SOLVING INTEGER LINEAR PROGRAMMING (ILP) PROBLEM

B: CUTTING-PLANE METHOD

NOTE:
Solving ILP model is much much harder than LP model.

Research continues to solve ILP model efficiently but still, solving ILP is a big big issue.

LP = No issue in solving

ILP = Big issue

Example:

$$\text{Max } Z = 5x_1 + 8x_2$$
$$x_1 + 2x_2 \leq 8$$
$$4x_1 + x_2 \leq 10$$
$$x_1, x_2 \geq 0 \text{ and } \underline{\text{Integers}}$$

Step I: Relax the integer requirement and solve the resulting relaxed model using Simplex Method.

$$\text{Max } Z = 5x_1 + 8x_2$$
$$x_1 + 2x_2 \leq 8$$
$$4x_1 + x_2 \leq 10$$
$$x_1, x_2 \geq 0 \quad (\text{Continuous Variables}) \text{ (Relaxed)}$$

STANDARD FORM

$$\text{Max } Z = 5x_1 + 8x_2$$
$$x_1 + 2x_2 + S_1 = 8$$
$$4x_1 + x_2 + S_2 = 10$$
$$x_1, x_2 \geq 0$$

	C_j	5	8	0	0		
C_B	Basic	x_1	x_2	s_1	s_2	Sol	Ratio
0	S_1	1	2	1	0	8	$8/2=4$ ←
0	S_2	4	1	0	1	10	$10/1=10$
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	5	8 ↑	0	0		
8	x_2	$1/2$	1	$1/2$	0	4	$4*2/1=8$
0	S_2	$7/2$	0	$-1/2$	1	6	$6*2/7=12/7$ ←
	Z_j	4	8	4	0	32	
	$C_j - Z_j$	1 ↑	0	-4	0		
8	x_2	0	1	$4/7$	$-1/7$	$22/7$	
5	x_1	1	0	$-1/7$	$2/7$	$12/7$	
	Z_j	5	8	$27/7$	$2/7$	$\frac{236}{7}$	
		0	0	$-27/7$	$-2/7$		

$(4 \;\; 1 \;\; 0 \;\; 1 \;\; 10) -$
$1(1/2 \;\; 1/2 \;\; 0 \;\; 4)$
$= (7/2 \;\; 0 \;\; -1/2 \;\; 1 \;\; 6)$

$(1/2 \;\; 1 \;\; 1/2 \;\; 0 \;\; 4) -$
$1/2(1 \;\; 0 \;\; -1/7 \;\; 2/7 \;\; 12/7)$

$(1/2 \;\; 1 \;\; 1/2 \;\; 0 \;\; 4) -$
$(1/2 \;\; 0 \;\; -1/14 \;\; 1/7 \;\; 6/7)$
$= (0 \;\; 1 \;\; 4/7 \;\; -1/7 \;\; \frac{22}{7})$

Solution: optimal

Solution: non-integer

Non-integer Solution

Step II:

STEP II: If all basic variables have integer Solution, STOP. This is your optimum Solution. If not, goto step III.

STEP III: For each basic variable with non-integer solution in the optimum iteration, find the fraction part f_i using

$$b_i = [b_i] + f_i \quad \text{where } [b_i] = \text{integer part}, \; f_i = \text{fractional part}$$

optimum iteration:

C_B	Basic	x_1	x_2	S_1	S_2	Sol
8	x_2	0	1	$4/7$	$-1/7$	$\frac{22}{7}$
5	x_1	1	0	$-1/7$	$2/7$	$12/7$

Basic	Sol (b_i)	$[b_i]$	f_i
x_1	$\frac{12}{7}$	1 +	$\frac{5}{7}$
x_2	$\frac{22}{7}$	3 +	$\frac{1}{7}$

STEP IV: Choose the largest fraction among various f_i (i.e $\max(f_i) = 5/7$).

~~theats~~

Treat the Constraint Corresponding to the maximum fraction (i.e., x_1-row) as the source row (equation)

Based on the Source equation, develop an additional contraint (Gomory's Constraint / fractional cut) as shown:

$$-f_i = S_i - \Sigma\,((f_i)(\text{Non-Basic Var.}))$$

Here Largest fraction = $5/7$ (Corresponding Constraint is x_1)

$\boxed{\text{Source row}}$

$$12/7 = x_1 + 0 \underbrace{- 1/7\, S_1 + 2/7\, S_2}_{\text{Non-Basic}}$$

$$1 + \frac{5}{7} = x_1 + (-1 + 6/7)\, S_1 + (0 + \frac{2}{7})\, S_2$$

f_i associated with x_1 | f_i associated with S_1 | f_i associated with S_2

The Corresponding fractional Cut:

$$-f_i = S_i - \Sigma\,((f_i)(\text{Non-Basic Var.}))$$

$$-5/7 = S_3 - 6/7\, S_1 - 2/7\, S_2$$

STEP V: Add the fractional Cut as the last row in the latest optimal iteration and proceed further using dual simplex method, and find the new optimum. If new optimum solution is integer then STOP, if not, go to step III.

		5	8	0	0	0	
C_B	Basic	x_1	x_2	S_1	S_2	S_3	Sol
8	x_2	0	1	4/7	-1/7	0	22/7
5	x_1	1	0	-1/7	2/7	0	12/7
0	S_3	0	0	-6/7 ✓	-2/7	1	-5/7 ←
	Z_j	5	8	27/7	2/7	0	236/7
	$C_j - Z_j$	0	0	-27/7 ✓	-2/7 ↑	0	
$(C_j - Z_j / \text{Pivot row})$	Ratio	-	-	9/2	min ratio 1	0	
8	x_2	0	1	1	0	-1/2	7/2
5	x_1	1	0	-1	0	1	1
0	S_2	0	0	3	1	-7/2	5/2
	Z_j	5	8	3	0	1	33
	$C_j - Z_j$	0	0	-3	0	-1	

optimum = YES
feasible = No

apply Dual Simplex Method to obtain feasible solution

$\boxed{x_2:}$

$(0\ 1\ 4/7\ -1/7\ 0\ \tfrac{22}{7}) - $
$-1/7(0\ 0\ 3\ 1\ -7/2\ 5/2)$

$\boxed{x_1:}$

$(1\ 0\ -1/7\ 2/7\ 0\ 12/7) - $
$2/7(0\ 0\ 3\ 1\ -7/2\ 5/2)$

- optimal
- feasible but
- non-integer

STEP III :

$$b_i = [b_i] + f_i$$

		$[b_i] + f_i$
x_2	7/2	3 + 1/2
S_2	5/2	2 + 1/2

max $(1/2, 1/2) = 1/2$ (tie) x_2.
preference will be given to decision variable

Treat Constraint related to x_2 as the Source row :

$$7/2 = 0 + x_2 + S_1 \overbrace{+0 - 1/2\, S_3}^{\text{Non-Basic}} \Rightarrow 3 + \tfrac{1}{2} = x_2 + S_1 + (-1 + 1/2)\underset{\text{Non Basic}}{S_3}$$

Develope additional Constraint as shown :

$$-f_i = S_i - \Sigma\ ((f_i)(\text{Non-Basic var.}))$$

$$-1/2 = S_4 - 1/2\, S_3$$

STEP IV : Add this fraction cut as the last row in the latest optimal iteration and find new optimum using dual Simplex.

	c_j	5	8	0	0	0	0	
C_B	Basic	x_1	x_2	s_1	s_2	s_3	s_4	Sol
8	x_2	0	1	1	0	$-1/2$	0	$7/2$
5	x_1	1	0	-1	0	1	0	1
0	s_2	0	0	3	1	$-7/2$	0	$5/2$
0	s_4	0	0	0	0	$-1/2$	1	$-1/2$ ←
	z_j	5	8	3	0	1	0	33
	$c_j - z_j$	0	0	-3	0	-1	0	
	Ratio	—	—	—	—	↑2	0	
8	x_2	0	1	1	0	0	-1	4
5	x_1	1	0	-1	0	0	2	0
0	s_2	0	0	3	1	0	-7	6
0	s_3	0	0	0	0	1	-2	1
	z_j	5	8	3	0	0	2	32
	$c_j - z_j$	0	0	-3	0	0	-2	

optimum = YES
Feasible = YES
integer = YES

optimum Solution :

$x_1 = 0$

$x_2 = 4$

$Z = 32$

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row operations

$\underline{x_2}$: $(0 \;\; 1 \;\; 1 \;\; 0 \;\; -1/2 \;\; 0 \;\; 7/2) - (-1/2)(0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; -2 \;\; 1) = (0 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; -1 \;\; 4)$

$\underline{x_1}$: $(1 \;\; 0 \;\; -1 \;\; 0 \;\; 1 \;\; 0 \;\; 1) - (0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; -2 \;\; 1) = (1 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 2 \;\; 0)$

$\underline{s_2}$: $(0 \;\; 0 \;\; 3 \;\; 1 \;\; -7/2 \;\; 0 \;\; 5/2) - (-7/2)(0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; -2 \;\; 1) = (0 \;\; 0 \;\; 3 \;\; 1 \;\; 0 \;\; -7 \;\; 6)$