Cut Elimination is an important theorem

Let's go into the proof!

Sketch:

Define $\Gamma \vdash_{cf} \varphi$ to formalize

Prove the ability to remove individual cuts on $\Gamma \vdash_{cf} \varphi$

Simple Induction

So lets define $\Gamma \vdash_{cf} \varphi$!

$\Gamma \vdash_{cf} \varphi$ is the exact same as $\Gamma \vdash \varphi$, except for no cut rule

Lemma: $\Gamma \vdash_{cf} \varphi$ and

$\Gamma, \varphi \vdash_{cf} \psi$ implies

$\Gamma \vdash_{cf} \psi$

By induction over the derivation of

$\Gamma, \varphi \vdash \psi$

Case: the last rule is Axiom:

If $\psi \neq \varphi$ then $\Gamma = \Gamma' \cup \psi$

so $$\frac{}{\Gamma', \psi \vdash_{cf} \psi}$$

If $\psi = \varphi$ then recall, by assumption

$$\frac{}{\Gamma \vdash_{cf} \varphi} \text{Assumption}$$

Case: the last rule is $\wedge$-intro

$$\frac{\Gamma, \varphi \vdash_{cf} \psi_1 \qquad \Gamma, \varphi \vdash_{cf} \psi_2}{\Gamma, \varphi \vdash_{cf} \psi_1 \wedge \psi_2} \wedge \text{intro}$$

by IH, $\Gamma \vdash \psi_1$ and $\Gamma \vdash \psi_2$ so

$$\frac{\Gamma \vdash_{cf} \psi_1 \qquad \Gamma \vdash_{cf} \psi_2}{\Gamma \vdash_{cf} \psi_1 \wedge \psi_2} \wedge \text{ intro}$$

Case: the last rule is $\wedge$-elim_L

$$\frac{\Gamma, \varphi \vdash_{cf} \psi_1 \wedge \psi_2}{\Gamma, \varphi \vdash \psi_1}$$

by IH, $\Gamma \vdash_{cf} \psi_1 \wedge \psi_2$

$$\frac{\dfrac{}{\Gamma \vdash_{cf} \psi_1 \wedge \psi_2} \text{ IH}}{\Gamma \vdash_{cf} \psi_1} \wedge\text{-elim}_L$$

Case: the last rule is $\bot$-elim

$$\frac{\Gamma, \varphi \vdash_{rf} \bot}{\Gamma, \varphi \vdash_{rf} \psi}\ \bot\text{-elim}$$

$$\dfrac{\dfrac{}{\Gamma \vdash_{rf} \bot}\ \text{IH}}{\Gamma \vdash_{rf} \psi}\ \bot\text{-elim}$$

Case: the last rule is $\top$-intro

$$\frac{}{\Gamma, \varphi \vdash_{rf} \top}\ \top\text{-intro}$$

$$\frac{}{\Gamma \vdash_{cf} \top}\ \top\text{-intro}$$

Remaining cases are similar

Thm: $\Gamma \vdash \varphi \Rightarrow \Gamma \vdash_{cf} \varphi$

By induction

Case: Cut

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

By IH $\Gamma \vdash_{cf} \varphi$ and $\Gamma, \varphi \vdash_{cf} \psi$

By Lemma $\Gamma \vdash_{cf} \psi$

Other Cases: straightforward induction

Corollaries!

Consistency: $\nvdash \bot$

Pf: Assume $\vdash \bot$. Then $\vdash_{cf} \bot$

Inversion on derivation of $\vdash_{cf} \bot$

There is no possible rule (no $\bot$-intro)

$\times$

No LEM: $\nvdash \varphi \vee \neg \varphi$ for arbitrary φ

Much more structure!

When we apply inversion to $\vdash \varphi_1 \wedge \varphi_2$,
the only rules that apply are $\wedge$-intro
And all elimination rules

Proof of $P \to P$:

$$
\dfrac{\dfrac{\dfrac{}{P \vdash P}}{\vdash P \to P} \qquad \dfrac{\dfrac{}{Q \vdash Q}}{\vdash Q \to Q}}{\dfrac{\vdash (P \to P) \wedge (Q \to Q)}{\vdash P \to P}}
$$

Hmm, what's going on here?

It seems like elimination rules can sometimes be useless

But they are sometimes necessary

$$\dfrac{\dfrac{P \wedge Q \vdash P \wedge Q}{P \wedge Q \vdash P}}{\vdash (P \wedge Q) \to P}$$

What's going on here?

Elimination rules are necessary, but they also get in the way of just applying inversion

We'd like to reduce ATP to just repeated inversion

But right now, there's an infinite number of proofs of even just the trivial

$\vdash P \to P$

We would like to fix this... eventually

Next time: we start talking about programs