

Let $(a, b) \in \mathbb{N}^2, b \neq 0$. Then $\exists!(q, r) \in \mathbb{N}^2$ such as:

1. $a = bq + r$
2. $0 \leq r < b$

Proof(informal):

$$(a, b) \in \mathbb{N}^2$$

$$\Rightarrow \exists k^{(1)} \in \mathbb{N} \mid a = b + k^{(1)}$$

$$\Rightarrow (\exists k^{(1)} \in \mathbb{N} \mid (k^{(1)} < b \text{ and } a = b + k^{(1)})) \text{ or } (\exists k^{(2)} \in \mathbb{N} \mid a = b + b + k^{(2)})$$

Continuing until $0 \leq k^{(q)} < b$, yields

$$\Rightarrow \exists(q, r) \in \mathbb{N}, r = k^{(q)} \mid a = qb + r \text{ and } 0 \leq r < b$$

$$\text{Let } ((q, r), (q', r')) \in \mathbb{N}^4 \mid a = bq + r \text{ and } a = bq' + r'$$

$$\Rightarrow r - r' = b(q - q')$$

$$\text{Having } -b < r - r' < b$$

$$\Rightarrow -1 < q - q' < 1$$

$$\Rightarrow q = q' \text{ and } r = r'$$

Proof 2:

$$\text{Let } A = \{k \in \mathbb{N} : \exists x \in \mathbb{N} : k = xb \leq a\}$$

$$\text{Having } 0 \in A, (\forall k \in A, k \leq a) \text{ and } A \subset \mathbb{N}$$

$$\Rightarrow \exists!q \in \mathbb{N} \mid \max_A = qb$$

$$\Rightarrow qb \leq a$$

$$\Rightarrow \exists!r \in \mathbb{N}, r \geq 0 \mid qb + r = a$$

$$\text{Having } qb + b > a$$

$$\Rightarrow qb + b > qb + r$$

$$\Rightarrow b > r$$