4. Convex optimization problems

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
- vector optimization

کلی ترین شکل بهینه سازی محدب

$$\underbrace{\min_{x \in C} f_0(x)}$$

تابع هدف : f_0 محدب

و C مجموعه ای محدب

$f_0(x)$: تابع هدف محدب

$x \in C$: قید

$x \in \{0, 1\}$: قید غیر محدب

Optimization problem in standard form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

optimal value:

$$
p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}
$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Optimal and locally optimal points

x is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

X_{opt} ناتهی است

a feasible x is **optimal** if $f_0(x) = p^\star$; X_{opt} is the set of optimal points

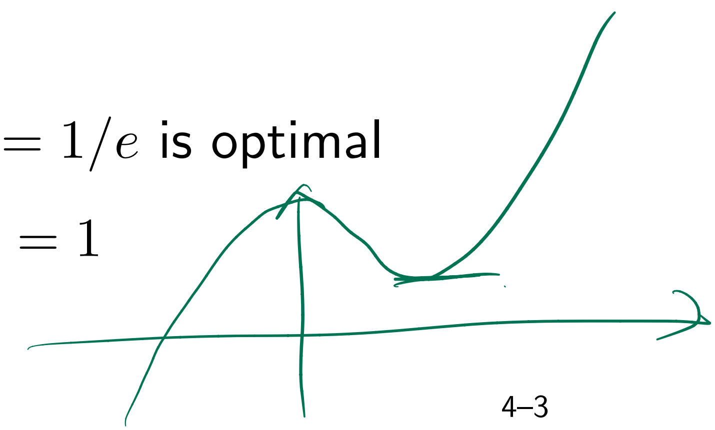
اگر $\inf = \min$

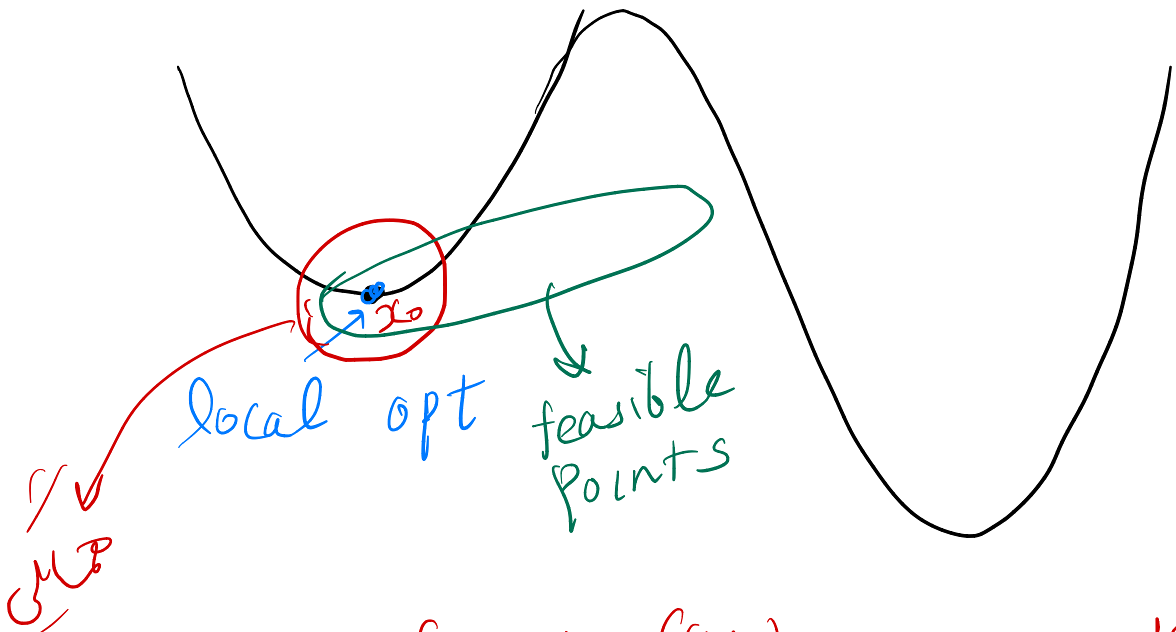
x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$
\begin{array}{ll}
\text{minimize (over } z\text{)} & f_0(z) \\
\text{subject to} & f_i(z) \le 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\
& \|z - x\|_2 \le R
\end{array}
$$

examples (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -\infty$
- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^\star = -\infty$, local optimum at $x = 1$





$B_\varepsilon(x_0)$: $f(x) \geq f(x_0)$ $\longrightarrow$ نقطه مینیمم کلی x_0

$\forall x \in B_\varepsilon(x_0)$

زمانیکه قید داریم

$\forall x \in B_\varepsilon(x_0) \cap \{\text{feasible points}\}$

$f(x) \geq f(x_0)$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the **domain** of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints ($m = p = 0$)

example:

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$
\begin{array}{ll}
\text{find} & x \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$
\begin{array}{ll}
\text{minimize} & 0 \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Convex optimization problem

standard form convex optimization problem

کلی‌ترین حالت

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

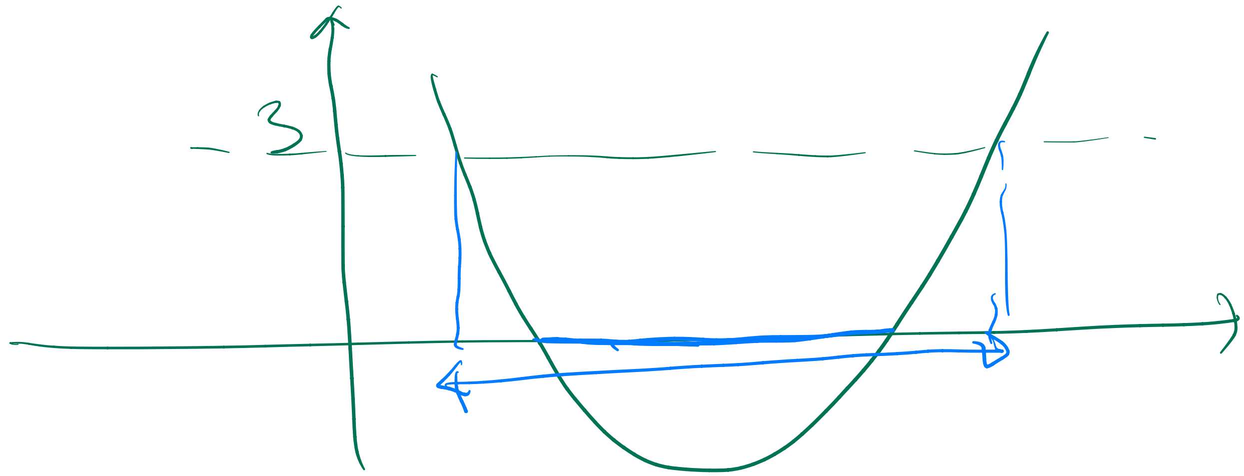
$$\begin{cases} \min f_0(x) \\ x \in C \end{cases}$$

- $f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine
- ~~problem is *quasiconvex* if f_0 is quasiconvex (and $f_1, \ldots, f_m$ convex)~~

often written as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

important property: feasible set of a convex optimization problem is convex



<u>یادآوری</u> :

f محدب : مجموعه زیرتراز سطح c $= \{ x : f(x) \le c \}$ → محدب

example

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

تابع محدب

قید خطی نامساوی (محدب)

قید تساوی خطی

Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

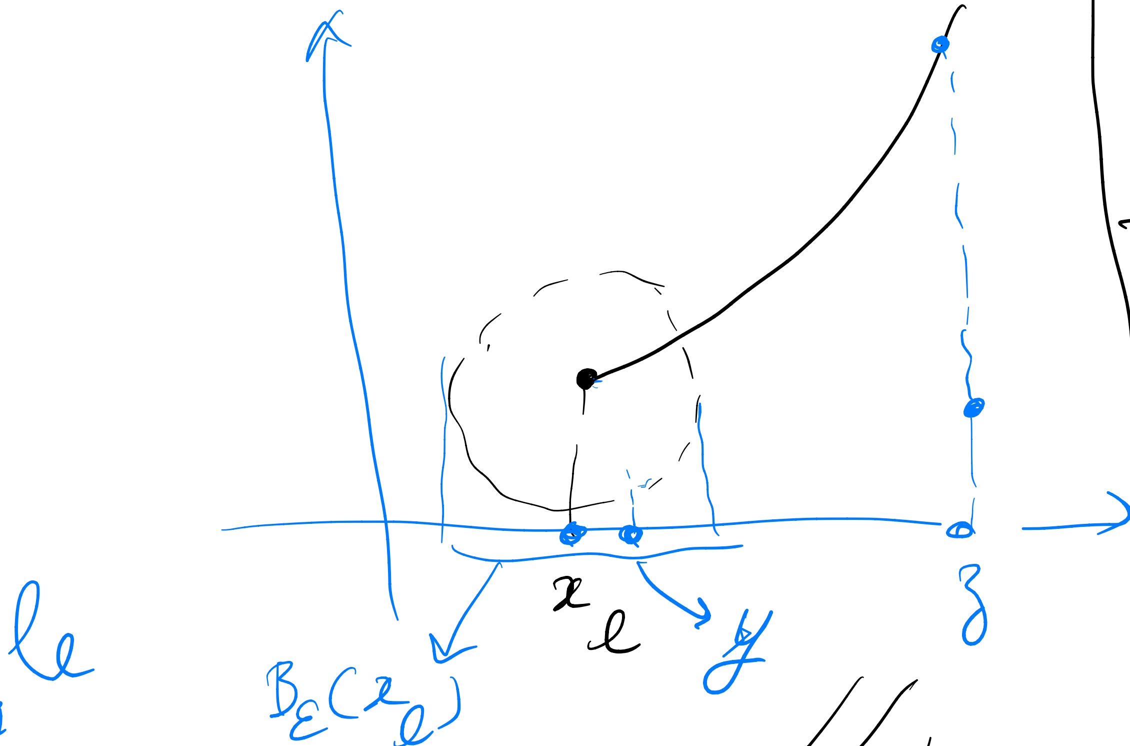
$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and

$$f_0(z) \leq \theta f_0(y) + (1 - \theta) f_0(x) < f_0(x)$$

which contradicts our assumption that x is locally optimal



$$\min_{x \in \mathcal{C}} f(x)$$

$x_0 : \; x_0 \in \mathcal{C}$

$\forall y \in \mathcal{C} \cap B_\varepsilon(x_\ell)$

$f(y) \geq f(x_\ell)$

$\Downarrow$

$\forall z \in \mathcal{C} : \; f(z) \geq f(x_\ell)$

$y = \theta z + (1-\theta) x_\ell$: θ به اندازه کافی کوچک

$$\|y - x_\ell\| = \|\theta(z - x_\ell)\|$$
$$= \theta \|z - x_\ell\| < \varepsilon$$

$$\bar{\theta} \cancel{f(x_\ell)} + \theta f(z) \geq f(\bar{\theta} x_\ell + \theta z) = f(y) \geq \cancel{f(x_\ell)}$$

$$\rightarrow \quad f(z) \geq f(x_\ell)$$

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

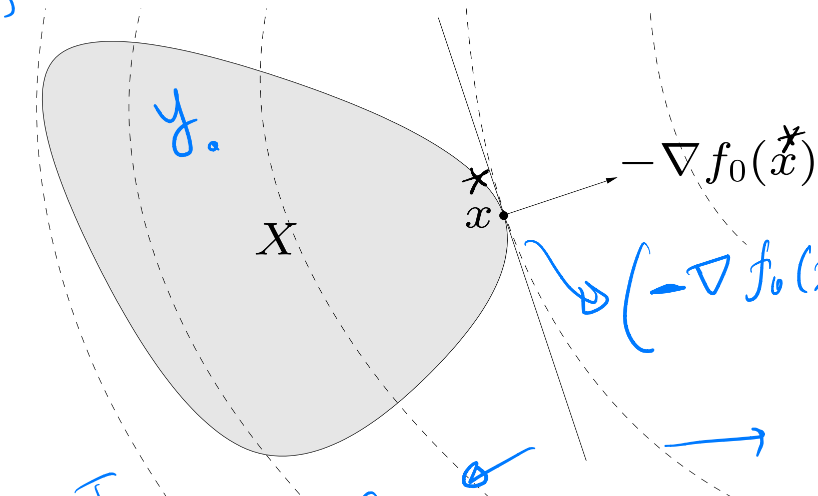
$$\nabla f_0(x)^T(y-x) \geq 0 \quad \text{for all feasible } y$$

$\min \; f_0(x)$

$x \in X$

$N_C(x) = \{n : n^T(y-x) \geq 0 \;\; \forall y \in C\}$

$f_0 = 3 \quad f_0 = 2 \quad f_0 = 1 \quad f_0 = 0$

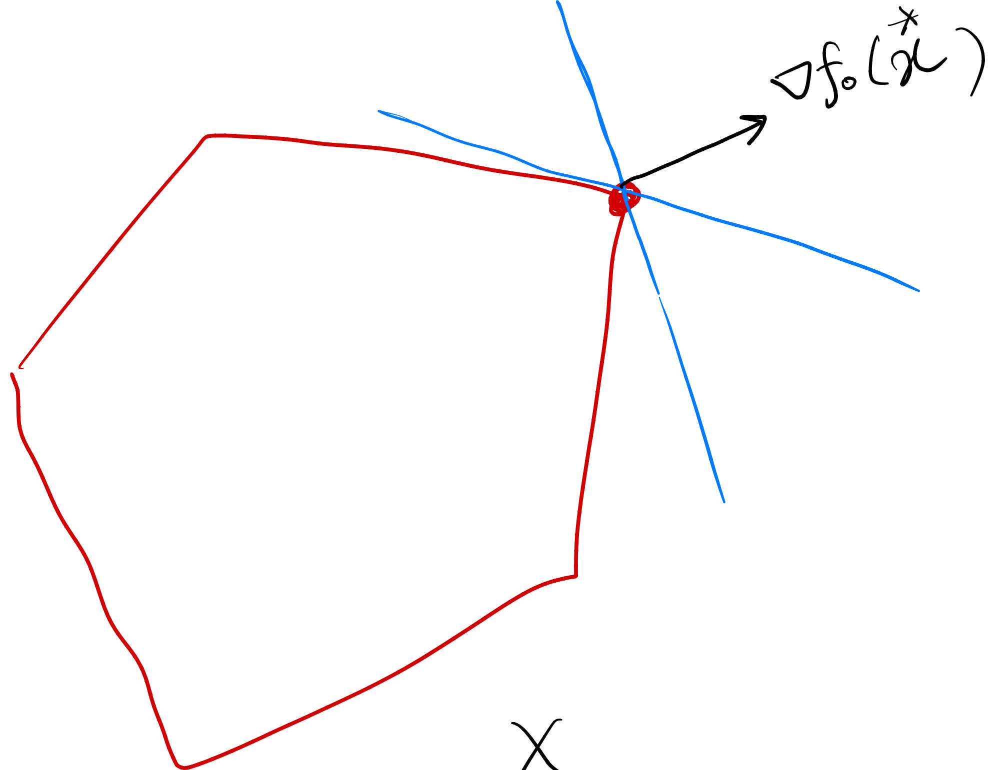


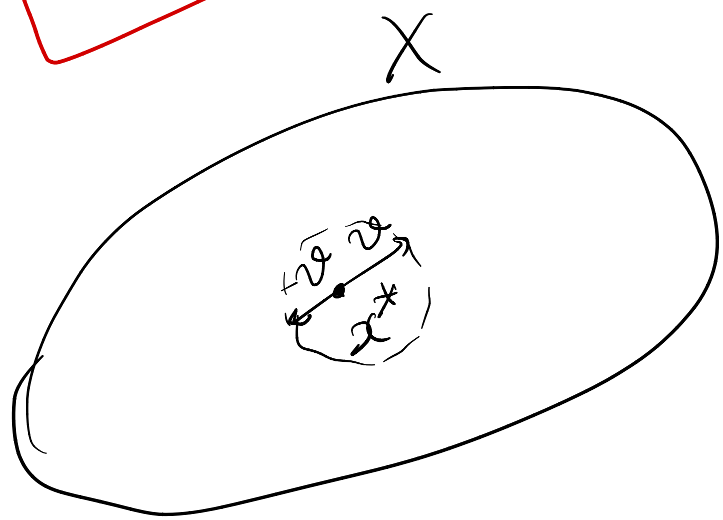
$(-\nabla f_0(x))^T x = c$

$(-\nabla f_0)\, y > c$

$(-\nabla f_0)^T y < c$

if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x





$x^* \in \mathrm{Int}(X)$

$$\langle \nabla f_0(x^*), \overbrace{y - x}^{v} \rangle \geq 0$$

$\forall y$

$$\rightarrow \begin{cases} \langle \nabla f_0(x), v \rangle \geq 0 \\ \langle \nabla f_0(x), -v \rangle \geq 0 \end{cases}$$

$$\rightarrow \langle \nabla f_0(x), v \rangle = 0$$

$$\rightarrow \nabla f_0(x) = 0$$

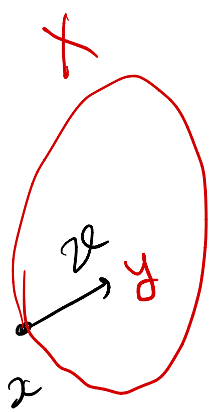
$$\langle \nabla f_0(x), y-x \rangle \geq 0 \qquad \forall y \in X$$

$$\xrightarrow{?} \quad f_0(y) \geq f_0(x) \qquad \forall y \in X$$

$$f_0(y) \geq f_0(x) + \underbrace{\langle \nabla f_0(x), y-x \rangle}_{\geq 0}$$ شرط اول تحدب با مشتق :

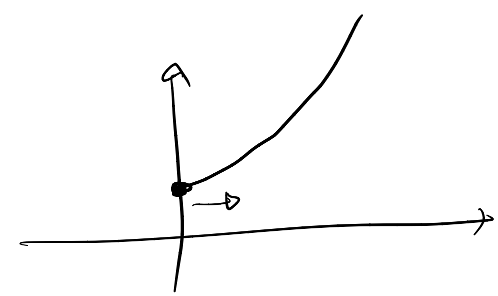
$$\geq f_0(x)$$

$$f_0(y) \geq f_0(x) \implies \quad < \qquad > \geq 0$$



$$g_v(t) = f_0(x^* + tv) \qquad t \geq 0$$

$$\implies \quad g_v(0) \leq g_v(t)$$



$$g'_{v,+}(0) \geq 0$$

$$\Downarrow$$

$$\langle \nabla f_0(x), v \rangle \geq 0$$

$$v = y - x$$

- **unconstrained problem**: x is optimal if and only if

مگر X کل فضا را بگیرد

$$x \in \overset{\text{int}}{\mathbf{dom}}\, f_0, \qquad \nabla f_0(x) = 0$$

int(X)

- **equality constrained problem**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad Ax = b$$

$$\begin{cases} a_1^T x = b_1 \\ \vdots \\ a_m^T x = b_m \end{cases}$$

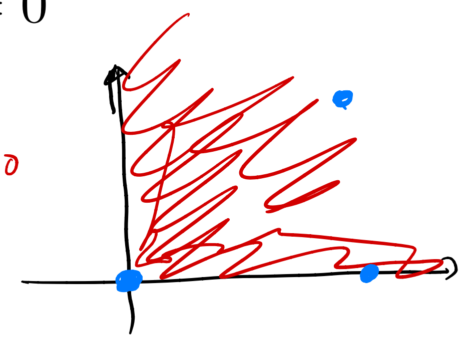
x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\, f_0, \qquad Ax = b, \qquad \nabla f_0(x) + A^T \nu = 0$$

- **minimization over nonnegative orthant**

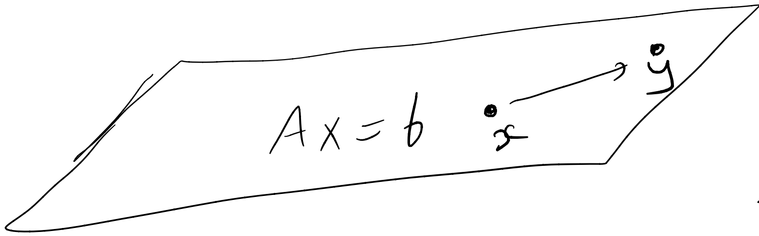
همه درایه‌های x ≥ 0

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad x \succeq 0$$



x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad x \succeq 0, \qquad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$



$$\nabla f(x)^T \cdot (y - x) \geq 0 \; ; \quad \forall y \in Ay = b$$

$A x = b$

$A y = b$ $\rightarrow$ $A(y - x) = 0$

$y - x \in Null$

$$\rightarrow \quad \nabla f(x)^T \cdot z = 0$$

$\forall z \in Null(A)$

or $\forall z : Az = 0$

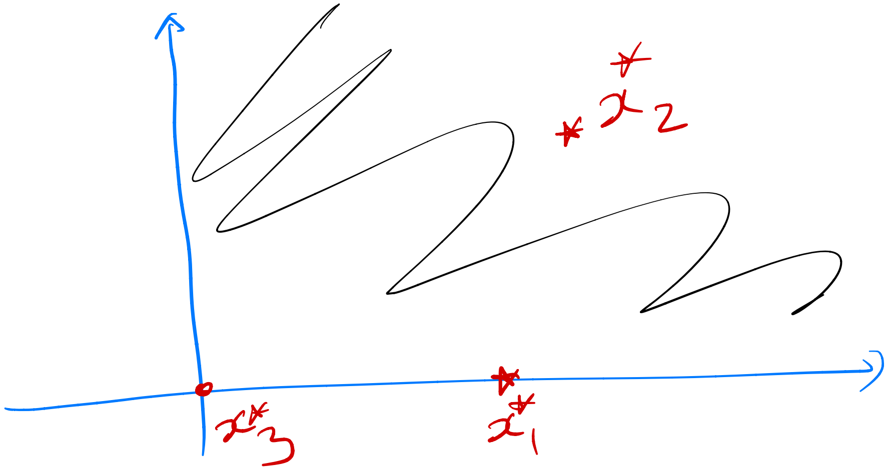
$$Null(A) = (\text{فضای سطری } A)^{\perp} \longleftrightarrow Null^{\perp} = \text{فضای سطری } A$$

$$\Rightarrow \quad \nabla f(x) \in \text{فضای سطری } A$$

$$\Rightarrow \quad \nabla f(x) = \underbrace{-A^T v}_{\text{ترکیب خطی از سطرهای } A}$$

وجود دارد بردار v به طوری که ;

$$\nabla f(x) = v_1 (\text{سطر اول})^T + v_2 (\text{سطر دوم})^T + $$



$$\langle \nabla f_0(x), y-x \rangle \geq 0$$

$\forall y \in$ کنج مثبت

اگر x_2^* مینیمم ; $\nabla f_0(x_2) = 0$

اگر x_1^* مینیمم باشد محور x

$$\langle \nabla f_0(x_1), y-x_1 \rangle \geq 0$$

در راستای محور x حرکت $\quad \langle \nabla f_0(x_1^*), e_1 \rangle = 0$

y ـ $\quad \langle \nabla f_0(x^*), e_2 \rangle \geq 0$

$$\frac{\partial f_0}{\partial x_1}(x_1^*) = 0 \; ; \quad \frac{\partial f_0(x_1^*)}{\partial x_2} \geq 0$$

در نقطه x_3^*

$$\frac{\partial f_0}{\partial x_i} \geq 0 \quad ; \; i = 1, 2$$

Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **eliminating equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  is equivalent to

$$\begin{array}{ll} \text{minimize (over } z\text{)} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m \end{array}$$

z از فضای با بعد کرنل A
که می‌تواند فضای کوچکتر از تعداد x باشد

  where F and x_0 are such that

$$Ax = b \quad \Longleftrightarrow \quad x = Fz + x_0 \text{ for some } z$$

$$Ax_0 = b \longrightarrow x = x_0 + v \; ; \quad v \in Null(A)$$

$$F = \left(f_1 \middle| \cdots \middle| f_k \right) \; ; \quad Null(A) = span\{f_1, \cdots, f_k\}$$

$$\longrightarrow \quad v = Fz$$

- **introducing equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(A_0 x + b_0) \\ \text{subject to} & f_i(A_i x + b_i) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x,\ y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\ & y_i = A_i x + b_i, \quad i = 0, 1, \ldots, m \end{array}$$

- **introducing slack variables for linear inequalities**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x,\ \vec{s}) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\ & s_i \geq 0, \quad i = 1, \ldots m \end{array}$$

منظور از معادل بودن :

۱- مقدار مینیمم هر دو مسئله یکی باشد

۲- جواب‌ها در دو مسئله مرتبط باشند.

P_1

$$\min_{x} \quad f_0(x)$$
$$a^T x \leq b$$

P_2

$$\min_{x,s} \quad f_0(x)$$
$$a^T x + s = b$$
$$s \geq 0$$

$x \in P_1$ از مجموعه شدنی $\longrightarrow$ $\exists s \geq 0$ بطوریکه $(x, s) \in P_2$

$x \in P_1 \longleftarrow (x, s) \in P_2$

- **epigraph form**: standard form convex problem is equivalent to

$$\begin{array}{ll} \text{minimize (over } x,\ t) & t \\ \text{subject to} & f_0(x) - t \le 0 \\ & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

min $f_0(x)$
$f_i(x) \le 0$
$Ax = b$

- **minimizing over some variables**

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \le 0, \quad i = 1, \ldots, m \end{array}$$

$\exists x_2 \; f_0(x_1^*, x_2^*) > f_0(x_1^*, x_2)$

is equivalent to (Convex opt.)

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \le 0, \quad i = 1, \ldots, m \end{array}$$

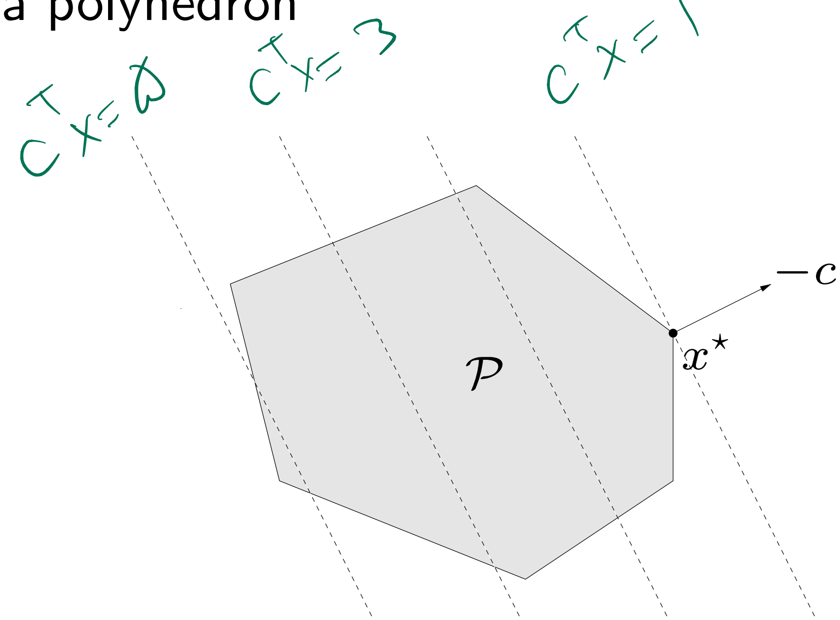
where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$ $\left(\min_{x_2 \in \mathcal{C}} f(x_1, x_2) = \tilde{f}(x_1) : \ldots \right)$

Linear program (LP)

$$
\begin{array}{ll}
\text{minimize} & c^T x \cancel{+ d} \\
\text{subject to} & Gx \preceq h \\
 & Ax = b
\end{array}
\longrightarrow
\left\{
\begin{bmatrix} g_1^T \\ \vdots \\ g_k^T \end{bmatrix} x \preceq \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_k \end{bmatrix}
\right.
$$

$$
g_1^T x \le h_1 \quad \ldots \quad g_k^T x \le h_k
$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Examples

diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\ldots,m}(a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

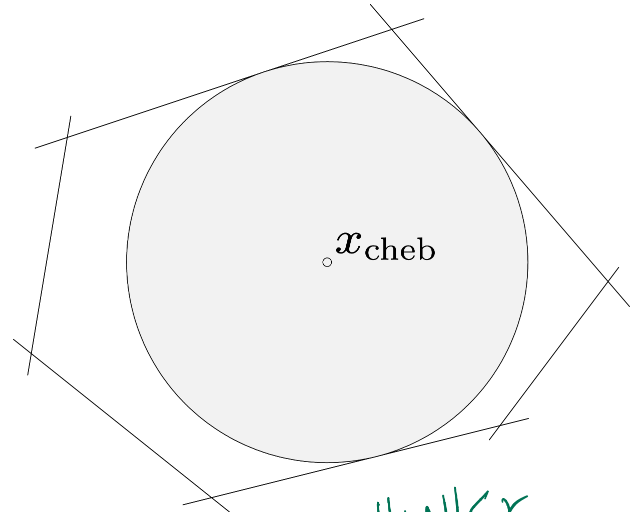
x, t

Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, m\}$$

is center of largest inscribed ball



$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\}$$

(x_c, r)

$\|u\| \le r$

c.s.

$\langle u, a_i \rangle \le \|u\| \, \|a_i\|$

$\le r \, \|a_i\|$

- $a_i^T x \le b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) \mid \|u\|_2 \le r\} = a_i^T x_c + r\|a_i\|_2 \le b_i$$

- hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

Linear-fractional program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

linear-fractional program

$$f_0(x) = \frac{(c^Tx + d)z}{(e^Tx + f)z}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e^Tx + f > 0\}$$

- ~~a quasiconvex optimization problem; can be solved by bisection~~
- ~~also~~ equivalent to the LP (variables y, z)

$$\begin{array}{ll} \text{minimize} & c^Ty + dz \\ \text{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^Ty + fz = 1 \\ & z \geq 0 \end{array}$$

تغيير متغير

$y = xz$

generalized linear-fractional program

$$f_0(x) = \max_{i=1,\ldots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e_i^T x + f_i > 0,\ i = 1, \ldots, r\}$$

a quasiconvex optimization problem; can be solved by bisection

example: Von Neumann model of a growing economy

$$\begin{array}{ll} \text{maximize (over } x,\ x^+) & \min_{i=1,\ldots,n} x_i^+/x_i \\ \text{subject to} & x^+ \succeq 0, \quad Bx^+ \preceq Ax \end{array}$$

- $x, x^+ \in \mathbf{R}^n$: activity levels of n sectors, in current and next period
- $(Ax)_i$, $(Bx^+)_i$: produced, resp. consumed, amounts of good i
- x_i^+/x_i: growth rate of sector i

allocate activity to maximize growth rate of slowest growing sector

Quadratic program (QP)

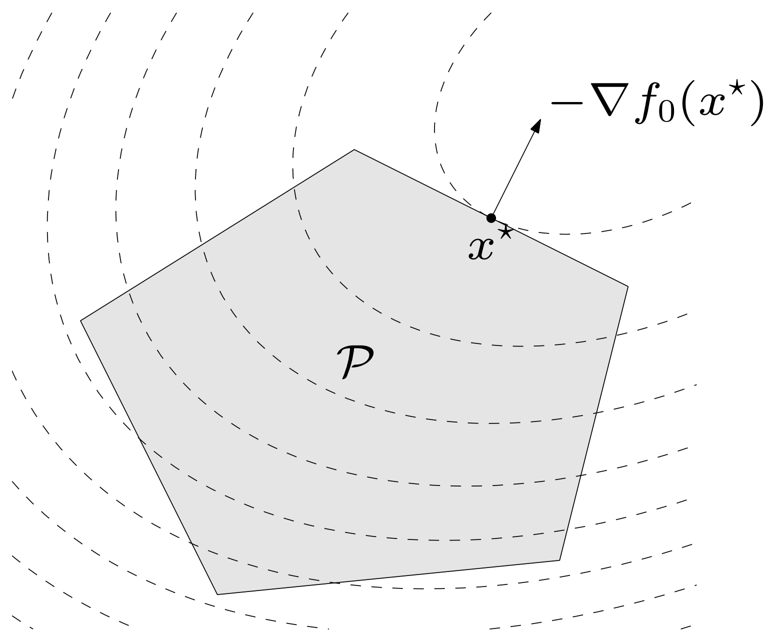
$x = y + P^{-1}q$

$\frac{1}{2}\|P^{-1}y\| \le$ عدد ثابت

P

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x + r \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- $P \in \mathbf{S}_+^n$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



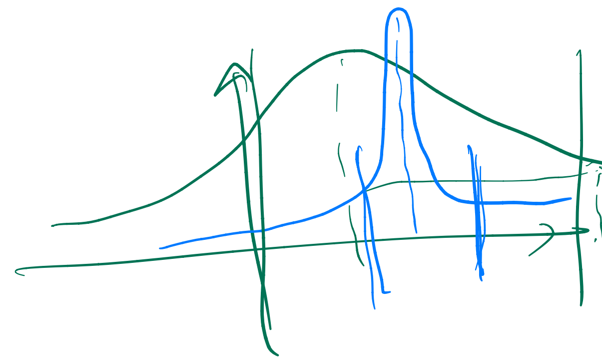
Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

$= x^T(A^TA)x - 2b^TAx + b^Tb$

- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, *e.g.*, $l \preceq x \preceq u$



linear program with random cost

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

$= \sum c_i x_i$

- c is random vector with mean $\bar{c}$ and covariance Σ
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

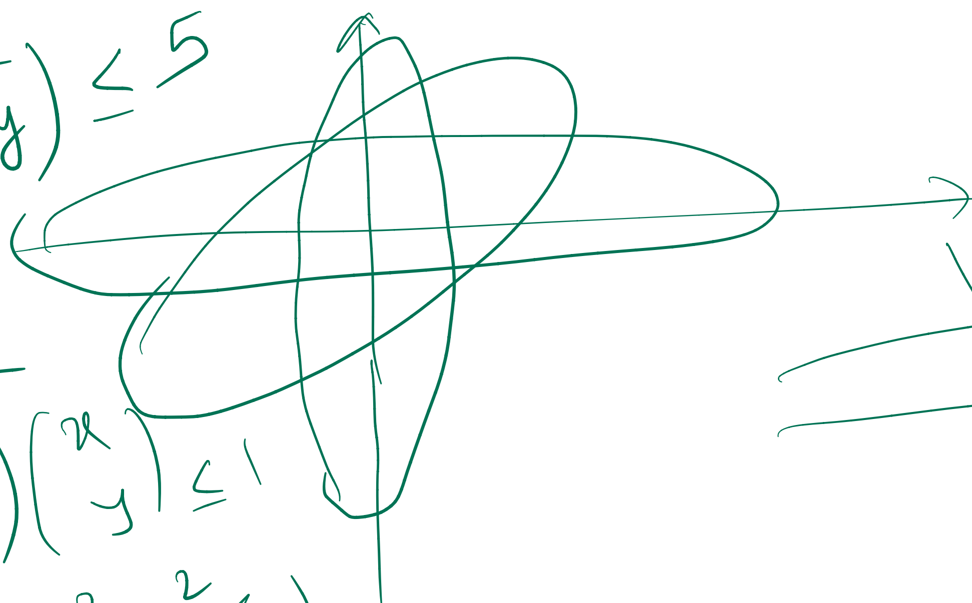
Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of m ellipsoids and an affine set

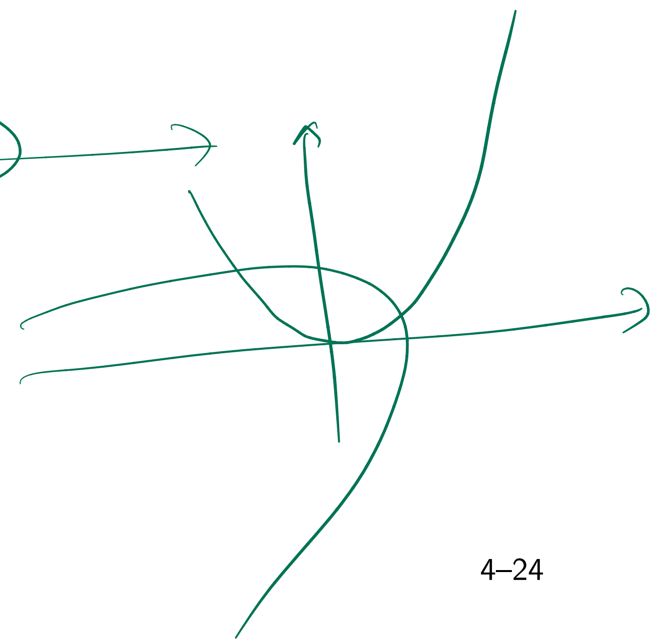
$$\begin{pmatrix}x\\y\end{pmatrix}^T \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \le 5$$

$$2x^2+3y^2 \le 5$$

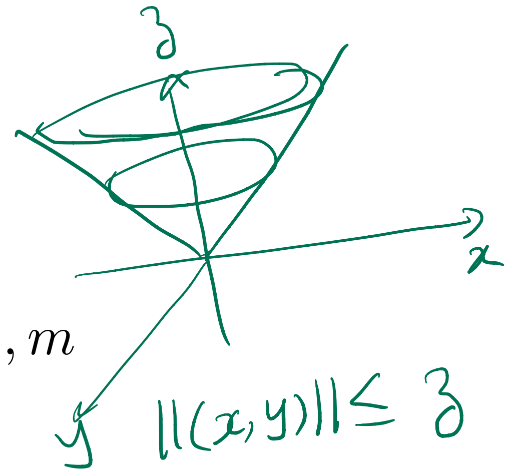


$$\begin{pmatrix}x\\y\end{pmatrix}^T \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \le 1$$

$$x^2-y^2 \le 1$$



Second-order cone programming



$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\
& Fx = g
\end{array}
$$

$(A_i \in \mathbf{R}^{n_i \times n}$, $F \in \mathbf{R}^{p \times n})$

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbf{R}^{n_i+1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP

  $\|A_i x + b_i\| \le d_i$

- more general than QCQP and LP

Constraint : $x_1^2 \le x_2$ تمرین

$x = (x_1, x_2)$

$$x_1^2 - x_2 \le 0$$

$$\Downarrow$$

$$\left\| A\begin{bmatrix}x_1\\x_2\end{bmatrix} + b \right\| \le c^T\begin{bmatrix}x_1\\x_2\end{bmatrix} + d$$

$$(x_1 \;\; x_2)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + (0 \;\; -1)\begin{pmatrix}x_1\\x_2\end{pmatrix} \le 0$$

$$\Longleftarrow \left\| A\begin{bmatrix}x_1\\x_2\end{bmatrix} + b \right\| \le$$

$$\longrightarrow \quad x_2 = \left(x_2 + \frac{1}{4}\right)^2 - \left(x_2 - \frac{1}{4}\right)^2$$

$$\underbrace{x_1^2 + \left(x_2 - \frac{1}{4}\right)^2}_{\left\| \begin{bmatrix}x_1\\ x_2 - \frac{1}{4}\end{bmatrix} \right\|} \le \left(x_2 + \frac{1}{4}\right)^2$$

$$\left\| \begin{bmatrix}x_1\\ x_2 - \frac{1}{4}\end{bmatrix} \right\| \le x_2 + \frac{1}{4}$$

Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

there can be uncertainty in c, a_i, b_i

two common approaches to handling uncertainty (in a_i, for simplicity)

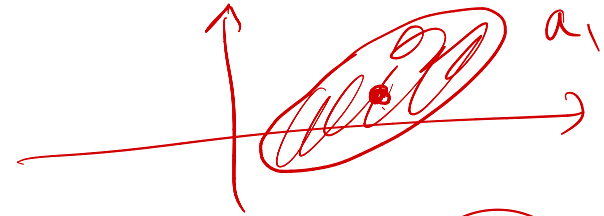
- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

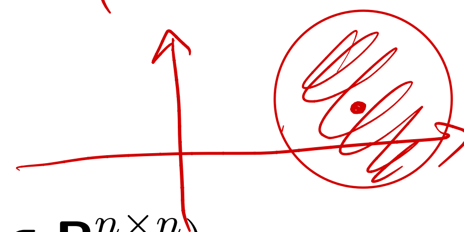
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

deterministic approach via SOCP





- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\} \qquad (\bar{a}_i \in \mathbf{R}^n, \quad P_i \in \mathbf{R}^{n \times n})$$

center is $\bar{a}_i$, semi-axes determined by singular values/vectors of P_i

$a_1 =$ [مقدار ... ، مقدار ...]

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

$a_2 =$ [...]

is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

(follows from $\sup_{\|u\|_2 \le 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

$u^T P_i^T x = \langle u_i, P_i^T x \rangle \le \|u_i\| \, \|P_i^T x\|$

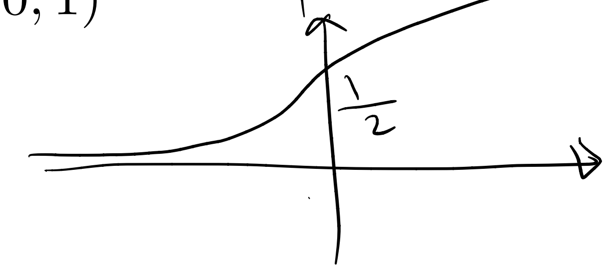
stochastic approach via SOCP

- assume a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)
- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence

$$\mathbf{prob}(\underbrace{a_i^T x}_{z} \leq b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

$$\frac{z - \mu}{\sigma} \qquad \sqrt{x^T \Sigma_i x} \qquad \Phi$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is CDF of $\mathcal{N}(0,1)$



- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m, \end{array}$$

with $\eta \geq 1/2$, is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

$$\longrightarrow \quad \frac{b_i - a_i^T x}{\|\ \ \|} \leq \varphi^{-1}(\eta)$$

$$Z = a_i^T x \quad ; \quad \mathbb{E}[\,---\,]$$

بدون کم شدن از کلیت

$$a_i \sim \mathcal{N}(0, \Sigma)$$

$$\mathrm{Var}(a_i^T x) = \mathbb{E}\left[(a_i^T x)^2\right]$$

$$= \mathbb{E}\left[(x^T a_i)(a_i^T x)\right]$$

$$= x^T \underbrace{\mathbb{E}\left[a_i a_i^T\right]}_{\Sigma_i} x$$

$$\mathbb{E}\left(\begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} (a_{i1} \; \cdots \; a_{in})\right) = \mathbb{E}\begin{pmatrix} & & & \\ & & a_{i,k} a_{i,l} & \\ & & & \end{pmatrix} \to \mathbb{E}[a_{i,k} a_{i,l}]$$

$$k$$

$$= \Sigma$$

$$x^T \Sigma x = \underbrace{x^T \Sigma^{\frac{1}{2}}}_{(\Sigma^{1/2} x)^T} \Sigma^{\frac{1}{2}} x = \|\Sigma^{1/2} x\|^2$$

Geometric programming

monomial function

$$f(x) = cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

with $c > 0$; exponent a_i can be any real number

posynomial function: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

geometric program (GP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \\ & h_i(x) = 1, \quad i = 1, \ldots, p \end{array}$$

with f_i posynomial, h_i monomial

$y_i = \log x_i$

$\mathbb{R}_{++} \longrightarrow \mathbb{R}$

Geometric program in convex form

change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(x) = c x_1^{a_1} \cdots x_n^{a_n}$ transforms to

$c\, e^{a_1 y_1} e^{a_2 y_2} \cdots$

$$\log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \qquad (b = \log c)$$

- posynomial $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \log \left(\sum_{k=1}^K e^{a_k^T y + b_k} \right) \qquad (b_k = \log c_k)$$

- geometric program transforms to convex problem

$$\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^K \exp(a_{0k}^T y + b_{0k}) \right) \\ \text{subject to} & \log \left(\sum_{k=1}^K \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \\ & Gy + d = 0 \end{array}$$

مثال: (کمینه کردن نرم فروبنیوس)

$$y = Ax$$

A: نمایش استاندارد نگاشت خطی

$$A = \begin{pmatrix} a_{11} & & \\ & a_{ij} & \\ & & a_{nn} \end{pmatrix}$$

; $\|A\|_{F,HS} = \left(\sum a_{ij}^2\right)^{\frac{1}{2}}$

$$\tilde{x}_i = d_i x_i$$

$$\tilde{y}_i = d_i y_i$$

$$y = Ax$$

$$\vec{\tilde{y}} = \begin{pmatrix} d_1 y_1 \\ \vdots \\ d_n y_n \end{pmatrix} = \underbrace{\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}}_{D} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = D\vec{y}$$

$$\vec{\tilde{x}} = D\vec{x} \longrightarrow x = D^{-1}\tilde{x}$$

$$\tilde{y} = Dy = DAx = (DAD^{-1})\tilde{x}$$

$$\min \|DAD^{-1}\|_2 = \min_{\{(d_i)\}} \sum \frac{d_i^2}{d_j^2} A_{ij}$$

D: ماتریس قطری با درایه

$$\|DAD^{-1}\|_2^2 = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \begin{pmatrix} & & \end{pmatrix} \begin{pmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_n} \end{pmatrix}$$

$$= \left\| \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} & & \\ & \boxed{\frac{d_i}{d_j} A_{ij}} & \\ & & \end{pmatrix} \end{array} \right\|_2^2 = \sum_{i,j} \frac{d_i^2}{d_j^2} A_{ij}$$

Minimizing spectral radius of nonnegative matrix

Perron-Frobenius eigenvalue $\lambda_{\mathrm{pf}}(A)$

- exists for (elementwise) positive $A \in \mathbf{R}^{n\times n}$
- a real, positive eigenvalue of A, equal to spectral radius $\max_i |\lambda_i(A)|$
- determines asymptotic growth (decay) rate of A^k: $A^k \sim \lambda_{\mathrm{pf}}^k$ as $k \to \infty$
- alternative characterization: $\lambda_{\mathrm{pf}}(A) = \inf\{\lambda \mid Av \preceq \lambda v \text{ for some } v \succ 0\}$

minimizing spectral radius of matrix of posynomials

- minimize $\lambda_{\mathrm{pf}}(A(x))$, where the elements $A(x)_{ij}$ are posynomials of x
- equivalent geometric program:

$$
\begin{array}{ll}
\text{minimize} & \lambda \\
\text{subject to} & \sum_{j=1}^{n} A(x)_{ij} v_j/(\lambda v_i) \leq 1, \quad i = 1, \ldots, n
\end{array}
$$

variables λ, v, x

Semidefinite program (SDP)

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\
& Ax = b
\end{array}
$$

with $F_i,\ G \in \mathbf{S}^k$

LP

$$
\begin{array}{l}
\min \ c^T x \\
Ax = b \\
Gx \leq \vec{d}
\end{array}
$$

$$
\Big(\quad\Big)\Big(\ \Big) \leq \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix}
$$

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

$$
x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \qquad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0
$$

is equivalent to single LMI

$$
x_1 \underbrace{\begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix}}_{F_1} + x_2 \underbrace{\begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix}}_{F_2} + \cdots + x_n \underbrace{\begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix}}_{F_n} + \underbrace{\begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix}}_{G} \preceq 0
$$

$$\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \succeq 0$$

$$\iff M \succeq 0 \ ; \ N \succeq 0$$

$$\begin{pmatrix} u^T & v^T \end{pmatrix} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \succeq 0 \quad \forall u, v$$

$$u^T M u + v^T N v \succeq 0$$

$$\iff \begin{matrix} u^T M u \succeq 0 \\ v^T N v \succeq 0 \end{matrix}$$

LP and SOCP as SDP

LP and equivalent SDP

LP: minimize $c^T x$
subject to $Ax \preceq b$

$Ax - b \preceq 0$

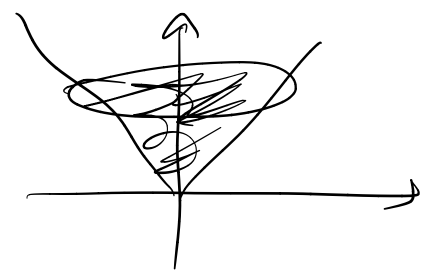
SDP: minimize $c^T x$
subject to $\mathbf{diag}(Ax - b) \preceq 0$

$v \leq 0 \iff \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{pmatrix} \preceq 0$

(note different interpretation of generalized inequality $\preceq$)

SOCP and equivalent SDP

SOCP: minimize $f^T x$
subject to $\|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m$



SDP: minimize $f^T x$
subject to $\begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m$

$$(c_i^T x + d_i)\begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} + \begin{pmatrix} 0 & A_i x + b_i \\ (\quad)^T & 0 \end{pmatrix}$$

$\rightarrow \quad \|u\| \leq v \iff \begin{bmatrix} v & u^T \\ u & vI \end{bmatrix} \succeq 0$

Schu

$\rightarrow \quad \begin{bmatrix} A & B^T \\ B_{k\times l} & C_{k\times k} \end{bmatrix} \succeq 0 \iff C \succeq 0, \quad A - B^T C^{-1} B \succeq 0$

$\rightarrow \quad v \succeq 0 \; ; \; vI - \frac{1}{v} u u^T \succeq 0$

$v^2 I \succeq \left(u u^T = \|u\|^2 \cdot \tilde{u}\tilde{u}^T \right)$

$C \leftarrow vI \qquad vI \succeq 0$

$\rightarrow \quad A \leftarrow v$

$v - \frac{1}{v} u^T u \succeq 0$

$\rightarrow \quad v^2 \geq u^T u = \|u\|^2$

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{S}^k$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

$$\left| \begin{array}{ll} \min & t \\ \text{s.t.} & \lambda_{\max}(A) \leq t \end{array} \right.$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$
- follows from

$$\lambda_{\max}(A) \leq t \iff A \preceq tI$$

$$Q \begin{pmatrix} \lambda_{\max} & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} Q^T \preceq Q \begin{pmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{pmatrix} Q^T = \begin{pmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{pmatrix}$$

$$A \leq B$$

$$\longrightarrow \quad C^T A C \leq C^T B C$$

SDP - شکل استاندارد

SDP - شکل ماتریسی

$$\min_{X} \; Tr[C^T X]$$
$$Tr[A_i X] = b_i$$
$$X \succeq 0 .$$

LP

$$\min_{\vec{x}} \; \vec{c}^{\,T} \vec{x}$$
$$Ax = b$$
$$x \geq 0$$

مثال: بزرگترین مقدار ویژه برای ماتریس متقارن

$$\lambda_{max}(A) = \max_{u:\|u\|=1} u^T A u$$

$$u^T A u = \mathrm{Tr}[u^T A u] = \mathrm{Tr}[A u u^T]$$

$$\lambda_{max}(A) = \max_{u:\|u\|=1} \mathrm{Tr}[A \underbrace{u u^T}_{U}]$$

$$= \max_{U \succeq 0} \mathrm{Tr}[A U] \; ; \; \mathrm{rank}(U) = 1 \; ; \; \mathrm{Tr}[U] = \|u\|^2 = 1$$

مشکل

non-convex

$$\overset{\text{relax}}{\leq} \max_{U \succeq 0 \; ; \; \mathrm{Tr}[U]=1} \mathrm{Tr}[A U]$$

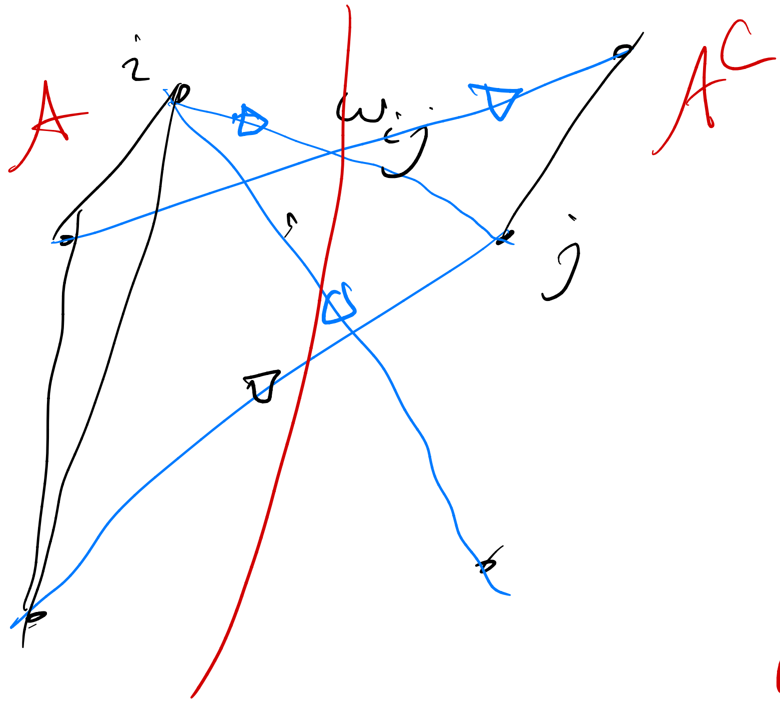
* نکته جالب :

$$\lambda_{max}(A) = \max_{U} \; Tr[AU] \quad : \quad U \succeq 0, \quad Tr[U] = 1$$

(اثبات : تمرین - راهنمایی قطری کردن U)

$$U = \sum \lambda_i u_i u_i^T$$

$$Tr[U] = 1 \; : \; \rightarrow \boxed{\sum \lambda_i = 1}$$



$w_{ii} = 0$

• مثال : گراف

MAX-CUT

MAX-CUT : $\max\limits_{\text{کات}}$ (مجموع وزن یال های در کات)

$Cut = (A, A^c)$; $A \subseteq V$

• حرکات :

$$Cut \longleftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_v \end{pmatrix} = \begin{cases} x_i = 1 & ; i \in A \\ x_i = -1 & ; i \in A^c \end{cases}$$

$$\frac{1}{4} \sum_{i,j} (1 - x_i x_j) w_{ij}$$

$\rightarrow$ $\max\limits_{(x_1, \dots, x_v) : x_i = \pm 1}$

$$\sum x_i x_j w_{ij}$$

$\Rightarrow$ $\min\limits_{(x_1, \dots, x_v) : x_i = \pm 1}$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_v \end{pmatrix} (x_1 \; \cdots \; x_v) = \begin{pmatrix} x_i^2 & & \\ & & \\ & & x_i x_j \end{pmatrix}$$

X قطری $= 1$ $\qquad W = (w_{ij})$

$$\min \quad \mathrm{Tr}[X W]$$

$\mathrm{rank}(X) = 1$

$X_{ii} = 1$

$X \succcurlyeq 0$

ریلکس شده

relax: $\quad \min \quad \mathrm{Tr}[XW]$

$X_{ii} = 1$

$X \succcurlyeq 0$

$$\longrightarrow \quad \max \quad \frac{1}{4}\sum w_{ij} - \frac{1}{4}\mathrm{Tr}[XW]$$

$X_{ii} = 1$

$X \succcurlyeq 0$

کران ۰٫۸۷ برای تعداد MAX-CUT

LIN Relax $\leq$ MAX_CUT $\leq$ Relax