4.4 Automorphisms

April 1, 2018

Exercise 4.4.1. If $\sigma \in \mathrm{Aut}(G)$ and φ_g is a conjugation by g prove $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)}$. Deduce that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. (The group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G.)

Proof: For any $x \in G$ we have $\sigma\varphi_g\sigma^{-1} = \sigma gxg^{-1}\sigma^{-1} = (\sigma(g))x(\sigma(g))^{-1} = \varphi_{\sigma(g)}$. Now, since we've proved that for any $\sigma \in \mathrm{Aut}(G)$ we have $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)} \in$ Inn(G) and since $\mathrm{Inn}(G) = \{\varphi_g|$ for all $g \in G\}$, then, by the definition of a normal subgroup, $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

Exercise 4.4.3. Prove that under any automorphism of D_8, r has at most 2 possible images and s has at most 4 possible images (r and s are the usual generators). Deduce that $|\mathrm{Aut}(D_8)| \leq 8$.

Proof: We start by recalling that $|r| = 4$ and since an isomorphism must preserve the order of an element, then the only possible images of an automorphism σ are the 2 images r and r^3.
For s there are 5 elements of its order, but we notice that for any automorphism σ we have $\sigma(r^2) = \sigma(r)^2$ which either equals $(r)^2 = r^2$ or $(r^3)^2 = r^6 = r^2$ as we've established from the possible images of r, thus $\sigma(s) \neq r^2$ since it's 1-to-1, which leads us to conclude that there are only 4 possible images for s at most. We thus conclude that there are at most $2 \cdot 4 = 8$ possiblities for σ, that is $|\mathrm{Aut}(D_8)| \leq 8$.

Exercise 4.4.5. Use the fact that $D_8 \trianglelefteq D_{16}$ to prove that $\mathrm{Aut}(D_8) \cong D_8$.

Proof: By Collary 15 we have $G \cong N_{D_{16}}(D_8)/C_{D_{16}}(D_8)$ where $G \leq \mathrm{Aut}(D_8)$. Since we know that $D_8 \trianglelefteq D_{16}$ then $N_{D_{16}}(D_8) = D_{16}$, and since it is easy to see that $C_{D_{16}}(D_8) = Z(D_{16}) = \langle r^4 \rangle$, we deduce that $G \cong D_{16}/Z(D_{16})$ which is of order $\frac{|D_{16}|}{|Z(D_{16})|} = \frac{16}{2} = 8$, and since we've deduced that $|\mathrm{Aut}(D_8)| \leq 8$ we see that $G = \mathrm{Aut}(D_8)$.
Now, we need to show that $D_8 \cong D_{16}/Z(D_{16})$. We start by noting that for any $d^{'} \in D_{16}$ we either have $d^{'} = d$ or $d^{'} = dr^4$ for some $d \in D_8$ and that $dr^4\langle r^4 \rangle = d\langle r^4 \rangle$. We now construct a homomorphism $\varphi : D_{16}/Z(D_{16}) \to D_8$ s.t. $d\langle r^4 \rangle \to d$, it is easy to see that it's well defined and bijective, thus we conclude that $D_{16}/Z(D_{16}) \cong D_8$.

Since $\mathrm{Aut}(D_8) \cong D_{16}/Z(D_{16}) \cong D_8$ we have $\mathrm{Aut}(D_8) \cong D_8$.