5.2 Derivatives and The Intermediate Value Property

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Exercise 5.2.2. **a)** Use Definition 5.2.1 to produce the proper formulat for the derivative of $f(x) = \frac{1}{x}$.

Answer: $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} = \lim_{x\to c} \frac{c-x}{xc(x-c)} = \lim_{x\to c} \frac{-1}{xc} = \frac{-1}{c^2}$.

b) Combine the result in part a) with the chain rule (Theorem 5.2.5) to supply a proof of part iv) of Theorem 5.2.4.

Proof: Take $h(x) = \frac{1}{x}$, then we have $(h \circ g)(x) = \frac{1}{g(x)}$. By the chain rule and part a), we have $(h \circ g)'(x) = \frac{-g'(x)}{g(x)^2}$.
Now by part iii) of Theorem 5.2.4, we have

$$\begin{aligned}
(\frac{f}{g})'(x) = (f(x)(h\circ g)(x))' &= f'(x)(h\circ g)(x) + f(x)(h\circ g)'(x)\\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}\\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

Exercise 5.2.3. By imitating the Dirichlet constructions in Section 4.1, construct a function on $\mathbb{R}$ that is differentiable at a single point.

Answer: Take the function

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

We can see that such a function is continuous at 0. Now, we have $h'(0) = \lim_{x\to 0} \frac{h(x)}{x}$. Given $\epsilon > 0$, take $\delta = \epsilon$. We have $\frac{|h(x)|}{|x|} < x$, we see that $\frac{|h(x)|}{|x|} < \epsilon$ when $|x| < \delta$ and thus h is differentiable at 0 and $h'(0) = 0$.

Exercise 5.2.5. Let

$$g_a(x) = \begin{cases} x^a \sin(\frac{1}{x}) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0, 1]$.
Answer: For $a = 0.5$ it's easy to see that the derivative blows up as it approaches 0.

b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.
Answer: For $a = 2.5$ the second derivative doesn't exist at 0.

Exercise 5.2.8. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

a) If a derivative function is not constant, then the derivative must take on some irrational values.
Proof: Since it satisfies the Intermediate Value Property and it has two distinct values, then it must attain every number in between, which also includes irrationals.

b) If f' exists on an open interval, and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.
Counterexample: The function

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$f'(0) = \frac{1}{2}$ while $f'(x)$ is negative everywhere else.