3.3 Compact Sets

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Exercise 3.3.2. Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded then it is compact.
Proof: We shall assume K is not compact. Consider the sequence $(x_n) \subseteq K$ whose subsequence (x_{n_k}) converges to some point $x \notin K$. Such a sequence cannot exist because K is closed and as such it must contain its limit point x for its converging subsequence (x_{n_k}). Because K is bounded then by the Bolzano-Weierstrass Theorem, every sequence in K must contain a converging subsequence. This leads to a contradiction which proves that K must be Compact.

Exercise 3.3.3. Show that the Cantor set defined in Section 3.1 is a compact set.
Proof: The Cantor C is defined as follows:

$$C = \bigcap_{a=0}^{\infty} C_n$$

Given that each C_n for some $n \in \mathbb{N}$ is the union of a finite number of closed subsets then C_n is closed for every $n \in \mathbb{N}$. By Theorem 3.2.14 we know that C is closed as well since it is the the result of an arbitrary intersection of closed sets and since it is also bounded, and as such by Theorem 3.3.4, C is compact.

Exercise 3.3.7. Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) An arbitrary intersection of compact sets is compact.
Proof: Consider the collection of compact sets $K = \{K_\lambda : \lambda \in \Lambda\}$. By Theorem 3.3.4, we know that every K_λ closed and bounded, and by Theorem 3.2.14 we know that $\bigcap_{\lambda\in\Lambda} K_\lambda$ is also closed. This leaves us with proving that $\bigcap_{\lambda\in\Lambda} K_\lambda$ is also bounded. Consider $k = max\{k_\lambda : k_\lambda = max\{|x| : x \in K_\lambda\}\}$, $[-k, k]$ acts as a bound. Hence, $\bigcap_{\lambda\in\Lambda} K_\lambda$ is both bounded and closed, thus compact.

(b) Let $A \subseteq \mathbb{R}$ be arbitrary, and let $K \subseteq \mathbb{R}$ be compact. Then the intersection $A \cap K$ is compact.

Counterexample: Consider $A = (0, 1)$ and $K = [0, 1]$.

(c) If $F_1 \supseteq F_2 \supseteq F_3 \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \phi$.
Counterexample: Consider $F_n = [n, \infty)$, each F_n is closed yet their intersection is empty.

(d) A finite set is always compact.
Proof: As was proven before, every finite set is closed, and since the set is trivially bounded, then by the Heine-Borel Theorem we the set is compact.

(e) A countable set is always compact.
Counterexample: The countable set of rational numbers $\mathbb{Q}$.