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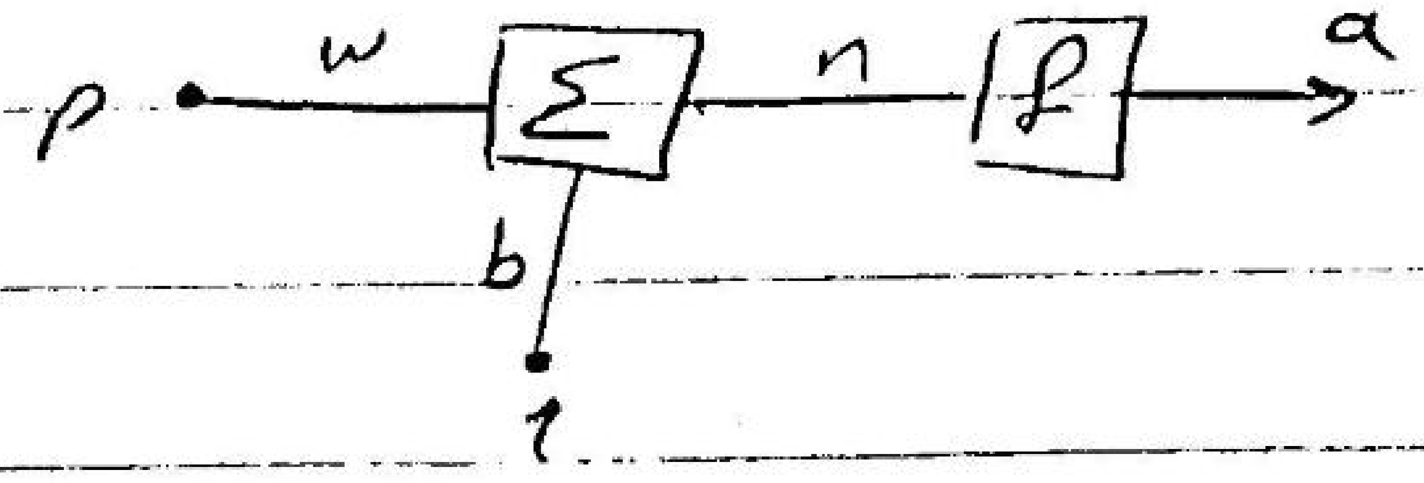
۴۰۰۳۴۲۴۰۱۴ تمارین شبکه های عصبی

E2.1




$w = 1.3 \quad b = 3.0$

$$f = wp + b = (1.3)p + 3.0$$

(i) 1.6 → purelin, logsig, tansig, poslin
(ii) 1.0 → hardlim, hardlims, purelin, satlin, satlins, logsig, transig, poslin, compet
(iii) 0.9963 → purelin, satlin, satlins, logsig, transig, poslin
(iv) −1.0 → hardlims, purelin, satlins, logsig, transig

E2.2

single neuron with bias $\text{output} = \begin{cases} -1 & \text{input} < 3 \\ +1 & \text{input} \geq 3 \end{cases}$

(i) hardlims function can be applied to this problem

(ii) if we write the single neuron with hardlims transfer-function formula:

$$out = \text{hardlims}(wp + b)$$

the parameters to get the requested output can be: $b = -3$, weight $> 0$

E2.2 Continues

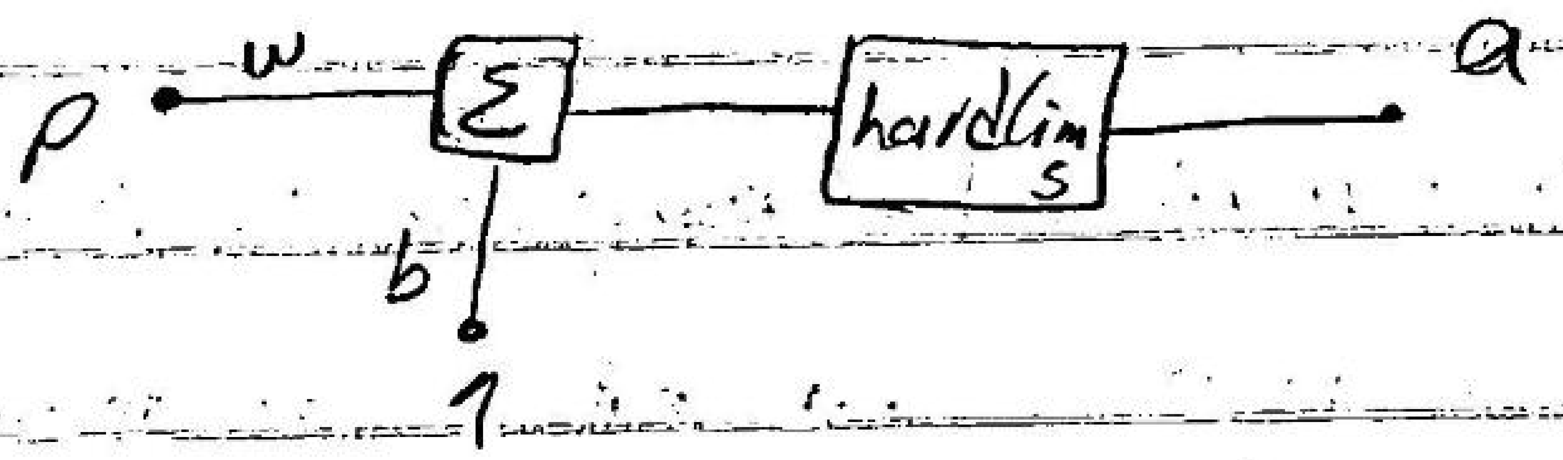
(ii) The bias can be related to input weights with the condition of

(a) first of all weight must be not zero!

(b) second, if weight is set to positive numbers then bias can be $-3$,

(c) third, if weight is set to a negative number, then any bias won't satisfy our problem.

(iii)




$$a = \text{hardlims}(wp + b)$$

Code is attached in the file named E212.py

E2.3

two neuron $w=[3\ \ 2]$

$p=[-5\ \ 7]^T$

$a=0.5$ $\quad a=f(wp+b)$

(i) if bias = 0 $\longrightarrow$

$$a=f\left([3\ \ 2]\begin{bmatrix}-5\\7\end{bmatrix}+0\right)=f(-15+14)=f(-1)$$

X there is ~~NO~~ transfer function defined in chapter 2 that outputs 0.5 for the input number −1.

(ii) transfer function purelin

Yes

$$a=purelin(wp+b)=purelin(-1+b)=0.5$$

if we define $b=1.5 \longrightarrow$ then purelin will work.

(iii) transfer function logsig

NO

$$a=logsig(wp+b)=logsig(-1+b)=0.5$$

$$\frac{1}{1+e^{(1-b)}}=0.5 \Rightarrow \frac{1}{2}+\frac{1}{2}e^{(1-b)}=1 \Rightarrow e^{1-b}=-1 \text{ (A)}$$

(A) $\longrightarrow$ e to the power of any number cannot be a negative value, so there is no answer for b.

X in result we cannot define log sigmoid transfer function here.

E2.3 continue:

(iv) transfer function, hardlims

NO

if we look at the hardlims transfer function closely, we can find that there is no way to use this function in this problem. because the possible outputs in this function are -1, 0, 1.

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E2.4

(i) we have four input and six output

(a) for the second layer six neron are sufficient.

(b) for the first layer any number of neurons between 1 to 4 can be used.

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(ii) first layer weight matrix:

rows: count of neurons → 1 to 4 = Ⓐ

columns: " " inputs: 4

size(first layer weight matrix) = Ⓐ, 4

second layer weight matrix:

rows: count of the layer neurons: 6

columns: " " neurons in the last layer → 1 to 4

size(second layer weight matrix) = 6, Ⓐ

E2.4 continue:

(iii) for the second layer logsig function can be applied here.
for the first layer any transfer function can be applied!

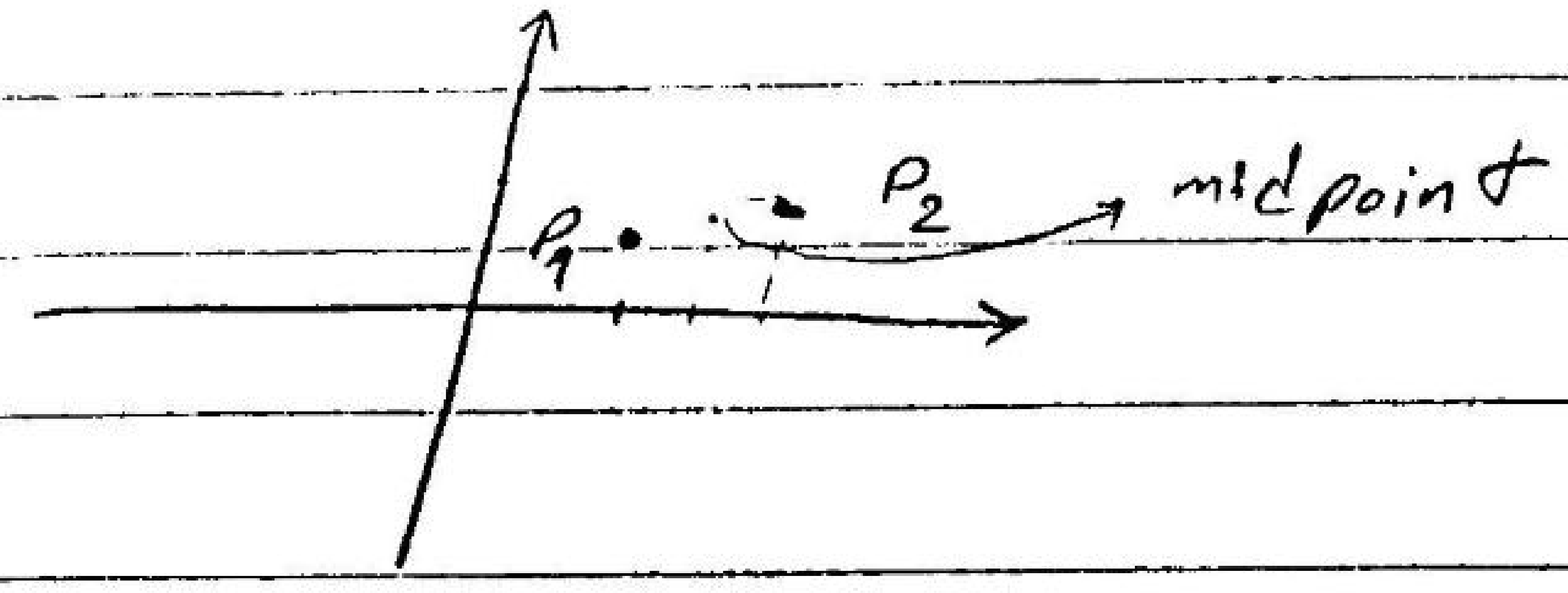
(iv) with respect to the first layer transfer function and if more information was given, we could answer this part.

# Neural Network

E3.

$P_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad P_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(i)




$a = hardlims(w^T p + b)$

we must make a boundary that seperate P1 and P2 well. we will try the graphical method used in book.

for $P_1$ $a = -1$ then $wP_1 + b < 0$
for $P_2$ $a = 1$ $wP_2 + b \geq 0$

$\Rightarrow [w_{11} \quad w_{12}] \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + b < 0 \Rightarrow w_{11} + \frac{1}{2} w_{12} + b < 0$

$\Rightarrow [w_{11} \quad w_{12}] \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \geq 0 \rightarrow 2w_{11} + w_{12} + b \geq 0$

with bias $= 0 \Longrightarrow$ $w_{11} + \frac{1}{2} w_{12} < 0$

$2w_{11} + w_{12} \geq 0$

computing equations : $\begin{cases} w_{11} + \frac{1}{2} w_{12} = 0 \\ 2w_{11} + w_{12} = 0 \end{cases}$ $\rightarrow$ there is no answer for the equation because two equations are both equal

so we try a different bias

As we know from the mid point method the line must go through the point $\begin{bmatrix} 1.5 \\ 0.75 \end{bmatrix}$

$P_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ $\quad P_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$x = \frac{2+1}{2} = 1.5$

$y = \frac{1+0.5}{2} = 0.75$

So we can define the bias as $-1.5$ beacuse of the mid point

let's try it

$b = -1.5$

$$[w_{11} \;\; w_{12}] \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + b < 0$$

$$[w_{11} \;\; w_{12}] \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \geq 0$$

$$w_{11} + \frac{1}{2} w_{12} - 1.5 < 0$$

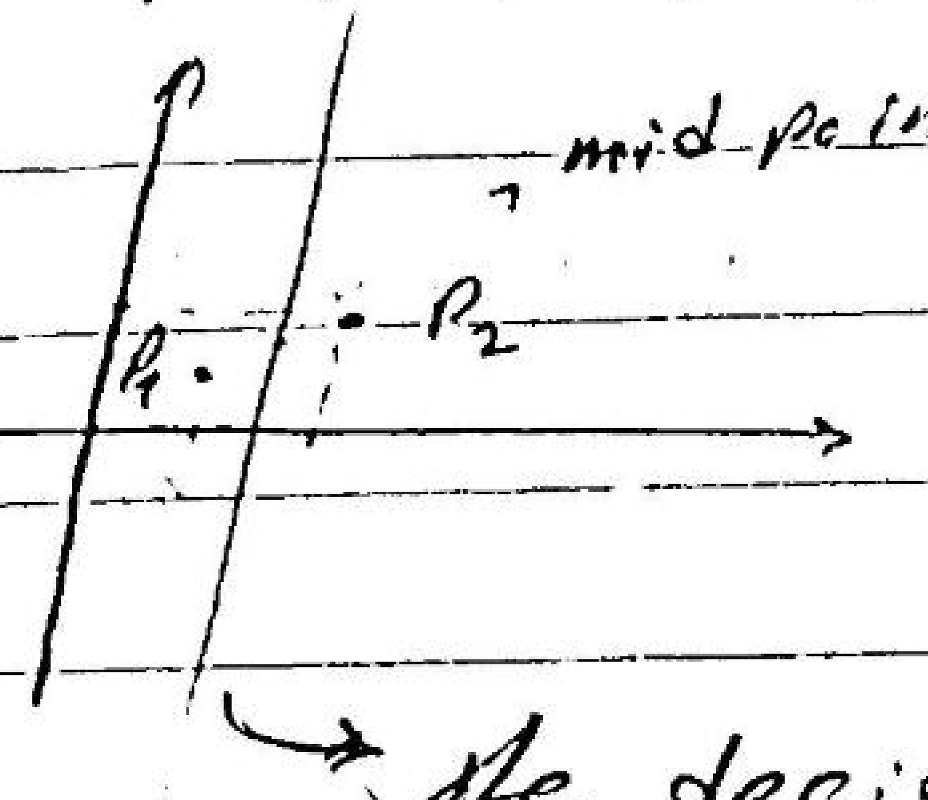
$$2w_{11} + w_{12} - 1.5 \geq 0$$

to find the points overlap:

$$\begin{cases} w_{11} + 0.5\, w_{12} - 1.5 = 0 \\ 2w_{11} + w_{12} - 1.5 = 0 \end{cases} \xrightarrow{\times(-2)} \begin{aligned} -2w_{11} - w_{12} + 3 = 0 \\ 2w_{11} + w_{12} - 1.5 = 0 \end{aligned}$$

Again there is no way to find an answer here.

So we try to look at the diagram and plot a decision boundary for it.




plot it again
we can plot a decision boundary just by the look of the eye!

the decision boundary

(ii)

To find the weights we first write the line equation

$x = +1.5$ ⇝ the line

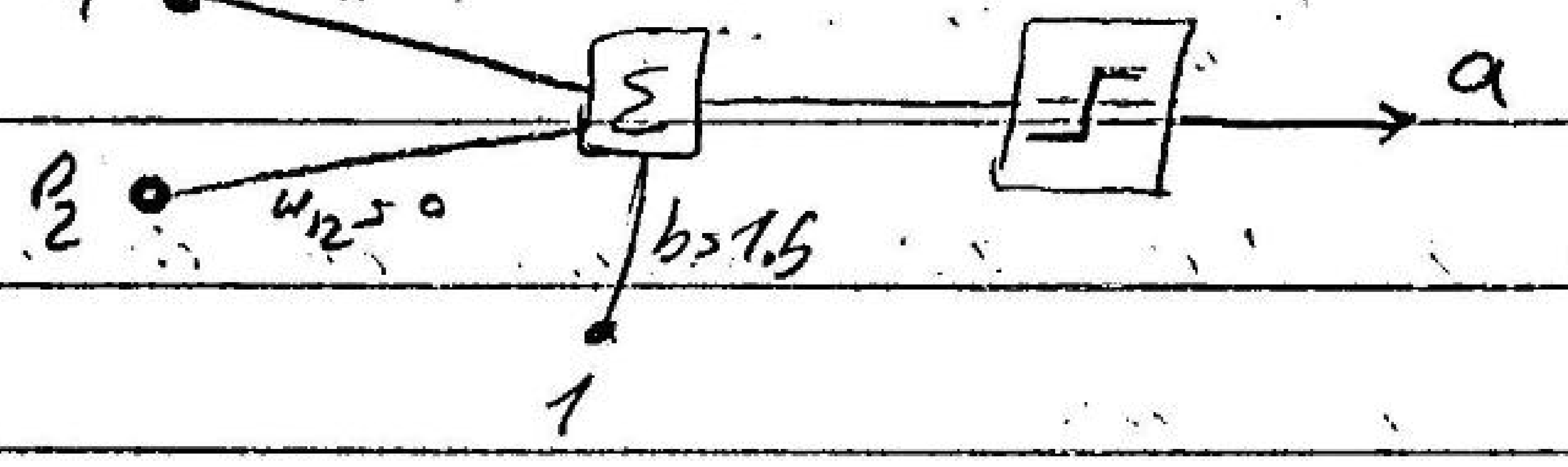
so it can be formulated with weight and biases as below:

(suppose $w_{11}$ be the coefficient $x$
and $w_{12}$ be the " $y$
and bias is just a number)

So $w_{11} = 1$, $w_{12} = 0$, bias $= 1.5$

$w_{11} - 1.5 = 0$

to sketch the network diagram




(iii)

$$\underbrace{[1 \quad 0]}_{w} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{p} - \underbrace{1.5}_{bias} = -0.5 \rightsquigarrow hardlims(-0.5) = -1$$

the point $p$ is in the left side of decision boundary, So the expected output must be $-1$

And by the decision rule we made, the network response was reasonable

E3.4

$W = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\qquad$ w.shape = (2,2) $S \times R$

$b = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ $\qquad$ b.shape = (2,1)

$S \rightarrow$ neuron count $\Rightarrow$ 2 nerons we have

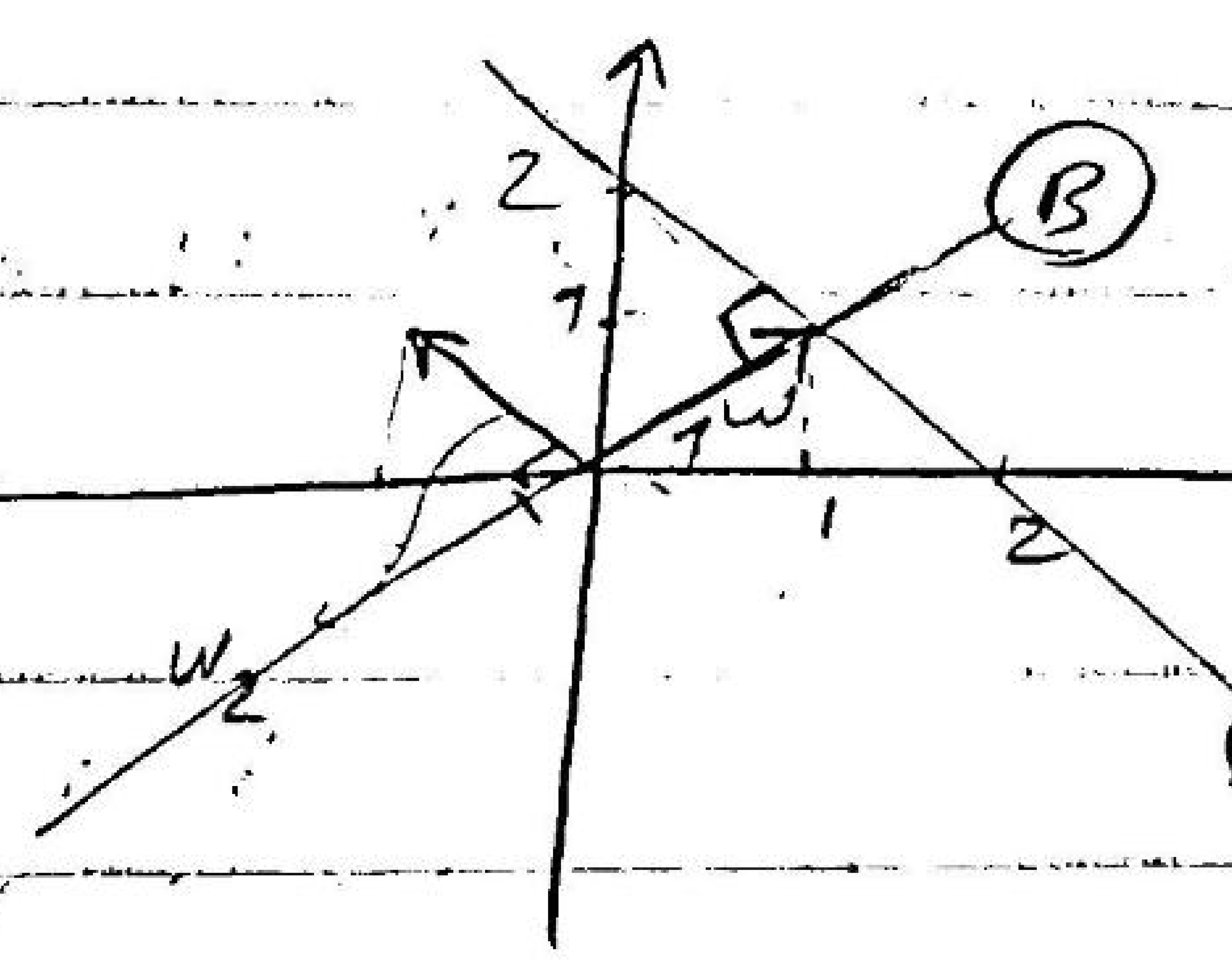
$R \rightarrow$ input count $\rightarrow$ 2 inputs " "

2 neurons can classify up to $2^2$ classes.

(i) so we have 4 classes!

$W = \begin{bmatrix} {}_1w^T \\ {}_2w^T \end{bmatrix}$

(ii)




${}_1w^T = [1 \ 1]$

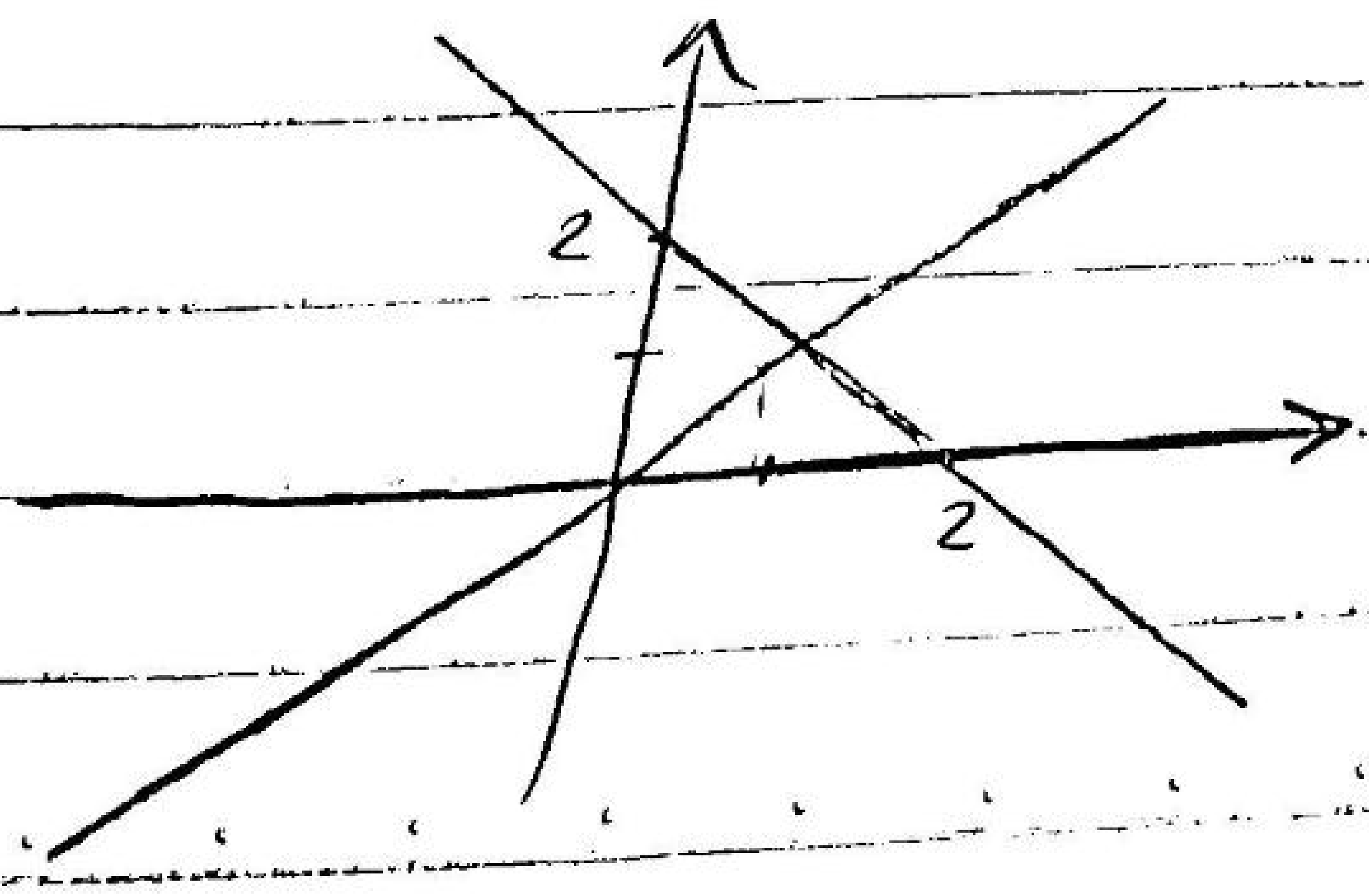
${}_2w^T = [-1 \ 1]$

(A) bias for ${}_1w$ is $-2$

(A) the first boundary $\rightarrow$ $x + y - 2 = 0$ $\quad {}_1w = [1 \ 1]$

(B) bias for the second line is zero, So it will go through the origin. $-x + y = 0 \Rightarrow x = y$

draw the decision boundaries without weights to make it more transparent



(iii)

$$a = hardlims(Wp + b)$$

$$a = hardlims\left(\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\begin{bmatrix} 0 \\ -2 \end{bmatrix}} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = hardlims\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

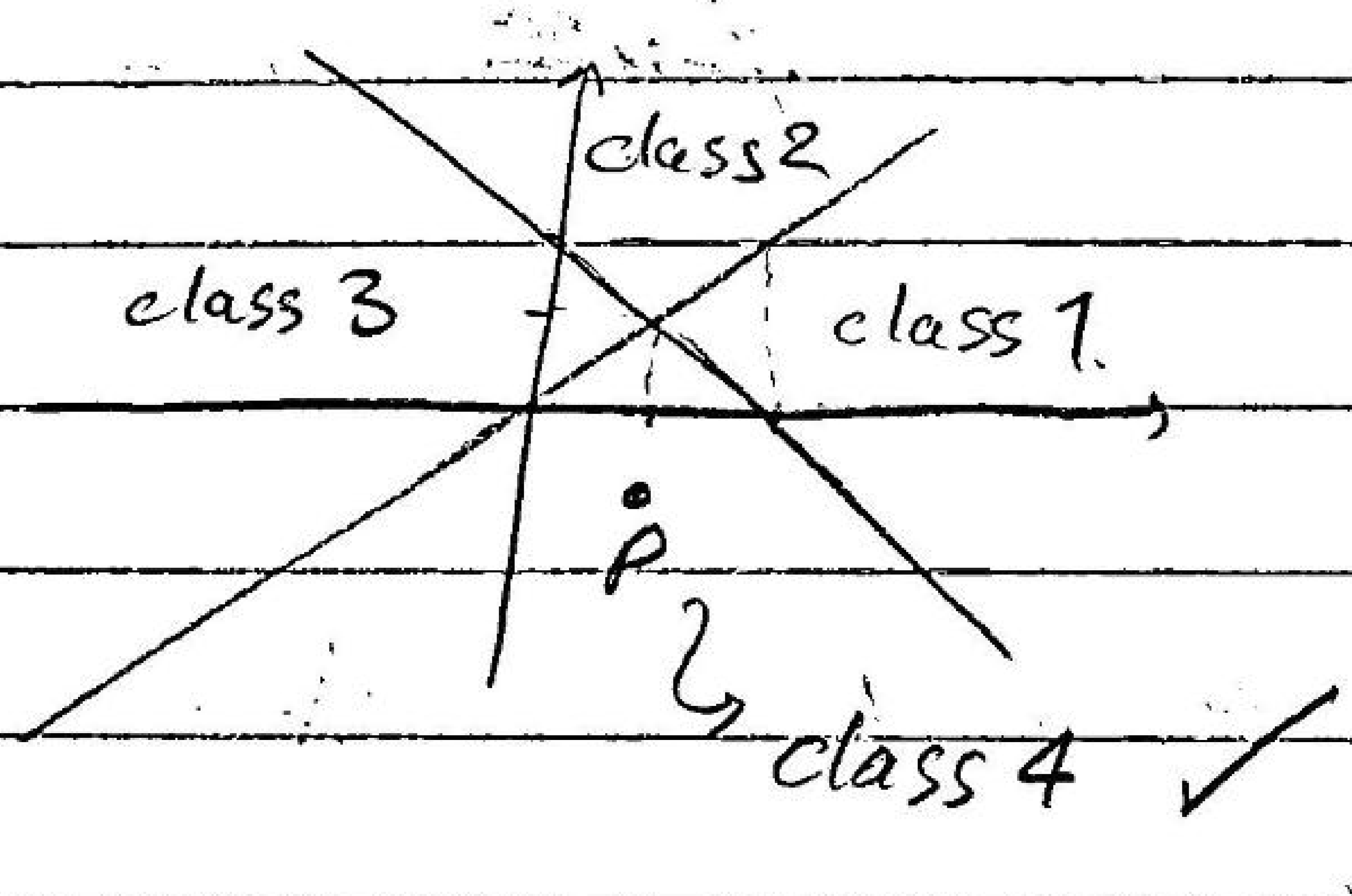
If we define the classes lable as below

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow$ class 1 $\quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ class 2 $\quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ class 3

$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \rightarrow$ class 4

the label for input P will be class 4

(iv)




It is correctly in its labeled region

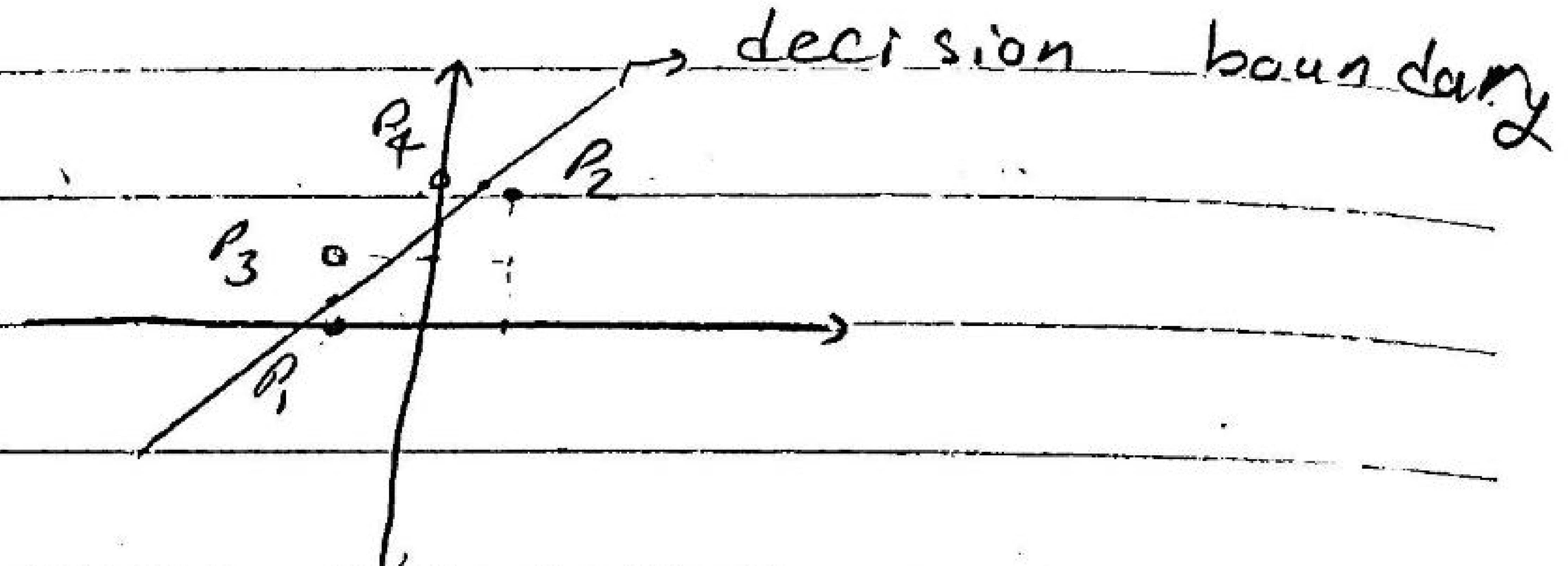
E3.5)

we can define hardlims function so its output (-1 and 1) is equal to our need.

(i)

$P_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $P_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $P_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$




(ii)

we can define the decision boundaries in the midpoint of $P_1, P_3$ and $P_2, P_4$

$$\text{mid point}(P_1, P_3) = \begin{bmatrix} \frac{P_1^x + P_3^x}{2} \\ \frac{P_1^y + P_3^y}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}$$

$$\text{mid point}(P_2, P_4) = \begin{bmatrix} \frac{P_2^x + P_4^x}{2} \\ \frac{P_2^y + P_2^y}{2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix} \text{Ⓐ}$$

slope $m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1.5}{1.5} = 1$

so the slope of the weight matrix is inverse negative of the decision boundary $\rightarrow$ slope weight $= -1$

decision boundary equation $\Rightarrow$ $(y - 2) = 1(x - \frac{1}{2}) \Rightarrow y = x + \frac{3}{2}$ Ⓐ

weight equation $\rightarrow$ $(y - 0) = -1(x - 0) \Rightarrow y = -x$

(it goes from the origin) $w = [-1 \ \ 1]$

(iii) to find the bias

$w^T p + b = 0 \Rightarrow b = -w^T p \sim b = -[-1 \ \ 1]\begin{bmatrix} -1 \\ 0.5 \end{bmatrix}$

$\Rightarrow b = -(1 + 0.5) = -1.5$

$a = \text{hardlims}(w^T p + b) = \text{hardlims}(-P_x + P_y - 1.5)$

(iv)

$a$ must be 1 for $P_1$, $P_2$

lets test it $P_1 \rightarrow a = \text{hardlims}(-(-1)+0) = 1$ ✓

$P_2 \rightarrow a = \text{hardlims}(-1+2) = 1$ ✓

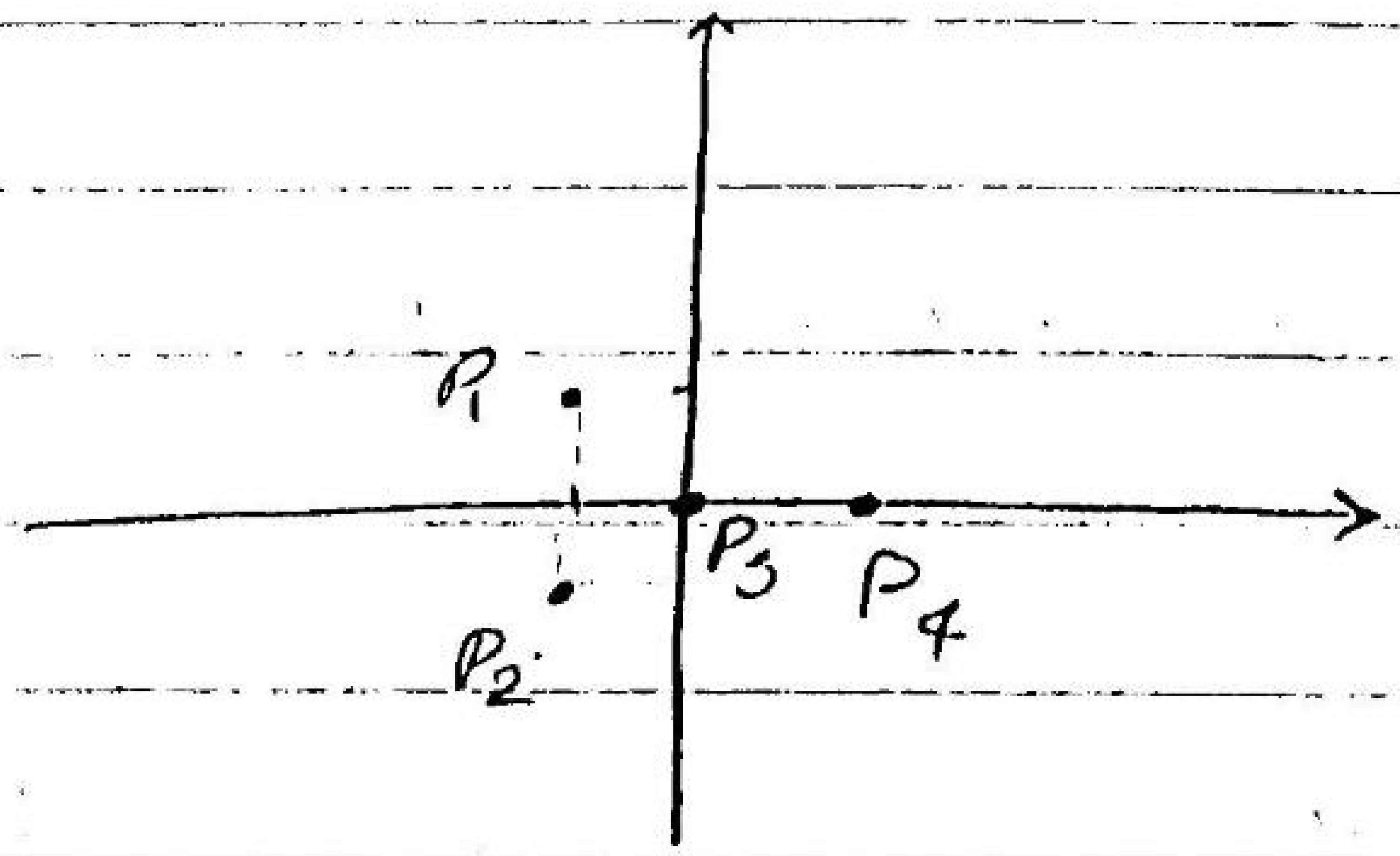
$a$ must be $-1$ for $P_3$, $P_4$

lets test is :

$P_3 \longrightarrow a = \text{hardlims}(1+1) = 1$ ✗ ⇝ غلط زي ما!

## E 4.21

(i)



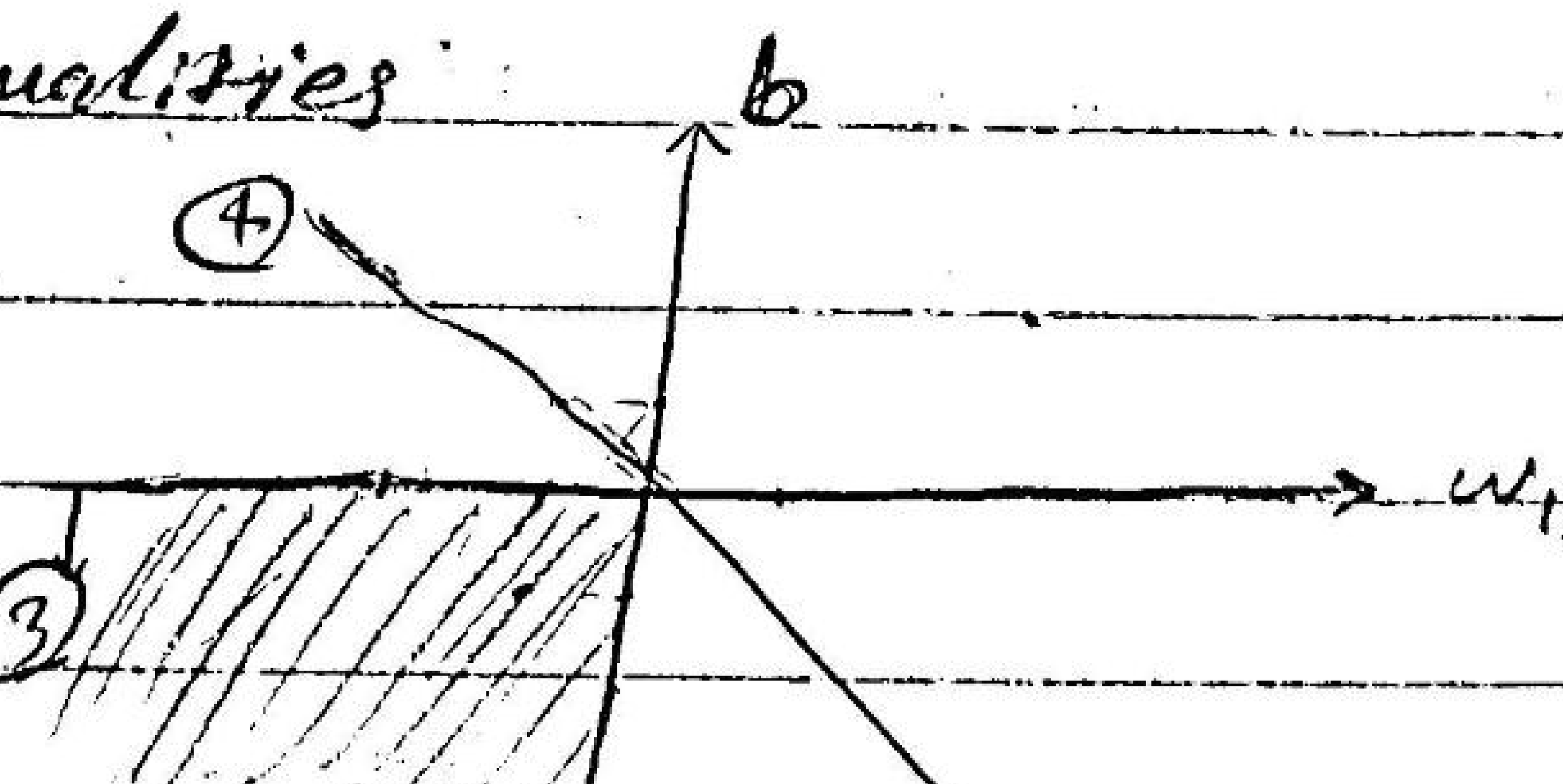

it can be known as the outputs for neurons are 1 and zero the activation function that can be used here is hardlim

$a = \text{hardlim}(w^T p + b)$

for $P_1, P_2$ the $w^T p + b$ must be semi-positive
for $P_3, P_4$ " $w^T p + b$ " " negative

$$\left.\begin{array}{l} P_1 \longrightarrow w_{11} P_{1x} + w_{12} P_{1y} + b \geq 0 \\ P_2 \longrightarrow w_{11} P_{2x} + w_{12} P_{2y} + b \geq 0 \\ P_3 \longrightarrow w_{11} P_{3x} + w_{12} P_{3y} + b < 0 \\ P_4 \longrightarrow w_{11} P_{4x} + w_{12} P_{4y} + b < 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} -w_{11} + w_{12} + b \geq 0 & (1) \\ -w_{11} - w_{12} + b \geq 0 & (2) \\ 0 + 0 + b < 0 & (3) \\ w_{11} + 0 + b < 0 & (4) \end{array}\right.$$

plotting the inequalities




(A) $\left\{\begin{array}{l} w_{11} = -2 \\ b = -1 \end{array}\right.$

$(A) \xrightarrow{(2)} +2 - w_{12} - 1 \geq 0 \Rightarrow w_{12} \leq 1$

$(1) \longrightarrow +2 + w_{12} - 1 \geq 0 \Rightarrow w_{12} \geq -1$ ↳ we define it as $w_{12} = 0$

So the results are:

$$W^T = [-2 \ \ 0] \qquad b = -1$$

(ii)
test the results

$p_1 \to \text{hardlim}(W^T p + b) = \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}-1\\1\end{bmatrix} - 1\right) = \text{hardlim}(1) = 1$ ✓

$p_2 \to \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}-1\\-1\end{bmatrix} - 1\right) = \text{hardlim}(1) = 1$ ✓

$p_3 \to \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}0\\0\end{bmatrix} - 1\right) = \text{hardlim}(-1) = 0$ ✓

$p_4 \to \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}1\\0\end{bmatrix} - 1\right) = \text{hardlim}(-3) = 0$ ✓

So training set had no errors.

(iii)

$p_5 \to \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}-2\\0\end{bmatrix} - 1\right) = \text{hardlim}(3) = 1$

$p_6 \to \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}1\\1\end{bmatrix} - 1\right) = \text{hardlim}(-3) = 0$

$p_7 \to \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}0\\1\end{bmatrix} - 1\right) = \text{hardlim}(-1) = 0$

$p_8 \to \text{hardlim}\left([-2 \ \ 0]\begin{bmatrix}-1\\-2\end{bmatrix} - 1\right) = \text{hardlim}(1) = 1$

(iv)
$p_7$ depends on the bias because the inner product of it with weights will be always zero.

E4.3

the Answer to E4.2 question was using inequality method. we can define a range of values [-1, 1] for $w_{12}$ until $w_{11} = -2$ and $b = 1$.

So we try $W = [-2 \ -1]$ , $b = -1$

$p_1 \rightarrow a = hardlim\left([-2 \ -1]\begin{bmatrix}-1\\1\end{bmatrix} - 1\right) = hardlim(0) = 1$ ✓

$p_2 \rightarrow a = hardlim\left([-2 \ -1]\begin{bmatrix}-1\\-1\end{bmatrix} - 1\right) = hardlim(2) = 1$ ✓

$p_3 \rightarrow a = hardlim\left([-2 \ -1]\begin{bmatrix}0\\0\end{bmatrix} - 1\right) = hardlim(-1) = 0$ ✓

$p_4 \rightarrow a = hardlim\left([-2 \ -1]\begin{bmatrix}1\\0\end{bmatrix} - 1\right) = hardlim(-3) = 0$ ✓

we can define other values for weights and bias since we have inequalities.

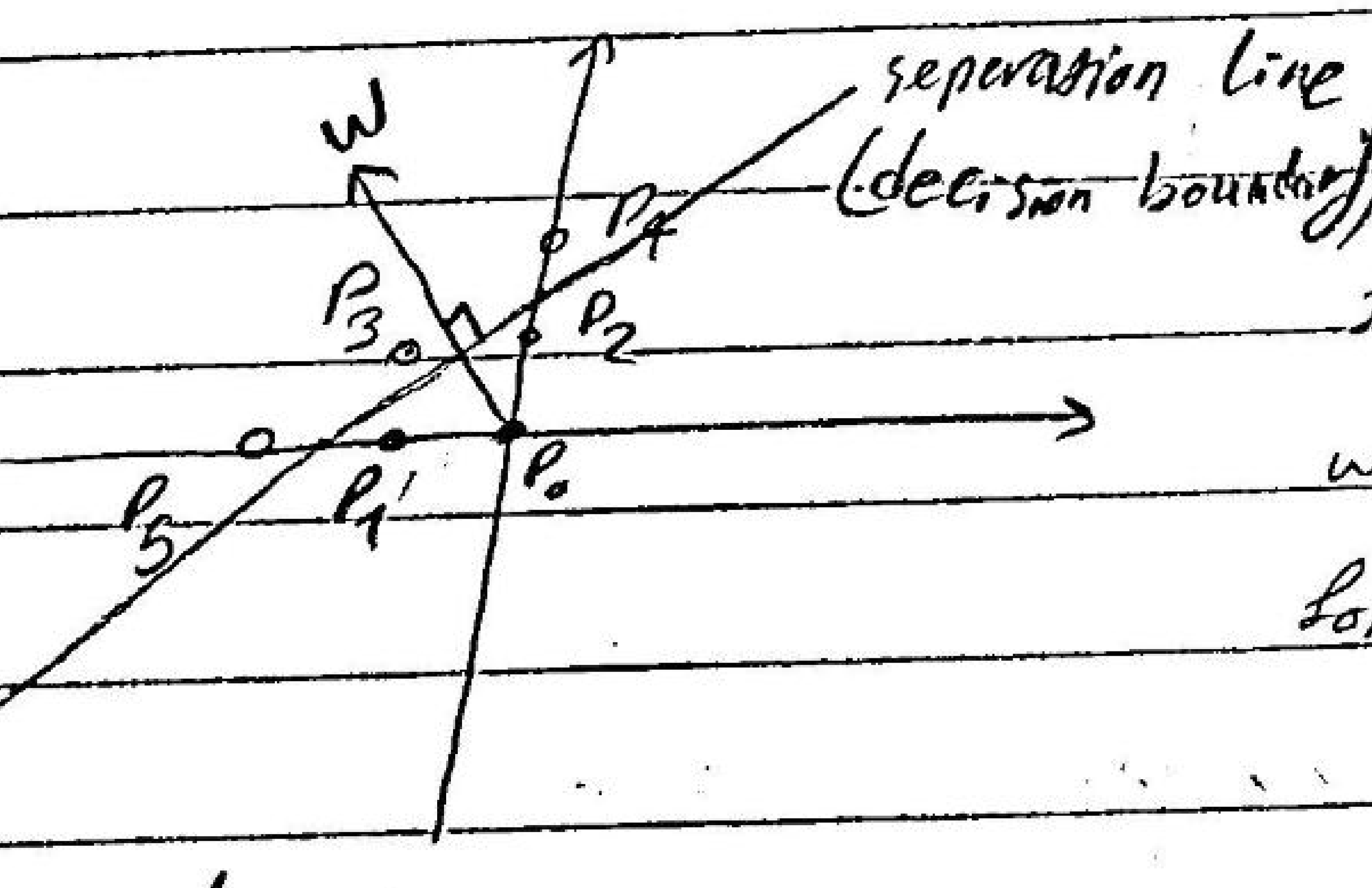
E4.7

To go with perceptron learning rule it would be cost alot of time. So we will try the graphical method.

At first we sketch the points and try to find a seperation line in the middle of the classes.

class I points $\rightarrow$ $P_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $P_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $P_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

class II points $\rightarrow$ $P_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $P_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $P_5 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$




we define output 1 for class II

we define output 0 for class I

from the plot it's easy to find the seperation line. The seperation line is between the two classes and for that we can say it goes through $\begin{bmatrix} -1.5 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$ so the line can be formulated

as: $m_{line} = \frac{0 - 1.5}{-1.5 - 0} = +1 \Rightarrow (y - 0) = -1(x - (-1.5))$

$y = +x - 1.5$

weights are orthogonal so its slope is $-1$

weights equation : $y = -x$

E4.7 continue

So the equation of weights is

$y = -x$ or $w_{11} = -w_{12}$

$w^T = [-1 \ \ 1]$

to find the biases we try one of the points on the decision boundary

$w^T p + b = 0 \Rightarrow b = -w^T p$

$b = -[-1 \ \ 1]\begin{bmatrix} -1.5 \\ 0 \end{bmatrix} = -1.5$

So lets test $w^T = [-1 \ \ 1]$, bias $= -1.5$

Before we start the test it's important to note that we use hardlim transfer function and the output 0 for class I and 1 for class II

$a_0 = \text{hardlim}(\overbrace{[-1 \ \ 1]\begin{bmatrix} 0 \\ 0 \end{bmatrix} - 1.5}^{-1.5}) = 0$ ✓

$a_1 = \text{hardlim}(\underbrace{[-1 \ \ 1]\begin{bmatrix} -1 \\ 0 \end{bmatrix} - 1.5}_{-0.5}) = 0$ ✓

$a_2 = \text{hardlim}(\underbrace{[-1 \ \ 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1.5}_{-0.5})$

$\Rightarrow a_2 = 0$ ✓

$a_3 = \text{hardlim}(\underbrace{[-1 \ \ 1]\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 1.5}_{0.5}) = 1$ ✓

$a_4 = \text{hardlim}(\underbrace{[-1 \ \ 1]\begin{bmatrix} 0 \\ 2 \end{bmatrix} - 1.5}_{0.5}) = 1$ ✓

$a_5 = \text{hardlim}([-1 \ \ 1]\begin{bmatrix} -2 \\ 0 \end{bmatrix} - 1.5) = 1$ ✓

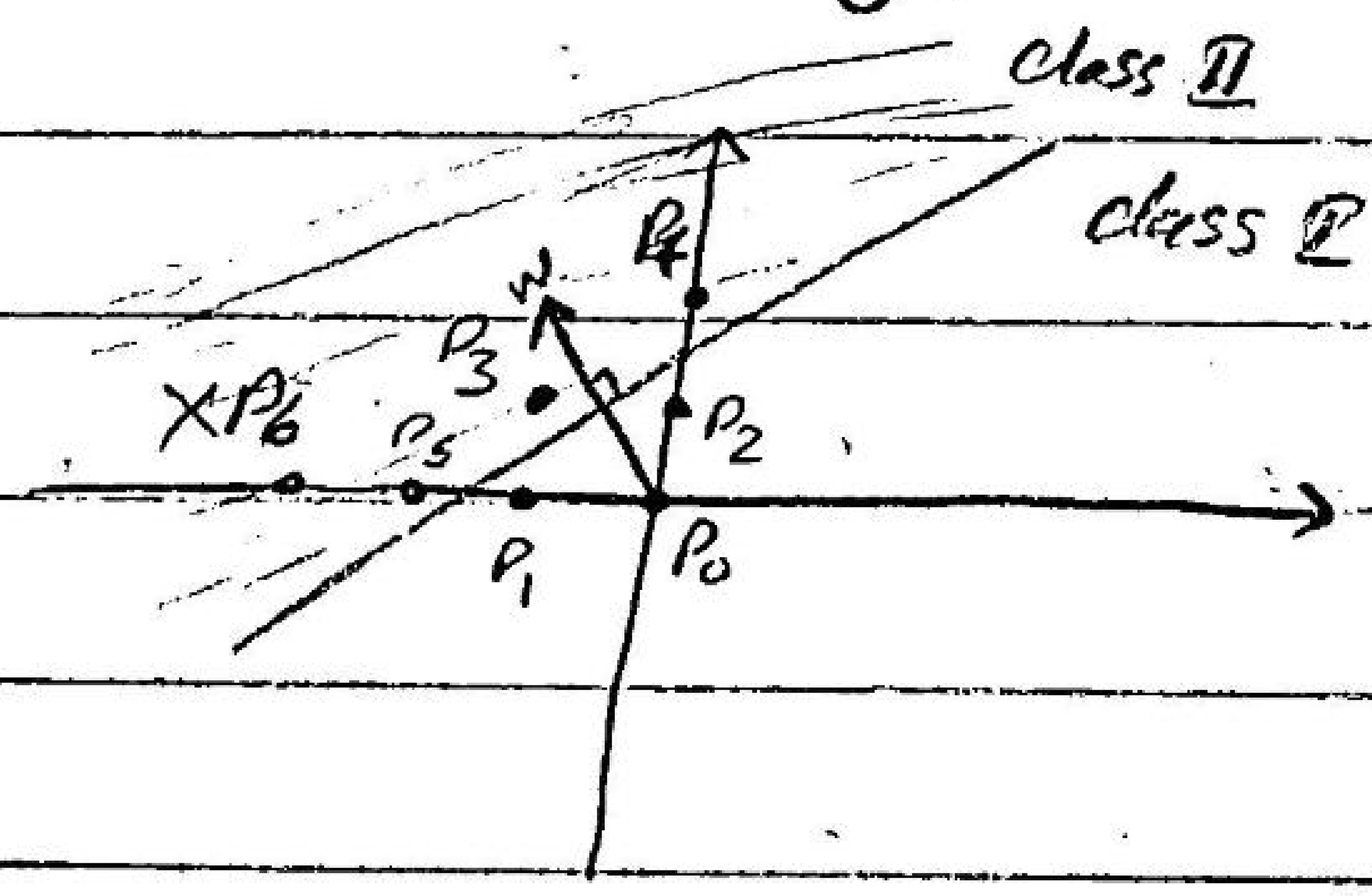
E4.7 continue

(iv) $P_6 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

$$a_6 = \text{hardlim}\left( \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} - 1.5 \right) = \text{hardlim}(1.5) = 1$$

the output was incorrect!

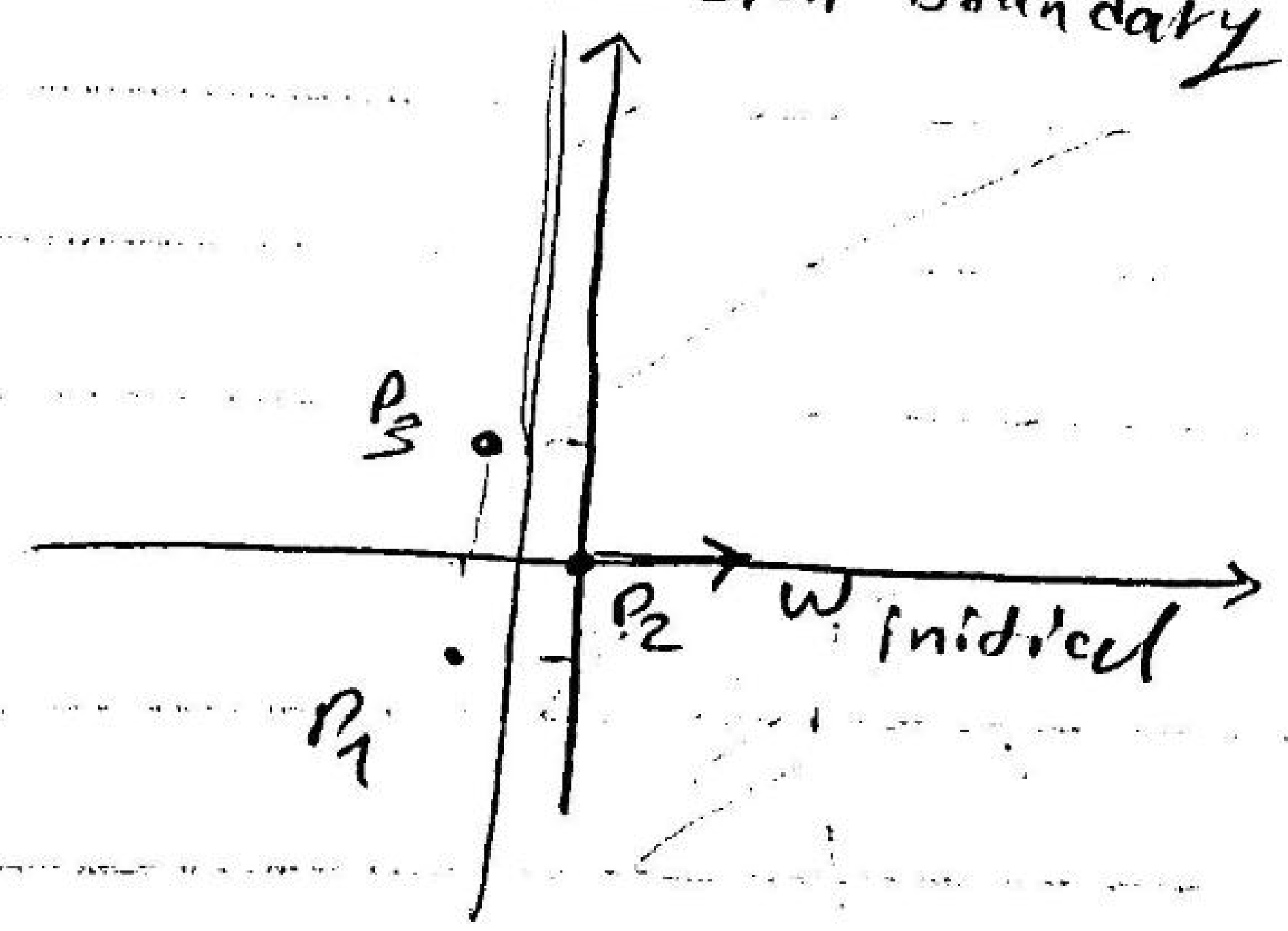
(v) To demonstrate the condition we first plot the points and decision boundary




if we include $P_6$ to the dataset, they could not be able to be linearly seperated, So the answer is No, we cannot classify $P_6$ correctly while the others ouputs are classified correct!

E 4.8

initial diecision boundary




(i)

$P_1$ is correctly classified and $P_3, P_2$ are misclassified

(ii)

from the training set it is obvious that hardlim function is used.

$a_1 = hardlim(W^T P_1 + b) = hardlim(\underbrace{[1\ 0]\begin{bmatrix}-1\\-1\end{bmatrix} + 0.5}_{-0.5}) = 0$ ✓

$e = t - a = 0 - 0 = 0 \Rightarrow w^{new} = w^{old} + \cancel{eP_1}$

$b^{new} = b^{old} + \cancel{e}$

$a_2 = hardlim(\underbrace{[1\ 0]\begin{bmatrix}0\\0\end{bmatrix} + 0.5}_{0.5}) = 1$ X

$e = 0 - 1 = -1$

$w^{new} = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

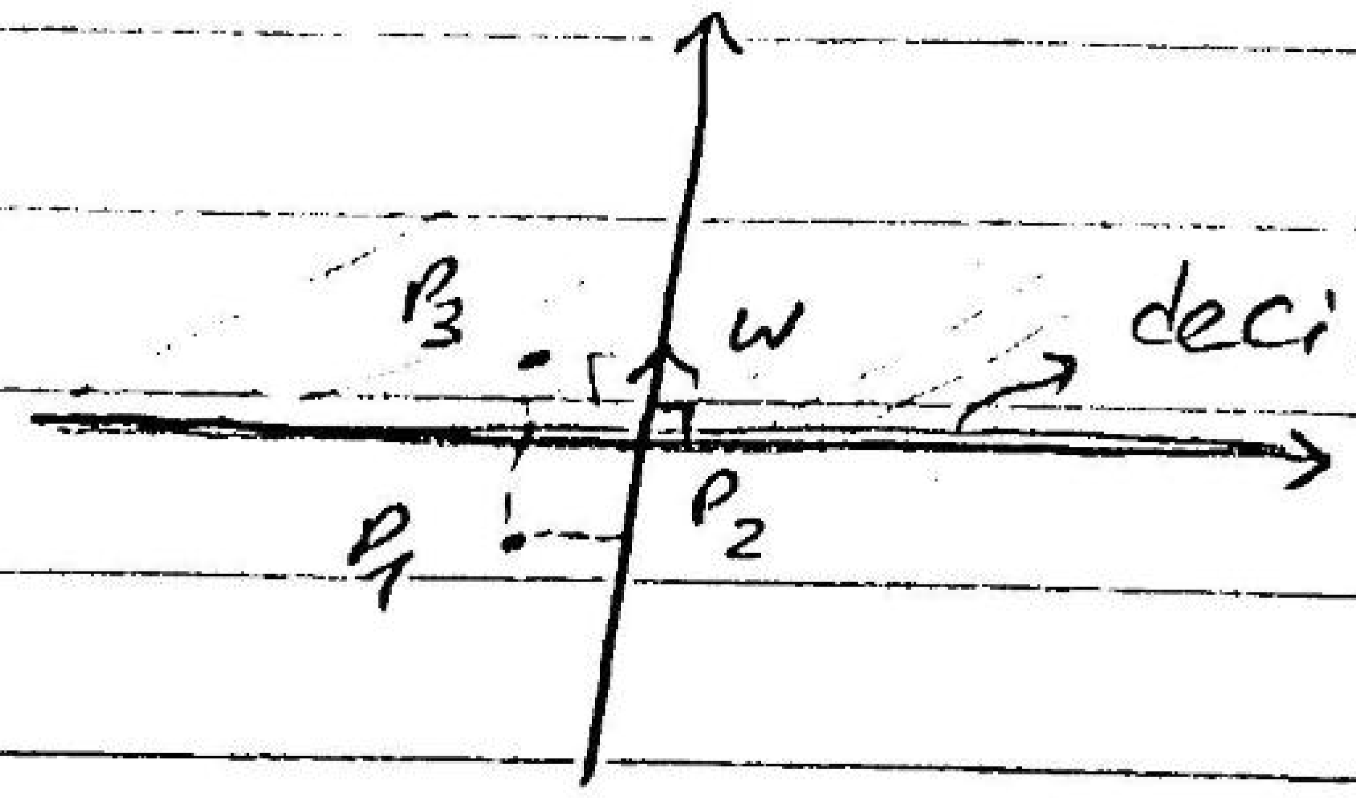
$b^{new} = 0.5 - 1 = -0.5$

$a_3 = hardlim(\underbrace{[1\ 0]\begin{bmatrix}-1\\1\end{bmatrix} - 0.5}_{-1.5}) = 0$ X $\quad e = 1 - 0 = 1$

$w^{new} = \begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad b^{new} = -1 + 1 = 0$

E4.8 Continue (

(iii)




decision boundary is the $x = 0$

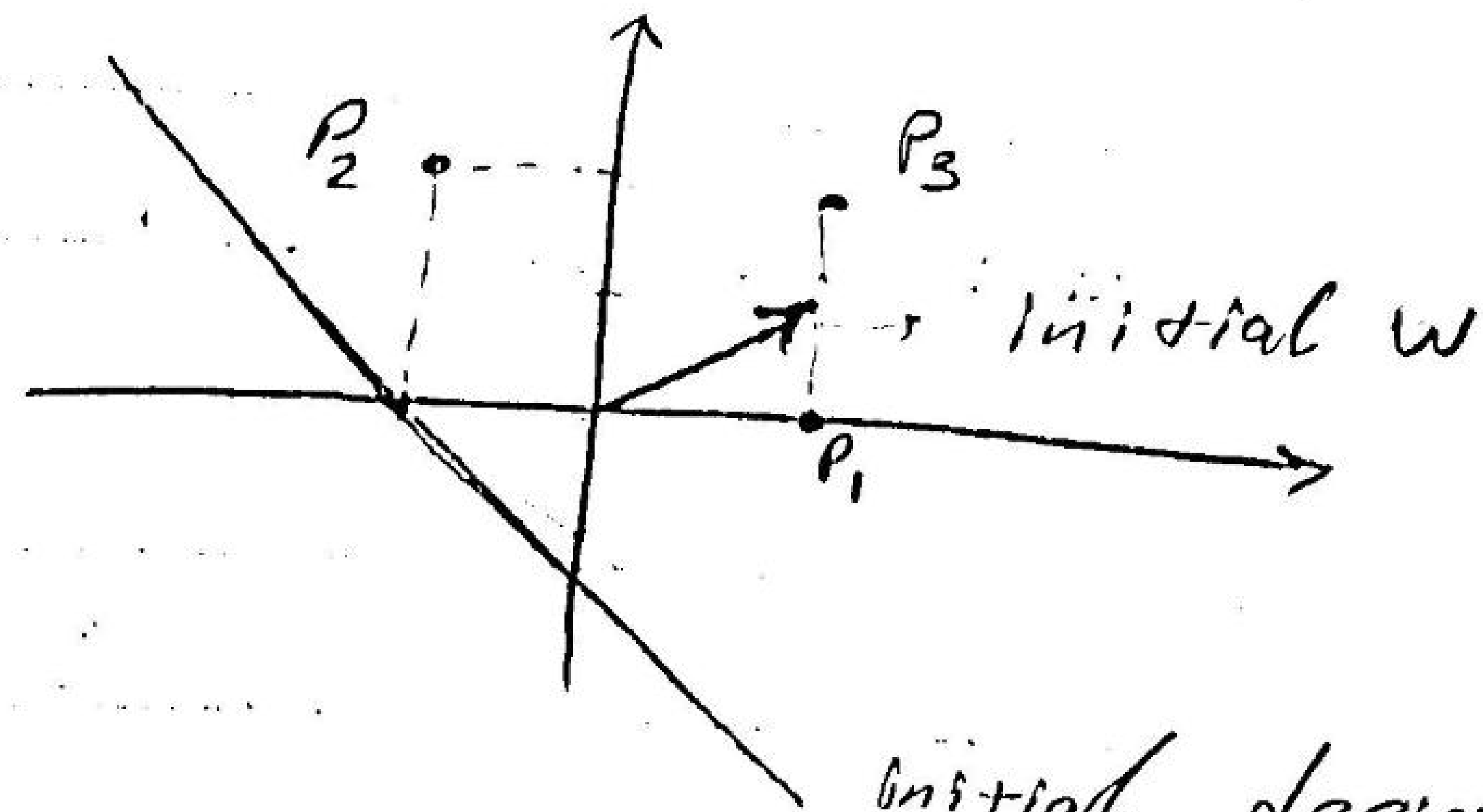
here $P_3$ and $P_1$ was correctly classified But $P_2$ wasn't Because it is on the decision boundary.

(iv)

of course it will learn the pattern. Becuase the training data is linearly seperable.

The answer to the question here is also Yes. No matter the initial weights are, eventually the perceptron rule will learn the data patterns The difference between various weights are the learning process may take longer time or less.

E4.? (i)




$P_1 = \begin{bmatrix}1\\0\end{bmatrix}, t_1 = 0$

$P_2 = \begin{bmatrix}-1\\2\end{bmatrix}, t_2 = 0$

$P_3 = \begin{bmatrix}1\\2\end{bmatrix}, t_3 = 1$

initial decision boundary

$$w^T P_3 + b = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} + 1 = 3 \xrightarrow[\text{function}]{\text{hardlim}} a_3 = 1$$

Just the third point is correctly classified

$$w^T P_2 + b = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} + 1 = 3 \xrightarrow{\text{hardlim}} a_2 = 1 \quad X$$
$$w^T P_1 + b = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + 1 = 1 \xrightarrow{\text{hardlim}} a_1 = 1 \quad X$$

does not correctly classified

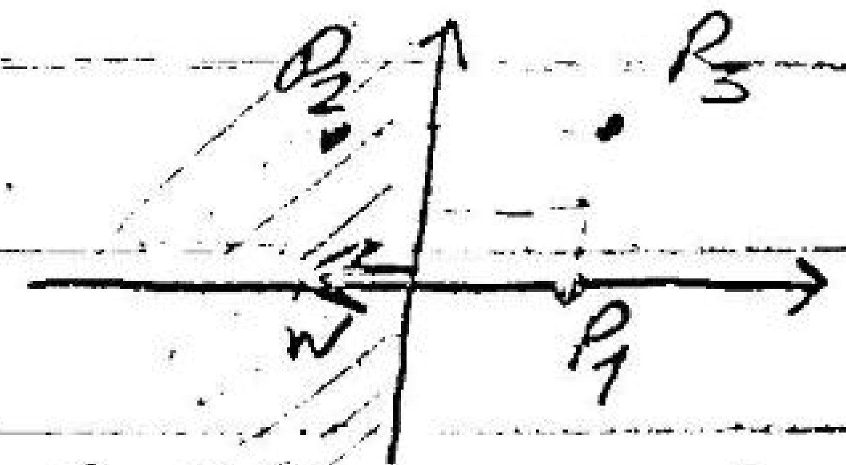
(ii) to reform weight vector we define the weight agian by the perceptron learning rule

$$W^{new} = W^{old} + e\, p$$

$e = t - a$

for $P_1$:

$$W^{new} = \begin{bmatrix}0\\1\end{bmatrix} + (0-1)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$



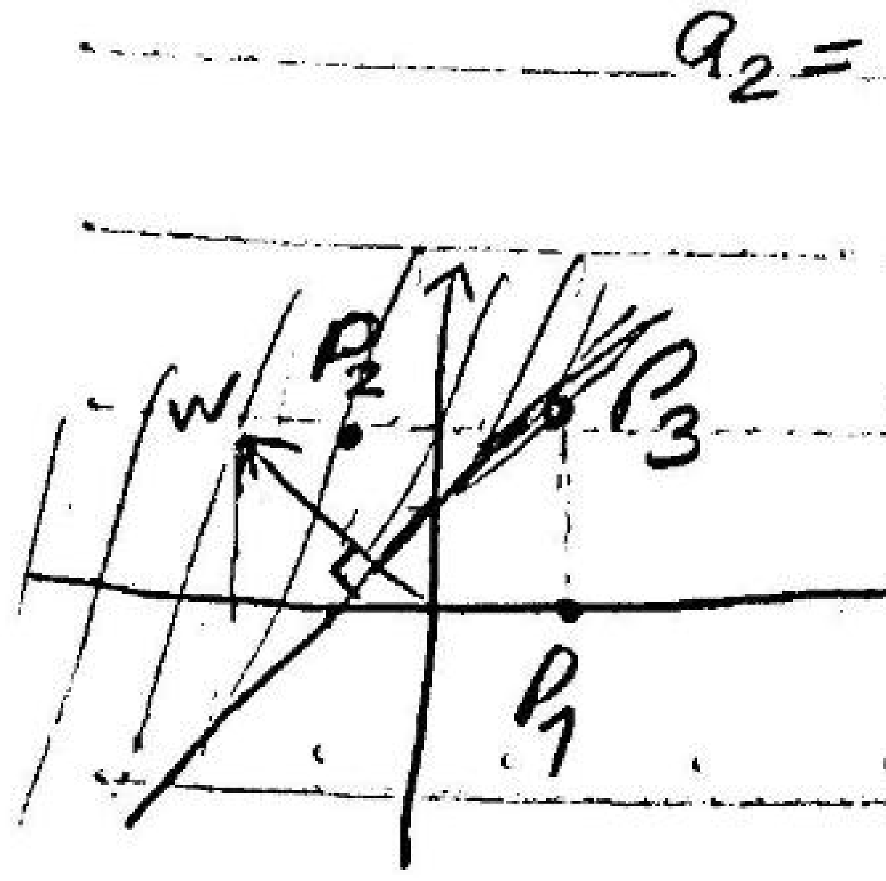

(iii) $b^{new} = b^{old} + e = 1 - 1 = 0$

$P_1, P_2, P_3$ are misclassified!

then try P2:

$$a_2 = \text{hardlim}\left(\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} + 0\right) = \text{hardlim}(1) = 1$$

$$W^{new} = \begin{bmatrix}-1\\0\end{bmatrix} + (1-0)\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$




$b = 0 + 1 = 1$

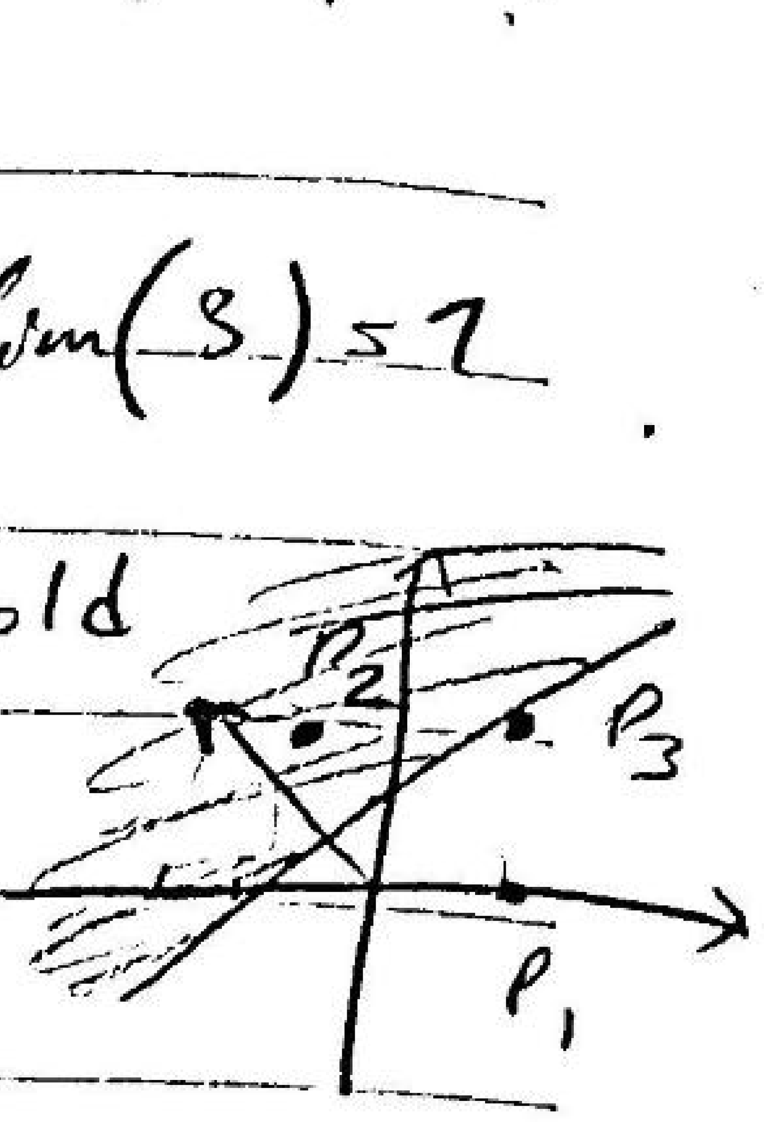
$P_2$ and $P_1$ are misclassified

(iv) try $P_3$ :

$$a_3 = hardlim\left(\begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}+1\right) = hardlim(3) = 1$$

$$e = t - a = 1 - 1 = 0 \Rightarrow w^{new} = w^{old}$$

$$b^{new} = b^{old}$$




(v) → $P_1$ and $P_2$ are misclassified

retry the points :

$P_1$ :

$$a_1 = hardlim\left(\begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}+1\right) = hardlim(-1) = 0$$

$$e = t - a = 0 - 0 = 0 \rightarrow w^{new} = w^{old} , b^{new} = b^{old}$$

$P_2$ :

$$a_2 = hardlim\left(\begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}+1\right) = hardlim(6) = 1$$

$$e = 0 - 1 = -1 \Rightarrow w^{new} = \begin{bmatrix}-2\\2\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$b^{new} = b^{old} + e = 1 - 1 = 0$$

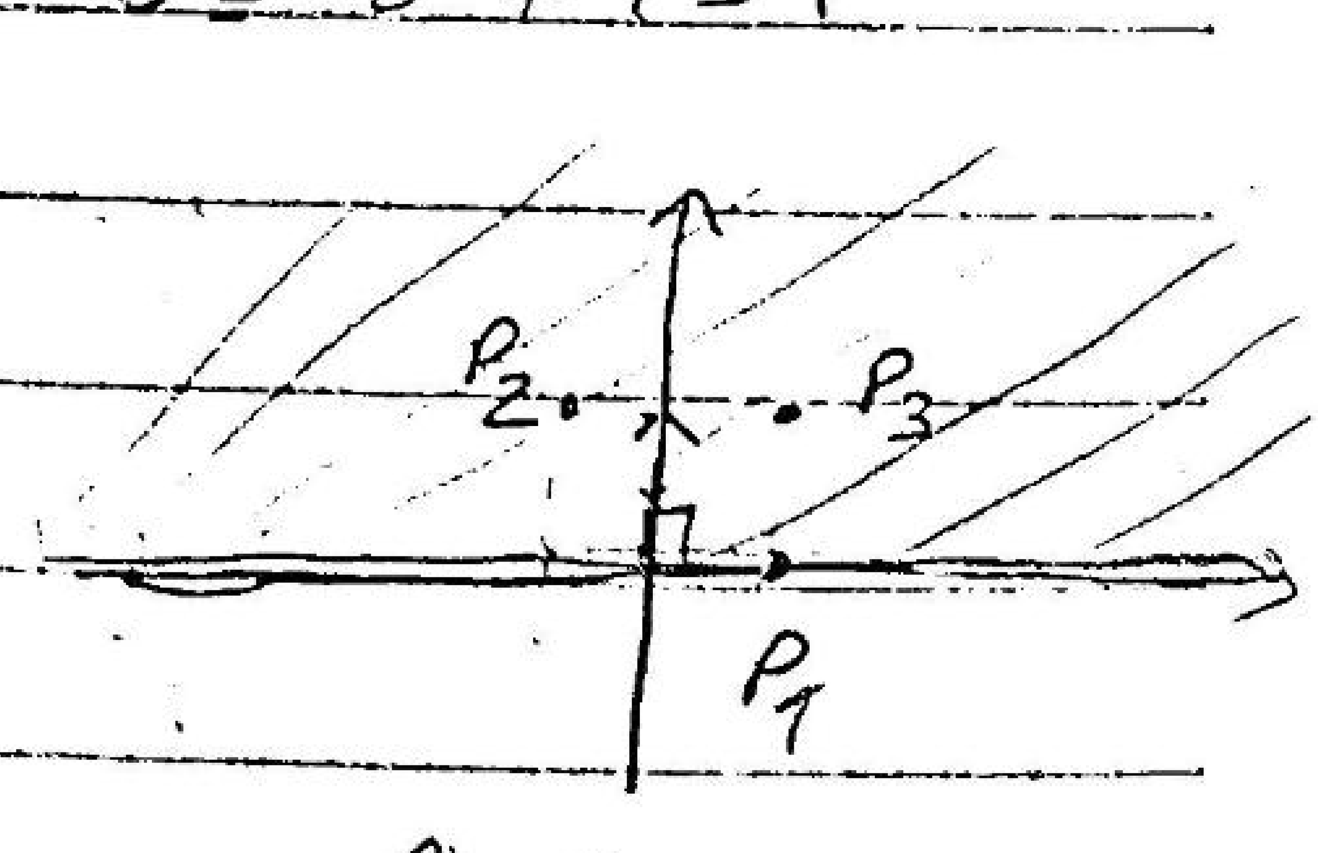
$P_3$ :

$$a_3 = hardlim\left(\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}+0\right) = hardlim(-1) = 0$$

$$e = t - a = 1 - 0 = 1$$

$$w^{new} = \begin{bmatrix}-1\\0\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} \qquad b^{new} = 0 + 1 = 1$$

$P_2$, $P_1$ → are misclassified




(vi)

If we continue the loop of learning eventually we can find the answer.

Because the points can be linearly seperated!

6.8
E [illegible]

to test that any set of vectors are independent we must prove that the matrix has inverse and coefficients are set to zero.

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ so

A

$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{vmatrix} = 2-2=0$

as we can see here the determinante of A is zero! so its values meaning the vectors in part (i) are correlated. we can say they are dependent.

(ii)

$\sin t, \cos t, \cos 2t$

we can say that if we find one of the vectors that can be the result of sum of the other vectors, then there is an aswer that we can project vectors into less vector space

$\cos(2t) = \cos(t+t) = \cos(t)\cos(t) - \sin(t)\sin(t)$

$= \cos^2(t) - \sin^2(t)$

the combination is not linear so we cannot find a vector from the others. therefor vectors are independent

6 6.8 continue(

(iii) $1+t$ $1-t$

there is no way we could find one from the other, so we can say these vectors are independent.

(iv) $x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix}$

we create the gramian matrix

Gramian matrix $\rightsquigarrow$ $A = \begin{bmatrix} x_1 \cdot x_1 & x_1 \cdot x_2 & x_1 \cdot x_3 \\ x_2 \cdot x_1 & x_2 \cdot x_2 & x_2 \cdot x_3 \\ x_3 \cdot x_1 & x_3 \cdot x_2 & x_3 \cdot x_3 \end{bmatrix}$

(it is the inner product of each vector)

$A = \begin{bmatrix} 10 & 2 & 22 \\ 2 & 2 & 6 \\ 22 & 6 & 50 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 10 & 2 & 22 \\ 2 & 2 & 6 \\ 22 & 6 & 50 \end{vmatrix} \begin{vmatrix} 2 & 22 \\ 2 & 6 \\ 6 & 50 \end{vmatrix} =$

$= (1000 + 72 + 2200) - (\cancel{44} * 22 + 72 + 2200) \neq 0$

So as we can see determinant is not zero in result it can be said that vectors are independent!

E 5.10 |

$$y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad y_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad y_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In Gram-schmidt orthogonalization the first vector was equal to $y_1$, so

$$v_1 = y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

then to find other vectors:

$$v_k = y_k - \sum_{i=1}^{k-1} \frac{v_i \cdot y_k}{v_i \cdot v_i} v_i$$

→ we remove the projection of the other vectors!

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{v_1^t y_2}{v_1^t \cdot v_1} v_1 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{[1\ 0\ 0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{[1\ 0\ 0]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = y_3 - \frac{v_2^t y_3}{v_2^t v_2} v_2 - \frac{v_1^t y_3}{v_1^t v_1} v_1$$

$$\Rightarrow v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{[0\ 1\ 0]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{[0\ 1\ 0]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{[1\ 0\ 0]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So the $y_1, y_2, y_3$ was projected onto $v_1, v_2, v_3$

$E6.11$

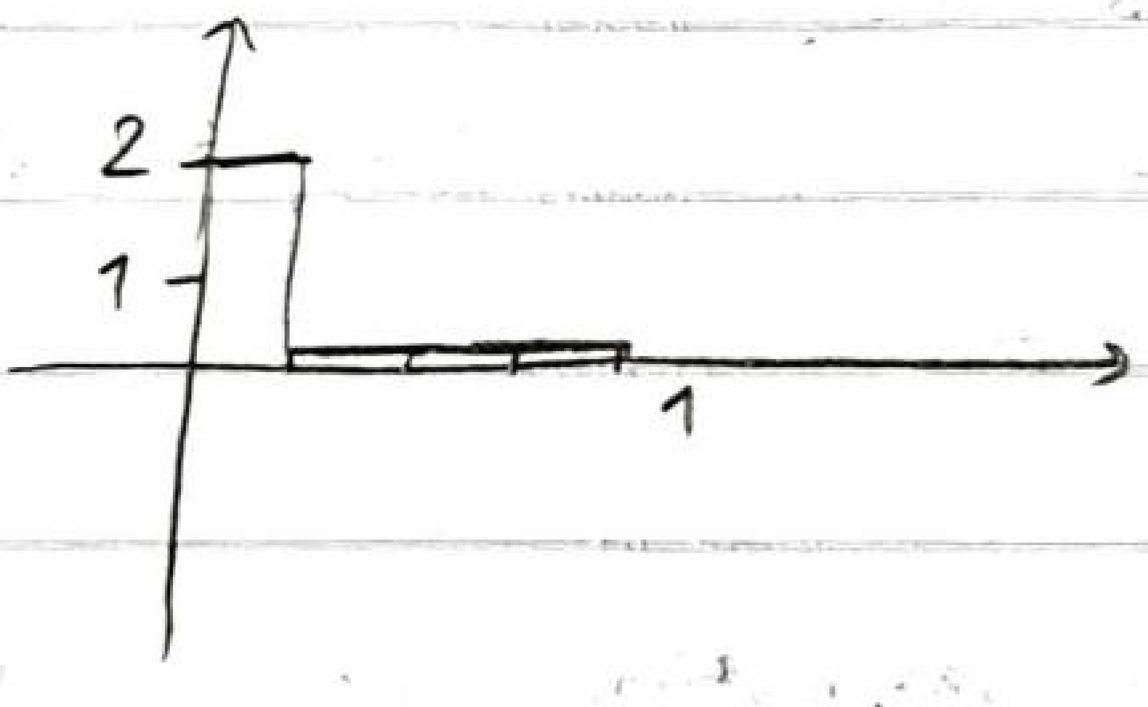
(i) to show that $f_1, f_2, f_3$ are linearly independent, we must prove that one of them cannot be calculated from the other.

there are 3 combination of $f$ functions than can show they are dependent.

$a_1 f_1(t) + a_2 f_2(t) = f_3(t)$ (1)
$a_2 f_2(t) + a_3 f_3(t) = f_1(t)$ (2)
$a_1 f_1(t) + a_3 f_3(t) = f_2(t)$ (3)

we try (1) with coeficients equal to 1

$f_1(t) + f_2(t)$




in general with unknown coefficients $a_1 f_1 + a_2 f_2$ equation will be:

$$a_1 f_1(t) + a_2 f_2(t) = \begin{cases} a_1 + a_2 & 0 \leq t \leq \frac{1}{4} \\ a_1 - a_2 & \frac{1}{4} < t \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_3(t) = \begin{cases} 1 & 0 \leq t < \frac{3}{4} \\ -1 & \frac{3}{4} < t \leq 1 \end{cases}$$

there is no linear combination for (1) equation

$$0 \leq t \leq \frac{3}{4} \longrightarrow \begin{cases} a_1 + a_2 = 1 \\ a_1 + a_2 = -1 \end{cases}$$

E 5.11 continue!

The equation (1) would not be ever satisfied so we try the (2)nd

$$a_2 f_2(t) + a_3 f_3(t) = \begin{cases} a_2 + a_3 & 0 \leq t \leq \frac{1}{4} \\ a_3 - a_2 & \frac{1}{4} < t \leq \frac{3}{4} \\ -a_3 - a_2 & \frac{3}{4} < t \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{o.w.} \end{cases}$$ , so the equations below must be satisfied:

$$\begin{cases} a_2 + a_3 = 1 \\ a_3 - a_2 = 1 \\ -a_3 - a_2 = 1 \Rightarrow a_2 + a_3 = -1 \end{cases}$$

$\times$ there is no answer for $a_3$ and $a_2$

So until now equations (1) and (2) was incorrect! we try the 3rd to show the functions are dependent,

$$(2) \to a_1 f_1(t) + a_3 f_3(t) = \begin{cases} a_1 + a_3 & 0 \leq t \leq \frac{3}{4} \\ a_1 - a_3 & \frac{3}{4} < t \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{4} \\ -1 & \frac{1}{4} < t \leq 1 \\ 0 & \text{o.w} \end{cases}$$

So for $a_1 + a_3$ we find 1 and -1

Again there is no answer for this equation

E6.11 Continue

we proved that there is no answer for (1), (2) and (3). In the results we can determine that the set $\{f_1, f_2, f_3\}$ are linearly independent.

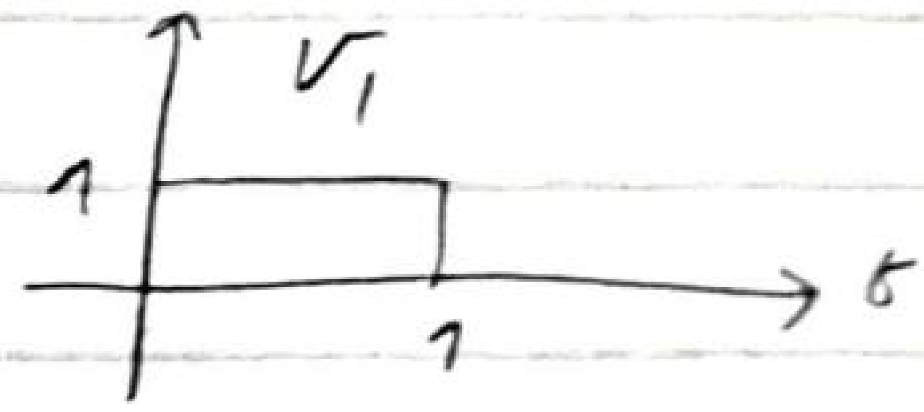
(ii)

Gram-Schmit procedure

$$v_1 = y_1 \qquad v_k = y_k - \sum_{i=1}^{k-1} \frac{v_i \cdot y_k}{v_i \cdot v_i} v_i \qquad k > 1$$

as the question said $\rightarrow$ $(f,g) = \int_0^1 f(t)\, g(t)\, dt$

So for $v_1 = f_1(t)$

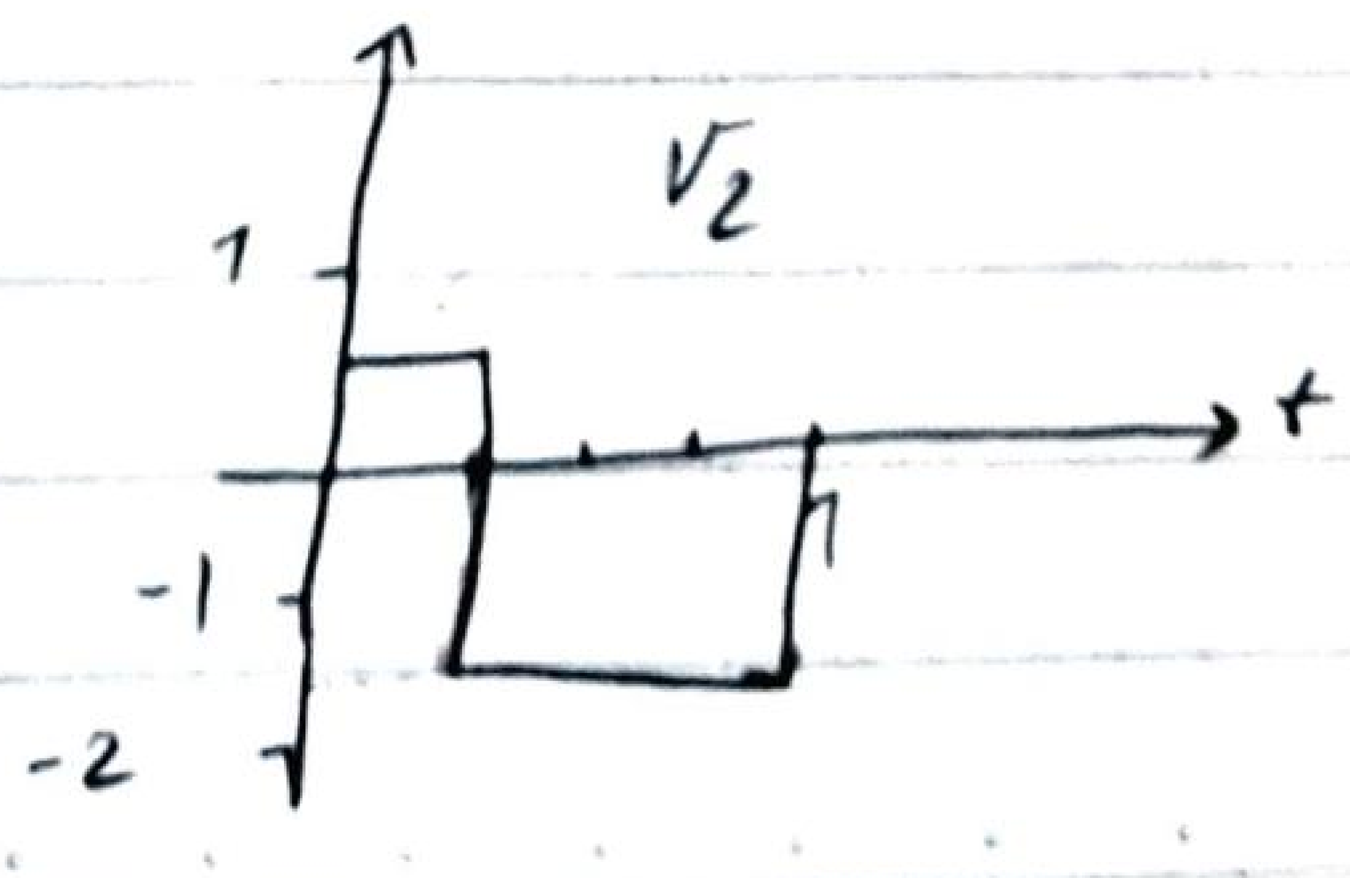



$$v_2 = f_2(t) - \frac{v_1 \cdot f_2(t)}{v_1 \cdot v_1} \cdot v_1$$

$$v_1 \cdot f_2(t) = \int_0^1 f_1(t)\, f_2(t)\, dt = \int_0^{\frac{1}{4}} 1\, dt + \int_{\frac{1}{4}}^{1} 1 \times -1\, dt$$

$$= \frac{1}{4} + \left(-1 + \frac{1}{4}\right) = -\frac{1}{2}$$

$$v_1 \cdot v_1 = \int_0^1 1.1\, dt = 1 \qquad v_2 = f_2(t) - \frac{1}{2} v_1$$



E6.11 continued

$$v_3 = f_3(t) - \frac{v_1 f_3(t)}{v_1 \cdot v_1} v_1 - \frac{v_2 f_3(t)}{v_2 \cdot v_2} \cdot v_2$$

$$v_1 \cdot f_3(t) = \int_0^1 v_1 f_3(t)\, dt = \int_0^{3/4} 1.1\, dt + \int_{\frac{3}{4}}^1 1\times(-1)\, dt$$

$$= \frac{3}{4} + \left(-1 - \left(-\frac{3}{4}\right)\right) = \frac{2}{4} = \frac{1}{2}$$

$v_1 . v_1 = 1 \rightsquigarrow$ as we calculated before.

$$v_2 f_3(t) = \int_0^1 v_2 \times f_3(t)\, dt = \int_0^{1/4} 1.1\, dt + \int_{\frac{1}{4}}^{\frac{3}{4}} -\frac{3}{2} \cdot 1\, dt$$

$$+ \int_{\frac{3}{4}}^1 -\frac{3}{2} \cdot -1\, dt = \frac{1}{4} - \frac{3}{2}\left(\frac{3}{4} - \frac{1}{4}\right) + \frac{3}{2}\left(1 - \frac{3}{4}\right)$$

$$\Rightarrow v_2 . f_3(t) = \frac{1}{4} - \frac{3}{4} + \frac{3}{8} = \frac{2-6+3}{8} = -\frac{1}{8}$$

$$v_2 . v_2 = \int_0^1 v_2 . v_2\, dt = \int_0^{1/4} +1\, dt + \int_{\frac{1}{4}}^1 -\frac{3}{2} \cdot -\frac{3}{2}\, dt =$$

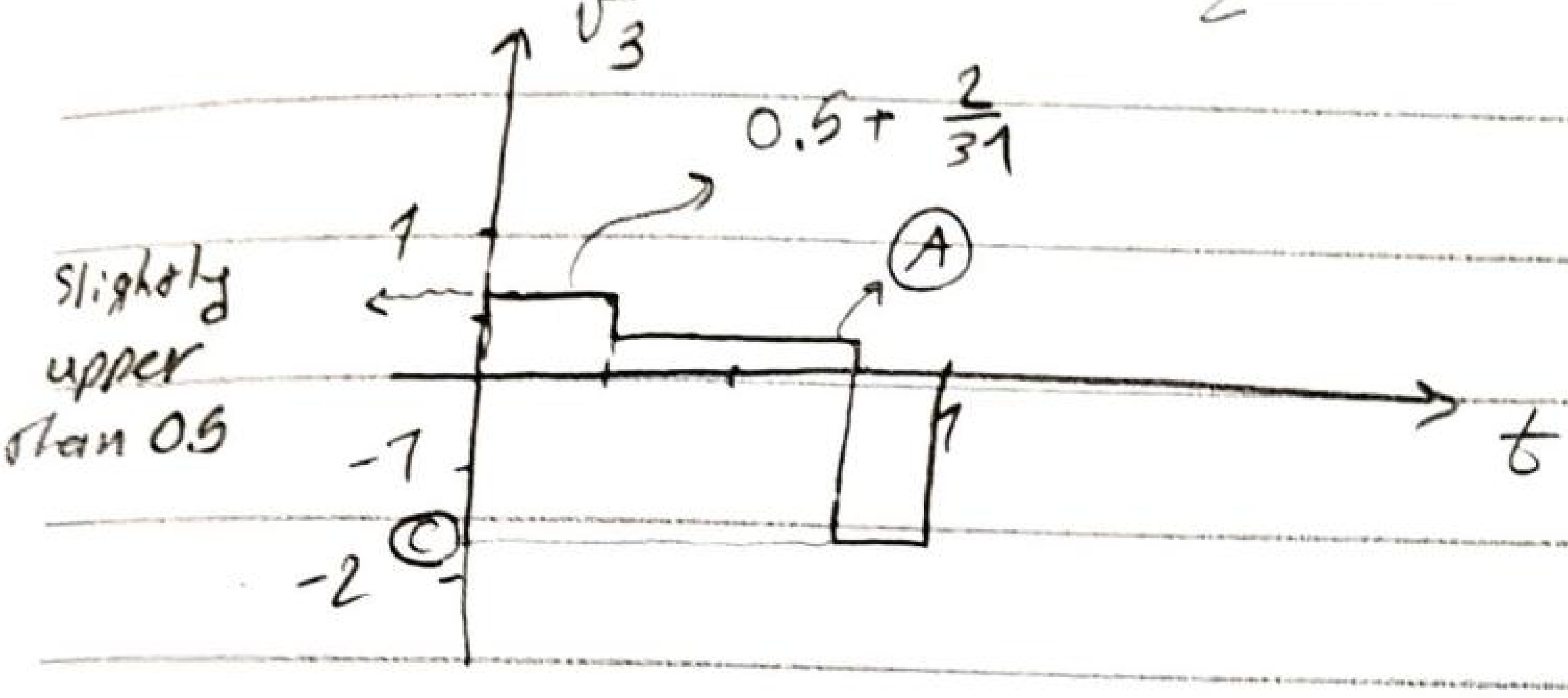
$$= \frac{1}{4} + \frac{9}{4}\left(1 - \frac{1}{4}\right) = \frac{1}{4} + \frac{27}{16} = \frac{31}{16}$$

So $v_3$ now can be found as

$$v_3 = f_3(t) - \frac{1}{2} v_1 - \frac{-\frac{1}{8}}{\frac{31}{16}} v_2 \Rightarrow \text{next page}$$

## E6.11 / Continue

$$v_3 = f_3(t) - \frac{1}{2} v_1 + \frac{\frac{1}{8}}{\frac{31}{16}} v_2 \qquad \frac{\frac{1}{8}}{\frac{31}{16}} = \frac{2}{31}$$




$-\frac{3}{2} \times \frac{2}{31} = -\frac{3}{31}$

A $\rightarrow 0.5 - \frac{3}{31}$

C $\rightarrow -\frac{3}{2} - \frac{2}{31} \times \frac{3}{2}$

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## E6.15 /

(i) In Gram Schmit method the first orthogonal vector is the same as the first vector of the question. So we have

$g_1 = f_1$ $\qquad\qquad g_k = f_k - \sum_{i=1}^{k-1} \frac{g_i \cdot f_k}{g_i \cdot g_i} g_i$

$$g_2 = f_2 - \frac{g_1 \cdot f_2}{g_1 \cdot g_1} g_1$$

$$g_1 \cdot f_2 = \int_0^1 g_1(t) f_2(t)\, dt = \int_0^1 1 (1-t)\, dt = t - \frac{t^2}{2} \Big]_0^1$$

$$\Rightarrow g_1 \cdot f_2 = \left(1 - \frac{1}{2}\right) - 0 = \frac{1}{2}$$

$$g_1 \cdot g_1 = \int_0^1 g_1(t) g_1(t)\, dt = \int_0^1 1\, dt = 1$$

$$g_2(t) = \begin{cases} 1 - t - \frac{1}{2}(1) = 0.5 - t & 0 < t < 1 \\ 0 & \text{o.w.} \end{cases}$$

E5.14]

(i) $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

transformation matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$x = v_1 + v_2 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

expansion for $x \longrightarrow \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\Rightarrow x = 2s_1 + 0(s_2)$

(ii)

$y = 1s_1 + 1s_2 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A^{-1} = \frac{1}{1-(-1)} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

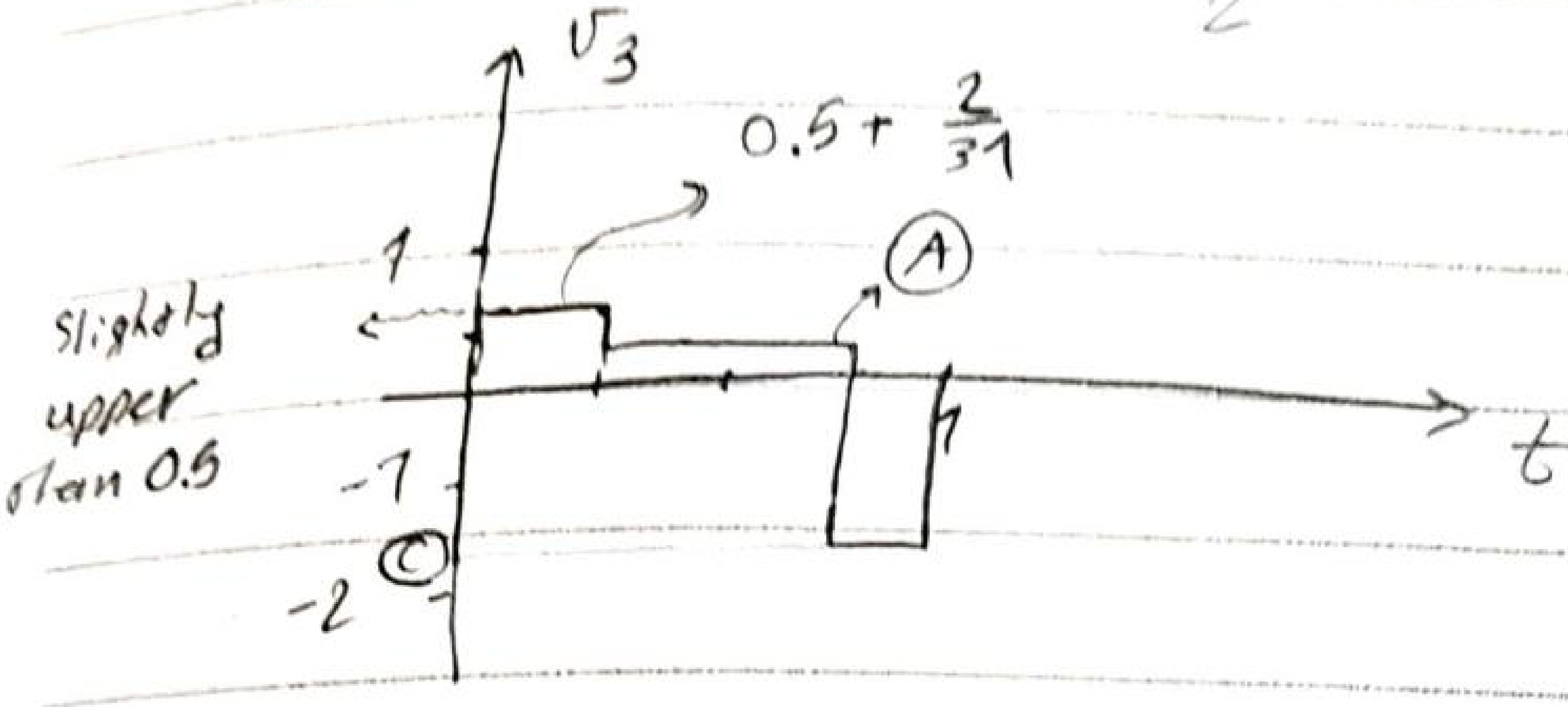
$A^{-1}y = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} \Rightarrow y = (0)v_1 + \frac{3}{2}v_2$

E6.11| continue

$$\frac{2}{31}$$

$$v_3 = f_3(t) - \frac{1}{2} v_1 + \frac{\frac{1}{8}}{\frac{31}{16}} v_2$$

$2$




$$-\frac{3}{2} \times \frac{2}{31} = -\frac{3}{31}$$

A $\rightarrow 0.5 - \frac{3}{31}$

C $\rightarrow -\frac{3}{2} - \frac{2}{31} \times \frac{3}{2}$

---

E6.15|

(i) In Gram Schmit method the first orthogonal vector is the same as the first vector of the question. So we have

$$g_1 = f_1 \qquad g_k = f_k - \sum_{i=1}^{k-1} \frac{g_i f_k}{g_i \cdot g_i} g_i$$

$$g_2 = f_2 - \frac{g_1 \cdot f_2}{g_1 \cdot g_1} g_1$$

$$g_1 \cdot f_2 = \int_0^1 g_1(t) f_2(t)\, dt = \int_0^1 1(1-t)\, dt = t - \frac{t^2}{2} \Big]_0^1$$

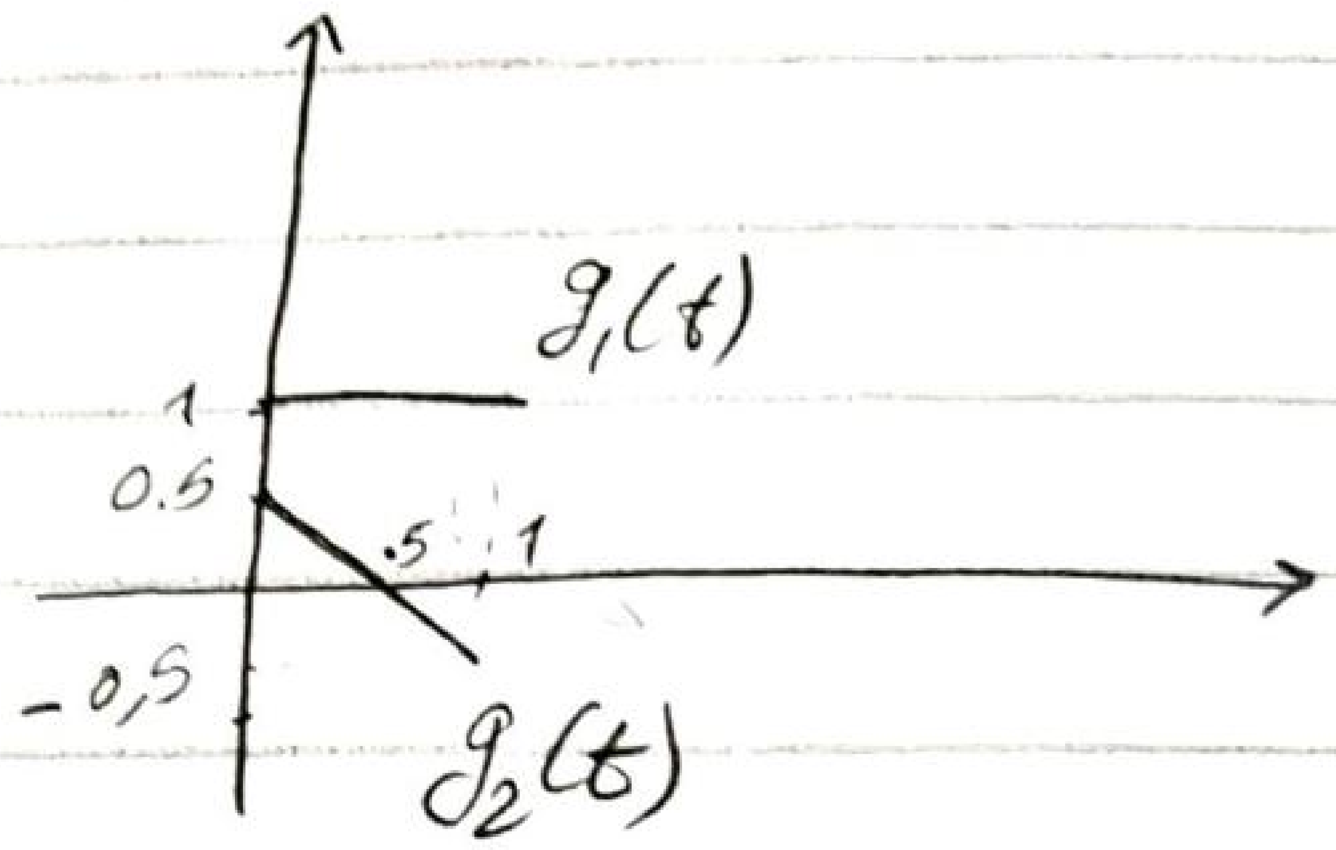
$$\Rightarrow g_1 \cdot f_2 = \left(1 - \frac{1}{2}\right) - 0 = \frac{1}{2}$$

$$g_1 \cdot g_1 = \int_0^1 g_1(t) g_1(t)\, dt = \int_0^1 1\, dt = 1$$

$$g_2(t) = \begin{cases} 1 - t - \frac{1}{2}(1) = 0.5 - t & 0 < t < 1 \\ 0 & \text{o.w.} \end{cases}$$

Continue

E5.15 (




(ii)

$$x_j = \frac{v_j \cdot x}{(v_j \cdot v_j)}$$

$$h_1(t) = \frac{g_1(t) \cdot h(t)}{g_1(t) \cdot g_1(t)} \qquad h_2(t) = \frac{g_2(t) \cdot h(t)}{g_2(t) \cdot g_2(t)}$$

$$g_1(t) \cdot h(t) = \int_0^1 (t-1) \cdot 1 \; dt = \frac{t^2}{2} - t \Big]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$g_1(t) \cdot g_1(t) = \int_0^1 1 \cdot 1 \; dt = t \Big]_0^1 = 1$$

$$h_1(t) = \frac{-\frac{1}{2}}{1} = -\frac{1}{2} \checkmark$$

$$g_2(t) \cdot h(t) = \int_0^1 \left(\tfrac{1}{2} - t\right)(t-1) \; dt = \int_0^1 -t^2 + \frac{3}{2} t - \frac{1}{2} \; dt$$

$$= -\frac{t^3}{3} + \frac{3t^2}{4} - \frac{t}{2} \Big]_0^1 = \left(-\frac{1}{3} + \frac{3}{4} - \frac{1}{2}\right) - 0 = \frac{-4+9-6}{12} = -\frac{1}{12}$$

$$g_2(t) \cdot g_2(t) = \int_0^1 \left(\tfrac{1}{2} - t\right)\left(\tfrac{1}{2} - t\right) dt = \int_0^1 \frac{1}{4} + t^2 - t \; dt = \frac{t}{4} + \frac{t^3}{3} - \frac{t^2}{2} \Big]_0^1$$

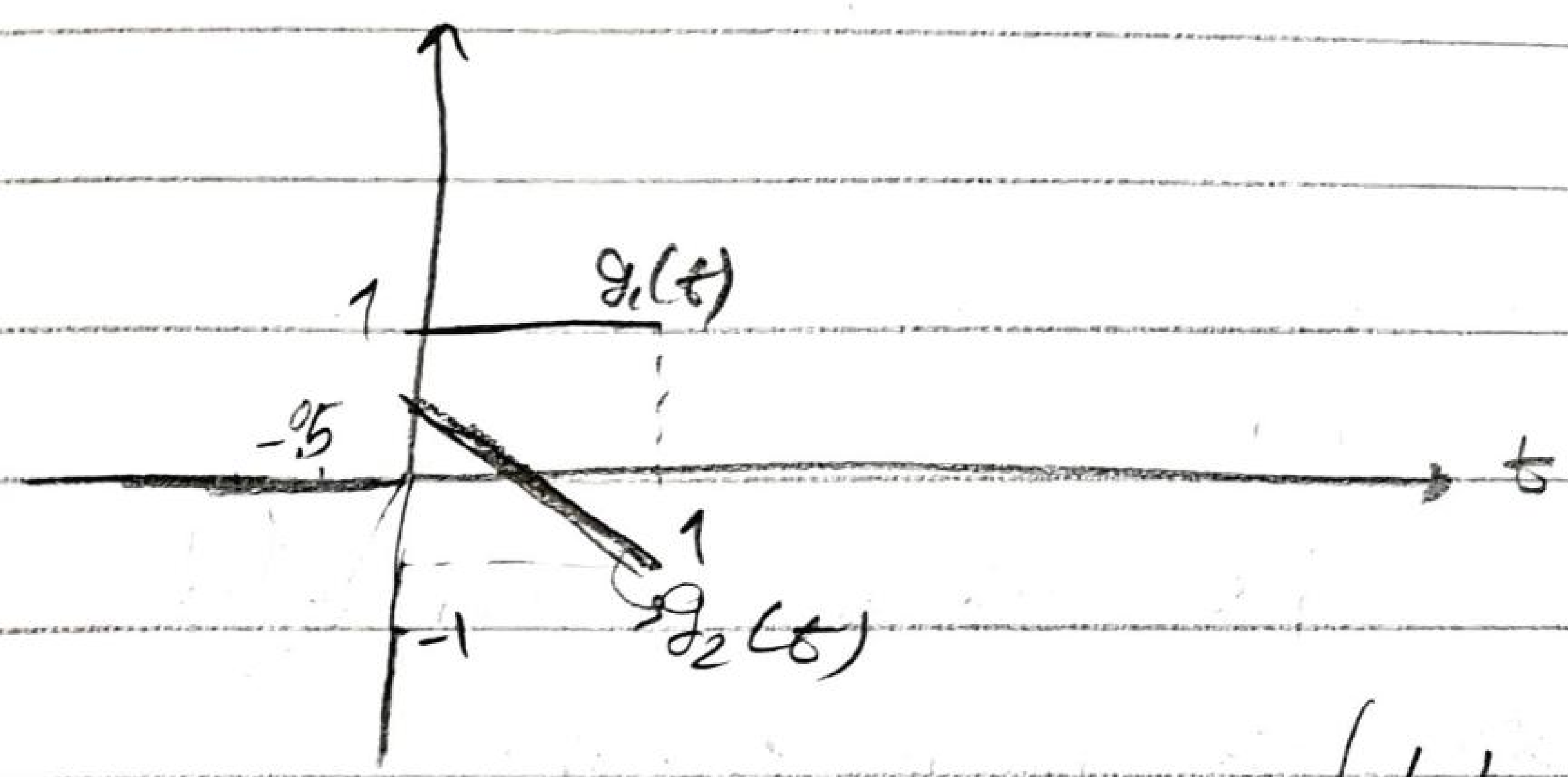
$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{2} = \frac{3+4-6}{12} = \frac{1}{12}$$

$$\Rightarrow h_2(t) = \frac{-1/12}{1/12} = -1$$

E 6.15 continue | we have found:

$h_1 = -\frac{1}{2}$ $h_2 = -1$

$g_1(t) = \begin{cases} 1 & 0<t<1 \\ 0 & o.w. \end{cases}$ $g_2(t) = \begin{cases} \frac{1}{2}-t & 0<t<1 \\ 0 & o.w. \end{cases}$




$h_{new}(t) = -0.5\, g_1(t) - g_2(t) \Rightarrow h_{new}(t) = \begin{cases} -\frac{1}{2}-\frac{1}{2}+t = t-1 & 0<t<1 \\ 0 & o.w. \end{cases}$

As we can see we again find "t-1" for h function! this shows that our calculation is completely correct!

E 5.17 (

$S_1^t = [1 \quad 0] \qquad S_2^t = [0 \quad 1]$

$u_1^t = [2 \quad 0] \qquad u_2^t = [-1 \quad -1] \qquad x = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

(i)

$x_1 = \dfrac{S_1^t \cdot x}{S_1^t \cdot S_1} \longrightarrow x_1 = \dfrac{[1 \quad 0]\begin{bmatrix} -2 \\ 2 \end{bmatrix}}{[1 \quad 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \dfrac{-2}{1}$

$x_2 = \dfrac{S_2^t \cdot x}{S_2^t \cdot S_2} = \dfrac{[0 \quad 1]\begin{bmatrix} -2 \\ 2 \end{bmatrix}}{[0 \quad 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \dfrac{2}{1}$

$\Big\} \rightarrow x = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

we can see that the ~~r~~expansion with basis set would result the same vector.

(ii)

$u_{11} = \dfrac{S_1^t u_1}{S_1^t S_1} = \dfrac{[1 \quad 0]\begin{bmatrix} 2 \\ 0 \end{bmatrix}}{[1 \quad 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \dfrac{2}{1} \qquad u_{12} = \dfrac{S_2^t \cdot u_1}{S_2^t S_2} = \dfrac{[0 \quad 1]\begin{bmatrix} 2 \\ 0 \end{bmatrix}}{[0 \quad 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = 0$

$u_1^t = [u_{11} \quad u_{12}] = [2 \quad 0] \rightsquigarrow$ the same as before

$u_{21} = \dfrac{S_1^t u_2}{S_1^t S_1} = \dfrac{[1 \quad 0]\begin{bmatrix} -1 \\ -1 \end{bmatrix}}{[1 \quad 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \dfrac{-1}{1} = -1$

$u_{22} = \dfrac{S_2^t u_2}{S_2^t S_2} = \dfrac{[0 \quad 1]\begin{bmatrix} -1 \\ -1 \end{bmatrix}}{[0 \quad 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \dfrac{-1}{1} = -1$

$\Big\} \rightarrow u_2^t = [-1 \quad -1]$ the same as before

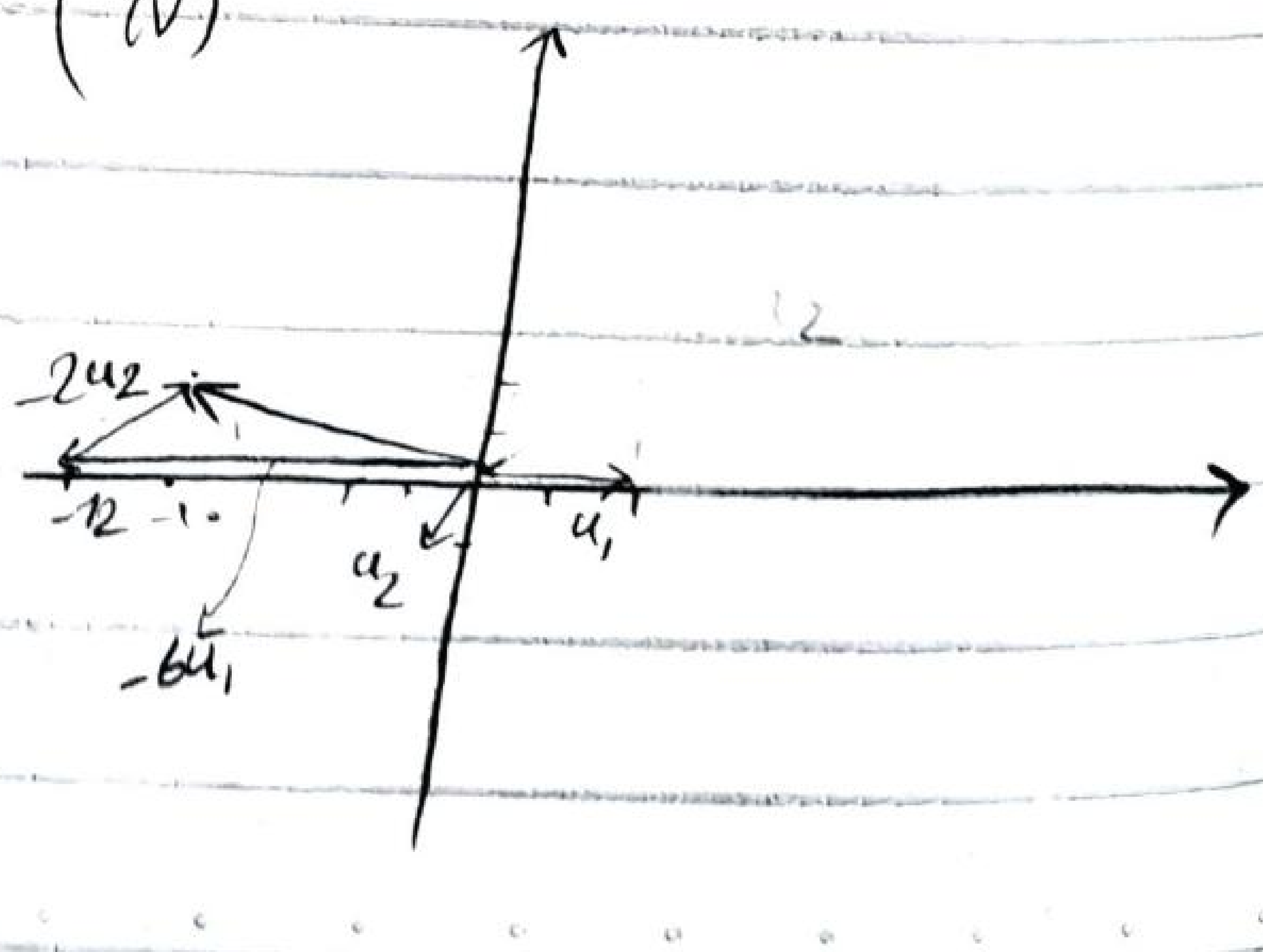
(iii)

$$R^T = B^{-1} = \begin{array}{c} \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} \\ u_1 \quad u_2 \end{array}$$

$$x_{projected} = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$$

$x_{projected} = -6u_1 - 2u_2$

(iv)



E6.3)

(i)

$A(v_1) = S_1 + S_2 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A(v_2) = -S_2 + (0)S_1 \rightsquigarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

(ii)

$v_1 = 2S_1 + S_2$

$v_2 = -S_1 + 2S_2 \quad \rightsquigarrow \quad B_t = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

$B_w = B_t$

$A' = [B_w^{-1} A B_t]$

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \xrightarrow{\theta = 90} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \qquad |B| = 4-(-1) = 5$

$\Rightarrow B^{-1} = \begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix}$

$B^{-1} A = \begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

$B^{-1} A B = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ \frac{3}{5} & 0 \end{bmatrix}$

$\Rightarrow A' = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad x' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y = Ax = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$y' = A'x' = \begin{bmatrix} 4/5 & 3/5 \\ \frac{3}{5} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ \frac{3}{5} \end{bmatrix}$

E6.3 continued

To verify the results we calculate $y' = B^{-1}y$

As we have before

$\hookrightarrow B^{-1} = \begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix}$

$$y' = B^{-1}y \Rightarrow y' = \begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$$

---

E6.4)

$s_1 = 1 + j$ $\qquad A(s_1) = (1+j)(1+j) = \underbrace{j^2 + 1}_{=-1+1} + 2j = 2j$

$s_2 = 1 - j$

(i) $\qquad A(s_2) = (1+j)(1-j) = 1 - j^2 = 1 - (-1) = 2$

$$\Rightarrow \begin{matrix} A(s_1) = 2j \\ A(s_2) = 2 \end{matrix} \xrightarrow[\text{written as}]{\text{can be}} \begin{matrix} A(s_1) = s_1 - s_2 \\ A(s_2) = s_1 + s_2 \end{matrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(ii)

from the formula $Az = \lambda z$ we can calculate both eigen values and eigen vectors.

$(A - \lambda I)z = 0 \Rightarrow |A - \lambda I| = 0$

$$\left|\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right| = 0 \Rightarrow (1-\lambda)^2 - (-1) = 0 \Longrightarrow \lambda^2 - 2\lambda + 1 + 1 = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$\Delta = 4 - 4(1)(2) = -4 \rightarrow$

$$\lambda = \frac{2 \pm 2j}{2} \Rightarrow \lambda_1 = 1 + j$$

$$\lambda_2 = 1 - j$$

6.6.4 continued $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

for $\lambda_1 = 1+j$

$$\begin{bmatrix} \overbrace{1-(1+j)}^{-j} & 1 \\ -1 & \underbrace{1-(1+j)}_{-j} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} -e_1 j + e_2 = 0 \xrightarrow{\times(j)} e_1 + e_2 j = 0 \\ -e_1 - e_2 j = 0 \rightarrow -e_1 - e_2 j = 0 \end{cases}$$

both equations will give the same results. so we use one of them to find the suitable answer.

$e_1 = -e_2 j \xrightarrow[e_2 = 1]{} \begin{bmatrix} -j \\ 1 \end{bmatrix} \sim$ first eigen vector

for $\lambda_2 = 1-j$

$$\begin{bmatrix} \overbrace{1-1+j}^{j} & 1 \\ -1 & \underbrace{1-1+j}_{j} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \rightarrow \begin{cases} je_1 + e_2 = 0 \xrightarrow{\times j} -e_1 + je_2 = 0 \\ -e_1 + je_2 = 0 \end{cases}$$

So again the two equations are dependent So one of them is use to find from the infinite solution set.

$$-e_1 + je_2 = 0 \Rightarrow e_1 = je_2 \rightarrow \begin{bmatrix} j \\ 1 \end{bmatrix}$$

so the second eigen vector was calculated as $\begin{bmatrix} j \\ 1 \end{bmatrix}$ by $e_2 = 1$

continue

C6.41

(iii)

$$A = [B^{-1} \hat{A} B]$$

$$B = \begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix} \qquad B^{-1} = \frac{1}{-2j} \begin{bmatrix} 1 & -j \\ -1 & -j \end{bmatrix} =$$

the eigen vectors matrix

$$A = -\frac{1}{2j} \begin{bmatrix} 1 & -j \\ -1 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow A = -\frac{1}{2j} \begin{bmatrix} 1+j & 1-j \\ j-1 & -1-j \end{bmatrix} \begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow A = -\frac{1}{2j} \begin{bmatrix} -j-j^2+1-j & j^2+j+1-j \\ -j^2+j-1-j & j^2-j-1-j \end{bmatrix} = \begin{bmatrix} 1+j & 0 \\ 0 & 1-j \end{bmatrix}$$

we know that $j^2 = -1$

And the eigen values is found!

E6. 8 (i)

$$D(1+e^{2t}) = 2e^{2t} = 0(\alpha) + (2)e^{2t}$$
$$D(1-e^{2t}) = -2e^{2t} = 0(\alpha) + (-2)e^{2t}$$

$$A = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$$

(ii)

$$D(S_1) = 0(0) + 1(2e^{2t})$$
$$D(S_2) = 0(0) + (-1)(2e^{2t}) \longrightarrow \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

(iii)

$$AZ = \lambda Z \rightarrow (A - \lambda I)Z = 0 \rightarrow |A - \lambda I| = 0$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)(-\lambda) = 0$$

$$\Rightarrow \lambda(\lambda+2) = 0 \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -2 \end{cases}$$

for $\lambda_1 = 0$

$$\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \Rightarrow 2e_1 - 2e_2 = 0 \rightarrow e_1 = e_2$$

first eigen vector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda_2 = -2$

$$\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \rightarrow \begin{matrix} -2e_1 = 0 \\ 2e_1 = 0 \end{matrix} \rightarrow e_1 = 0$$

we define $e_2$ as 1 so $\rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

E 6.8 continue

(iv) eigen vectors matrix $\rightarrow B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$B^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A' = [B^{-1}AB] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$\Downarrow$

we can see that the eigen values is on the main diagonal!

E 6.10

$S_1 = 1$ $A(S_1) = t = v_1$

$S_2 = t$ $A(S_2) = \frac{1}{2}t^2 = v_2$

(i)

$v_1 = 0\,S_1 + 1\,S_2$

$v_2 = 0\,S_1 + \frac{1}{2}t\,S_2$

$A = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2}t \end{bmatrix}$

(ii)

$q = 6 + 8t$

$A(q) = 6t + 4t^2$

$\begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2}t \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 4t \end{bmatrix} \rightsquigarrow 6t + 4t^2$ ✓

(iii)

$A(S_1) = t + \frac{1}{2}t^2 = v_1 \Rightarrow v_1 = S_1 + \frac{1}{2}t\,S_2$

$A(S_2) = t - \frac{1}{2}t^2 = v_2 \Rightarrow v_2 = S_1 - \frac{1}{2}t\,S_2$

$A = \begin{bmatrix} 1 & 1 \\ \frac{1}{2}t & -\frac{1}{2}t \end{bmatrix}$

E 6.11

(i) $D(f) = \frac{df}{dt}$

$\underbrace{D(u_1)}_{\substack{\shortparallel \\ v_1}} = 5e^{5t}$ $\underbrace{D(u_2)}_{\substack{\shortparallel \\ v_2}} = e^{5t} + 5te^{5t}$

$v_1 = 5\,u_1 + (0)u_2$ $v_2 = u_1 + 5u_2$

we can see that $v_1$ and $v_2$ are calculated from $u_1$ and $u_2$ by coefficients that are numbers. So the transformation $D$ is a linear transformation.

(ii)

As we had $v_1$ and $v_2$ in before the combination of $u_1$ and $u_2$. we can find the transformation matrix by the coefficients As:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

(iii) from the formula $Az = \lambda z$ we can get

$(A - \lambda I)z = 0 \Rightarrow |A - \lambda I| = 0$

$$\left|\begin{bmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{bmatrix}\right| = 0 \Rightarrow (5-\lambda)^2 - 0 = 0 \Rightarrow \lambda = 5$$

the eigenvectors: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \Rightarrow e_2 = 0$

so the eigenvectors $\begin{bmatrix} e_1 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$