# Assignment 8 Solutions

## 1

We need to prove the claim: $(\exists m, n \in \mathbb{Z})(\exists k \in \mathbb{Z})[p = k^2 \wedge m^2 + mn + n^2 = p]$
Checking with $m = n = k = 0$:
$0 + 0 + 0 = 0 = 0^2$, which proves the statement.

## 2

We need to prove the claim:
$(\forall m \in \mathbb{Z}^+)(\exists n \in \mathbb{Z}^+)(\exists k \in \mathbb{Z})[p = k^2 \wedge mn + 1 = p]$
$mn + 1 = p = k^2$
$m = \frac{k^2-1}{n} = \frac{(k-1)(k+1)}{n}$
So if n = k - 1, then m would be k + 1, so the initial claim is true for n = k - 1, because:
$m = \frac{(k-1)(k+1)}{k-1} = k + 1$ given k = n + 1, where $n \geq 0$ then $k \geq 1$, which is subset of $\mathbb{Z}^+$, which we need for m.

## 3

claim: $(\forall n \in \mathbb{Z}^+)(\exists b, c \in \mathbb{Z}^+)(\exists p, q \in \mathbb{N})[f(n) = n^2 + bn + c = p * q]$
Trying to prove by brute forcing:
1. $b = c = 1$
$n^2 + n + 1$, which we can prove that is odd, but it might be prime or composite.
2. $b = 1, c = 2$
$n^2 + n + 2 = n(n + 1) + 2$, which I proved is even in some of the previous assignments, so this proves the claim for $f(n)$.

## 4

Claim: even number $> 2 = p + q \implies$ odd number $> 5 = a + b + c$, where p, q, a, b, c are primes.
Represent every even number $> 2$ as 4k, where $k \geq 1$ and every odd number $> 5$ as $4k + 3$, where $k \in \mathbb{N}$.
Assuming the Goldbach conjecture: $4k = p + q$, where p and q are primes.
Then odd number in the form 4k + 3 = p + q + 3 (by substituting 4k with the assumption), proves the implication.

## 5

Claim: $\sum_1^n 2(n-1) + 1 = n^2$
Using proof by induction:
Base case is true:
n = 1
$2(1-1) + 1 = 1$
Induction hypothesis:
assume it's true for n = k
$\sum_1^k 2(k-1) + 1 = k^2$
Then prove for $n = k + 1$, that
$\sum_1^{k+1} 2(n-1) + 1 = (k+1)^2$
Using the definition of summation:
$\sum_1^{k+1} 2(n-1) + 1 = \sum_1^k (2(n-1) + 1) + 2(k + 1 - 1) + 1$
Using the assumption:
$= k^2 + 2k + 1 = (k+1)^2$, so by principle of math induction the claim is proved.

## 6

Claim: $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
Using proof by induction:
Base case:
n = 1
$1 = \frac{1}{6}1(2)(3) = 1$, so it's true for the base case.
Induction hypothesis:
assume it's true for r = k:
$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$
Then check for $n = k+1$
By definition of summation:
$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$
Substitute the part that we assumed:
$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$
$(k+1)(\frac{1}{6}k(2k+1) + k + 1) = \frac{1}{6}(k+1)(k+2)(2k+3)$, which proves the claim by the principle of math induction.

# Optional

## 7

The sum is:
$1 + 2 + ... + n - 1 + n = N$ Listing the members backwards:
$n + n - 1 + ... + 2 + 1 = N$
both have equal number of members, so summing them would be:
$(n+1) + (n-1+2) + ... + (n-1+2) + (n+1) = 2N$
we have n number of members, so:
$n(n+1) = 2N$ /divide both sides by 2
$\frac{1}{2}n(n+1) = N$

## 8

Claim: Any finite collection of points in the plane, not all collinear, there is a triangle having three points as its vertices, which contains none of the other points in its interior.
The base case is n = 3, where we have 3 points. 3 points are going to form a triangle (let's denote it as $\triangle ABC$) with no points inside.
If we add one more point there are two cases:
1. the point is inside $\triangle ABC$ formed by the previous 3 points.
Then if we connect this point to any of the other 2 points, we'll be able to construct a triangle that does not have any points inside.
2. the point is outside $\triangle ABC$ formed by the previous 3 points.
Then if we connect this point to any of the other 2 points, we'll still have $\triangle ABC$ with no points inside.
Let's assume that it's true for n = k.
Then check for n = k + 1. By adding new point we are going to have one of the cases above: inside a triangle or outside of the triangle, so by principle of math induction the claim is proved.

## 9

a.
Claim: $4^n - 1$ is divisible by 3.
if $4^n - 1$ is divisible by 3, it can be represented as $4^n - 1 = 3k$
Using proof by induction:

Base case:
n = 1
4 - 1 = 3, which is divisble by 3 and that makes the base case true.
Induction hypothesis:
Assume it's true for n = k:
$4^k - 1 = 3r$
Then check for $n = k + 1$
$4^{k+1} - 1 = 4^{k+1} - 4 + 3 = 4(4^k - 1) + 3$
Substituting the assumption:
$43r + 3 = 3(4r + 3)$, which is divisible by 3. By principle of math induction the claim is proved.

b.
Claim: $(\forall n \geq 5)[(n + 1)! > 2^{n+3}]$
Using proof by induction:
Base case:
n = 5
$6! > 2^{5+3} \equiv 360 * 2! > 256$, which makes the base case true.
Induction hypothesis:
Assume it's true for n = k:
$(k + 1)! > 2^{k+3}$
Then check for $n = k + 1$
$(k + 2)! > 2^{k+4}$
$(k + 2)(k + 1)! > 2^{k+3}2$
By the assumption we know that:
$(k + 1)! > 2^{k+3}$
We need to prove that
$k + 2 > 2$, which is true, because k is always $k > 0$

c.
Claim: $\forall n \in \mathbb{N} : \sum_{r=1}^{n} rr! = (n + 1)! - 1$
Using proof by induction:
Base case:
n = 1
$1 * 1! = 2! - 1$, which makes the base case true.
Induction hypothesis:
Assume it's true for n = k:
$\sum_{r=1}^{k} rr! = (k + 1)! - 1$
Then check for $n = k + 1$
$\sum_{r=1}^{k+1} rr! = \sum_{r=1}^{k} rr! + (k + 1)(k + 1)!$
Substituting the assumption:
$(k + 1)! - 1 + (k + 1)(k + 1)! = (k + 1)!(k + 2) - 1 = (k + 2)! - 1$, which is the summation for $n = k + 1$, this proves the claim by math induction principle.