5.3 The Mean Value Theorem

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Exercise 5.3.3. Let h be a differentiable function defined on the interval $[0,3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

a) Argue that there exists a point $d \in [0,3]$ where $h(d) = d$.
Proof: Take the function $g(x) = h(x) - x$. Given that $g(1) = 1$ and $g(3) = -1$, then by the Intermediate Value Theorem, $g(d) = 0$ for some d which implies that $h(d) = d$ as desired.

b) Argue that at some point c we have $h'(c) = \frac{1}{3}$.
Proof: By the Mean Value Theorem we have $h'(c) = \frac{h(3)-h(0)}{3-0} = \frac{2-1}{3} = \frac{1}{3}$ for some $c \in [0,3]$.

c) Argue that $h'(x) = \frac{1}{4}$ at some point in the domain.
Proof: By Rolle's Theorem we know that $g'(c) = 0$ for some $c \in [1,3]$ and by **b)** we know that $g'(d) = \frac{1}{3}$ for some $d \in [0,3]$. Now, by Darboux's Theorem on the interval $[0,3]$, we can conclude that $h'(t) = \frac{1}{4}$ for some $t \in [0,3]$.

Exercise 5.3.5. A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Proof: Assume that f has two fixed points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. By the Mean Value Theorem we have $f'(c) = \frac{f(x_1)-f(x_2)}{x_1-x_2} = 1$ which leads to a contradiction. Thus, f can have at most one fixed point.

Exercise 5.3.7. **a)** Recall that a function $f : (a,b) \to \mathbb{R}$ is increasing on (a,b) if $f(x) \leq f(y)$ whenever $x < y$ in (a,b). Assume f is differentiable on (a,b). Show that f is increasing on (a,b) if and only if $f'(x) \geq 0$ for all $x \in (a,b)$.

Proof: $\Longrightarrow$ Assume that f is increasing. Now, suppose that for some c in the domain we have $f'(c) < 0$. By the Mean Value Theorem we know that $f'(c) = \frac{f(e)-f(d)}{e-d}$ for some $a \leq d < e \leq b$. We now have $f(e) - f(d) < 0$, but this is a contradicition since that'd imply $f(e) < f(d)$. Thus, if $f(x)$ is increasing then $f'(x) \geq 0$.
$\Longleftarrow$ Assume that $f'(x) \geq 0$ for all $x \in (a,b)$. Now suppose that $f(e) < f(d)$

for some $a \leq d < e \leq b$. By employing the technique shown in the previous direction, we can easily see that this would imply $f'(c) < 0$ for some $c \in (a, b)$, which is a contradiction. Thus, we can see that if $f'(x) \geq 0$ then $f(x) \leq f(y)$ for all $a \leq x < y \leq b$. This completes from the other direction and we can now conclude that f is increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

b) Show that the function

$$g(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Now, prove that g is not increasing over any open interval containing 0.

Proof: By the definition of the derivative, we have $g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to 0} \frac{1}{2} + x \sin(\frac{1}{x})$. Now, by the Algebraic Limit Theorem, we can see that $g'(0) = \frac{1}{2}$.

For $x \neq 0$ we have $g'(x) = \frac{1}{2} - \cos(\frac{1}{x}) + 2x \sin(\frac{1}{x})$. Now, we need to find a sequence (x_n) converging to 0 such that $g'(x_n) < 0$, the sequence $x_n = \frac{1}{2n\pi}$ satisfies this. Thus, there is no open interval around 0 where $g'(x) \geq 0$, and by the previous proof, g' is not increasing on any intervak containing 0.