Transition-Based Coding and Formal Language Theory for Ordered Digraphs

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Embedding Arbitrary Graphs to the String Space
to Support Succinct Representation of Infinite Sets of Graphs

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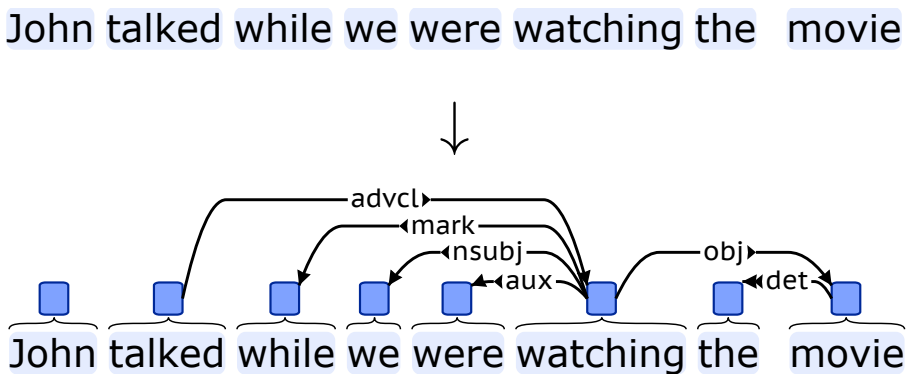
(1) FSMNLP, September 23–25, 2019, Dresden
(2) LT Research Seminar, October 24, 2019, Helsinki
(3) Grupo LYS, November 12, 2019, A Coruña

(4) NLP Reading Club, December 16, 2019, Jerusalem

Parsing produces Graphs



Transition-Based Parsing

$\mathbf{x} = x_1 \ldots x_m \in \Sigma^*$: input string,

C : set of configurations, $C_I \subseteq C$: set of intial configurations

$c_0 \in C_I$: initial configuration

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$P(c_0 \xrightarrow{a_1} c_1 \mid \mathbf{x}, c_0)$

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$$P(c_2 \xrightarrow{a_3} c_3 \mid \mathbf{x}, c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_1} c_2)$$

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$$P(c_2 \overset{a_3}{\longrightarrow} c_3 \mid \mathbf{x}, c_0 \overset{a_1}{\longrightarrow} c_1 \overset{a_1}{\longrightarrow} c_2)$$

$$\vdots$$

$$P(c_n \in C_{F_\mathbf{x}})\ P(c_{n-1} \overset{a_n}{\longrightarrow} c_n \mid \mathbf{x}, c_0 \overset{a_1}{\longrightarrow} c_1 \overset{a_2}{\longrightarrow} ... \overset{a_{n-1}}{\longrightarrow} c_{n-1})$$

Transition-Based Parsing

$\mathbf{x} = x_1 ... x_m \in \Sigma^*$: input string,

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$c_0 \xrightarrow{a_1 ... a_n} c_n, c_0 \in C_I, c_n \in C_{F_\mathbf{x}}$: valid action sequences V

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giving $P(o(a_1 ... a_n) \mid \mathbf{x})$ where $o : V \rightarrow GRAPH$

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$c_0 \xrightarrow{a_1 ... a_n} c_n, c_0 \in C_I, c_n \in C_I$: valid action sequences V

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$$P(c_n \in C_I \quad) \; P(c_{n-1} \xrightarrow{a_n} c_n \mid \quad c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} ... \xrightarrow{a_{n-1}} c_{n-1})$$

giving $P(o(a_1 ... a_n) \quad)$ where $o : V \rightarrow GRAPH$

Universal Transition System?

The problem of this paper is to find a **universal** transition system that can produce **arbitrary digraphs**.

> *'I found it quite surprising that there should be a (finite) transition system for arbitrary digraphs.*
> *I would have liked to see a discussion of this seeming contradiction between the present work and the existing literature.*
> *'*
>
> Reviewer

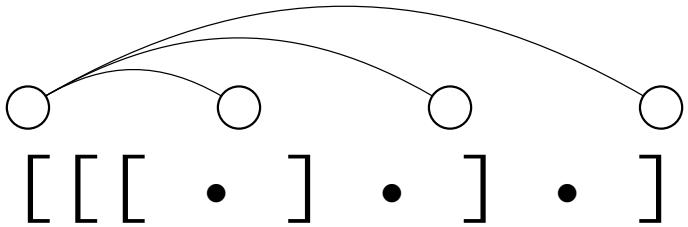
Prior art: no finite axiomatisation for graphs is known.

> *M. Ogawa. Complete axiomatization of an algebraic construction of graphs. In Y. Kameyama and P. J. Stuckey, editors,* Functional and Logic Programming, *pages 163–179, Berlin, Heidelberg. Springer Berlin Heidelberg, 2004*

This paper is rather about **representations** for graphs. Clearly, an adjacency matrix is one possible representation.

Bracketed Representations: Prior Art

Bracketing of Noncrossing Graphs



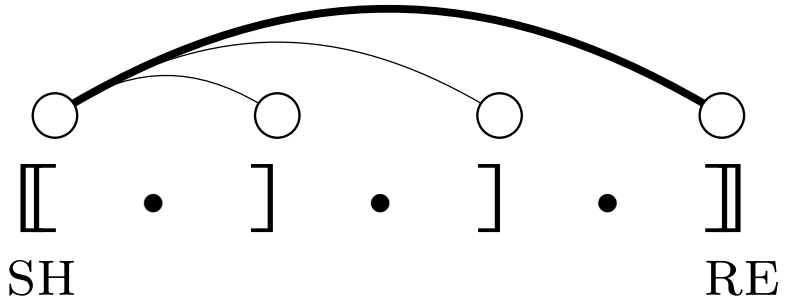
– A. Yli-Jyrä. Approximating dependency grammars through intersection of star-free regular languages. *International Journal of Foundations of Computer Science*, 16(3):565–579, 2005

– A. Yli-Jyrä. On dependency analysis via contractions and weighted FSTs. In D. Santos et al., editors, *Shall We Play the Festschrift Game?, Essays on the Occasion of Lauri Carlson's 60th Birthday*, pages 133–158. Springer, 2012

– A. Yli-Jyrä and C. Gómez-Rodríguez. Generic axiomatization of families of noncrossing graphs in dependency parsing. In *ACL 2017 (Vancouver)*, pages 1745–1755, 2017

Shared Brackets: Prior Art

Bracketing with Shared Endpoints



– A. Yli-Jyrä. Bounded-depth high-coverage search space for noncrossing parses. In *FSMNLP 2017*, pages 30–40, 2017
– A. Yli-Jyrä. How to embed noncrossing trees in Universal Dependencies treebanks in a low-complexity regular language. *Journal of Language Modelling*, 2019. URL: https://doi.org/10.15398/jlm.v7i2.213

Shared Endpoints and Crossing

– E. Pitler et al. Finding optimal 1-endpoint-crossing trees. *TACL*, 1:13–24, 2013
– R. Kurtz and M. Kuhlmann. Exploiting structure in parsing to 1-endpoint-crossing graphs. In *IWPT 2017 (Pisa)*, pages 78–87, 2017
– J. K. Kummerfeld and D. Klein. Parsing with traces: an o(n^4) algorithm and a structural representation. *TACL*, 5:441–454, 2017

The Method

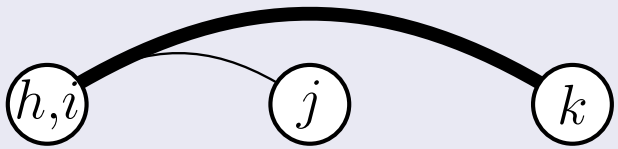
1. **Proper Rope Cover (PRC)**
2. **Rope Decomposition**
3. **Transition System**
4. **Balanced Bracketing**

Preliminaries: Covering Edge

Definition

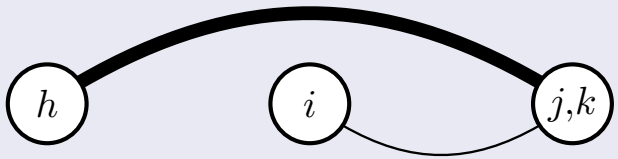
Let $([1, ..., n], E)$ be an ordered graph with $E \subseteq \{(i, j) \mid 1 \leq i < j \leq n\}$. Edge (h, k) is a (properly-longer shared-endpoint) *covering edge* for edge (i, j) if either

- $h = i$ and $i < j < k$



or

- $j = k$ and $h < i < j$.



Denote this situation by $(h, k) : (i, j)$.

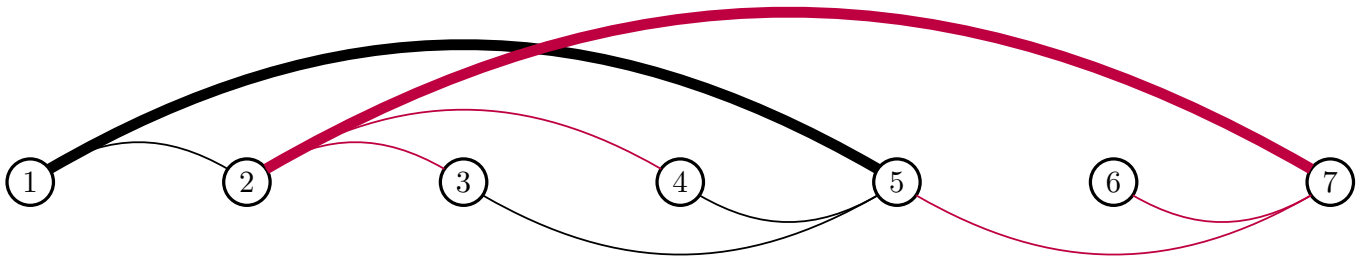
Proper Rope Cover (PRC) – a Novel Concept

Definition

A subset $R \subseteq E$ is a *rope cover* of the graph (V_n, E) if, for every edge $e \in E \backslash R$, there is an edge $c \in R$ such that $c : e$. Moreover, R is a *proper rope cover (PRC)* if is there is no edges $c_1, c_2 \in R$ s.t. $c_1 : c_2$.

Lemma

Every graph has a unique PRC R, and for every $(i, j), (i, k) \in R$, we have $j = k$.



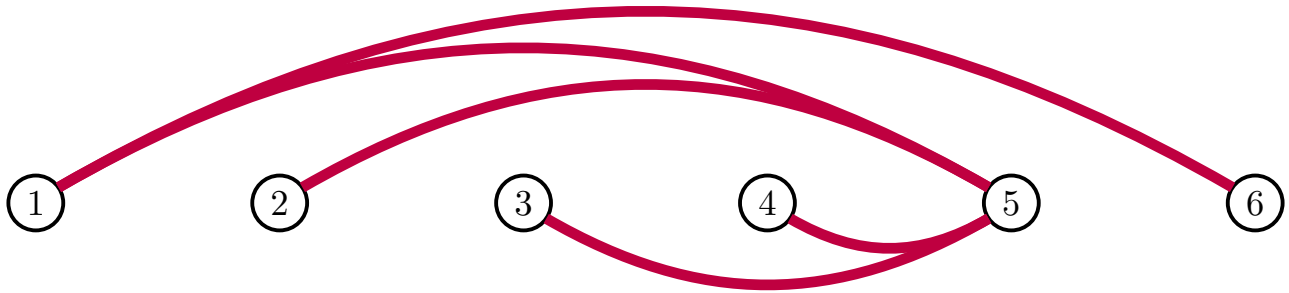
PRC $= \{(1, 5), (2, 7)\}$

Computation of Proper Rope Cover

Lemma

For any graph of n vertices, its PRC and rope-thickness can be computed in $O(n^3)$.

For graph $([6], \{(1,6), (1,5), (2,5), (3,5), (4,5)\})$



we compute PRC as follows:

PRC layer	*Remaining edges*
$R_0 = \{\}$	$E_0 = \{(1,6), (1,5), (2,5), (3,5), (4,5)\}$

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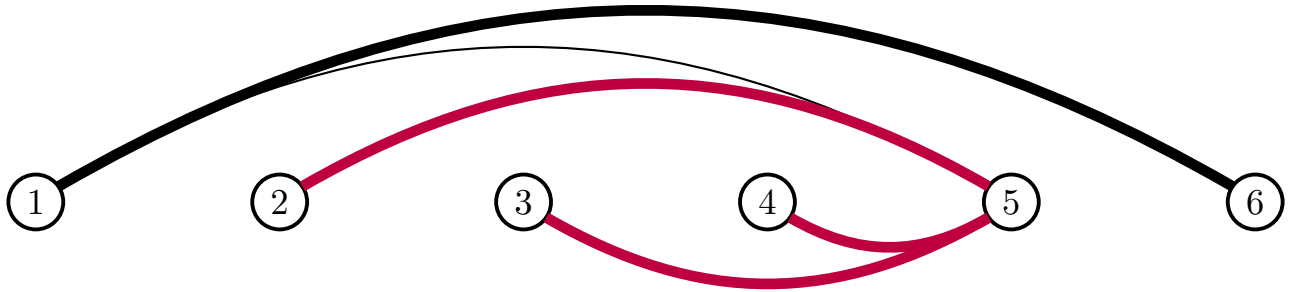

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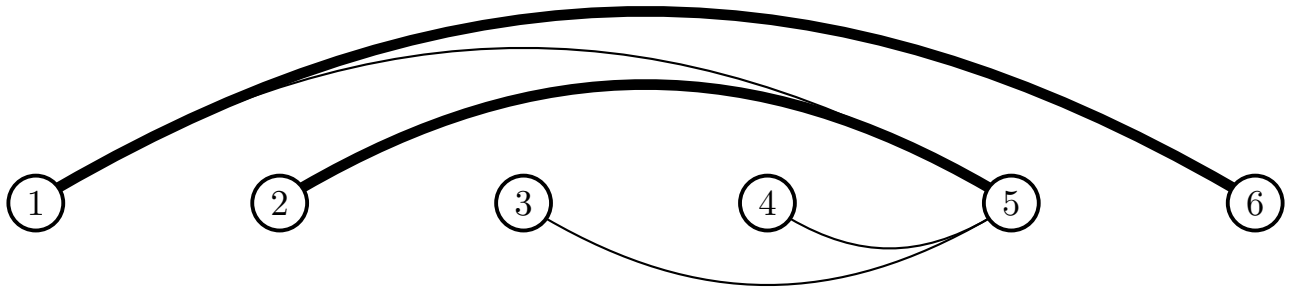

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$R_1 = \{(1,6)\}$	$E_1 = \{(2,5), (3,5), (4,5)\}$
$R_2 = \{(2,5)\}$	$E_2 = \{\}$ (algorithm terminates)

The Method

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4.

Unique Assignment

Theorem

Some edges can have two distinct covering edges in the PRC.

The edge $(2, 3)$ is covered by the edges $(1, 3)$ and $(2, 4)$.

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(1)

Convention ("EARLIEST")

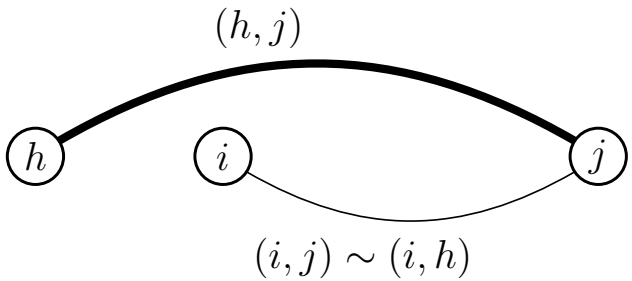
We adopt a convention according to which the ambiguity between two possible covering edges is resolved by assigning the edge to the earliest available covering edge.

Thus, edge $(2, 3)$ is assigned to this EARLIEST covering edge.

Rope Decomposition

Convention ("Indirect Edges")

When an edge (i, j) has a covering edge (h, j), $h < i < j$, we refer to the edge (i, j) indirectly, via the pair (i, h) where h is the LEFT INDEX of the covering edge.



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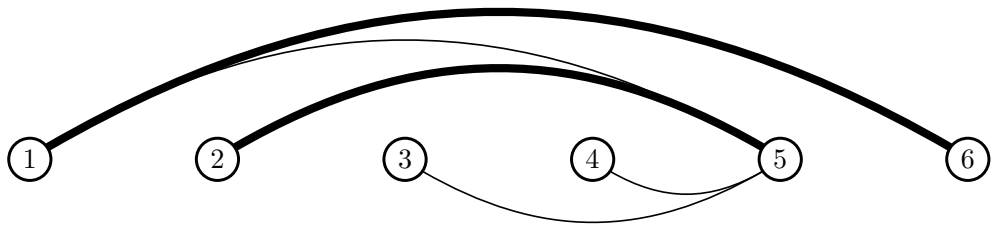

Rope Decomposition of this graph:

- proper rope cover $=\{(1, 6), (2, 5)\}$
- indirect edges $(3, 2)$, $(4, 2)$ referring to edges $(3, 5)$, $(4, 5)$
- other edge $(1, 5)$.

The Method

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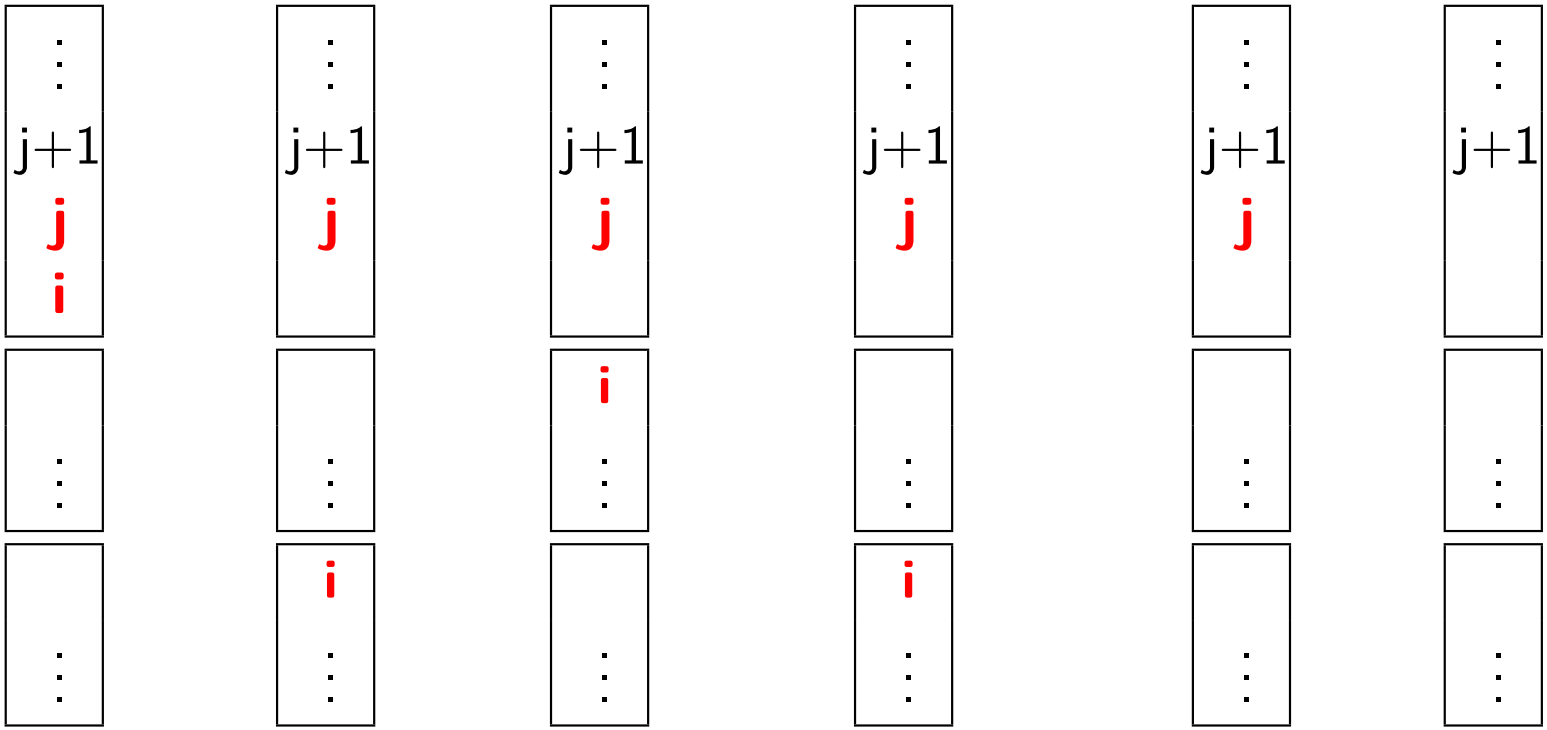
Novel Transition System (Simplified)

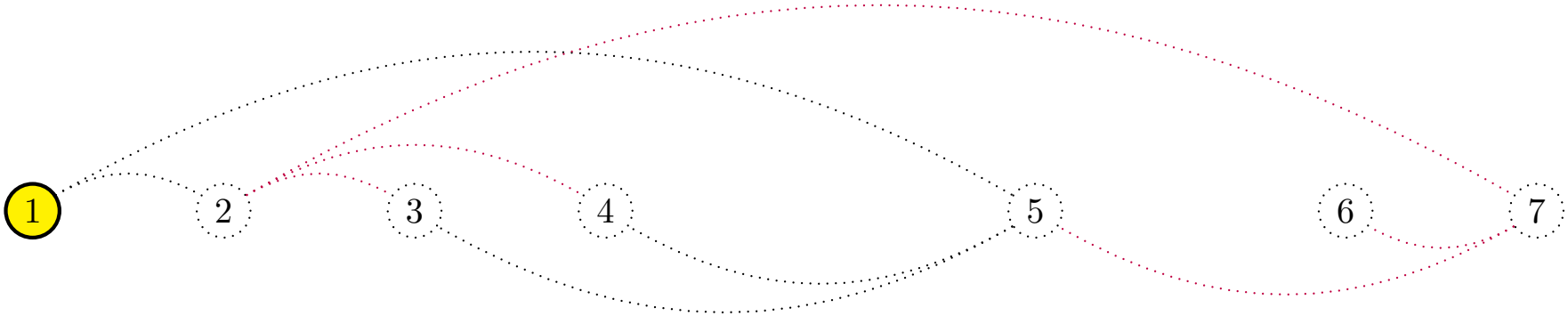


SHift PASS INSert REduce NeXt

$[\![\bullet$ $]$ $[$ $]\!]$ $\bullet$

In a *digraph*, the edges/some actions have directions: $<$, $>$, $<>$, 0 (dummy action)

From Transitions to Rope Decomposition



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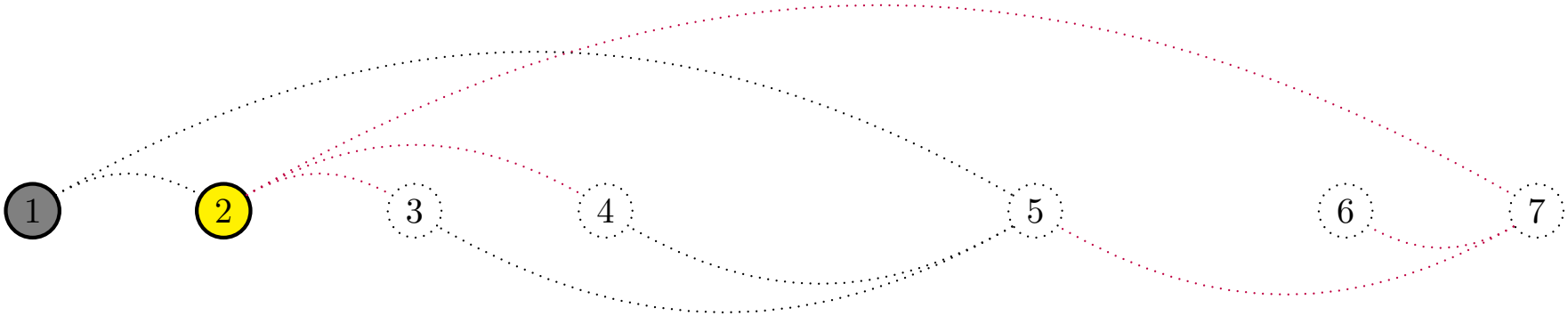
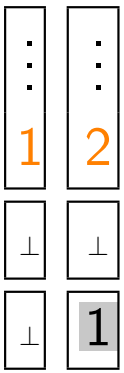


$[\![\bullet$



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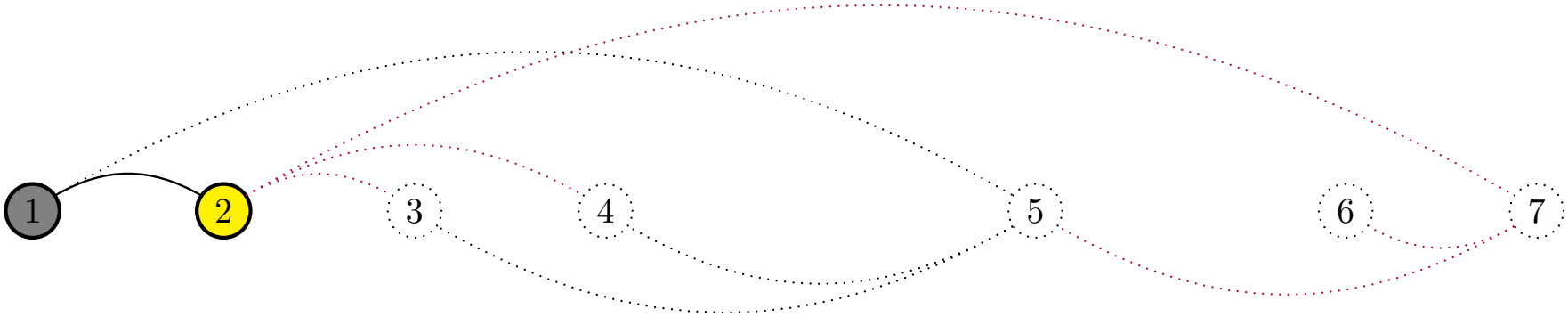
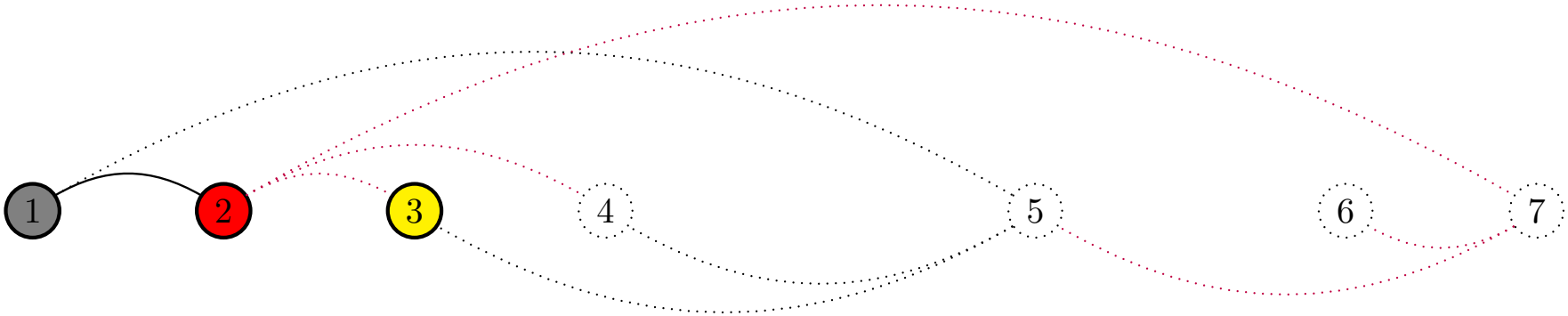
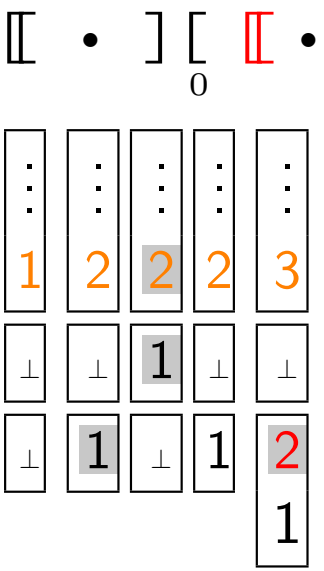


$[\![\; \bullet \;]\!]_0$

⋮	⋮	⋮	⋮
1	2	2	2
⊥	⊥	1	⊥
⊥	1	⊥	1

From Transitions to Rope Decomposition





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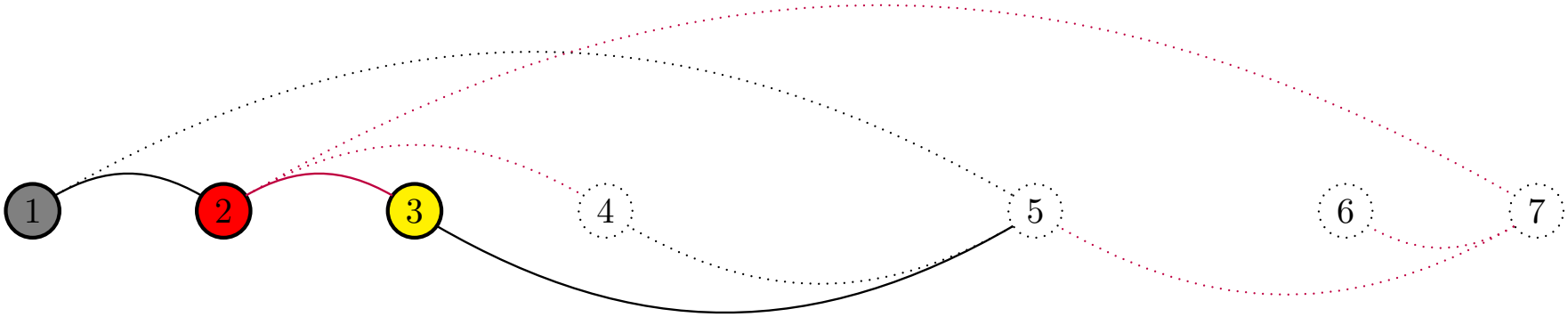

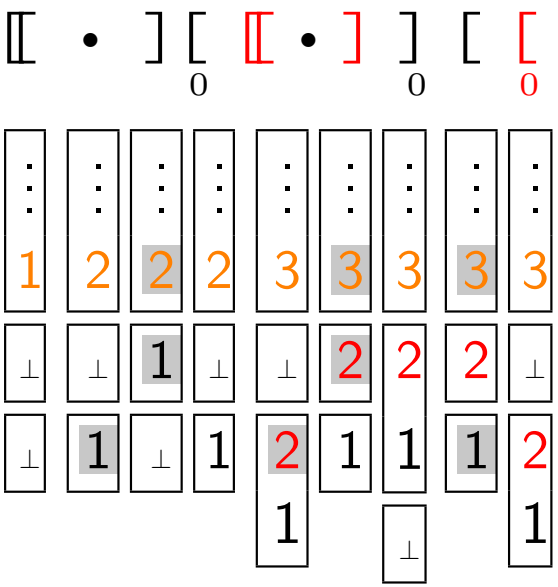

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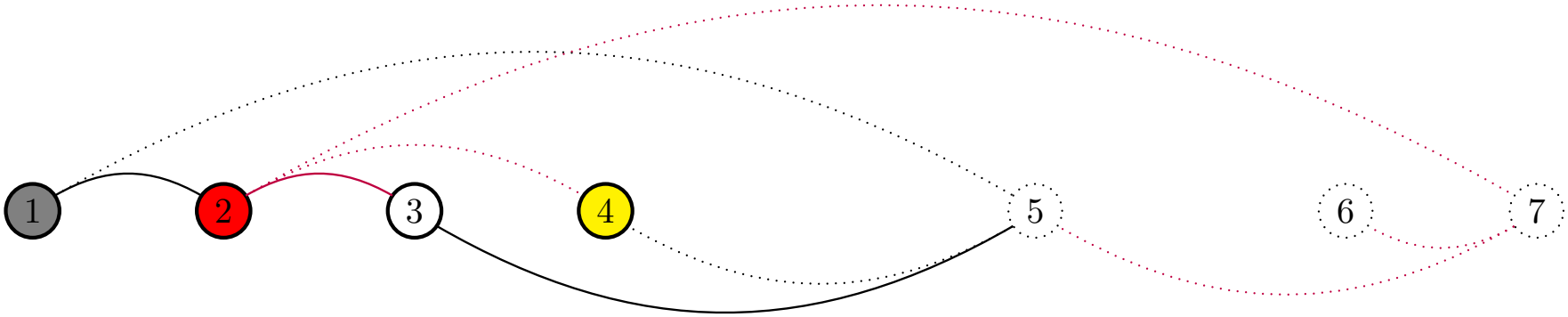

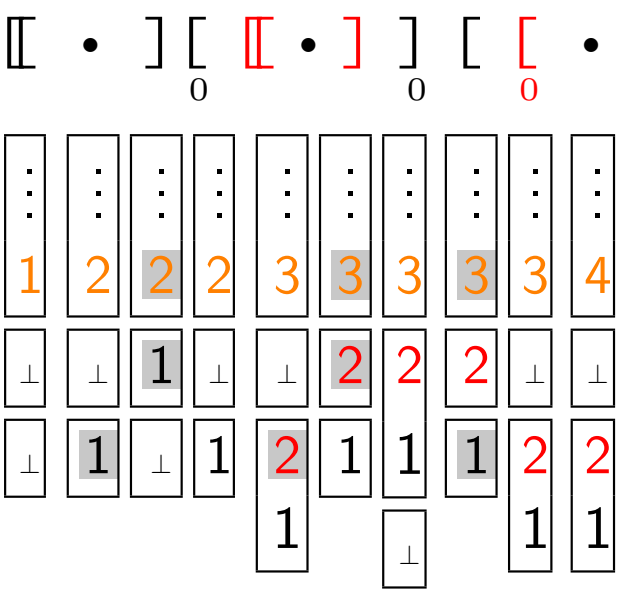

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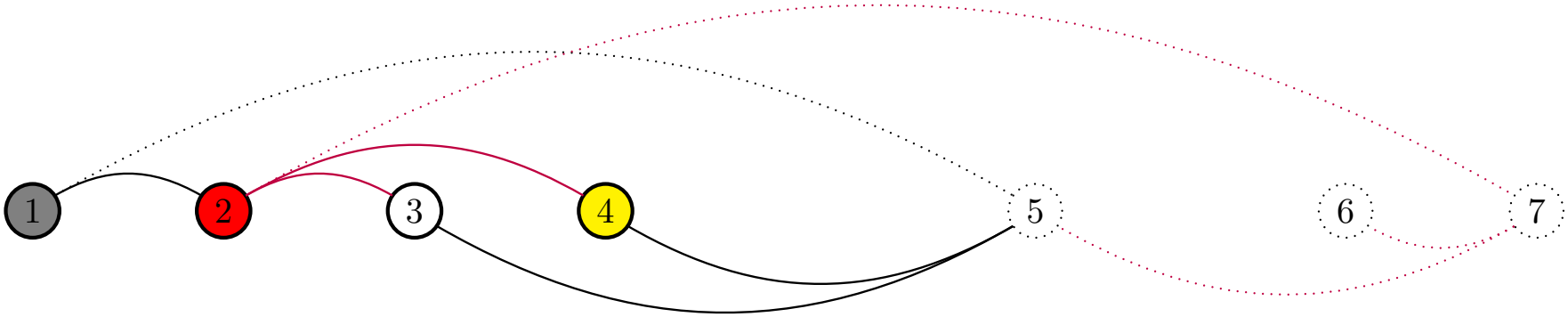

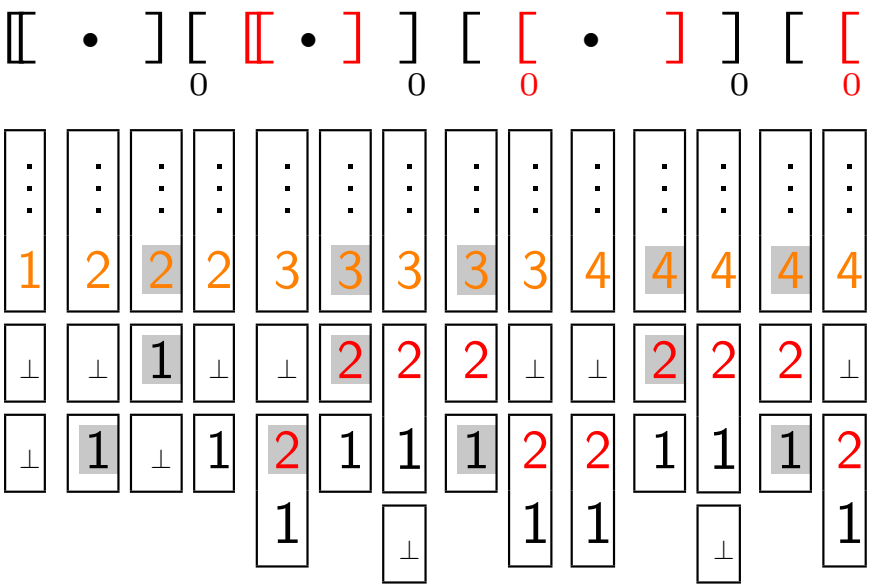

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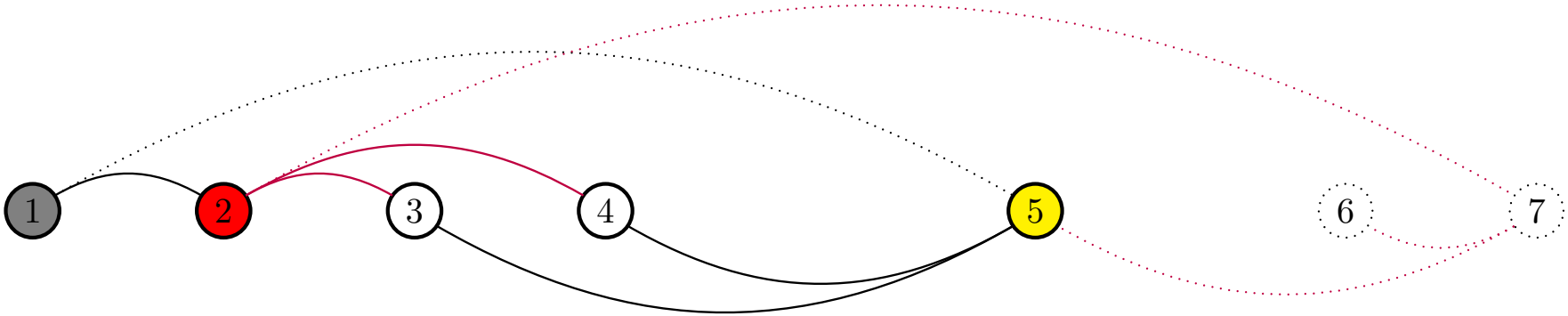
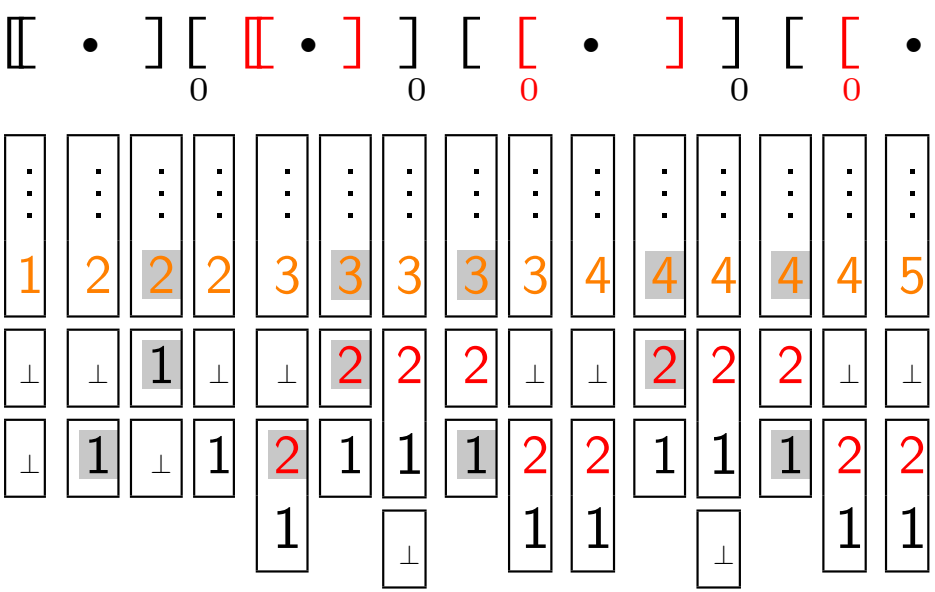
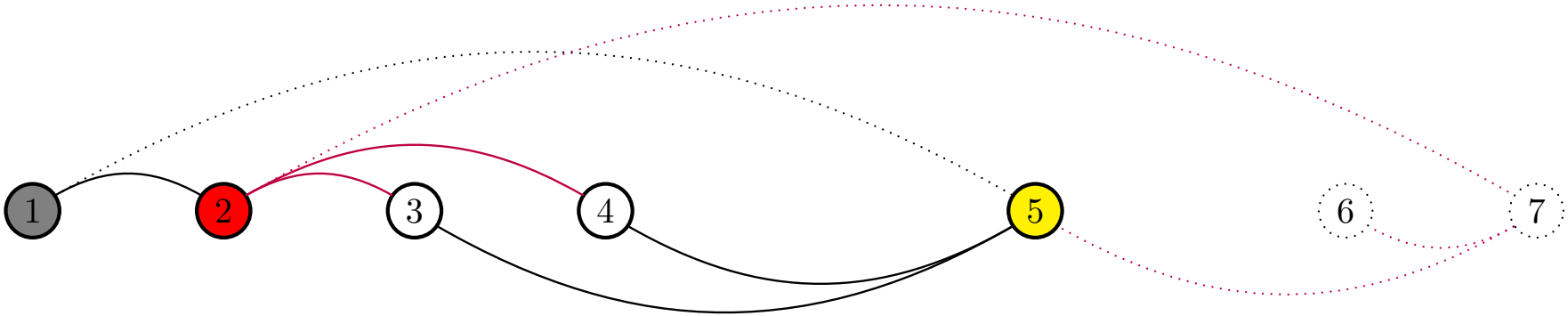
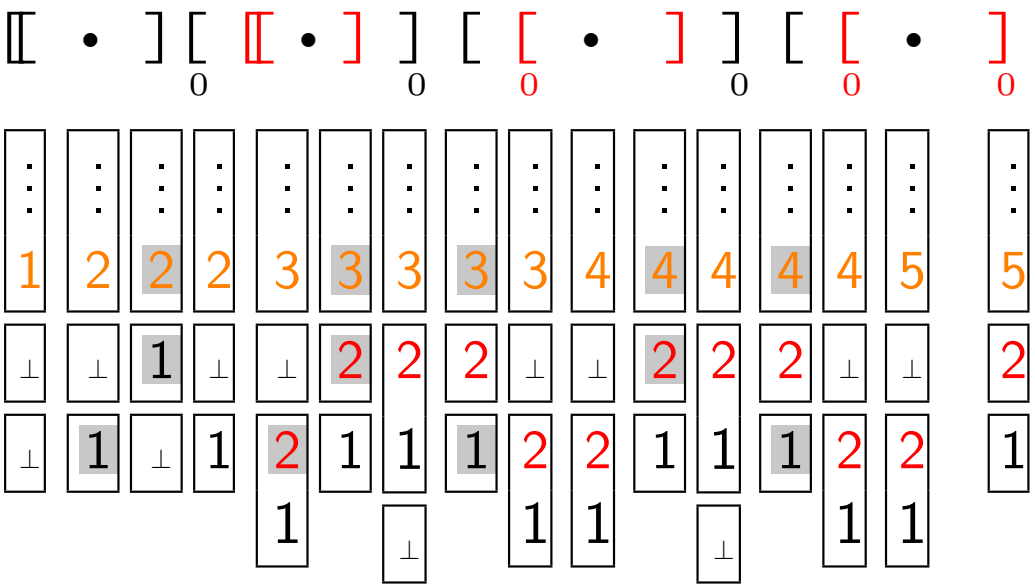




From Transitions to Rope Decomposition





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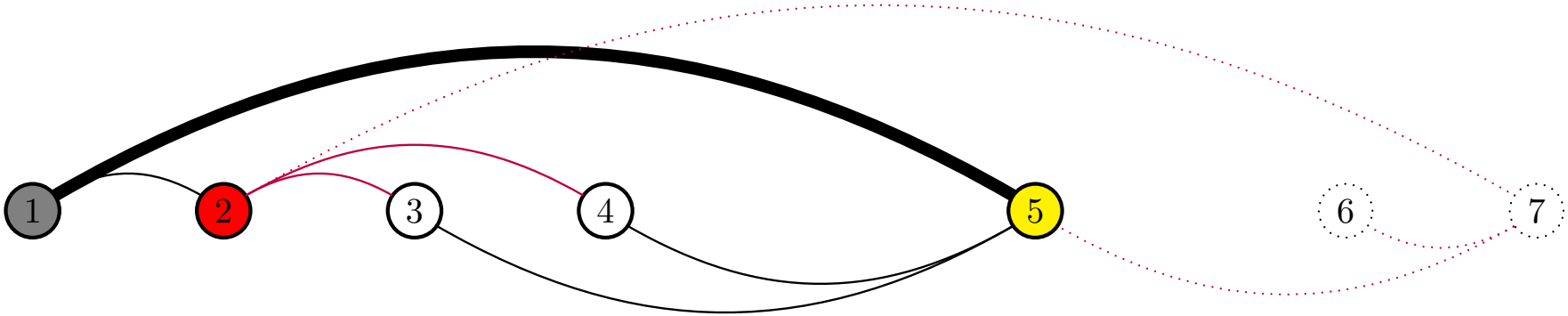

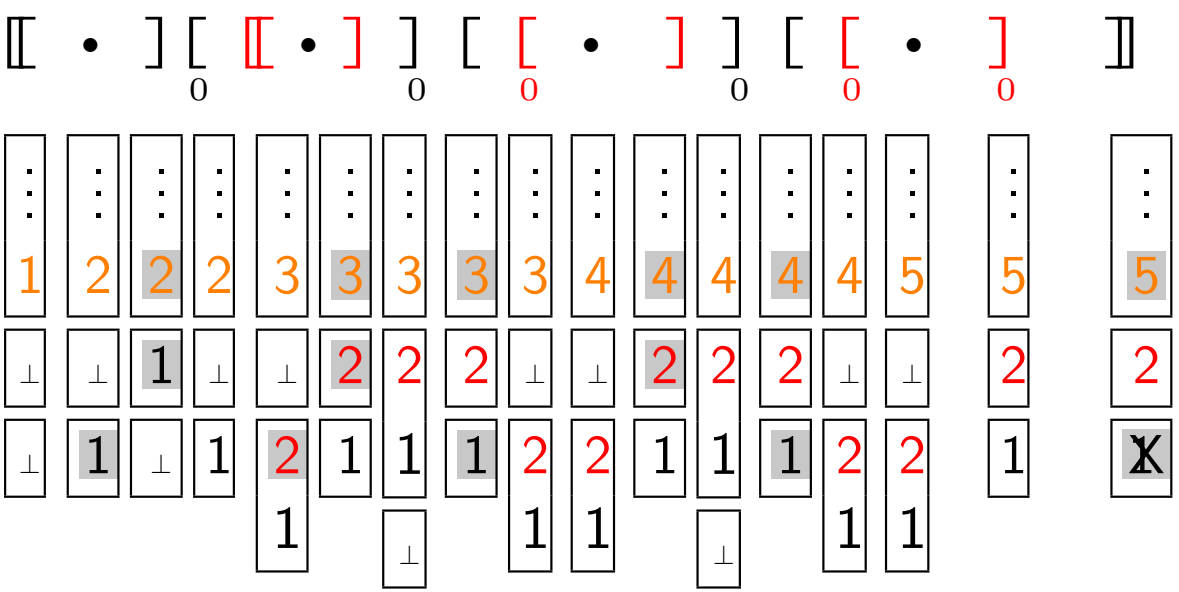

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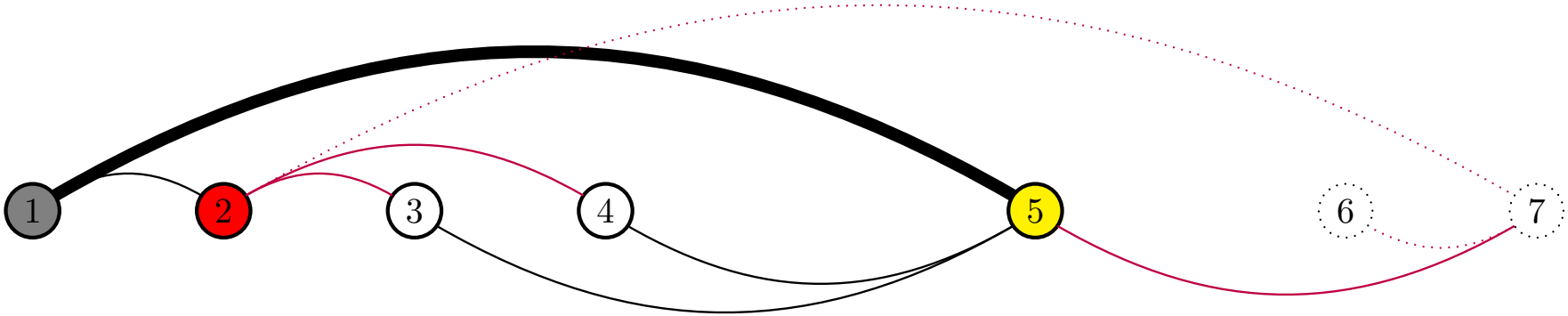

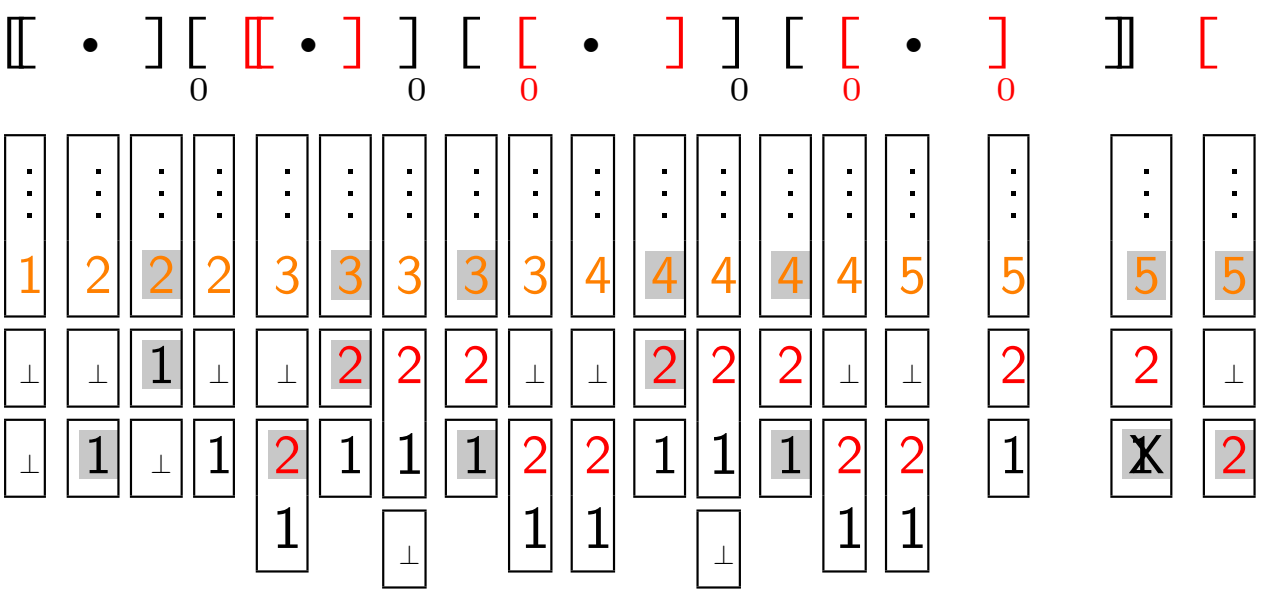

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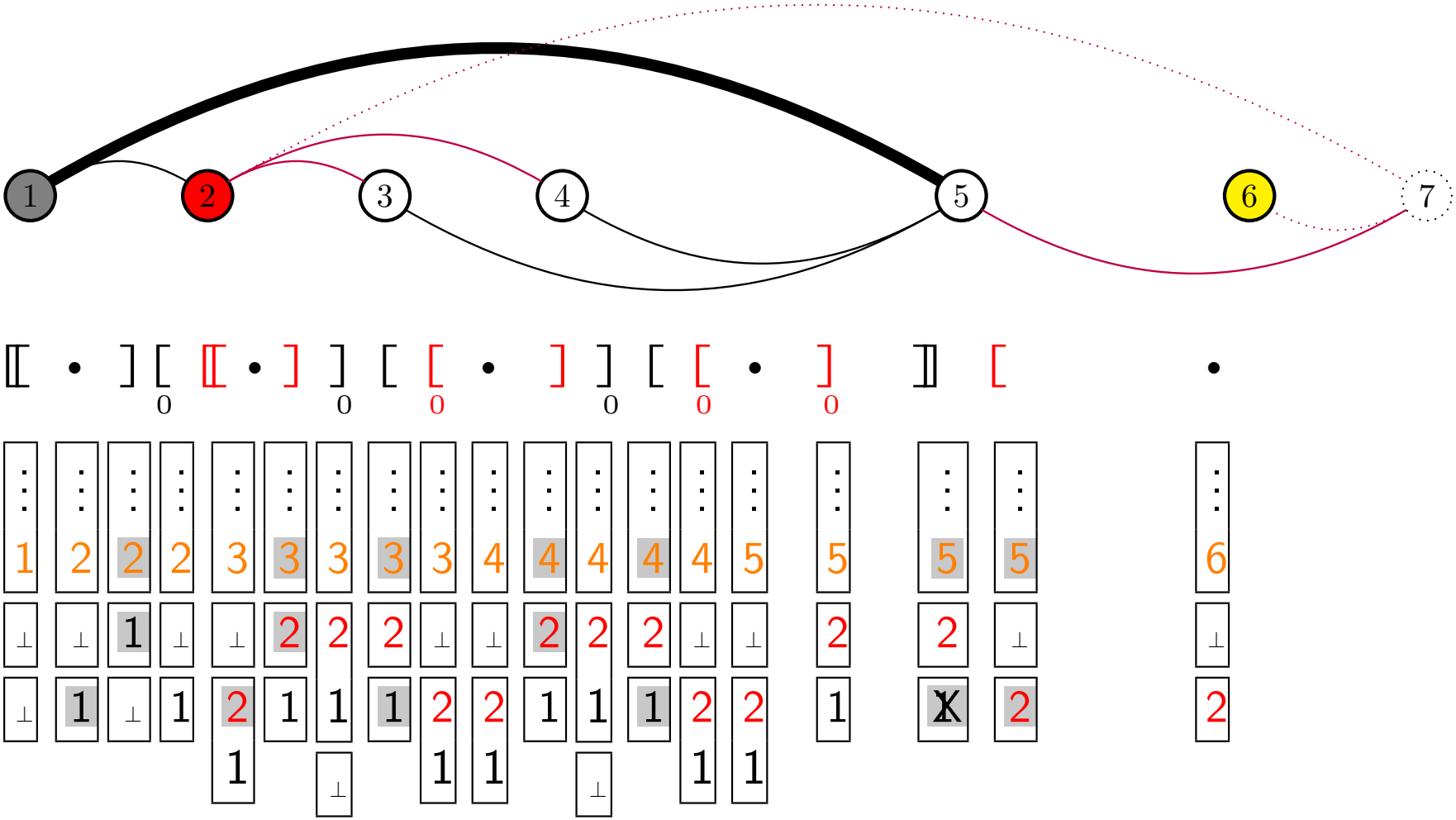

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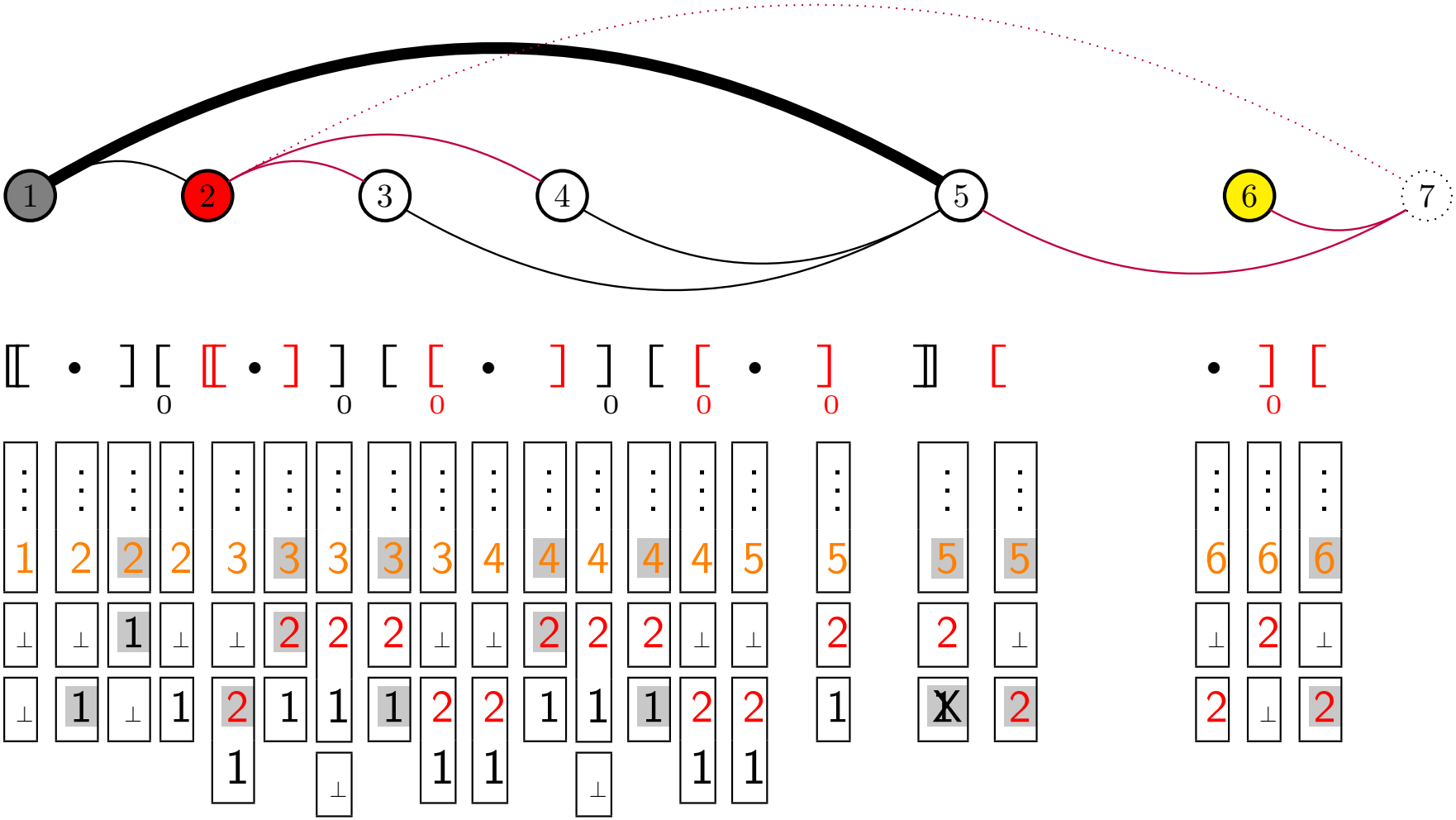

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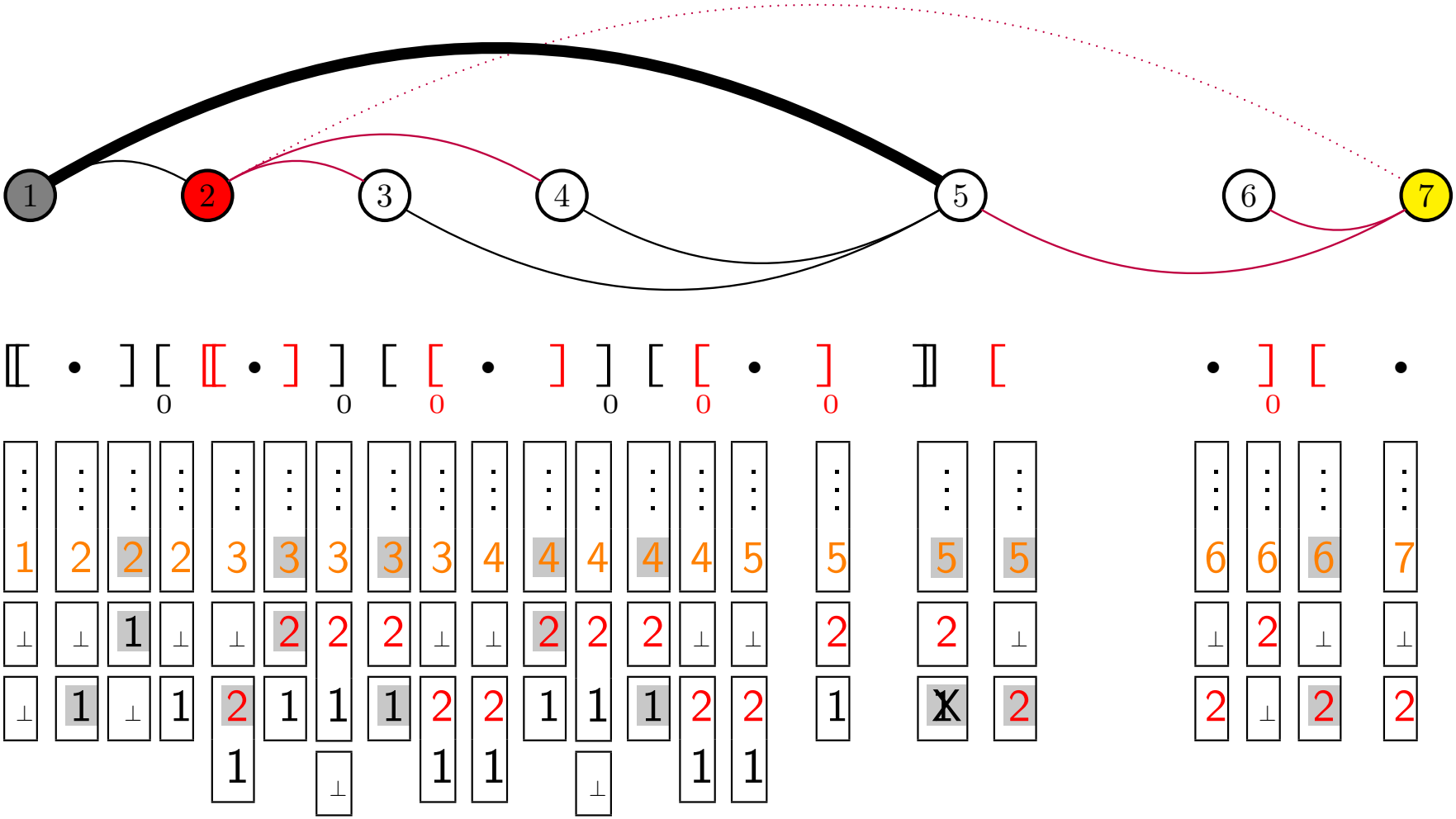
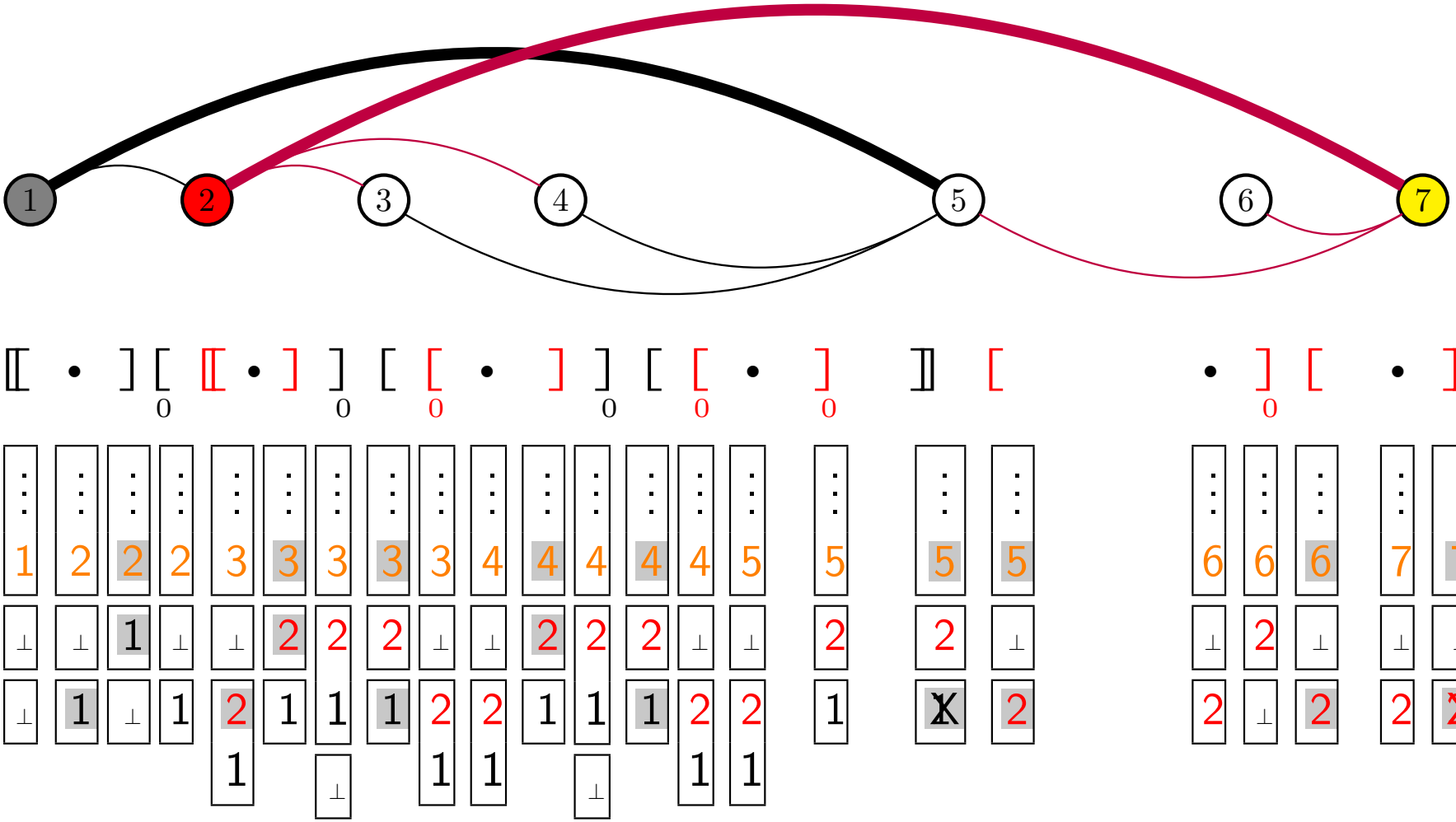


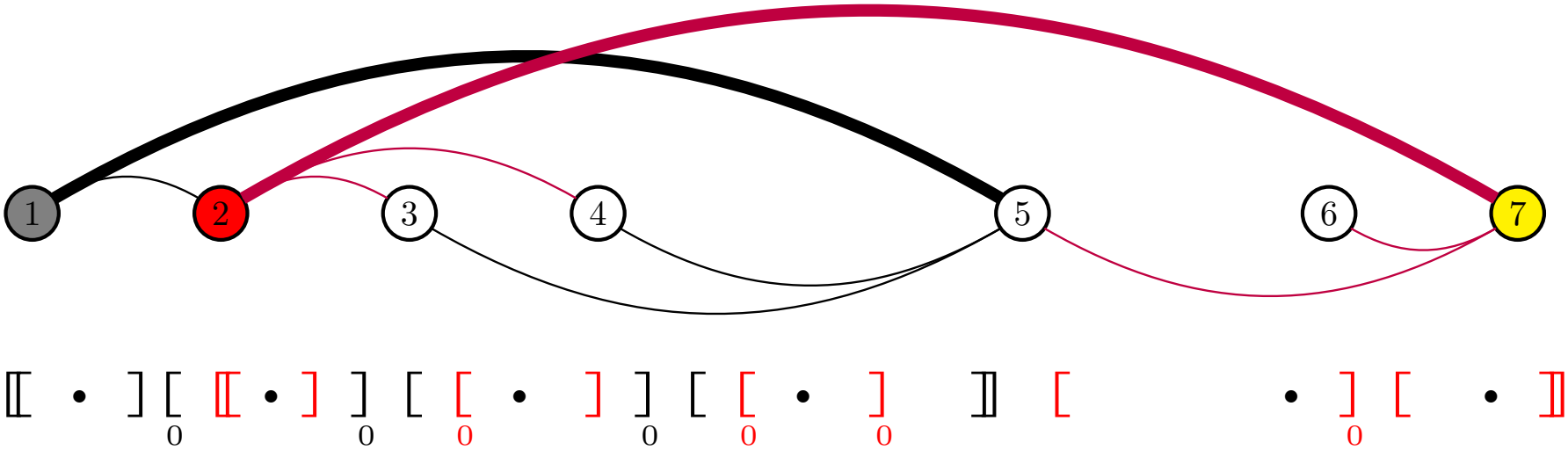
From Transitions to Rope Decomposition



The Method

1. **Proper Rope Cover (PRC)**
2. **Rope Decomposition**
3. **Transition System**
4. **Bracketing**

Bracketing (1/2)



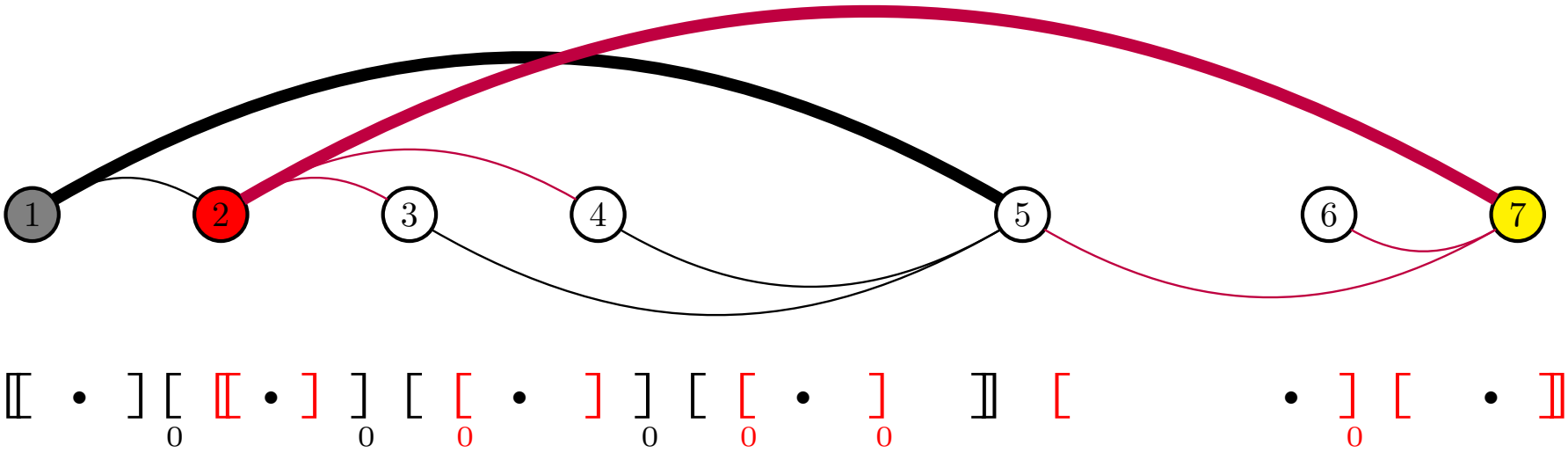
(i) Locally balanced brackets:

$$\mathcal{L} = W \bullet \cdots \bullet W$$

where

$$W = \underbrace{\underset{0}{]}^{*}}_{m} \quad \underbrace{\underset{\mathrm{d}}{]\!]}}_{\text{optional}} \quad \underbrace{\underset{\mathrm{d}}{]}^{*}}_{n} \quad \underbrace{\underset{\mathrm{d}}{[}^{*}}_{m+n} \quad \underbrace{[\![}_{\text{optional}}$$

Bracketing (2/2)



(ii) Globally balanced brackets

$$S \rightarrow \epsilon \mid \bullet \mid S\,S \mid \llbracket S \rrbracket_{d_1} \mid [\,S\,]_{d_1}^{d_1} \mid [\,S \rrbracket_{d_2}^{e} \mid \llbracket S \rrbracket_{e}$$

where

$$d_1, d_2 \in \{<, >, <>, 0\}, e \in \{<, >, <>\}$$

The Results

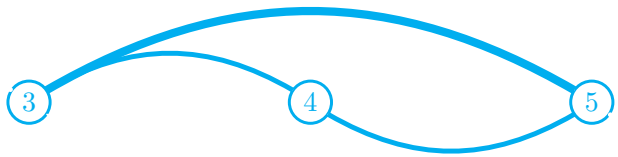
1. Relevance to natural language complexity
2. Bijection $\mathrm{DIGRAPH} \leftrightarrow L_{\mathrm{DIGRAPH}}$
3. Subsets in formal language theory
4. Succinct graph representation

Rope-Thickness

Definition

For graph (V_n, E) with a PRC R, the *rope-thickness of a vertex* $i \in V_{n-1}$ is the number of edges $(h, j) \in R$ satisfying $h \leq i < j$. The *rope-thickness of the graph* is the maximum over the rope-thicknesses of all vertices $i \in V_{n-1}$ in the graph.

Example. Complete graph of 3 vertices and 3 edges:



This graph has rope thickness 1

Rope-Thickness

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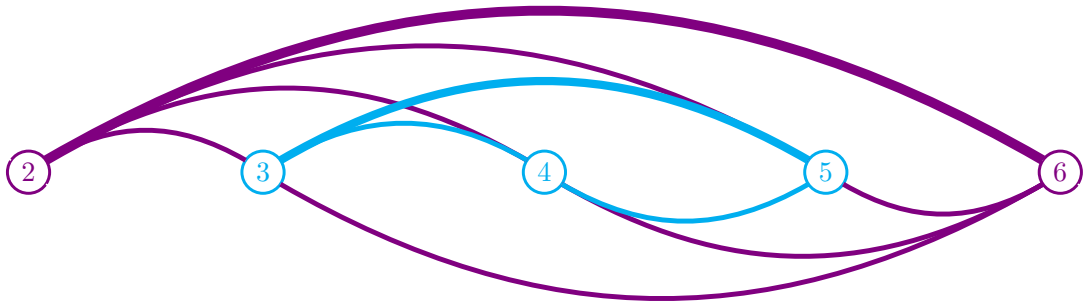
Example. Complete graph of 3+2 vertices and 3+7 edges:



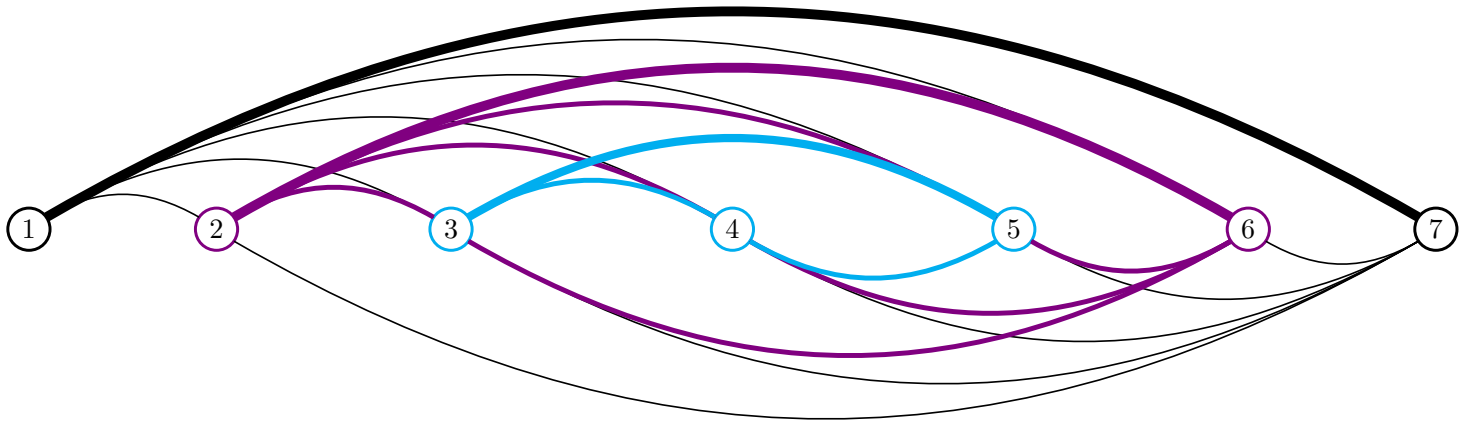
This graph has rope thickness 1+1

Rope-Thickness

Definition

For graph (V_n, E) with a PRC R, the *rope-thickness of a vertex* $i \in V_{n-1}$ is the number of edges $(h, j) \in R$ satisfying $h \leq i < j$. The *rope-thickness of the graph* is the maximum over the rope-thicknesses of all vertices $i \in V_{n-1}$ in the graph.

Example. Complete graph of 3+2+2 vertices and 3+7+11 edges:



This graph has rope thickness 1+1+1

Rope-Thickness in Universal Dependencies v2.4

language	trees	rope-th. 1	2	3	4	5	6	7	8
Arabic	28,402	4.9%	20.5%	63.5%	94.0%	99.64%	99.986%	100.000%	100.000%
Czech	127,507	13.3%	48.3%	89.3%	98.8%	99.91%	99.992%	100.000%	100.000%
Dutch	20,735	21.3%	63.4%	92.9%	98.9%	99.92%	99.986%	100.000%	100.000%
English	34,601	15.6%	54.9%	92.3%	99.3%	99.98%	99.997%	100.000%	100.000%
Finnish	34,641	37.0%	78.7%	96.8%	99.6%	99.97%	99.994%	100.000%	100.000%
German	191,757	10.1%	49.2%	90.5%	99.2%	99.96%	99.997%	100.000%	100.000%
Hindi	19,545	2.9%	44.4%	85.3%	98.3%	99.87%	99.980%	100.000%	100.000%
Italian	34,057	5.6%	49.9%	90.3%	98.8%	99.89%	99.997%	100.000%	100.000%
Korean	34,702	15.5%	65.6%	94.7%	99.5%	99.95%	99.994%	100.000%	100.000%
Latvian	12,558	12.7%	50.0%	89.8%	98.8%	99.90%	99.992%	99.992%	100.000%
Russian	87,377	15.0%	55.5%	90.7%	98.9%	99.91%	99.994%	99.999%	100.000%
Chinese	18,628	46.1%	73.7%	91.9%	98.6%	99.89%	99.989%	100.000%	100.000%
all 83	1,232,262	15.1%	54.8%	90.5%	99.0%	99.93%	99.995%	100.000%	100.000%

Processing the UD v2.4 trees requires only finite-state memory!
Preliminary results on graph banks are similar.

Certainly prompts further research on language complexity

Encoded Graph Languages

Given

- $\mathcal{L}$ – $\bullet$-marked Kleene star of W, the vertex bracketing lexicon
- $\mathcal{D}$ – the Dyck language of balanced brackets in encoded graphs

We obtain **language representations à la Chomsky and Schützenberger**

1. All ordered digraphs

$$L_{\mathrm{DIGRAPH}} = h(h^{-1}(\mathcal{L}) \cap \mathcal{D}) \in \mathcal{INDEXED}$$

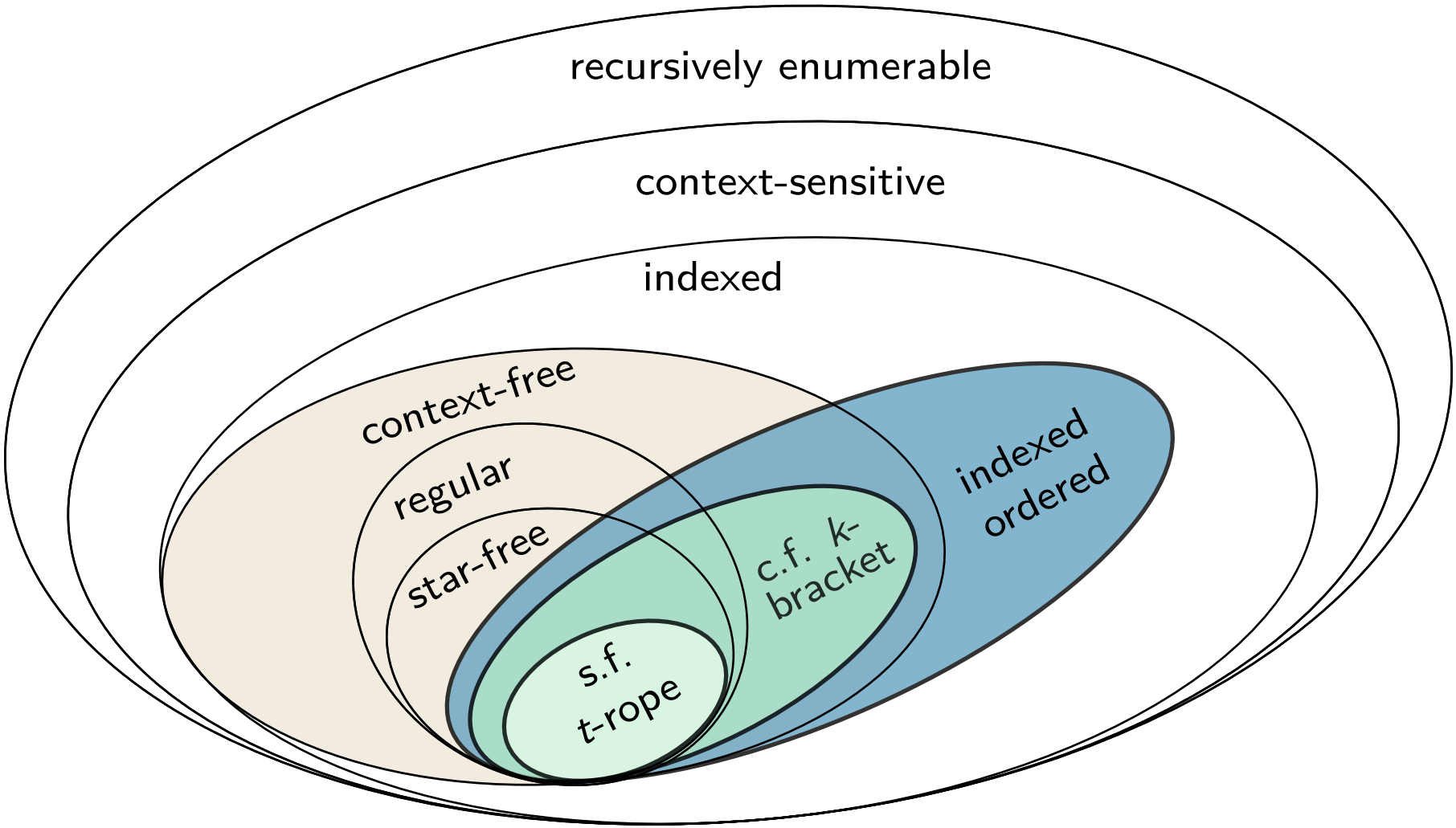
2. k-bounded brackets in vertex lexicon

$$L_{\mathrm{DIGRAPH},k} = h(h^{-1}(\mathcal{L}_{<k}) \cap \mathcal{D}) \in \mathcal{CF}$$

3. t-bounded rope thickness

$$L_{\mathrm{DIGRAPH},2t,t} = h(h^{-1}(\mathcal{L}_{<2t}) \cap \mathcal{D}_t) \in \mathcal{REGULAR}$$

String Language Subsets of All Encoded Digraphs



Succinct Graph Representation

Definition (Turán 1984)

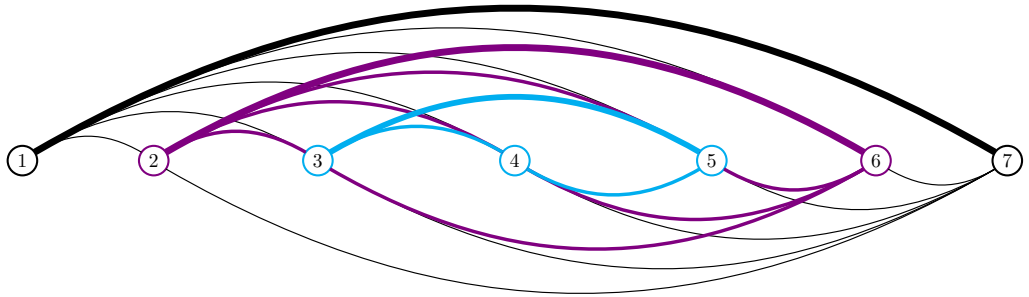
Given a class C of graphs how many bits and how much time are needed to encode a graph $G \in C$ into a binary string s, and to decode s to reconstruct G?

class of graphs	bits	author
planar graph	$3.58m$	Keeler&Westbrook 1995
planar graph	$2m+8n+o(n)$	Munro&Raman 1997
planar graph	$2m+(5+1/k)n+o(n)$	Chuang et al. 1998
planar graph	$2m+2n+o(n)$	Chiang,Lin,Lu 2005
arb. graphs	$1.5n^2+3n$	**this work**
arb. digraphs	$2n^2+4n$	**this work**
arb. digraphs	$lg \binom{n^2}{m}$	an inf. theoretic lower bound (Farzan&Munro 2013)
arb. graphs	$n(n-1)/2$	numbering, log of A006125
arb. digraphs	$n(n-1)$	numbering, log of A000088

Now $1.5n^2 + 3n$b/compl.graph, $2n^2 + 4n$b/compl.digraphs

Proposition

The current representation needs $O(n^3)$ encoding time, $O(n + |E|)$ decoding time, and at most $1.5n^2 + 3n$ bits (with block code).



$[[\bullet][[\bullet]][[[\bullet]]][[[\bullet]]][[\bullet]][\bullet]]$(*eos*)

symb	count	bits		
$]/]_0/[/[_0$	18	54	$6\lfloor n/2 \rfloor \lceil n/2 - 1 \rceil$	
$]]$	3	9	$3\lfloor n/2 \rfloor$	
$[[\bullet/\bullet$	6	18	$3n - 3$	
(eos)	1	3	3	
total	28	84	$3\lfloor n/2 \rfloor (2\lceil n/2 \rceil - 1) + 3n$	$\approx 1.5n^2 + 3n$

Useful for Parsing?

'Does the (context-freeness of the) encoding provide access to methods and results from classical formal language theory that can be used to solve problems in, say, non-projective dependency parsing more efficiently or elegantly than by working directly on the graphs?' a reviewer

Useful for Parsing?

'... these rope decompositions look related to the endpoint-crossing definitions...' a colleague in NLP, p.c.

- Turns string FSTs into string-to-graph transducers
- $O(n^3)$ parsing for noncrossing graphs:
  - Chomsky-Schützenberger parsing
  - Parsing by contractions
- Neural parsing
  - Supertagging, parsing as labeling
  - Encoder-decoder parsing

Conclusion

- **Problem:** transition system for all graphs?
- **New method:** *proper rope cover* – unique and minimal
- **Results:** universal transition system
  - empirical relevance: 6-rope covers 99.995% of UD v2.4
  - succinct representation $\mathrm{DIGRAPH} \leftrightarrow L_{\mathrm{DIGRAPH}}$
  - **conllenc.py** at `https://github.com/amikael/coding`
- Related to existing parsing frameworks
- Further work
  - decidable MSO logic fragments over graphs
  - semantic graphs
  - maximum subgraph problems
  - enumerative graph theory