j \Sigma_i|\epsilon_{ij}||a_{ij}| \leq \max_j \Sigma_i\epsilon|a_{ij}| = \epsilon||A||_1 \tag{4}$$

   (c) Let's write down the equation of each norm:

$$||\delta A||_\infty = \max_i \Sigma_j|\delta a_{ij}| \leq \max_i \Sigma_j|\epsilon_{ij}||a_{ij}| \leq \max_i \Sigma_j\epsilon|a_{ij}| = \epsilon||A||_\infty \tag{5}$$

3. (P53)

   Solution:

   (a) We want to compute the 2-norm of the vector **a21** and add 1 to it.

   The vector **a21** is defined as:

$$\mathbf{a21} = \left[\sqrt{\pi+1}, \sqrt{\pi+2}, \ldots, \sqrt{\pi+(n-1)}\right]^T$$

   To calculate the 2-norm (Euclidean norm) of a vector, we square each element, sum the squares, and then take the square root of the sum:

$$\begin{aligned}||\mathbf{a21}||_2 &= \sqrt{\left(\sqrt{\pi+1}\right)^2 + \left(\sqrt{\pi+2}\right)^2 + \ldots + \left(\sqrt{\pi+(n-1)}\right)^2} \\ &= \sqrt{(\pi+1) + (\pi+2) + \ldots + (\pi+(n-1))}\end{aligned}$$

Simplifying the sum inside the square root, we get:

$$||\mathbf{a21}||_2 = \sqrt{(n-1)\pi + 1 + 2 + \ldots + (n-1)}$$

So, the sum simplifies to:

$$||\mathbf{a21}||_2 = \sqrt{(n-1)\pi + \frac{n(n-1)}{2}}$$

Finally, we can compute $a11$ by adding 1 to this value:

$$a_{11} = ||\mathbf{a21}||_2 + 1 = \sqrt{(n-1)\pi + \frac{n(n-1)}{2}} + 1$$

4. (P54)

Solution:

(a) According to P53, we have

$$(a_{11} - a_{21}^T a_{21})x_1 = 1 \tag{6}$$
$$(a_{11} - (a_{11} - 1))x_1 = 1 \tag{7}$$
$$x_1 = 1 \tag{8}$$

and also for x_2, we have

$$x_2 = -a_{21}x_1 \tag{9}$$
$$x_2 = -a_{21} \tag{10}$$
$$x_2 = \left[-\sqrt{\pi+1}, -\sqrt{\pi+2}, \ldots, -\sqrt{\pi+(n-1)}\right]^T \tag{11}$$

5. (P55) **Solution:**

(a) Let $U \in \mathbb{R}^{2\times 2}$ be such that $U^T U = I$. Consider the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$[y_i, y_j]^T = U[x_i, x_j]^T$$

and $y_k = x_k$ for $k \neq i, k \neq j$. That is, the matrix U operates on rows i and j of vector x, while all other rows are left untouched. To prove that f is a linear mapping, we need to show that it satisfies the properties of additivity and homogeneity.

- **Additivity**:
  We need to show that for any two vectors x and z, the mapping f applied to their sum equals the sum of the mappings of each vector:

  $$f(x+z) = f(x) + f(z)$$

  Let's calculate $f(x+z)$ and $f(x) + f(z)$ and check if they are equal.
  For $f(x+z)$:

  $$[y_i, y_j]^T = U[x_i + z_i, x_j + z_j]^T \tag{12}$$
  $$= U[x_i, x_j]^T + U[z_i, z_j]^T \tag{13}$$

  Moreover note that for all $t \neq i,\ j$, we have $f(x+z)_t = (x+z)_t = x_t + z_t$.
  Now, for $f(x) + f(z)$:

  $$[y_i, y_j]^T = U[x_i, x_j]^T + U[z_i, z_j]^T \tag{14}$$

  Moreover note that for all $t \neq i,\ j$, we have $f(x)_t + f(z)_t = x_t + z_t$, and since $U[x_i, x_j]^T + U[z_i, z_j]^T = U[x_i + z_i, x_j + z_j]^T$, we have shown that f satisfies additivity.

- **Homogeneity**:
  We need to show that for any scalar α and vector x, the mapping f applied to the scaled vector αx equals the scaled mapping of the vector:
  $$f(\alpha x) = \alpha f(x)$$
  Let's calculate $f(\alpha x)$ and $\alpha f(x)$ and check if they are equal.
  For $f(\alpha x)$:
  $$\begin{aligned}[y_i, y_j]^T &= U[\alpha x_i, \alpha x_j]^T \\ &= \alpha U[x_i, x_j]^T\end{aligned}$$
  Moreover note that for all $t \neq i,\ j$, we have $f(\alpha x)_t = (\alpha x)_t = \alpha x_t$.
  Now, for $\alpha f(x)$:
  $$[y_i, y_j]^T = \alpha U[x_i, x_j]^T$$
  Moreover note that for all $t \neq i,\ j$, we have $\alpha f(x)_t = \alpha x_t$, and since $\alpha U[x_i, x_j]^T = \alpha U[x_i, x_j]^T$, we have shown that f satisfies homogeneity.
  Therefore, we have demonstrated that the mapping f is linear because it satisfies both additivity and homogeneity properties.

(b) The right side of the equation is given by:

$$x^T y = \sum_t x_t y_t$$

For the left side of the equation, we note that the elements of $f(x)$ and $f(y)$ are the same as x and y except in coordinates i and j. Therefore, we have:

$$\begin{aligned}f(x)^T f(y) &= \left(U[x_i, x_j]^T\right)^T \left(U[y_i, y_j]^T\right) + \sum_{t\neq i,j} x_t y_t \\ &= \left([x_i, x_j]U^T U[y_i, y_j]^T\right) + \sum_{t\neq i,j} x_t y_t \\ &= \left([x_i, x_j] I [y_i, y_j]^T\right) + \sum_{t\neq i,j} x_t y_t \\ &= x_i y_i + x_j y_j + \sum_{t\neq i,j} x_t y_t \\ &= \sum_t x_t y_t\end{aligned}$$

Hence, we have demonstrated that both sides of the equation are equal.

(c) Let's consider the mapping $f(x) = Gx$. Since it satisfies equation (1.95), we have:

$$\begin{aligned}f(x)^T f(y) &= x^T y \\ (Gx)^T G y &= x^T G^T G y = x^T y\end{aligned}$$

Now, let's consider $A = G^T G$, and we want to obtain A_{ij} for all $1 \leq i, j \leq n$. Since the above equation holds for all x and y, we can define x and y in such a way that the left side of the equation becomes equal to A_{ij}.

Define x as follows:

$$x_t = \begin{cases} 1, & \text{if } t = i \\ 0, & \text{otherwise} \end{cases}$$

Define y as follows:

$$y_t = \begin{cases} 1, & \text{if } t = j \\ 0, & \text{otherwise} \end{cases}$$

Now, $x^T A$ will only keep row i of A, and similarly, $x^T A y$ will only keep row i and column j.
Note that if $i \neq j$, the left side of the equation is equal to zero, and if $i = j$, the left side of the equation is equal to one.
Thus, we find that $A_{ii} = 1$ and $A_{ij} = 0$ for $i \neq j$. Therefore, we conclude that $A = G^T G = I$.
Hence, G is unitary.

6. (P56)

Solution:

(a) We know that

$$UA = \begin{bmatrix} r_1 & r_2 \\ 0 & r_3 \end{bmatrix} \to A = U^T \begin{bmatrix} r_1 & r_2 \\ 0 & r_3 \end{bmatrix}$$

Now we need to find QR-decomposition of A.

Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Step 1: Initialize Q and R: Set Q as an empty matrix and R as a matrix of zeros.

Step 2: Orthogonalize the first column of the matrix: For the first column of A, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, normalize it to obtain the first column of Q:

$$v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1^2 + 3^2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Now, set the first column of Q to be q_1 and update R:

$$Q = \begin{bmatrix} q_1 \end{bmatrix}, \quad R = \begin{bmatrix} \langle v_1, q_1 \rangle \end{bmatrix}$$

Step 3: Orthogonalize the second column of the matrix: For the second column of A, $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, subtract its projection onto q_1 from itself:

$$\begin{aligned} v_2 &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \langle \begin{bmatrix} 2 \\ 4 \end{bmatrix}, q_1 \rangle q_1 \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \frac{14}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \frac{14}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} \frac{14}{10} \\ \frac{42}{10} \end{bmatrix} \\ &= \begin{bmatrix} 2 - \frac{14}{10} \\ 4 - \frac{42}{10} \end{bmatrix} = \begin{bmatrix} \frac{-6}{10} \\ \frac{-2}{10} \end{bmatrix} \end{aligned}$$

Now, normalize v_2 to obtain q_2:

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\left(\frac{-6}{10}\right)^2 + \left(\frac{-2}{10}\right)^2}} \begin{bmatrix} \frac{-6}{10} \\ \frac{-2}{10} \end{bmatrix} = \frac{1}{\sqrt{\frac{2}{5}}} \begin{bmatrix} \frac{-6}{10} \\ \frac{-2}{10} \end{bmatrix}$$

Now, update Q:

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$$

Moreover we define U, transpose of obtained Q.

(b) Since the second row remains unchanged after the multiplication, and only the first and third rows are affected, we can consider U as a similar linear map to the previous problem. Assuming that the matrix U can be represented as:

$$U = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}$$

To determine the values of a, b, c, and d, we can perform the following calculations:

$$\begin{aligned} c + 3d &= 0 \quad \text{(from the original problem)} \\ d &= \frac{-c}{3} \end{aligned}$$

On the other hand, since U is unitary, we have:

$$U^T U = UU^T = I$$

This implies:

$$\begin{aligned} a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1 \\ a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \end{aligned}$$

So we have either $c = \frac{3}{\sqrt{10}}$ or $c = \frac{-3}{\sqrt{10}}$. Assuming the first case, we find $d = \frac{-1}{\sqrt{10}}$. Then, by computation, we conclude that a is either $\frac{1}{\sqrt{10}}$ or $\frac{-1}{\sqrt{10}}$, and b is either $\frac{3}{\sqrt{10}}$ or $\frac{-3}{\sqrt{10}}$.

A possible matrix representation for U is:

$$U = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{-1}{\sqrt{10}} \end{bmatrix}$$