Exercise 3.2.4. Prove that the converse of Theorem 3.2.5 by showing that if $x = \lim_{n\to\infty} a_n$ for some sequence $\{a_n\}$ contained in A satisfying $a_n \neq x$, then x is a limit point of A.
Proof: since a_n is a converging sequence in A then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ s.t. for any $n \geq N$, $|a_n - x| < \epsilon$ and as such $a_n \in V_\epsilon(x)$ for any $\epsilon > 0$ and since $a_n \neq x$ by assumption, then x is a limit point of A by Definition 3.2.4.

Exercise 3.2.8. Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.
Proof: Let L be the set of limit points of A, and suppose that x is a limit point of L, we want to show that x is an element of L; in other words, that x is a limit point of A. Let $V_\epsilon(x)$ be arbitrary. By the definition of a limit point, $V_\epsilon(x)$ intersects L at a point $l \in L$, where $l \neq x$. Now choose $\epsilon^{'} > 0$ small enough so that $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$. Since $l \in L$, l is a limit point of A and so $V_{\epsilon'}(l)$ intersects A. This implies $V_\epsilon(x)$ intersects A at a point different than x, and therefore x is a limit point of A and thus an element of L.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to fernish a proof for Theorem 3.2.12.
Proof: By definition, x is either a limit point of A or L. If x is a limit point of A, then we're done, but if x is a limit of L we use the same argument employed above to prove that x is a limit of A. We can now conclude that $A \cup L$ does not produce any new limits $x \notin A$.

Exercise 3.2.12. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.
(I shall use **Proof** and **Counterexample** to indicate **True** and **False** respectively.)

(a) For any set $A \subseteq \mathbb{R}$, $\overline{A}^c$ is open.
Proof: By Theorem 3.2.12, we know that $\overline{A}$ is closed, and by Theorem 3.2.13, we know that the compliment of a closed set is an open set and as such we conclude that $\overline{A}^c$ is open since its compliment $\overline{A}$ is closed.

(b) If a set A has an isolated point, it cannot be an open set.
Proof: For A to be an open set then every point a must have an ϵ-neighborhood $V_\epsilon(x) \subseteq A$, but since there exists an x for which $V_\epsilon(x) \cap A = \{x\}$, which means that $V_\epsilon(x) \not\subseteq A$ and as such, any $\epsilon^{'}$ where $0 < \epsilon^{'} < \epsilon$, $V_{\epsilon'}(x) \not\subseteq A$ thus A is not an open set.

(c) A set A is closed if and only if $\overline{A} = A$.
Proof: $\Rightarrow$ By definition, A is closed if and only if it contains its limits $l \in L$ and as such $\overline{A} = A \cup L = A$.
$\Leftarrow$ If $\overline{A} = A$ then $A = A \cup L$, i.e. A contains its limits which, by definition,

means A is closed.

(d) If A is a bounded set, then $s = \sup A$ is a limit point of A.
Proof: Given that the set A is bounded and by using the Monotone Convergence and the Bolzano-Weierstrass Theorems, we can probably construct a convergent sequence that converges to s and by Theorem 3.2.9 we can conclude that s is indeed a limit point of A.

(e) Every finite set is closed.
Proof: Suppose we have a limit point l of A, by definition, every $V_\epsilon(l)$ must intersect A at a value $a \neq l$. Suppose we have $\epsilon^{'} = min\{|l - a_n| : a_n \in A\}$, we can see that $V_{\epsilon'}(l)$ does not intersect A, which leads to a contradicition, thus we can deduce that A has no limit points, i.e. $L = \phi$, and since $\phi \subseteq A$ then A is closed.

(f) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.
Counterexample: $(-\infty, x) \cup (x, \infty)$ where $x \notin \mathbb{Q}$.