

4.2 Groups Acting on Themselves by Left Multiplication - Cayley's Theorem

April 1, 2018

Exercise 4.2.6. Let r and s be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of N in D_8 as $1N, rN, sN$ and srN . Label these cosets with the integers 1, 2, 3, 4 respectively. Exhibit the image of each element of D_8 by left multiplication on the set of 4 left cosets of N in D_8 . Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.

The images are:

$$\begin{aligned}\sigma_r(1) &= 2 \\ \sigma_r(2) &= r^2N = N = 1 \\ \sigma_r(3) &= rsN = sr^{-1}N = sr^3N = sr(r^2N) = srN = 4 \\ \sigma_r(4) &= rsrN = sN = 3 \\ \sigma_s(1) &= sN = 3 \\ \sigma_s(2) &= srN = 4 \\ \sigma_s(3) &= ssN = N = 1 \\ \sigma_s(4) &= ssrN = rN = 2\end{aligned}$$

Because σ is a homomorphism and we know the images of the generators of D_8 (r and s) we can deduce the image of every other element in D_8 through the multiplication of images (e.g., $\sigma_{rs} = \sigma_r\sigma_s$).

It is easy to see that r^2 is in the kernel of the action and thus we deduce that the representation is not faithful.

As we've shown, the subgroup of S_{D_8} that π_N maps to can be generated by $\sigma_r = (1\ 2)(3\ 4)$ and $\sigma_s = (1\ 3)(2\ 4)$ which is the exact same subgroup that we've seen the Klein 4-group isomorphic to, thus $\pi_N \cong K_4$ where K_4 is the Klein 4-group.