3.4 Perfect Sets and Connected Sets

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Exercise 3.4.6. Prove that a set $E \subseteq \mathbb{R}$ is connected iff, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.
Proof: $\Rightarrow$ Assume E is connected. Since E is connected this means that $\overline{A} \cap B \neq \phi$ with $A \neq \overline{A}$, otherwise that'll lead to a contradiciton. This means that B contains at least one of A's limit points $x \notin A$, thus we can always find a convergent sequence $(x_n) \subseteq A$ s.t. $(x_n) \to x$, thus completing the proof in one direction.
$\Leftarrow$ Assume $(x_n) \to x$ is a convergent sequence in A with $x \in B$. Because x is a limit point of A, then we have $x \in \overline{A}$. Since $x \in B$, then $\overline{A} \cap B = x \neq \phi$. Hence, E is connected, completing the proof in the other direction.
The proof works just as well if we swap A and B, thus completing the proof.

Exercise 3.4.7. (a) Find an example of a disconnected set whose closure is connected.
Example: Any set of the form $S = (a, b) \cup (b, c)$ will do.