Option pricing methods

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Exercise 1

The geometric Brownian motion S, solves

$$dS(t) = rS(t)dt + \sigma S(t)dW(t),$$
$$S(0) = S_0.$$

where, in the context of option pricing, r and σ are constants representing the short interest rate and the volatility

We denote $Y_t = \log S_t$.
By using the Itô formula for $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ a given bounded function in $C^2([0,T]\times\mathbb{R})$ with $f(t,x) = \log(x)$
Then $Y(t) \equiv f(t, S(t))$ satisfies the SDE

$$dY(t) = (\partial_t f + \partial_x f a + .\partial_x^2 f \frac{b^2}{2})(t, S(t))dt + (\partial_x f b)(t, S(t))dW(t)$$

With

$$a = rS(t)$$

$$b = S(t)\sigma$$

$$dY(t) = rS(t)\frac{1}{S(t)}dt + \frac{S(t)^2}{2}\frac{-1}{S(t)^2}\sigma^2 dt + S(t)\sigma\frac{1}{S(t)}dW(t)$$

$$dY(t) = rdt - \frac{\sigma^2}{2}dt + \sigma dW(t)$$

$$Y(T) = Y_0 + rT - \frac{\sigma^2}{2}T + \sigma W(T)$$

Since $Y(T) = \log S(T)$, we finally get the exact solution for (1) is

$$S(T) = \exp\left(\left(r - \sigma^2/2\right)T + \sigma W(T)\right)S_0$$

We want to compute a numerical approximation to the option value

$$\pi = e^{-rT}E[g(S(T))]$$

First we consider M independent samples $S^{(m)}(T)$, $m = 1, .., M$ of the random variable

$$S(T) = \exp\left(\left(r - \sigma^2/2\right)T + \sigma W(T)\right)S_0$$

and we use the Monte Carlo estimator

$$\tilde{\pi} = e^{-rT}M^{-1}\sum_{m=1}^{M} g(S^{(m)})$$

We have 3 payoff functions :

1. g(x) = max(x - K,0)
2. g(x) = 1 if $\frac{K}{2}$<x<K and 0 else
3. g(x) = $\frac{x}{\frac{x}{K}^2+1}$

Exact payoff calculations for a call

1. We already know the value of a call options from previous homeworks the formula is below

$$C = \Phi(d_1) S_t - \Phi(d_2) K e^{-r(T-t)}$$
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

cumulative function of standard normal distribution

Numerical application we have C = 8.4333

2.

$$c = e^{-rT}\int_{\frac{k}{2}<s(x)<k} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2}\frac{x^2}{T}} \cdot dx$$

$$\frac{K}{2} < S(x) < K$$
$$\frac{K}{2} < S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma x} < K$$
$$\frac{K}{2S0} < e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma x} < (\frac{K}{S0}$$
$$ln(\frac{K}{2S0}) < r - \frac{\sigma^2}{2}T + \sigma x < ln((\frac{K}{S0}))$$
$$\frac{ln(\frac{K}{2S0}) - r + \frac{\sigma^2}{2}T}{\sigma} < x < \frac{ln(\frac{K}{S0}) - r + \frac{\sigma^2}{2}T}{\sigma}$$

we pose :

$$\alpha_1 = \frac{ln(\frac{K}{2S0}) - r + \frac{\sigma^2}{2}T}{\sigma}$$

$$\alpha_2 = \frac{ln(\frac{K}{S0}) - r + \frac{\sigma^2}{2}T}{\sigma}$$

back to our integral :

$$c = e^{-rT}\int_{alpha_1<x<alpha_2} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2}\frac{x^2}{T}} \cdot dx$$

$$c = e^{-rT}\mathbb{P}(\frac{alpha_1}{\sqrt{T}} < Z < \frac{alpha_2}{\sqrt{T}})$$

With Z a standard normal random variable so that can be expressed in term of the cumulative function of standard normal distribution as follow :

$$c = e^{-rT}(1 - \Phi(\frac{alpha_1}{\sqrt{T}}) - \Phi(-\frac{alpha_2}{\sqrt{T}}))$$

Numerical application C = 0.5144504433957848

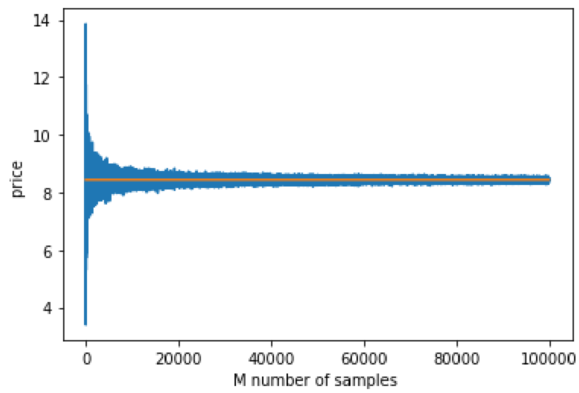
3. complicated, so we use a numerical integration method Numerical application using python to evaluate the option price is 48.556064546301315

To mention that in the next plots the statistcal error formula is a follow :

$$3\sqrt{\frac{Var(g(S)}{M}}$$

where M is the number of samples

Simulation results Exact error and Statistical error



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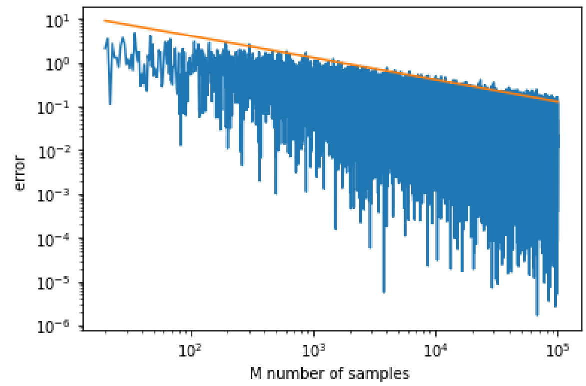

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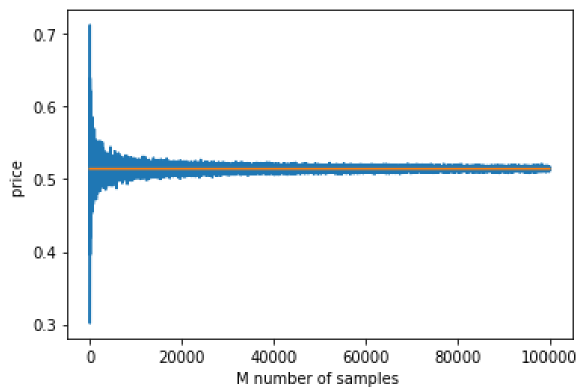
Payoff function 1



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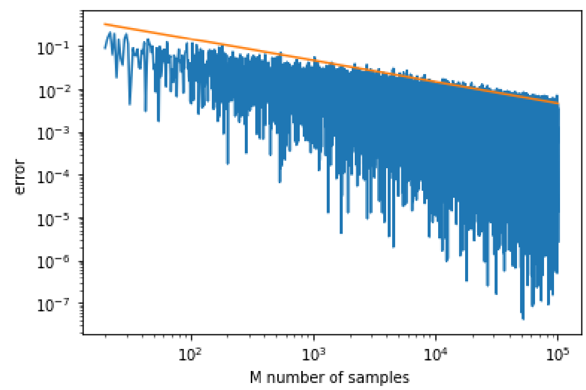

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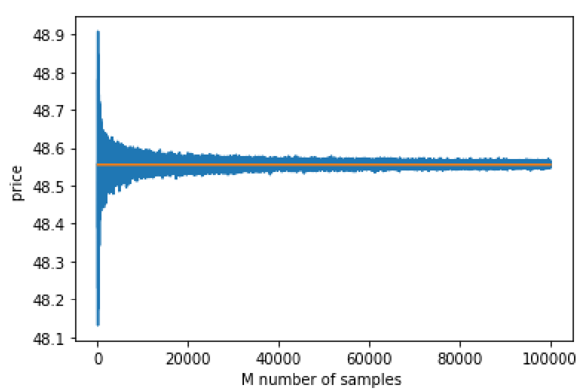
Payoff function 2



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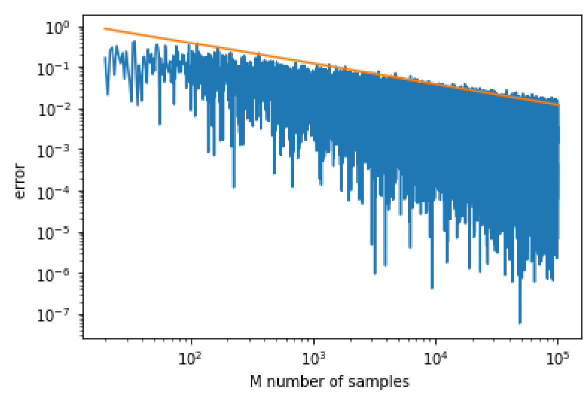

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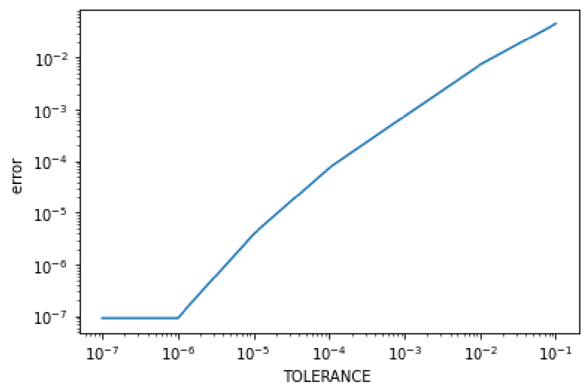
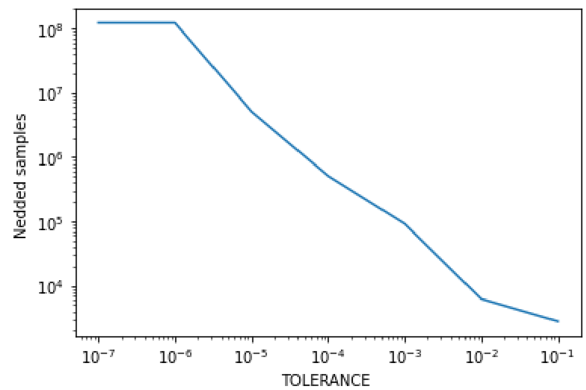
Payoff function 3

Figure 1: MC simulaton with the exact price for each payoff function on the left.The statistical error and the exact error on the right

Simulation results Exact error and Statistical error



(a)



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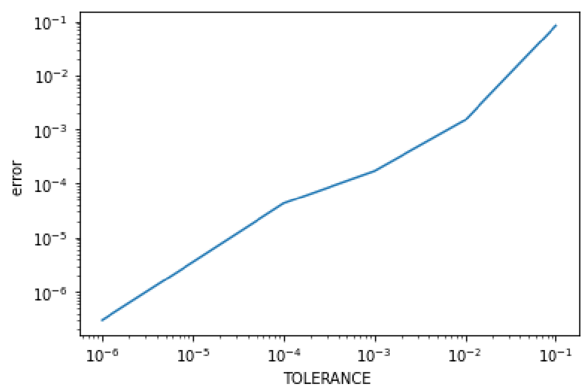
Payoff function 1



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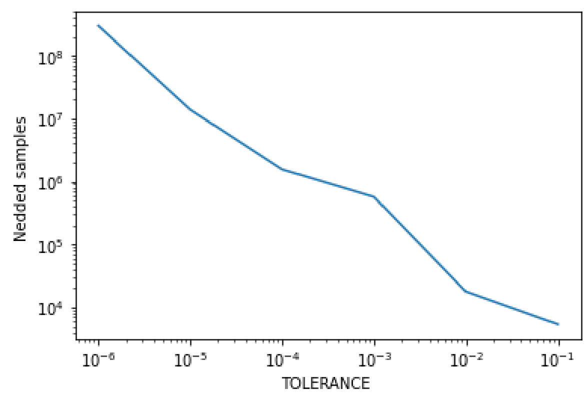

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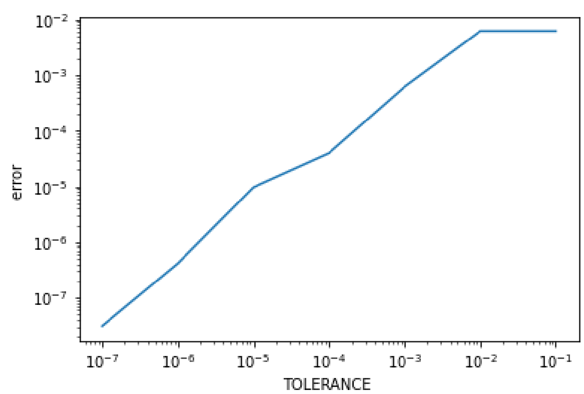
Payoff function 2



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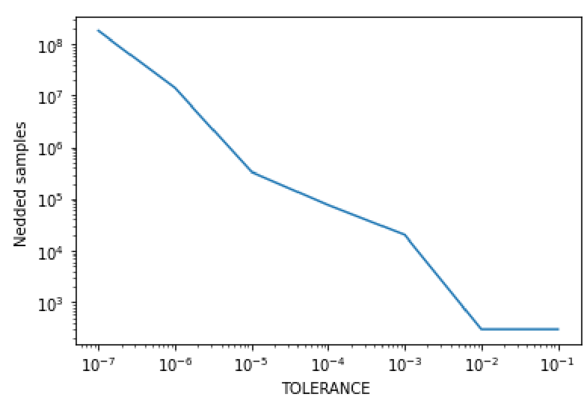

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Payoff function 3

Figure 2: MC simulaton Error vs desired tolerance for each payoff function on the left.The needed number of Samples to achieve a certain toleranceon the right

Exercise 2

We want to solve Exercise 1 by instead approximating the corresponding Black-Scholes PDE

$$\partial_t f + rS\partial_s f + \frac{1}{2}\sigma^2 S^2 \partial_{ss} f - rf = 0$$
$$f(T, s) = g(s)$$

a.

Backward Euler

$$\partial_t f = \frac{f_i^n - f_i^{n-1}}{\Delta t} + \mathcal{O}(\Delta t)$$
$$\partial_s f = \frac{f_{i+1}^{n-1} - f_{i-1}^{n-1}}{2\Delta S} + \mathcal{O}(\Delta S^2)$$
$$\partial_{ss} f = \frac{f_{i+1}^{n-1} - 2f_i^{n-1} + f_{i-1}^{n-1}}{\Delta S^2} + \mathcal{O}(\Delta S^2)$$

with $f_i^n = f(n\Delta t, i\Delta S)$

So $\partial_t f + rS\partial_s f + \frac{1}{2}\sigma^2 S^2 \partial_{ss} f - rf = 0$ is written as:

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} + \frac{ri}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1}) + \frac{\sigma^2 i^2}{2}(f_{i+1}^{n-1} - 2f_i^{n-1} + f_{i-1}^{n-1}) - rf_i^{n-1} = 0$$

$$f_i^n = f_i^{n-1}(1 + \sigma^2 i^2 \Delta t + r\Delta t) + f_{i+1}^{n-1}(-ri - \sigma^2 i^2)\frac{\Delta t}{2} + f_{i-1}^{n-1}(ri - \sigma^2 i^2)\frac{\Delta t}{2}$$

Finally

$$f_i^n = a_i f_{i-1}^{n-1} + b_i f_i^{n-1} + c_i f_{i+1}^{n-1}$$

with

$$a_i = (ri - \sigma^2 i^2)\frac{\Delta t}{2}$$
$$b_i = (1 + \sigma^2 i^2 \Delta t + r\Delta t)$$
$$c_i = (-ri - \sigma^2 i^2)\frac{\Delta t}{2}$$

So

$$f^n = Af_{n-1}$$

$$f_n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_i^n \\ \vdots \\ f_n^{M-1} \end{bmatrix} A = \begin{pmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & 0 & a_{M-1} & b_{M-1} \end{pmatrix}$$

Crank-Nicholson

We denote

$$\begin{aligned} f_i^n &= f(n\Delta t, i\Delta S) \\ f_i^{n-\frac{1}{2}} &= f((n-\frac{1}{2})\Delta t, i\Delta S) \end{aligned}$$

$$\partial_t f = \frac{f_i^n - f_i^{n-1}}{\Delta t} + \mathcal{O}(\Delta t^2)$$

$$\begin{aligned} \partial_s f &= \frac{1}{2}(\partial_s f_i^{n-1} + \partial_s f_i^n) \\ &= \frac{1}{2}(\frac{f_{i+1}^{n-1} - f_{i-1}^{n-1}}{2\Delta S} + \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta S}) + \mathcal{O}(\Delta S^2) \end{aligned}$$

$$\begin{aligned} \partial_{ss} f &= \partial_{ss} f_i^{n-\frac{1}{2}} \\ &= \frac{1}{2}(\partial_{ss} f_i^{n-1} + \partial_{ss} f_i^n) \\ &= \frac{1}{2}(\frac{f_{i+1}^{n-1} - 2f_i^{n-1} + f_{i-1}^{n-1}}{\Delta S^2} + \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta S^2}) + \mathcal{O}(\Delta S^2) \end{aligned}$$

So $\partial_t f + rS\partial_s f + \frac{1}{2}\sigma^2 S^2 \partial_{ss} f - rf = 0$ is written as:

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} + rS\frac{1}{2}(\frac{f_{i+1}^{n-1} - f_{i-1}^{n-1}}{2\Delta S} + \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta S}) + \frac{1}{2}\sigma^2 S^2 \frac{1}{2}(\frac{f_{i+1}^{n-1} - 2f_i^{n-1} + f_{i-1}^{n-1}}{\Delta S^2} + \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta S^2}) - r\frac{f_i^n + f_i^{n-1}}{2} = 0$$

$$(-\frac{ri}{4} + \frac{\sigma^2 i^2}{4})f_{i-1}^n + (\frac{1}{\Delta t} - \frac{r}{2} - \frac{\sigma^2 i^2}{2})f_i^n + (\frac{ri}{4} + \frac{\sigma^2 i^2}{4})f_{i+1}^n = (\frac{ri}{4} - \frac{\sigma^2 i^2}{4})f_{i-1}^{n-1} + (\frac{1}{\Delta t} + \frac{r}{2} + \frac{\sigma^2 i^2}{2})f_i^{n-1} + (-\frac{ri}{4} - \frac{\sigma^2 i^2}{4})f_{i+1}^{n-1}$$

So

$$a_i f_{i-1}^n + b_i f_i^n + c_i f_{i+1}^n = -a_i f_{i-1}^{n-1} + d_i f_i^{n-1} - c_i f_{i+1}^{n-1}$$

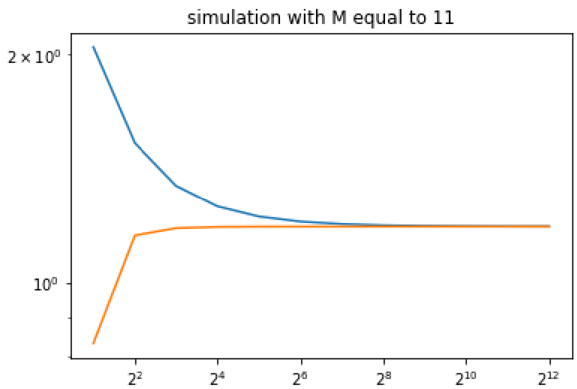
with

$$\begin{aligned} a_i &= (-\frac{ri}{4} + \frac{\sigma^2 i^2}{4}) \\ b_i &= (\frac{1}{\Delta t} - \frac{r}{2} - \frac{\sigma^2 i^2}{2}) \\ c_i &= (\frac{ri}{4} + \frac{\sigma^2 i^2}{4}) \\ d_i &= (\frac{1}{\Delta t} + \frac{r}{2} + \frac{\sigma^2 i^2}{2}) \end{aligned}$$

finally,

$$Af_n = Bf_{n-1}$$

$$f_n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_n^i \\ \vdots \\ f_n^{M-1} \end{bmatrix} A = \begin{pmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & 0 & a_{M-1} & b_{M-1} \end{pmatrix} B = \begin{pmatrix} d_1 & -c_1 & 0 & 0 & \cdots & 0 \\ -a_2 & d_2 & -c_2 & 0 & \cdots & 0 \\ 0 & -a_3 & d_3 & -c_3 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & -c_{M-2} \\ 0 & 0 & \cdots & 0 & -a_{M-1} & d_{M-1} \end{pmatrix}$$



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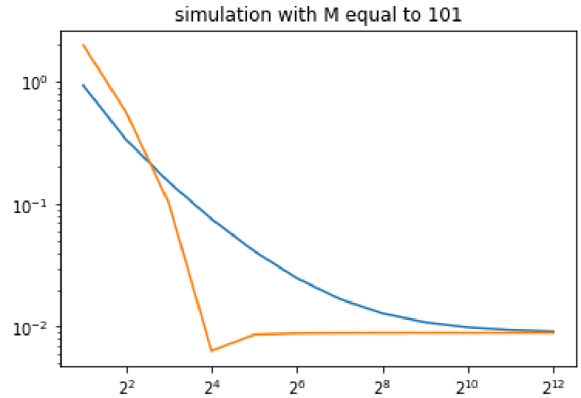

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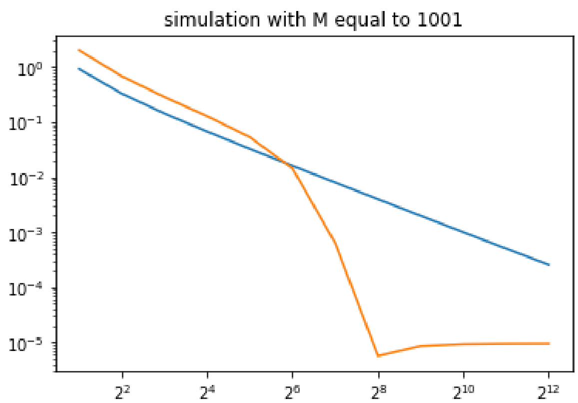

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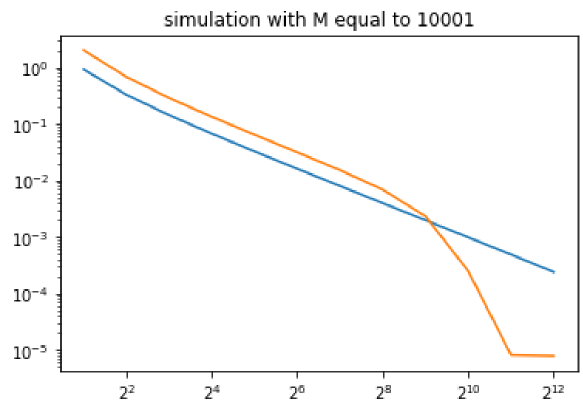

(d)

The error:

$$\epsilon = c_1 \Delta t^p + c_2 \Delta S^q \leq TOL$$
$$c_1 \Delta t^p = \frac{TOL}{2}$$
$$c_2 \Delta S^q = \frac{TOL}{2}$$

So

$$\Delta t = (\frac{TOL}{2c_1})^{\frac{1}{p}}$$
$$\Delta S = (\frac{TOL}{2c_2})^{\frac{1}{q}}$$

c1 and c2 are derived using the Richardson extrapolation The principle is as follow

$$\begin{cases} \bar{X}_{\Delta t} = X + a_1.\Delta t^p + a_2.\Delta s^r + h.o.t \\ \bar{X}_{\frac{\Delta t}{2}} = X + a_1.(\frac{\Delta s}{)}{}^p + a_2.(\frac{\Delta s}{2})^r + h.o.t \end{cases} \tag{1}$$

$$a_1 \approx \frac{X_{\Delta t} - X_{\frac{\Delta t}{2}}}{\Delta t^p - (\frac{\Delta t}{2})^p}$$

Using the same principle we can derive an approximation for the space constant and we get

$$a_2 \approx \frac{X_{\Delta s} - X_{\frac{\Delta s}{2}}}{\Delta s^p - (\frac{\Delta s}{2})^p}$$

b.
We want to prove that

$$Work(MC) = \mathcal{O}(TOL^{-3})$$
$$Work(PDE) = \mathcal{O}(TOL^{-\frac{d}{q}-\frac{1}{p}})$$

where d is the number of stock .
For MC Euler estimator:

$$\tilde{\pi} = e^{-rT} M^{-1} \sum_{m=1}^{M} g(\bar{S}_{\Delta t}^{(m)})$$

Where $\bar{S}_{\Delta t}$ is the numerical solution of the SDE by using the Euler method with time step Δt.
So the MC error: $\epsilon_{MC}(M, \Delta t) = \mathcal{O}(M^{\frac{-1}{2}} + \Delta t)$
The asymptotic average work for MC Euler is obtained by solving the minimization problem for a given tolerance $TOL > 0$

$$M^*, \Delta t^* = \arg\min_{M,\Delta t} M(\Delta t)^{-1}$$
$$\text{s.t} \quad M^{\frac{-1}{2}} + \Delta t < TOL$$

The Lagrangian with $\lambda > 0$

$$\mathcal{L}(M, \Delta t, \lambda) = M(\Delta t)^{-1} + \lambda(M^{\frac{-1}{2}} + \Delta t - TOL)$$
$$\frac{\partial \mathcal{L}}{\partial \Delta t} = -M(\Delta t)^{-2} + \lambda = 0 \Longrightarrow M = \lambda(\Delta t)^2$$
$$\frac{\partial \mathcal{L}}{\partial M} = (\Delta t)^{-1} - \frac{1}{2}\lambda M^{\frac{-3}{2}} = 0 \Longrightarrow \Delta t = \frac{2}{\lambda}(M)^{\frac{3}{2}}$$
$$\frac{\partial \mathcal{L}}{\partial M} = (\Delta t)^{-1} - \frac{1}{2}\lambda M^{\frac{-3}{2}} = 0 \Longrightarrow \Delta t = \frac{2}{\lambda}(M)^{\frac{3}{2}}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = M^{\frac{-1}{2}} + \Delta t - TOL = 0$$

$\Longrightarrow$

$$\Delta t^* = \mathcal{O}(TOL)$$
$$M^* = \mathcal{O}(TOL^{-2})$$

So $Work^*(MCE) = M^*(\Delta t^*)^{-1} = \mathcal{O}(TOL^{-3})$

For the PDE:
The asymptotic average work for the PDE approach is obtained by solving the minimization problem for a given tolerance $TOL > 0$

$$\Delta S^*, \Delta t^* = \arg\min_{\Delta S,\Delta t} (\Delta S)^{-d}(\Delta t)^{-1}$$

s.t

$$\Delta t^p + \Delta S^q < TOL$$
$$\Delta t^p < \Delta S^q$$

The Lagrangien with $\lambda > 0$

$$\mathcal{L}(\Delta t, \Delta S, \lambda) = (\Delta S)^{-d}(\Delta t)^{-1} + \lambda(\Delta t^p + \Delta S^q - TOL)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (\Delta t^p + \Delta S^q - TOL) = 0$$

$\Longrightarrow$

$$\Delta S = \mathcal{O}(TOL^{\frac{1}{q}})$$
$$\Delta t = \mathcal{O}(TOL^{\frac{1}{p}})$$

So the asymptotic average work is

$$(\Delta S^*)^{-d}(\Delta t^*)^{-1} = \mathcal{O}(TOL^{\frac{-d}{q}})\mathcal{O}(TOL^{\frac{-1}{p}})$$

$$Work(PDE) = \mathcal{O}(TOL^{-\frac{d}{q}-\frac{1}{p}})$$

We can see that the work by the PDE approach suffers from the dimentionality curse as the work grows exponentially with d.
So when $d >> 1$ MC Euler method has a better assymptotic accuracy because $\frac{d}{q} + \frac{1}{p} > 3$
c.
MLMC
Let$\Delta t_\ell = \Delta t_0 2^{-\ell}, \ell = 0, \ldots, L$ and $\bar{X}\left(T; \Delta t_\ell\right)$ be a F. Euler approximation computed with uniform step Δt_ℓ and $g_\ell = g\left(\bar{X}\left(T; \Delta t_\ell\right)\right)$. We write the telescopic sum

$$\mathbb{E}\left[g_L\right] = \mathbb{E}\left[g_0\right] + \sum_{\ell=1}^{L} \mathbb{E}\left[g_\ell - g_{\ell-1}\right]$$

The Forward Euler has weak order 1

$$|E[g_\ell - g]| \leq \Delta t_\ell$$

andthe bias form considering levels up to L is to ensure $\left(E\left[g_L - g(X(T))\right] = \mathcal{O}(TOL)\right)$

$$TOL \leq \Delta t_L = \Delta t_0 2^{-L}$$
$$L = \mathcal{O}\left(\log_2\left(\frac{\Delta t_0}{TOL}\right)\right)$$

We use $L+1$ batches, each with M_ℓ independent realizations, $\ell = 0, \ldots, L$ to create the estimator

$$A = \frac{1}{M_0}\sum_{j=1}^{M_0} g_0\left(\omega_j^0\right) + \sum_{\ell=1}^{L}\frac{1}{M_\ell}\sum_{j=1}^{M_\ell}\left(g_\ell - g_{\ell-1}\right)\left(\omega_j^\ell\right)$$

And we compute its variance:

$$\operatorname{Var}(A) = \frac{\operatorname{Var}\left(g_0\right)}{M_0} + \sum_{\ell=1}^{L}\frac{\operatorname{Var}\left(g_\ell - g_{\ell-1}\right)}{M_\ell}$$

Since F. Euler has strong order 1/2 we have

$$\operatorname{Var}(A) = \frac{\mathcal{O}(1)}{M_0} + \sum_{\ell=1}^{L}\frac{\mathcal{O}\left(\Delta t_\ell\right)}{M_\ell}$$

Now we take

$$\frac{\Delta t_\ell}{M_\ell} = \frac{\Delta t_0}{M_0}, \ell = 1, \ldots, L$$

This implies $M_\ell = M_0 2^{-\ell}$ and

$$\operatorname{Var}(A) = \frac{\mathcal{O}(1)}{M_0}\left(1 + \Delta t_0 L\right)$$

To achieve accuracy TOL we need $\mathrm{Var}(A) = \mathcal{O}\left(TOL^2\right)$ and thus

$$\begin{aligned} M_0 &= \mathcal{O}\left(TOL^{-2}\left(1 + \Delta t_0 L\right)\right) \\ &= \mathcal{O}\left(TOL^{-2}\left(1 + \Delta t_0 \log_2\left(\Delta t_0/TOL\right)\right)\right) \end{aligned}$$

So the computational work of computing the estimator A to achieve a given accuracy TOL

$$\begin{aligned} Work &= \sum_{\ell=1}^{L} \frac{M_\ell}{\Delta t_\ell} \\ &= (1+L)\frac{M_0}{\Delta t_0} \\ &= \mathcal{O}\left((1+\left(\log_2\left(\frac{\Delta t_0}{TOL}\right)\right))(1+\left(\log_2\left(\frac{\Delta t_0}{TOL}\right)\right)\frac{TOL^{-2}}{\Delta t_0})\right) \\ &= \mathcal{O}\left(\left(\log_2\left(\frac{\Delta t_0}{TOL}\right)TOL^{-2}\right)\right) \end{aligned}$$

So the Multi Level F. Euler Monte Carlo method improves the cost from $\mathcal{O}(TOL^{-3})$ to $\mathcal{O}((log(TOL)TOL^{-1})^2)$