a)T_n has $n(n-1)/2$ vertices.

T_n has vertices created out of two element subsets with n elements where ordering doesn't matter. This can be represented by the choose function $\binom{n}{2}$, which expanded is:

$$\frac{n!}{(n-1)!\cdot 2!}=\frac{n(n-1)(n-2)(n-3)...}{(n-2)(n-3)...1\cdot 2}$$

Crossing out similar terms, we get:

$$\frac{n(n-1)}{2}$$

b) Vertext of T_n has degree $2n-4$

There are $n-1$ vertices containing the same element, as they are produced by pairing that element with the remaining elements in the set. As a vertex cannot connect with itself, it can pair with $n-2$ vertices containing that same element in the graph. As there are 2 elements in a vertex, there are $2\cdot(n-2)$ vertices that that vertex can pair with, or $2n-4$

c)If two vertices x and y are adjacent to each other in T_n, then there are $n-2$ vertices that are adjacent to both.

Two vertices, say $\left[a,b\right]$ and $\left[a,c\right]$ can connect with all vertices containing the similar element a, as they also connect with the vertex containing their opposing element $\left[b,c\right]$. As we've determined, there are $n-1$ vertices containing an element a, subtract the vertices we are looking at: x and y, and adding the one vertex containing their opposing elements, there are $n-2$ vertices adjacent to both.

d) If two vertices x and y are *not* adjacent to each other in T_n, then there are 4 vertices that are adjacent to both.

If there are two vertices $\left[a,b\right]$ and $\left[c,d\right]$, the only vertices adjacent to both are vertices formed for pairs of each other's variables: $\left[a,c\right]$, $\left[a,d\right]$, $\left[b,c\right]$, $\left[b,d\right]$. There are no other vertices both can connect to.

e) Are there any $T_n\left(n\geq 3\right)$ for which T_n is bipartite? Justify your answer.
No.

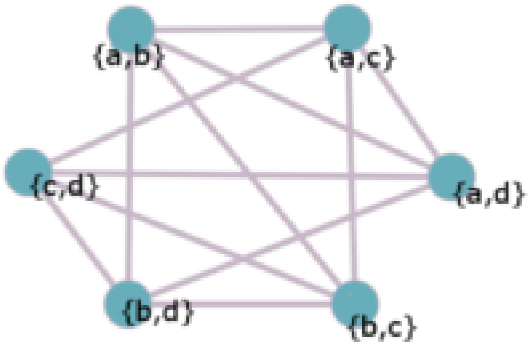
If $n=3$, each vertex is adjacent to each other and would require 3 colors. On any graph larger than that we've proven above that any 2 adjacent vertices will have $n-2$ vertices in common, thus there will always be a need for 3 or greater number of colors.

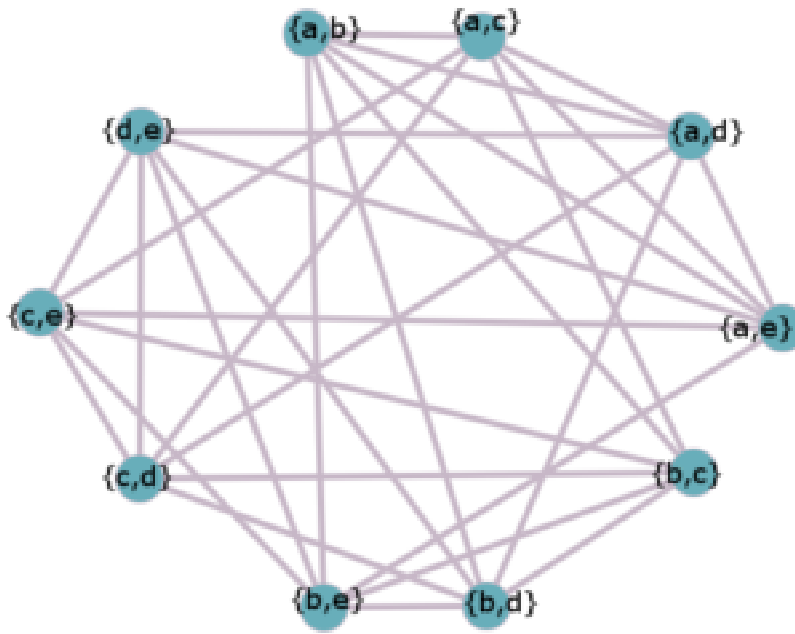
f) Prove that T_n is connected for any n. In particular, what is the longest distance between two vertices in T_n?

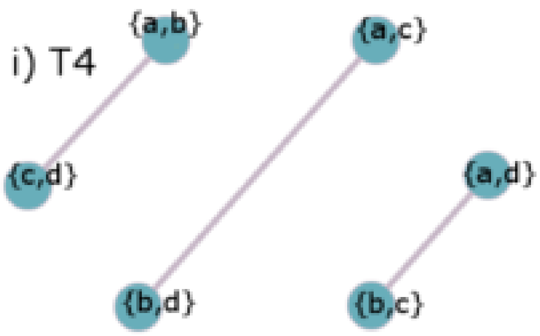
The longest distance between any two vertices is 2. As we've proven, for any two vertices not adjacent to each other, there are 4 adjacent to both. In this way there are only 2 edges max one needs to travel down to get to another vertex, otherwise the vertices are already adjacent.

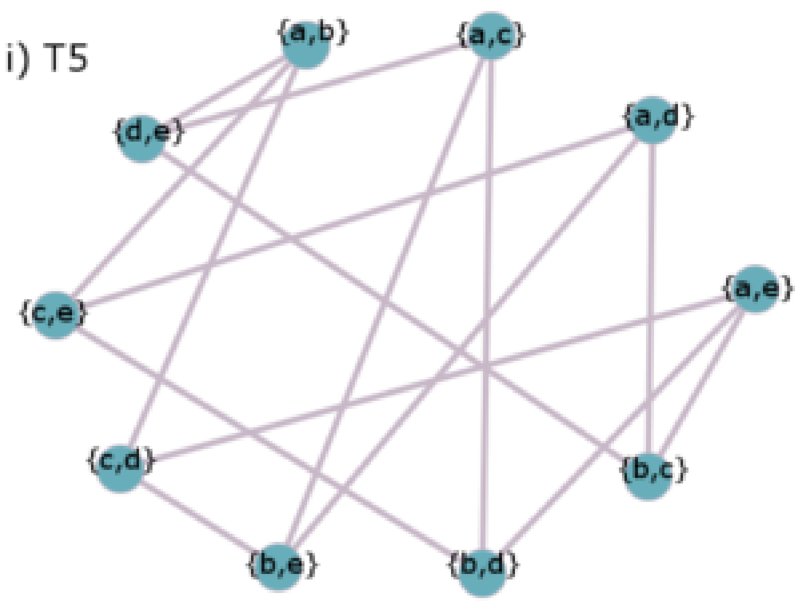
h) Conjecture and prove: what is $\left(T_n\right)$

will be $n-1$. Every vertex with the same element as part of it's component will be able to connect with each other. As a single veriable can be in $n-1$ different pairs, there will be a clique of $n-1$. The only exception to this is $n = 3$, as $[a, b]$, $[a, c]$, and the only other vertex connects to both of those vertices, $[b, c]$.









i)

j) For what n values is $|E(T_n)| > |E(\overline{T_n})|$?

$\binom{n}{2} \cdot (2n-4)$ $\left(\binom{\binom{n}{2}}{2}\right) - \binom{n}{2}(2n-4)$

$2\binom{n}{2}(2n-4) > \left(\binom{\binom{n}{2}}{2}\right)$ $k = \binom{n}{2}$

$2k(2n-4) > \binom{k}{2}$

$2k(2n-4) > \frac{k(k-1)}{2}$

$4k(2n-4) > k(k-1)$

$4(2n-4) > k-1$

$8n-15 > k$

$8n-15 > \frac{n(n-1)}{2}$

$16n-30 > n^2-n$

$0 > n^2-17n+30$

$0 > (n-15)(n-2)$

When $n=16$
$0 > 14$
When $n=14$
$0 > -12$
When $n=3$
$0 > -12$
When $n=1$
$0 > 14$

For all $2 < n < 15$, $|E(T_n)| > |E(\overline{T_n})|$

k) Is T_n an induced subgraph of T_{n+1}?
$T_4 T_5$
Yes. As $n+1$ contains all of the elements in n as well as 1 additional element, it will contain all the vertices in T_n, and as the adjacency rules remain the same, the same edges will form. As such, T_n is a subgraph of T_{n+1}