Suppose $x + \frac{1}{x}$ is an integer. Prove by induction that $x^n + \frac{1}{x^n}$ for every $n \geq 1$ would also be an integer.

Before we start, let us note that we assume that x is a valid value in order for instances of $x + \frac{1}{x}$ to yield a defined integer. We have also chosen to re-represent the problem using the recurrence relation that we found: $a_{n+1} = a_n * a_1 - a_{n-1}$ with $a_0 = 2$ and $a_1 = x + \frac{1}{x}$.

For the first base case: If $n = 1$, then $a_1 = x^{n=1} + \frac{1}{x^{n=1}} = x + \frac{1}{x}$; an integer.

For the second base case, let us consider the case when $n = 2$. If $n = 2$, then $a_{n=2} = a_1^2 - a_0$, which is $(x + \frac{1}{x})^2 - 2$, expanded as: $x^2 + \frac{1}{x^2} + 2 - 2$. Notice that $(x^2 + \frac{1}{x^2})$ is essentially a_1^2 so we can rewrite the expanded form further as $a_1^2 = a_2^2 + 2$ or $a_2^2 = a_1^2 - 2$. Since the first base case is an integer, an $integer^2 + 2$ or $integer * integer + 2$ will result an integer.

We will now prove the hypothesis by induction, showing that a_{n+1} (when $n = n + 1$) is also an integer.

- $a_{n+1} = a_n * a_1$, so $a_{n+1} = (x^n + \frac{1}{x^n}) * (x^{n+1} + \frac{1}{x^{n+1}}) =$

  $(x^{n+1} + \frac{1}{x^{n+1}}) + (x^{n-1} + \frac{1}{x^{n-1}})$
- We can rewrite it as $a_n * a_1 = a_{n+1} + a_{n-1}$ or $a_{n+1} = a_n * a_1 - a_{n-1}$.
- Testing for a_2, $a_{1+1} = a_1 * a_1 - a_0$. $a_0 = x^0 + \frac{1}{x^0} = 2$ when the x values result in an integer. This is equivalent to an $integer * integer - 2$; which results in an integer.
- Then for every increment of a_n, it will always calculate an $integer * integer - integer$, resulting in an integer.

Thus, $x^n + \frac{1}{x^n}$ is an integer, for every $\boldsymbol{n} \geq 1$.