7.1 Introduction to Rings

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Exercise 7.1.1. Show that $(-1)^2 = 1$ in R.

Proof: $(-1)(-1) = 1 \cdot 1 = 1$.

Exercise 7.1.2. Prove that if u is a unit in R then so is $-u$.

Proof: Since u us a unit then there exists a multiplicative inverse v of it. The multiplicative inverse, by the axioms of a group, must have an additive inverse and thus we have $-v \in R$. Now, since $(-u)(-v) = uv = 1$ then $-u$ is a unit.

Exercise 7.1.7. The center of a ring R is $\{z \in R \mid zr = rz \text{ for all } r \in R\}$ (i.e., the set of all elements which commute with every element in R). Prove that the center of a ring is a subring that contains the identity. Prove that the center of a division ring is a field.

Proof: To prove that the center is indeed a subring it suffices to show that it has the identity and is closed under addition and multiplication. We shall denote the center by C. Since $0, 1$ commutes with every element then $0, 1 \in C$. Consider any $x, y \in C$, it is easy to see that $xyz = zyx$ and $(x+y)z = xz+yz = zx + zy = z(x + y)$ for any $z \in C$, thus C is closed under addition and multiplication and is indeed a subring.
Since for any $z \in C$ and $r \in R$ we have $zr = rz \iff zrz^{-1} = r \iff rz^{-1} = z^{-1}r$ then for any C is a division ring. Now, since C is a commutative division ring then it is a field.

Exercise 7.1.12. Prove that any subring of a field which contains the identity is an integral domain.

Proof: We shall denote the subring in question by R. Given that R is a subring of a field then it is commutative and has no zero divisors, thus it is a an integral domain.