Groups Acting on Themselves by Left Multiplication - Cayley's Theorem

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Exercise 4.2.6. Let r and s be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of N in D_8 as $1N, rN, sN$ and srN. Label these cosets with the integers 1, 2, 3, 4 respectively. Exhibit the image of each element of D_8 by left multiplication on the set of 4 left cosets of N in D_8. Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.
The images are:

$\sigma_r(1) = 2$
$\sigma_r(2) = r^2N = N = 1$
$\sigma_r(3) = rsN = sr^{-1}N = sr^3N = sr(r^2N) = srN = 4$
$\sigma_r(4) = rsrN = sN = 3$
$\sigma_s(1) = sN = 3$
$\sigma_s(2) = srN = 4$
$\sigma_s(3) = ssN = N = 1$
$\sigma_s(4) = ssrN = rN = 2$

Because σ is a homomorphism and we know the images of the generators of D_8 (r and s) we can deduce the image of every other element in D_8 through the multiplication of images (e.g., $\sigma_{rs} = \sigma_r\sigma_s$).
It is easy to see that r^2 is in the kernel of the action and thus we deduce that the representation is not faithful.
As we've shown, the subgroup of S_{D_8} that π_N maps to can be generated by $\sigma_r = (1\ 2)(3\ 4)$ and $\sigma_s = (1\ 3)(2\ 4)$ which is the exact same subgroup that we've seen the Klein 4-group isomorphic to, thus $\pi_N \cong K_4$ where K_4 is the Klein 4-group.