

# Automorphisms

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**Exercise 4.4.1.** If  $\sigma \in \text{Aut}(G)$  and  $\varphi_g$  is a conjugation by  $g$  prove  $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)}$ . Deduce that  $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$ . (The group  $\text{Aut}(G)/\text{Inn}(G)$  is called the *outer automorphism group* of  $G$ .)

**Proof:** For any  $x \in G$  we have  $\sigma\varphi_g\sigma^{-1} = \sigma g x g^{-1} \sigma^{-1} = (\sigma(g))x(\sigma(g))^{-1} = \varphi_{\sigma(g)}$ . Now, since we've proved that for any  $\sigma \in \text{Aut}(G)$  we have  $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)} \in \text{Inn}(G)$  and since  $\text{Inn}(G) = \{\varphi_g \mid \text{for all } g \in G\}$ , then, by the definition of a normal subgroup,  $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$ .

**Exercise 4.4.3.** Prove that under any automorphism of  $D_8$ ,  $r$  has at most 2 possible images and  $s$  has at most 4 possible images ( $r$  and  $s$  are the usual generators). Deduce that  $|\text{Aut}(D_8)| \leq 8$ .

**Proof:** We start by recalling that  $|r| = 4$  and since an isomorphism must preserve the order of an element, then the only possible images of an automorphism  $\sigma$  are the 2 images  $r$  and  $r^3$ .

For  $s$  there are 5 elements of its order, but we notice that for any automorphism  $\sigma$  we have  $\sigma(r^2) = \sigma(r)^2$  which either equals  $(r)^2 = r^2$  or  $(r^3)^2 = r^6 = r^2$  as we've established from the possible images of  $r$ , thus  $\sigma(s) \neq r^2$  since it's 1-to-1, which leads us to conclude that there are only 4 possible images for  $s$  at most. We thus conclude that there are at most  $2 \cdot 4 = 8$  possibilities for  $\sigma$ , that is  $|\text{Aut}(D_8)| \leq 8$ .

**Exercise 4.4.5.** Use the fact that  $D_8 \trianglelefteq D_{16}$  to prove that  $\text{Aut}(D_8) \cong D_8$ .

**Proof:** By Collary 15 we have  $G \cong N_{D_{16}}(D_8)/C_{D_{16}}(D_8)$  where  $G \leq \text{Aut}(D_8)$ . Since we know that  $D_8 \trianglelefteq D_{16}$  then  $N_{D_{16}}(D_8) = D_{16}$ , and since it is easy to see that  $C_{D_{16}}(D_8) = Z(D_{16}) = \langle r^4 \rangle$ , we deduce that  $G \cong D_{16}/Z(D_{16})$  which is of order  $\frac{|D_{16}|}{|Z(D_{16})|} = \frac{16}{2} = 8$ , and since we've deduced that  $|\text{Aut}(D_8)| \leq 8$  we see that  $G = \text{Aut}(D_8)$ .

Now, we need to show that  $D_8 \cong D_{16}/Z(D_{16})$ . We start by noting that for any  $d' \in D_{16}$  we either have  $d' = d$  or  $d' = dr^4$  for some  $d \in D_8$  and that  $dr^4\langle r^4 \rangle = d\langle r^4 \rangle$ . We now construct a homomorphism  $\varphi : D_{16}/Z(D_{16}) \rightarrow D_8$  s.t.  $d\langle r^4 \rangle \rightarrow d$ , it is easy to see that it's well defined and bijective, thus we conclude that  $D_{16}/Z(D_{16}) \cong D_8$ .

Since  $\text{Aut}(D_8) \cong D_{16}/Z(D_{16}) \cong D_8$  we have  $\text{Aut}(D_8) \cong D_8$ .