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$$I(x) = \int_0^\infty n(1-x)^{n-1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{n-\mu}{\sigma}\right)^2} dn$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty n\, e^{(n-1)\log(1-x)} e^{-\frac{1}{2\sigma^2}(n-\mu)^2} dn$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty n\, e^{u(n)} dn$$

$$u(n) = -\frac{1}{2\sigma^2} n^2 + n\left[\log(1-x) + \frac{\mu}{\sigma^2}\right] - \left[\log(1-x) + \frac{\mu^2}{2\sigma^2}\right]$$

$$u(n) = -a n^2 + b(x) n + c(x)$$

with $a = \frac{1}{2\sigma^2}$

$$b(x) = \log(1-x) + \frac{\mu}{\sigma^2}$$

$$c(x) = \log(1-x) + \frac{\mu^2}{2\sigma^2}$$

$$I(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty n\, e^{-an^2+bn-c} dn$$

$$= \frac{e^{-c(x)}}{\sigma\sqrt{2\pi}} \underbrace{\int_0^\infty n\, e^{-an^2+bn} dn}_{J}$$

Solving for $J = \int_0^\infty n\, e^{bn-an^2} dn$

Write n as $\left(-\frac{1}{2a}(b-2an) + \frac{b}{2a}\right)$ and split

$$J = \int_0^\infty n\, e^{-an^2+bn} dn = \frac{b}{2a} \underbrace{\int_0^\infty e^{+bn-an^2} dn}_{L} + \frac{1}{2a} \underbrace{\int_0^\infty (2an-b)\, e^{bn-an^2} dn}_{K}$$

Solving for $K = \int_0^\infty (2an-b)\, e^{bn-an^2} dn$

substitution $u = bn - an^2 \quad du = (b-2an)dn$

$$K = -\int_0^\infty e^u du = -e^u\Big]_0^\infty = -e^{bn-an^2}\Big]_0^\infty = -1$$

Solving for $L = \int_0^\infty e^{bx - ax^2} dx$

Completing the square

$$L = \int_0^\infty e^{\frac{b^2}{4a} - \left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)^2} dx$$

Substitute $u = \frac{2ax - b}{2\sqrt{a}}$ $\quad du = \sqrt{a}\, dx$

$$L = \frac{e^{\frac{b^2}{4a}}}{\sqrt{a}} \frac{\sqrt{\pi}}{2} \int_0^\infty \frac{2e^{-u^2}}{\sqrt{\pi}} du$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

donc $L = \frac{\sqrt{\pi}\, e^{\frac{b^2}{4a}}}{2\sqrt{a}} \underbrace{\left[\text{erf}(x)\right]_0^\infty}_{(1-0)} = \frac{\sqrt{\pi}\, e^{b^2/4a}}{2\sqrt{a}}$

Put everything together

$$J = \frac{b}{2a} L + \frac{1}{2a} K = \frac{b}{2a} \frac{\sqrt{\pi}}{2} \frac{e^{b^2/4a}}{\sqrt{a}} - \frac{1}{2a}$$

$$I(x) = \frac{e^{-c}}{\sigma\sqrt{2\pi}} J = \frac{e^{-c}}{\sigma\sqrt{2\pi}} \frac{1}{2a} \left[b \frac{\sqrt{\pi}}{2} e^{b^2/4a} - 1 \right]$$

b and c are functions of x

$$e^{-c} = e^{-\log(1-x)} e^{-\frac{\mu^2}{2\sigma^2}} = \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{1-x}$$

$$\frac{1}{2a\sqrt{2\pi}\,\sigma} = \frac{\sigma}{\sqrt{2\pi}}$$

$$\frac{e^{-c}}{2a\sigma\sqrt{2\pi}} = \frac{k_1}{1-x} \quad \text{with } \boxed{k_1 = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}}}$$

$$\frac{b^2}{4a} = \left[\log(1-x) + \frac{\mu}{\sigma^2}\right]^2 \times \frac{\sigma^2}{2} = \frac{\sigma^2}{2}\left[\log^2(1-x) + \frac{\mu^2}{\sigma^2} + \frac{2\mu}{\sigma^2} \log(1-x)\right]$$

$$= \frac{\sigma^2}{2} \log^2(1-x) + \frac{\mu^2}{2} + \mu \log(1-x)$$

$$e^{\frac{b^2}{4a}} = e^{\frac{\mu^2}{2}} \times (1-x)^\mu \times e^{\frac{\sigma^2}{2}\log^2(1-x)}$$

$$e^{\frac{\sigma^2}{2}\log^2(1-x)} = e^{\left[\log(1-x)\sigma/\sqrt{2}\right]^2} = \left[e^{\log(1-x)\sigma/\sqrt{2}}\right]^{\log(1-x)\sigma/\sqrt{2}} = \left[(1-x)^{\sigma/\sqrt{2}}\right]^{\log(1-x)\sigma/\sqrt{2}}$$

$$= (1-x)^{\frac{\sigma}{\sqrt{2}} \log(1-x)^{\sigma/\sqrt{2}}}$$

$$e^{\frac{b^2}{4a}} = e^{\frac{\mu^2}{2}} (1-x)^{\mu + \frac{\sigma}{\sqrt{2}} \log(1-x)^{\sigma/\sqrt{2}}}$$

$$\frac{\sqrt{\pi}}{2} b e^{\frac{b^2}{4a}} = \frac{\sqrt{\pi}}{2}\left[\log(1-x) + \frac{\mu}{\sigma^2}\right] e^{\frac{\mu^2}{2}} (1-x)^{\mu + \frac{\sigma}{\sqrt{2}} \log(1-x)^{\sigma/\sqrt{2}}}$$

Thus

$$I(x) = \frac{k_1}{(x-1)}\left[\frac{\sqrt{\pi}}{2}\left[\log(1-x) + \frac{\mu}{\sigma^2}\right] e^{\frac{\mu^2}{2}} (1-x)^{\mu + \frac{\sigma}{\sqrt{2}} \log(1-x)^{\sigma/\sqrt{2}}} - 1\right]$$

$$I(x) = k_1 \frac{\sqrt{\pi}}{2} e^{\frac{\mu^2}{2}} \left(\log(1-x) + \frac{\mu}{\sigma^2}\right) (1-x)^{\mu - 1 + \frac{\sigma}{\sqrt{2}} \log(1-x)^{\sigma/\sqrt{2}}} - \frac{k_1}{1-x}$$

$$\boxed{I(x) = k_2 (\log(1-x) + k_3) (1-x)^{\mu - 1 + \frac{\sigma}{\sqrt{2}} \log(1-x)^{\sigma/\sqrt{2}}} - \frac{k_1}{1-x}}$$

with

$$\begin{cases} k_1 = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} \\ k_2 = k_1 \frac{\sqrt{\pi}}{2} e^{\frac{\mu^2}{2}} = \frac{\sigma}{2\sqrt{2}} e^{\frac{\mu^2}{2} - \frac{\mu^2}{2\sigma^2}} = \frac{\sigma}{2\sqrt{2}} e^{\frac{\mu^2}{2}\left[1 - \frac{1}{\sigma^2}\right]} \\ k_3 = \frac{\mu}{\sigma^2} \end{cases}$$

$I(0) = k_2 k_3 - k_1$ $\quad k_2 k_3 \gg k_1 \quad k_2 k_3 - k_1 \sim \frac{\sigma}{2\sqrt{2}} e^{\frac{\mu^2}{2}}$

$I(1^-) \to \infty \, 0^{\infty} - \infty$ jolie forme indeterminée.

il faudrait que $I(1^-)$ tende vers 0