

## 3.3 Compact Sets

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**Exercise 3.3.2.** Prove the converse of Theorem 3.3.4 by showing that if a set  $K \subseteq \mathbb{R}$  is closed and bounded then it is compact.

**Proof:** We shall assume  $K$  is not compact. Consider the sequence  $(x_n) \subseteq K$  whose subsequence  $(x_{n_k})$  converges to some point  $x \notin K$ . Such a sequence cannot exist because  $K$  is closed and as such it must contain its limit point  $x$  for its converging subsequence  $(x_{n_k})$ . Because  $K$  is bounded then by the Bolzano-Weierstrass Theorem, every sequence in  $K$  must contain a converging subsequence. This leads to a contradiction which proves that  $K$  must be Compact.

**Exercise 3.3.3.** Show that the Cantor set defined in Section 3.1 is a compact set.

**Proof:** The Cantor set  $C$  is defined as follows:

$$C = \bigcap_{n=0}^{\infty} C_n$$

Given that each  $C_n$  for some  $n \in \mathbb{N}$  is the union of a finite number of closed subsets then  $C_n$  is closed for every  $n \in \mathbb{N}$ . By Theorem 3.2.14 we know that  $C$  is closed as well since it is the result of an arbitrary intersection of closed sets and since it is also bounded, and as such by Theorem 3.3.4,  $C$  is compact.

**Exercise 3.3.7.** Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) An arbitrary intersection of compact sets is compact.

**Proof:** Consider the collection of compact sets  $K = \{K_\lambda : \lambda \in \Lambda\}$ . By Theorem 3.3.4, we know that every  $K_\lambda$  is closed and bounded, and by Theorem 3.2.14 we know that  $\bigcap_{\lambda \in \Lambda} K_\lambda$  is also closed. This leaves us with proving that  $\bigcap_{\lambda \in \Lambda} K_\lambda$  is also bounded. Consider  $k = \max\{k_\lambda : k_\lambda = \max\{|x| : x \in K_\lambda\}\}$ ,  $[-k, k]$  acts as a bound. Hence,  $\bigcap_{\lambda \in \Lambda} K_\lambda$  is both bounded and closed, thus compact.

(b) Let  $A \subseteq \mathbb{R}$  be arbitrary, and let  $K \subseteq \mathbb{R}$  be compact. Then the intersection  $A \cap K$  is compact.

**Counterexample:** Consider  $A = (0, 1)$  and  $K = [0, 1]$ .

(c) If  $F_1 \supseteq F_2 \supseteq F_3 \cdots$  is a nested sequence of nonempty closed sets, then the intersection  $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$ .

**Counterexample:** Consider  $F_n = [n, \infty)$ , each  $F_n$  is closed yet their intersection is empty.

(d) A finite set is always compact.

**Proof:** As was proven before, every finite set is closed, and since the set is trivially bounded, then by the Heine-Borel Theorem the set is compact.

(e) A countable set is always compact.

**Counterexample:** The countable set of rational numbers  $\mathbb{Q}$ .