4.5 The Intermediate Value Theorem

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Exercise 4.5.3. Is there a continuous function on all of $\mathbb{R}$ with range $f(\mathbb{R}) = \mathbb{Q}$?

Answer: No, $\mathbb{Q}$ is not connected. If the function had 1 and 2 in its domain, then by the IVT, its range must contain $\sqrt{2}$ (or any other irrational point.).

Exercise 4.5.4. A function f is increasing on A if $f(x) < f(y)$ for all $x < y$ in A. Show that the IVT does have a converse if we assume f is increasing on $[a, b]$.

Proof: We want to show that this function is continuous given that it satisfies the IVP. Since f is increasing. then we have $f(a) < f(c)$. If $f(c) - \epsilon < f(a)$, then set $x_1 = a$, if $f(c) - \epsilon \leq f(a)$, then, by the IVP, we know that there exists a $x_1 < c$ s.t $f(x_1) = f(c) - \epsilon$. In either case we have for $x \in (x_1, c]$

$$f(c) - \epsilon \leq f(x_1) \leq f(x) \leq f(c)$$

In a similar fashion we can find $x_2 > c$, s.t. for $x \in [c, x_2)$ we have

$$f(c) \leq f(x) \leq f(x_2) \leq f(c) + \epsilon$$

Now we choose $\delta = \min[c - x_1, x_2 - c]$, we thus have

$$|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon$$

We can thus conclude that f is continuous and as such the converse is true.

Exercise 4.5.5. Finish the proof of the IVT using the AoC started previously.

Proof: Suppose $f(c) > 0$. Set $\epsilon_0 = f(c)$, then the continuity of f implies that there exists a δ_0 for which $x \in V_{\delta_0}(c)$ implies $f(x) \in V_{\epsilon_0}(c)$, but that in turn implies that $f(x) > 0$ and thus $x \notin K$ for all $x \in V_{\delta_0}(c)$. This means that if c is an upper bound on K, then $c - \epsilon$ is a smaller upper bound, which violates the defintion of a supremum, thus $f(c) \not> 0$.
Now, suppose $f(c) < 0$. The continuity of f allows to construct a neighborhood $V_{\delta_0}(c)$ where $x \in V_{\delta_0}(c)$ implies $f(x) < 0$. But this implies that there exists a

point s.t. $c + \frac{\delta}{2}$ is an element of K, violating the fact that c is an upper bound for K. Thus, $f(c) < 0$ is impossible.
We conclude that $f(c) = 0$ as desired.
This proves the IVT for the the special case $L = 0$. To prove the more general version, we use an auxiliary function $h(x) = f(x) - L$. We know that $h(c) = 0$ for some $c \in (a, b)$ from which it follows that $f(c) = L$.

Exercise 4.5.7. Let f be a continuous function on a closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show that $f(x) = x$ for at least one value $x \in [0, 1]$.

Proof: We start by constructing an auxiliary function $g(x) = f(x) - x$. Now, since the range of f is contained in $[0, 1]$, then $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. By the IVT, we can find an x s.t. $g(x) = 0$, which completes the proof.