Division Theorem/Euclidean Division

Let a, b be integers, $b > 0$. Then there exist unique integers q and r such that $a = qb + r$ and $0 \leq r < b$.

Proof:
Split the problem into proving existance and proving uniqueness.

Proof of existance of q and r:

Expressing r through the given formula: $a = qb + r$. We have: $r = a - qb$.
We also know that $r \geq 0$
Let's consider the set of all reminders in the form $a - kb \geq 0$, where $k \in \mathbb{Z}$
$S = \{a - kb, a' - k'b', a'' - k''b'', ...\}$, where all members are non-negative numbers (because we need to have remainders greater or equal to zero $r \geq 0$).
If we can prove that this set is non-empty then we can prove that there exist remainder in the form $a - kb \geq 0$ where $b > a - kb \geq 0$.

There are such integers:
$b > a - kb \geq 0 \implies b > 0$
$b > 0 \land b \in \mathbb{N} \implies b \geq 1$
For example let's take $b = 1$ and $k = -|a|$, so
$a - kb \geq 0$
$a + |a| \geq 0$, which is true for each a

Now we know that the set S has members. We also know that the members are all non-negative, so by WOP (axiom), we can conclude that the set S has a smallest element. (1)
Let's say this element is $r = a - qb$.
In order to finish the proof of existance we should prove that $r < b$
Using proof by contradiction, let's assume $r \geq b$
$a - qb \geq b$ (by substituting r)
$a - qb - b \geq 0$
$a - b(q + 1) \geq 0$
this looks like the form $a - bk$, where k = q + 1. It's also non-negative, so it must be a member of the set of reminders S.
Let's compare $r = a - bq$ and $a - b(q + 1)$
$a - b(q + 1) < a - bq$, because $b(q + 1) > bq$
but this is contradiction to (1), so $r < b$

This concludes the proof of existance: There exist q and r such that $a = qb + r$ and $0 \leq r < b$.

Proof of uniqueness of q and r:

To prove the uniqueness we are gonna try to show that there's q' and r', but they are equal to q and r.
Let's assume that we can represent a in two ways:
$a = bq + r = bq' + r'$
$r - r' = bq' - bq$
$r - r' = b(q' - q)$

From the existance proof we know that $0 \leq r' < b$ and $0 \leq r < b$.
Representing $-r'$ using $0 \leq r' < b$, we get $0 \geq -r' > -b$
let's sum both inequalitites:
$-b < -r' \leq 0$ /+
$0 \leq r < b$

$-b < r - r' < b$, which means that $|r - r'| < b$
$|b(q' - q)| < b$ (move b out of the module because $b > 0$)
$b|q' - q| < b$ dividing by $b \geq 1$
$|q' - q| < 1$
this means $-1 < q' - q < 1$. Given the fact that both q' and q $\in \mathbb{N}$ the result of their subtraction should also be in $\mathbb{N}$
The only way for that to be possible $q' - q = 0 \equiv q' = q$, which proves the uniqueness of q and r.