1.B Definition of a Vector Space

May 7, 2018

Exercise 1.B.1. Prove that $-(-v) = v$ for every $v \in V$.

Proof: By definition we have:

$$(-v) + (-(-v)) = 0 \quad \text{and} \quad (-v) + v = 0$$

This implies that both v and $-(-v)$ are additive invereses to $-v$, by the uniqueness of additive inverses, we have $v = -(-v)$.

Exercise 1.B.5. Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = \mathbf{0}.$$

Proof: $\Longrightarrow$ If we assume $0v = \mathbf{0}$ then we can see that $v + (-v) = 1v + (-1)v = (1 + (-1))v = 0v = \mathbf{0}$.
$\Longleftarrow$ If we assume the additive inverse then we can see that $0v = (1 + (-1))v = v + (-v) = \mathbf{0}$.

Exercise 1.B.6. Let $-\infty$ and ∞ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & t < 0, \\ 0 & t = 0, \\ \infty & t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & t < 0 \\ 0 & t = 0 \\ -\infty & t > 0 \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.

No, consider $\infty = (2 + (-1))\infty = 2\infty + (-\infty) = \infty + (-\infty) = 0$, which is a contradiction.