Fuzzy Systems Homeworks #1

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Fuzzy sets and related topics

Excersise 1. Answer to the question 4 of chapter 1 from the book Ross, 4th ed. 2017

The question asked us to find a membership function to terms “half-full”, “half-empty” and “full” while pointing to a glass of water.

Answer: The terms half-full or half-empy are ambiguous words. For example For the term “half-empty”:

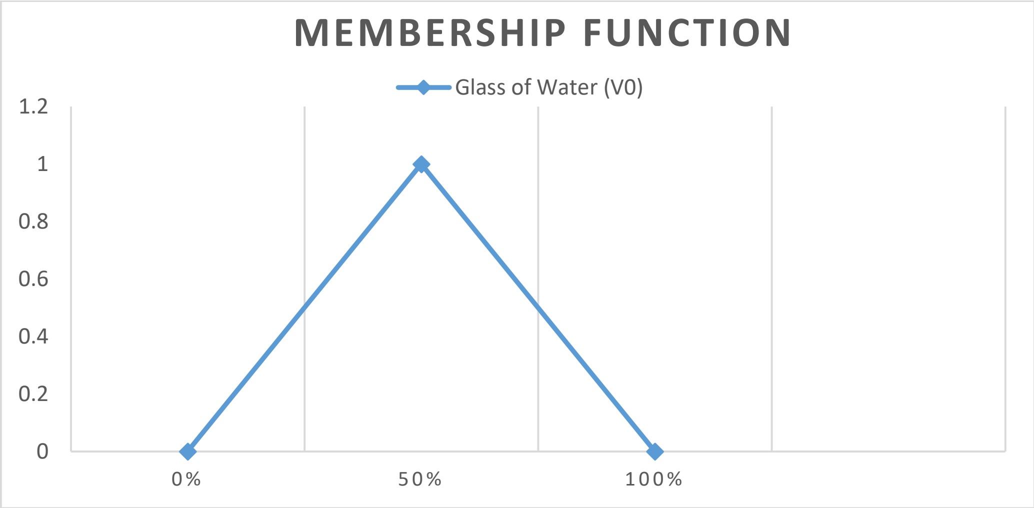
- when our glass of water is full, we can assign zero value to half-empty meaning there is no have empty.
- Or if the glass is empty it’s possible to assign zero value to half-empy.

And for the term “half-full”:

- when our glass of water is full, we can assign zero value to “half-full”.
- Or if the glass is empty, again we can assign zero value to “half-full”.

With these assumptions above we can aswer the question “Does half-full and half-empty have identical membership function?” As below

Because both assumptions have same value assigned for an empty glass of water we can say that both have identical membership function. Below a triangular function is an example of membership function for these terms.



But for full we can assign another membership function such as S-type membership function.

And for conclusion it can be said that, the answer does not solve this ageless riddle.

Excersise 2. Use a programming language to answer to the question 2 of the second chapter of Zimmermann, 4^{th} ed, 2001 book.

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 3. Which of the fuzzy sets in question 2 of the second chapter of Zimmermann, 4^{th} ed, 2001 book, are Convex?

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 4. Answer the question 2 of chapter 4 from the book Ross, 4th ed. 2017

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 5. Answer Question 4 of the second chapter of Zimmermann, 4th ed, 2001 book

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Fuzzy Operations

Excersise 6. Answer to the question 8, chapter 2 from the book Ross, 4th ed. 2017

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 7. Answer to the question 5, chapter 2 from the book Ross, 4th ed. 2017

Answer: To answer this question we must first analyse the chances of not having rainfall for each cities, then to evaluate the best week between cities, we would like to find the most value for the chance of not having rainfall.

So first we find the highest value of not having rain in each city.

$\max\left(\mu_{\widetilde{City1}}\right) = 0.8$ then the set with maximum membership value is $\{\frac{0.8}{week1} + \frac{0.8}{week4}\}$

$\max\left(\mu_{\widetilde{City2}}\right) = 0.9$ then the set with maximum membership value is $\{\frac{0.9}{week4}\}$

$\max\left(\mu_{\widetilde{City3}}\right) = 0.9$ then the set with maximum membership value is $\{\frac{0.9}{week2}\}$

And then we need to choose the highest between the highest membership values between cities. It's obvious that both $\widetilde{City}2$ and $\widetilde{City}3$ have the highest values. So to hold the event we can either choose $\underline{\text{week4 in } \widetilde{City}2}$ or $\underline{\text{week2 in } \widetilde{City}3}$.

Fuzzy Numbers

Excersise 8. Find the Answer to question 3, chapter five of Zimmermann, 4th ed, 2001 book

Answer:

(a) for this part, it's obvious that $\mu_{\tilde{A}}$ is similar to a LR type fuzzy number. First the parameters can be $m = 5,\ \alpha = 2,\ \beta = 3$. And we can write the L and R functions as

$$L(x) = (1 + x^2)^{-1}$$

$$R(x) = (1 + |2x|)^{-1}$$

And to prove that these L-R functions would produce the Left and right functions represented for $\mu_{\tilde{A}}$, we can write them down.

$$L\left(\frac{m - x}{\alpha}\right) = L\left(\frac{5 - x}{2}\right) = \left(1 - \left(\frac{5 - x}{2}\right)^2\right)^{-1}$$

$$L\left(\frac{x - m}{\beta}\right) = L\left(\frac{x - 5}{3}\right) = \left(1 + \left|\frac{2(x - 5)}{3}\right|\right)^{-1}$$

Until here we defind the L and R functions but we didn't actually prove that these functions are descending, symmetric and have the initial values as one. In order to prove these conditions we write the equations below

- **Initial Value is one $\boldsymbol{f(0) = 1}$**

$$L(0) = (1 + 0^2)^{-1} = 1$$

$$R(0) = (1 + |2(0)|)^{-1} = 1$$

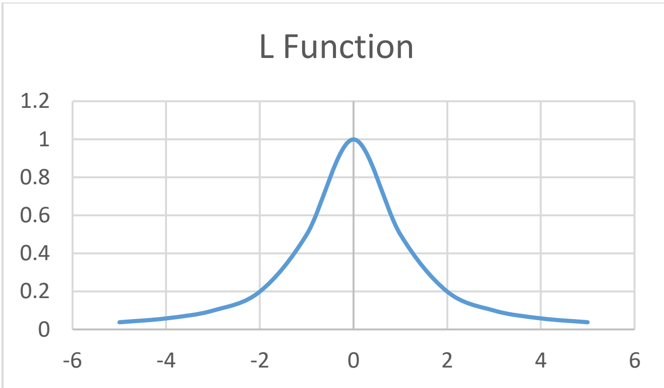
  Both conditions are correct, meaning the functions have initial value one. ✓

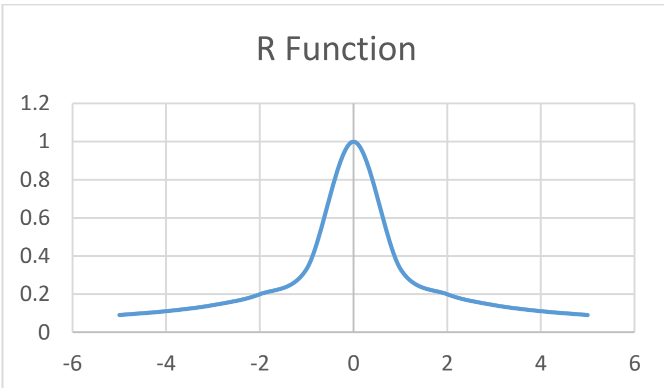
- **Symetric**

$$\boldsymbol{L(x) = L(-x) \Rightarrow (1 + x^2)^{-1} = (1 + (-x)^2)^{-1}}$$
$$\boldsymbol{R(x) = R(-x) \Rightarrow (1 + |2x|)^{-1} = (1 + |-2x|)^{-1}}$$

  Both Conditions are correct, meaning the functions are symmetric. ✓

- **Descending for positive values**

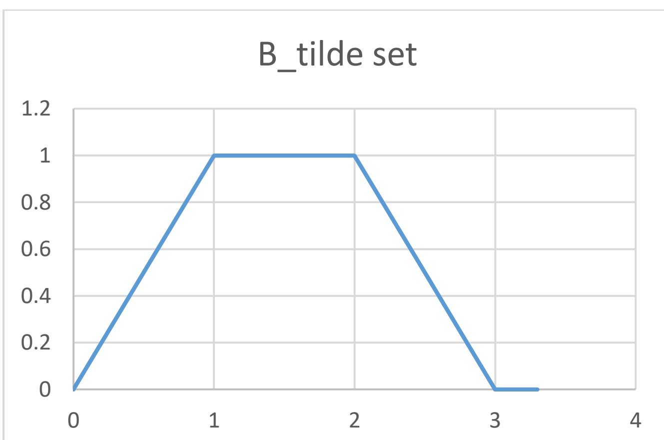




So with the charts plotted, we can see that L and R functions are Descending for positive values. ✓

And the answer to part (a) is correct!

(b) To answer this part we first plot the $\mu_{\tilde{B}}(x)$



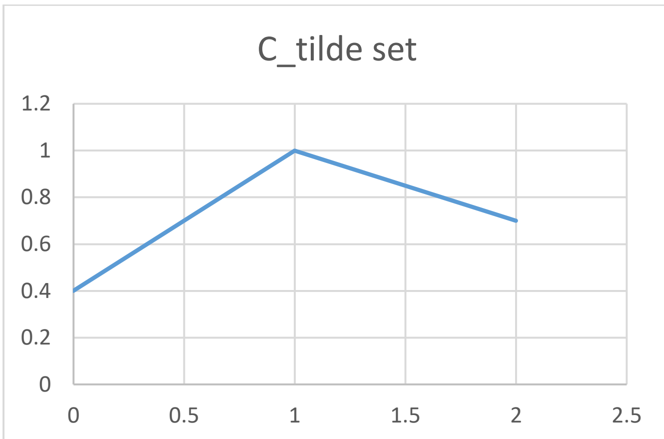
Here to finding the LR functions and testing its conditions are not an easy task, so we will go through another way. Here we would like to test the conditions of a fuzzy number not conditions of LR functions. The condtions for a fuzzy number are to be convex and to be piecewise linear.

- **Convex**
  It’s obvious from the plotted membership function that the membership function is convex on a sample data. ✓
- **Have Unique Core**

  As we can see from the equation in the question and also in the plotted membership function, the core have unique values. ✓

So here we can conclude that $\tilde{B}$ is a set of fuzzy numbers.

(c) Here to check wether $\tilde{C}$ is a set of fuzzy numbers or not, we would go through the way we used in (b). We plot the set as below



It can be seen from this function that $\tilde{C}$ is convex and piecewise linear and So it’s fuzzy numbers set. Of course we could conclude this without plotting it but for better view we plotted it.

Excersise 9. Find the Answer to question 4, chapter five of Zimmermann, 4th ed, 2001 book

Answer: To answer this question we must apply three conditions to each item to prove that they can be a reference function or not. Conditions are have initial value one, be a descending function in positive domain and be symmetric.

(a) $f_1(x) = |x + 1|$

Checking the conditions:

- Initial value is one, $f(0) = 1$

$$f_1(0) = |0 + 1| = 1$$

This condition is satisfied. ✓

- Be a descending function in positive domain

$$for\ x \in [0.\infty): x_1 \le x_2 \Longrightarrow f_1(x_1) = |1 + x_1| \ngeq |1 + x_2| = f_1(x_2)$$

So this condition is not satisfied and we cannot use $f_1(x)$ as a reference function. ××

(b) $f_2(x) = \frac{1}{1+x^2}$

Checking the conditions:

- Initial value is one, $f(0) = 1$

$$f_2(0) = \frac{1}{1+0^2} = 1$$

So this condition is satisfied.

- Be a descending function in positive domain

$$for\ x \in [0.\infty): x_1 \le x_2 \Longrightarrow f_2(x_1) = \frac{1}{1+x_1^2} \ge \frac{1}{1+x_2^2} = f_2(x_2)$$

This condition is satisfied too.

- Be a symmetric function

$$f(x) = f(-x) \Longrightarrow \frac{1}{1+x^2} = \frac{1}{1+(-x)^2}$$

The last condition is satisfied and function in part (b) can be a reference function.

(c) $f_3(x) = \begin{cases} \frac{1}{2}x + 1 & x \in [-2.0] \\ -2x + 1 & x \in [0.\frac{1}{2}] \\ 0 & else \end{cases}$

Applying the conditions would produce

- Have initial value one

$$f_3(0) = 1$$

This condition is satisfied. ✓

- Be descending function for positive values

For this condition because we just have values from zero to half in positive values, we just evaluate this interval

$$for\ x \in [0.\frac{1}{2}]\ x_1 \le x_2 \Longrightarrow f_3(x_1) = -2x_1 + 1 \ge\ 2x_2 + 1 = f_3(x_2)$$

And because of the negative sign we could conclude that this condition is satisfied too. ✓

- Be a symmetric function

  An example here shows that this function is not symmetric.

$$f_3(1) = 0 \neq \frac{1}{2} = f_3(-1)$$

  So here this function in part (c) cannot be a reference function. ××

(d) $f_4(x) = \frac{1}{1+a|x|^p}\ p \geq 1$

Checking the conditions:

- Have initial value one

$$f_4(0) = \frac{1}{1 + a|0|^p} = 1$$

  This condition is satisfied. ✓

- Be descending function for positive values

$$for\ x \in [0. \infty): x_1 \leq x_2 \Longrightarrow f_4(x_1) = \frac{1}{1 + ax_1^p} \geq \frac{1}{1 + ax_2^p} = f_4(x_2)$$

  So this condition is satisfied too, because of the division. ✓

- Be a symmetric function

$$f_4(x) = f_4(-x) \Longrightarrow \frac{1}{1 + a|x|^p} = \frac{1}{1 + a|-x|^p}$$

  This condition is satisfied because of absolute function surrounding x. So the function in (d) can be a reference function. ✓

Fuzzy Extensions and their properties

Excersise 10. Answer the question 3, chapter 3 of Zimmermann, 4th ed, 2001 book.

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 11. Answer the question 5, chapter 3 of Zimmermann, 4th ed, 2001 book.

Answer: To be able to answer this question we first need to take a look at yager's union operator.

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = min\left\{1, \left(\mu_{\tilde{A}}(x)^p + \mu_{\tilde{B}}(x)^p\right)^{1/p}\right\}, p \geq 1$$

So now we will go through each part of this question.

(a) $\mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{A}}(x) \quad for \quad \mu_{\tilde{B}}(x) = 0$

It's easy to prove this with yager's union equation. From the equation we wrote in the first place, we replaces $\mu_{\tilde{B}}(x)$ with 0 as below

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = min\left\{1, (\mu_{\tilde{A}}(x)^p + 0^p)^{1/p}\right\} = min\{1, \mu_{\tilde{A}}(x)\} = \mu_{\tilde{A}}(x)$$

Knowing that $\mu_{\tilde{A}}(x)$ is normalized, it can give that all values are less equal than 1, so the answer of the minimum operator would be $\mu_{\tilde{A}}(x)$.

(b) $\mu_{\tilde{A}\cup\tilde{B}}(x) = 1 \quad for \quad \mu_{\tilde{B}}(x) = 1$

We again re-write yager's union equation replacing $\mu_{\tilde{B}}(x)$ with 1.

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = min\left\{1, (\mu_{\tilde{A}}(x)^p + 1^p)^{1/p}\right\}$$

The result of <u>the equation $\mu_{\tilde{A}}(x)^p + 1^p$ gives a value greater equal than 1 and less equal than 2</u>. The reason behind this is, while p has the value greater equal than 1, the result of $\mu_{\tilde{A}}(x)^p$ is an array with values less than 1($\tilde{A}$ is a normalized fuzzy set, meaning all the values are between 0 and 1).

So with having the assumption underlined in the paragraph above, we can conclude that the value of $(\mu_{\tilde{A}}(x)^p + 1^p)^{1/p}$ is always between 1 and 2. Knowing this can represent that the minimum value between 1 and $(\mu_{\tilde{A}}(x)^p + 1^p)^{1/p}$ is always 1. So the answer of yager's union operation in part (b) is proved!

(c) $\mu_{\tilde{A}\cup\tilde{B}}(x) \geq \mu_{\tilde{A}}(x) \quad for \quad \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$

With knowing the condition $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, we can replace $\mu_{\tilde{B}}(x)$ with $\mu_{\tilde{A}}(x)$ in yager's union equation. Replacing that would gives us the equation below

$$\begin{aligned}\mu_{\tilde{A}\cup\tilde{B}}(x) &= min\left\{1, (\mu_{\tilde{A}}(x)^p + \mu_{\tilde{B}}(x)^p)^{1/p}\right\} = min\left\{1, (\mu_{\tilde{A}}(x)^p + \mu_{\tilde{A}}(x)^p)^{1/p}\right\} \\ &= min\left\{1, (2 \times \mu_{\tilde{A}}(x)^p)^{1/p}\right\} = min\left\{1, 2^{1/p} \times (\mu_{\tilde{A}}(x)^p)^{1/p}\right\} \\ &= min\left\{1, 2^{1/p} \times \mu_{\tilde{A}}(x)\right\} = \begin{cases} 1, & 2^{1/p} \times \mu_{\tilde{A}}(x) \geq 1 \\ 2^{1/p} \times \mu_{\tilde{A}}(x), & 2^{1/p} \times \mu_{\tilde{A}}(x) < 1 \end{cases}\end{aligned}$$

Here we derive to an important condition. Investigating these condtions would arrive us these answers

- First condition

$$2^{1/p} \times \mu_{\tilde{A}}(x) \geq 1 \Rightarrow \mu_{\tilde{A}}(x) \geq \frac{1}{2^{1/p}}$$

If p = 1, then the $\mu_{\tilde{A}}(x)$ values are in the interval [0.5, 1]. But we know that in this condtion the value of $\mu_{\tilde{A}\cup\tilde{B}}(x)$ is always 1, So with the condition of p=1, the property mentioned in

(c) is proved. Also having $p > 1$, would gives us values less than 1 for $\mu_{\tilde{A}}(x)$. So again the property is satisfied.

- The second condition

$$2^{1/p} \times \mu_{\tilde{A}}(x) \leq 1$$

  In this conditions the values given for $\mu_{\tilde{A}\cup\tilde{B}}(x)$ is always slightly more than $\mu_{\tilde{A}}(x)$, because of the multiplication of $2^{1/p}$. So again the property in (c) is satisfied!

Knowing the satisfactions in two conditions would lead us to proof of this part of the question. So the answer here proved that the equations in part (c) is satisfied!

(d) For $p \to 0$, the yager's union operator reduces to drastic sum.

Re-writing yager's union operator with $p = 0$, would lead us to 1 for all membership values.

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = min\left\{1, (\mu_{\tilde{A}}(x)^p + \mu_{\tilde{B}}(x)^p)^{1/p}\right\} = \; min\left\{1, (\mu_{\tilde{A}}(x)^0 + \mu_{\tilde{B}}(x)^0)^{1/0}\right\}$$
$$= min\{1, (1+1)^{\infty}\} = 1$$

So it is shown that having $p = 0$ represent the maximum membership values. We need to know that in the assumption we had the each membership values cannot be zero! If one membership values was zero and the other was a number less equal than 1, we can conclude that the floating number in power of infinity would represent a small number and the minimum would be the second membership value (not the zero).

With the assumptions we had we can conclude that having p value near zero can represent the drastic sum.

Excersise 12. Answer the question 6, chapter 3 of Zimmermann, 4th ed, 2001 book.

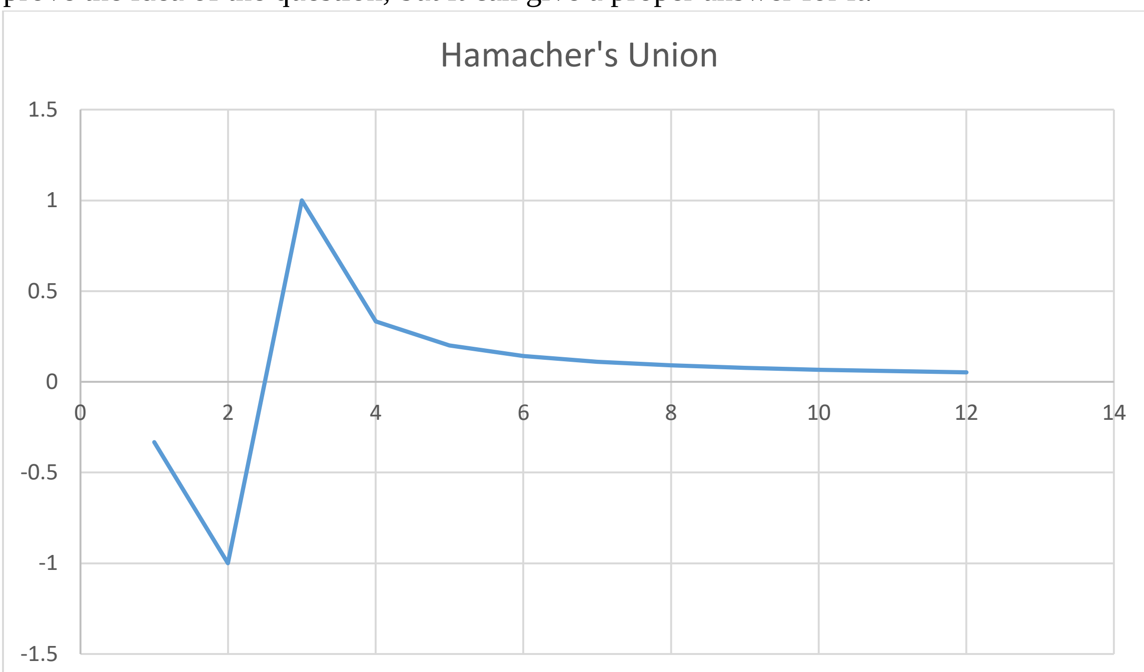
Answer: The question asked to prove that having increase in parameters of Yager and Hamacher operations would decrease the value return by these operations.

- So First we go through Yager's operations.

  The main thing in Yager's operations is we calculate the power of each membership values (power to p) and then we apply the power of inverse of the parameters (power to $1/p$).

  Having the idea of yager's operations in mind would gives us the lead that with the power of each normalized membership values, the returned value is always less equal than the original membership values. And again applying the power of inverse of the parameter ($1/p$), just can slightly increase the values, But the main area shows that calculating the power of normalized values with higer parameters can decrease the values found for each memberships.

- And for Hamacher's operation
  To prove the idea of the question we seperate our explanations for hamacher's union and intersection.
  Intersection: Having parameters γ below the division shows that giving higher values to this parameter can represent lower values for the answer. To explain more we can say that, the membership multiplications in above the division would have always a value less equal than 1 and summing each membership values can have bigger value than the multiplication (So the second term below the division is always positive).
  With the assumptions in above paragraph in mind we can conclude that the multiplication of $1 - \gamma$ in the division will gives us a positive number and summing it with γ would also have bigger number. While increasing γ with the knowledge we explained the value of the hamacher intersection would decrease.
  So this was the proof for hamacher's intersection.
  Union: To answer this part we've plotted it with membership values 1 for each fuzzy sets. The x axis is the γ' and y-axis is our hamacher's union value. While plotting may not fully prove the idea of the question, but it can give a proper answer for it.



Measure of Fuzziness of fuzzy sets

Excersise 13. Answer to the question 1, chapter 4 of Zimmermann, 4th ed, 2001 book.

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 14. Answer to the question 2, chapter 4 of Zimmermann, 4th ed, 2001 book.

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Fuzzy Relations and Operations

Excersise 15. Answer to the question 8, chapter 6 of Zimmermann, 4th ed, 2001 book.

Answer: To answer this question we must bring values to x and y variables. So we would do it as below.

- Variables for $\tilde{R}$

  First we analyze the membership values in order to give x and y some number.

  1. All the membership values in the first row is greater than 0.5 except y_3, so the values assigned to x_1 sould be greater than all then values of y except y_3.
  2. Analyzing the second row shows that x_2 can be greater than y_2 and is less than all other y variables.
  3. Analyzing the third row shows that x_3 is greater than all the y variables.

  So our answer is based on all three assumptions above.

  $$x_1 = 7$$

  $$y_1 = 2. y_2 = 0. y_3 = 6.\ y_4 = 3$$

  After assigning values to these variables the others must obey these values and assumptions.

  $$x_2 = 1. x_3 = 9$$

- Variables for $\tilde{Z}$

  We analyze each columns instead of rows of the matrix in order to give x and y some value.

  1. The first column shows that y_1 should have a value near x_2, and bit more far with $x_1. x_3$.

2. From the second column we can find out that y_2 cannot have a close number with $x_1 . x_3$ but the value can be a bit close to x_2.
3. For the third column we can assign some numbers for each variables (Assumption is based on membership values)
4. And For the fourth column their again somehow each variable y near the x variables.

Based on assumptions and for more ease we first assign values to x variables.

$$x_1 = 0.\ x_2 = 5\ .x_3 = -1$$

$$y_1 = 6$$

And we go on for the second, third and fourth column using the assigned values.

$$y_2 = 11. y_3 = 1. y_4 = 3$$

So the values are assigned and the anwer for this question is found (Note: All the values was in integer but we could use floating points either).

To better understand why we chosed these answers for each variables have the difference of each variables in mind and again read the answer.

Excersise 16. Answer to the question 3, chapter 6 of Zimmermann, 4th ed, 2001 book.

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 17. Answer to the question 12, chapter 3 from the book Ross, 4th ed. 2017

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Linguistic Variables

Excersise 18. Answer to the question 1, chapter 9 of Zimmermann, 4th ed, 2001 book.

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 19. Answer to the question 19, chapter 5 from the book Ross, 4th ed. 2017

Answer: The answer is in *main.ipynb* file or the exported form of it *main.pdf*

Excersise 20. Answer to the question 2, chapter 9 of Zimmermann, 4th ed, 2001 book.

Answer: To anwer this question we need the function or the operators for each specific word.

- *Very True*
  Very would apply concentration operator on membership function. The concentration operator tries to power all the membership values to 2. So the membership function would looks like as

$$T(v;\alpha.\beta.\gamma)=\begin{cases}0 & if\ v\le\alpha\\ 4\left(\dfrac{v-\alpha}{\gamma-\alpha}\right)^4 & if\ \alpha\le v\le\beta\\ \left(1-2\left(\dfrac{v-\gamma}{\gamma-\alpha}\right)^2\right)^2 & if\ \beta\le v\le\gamma\\ 1 & if\ v\le\gamma\end{cases}$$

- *False*
  To compute False we can complement the original membership function.

$$1-T(v;\alpha.\beta.\gamma)=\begin{cases}1 & if\ v\le\alpha\\ 1-2\left(\dfrac{v-\alpha}{\gamma-\alpha}\right)^2 & if\ \alpha\le v\le\beta\\ 2\left(\dfrac{v-\gamma}{\gamma-\alpha}\right)^2 & if\ \beta\le v\le\gamma\\ 0 & if\ v\le\gamma\end{cases}$$

- *Very False*
  The concentration operator is applied on above membership function.

$$CON(1-T(v;\alpha.\beta.\gamma))=\begin{cases}1 & if\ v\le\alpha\\ \left(1-2\left(\dfrac{v-\alpha}{\gamma-\alpha}\right)^2\right)^2 & if\ \alpha\le v\le\beta\\ 4\left(\dfrac{v-\gamma}{\gamma-\alpha}\right)^4 & if\ \beta\le v\le\gamma\\ 0 & if\ v\le\gamma\end{cases}$$

- *Rather true*

  To apply the term “Rather”, we would like to use the membership function of “fairly”. For fairly the operator dialation is needed to be applied (due to the equations at the bottom of page 154 of Zimmerman reference book). So the function T would looks like as below

$$T(v;\alpha.\beta.\gamma)=\begin{cases} 0 & if\ v\le\alpha \\ \left(2\left(\frac{v-\alpha}{\gamma-\alpha}\right)^2\right)^{1/2} & if\ \alpha\le v\le\beta \\ \left(1-2\left(\frac{v-\gamma}{\gamma-\alpha}\right)^2\right)^{1/2} & if\ \beta\le v\le\gamma \\ 1 & if\ v\le\gamma \end{cases}$$