4.5 Sylow's Theorem

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Exercise 4.5.8: Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.

Answer: We take $\langle(1234)(13)\rangle$ and $\langle(2345)(35)\rangle$ as the wanted Sylow 2-subgroups and conjugating one of them by (15) yields the other.

Exercise 4.5.13: Prove that a group of order 56 has a normal Sylow p-subgroup for some p dividing its order.

Proof: We first note that $56 = 2^3 \cdot 7$. Suppose $n_7 = 1$, then the Sylow 7-subgroup is the normal subgroup we desire. Now suppose $n_7 \neq 1$, $n_7 \mid 8$ and $n_7 \equiv 1 \bmod 7$, then $n_7 = 8$. We note that each of the 8 distinct Sylow 7-supgroups contains 6 elements of order 7 while having the identity element in common, i.e., the number of elements of order 7 is $6 \cdot 8 = 48$, this leaves us with 7 unaccounted for elements and the identity. Now $n_2 \mid 7$ and $n_2 \equiv 1 \bmod 2$, this means either $n_2 = 1$ or $n_2 = 7$. Since we proved that there are only 7 elements not in any of the Sylow 7-subgroups and the identity exists in any of the $Syl_2(G)$ elements and since each Sylow 2-subgroup contains 8 elements, then we can deduce that $n_2 = 1$, i.e., the Sylow 2-subgroup is normal in G completing the proof.

Exercise 4.5.30: How many elements of order 7 must there be in a simple group of order 168?

Answer: We first note that $168 = 7 \cdot 24$. Since G is simple, then $n_7 \neq 1$ and since $n_7 \mid 24$ and $n_7 \equiv 1 \bmod 7$, then $n_7 = 8$. Since we have 8 Sylow 7-subgroups with the maximum power of 7 being 1, then the number of elements of order 7 is $8 \cdot 6 = 48$.