4.3 Combinations of Continuous Functions

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Exercise 4.3.2. **a)** Supply a proof for Theorem 4.3.9 using $\epsilon - \delta$ characterization of continuity.

Proof: Since g is continuous at $f(c)$, then for any $\epsilon > 0$ there exists a $\delta_f > 0$ s.t. $|y - f(c)| < \delta_f$ implies $|g(y) - g(f(c))| < \epsilon$. Now, since f is continuous at c, for this δ_f, there exists a $\delta > 0$ s.t. $|x - c| < \delta$ implies $|f(x) - f(c)| < \delta$, thus for any $\epsilon > 0$ there exists a $\delta > 0$ s.t. $|x - c| < \delta$ implies $|g(f(x)) - g(f(c))| < \epsilon$. We conclude that $g \circ f$ is indeed continuous at c.

b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iv)).

Proof: Assume $(x_n) \to c$. Since f is continuous at c, then $f(x_n) \to f(c)$. Now, since g is continuous at $f(c)$ and $f(x_n) \to f(c)$, then $g(f(x_n)) \to g(f(c))$. Therefore, $g \circ f$ is continuous at c.

Exercise 4.3.4. **a)** Show using Definition 4.3.1 that any function f with domain $\mathbb{Z}$ will necessarily be continuous at every point in its domain.

Proof: For all $\epsilon > 0$ take the point $c \in \mathbb{N}$. Now, consider $\delta = 1$, we note that the only x satisfying $|x - c| < \delta$ is $x = c$, thus $|f(x) - f(c)| = 0 < \epsilon$. By Theorem 4.3.1 (i), f is convergent at every point $x \in \mathbb{Z}$.

b) Show in general that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Proof: Given that c is an isolated point of A, then there exists a δ-neighborhood $V_\delta(c)$ that intersects A at c only. Thus, trivially, any point $x \in V_\delta(c) \cap A$ implies that $x = c$ and thus $f(x) = f(c) \in V_\epsilon(f(c))$. By Theorem 4.3.2 (iii), we deduce that f is continuous at c.

Exercise 4.3.7. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Proof: Suppose c is a limit point of K, then there exists a sequence $(x_n) \to c$ in K. Now, since $(x_n) \in K$, then $h(x_n) = 0$. Since h is continuous, then $\lim h(x_n) = h(c)$. We finish by noting that $h(x_n) = 0$ for all x_n implies that $\lim h(x_n) = 0$, thus $\lim h(c) = \lim h(x_n) = 0$, from which we can deduce that $c \in K$ as desired.

Exercise 4.3.12. Let C be the Cantor set constructed in Section 3.1. Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C. \end{cases}$$

a) Show that g fails to be continuous for any point $c \in C$.
Proof: Fix $c \in C$. Now, take the converging sequence (x_n) where $x_n = \frac{c}{\sqrt[n]{2}}$ where each $x_n \notin C$. Since $(\frac{1}{\sqrt[n]{2}}) \to 1$ and, by the Algebraic Limit Theorem, $\lim c \cdot (\frac{1}{\sqrt[n]{2}}) = c \cdot 1 = c$, then this sequence converges to c, yet $\lim g(x_n) \neq g(c)$, thus $g(x)$ is not continuous at any $c \in C$.

b) Prove that g is continuous at every point $c \notin C$.
Proof: Fix $c \notin C$ and let ϵ be arbitrary. Since C is closed then C^c is open, thus there exists a δ where $V_\delta(c) \subseteq C^c$. Now, consider $x \in V_\delta(c)$, then $x \in C^c$ implies that $g(x) \in V_\epsilon(g(c))$. Now, by Theorem 4.3.2 (iii), we see that $x \in V_\delta(c)$ implies that $g(x) \in V_\epsilon(g(c))$, thus g is continuous at every $c \in C^c$.