1.C Subspaces

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Exercise 1.C.1. For each of the following, determine whether it is a subspace of $\mathbb{F}^3$:

a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$.
Answer: We'll name this subset U. $\mathbf{0} \in U$. For any scalar $k \in F$, we have $ku = k(x_1, x_2, x_3) = (kx_1, kx_2, kx_3)$, we can thus see that $kx_1 + 2kx_2 + 3kx_3 = k(x_1 + 2x_2 + 3x_3) = 0$, thus U is closed under scalar multiplication. Now, take $u_1, u_2 \in U$. We can see that $(x_{1,1} + x_{1,2}) + 2(x_{2,1} + x_{2,2}) + 3(x_{3,1} + x_{3,2}) = (x_{1,1} + 2x_{2,1} + 3x_{3,1}) + (x_{1,2} + 2x_{2,2} + 3x_{3,2}) = 0 + 0 = 0$, where $u_1 = (x_{1,1}, x_{2,1}, x_{3,1})$ and $u_2 = (x_{1,2}, x_{2,2}, x_{3,2})$, thus U is closed under addition and as such it is a subspace.
b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$.
Answer: No, since $(0, 0, 0)$ is not in this subspace.

Exercise 1.C.4. Suppose that $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ iff b = 0.

Proof: Denote the subspace of $\mathbb{R}^{[0,1]}$ of continuous functions satisfying $\int_0^1 f = b$ as U_b.
Now, because $f \in U_b$ which is a subspace, then $kf \in U_b$. This can only be if $\int_0^1 kf = k \int_0^1 f = kb = b$, which is only the case iff $b = 0$.
We now prove the rest of the criteria for U_b to be a subspace when $b = 0$. Take $f, g \in U_b$, we have $\int_0^1 f + g = \int_0^1 f + \int_0^1 g = 0 + 0 = 0$. Now take $f \in U_b$ s.t. $f = 0$, then we have $\int_0^1 f = 0$ as desired. Thus, U_b is indeed a subspace iff $b = 0$.

Exercise 1.C.6.

a) Is $\{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?
Answer: Yes, since $a^3 = b^3$ iff $a = b$ in $\mathbb{R}$ and it's easy to see that it forms a subspace.
b) Is $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?
Answer: No, consider the vectors $\big(-\sqrt[3]{-1}z_2, \frac{-1}{2}i(\sqrt{3}z_1 - iz_1), c\big)$ and $\big(\sqrt[3]{-1^2}z_2, \frac{1}{2}i(\sqrt{3}z_1 + iz_1), c\big)$, if we add them we find that $a^3 \neq b^3$.

Exercise 1.C.8. Give an example of a subset U of $\mathbb{R}^2$ s.t. U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

Answer: Take the subset $U = \{(x,y) \in \mathbb{R} : x = 0 \text{ or } y = 0\}$. Clearly it is closed under scalar multiplication, yet if we take $u_1 = (1,0), u_2 = (0,1) \in U$ we can see that $u_1 + u_2 \notin U$, thus, U is not a subspace.

Exercise 1.C.11. Prove that the intersection of any collection of subspaces of V is a subspace of V.

Proof: We'll start by examining the intersection of 2 subspaces of $U_1 \cap U_2 \subseteq V$. Since $0 \in U_1$ and $0 \in U_2$, then $0 \in U_1 \cap U_2$. Now, take any vector $u \in U_1 \cap U_2$, since this vector exists in both subspaces then any scalar multiple of it ku is also in both subspaces, thus, $ku \in U_1 \cap U_2$. Finally, take any two vectors $u_1, u_2 \in U_1 \cap U_2$, it is easy to see that since $u_1 + u_2$ exists in both subspaces then so must it exist in $U_1 \cap U_2$.
This proves that the intersection of any 2 subspaces yields another subspcae, to prove this for the general case we proceed by induction, i.e., $U = U_1 \cap U_2$, which is a subspace, and then take its intersection with another subspace U_3 and proceed inductively from there, sitting $U = U \cap U_n$.

Exercise 1.C.12. Prove that the union of two subspaces of V is a subspace of V iff one of them is contained in the other.

Proof: It is easy to see that $U \cup W$ has the $\mathbf{0}$ element and all the scalar multiples of any vector in U or W, thus we are left with checking the the closure of addition. Suppose $U \cup W$ is a subspace and $U \nsubseteq W$ and $W \nsubseteq U$. Now, take $u \in U \backslash W$ and $w \in W \backslash U$, since $U \cup W$ is a subspace, then $u + w \in U \cup W$. Now, either $u + w \in U$ or $u + w \in W$. If $u + w \in U$, then $u + w - u = w \in U$, which is a contradiction, and if $u + w \in W$, then $u + w - w = u \in W$, which is, again, a contradiciton. So, either $U \subseteq W$ or $W \subseteq U$, in either case we can see that the result of the union is the bigger set which is already, by assumption, a subspace.

Exercise 1.C.17. Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

Answer: Yes, we first note that for any vectors $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$. Now, we can easily see that any element $x + (y + z) \in U_1 + (U_2 + U_3)$ can be expressed by an element $(x + y) + z \in (U_1 + U_2) + U_3$ and the opposite is true, thus, $U_1 + (U_2 + U_3) = (U_1 + U_2) + U_3$.

Exercise 1.C.18. Does the operation of addition on the subspace of V have an additive identity? Which subspaces have additive inverses?

Answer: Yes, the additive identity in this case would be the zero vector $\mathbf{0}$ which is trivially a subspace of V. For a subspace U to have an additive inverse then $U + W = 0$ which is only the case if $U = W = 0$.