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A THEORY OF THE CALCULUS OF VOTING*

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Much recent theorizing about the utility of voting concludes that voting is an irrational act in that it usually costs more to vote than one can expect to get in return.[1] This conclusion is doubtless disconcerting ideologically to democrats; but ideological embarrassment is not our interest here. Rather we are concerned with an apparent paradox in the theory. The writers who constructed these analyses were engaged in an endeavor to explain political behavior with a calculus of rational choice; yet they were led by their argument to the conclusion that voting, the fundamental political act, is typically irrational. We find this conflict between purpose and conclusion bizarre but not nearly so bizarre as a non-explanatory theory: The function of theory is to explain behavior and it is certainly no explanation to assign a sizeable part of politics to the mysterious and inexplicable world of the irrational.[2] This essay is, therefore, an effort to reinterpret the voting calculus so that it can fit comfortably into a rationalistic theory of political behavior. We describe a calculus of voting from which one infers that it is reasonable for those who vote to do so and also that it is equally reasonable for those who do not vote not to do so. Furthermore we present empirical evidence that citizens actually behave as if they employed this calculus.[3]

[1] The two most interesting presentations of this analysis are Anthony Downs, *An Economic Theory of Democracy* (New York: Harper, 1957), pp. 36–50, 260–276 and Gordon Tullock, *Toward a Mathematics of Politics* (University of Michigan Press, 1968), Chapter 7.

[2] In Downs' theory, which he characterizes as "positive, but not descriptive," there is no reason to expect descriptive accuracy, although in science one would expect to discard positive theories that are inadequate as descriptions. Tullock's theory is, however, intended to be descriptive.

[3] There is continuing interest in political science in the question of whether or not the

I. THE BASIC VOCABULARY

We start with the variables customarily used in the analysis of the calculus of voting:[4]

N: set of eligible voters for a particular election

n: number of members of N

V: set of actual voters in a particular election

v: number of members of V

R: the reward, in utiles, that an individual voter receives from his act of voting

B: the differential benefit, in utiles, that an individual voter receives from the success of his more preferred candidate over his less preferred one

P: the probability that the citizen will, by voting, bring about the benefit, B; of course, $0 \leq P \leq 1$

C: the cost to the individual of the act of voting.

The expected utility hypothesis expresses the calculus of voting thus:

$$(1) \qquad R = (BP) - C$$

so that, if $R > 0$, it is reasonable to vote; while, if $R \leq 0$, it is not reasonable. Since in any election where n and v are as large as they typically are in national elections in the United States P is a very small number (e.g. 10^{-8}) and since it is usually thought that $C > 0$, it follows that B

substantive decisions in voting are rational: e.g., B. R. Berelson, P.F. Lazarsfeld, and W. N. McPhee, *Voting* (Chicago, 1954); and V. O. Key, *The Responsible Electorate* (Cambridge, Mass., 1966), the former of which takes the position that substantive decisions are irrational while the latter argues that they are rational. See also W. H. Riker, *Democracy in the United States* (2nd ed., New York, 1965), pp. 47–49. This essay may be taken as supporting the position of Riker and Key on the procedural level, i.e., that voters are rational in deciding whether or not to cast a substantive ballot.

[4] When we say "customarily," we are referring not only to the work of Downs and Tullock, but also to the numerous unpublished analyses we have read or heard from political scientists and economists over the past twenty years.

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must be a very large number (as compared to C) when R is positive. The size of B when R is positive does indeed stagger the imagination. Hence it is concluded that R is typically negative, even when the citizen votes, which is of course irrational.

Accepting (1) as a reasonable initial expression of the calculus, the problem for those (like ourselves) who wish to render the inferences from it also reasonable in the explanation of behavior is to examine each element of (1) in order to determine an appropriate method for estimating its parameters. So we shall examine R, C, P, and B successively in order to understand and interpret (1).

Since, in (1), R is a consequence of three other variables, it cannot be examined directly. But it can be examined in relation to the real world, for the actions of real people in the real world set constraints on R, if we wish R to be descriptive. The problem for the theory is to explain the fact that in the real world some people vote and others, who are eligible, do not. Hence if R is to be descriptive, it must be the case that, for an individual voter, i,

(2)   if $i \in N$ and $i \in V$, then $R_i > 0$ and if $i \in N$ and $i \notin V$, then $R_i \leq 0$.

We know from empirical investigation that there are classes of eligible voters who vote in very high proportions (e.g. about 92 percent of the white prosperous males aged 46–64 in the midwest voted in 1964) and there are other classes of eligible voters who vote in very low proportions (e.g. about 20 percent of the Negro, poor, females aged 21–24 in the South voted in 1964).[5] Surely any theory of the utility of voting must reflect and explain this difference. And this is what statement (2) demands by insisting that, for each individual, R be calculated in such a way that it is positive for voters and zero or negative for non-voters.

Empirical investigation has revealed numerous other facts about R. Not all of these appear to be relevant here; but at least one certainly is, so we point it out as an additional constraint on R. In a descriptive theory, R must be consonant with facts we already know. Hence this prior knowledge is a kind of constraint on the way R is calculated.

The constraint arises from the following fact: In election districts in which one party is completely dominant (e.g., the solid South as it existed from ca. 1880 to 1952), it is typically

the case that many more people vote in the primary of the dominant party then in the general election. Signifying the primary with the subscript "1" and the general election with the subscript "2", this fact can be expressed

(3) $v_1 > v_2$.

For at least those voters included in V_1 but not in V_2, it must be the case that

(4) $R_1 > R_2$.

It is probably true that (4) holds for those who vote in both elections as well; but we have no way of knowing for sure. And of course about those who do not vote in either election we know only that $R_1 \leq 0$ and $R_2 \leq 0$. But we can be sure that (4) holds for some voters.

The constraints on R, (2) and (4), are obvious; but they are not trivial for they permit some methods of calculating R and prohibit others.

II. A REINTERPRETATION OF B AND C

It has usually been assumed that B, representing the gain from the success of one's favored candidate, is always positive and C, representing the cost of voting, is always a subtraction in equation (1). These assumptions, $B > 0$ and $(-C) < 0$, seem needlessly restrictive and perhaps unrealistic and so we reopen the discussion of them.

In Downs' analysis B is calculated for an individual, i, thus:

(5) $B_i = E(U^1{}_{t+1})_i - E(U^2{}_{t+1})_i$

where

U^x is the utility for the individual of the election of candidate x,

$$x = 1, 2$$
$$(U^1)_i > (U^2)_i$$

t is a time period

$E(U)$ is expected value of U

so that B is simply the difference in expected utility for the individual of the election of his favored candidate over the election of his disfavored one. As such, of course, it is always positive or zero.[6] Similarly C is the collection of time spent on the voting decision, the act of voting itself, etc. Since time itself is always costly, C is always to be subtracted in equation (1). Unfortunately, this incomplete analysis of the act of voting has been followed by most other analysts.[7]

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[5] U. S. Bureau of the Census, *Current Population Reports* Series P-20 #143, "Voter Participation in the National Election: November 1964," (Washington, 1965).

[6] Downs refines (5) considerably and we refer the reader to his work cited, pp. 38–47, for the full presentation.

[7] Downs, of course, recognizes that his is an incomplete analysis of the act of voting. Because

TABLE 1. CATEGORIZATION OF EFFECTS ON THE EXPECTED UTILITY OF VOTING

(6)	Effects on the Expected Utility of Voting	Effects for which the magnitude is dependent on the individual contribution to the outcome	negative (a)—absent from (1)
			positive (b)—B in (1)
		Effects for which the magnitude is independent of the individual's contribution to the outcome	negative (c)—C in (1)
			positive (d)—absent from (1)
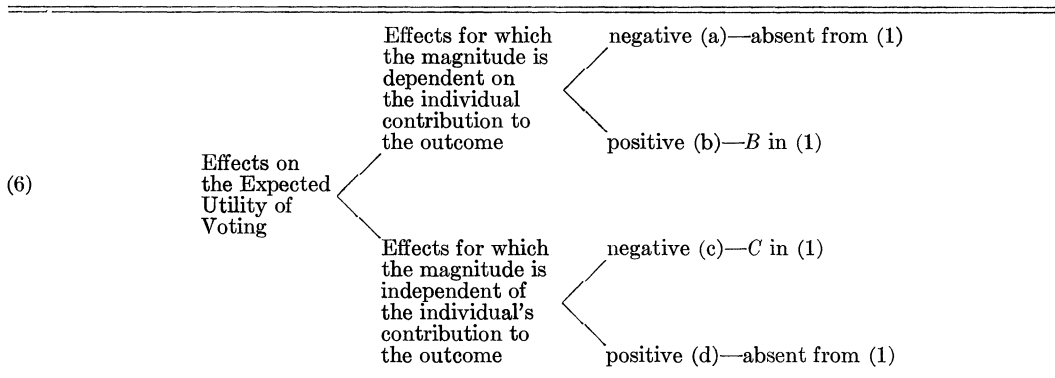
If, however, we consider voting as a self-contained act with a set of associated effects on expected utility, some of which are positive and some negative, then we can approach the matter more realistically, for we have put the act of voting in a fuller context. In (1), we apparently distinguish between those parameters of the utility function for which the magnitude is dependent on the individual's contribution to the occurrence of a favorable outcome and those parameters of the utility function for which the magnitude is independent of his contribution. That is, B is multiplied by P, while C is not. This suggests a categorization of the effects on expected utility into those which are dependent on the outcome and those which are not. And each of these categories can be further subdivided into positive and negative effects. Hence we get Table 1.

of the particular theory of rationality he adopted, however, he was unable to include other considerations. Rationality may be interpreted broadly as the ability to order preferences and to choose the more preferred action over the less preferred. In this sense, almost all behavior is rational and the assumption of rationality is close to tautological. Because he thinks the tautology is sterile, Downs rejects this definition of rationality, which is the one customarily used by economists, for a narrower interpretation in which rational behavior is only that behavior directed toward the goals that the theorist postulates as appropriate for a particular realm of action. As a result Downs cannot include "non-political" benefits (i.e. those calculated in (7) but not in (1)) in his analysis, although many of the benefits he cannot include are clearly political. It seems to us that he is unduly bothered by the tautology, which after all does affirm something, namely that people calculate about their actions and that

Since category (6 (b)) is what has hitherto been called "B" and category (6(c)) what has hitherto been called "C", we need not concern ourselves further with them and can concentrate on interpreting (6(a)) and (6(d)). Let $(6(a)) = A$ and $(6(d)) = D$.

Both A and C are kinds of costs, which have not previously been distinguished, probably because only the costs in C are universal. Nevertheless costs in A can exist. Consider an election without a secret ballot (as still occurs in one state) in which a Negro farm laborer votes for a candidate opposed by his white employer. The amount of reprisal the laborer can expect is presumably related to the amount of rancor the employer feels. And this will doubtless be greatest if the employer's favored candidate loses by a narrow margin. Hence it will probably not much hurt the laborer to vote for a candidate who loses or who wins by a wide margin; but it may hurt him considerably to vote for a candidate who just barely wins. Such circumstances of course exist even under the secret ballot if the employee reveals his preference to his employer or the part of the laborer is played by one spouse and of the employer by the other.

The variable A appears to be a rather com-

their calculations can be understood by others. It seems to us also that in a descriptive theory it is unwise for the theorist to impose his own interpretation of goals on the observed behavior. By so doing, he falls into the trap (that all the natural law theorists fall into) of saying that one goal is rational and another is not. Because it is not possible to judge the "rationality" of goals—unless one adopts some sort of natural law theory—we will adopt here the broader interpretation of rationality, recognizing its tautological character, in order to develop a theory that may more adequately describe behavior.

plex function of the election returns and P. Since the magnitude of A is, in the foregoing example and probably generally, dependent on the margin of victory of the employer's less preferred candidate, the expected utility loss to the employee can be written as $Pf(M)$, $A = f(M)$ and $M = $ margin of victory. Because A is probably both non-universal and complexly related to R, we think it likely that formal incorporation of A in the analysis will yield little payoff. We therefore neglect it from this point on.

Although the costs in A are probably rare in the real world, the benefits in D are substantial and their omission from (1) seriously impairs its adequacy. Among these elements in D are the following positive satisfactions:

1. the satisfaction from compliance with the ethic of voting, which if the citizen is at all socialized into the democratic tradition is positive when he votes and negative (from guilt) when he does not.
2. the satisfaction from affirming allegiance to the political system: For many people, this is probably the main rationale for voting. It is also a highly political motive and to leave it out of the calculus would be absurd.
3. the satisfaction from affirming a partisan preference: Voting gives the citizen the chance to stand up and be counted for the candidate he supports. For many voters this must be the most important and politically significant feature of voting. Why else vote so determinedly for a candidate whom the voter knows is almost certainly going to lose or, for that matter, going to win? In the United States, millions of Stevenson supporters did that in 1956 and millions of others in 1964 for Goldwater, two sets for which there was probably very little overlap. Some of these millions must surely have been clear-headed enough to see how poor were their hero's chances. And for this clear-headed group at least, the major satisfaction in voting must certainly have been this one. Of course, it is relevant for those who support a winning candidate also.
4. the satisfaction of deciding, going to the polls, etc.: These items are usually regarded as costs, but for those who enjoy the act of informing themselves for the decision, who get social satisfactions out of going to the polling booth, etc., these supposed costs are actually benefits.
5. the satisfaction of affirming one's efficacy in the political system: The theory of democracy asserts that individuals and voting are meaningful and for most people

the only chance to fulfill this role is in the voting booth.

Doubtless there are other satisfactions that do not occur to us at the moment; but this list is sufficient to indicate the nature of D. It should be noted that most of the items, and the most significant items, are political satisfactions or benefits and therefore must be included in any consideration of the political rewards of voting.

Equation (1) can now be rewritten:

$$(7) \qquad R = PB - C + D$$

III. CONSTRAINTS ON P

For many voters, the weightiest parameters of R are C and D. It may be tempting, therefore, to disregard P and B. But this is unreasonable, as the following illustration suggests: Consider a primary election in which the only candidates are those for a non-partisan judicial office. (Some states actually conduct such trivial elections.) Naturally the participation is very low, but still as much as ten percent of those registered vote. The usual incentives summarized by D are far less influential, so that for most citizens $C > D \geq 0$. For those who do not vote, it must be that $C > (D + PB)$. For those who do vote, however, it is probably also true that D is very small or zero, since voters are subject to the same influences rendering D small as are non-voters. So, for at least some voters it must be the case that $C > D$. Yet they can become voters, given the constraint of (2) only if $PB > (C-D)$. We can be sure in such an election that P is vastly larger than usual in elections in the same jurisdiction. Hence, even if B is smaller than usual, PB may be relatively large. It is not unreasonable to suppose, therefore, that what impels the (admittedly small number of) voters in such elections to be voters is the size of PB. And if we can see the force of PB in elections like this one, it can be supposed to operate for some voters in other elections.

Even if we can have some confidence that voters exist for whom the size of (PB) is an incentive to vote, we are still faced with a difficult problem of calculating P and (PB). Although Downs' number, $E(U)$, in (5), may result from the method of calculating P that we shall subsequently propose, he does not discuss the operator E and we cannot be sure just what he intended. Tullock uses P, but does not explain his calculation so that we can only infer his method of calculation from his examples. Most of the unpublished analyses we have seen or heard, however, assume either:

$$(8) \qquad P = v^{-1}$$

which is attractive because it is the Shapley value when voters are weighted equally,[8] or

$$(9) \qquad P = \left(\frac{v+1}{2}\right)^{-1}$$

Statement (8) is the chance that a voter casts the last necessary vote for the winning candidate. Statement (9) is the chance that a voter on the winning side casts the deciding vote. Tullock apparently uses (9). Both (8) and (9) are unreasonable, however, as the following argument shows: Given, for a one-party state, a primary, 1, and a general election, 2, in which the same man wins both. We assume $v_1 > v_2$, for reasons indicated in our argument leading to (4). Since the same candidate is involved in both elections, for those who favor that candidate, either $B_1 = B_2$ or (since B is a differential benefit and thus likely to be larger in the general election) $B_1 < B_2$. (While it might be argued that in special cases $B_1 > B_2$, as when, for a conservative voter, a conservative defeats a liberal in a primary and faces another conservative in the general election, we can nevertheless assume that in most one-party states $B_1 \leq B_2$ if only because the party attachment is itself so strong—as in the solid South in the 1920's and 1930's.) We can easily assume that $C_1 = C_2$.[9] And similarly we can also assume D constant so that $D_1 = D_2$. We then have the pair of equations

$$(10) \qquad R_1 = P_1 B_1 - C + D$$
$$R_2 = P_2 B_2 - C + D$$

in which the C and D terms can be combined as a constant, k. Substituting (8) into (10) we get

$$(11) \qquad R_1 = v_1^{-1} B_1 + k$$
$$R_2 = v_2^{-1} B_2 + k$$

Since $v_1 > v_2$, it follows that $v_1^{-1} < v_2^{-1}$. We have two cases:

1. $B_1 = B_2$. In this case, $R_i = v_i^{-1} B_i + k$, so that $R_2 > R_1$.

[8] L. S. Shapley, "A Value for n-Person Games," *Annals of Mathematics Study #28* (Princeton, 1953), pp. 307–317. See also L. S. Shapley and M. Shubik, "A Method for Evaluating the Distribution of Power in a Committee System," this REVIEW, 48 (1954), 787–792.

[9] It might be argued that $C_1 > C_2$ owing to the fact that in an area of strong partisanship, the party label reduces the cost of decision in the general election to zero. If this is true then the argument in the text is even more persuasive because B_1 must then be even greater than B_2 for v_1 to be greater than v_2.

2. $B_1 < B_2$. In this case, $R_i = v_i^{-1} B_i + k$. Because R_2 is the product of the larger number of the pair (v_1^{-1}, v_2^{-1}) and the larger number from the pair (B_1, B_2), again $R_2 > R_1$. But to say $R_2 > R_1$ is a direct violation of condition (4), which is $R_1 > R_2$. Indeed any form of P in which the major component is $1/v$, as in (8) or (9), necessarily violates (4). Hence we may reject such methods of calculation of P as (8) and (9) as unrealistic.

Approaching the task of calculating P in a way that seems to us to accord with the thought processes of a citizen who is trying to decide whether or not to vote, we ignore D initially because it in part involves considerations arising outside of the context of the election at hand. The citizen has two alternatives, to vote and not to vote; and the state of the world can be that his preferred candidate either wins or loses. If a citizen votes and his candidate wins, he receives the utility to him of the victory of his candidate, U^1_{t+1}, to use Downs' terminology, less, of course, the cost of voting, C. So we can write, for a particular voter, $U^1_{t+1} - C$, which is the reward to that voter when he votes and his preferred candidate wins. When his preferred candidate loses, the voter's reward is $U^2_{t+1} - C$. But, of course, his candidate has only the subjectively estimated chance, q say, of winning, where $0 \leq q \leq 1$. Therefore the expected utility of voting is:

$$q(U^1_{t+1} - C) + (1-q)(U^2_{t+1} - C)$$

As for the citizen who does not vote, he receives (U^1_{t+1}) with the probability q' and (U^2_{t+1}) with the probability $(1-q')$; of course, he has no cost of voting. This information is summarized in Table 2, where the entries in the cells are the utility outcomes for the citizen times their probability of occurring, given his choice of alternatives and the state of the world.

TABLE 2. EXPECTED UTILITY FOR VOTING AND NON-VOTING

	Possible Results of Election	
	1 wins	2 wins
Citizen votes	$q(U^1_{t+1} - C)$	$(1-q)(U^2_{t+1} - C)$
Citizen does not vote	$q'(U^1_{t+1})$	$(1-q')(U^2_{t+1})$

The marginal reward the citizen, i, gets from voting is then the sum of the outcomes in row 1 less the sum of the outcomes in row 2.

(12) $R_i = [q(U^1{}_{t+1} - C) + (1 - q)(U^2{}_{t+1} - C)]$
$\qquad - [q'(U^1{}_{t+1}) + (1 - q')(U^2{}_{t+1})]$
$\qquad = qU^1{}_{t+1} - qC + U^2{}_{t+1} - C - qU^2{}_{t+1}$
$\qquad + qC - q'U^1{}_{t+1} - U^2{}_{t+1} + q'U^2{}_{t+1}$

(13) $R_i = (q - q')(U^1{}_{t+1} - U^2{}_{t+1}) - C$

Setting $P = (q-q')$ and $B = (U^1{}_{t+1} - U^2{}_{t+1})$, equation (13) is another form of equation (1). With the addition of D to the right side, it becomes (7).

Despite the similarity of (13) to (1) and (7), however, we have learned something new from the analysis, specifically,

(14) $\qquad\qquad P = q - q'$

and from (14) we have some notion of how P might be calculated.

But in order to calculate it exactly we need to know more about q and q' which are probability numbers, specifically probabilities that 1 will win when i does (or does not) vote. In order to define q and q' precisely, we offer the following definitions:

(15) "pr $[1; x]$" means "the probability that candidate 1 will receive exactly x votes;" (while pr $[1; x]$ is a subjective probability, we assume that it behaves like an objective probability so that

$$\sum_{x=1}^{v} \text{pr}[1; x] = 1);^{10}$$

where there are v voters, $0 < v \leq n$, the decision rule for winning is that the winner must receive at least m votes, where, if v is even, $m = (v/2) + 1$, and, if v is odd, $m = (v+1)/2$;

We now define q and q', when there are v voters in addition to voter i: if v is odd

(16) $q = \text{pr}\left[1; \dfrac{v+1}{2} + 1\right] + \text{pr}\left[1; \dfrac{v+1}{2} + 2\right]$
$\qquad + \cdots + \text{pr}[1; v+1];$

$q' = \text{pr}\left[1; \dfrac{v+1}{2}\right] + \text{pr}\left[1; \dfrac{v+1}{2} + 1\right]$
$\qquad + \cdots + \text{pr}[1; v];$

and if v is even

(17) $q = \text{pr}\left[1; \dfrac{v}{2} + 1\right] + \text{pr}\left[1; \dfrac{v}{2} + 2\right]$
$\qquad + \cdots + \text{pr}[1; v+1];$

$q' = \text{pr}\left[1; \dfrac{v}{2} + 1\right] + \text{pr}\left[1; \dfrac{v}{2} + 2\right]$
$\qquad + \cdots + \text{pr}[1; v];$

[10] In the general case, since we are dealing with subjective probabilities, we may write $\sum_{x=1}^{v}$ pr $[1; x] = A$, where A is any positive number.

or more concisely

(16a) $\qquad q = \displaystyle\sum_{x=(v+1)/2+1}^{v+1} \text{pr}[1; x];$
$\qquad\qquad\qquad\qquad\qquad$ (odd)
$\qquad q' = \displaystyle\sum_{x=(v+1)/2}^{v} \text{pr}[1; x];$

(17a) $\qquad q = \displaystyle\sum_{x=v/2+1}^{v+1} \text{pr}[1; x];$
$\qquad\qquad\qquad\qquad\qquad$ (even)
$\qquad q' = \displaystyle\sum_{x=v/2+1}^{v} \text{pr}[1; x].$

Note that, if v is odd, then q and q' both contain $(v+1)/2$ terms, while, if v is even, q' contains $v/2$ terms and q contains $(v/2)+1$ terms. This curious fact can be readily explained: When v is odd, the addition of the citizen who decides to vote increases both the number of voters and the number required to win by one, so that q and q' contain the same number of terms. But when v is even, the addition of the citizen who decides to vote increases the number of voters by one but does not change the number required to win, so that q has one more term than q'.

We now need to prove a modest lemma about the terms in the definition of q and q' and for this purpose we need the additional notation that "$\text{pr}_v [1; x]$" means "the probability, when there are v votes cast, that 1 receives x votes" and an axiom:

(18) The preference for a member of V between candidates 1 and 2 remains unchanged when the set of voters is expanded by voter i to $(V \cup [i])$.

This axiom is probably realistic when there are hundreds or millions of voters but it may not be totally realistic when, e.g., $v = 5$, since it is possible that a voter, j, change sides from 2 to 1 or 1 to 2 just and only because a voter, i, who prefers 1 is added to the system. Such bandwagon and repulsion effects are probably limited to small committees, however; and so we assume (18).

We now prove:

(19) If voter i intends to vote for candidate 1, and $p_1 = \text{pr}_v [1; x]$, and $p_2 = \text{pr}_{v+1} [1; x+1]$, then $p_1 = p_2$

Assume citizen i believes p_1 exists when he does not vote and then adds his own vote to the system to produce p_2. He knows, of course, that his vote will be cast for 1 for certain, while by (18) the probability distribution remains unchanged for other voters. Hence the probability, p_2, that 1 receives $x+1$ votes in the expanded system is at least as great as the probability, p_1, that 1 receive x votes in the smaller
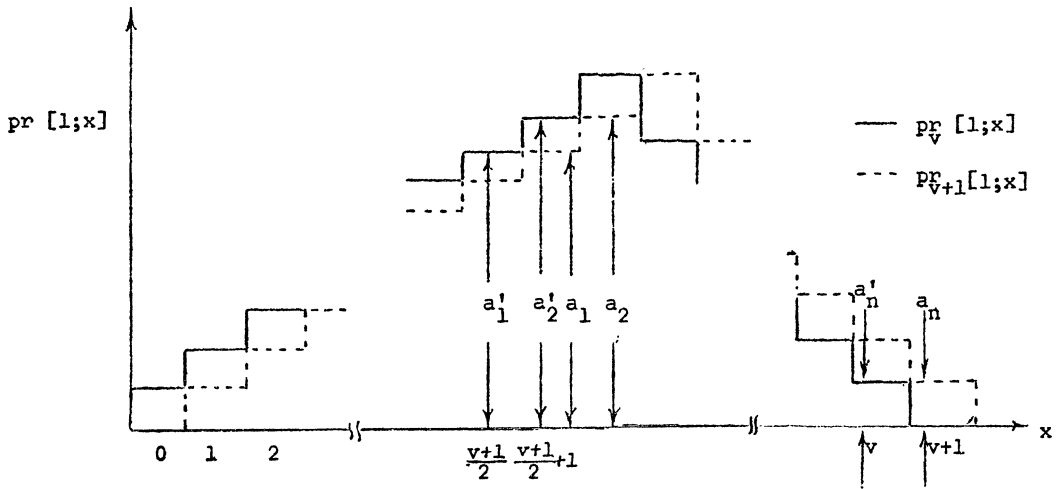
FIG. 1(a): v odd

system. That is, (18) forbids $p_1 > p_2$; hence at most, $p_1 = p_2$.

Assume $p_1 < p_2$; Then it follows that

(20) $$\sum_{x=0}^{v} \mathrm{pr}_v[1; x] < \sum_{x=0}^{v} \mathrm{pr}_{v+1}[1; x + 1].$$

But this is impossible because by the definition of probability numbers (15) both sides of (20) must equal one. The assumption, $p_1 < p_2$, thus leads to a contradiction. So it must be that $p_1 = p_2$ as stated in (19).

It is now possible to prove an identity about P:

(21) $$P = q - q' = \frac{1}{2}\mathrm{pr}_v\left[1; \frac{v}{2}\right]$$

There are two cases:

1. v is odd. In this case, $q-q' = 0$. Consider a typical term in q', e.g. $a' = \mathrm{pr}_v[1; k]$, where $(v+1)/2 \leq k \leq v$. By reason of (19) there must be in q a term, $a = \mathrm{pr}_{v+1}[1; k+1]$, such that $a = a'$. Since by (16) q and q' have the same number of terms, the expression, $q-q'$, must consist of a series of subtractions, $a-a' = 0$, so that $q-q' = 0$.

2. v is even. In this case, $q-q' = \mathrm{pr}_v[1; v/2]$. For all terms a' in q', there is, by (19), a matching term a in q such that $a-a' = 0$. But q contains one more term than q'. This additional term must be $b = \mathrm{pr}_{v+1}[1; v/2+1]$ because the term matched with it by (19) is $b' = \mathrm{pr}_v[1;v/2]$, which is not in q'. Hence it must be that $q-q' = b = b' = \mathrm{pr}_v[1; v/2]$.

Since case 1 may be expected to occur with a probability of $\frac{1}{2}$, we get, when we sum the outcomes in the two cases,

$$P = \frac{1}{2}[q - q' = 0] + \frac{1}{2}[q - q' = \mathrm{pr}_v[1; v/2]]$$
$$= \frac{1}{2}\mathrm{pr}_v[1; v/2]. \quad \text{Q.E.D.}$$

Statement (21) lends itself readily to a verbal interpretation: the probability that i will by voting influence the outcome is exactly one-half his subjective estimate of the chance that, if i does not vote, a tie will occur. His influence is thus a function of his chance to break a tie. This result can perhaps be conveyed more effectively from a geometric interpretation. In Figure 1(a), we have depicted the case where v is odd. On the horizontal axis is recorded the number of votes 1 can receive, x, where (0 or 1) $\leq x \leq$ (v or $v+1$). On the vertical axis is recorded the probability that 1 receive x votes. As is apparent from (21) and Figure 1(a), $a' = a$ and hence

$$\sum_{x=(v+1)/2}^{v} (a')_x = \sum_{x=(v+1)/2+1}^{v+1} (a)_x$$

and hence $q-q' = 0$.

Figure 1(b) depicts the case where v is even. In Figure 1(b), $a_1' = \mathrm{pr}_v[1; v/2+1] = a_1 = \mathrm{pr}_{v+1}[1; v/2+2]$ and so on to $a_n' = \mathrm{pr}_v[1; v] = a_v = \mathrm{pr}_{v+1}[1; v+1]$, just as in Figure 1(a). But in Figure (1b) there is an unmatched probability in q, namely $b = \mathrm{pr}_{v+1}[1; v/2+1] = b' = \mathrm{pr}_v[1; v/2]$. Clearly b' is not in q', while b is in q. Hence in the case where v is even, the addition of i makes a difference.

Statement (21) that $P = \frac{1}{2}\mathrm{pr}_v[1; v/2]$ permits us to make an additional statement about an important property of P. Let x_0 be the value of x for which $\mathrm{pr}_v[1; x]$ is at its maximum. Then, assuming the variance is constant, as x_0 approaches $v/2$, P increases. That is, the higher the probability of a tie without the voter and as subjectively estimated by the voter, the higher is P. In more conventional terms, the closer
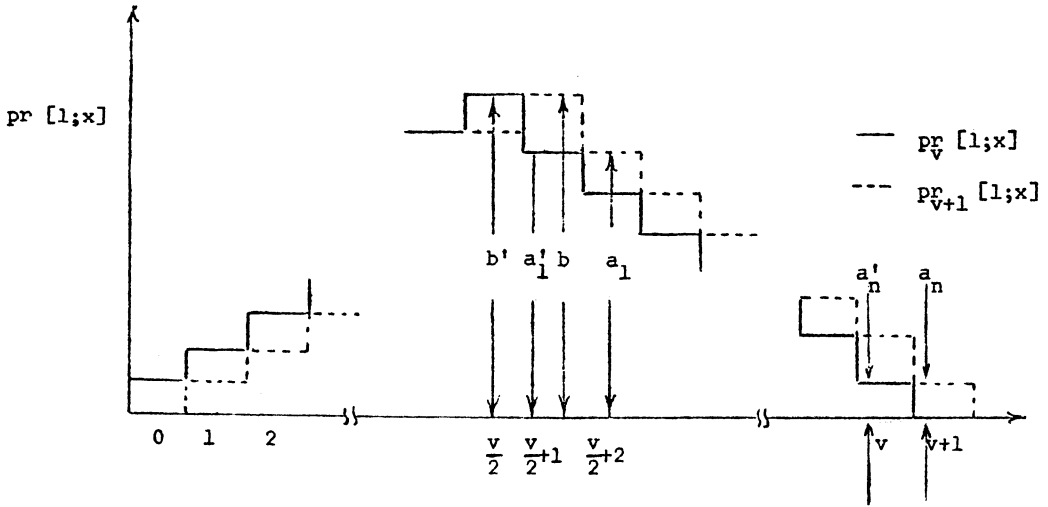
FIG. 1(b): v even

the anticipated outcome the higher is P. P is thus a function of the estimated closeness of the vote. This fact can be visualized as in Figure 2. There are two distributions in this figure, j and j', with approximately equivalent variances and with two maximum $pr_v[1; x]$ at x_0 and x'_0. Clearly P is greater in j than in j' because $r > r'$, where r and r' are the points of intersection between j and j' and a perpendicular line at point $v/2$. Note that this property of P, where $P = q - q'$, avoids the contradiction with (4) that occurred when P was calculated as in (8) and (9). In those equations, P was a function of the number of voters, regardless of the location of x_0, and this is what led to the contradiction. In (21), however, P is a function of the location of x_0 and the number of voters, thereby avoiding

the contradiction. For those voters who vote in the primary but not the general election, it may be the case that P_1 (in the primary) is substantially larger than P_2 (in the general election).

In the foregoing argument, we have assumed that the variance is constant. This need not be so. Indeed the probability density function for a particular voter is a function of certainty. Thus a voter who honestly says he doesn't know how the election will turn out has a density function as depicted in Figure 3(a), while a voter who says he is quite sure might have a density function like Figure 3(b). Therefore, for a voter, i, who believes the outcome less certain than does a voter j, i's P may be greater than j's, even though x_0^j may be closer

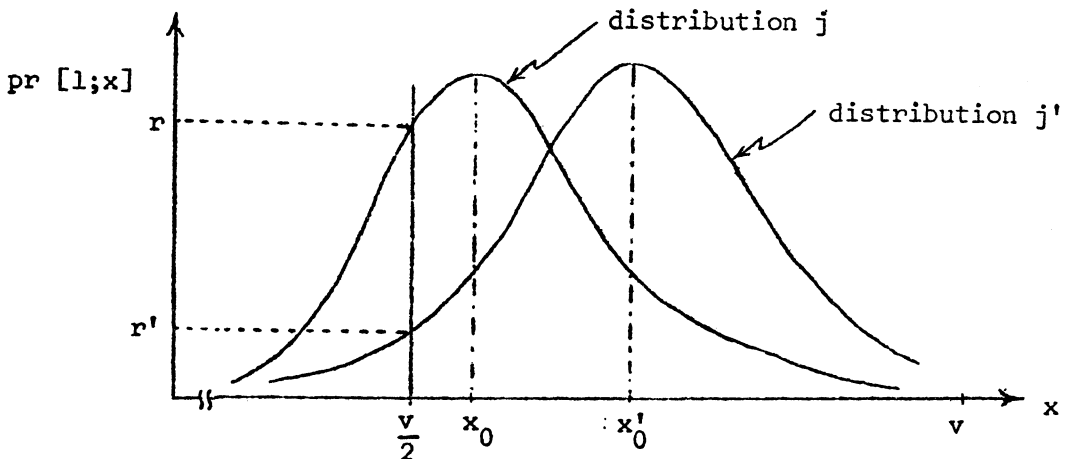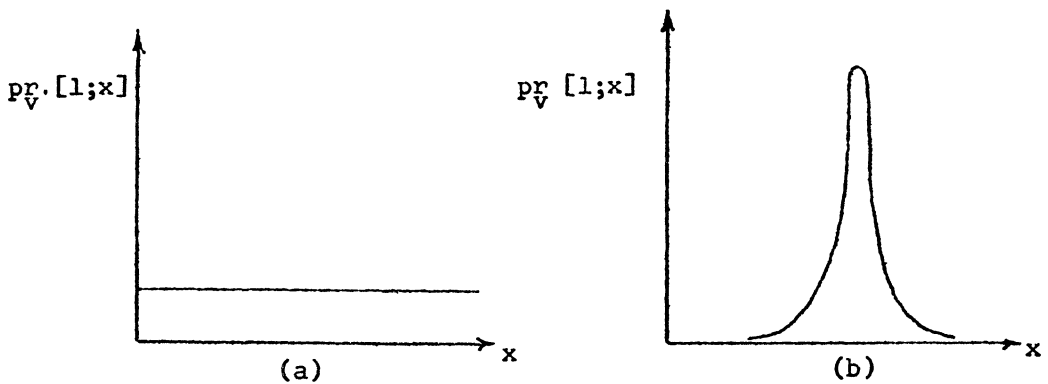
FIG. 2

FIG. 3

to $v/2$ than x_0^i. This situation is depicted in Figure 4. Clearly P is a very complicated number since it is a function of both x_0 and the subjective variance around x_0.[11]

In the proof of statement (21) we supposed the probabilities were discrete, while in the argument of the foregoing paragraph we have implied they are continuous, a confusion we must now clear up. In small electorates, a discrete probability density function is prob-

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[11] The variance is a function of certainty and certainty is in turn a highly complicated function of information such that increases in information may either increase or decrease certainty. For example, information about opinion surveys taken closer and closer to the election and showing a large and sustained majority for one candidate might increase certainty while information about opinion surveys showing an increase in undecided voters as the election approached might decrease certainty.

ably descriptively accurate, whereas in large societies like ours it may not be, for it is hard to imagine a voter estimating, e.g., pr [1; 37,492, 203]. (We are supposedly describing a rational voter and the reasonableness of such a calculation escapes us.) But if the voter assumes a continuous density function, then pr $[1; x] = 0$, even when $x = v/2$, because in general for continuous densities $\Pr[x] = f(x) = 0$. Hence statement (21) is rendered trivial in that $P = 0$. (Doubtless it was this fact about large electorates that led to the use of $P = v^{-1}$, as in statements (8) and (9).) There is, however, a way out of this contretemps. In probability theory it is assumed that, although $\Pr[x] = f(x) = 0$, still $0 \le \int_\delta f(x) \le 1$. That is, for some interval, δ, in the domain of $f(x)$, there will be a $\Pr[x] > 0$. Hence, for the continuous density shown in Figure 5, P would be calculated as approximately r rather than zero. This method of calculation is not behaviorally unrealistic. It assumes, however, that the citizen knows what
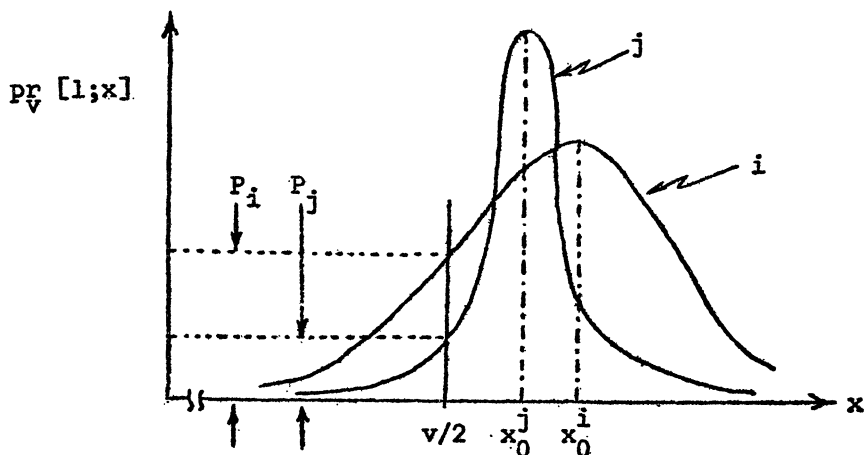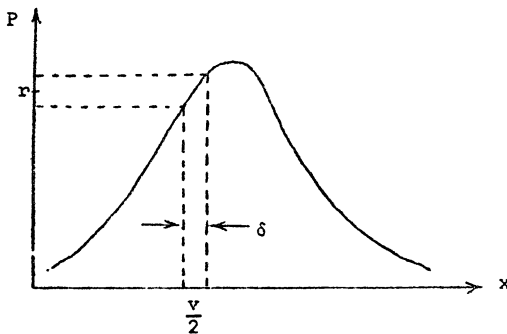

FIG. 4

FIG. 5

the turnout will be, which of course he does not. Behaviorally, it is probably more accurate to assume that his perceptions of electoral outcomes are not based on absolute vote totals but on percentages. Hence he probably evaluates P by estimating pr $[1; v_1/v = .5]$, where v_1 is the number of votes received by candidate 1. And when the horizontal axis is in terms of percentages rather than absolute numbers, it is reasonable to suppose that the points are thought of as discrete. Like Schelling's "tacit coordination points," certain points (e.g. .50, .505, .51, .55, .60, etc.) stand out as foci for citizens' perceptions. Hence we can return once more to our analysis which employed discrete density functions without loss of generality so that equation (21) becomes

(21') $P = \frac{1}{2} \operatorname{pr}_v[1; v/2] \cong \frac{1}{2} \operatorname{pr}[1; (v_1/v) = .5]$.

IV. A TEST OF THE THEORY—INTRODUCTORY COMMENTS

We now have a theory about the individual calculus of voting which is at least consistent with some known facts about the decision to vote and mildly compelling because it has been constructed out of a careful analysis of the voting calculus under the assumption of individual rationality. But it is still not intuitively convincing, for our only major difference in (7), i.e., $R = PB - C + D$, from (1), i.e., $R = PB - C$, is the addition of D in the former equation. Thus we have provided an excuse to render R positive for all voters. But we have said nothing to alter the intuitively persuasive supposition that $PB - C < 0$. And if this intuitive supposition is correct, then we have simply made voting rational by saying that the decision to vote by those who have been socialized to vote is a rational decision. So it appears that the rational calculation is based on habit and not on judgments about the candidates and the electoral situation.

By reason of our detailed analysis of P and B, however, we are in a stronger position than is suggested in the previous paragraph. Our theory makes it possible to infer statements about how voters behave in particular situations and these inferences can, of course, be subjected to an empirical test. In particular, our theory allows us to say something about the way voters calculate P and B so that we can discover whether or not P and B are ever large enough to determine a positive R. Our previous analysis shows that voters might reasonably: (1) Assume the probability density function is continuous and that $P = 0$, in which case the only components of the voting decision would be the magnitudes of D and C. (2) Assume that the function is continuous and evaluate P with a δ approximation or assume that it is discrete, defined over a domain of percentages, and evaluate $P = \sim\frac{1}{2} \operatorname{Pr} [1; (v_1/v) = .5]$. Whether or not voters follow the second procedure is a testable proposition, where the hypothesis tested is: In the United States in some recent elections some voters employ a calculation of P and B to decide whether or not to vote. In the remaining sections, therefore, we set forth the result of a test of this hypothesis, which indicates that P and B are significant components of the calculus of voting. To the degree that this evidence is persuasive, the whole equation is a useful description of behavior, while eliminating the implication of irrationality mentioned at the end of Section I.

The data we use in our test are questions asked by the Survey Research Center of the University of Michigan in surveys conducted in connection with the 1952, 1956, and 1960 Presidential elections.[12] We infer from our theory the kind of responses that should be expected to some of these questions. If the responses are in fact substantially those dictated by the inferences, as they indeed are, then we can regard the theory as verified. Incidentally we note that, historically, the development of the theory preceded by several months the test we are about to report. The relation of theory to experiment is, therefore, exactly that demanded in the notion of a deductive explanation of behavior.

Consider an individual citizen for whom we can assume D and C constant over several elections. Assuming P and B to vary, we then have three cases

[12] The data utilized in this study were made available by the Inter-University Consortium for Political Research and were originally collected by the Survey Research Center of the University of Michigan.
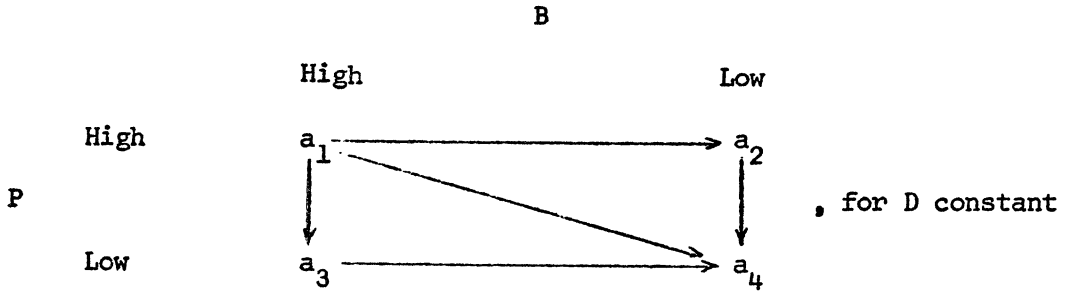
B

High   Low

FIG. 6

(a) $D > C$. Since $PB \geq 0$, R is positive and the individual always votes.

(b) $D \leq C$. For those elections in which $PB > C - D$, R is positive and the individual votes.

(c) $D \leq C$. For those elections in which $PB \leq C - D$, then $R \leq 0$ and the individual does not vote.

Unfortunately it is probably impossible to obtain empirical evidence about the calculus of voting for one individual, if only because D and C probably do not remain constant over time. So we attempt instead to make analogous predictions about groups of individuals. Consider a set of citizens who can be assigned to subsets such that, for those in each subset, D stands at about the same height on their preference scales. Suppose we have also some way of estimating the variables P and B for those in each subset. Then, we predict that, for those in a particular subset, those with a high (PB) would be more likely to vote than those with a low (PB). This prediction can be stated systematically thus: Let "pr $[a_i]$" signify "the probability that a citizen in cell a_i votes" so that from Figure 6 we obtain the following predictions as indicated by the arrows which can be read as signs of the inequality "greater than": pr $[a_1] >$ pr $[a_2]$; pr $[a_1] >$ pr $[a_3]$; pr $[a_1] >$ pr $[a_4]$; pr $[a_2] >$ pr $[a_4]$; pr $[a_3] >$ pr $[a_4]$.

In every cell we would expect the occurrence of some individuals covered by case (c), i.e. $D < C$, so that we would expect typically that pr $[a_i] < 1$. And we would expect some individuals covered by case (a) to appear in all cells so that typically Pr $[a_i] > 0$. But we would expect a greater probability of finding a case (b) individual in cell a_1 than in cells a_2, a_3, and a_4 and a lesser probability of finding a case (b) individual in cell a_4 than in cells a_2 and a_3. Note that we make no prediction about the relation of pr $[a_2]$ to pr $[a_3]$. Given equation (7) we should be able to make such a prediction; but our method of testing is far too gross to verify any prediction about this relation, so we

attempt none here. Nevertheless, if our predictions are verified with respect to the relations stated in Figure 6, we believe that we have shown that for persons in categories (b) and (c) the calculus in equation (7) is actually carried out.

V. THE OPERATIONAL ESTIMATORS OF P, B, C, D, AND pr $[a_i]$

For the operationalization of P, we have the pre-election interview responses to a question asked in each quadrennial survey from 1952 to 1960 about how close the respondent believed the outcome of the Presidential election would be. Since, as we have already shown, when $P = q - q' =$ pr$_v$ $[1; v/2]/2$ and the variance remains constant, P increases as x_0 approaches $v/2$, it follows that the closer the respondent believes the outcome will be the higher his subjective estimate of P. We assume that the variance is constant.[13] Since the SRC offered the respondents several degrees of closeness, we simplified the data by dichotomizing: All those who were in the two lowest categories of closeness, we said had a low P, while those in the other categories we said had a high P.[14]

For the operationalization of B, we have the pre-election interview responses to a question asked in the same surveys as to how much the respondent "cares" about the outcome of the

[13] An operationally equivalent assumption is that the variance of the probability density function is randomly distributed, independently of x_0. Hence those citizens with a subjective x_0 "close" to $v/2$ are expected to have a higher P than those whose x_0 is "far" from $v/2$.

[14] It is conceivable that some people who have a high P by this operationalization actually view the probability density as continuous so that for them $P = 0$ rather than $P = \frac{1}{2}$ pr $[1; v/2]$. If some "low" P persons are by this error of operationalization included in the "high" P category, our predictions are of course rendered more difficult to verify.

Presidential election. Since we have already assumed $B = U^1_{i+1} - U^2_{i+1}$, the greater the differential benefit from the election of his favored candidate, the more the voter can be expected to "care" about the outcome. The answers of this question are, therefore, estimates of B. Again, since the SRC offered the respondents several degrees of "caring", we simplified the data by dichotomizing: All those in the two highest categories of caring had, we said, a high B, while those in the remaining categories had a low B.

For the operationalization of pr $[a_i]$ we have the post-election interview responses to the question of whether or not the respondent recalled having voted. To estimate pr $[a_i]$ we can construct a ratio, v/n, where v is the number in cell a_i who said they recalled having voted and n is the number of persons in cell a_i as determined by their estimates of P and B. This ratio, v/n, is the percentage of voters in the cell and we can, grossly, interpret it as the probability that any individual in the cell is in fact a voter. So for purposes of estimating, we say pr $[a_i] = v/n$.

There is some ambiguity in the operationalization of P, B, and pr $[a_i]$; but we think it does not materially disturb the outcome. When respondents are asked whether or not they care about the outcome and whether or not they think it will be close, the question specifically refers to the Presidential election only. When citizens vote, however, they actually vote in several elections simultaneously. Each voter can be expected to have a vector $\vec{P}$ of P's and a vector $\vec{B}$ of B's, where each pair of elements in $\vec{P}$ and $\vec{B}$ relate to one office, e.g. President, governor, register of deeds, etc. In the voter's calculation of R, therefore, all the PB terms are summed. Since $P \geq 0$ and $B \geq 0$, the new marginal utility of voting, R', say, is greater than R. But $R' \geq R$ for voters in all cells, a_i. Hence, if we hold D constant as we intend to do, the inflation of R' should not materially affect the outcome of the test.

In the operationalization of P, B, and pr $[a_i]$, we have had a gross but intuitively unobjectionable procedure. Turning to the operationalization of D and C, however, we have a more difficult task with a more dubious result. For the operational estimate of D we have a so-called sense of citizen duty scale which the SRC constructed out of responses to four questions about the duty to vote asked in the pre-election interviews.[15] According to

Campbell, Gurin, and Miller, the scale of the sense of citizen duty has a high degree of internal consistency (i.e., an index of reproducibility of .96). Given this consistency, to the degree that the sense of citizen duty scale captures the essence of the notion of D, we can regard this scale as an operationalization of D.

The index of citizen duty yields five scale scores. The bulk of the respondents scored I or II, while the lower three scores each occur with considerably less frequency. In order to avoid practically empty cells, therefore, we collapsed the last three categories into one so that we had:

    I. high sense of citizen duty (score I on the scale)—high D

    II. medium sense of citizen duty (score II on the scale)—medium D

    III. low sense of citizen duty (scores III-V on the scale)—low D

We regarded the persons in each category of D as having approximately the same level of D on their scales of preference. This interpretation may seem to involve some interpersonal comparison of utility, which, of course, we wish to avoid. We think it does not, however, because we have simply categorized respondents into those for whom D is high, medium, or low on their own individual scales and thus we believe we avoid interpersonal comparison. In any event, we propose by this means to hold D constant in one of three categories.[16]

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[15] Unfortunately the SRC did not ask the appropriate questions in 1964 so that we cannot use other data from that year in our analysis. The sense of citizen duty scale is fully described in A. Campbell, G. Gurin and W. E. Miller, *The Voter Decides* (Evanston, 1954), pp. 194–199.

[16] The constancy of D is open to some technical question. The SRC ceased in 1964 to collect the necessary information, first, because the scale of citizen duty turned out to be very similar to the scale of the sense of political efficacy, which was more useful for SRC purposes, and, second, because it seemed possible that the four questions on the sense of citizen duty scale might be contaminated by a positive response set. Their first reason for the rejection of the scale is not immediately relevant to our concerns. Their second reason, if true, does, however, bias the data in favor of our predictions. Suppose some persons with an actually low D are included (by reason of response set) in our categories I and II of high and medium D. They will then lower the percentage of voters in these categories. And if low D is associated with not caring about the outcome of the election so that these low D's add disproportionately to the non-carers, their presence among the non-carers in category I or II will

The operationalization of C is somewhat arbitrary. We assume that C and D are not positively correlated and that C is constant in each category of D. In fact, it seems likely that C and D are negatively correlated. That is, if the halo effects of voting (D) are high, then the costs of voting (C) are low simply because the citizen who believes it is terribly important to vote is likely to minimize costs of voting while the citizen who thinks voting is unimportant is likely to maximize costs of voting.

VI. A TEST OF THE THEORY

Concerning the answers to questions in the three surveys, we offer the following predictions:

1. For each election in the set K, $k_1 = 1952$, $k_2 = 1956$, $k_3 = 1960$, and for each category of D, i.e., (I, II, III), it is the case that

$$\text{pr}[a_{i,I}]_{k_j} > \text{pr}[a_{i,II}]_{k_j} > \text{pr}[a_{i,III}]_{k_j}$$
$$j = 1, 2, 3$$

This prediction is essentially a test of our operationalization of D. If it were the case that the predicted relation did not hold, we would not be able to affirm that D had been held constant.

2. For each category of D, $d_1 = I$, $d_2 = II$, $d_3 = III$ and for each election, k_j, $j = 1, 2, 3$, it is the case that

$$\text{pr}[a_1]_{d_i k_j} > \text{pr}[a_2]_{d_i k_j}$$
$$\text{pr}[a_1]_{d_i k_j} > \text{pr}[a_3]_{d_i k_j}$$
$$\text{pr}[a_1]_{d_i k_j} > \text{pr}[a_4]_{d_i k_j}$$
$$\text{pr}[a_2]_{d_i k_j} > \text{pr}[a_4]_{d_i k_j}$$
$$\text{pr}[a_3]_{d_i k_j} > \text{pr}[a_4]_{d_i k_j} \quad \text{for } i = 1, 2, 3$$

The data for testing these predictions is set forth in Table 3 where each cell contains $v/n = \text{pr}[a_i]$.

As is indicated in Table 4, prediction 1 is true (T) in 23 out of 24 cases.[17] The one case in which it is false (F) involves a cell in which $n = 4$, by far the smallest n in any cell. We

make it easier to verify our hypothesis that those who care about the outcome are more likely to vote than those who do not care. If, however, this bias, for our purposes, was significant it would tend to wipe out differences among like a_i's between categories. Since, as will be shown later, this does not occur, we may suppose that the response-set contamination of the scale of citizen duty is minimal.

[17] This is the necessary confirmation of assumption, mentioned earlier, that the effect of response set is minimal. (See footnote 16.)

believe this one falsehood is well within the chance of sampling error and so we regard prediction 1 as well verified and we regard the operationalization of D as adequate. The test of prediction 2 is set forth in Table 5. As indicated in this table, the predicted differences occur in 42 out of 45 cases.

This high score for accuracy is, however, somewhat spurious, for the predictions are not independent. If pr $[a_1] > $ pr $[a_2]$ and if pr $[a_2] > $ pr $[a_4]$, then certainly pr $[a_1] > $ pr $[a_4]$. Similarly, if pr $[a_1] > $ pr $[a_3]$ and if pr $[a_3] > $ pr $[a_4]$, then certainly pr $[a_1] > $ pr $[a_4]$. So the prediction that pr $[a_1] > $ pr $[a_4]$ is not independent of the other predictions. We can allow for this fact by eliminating it, although this is probably unduly harsh. The situation is: Let α_1 stand for the pair of predictions pr $[a_1] > $ pr $[a_2]$ and pr $[a_2] > $ pr $[a_4]$. Let α_2 stand for the pair of predictions pr $[a_1] > $ pr $[a_3]$ and pr $[a_3] > $ pr $[a_4]$. Let α_3 stand for the prediction pr $[a_1] > $ pr $[a_4]$. Then

if α_1	and α_2,	then α_3
T	T	T
T	F	T
F	T	T
F	F	T or F

so that in 3 cases (i.e. if either α_1 or α_2 are true) the truth value of α_3 is determined. But when α_1 and α_2 are both false, the truth value of α_3 is not determined. Therefore, the total elimination of α_3 sets an unreasonably high standard for our prediction to meet. Nevertheless, if we do eliminate it, we still have 33 accurate predictions out of 36.

It is possible to calculate a statistical significance level for the results set forth in Table 5. Since our 45 predictions involve some interdependence, we will interpret each row of Table 5 as an observation. Then we can calculate the significance of obtaining seven out of nine observations which support the hypothesis. It is possible for an observation to have one of 18 forms. (For example, rows 1952-I, 1952-III, and 1956-II represent three of these forms.) If both P and B had no effect on a voter's calculus, we would expect on each observation any one of the 18 forms to occur with equal likelihood. Hence, for our null hypothesis, each form would occur with a probability of 1/18. Only one form (i.e., that of row 1952-I, for example) agrees with our theory, however, so the chance of a validating observation would be 1/18 and the chance of an invalidating observation 17/18. Assuming equiprobability of the occurrence of forms, the significance level of only 2 invalidating observations is the sum of the probabilities of obtaining exactly

TABLE 3. DATA FOR TEST OF PREDICTIONS

		I High D		II Medium D		III Low D	
		B High	B Low	B High	B Low	B High	B Low
1952							
P	High	$\frac{323}{362}=.89$	$\frac{104}{124}=.84$	$\frac{259}{310}=.84$	$\frac{95}{141}=.67$	$\frac{86}{150}=.57$	$\frac{73}{159}=.46$
	Low	$\frac{85}{99}=.86$	$\frac{21}{27}=.78$	$\frac{58}{75}=.77$	$\frac{17}{26}=.65$	$\frac{33}{54}=.61$	$\frac{16}{32}=.50$
1956							
P	High	$\frac{376}{421}=.89$	$\frac{160}{198}=.81$	$\frac{220}{274}=.80$	$\frac{89}{137}=.65$	$\frac{56}{98}=.57$	$\frac{64}{156}=.40$
	Low	$\frac{104}{129}=.81$	$\frac{49}{63}=.77$	$\frac{64}{87}=.74$	$\frac{22}{30}=.73$	$\frac{24}{46}=.52$	$\frac{10}{38}=.26$
1960							
P	High	$\frac{315}{335}=.94$	$\frac{128}{154}=.83$	$\frac{178}{198}=.90$	$\frac{79}{96}=.82$	$\frac{52}{70}=.74$	$\frac{42}{92}=.46$
	Low	$\frac{37}{41}=.90$	$\frac{8}{12}=.67$	$\frac{26}{33}=.79$	$\frac{3}{4}=.75$	$\frac{8}{11}=.73$	$\frac{5}{12}=.42$

2, 1, and 0 invalidating observations. These probabilities and the significance level may be calculated using binomial probability thus:

$$\text{significance level} = \sum_{x=0}^{2} \binom{9}{x} \left(\frac{1}{18}\right)^{9-x} \left(\frac{17}{18}\right)^{x}$$

Using this method, the level of significance of our result is approximately .00000005, so that there is about one chance in twenty million that we have falsely asserted our hypothesis to be true. Hence we regard prediction 2, our main theory, as validated.[18]

VII. CONCLUSION

We started out with an intuitively persuasive theory, equation (1), which many people

[18] If we drop from our calculations in Table 3 all those persons whom the SRC coded "not appropriate," "don't know," or "no answer" to the question about the closeness of the outcome (whom we have actually included in Table 3 in the "high" P category), then the results are somewhat improved. Two of the three bad predictions in Table 5 are eliminated and the level of significance increases.

believe to be false because it leads to counter-intuitive conclusions. We refined this theory, equations (7) and (21), so that we could draw from it a non-obvious inference, i.e., that P increases as x_0 approaches $v/2$. We then presented evidence that, given the non-obvious inference, people actually behaved as the theory predicts; so we conclude that the theory is nevertheless true. We have therefore described the rational calculus that does in fact occur in the act of deciding whether or not to vote. And we are able to affirm that the behavior of most people can be described by a theory of rational decision-making. This is what we set out to do.

Nevertheless, those who find the conclusions from the theory counter-intuitive are bound to be dissatisfied, for they will wonder how their intuition went wrong. For their sake we offer the following observations:

1. It is likely that, for many people, the subjective estimate of P is higher than is reasonable, given the objective circumstances. Subjected as we are to constant reminders that a few hundred carefully selected votes by non-voters could reverse the results of very close

TABLE 4. TEST OF PREDICTION 1

	$\text{pr } [a_1]_I > \text{pr } [a_1]_{II} > \text{pr } [a_1]_{III}$		$\text{pt } [a_2]_I > \text{pr } [a_2]_{II} > \text{pr } [a_2]_{III}$	
1952	T	T	T	T
1956	T	T	T	T
1960	T	T	T	T
	$\text{pr } [a_3]_I > \text{pr } [a_3]_{II} > \text{pr } [a_3]_{III}$		$\text{pr } [a_4]_I > \text{pr } [a_4]_{II} > \text{pr } [a_4]_{III}$	
1952	T	T	T	T
1956	T	T	T	T
1960	T	T	F	T

elections, such as the Presidential election of 1960, the subjectively estimated chance of a tie (i.e., P) may be as high as the propaganda urges it to be, even though in objective calculations the chance of a tie may be low.

2. It is likely that B is much higher for many people than anyone has heretofore supposed.

3. We have not, of course, completely eliminated instances of possible irrationality from the system. One wonders, for example, why, out of the 4294 respondents, the 104 who cared deeply, believed the outcome close, had a high sense of citizen duty, nevertheless did not vote (i.e., the non-voters of a_1, Category I, assuming—what may be false—that they were eligible to vote); and why the 31 who cared little, believed the outcome not close, had a low sense of citizen duty, nevertheless did note (i.e., the voters of a_4, Category III).

By the foregoing comments we believe we have explained most of the difficulties that led previous thinkers to decide that the intuitively satisfying equation (1) led to counter-intuitive results. Therefore, we reaffirm the validity of the intuitively satisfying equation (7).

The development of any science is characterized, in the beginning, by some conceptualization and a great deal of operationalization of notions relevant to the subject matter. This is followed by the formulation, inductively, of empirical generalizations and these become the hypotheses tested in elementary scientific research. We might call this stage 1 in the development. In further stages, the verified hypotheses from stage 1 are used to generate additional empirical generalizations but more importantly to generate a systematic theory— often stated in axiomatic form—which theory, when used to derive inferences that are subsequently verified, is then the corpus of the science itself. This we might call stage 2.

Political science is, in our opinion, now well into stage 1, proliferating hypotheses and beginning to learn how to test them. The leaders in this enterprise have mostly been those who are classified as "behavioralists," for example, the students of American voting

TABLE 5 TEST OF PREDICTION 2

	$\text{pr } [a_1] > \text{pr } [a_2]$	$\text{pr } [a_1] > \text{pr } [a_3]$	$\text{pr } [a_1] > \text{pr } [a_4]$	$\text{pr } [a_2] > \text{pr } [a_4]$	$\text{pr } [a_3] > \text{pr } [a_4]$	Form of Observation
1952						
I	T	T	T	T	T	R_1
II	T	T	T	T	T	R_1
III	T	F	T	F	T	R_2
1956						
I	T	T	T	T	T	R_1
II	T	T	T	F	T	R_3
III	T	T	T	T	T	R_1
1960						
I	T	T	T	T	T	R_1
II	T	T	T	T	T	R_1
III	T	T	T	T	T	R_1

behavior. What we have attempted in this essay, however, is to go somewhat beyond stage 1 and into stage 2 for one tiny aspect of politics. We have taken a few notions about voting, incorporated them into the larger theoretical system of rational decision-making, derived testable inferences from the theory, and tested them with the result that we now have confidence in the accuracy both of the inferences and of the theory.

APPENDIX

Many readers of an earlier version of this paper have suggested that an implicit assumption of our analysis misrepresents the voter's calculus. This criticism arises from the introspective belief that many voters vote either for causes which they know are lost or for candidates who are certain to win. In such a case, the utility of a vote is reflected not only by D but also by the effect on the outcome (defined not only by who wins but also by how much). Many citizens voted in the 1964 Presidential election, for example, knowing with almost certainty the inevitable outcome, since they desired Johnson to win by a landslide or they wanted the conservative voice of America to be counted.

We emphasize again that these benefits are not necessarily to be accounted for by D but are reflected in our PB term. The question then becomes: how do these additional considerations affect our analysis? Some additional notation must be introduced to answer this question.

Assume: (1) There is a utility function, $u(x)$, defined over the range of all possible electoral outcomes. (2) There are v voters in the system. (3) The equation, $u(v) = 1$, is the citizen's utility if his most preferred candidate receives all the votes. (4) The equation, $u(0) = 0$, is his utility if his most preferred candidate receives none of the votes. (5) There is a probability density function, $g(x - x_0)$, defined over the range of all possible electoral outcomes, where x_0 is the citizen's subjectivity estimated most probable outcome when he does not vote.

In terms of our previous analysis it was assumed:

$$u(x) = B \qquad x \geq m$$
$$u(x) = 0 \qquad x < m$$
$$m = v/2, \; v \text{ even}$$
$$m = \frac{v+1}{2} + 1, \; v \text{ odd.}[19]$$

[19] If the vote for the citizen's preferred party is $v/2$, v even, before he votes, then this outcome is

In other words, $u(x)$ was assumed to be a step function. For purposes of the present analysis we can normalize $u(x)$, but we would like to analyze PB when $u(x)$ is continuous throughout its range.

Given these definitions the expected value of the election, before the citizen votes, can be represented as,

$$(1A) \qquad EV = \int_0^v u(x)g(x - x_0)dx$$

Since we are applying the continuous calculus to variables that might best be described as discrete, the limitations of the assumption of continuity must be kept in mind. It is possible to represent $u(x)$ and $g(x - x_0)$ graphically as follows,
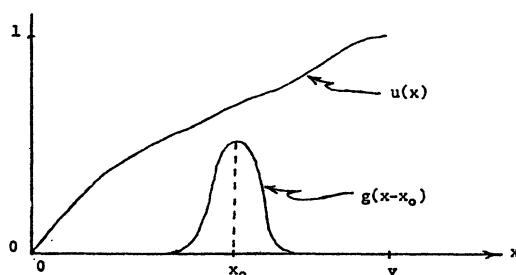

FIG. 1A

Hence equation (1A) is simply a means of representing the product of the value of an event and its subjectively estimated probability of occurrence, summed over all possible events. Furthermore, if we assume that $g(x - x_0)$ goes to zero sufficiently rapidly, we can write the change in EV with respect to an increase in x_0 (which would represent the citizen's decision to vote),[20]

worth B utiles since his vote gives his party $v/2 + 1$ votes, or enough to win. If, however, v is odd, then, assuming the voter does not value a tie, only until an outcome of $(v+1)/2 + 1$ votes is assured can the voter realize the desired outcome of his preferred candidate winning when he votes.

[20] It is assumed that $g(x - x_0)$ goes to zero sufficiently rapidly so that small changes in x_0 do not violate

$$\int_0^v g(x - x_0) = 1.$$

Also it is assumed that $u(x)$ does not change when one more voter is added to the electorate.

(2A) $\dfrac{\partial EV}{\partial x_0} = - \displaystyle\int_0^v u(x)g'(x - x_0)dx,$

where $g'(x - x_0) = \partial g(x - x_0)/\partial x$.

A special class of utility functions is that for which utility is a simple linear function of the number of votes the citizen's preferred candidate receives. In this case $u(x) = x/v$ and (2A) becomes,

(3A) $\dfrac{\partial EV}{\partial x_0} = - \dfrac{1}{v} \displaystyle\int_0^v xg'(x - x_0)dx.$

Integrating by parts (3A) becomes,[21]

(4A) $\dfrac{\partial EV}{\partial x_0} = - \dfrac{1}{v} [vg(v - x_0) - 1].$

But by our earlier assumption—that $g(x - x_0)$ goes to zero sufficiently rapidly—$g(v - x_0) = 0$. Hence:

(5A) $\dfrac{\partial EV}{\partial x_0} = \dfrac{1}{v}$

This is the Shapley value which we rejected earlier as a means for calculating P. We may conclude, therefore, that voters probably do not, in general, have linear monotonic utility functions for electoral outcomes.

Our original analysis, although recognizing the importance of taking into account the size of the electorate, indicates the importance of two additional factors of the voter's calculus, namely, x_0 and the variance of $g(x - x_0)$. Clearly, $\partial EV/\partial x_0 = 1/v$ is independent of these two factors. For nonlinear $u(x)$, however, it can be shown that $\partial EV/\partial x_0$ depends on these two factors. To prove this, take (2A) and integrate by parts.

(6A) $\dfrac{\partial EV}{\partial x_0} = \displaystyle\int_0^v u'(x)g(x - x_0)dx$

where $u'(x) = du(x)/dx$. Now assume that $u(x)$ can be described or approximated by a polynomial.[22] For polynomials of order n, the derivative of the polynomial will be of order $n - 1$. For the simple linear case the derivative will contain no factors in x and (6A) will reduce to (5A). For higher order polynomials, however, the first, second, etc. moments of

[21] Integration by parts is defined as follows:

$$\int u\,dv = uv - \int v\,du$$

In this particular case we let $x = u$ and $g'(x - x_0)dx = dv$.

[22] An n-order polynomial is any function of the following form:

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$

$g(x - x_0)$ will determine $\partial EV/\partial x_0$. The first moment is simply x_0. The second moment, with some simple manipulation, is the variance of $g(x - x_0)$. Hence, only for the special case of linear utility functions is $\partial EV/\partial x_0$ independent of how close the citizen feels the election will be or how certain he is of his prediction (the variance of $g(x - x_0)$ will not be a factor for quadratic utility functions, however).

A closer scrutiny of (6A) yields another important observation. The number $\partial EV/\partial x_0$ will be maximized if $u'(x)$ is greatest when $x = x_0$. The maximum amount $u(x)$ can change is 1 when it is a step function. Hence the maximum of $\partial EV/\partial x_0$ is $g(0) = \mathrm{pr}_v[1; x_0]$.[23] Hence the following constraints on P exist;

(7A) $0 \leq P \leq \mathrm{pr}_v[1; x_0]$[24]

which is identical to the constraints on P in our original analysis.

This leads to the following important observations: Although x_0 and $g(x - x_0)$ may be subjectively estimated by the citizen so that $\mathrm{pr}_v[1; v/2] = 0$, it is not necessarily the case that $\partial EV/\partial v_0 = 0$. This accounts for the vote for lost causes and inevitable winners. Both $\partial EV/\partial x_0$ and $\mathrm{pr}_v[1; v/2]$ are, however, bounded by the same limits. Hence, we cannot escape the conclusion that a voter's subjective P can appear and probably usually does appear greatly to exceed our objective calculation of it.

A utility function which decreases after some point (reflecting a desire on the part of the voter not to give a particular candidate a mandate), of course, complicates this analysis. For such utility functions and a given x_0 it may be the case that $\partial EV/\partial x_0 \leq 0$. In such circumstances the citizen obviously shouldn't vote. It might, moreover, be in his interest to dissuade others from voting. A more complete analysis of utility functions which have a

[23] This result differs by a factor of $\tfrac{1}{2}$ from our original formulation in equation (21) because we have implicitly assumed v even. Our original analysis showed however that when v is odd the voter cannot affect the outcome so as to bring about the victory of his preferred candidate. The gain associated with increasing his favored candidate's chance of winning is offset by increasing the chance of a tie (from 0 to $\mathrm{pr}_v[1; v+1/2]$). Given an equal likelihood for even and odd v we would once again let $P = \tfrac{1}{2}\mathrm{pr}_v[1; v/2]$. If we assumed the voter equally valued creating a tie to having his preferred candidate win then we would drop the $\tfrac{1}{2}$ once again.

[24] Or if one wishes $\tfrac{1}{2}\mathrm{pr}_v[1; v/2]$.

negative slope in some range is not intended here. All we wish to suggest is that an analysis of them can be incorporated into the present framework.

Finally, if we assume that S-shaped utility functions are the most general case we can make an important observation concerning the relationship of $\partial EV/\partial x_0$ and x_0. We can show graphically that the closer x_0 is to $v/2$ (assuming the inflection point of the utility function occurs at $v/2$) the greater is $\partial EV/\partial x_0$ (the step function is a special limiting case of the S-shaped function). Hence, our conclusion—the closer the citizen believes the election will be the larger will be P—receives additional justification.