1

$$I(x) = \int_0^\infty n(1-x)^{n-1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{n-\mu}{\sigma}\right)^2} dn$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty n\, e^{(n-1)\log(1-x)} e^{-\frac{1}{2\sigma^2}(n-\mu)^2} dn$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty n\, e^{u(n)} dn$$

$$u(n) = -\frac{1}{2\sigma^2} n^2 + n\left[\log(1-x) + \frac{\mu}{\sigma^2}\right] - \left[\log(1-x) + \frac{\mu^2}{2\sigma^2}\right]$$

$$u(n) = -a n^2 + b(x) n + c(x)$$

with $a = \frac{1}{2\sigma^2}$

$$b(x) = \log(1-x) + \frac{\mu}{\sigma^2}$$

$$c(x) = \log(1-x) + \frac{\mu^2}{2\sigma^2}$$

$$I(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty n\, e^{-an^2+bn-c} dn$$

$$= \frac{e^{-c(x)}}{\sigma\sqrt{2\pi}} \int_0^\infty \underbrace{n\, e^{-an^2+bn}}_{J} dn$$

Solving for $J = \int_0^\infty n\, e^{bn-an^2} dn$

write n as $\left(-\frac{1}{2a}(b-2an) + \frac{b}{2a}\right)$ and split

$$J = \int_0^\infty n\, e^{-an^2+bn} dn = \frac{b}{2a}\int_0^\infty \underbrace{e^{+bn-an^2} dn}_{L} + \frac{1}{2a}\int_0^\infty \underbrace{(2an-b)\, e^{bn-an^2} dn}_{K}$$

Solving for $K = \int_0^\infty (2an-b)\, e^{bn-an^2} dn$

substitution $u = bn - an^2$ $\quad du = (b-2an)dn$

$$K = -\int_0^\infty e^u du = -e^u = -e^{bn-an^2}$$

Solving for L $= \int_0^\infty e^{bn - an^2}\, dn$

Completing the square

$$L = \int_0^\infty e^{\frac{b^2}{4a} - \left(\sqrt{a}n - \frac{b}{2\sqrt{a}}\right)^2} dn$$

Substitute $u = \frac{2an - b}{2\sqrt{a}}$ $\quad du = \sqrt{a}\, dn$

$$L = \frac{\sqrt{\pi}\, e^{\frac{b^2}{4a}}}{2\sqrt{a}} \int_0^\infty \frac{2e^{-u^2}}{\sqrt{\pi}} du$$

$\int \frac{2e^{-u^2}}{\sqrt{\pi}}$ gauss error function $\operatorname{erf}(u)$

$$L = \frac{\sqrt{\pi}\, e^{\frac{b^2}{4a}}}{2\sqrt{a}} \operatorname{erf}\left(\frac{2an - b}{2\sqrt{a}}\right)$$

Put everything together

$$J = \frac{b}{2a} L + \frac{1}{2a} K$$

$$I(x) = \frac{e^{-c(x)}}{\sigma\sqrt{2\pi}} J$$

$$I(x) = \frac{e^{-c}}{\sigma\sqrt{2\pi}}\left[\frac{b}{2a}\frac{\sqrt{\pi}\, e^{\frac{b^2}{4a}}}{2\sqrt{a}} \operatorname{erf}\left(\frac{2an-b}{2\sqrt{a}}\right) - \frac{1}{2a} e^{bn - an^2}\right]$$

$$= \left[\frac{b\, e^{\frac{b^2}{4a} - c}}{\sqrt{2}\,\sigma\, 4a^{3/2}} \operatorname{erf}\left(\frac{2an-b}{2\sqrt{a}}\right) - \frac{e^{-c}}{2a\sqrt{2\pi}\,\sigma} e^{bn - an^2}\right]_0^\infty$$

$$= \frac{b\, e^{\frac{b^2}{4a} - c}}{\sigma\sqrt{\pi}\, 4a^{3/2}}\left[\operatorname{erf}(\infty) - \operatorname{erf}\left(\frac{-b}{2\sqrt{a}}\right)\right] - \frac{e^{-c}}{2a\sqrt{2\pi}\,\sigma}$$

$$= \frac{b\, e^{\frac{b^2}{4a} - c}}{\sigma\sqrt{\pi}\, 4a^{3/2}}\left[1 - \operatorname{erf}(-b/2\sqrt{a})\right] - \frac{e^{-c}}{2a\sqrt{2\pi}\,\sigma}$$

$$e^{-c} = e^{-\log(1-x) - \frac{\mu^2}{2\sigma^2}} = \frac{k_1}{1-x} \quad \text{with } k_1 = e^{-\frac{\mu^2}{2\sigma^2}}$$

$$e^{\frac{b^2}{4a} - c} = e^{\frac{b^2}{4a}} e^{-c}$$

$$b^2 = \left[\log(1-x) + \frac{\mu^2}{2\sigma^2}\right]^2 = \log^2(1-x) + \frac{\mu^4}{4\sigma^4} + \frac{\mu^2}{\sigma^2}\log(1-x)$$

(2)

$$4a = \frac{2}{\sigma^2} \qquad \frac{1}{4a} = \frac{\sigma^2}{2} \qquad 2a = \frac{1}{\sigma^2}$$

$$\frac{b^2}{4a} = \frac{\sigma^2}{2}\log^2(1-x) + \frac{\mu^4}{8\sigma^2} + \mu^2 \log(1-x)$$

$$e^{\frac{b^2}{4a}} = e^{\left(\frac{\sigma \log(1-x)}{\sqrt{2}}\right)^2} \times e^{\frac{\mu^4}{8\sigma^2}} * (1-x)^{\mu^2}$$

$$k_2 = e^{\frac{\mu^4}{8\sigma^2}}$$

$$e^{b^2/4a - c} = e^{b^2/4a} e^{-c} = k_2 (1-x)^{\mu^2} \frac{k_1}{(1-x)} e^{\left(\frac{\sigma \log(1-x)}{\sqrt{2}}\right)^2}$$

$$= k_1 k_2 (1-x)^{\mu^2 - 1} e^{\left[\frac{\sigma \log(1-x)}{\sqrt{2}}\right]^2}$$

$$\left(\log(1-x) + \frac{\mu}{\sigma^2}\right) k_1 k_2 (1-x)^{\mu^2-1} e^{\left[\frac{\sigma \log(1-x)}{\sqrt{2}}\right]^2}$$

$$4a^{3/2} = 4\left(\frac{1}{2\sigma^2}\right)^{3/2} = \frac{4}{\sqrt{8}\,\sigma^3}$$

$$4a^{3/2}\sigma\sqrt{\pi} = \frac{4}{\sqrt{8}\,\sigma^3}\sigma\sqrt{\pi} = \frac{\sqrt{2\pi}}{\sigma^2}$$

$$I(n) = \frac{k_1 k_2 \sigma^2}{\sqrt{2\pi}}\left[\log(1-x) + \frac{\mu}{\sigma^2}\right](1-x)^{\mu^2-1} e^{\left[\frac{\sigma \log(1-x)}{\sqrt{2}}\right]} \left[1 - \mathrm{erf}\left(-\frac{b}{2\sqrt{a}}\right)\right]$$

$$- \frac{k_1 \sigma}{1-x}\,\frac{1}{\sqrt{2\pi}}$$