MS-E1651 - Numerical Matrix Computations
Exercise 3

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1. (P40)

   Solution:

   (a) Given: $(A + uv^T)\hat{x} = b \quad (1)$ $Ax = b \quad (2)$

   We want to show that:

   $(A + uv^T)w = uv^T x \quad (3)$

   Now, let's evaluate $(A + uv^T)w$ on the left side of equation (3) using equation (4):

$$
\begin{aligned}
(A + uv^T)w &= (A + uv^T)x - (A + uv^T)\hat{x} \\
&= Ax + uv^T x - (A + uv^T)\hat{x} \\
&= b + uv^T x - b \\
&= uv^T x
\end{aligned}
$$

   (b) We know that:

$$
\begin{aligned}
(A + uv^T)w &= uv^T x \\
Aw + uv^T w &= uv^T x \\
A^{-1}Aw + A^{-1}uv^T w &= A^{-1}uv^T x \\
\rightarrow w &= A^{-1}uv^T x - A^{-1}uv^T w \\
w &= A^{-1}u(v^T x) - A^{-1}u(v^T w)
\end{aligned}
$$

   Both $(v^T w)$ and $(v^T x)$ are in $\mathbb{R}$ and they are scalar. Thus we can have above equation as follows:

$$
w = (v^T w - v^T x)A^{-1}u
$$

   and if we put $\alpha = (v^T w - v^T x)$ we would have the result that $w = \alpha A^{-1}u$ for some $\alpha \in \mathbb{R}$.

   (c) Now, let's find the value of α using these results.

   Starting from the equation in **(a)**:

$$
(A + uv^T)w = uv^T x
$$

   .

   Substituting $w = \alpha A^{-1}u$ from **(b)**:

$$
(A + uv^T)(\alpha A^{-1}u) = uv^T x
$$

Now, let's simplify this equation:

$$\alpha(A + uv^T)A^{-1}u = uv^T A^{-1}b$$

Both terms $\alpha(A + uv^T)A^{-1}$ and $v^T A^{-1}b$are scalar and both are coeficient of u in both side of equation, so they must be equal. Thus we have

$$\alpha(A + uv^T)A^{-1} = v^T A^{-1}b$$

$$\alpha = \frac{v^T A^{-1}b}{(A + uv^T)A^{-1}}$$

$$\alpha = \frac{v^T A^{-1}b}{(1 + uv^T A^{-1})}$$

Now, let's find $\hat{x}$ using the result from **(b)**:

$$w = \alpha A^{-1}u$$

$$w = \frac{v^T A^{-1}b}{1 + v^T A^{-1}u} A^{-1}u$$

Now, let's express $\hat{x}$ in terms of w:

$$\hat{x} = A^{-1}b - w$$

$$\hat{x} = A^{-1}b - \frac{v^T A^{-1}b}{1 + v^T A^{-1}u} A^{-1}u$$

and since $v^T A^{-1}b$ is a scalar we can write as follows:

$$\hat{x} = A^{-1}b - \frac{A^{-1}uv^T A^{-1}b}{1 + v^T A^{-1}u}$$

2. (P41) **Solution:**

(a) Based on the definition we have

$$(U\Sigma V^T - U\delta\Sigma V^T)e = b$$

Multiplying both side of equation by U^T from gives us

$$(\Sigma V^T - \delta\Sigma V^T)e = U^T b$$
$$(\Sigma - \delta\Sigma)V^T e = U^T b$$

Then we would have $(\Sigma - \delta\Sigma) = diag(\sigma_1 - \delta\sigma_1, \cdots, \sigma_n - \delta\sigma_n)$ and if we define matrix $\hat{\Sigma} = diag(\frac{\sigma_1^{-1}}{1-\sigma_1^{-1}\delta\sigma_1}, \cdots, \frac{\sigma_n^{-1}}{1-\sigma_n^{-1}\delta\sigma_n})$, we will have

$$\hat{\Sigma}(\Sigma - \delta\Sigma) = (\Sigma - \delta\Sigma)\hat{\Sigma} = I$$

So if multiply both side of equation by $\hat{\Sigma}$, we will have

$$V^T e = \hat{\Sigma} U^T b$$

Moreover it is enough to multiply by V from right in both side of equation, so we will have

$$e = V\hat{\Sigma} U^T b$$

(b) First of all, we know that inverse of A is

$$A^{-1} = V\Sigma^{-1}U^T$$

In this problem we are going to use unitarily invariant property, which is

$$||UA||_2 = ||A||_2$$

if U is unitary matrix. Since norm-2 of b is 1, we would have

$$e \leq \frac{||A^{-1}||_2}{1 - ||A^{-1}||_2\, ||\delta A||_2} \tag{1}$$

$$e \leq \frac{||V\Sigma^{-1}U^T||_2}{1 - ||V\Sigma^{-1}U^T||_2\, ||U\delta\hat{\Sigma}V^T||_2} \tag{2}$$

$$e \leq \frac{||\Sigma^{-1}U^T||_2}{1 - ||\Sigma^{-1}U^T||_2\, ||\delta\hat{\Sigma}V^T||_2} \tag{3}$$

$$e \leq \frac{||\Sigma^{-1}||_2}{1 - ||\Sigma^{-1}||_2\, ||\delta\hat{\Sigma}||_2} \tag{4}$$

So we would have

$$e \leq \frac{\max_i |\sigma_i|}{1 - (\max_i |\frac{1}{\sigma_i}|)(\max_i |\frac{\sigma_i^{-1}}{1-\sigma_i^{-1}\delta\sigma_i}|)}$$

3. (P43) **Solution:**

(a)

$$(345)_{10} = (101011001)_2 \approx (1.010)_2 \times 2^8$$

$$(\frac{1}{3})_{10} = (0.0101010101....)_2 \approx (1.011)_2 \times 2^{-2}$$

(b)

$$(1.010)_2 \times 2^8 = 320 \longrightarrow |\hat{x_1} - x_1| = 25$$

$$(1.011)_2 \times 2^{-2} = \frac{11}{32} \longrightarrow |\hat{x_2} - x_2| = \frac{1}{96}$$

(c)

$$u = \frac{1}{2^3} = \frac{1}{8}$$

(d)

$$\begin{aligned} \hat{x_1} + \hat{x_2} = (10100000000.00)_2 \times 2^{-2} + (1.011)_2 \times 2^{-2} &= (10100000001.011)_2 \times 2^{-2} && (5) \\ &= (1.010)_2 \times 2^8 && (6) \end{aligned}$$

4. (P47) **Solution:**

(a) In this part, we use the previous problem P46. Then, for all $i = 2, \cdots, n$, we write $1 + u$ instead of $1 + \delta_i$, and for all $k = 1, \cdots, n$, we write $1 + u$ instead of $1 + \hat{\delta}$. In the other word, we are trying to make $|fl(\Sigma x_i y_i) - \Sigma x_i y_i|$ greater, and also for each i, we consider the inner multiplication like $\Sigma_{k=1}^{n}(1 + \delta_i)$.

$$|fl(\Sigma x_i y_i) - \Sigma x_i y_i| \leq |\Sigma x_i y_i (1 + u)^n - x_i y_i| \tag{7}$$

Based on Lemma (1.5), we have:

$$\begin{aligned} |fl(\Sigma x_i y_i) - \Sigma x_i y_i| \leq |\Sigma x_i y_i (1 + u)^n - x_i y_i| \leq |\Sigma(1 + \frac{nu}{1 - nu})x_i y_i - \Sigma x_i y_i| &= \frac{nu}{1 - nu}|x^T y| && (8) \\ \rightarrow \frac{nu}{1 - nu}|x^T y| \leq \frac{nu}{1 - nu}||x|| \, ||y|| &\leq \frac{nu}{1 - nu} && (9) \end{aligned}$$

In the last part, we used Cauchy–Schwarz inequality to conclude the inequality.

(b) Note that since U and V are unitary matrix, each row and column of them has lenght exactly 1. Thus, if we name row of U, R_i^T and column of V, C_j, for all i and j,

$$||R_i||_2 = ||C_j||_2 = 1$$

So conditions of part (a) hold here, and we can use part (a). Also note that $(UV)_{ij} = R_i^T C_j$. So $|E_{ij}| = |fl(R_i^T C_j) - (UV)_{ij}| = |fl(R_i^T C_j) - R_i^T C_j|$. Moreover, based on part (a), we know that $|fl(R_i^T C_j) - R_i^T C_j| \leq \frac{nu}{1-nu}$. So we can write

$$fl(UV) = UV - E$$