

Let $(A, B) \in P(E)^2$

We define $A\Delta B = (A \cup B) \setminus (A \cap B)$

Express $\chi_{(A \cup B) \setminus (A \cap B)}$ in function of χ_A and χ_B

$$\begin{aligned}\chi_{A\Delta B} &= \chi_{(A \cup B) \cap (E \setminus (A \cap B))} \\ &= (\chi_A + \chi_B - \chi_A \chi_B) * (1 - \chi_A \chi_B) \\ &= \chi_A + \chi_B - \chi_A^2 \chi_B - \chi_A \chi_B^2 + (\chi_A \chi_B)^2 - \chi_A \chi_B\end{aligned}$$

Knowing that $\chi_Y^2 = \chi_Y, \forall Y$ a set

$$\Rightarrow \chi_{A\Delta B} = (\chi_A - \chi_B)^2$$

$$\begin{aligned}\chi_{(A\Delta B)\Delta C} &= (\chi_{A\Delta B} - \chi_C)^2 \\ &= ((\chi_A - \chi_B)^2 - \chi_C)^2 \\ &= \chi_A + \chi_B + \chi_C - 2\chi_A \chi_B - 2\chi_C \chi_A - 2\chi_C \chi_B + 4\chi_C \chi_B \chi_A \\ &= (\chi_B - \chi_C)^2 + \chi_A^2 - 2\chi_A (\chi_B - \chi_C)^2 \\ &= (\chi_A - \chi_{B\Delta C})^2 \\ &= \chi_{A\Delta(B\Delta C)} \\ \text{Then, } A\Delta(B\Delta C) &= (A\Delta B)\Delta C\end{aligned}$$