

## 3.2 Open and Closed Sets

March 8, 2018

**Exercise 3.2.4.** Prove the converse of Theorem 3.2.5 by showing that if  $x = \lim_{n \rightarrow \infty} a_n$  for some sequence  $\{a_n\}$  contained in  $A$  satisfying  $a_n \neq x$ , then  $x$  is a limit point of  $A$ .

**Proof:** Since  $a_n$  is a converging sequence in  $A$  then for any  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  s.t. for any  $n \geq N$ ,  $|a_n - x| < \epsilon$  and as such  $a_n \in V_\epsilon(x)$  for any  $\epsilon > 0$  and since  $a_n \neq x$  by assumption, then  $x$  is a limit point of  $A$  by Definition 3.2.4.

**Exercise 3.2.8.** Given  $A \subseteq \mathbb{R}$ , let  $L$  be the set of all limit points of  $A$ .

(a) Show that the set  $L$  is closed.

**Proof:** Let  $L$  be the set of limit points of  $A$ , and suppose that  $x$  is a limit point of  $L$ , we want to show that  $x$  is an element of  $L$ ; in other words, that  $x$  is a limit point of  $A$ . Let  $V_\epsilon(x)$  be arbitrary. By the definition of a limit point,  $V_\epsilon(x)$  intersects  $L$  at a point  $l \in L$ , where  $l \neq x$ . Now choose  $\epsilon' > 0$  small enough so that  $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$ . Since  $l \in L$ ,  $l$  is a limit point of  $A$  and so  $V_{\epsilon'}(l)$  intersects  $A$ . This implies  $V_\epsilon(x)$  intersects  $A$  at a point different than  $x$ , and therefore  $x$  is a limit point of  $A$  and thus an element of  $L$ .

(b) Argue that if  $x$  is a limit point of  $A \cup L$ , then  $x$  is a limit point of  $A$ . Use this observation to furnish a proof for Theorem 3.2.12.

**Proof:** By definition,  $x$  is either a limit point of  $A$  or  $L$ . If  $x$  is a limit point of  $A$ , then we're done, but if  $x$  is a limit of  $L$  we use the same argument employed above to prove that  $x$  is a limit of  $A$ . We can now conclude that  $A \cup L$  does not produce any new limits  $x \notin A$ .

**Exercise 3.2.12.** Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(I shall use **Proof** and **Counterexample** to indicate **True** and **False** respectively.)

(a) For any set  $A \subseteq \mathbb{R}$ ,  $\overline{A}^c$  is open.

**Proof:** By Theorem 3.2.12, we know that  $\overline{A}$  is closed, and by Theorem 3.2.13, we know that the complement of a closed set is an open set and as such we conclude that  $\overline{A}^c$  is open since its complement  $\overline{A}$  is closed.

(b) If a set  $A$  has an isolated point, it cannot be an open set.

**Proof:** For  $A$  to be an open set then every point  $a$  must have an  $\epsilon$ -neighborhood  $V_\epsilon(x) \subseteq A$ , but since there exists an  $x$  for which  $V_\epsilon(x) \cap A = \{x\}$ , which means that  $V_\epsilon(x) \not\subseteq A$  and as such, any  $\epsilon'$  where  $0 < \epsilon' < \epsilon$ ,  $V_{\epsilon'}(x) \not\subseteq A$  thus  $A$  is not an open set.

(c) A set  $A$  is closed if and only if  $\bar{A} = A$ .

**Proof:**  $\Rightarrow$  By definition,  $A$  is closed if and only if it contains its limits  $l \in L$  and as such  $\bar{A} = A \cup L = A$ .

$\Leftarrow$  If  $\bar{A} = A$  then  $A = A \cup L$ , i.e.  $A$  contains its limits which, by definition, means  $A$  is closed.

(d) If  $A$  is a bounded set, then  $s = \sup A$  is a limit point of  $A$ .

**Proof:** Given that the set  $A$  is bounded and by using the Monotone Convergence and the Bolzano-Weierstrass Theorems, we can probably construct a convergent sequence that converges to  $s$  and by Theorem 3.2.9 we can conclude that  $s$  is indeed a limit point of  $A$ .

(e) Every finite set is closed.

**Proof:** Suppose we have a limit point  $l$  of  $A$ , by definition, every  $V_\epsilon(l)$  must intersect  $A$  at a value  $a \neq l$ . Suppose we have  $\epsilon' = \min\{|l - a_n| : a_n \in A\}$ , we can see that  $V_{\epsilon'}(l)$  does not intersect  $A$ , which leads to a contradiction, thus we can deduce that  $A$  has no limit points, i.e.  $L = \phi$ , and since  $\phi \subseteq A$  then  $A$  is closed.

(f) An open set that contains every rational number must necessarily be all of  $\mathbb{R}$ .

**Counterexample:**  $(-\infty, x) \cup (x, \infty)$  where  $x \notin \mathbb{Q}$ .