4.2 Functional Limits

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Exercise 4.2.1. Use Definition 4.2.1 to supply a proof for the following statements.

a) $\lim_{x\to 2}(2x+4) = 8$
Proof: Let $\epsilon, \delta > 0$. Notice that $|f(x) - 8| = |2x + 4 - 8| = |2x - 4| = 2|x - 2|$. If we let $\delta = \frac{\epsilon}{2}$, then $0 < |x - 2| < \delta$ implies that $|f(x) - 8| < \epsilon$.

b) $\lim_{x\to 2} x^3 = 8$.
Proof: Let $\epsilon, \delta > 0$. Notice that $|f(x)-8| = |x^3-8| = |x-2||x^2+2x+4|$. We can make $|x-2|$ as small as we want, but we can't do so with $|x^2+2x+4|$ and as such we need to come up with an upper bound. Fix the maximum radius of the δ-neighborhood to be 1, the upper bound for the $|x^2+2x+4|$ in that case is 19. Now, choose $\delta = \min\{1, \frac{\epsilon}{19}\}$. If $0 < |x - 2| < \delta$, then $|x^3 - 8| = |x - 2||x^2 + 2x + 4| < \frac{\epsilon}{19}19 = \epsilon$.

Exercise 4.2.3. Use Collary 4.2.5 to show that each of the following limits does not exist.

b) $\lim_{x\to 1} g(x)$ where g is Dirichlet's function from Section 4.1.
Proof: Consider the sequences (x_n) and (y_n) where $x_n = \frac{n-1}{n}$ and $y_n = \sqrt[n]{2}$, it is easy to see that both converge to 1. By definition $x_n \in \mathbb{Q}$ for every $n \in \mathbb{N}$.
Now we prove that for every $y_n \notin \mathbb{Q}$ for every $n \in \mathbb{N}$. Suppose that $\sqrt[n]{2} = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and co-prime, then $2q^n = p^n$ which implies that p^n is even and thus so is p, we write $p = 2c$ and now $2q^n = 2^n c^n$, thus $q^n = 2^{n-1}c^n$ which leads to a contradiction since p and q are supposed to be coprime.
Now we have two sequences where the every element of the first one is a rational while every element of the other is irrational and both converge to 1, thus it is easy to see that $\lim_{x_n\to 1} g(x_n) = 1 \neq 0 = \lim_{y_n\to 1} g(y_n)$. By Collary 4.2.5, the limit does not exist.

Exercise 4.2.6. Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on $A \subseteq \mathbb{R}$(i.e., there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$). Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

Proof: We have an $M > 0$ where $|f(x)| \leq M$ for all $x \in A$. Let $\epsilon > 0$

be arbitrary, since we have $\lim_{x\to c} g(x) = 0$ then there exists a $\delta > 0$ s.t. $|g(x) - 0| = |g(x)| < \frac{\epsilon}{M}$ whenever $0 < |x - c| < \delta$, thus,

$$|g(x)f(x)| = |g(x)||f(x)| < \frac{\epsilon}{M}M = \epsilon$$

whenever $0 < |x - c| < \delta$ which proves that $\lim_{x\to c} g(x)f(x) = 0$.