University of Tehran

School of Electrical and Computer Engineering

Pattern Recognition

Assignment 1

Due Date: 15th of Esfand

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Esfand 97

PROBLEM 1

Consider Cauchy distributions in a two-class one-dimensional classification problem:

$$P(x|\omega_i)=\frac{1}{\pi b}\frac{1}{1+\left(\frac{x-a_i}{b}\right)^2} \qquad i=1,2 \qquad a_2>a_1$$

a) By explicit integration, show that the distributions are normalized.
b) Assuming $P(\omega_1)=P(\omega_2)$, show that $P(\omega_1|x)=P(\omega_2|x)$ if $x=(a_1+a_2)/2$. Use MATLAB to plot $P(\omega_1|x)$ and $P(\omega_2|x)$ on one axis for the case $a_1=3, a_2=5$ and $b=1$.
c) Show that the minimum probability of error is given by:

$$P(error)=\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|$$

d) What is the maximum value of $P(error)$ and under which conditions can this occur?
e) Design the Bayes minimum error classifier in terms of a_i and b if $P(\omega_1)=P(\omega_2)$. Plot the decision boundaries in this case. What is the probability of error?
f) Design the Bayes minimum risk classifier with the following error weights

$$\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}$$

Plot the decision boundaries in this case. What is the probability of error? Compare the result with that of (e).

PROBLEM 2

Consider the average risk minimization problem in a 2-Class case. Assume $\lambda_{ii}\neq 0$.

a) Calculate the decision region (R_1) corresponding to class one.
b) What is the upper bound for $\frac{f(x|\omega_2)}{f(x|\omega_1)}$ (hint: use the inequality concluded from part a)

PROBLEM 3

Calculate the discriminant function for two dimensional Neyman-Pearson criteria for two Cauchy distributions:

$$P(x|\omega_i)=\frac{1}{\pi c}\frac{1}{1+\left(\frac{x-a_i}{b}\right)^T\sum_i^{-1}\left(\frac{x-a_i}{b}\right)}; \qquad i=1,2$$

Consider the case where the covariance matrices for both of the classes are identical $\sum_i=\sum$.
Find the decision surfaces, plot the conjunctions of the two distributions and highlight the decision surfaces by implementing a MATLAB code.

PROBLEM 4

Consider a two class, one-dimensional problem with samples distributed according to the Rayleigh pdf in each class, that is,

$$p(x \mid \omega_i) = \begin{cases} \dfrac{x}{\sigma_i^2} \exp\left(\dfrac{-x^2}{2\sigma_i^2}\right) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Assume equal priors for both of the classes, compute the decision boundary point.

PROBLEM 5

Calculate the decision boundary for the two category two-dimensional data in the following figure. (Assume Gaussian distributions for both of the classes)

Let ω_1 be the set of ten black points, and ω_2 the set of nine red points.

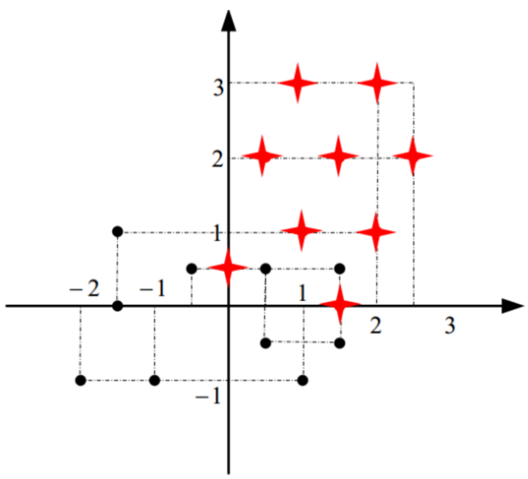

Figure 1

a) Find the mean and covariance for the Gaussian distributions of the two classes.
b) Assume equal prior probabilities, $P(\omega_1) = P(\omega_2) = 0.5$. Calculate decision boundary and plot it. Determine the empirical training error on your samples, i.e., the percentage of points misclassified.
c) Obtain and draw the Bayes decision boundary for minimum risk, given that

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2a \\ a & 0 \end{pmatrix} \qquad a > 0$$

d) Assume $P(\omega_1) = \dfrac{1}{3}$ and $P(\omega_2) = \dfrac{2}{3}$, then repeat part (b) considering the aforementioned priors.

PROBLEM 6 (Bonus)

Let the components of the vector $x = (x_1, \ldots, x_d)^t$ be binary valued (0 or 1) and $P(\omega_j)$ be the prior probability for the state of nature ω_j, $j = 1, \ldots, c$. Now define

$$p_{ij} = \mathrm{Prob}(x_i = 1 \mid \omega_j) \qquad \begin{matrix} i = 1, \ldots, d \\ j = 1, \ldots, c \end{matrix}$$

With the components of x_i being statistically independent for all x in each ω_j.

a) Interpret in words the meaning of p_{ij}.

b) Show that the minimum probability of error is achieved by the following decision rule. Decide ω_k if $g_k(x) \geq g_j(x)$ for all *j* and *k*, where

$$g_j(x) = \sum_{i=1}^{d} x_i \ln \frac{p_{ij}}{1 - p_{ij}} + \sum_{i=1}^{d} \ln(1 - p_{ij}) + \ln P(\omega_j)$$

PROBLEM 7

Consider a two-class classification problem with a risk matrix L. Show that if ε_1 is the probability of error corresponding to feature vectors from class ω_1 and ε_2 for those from class ω_2, then the average risk r is given by:

$$L = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

$$r = p(\omega_1)\lambda_{11} + p(\omega_2)\lambda_{22} + p(\omega_1)(\lambda_{12} - \lambda_{11})\varepsilon_1 + p(\omega_2)(\lambda_{21} - \lambda_{22})\varepsilon_2$$

PROBLEM 8

In many pattern classification problems one has the option either to assign a data point to one of c classes or to *reject* it as being unrecognizable. If the cost for rejects is not too high, rejection may be a desirable action.

a) Let

$$\lambda_{ij} = \begin{cases} 0 & i = j; i, j = 1, \ldots, c \\ \lambda_r & i = c + 1 \\ \lambda_s & otherwise \end{cases}$$

Where λ_r is the loss incurred for choosing the $(c+1)^{th}$ action, rejection, and λ_s is the loss incurred for making a substitution error. Show that the minimum risk is obtained if we decide ω_i when: (both conditions should be satisfied)

$$P(\omega_i \mid x) \geq P(\omega_j \mid x) \qquad \forall j = 1, \ldots, c$$

$$P(\omega_i \mid x) \geq 1 - \frac{\lambda_r}{\lambda_s}$$

and reject otherwise (class c+1).

What happens if $\lambda_r = 0$? What happens if $\lambda_r > \lambda_s$?

Use the formulation for minimum risk discussed in class.

b) Use the results of part (a) to show that the following discriminant functions are optimal for such problems:

$$g_i(x) = \begin{cases} f_x(x \mid \omega_i) P(\omega_i) & i = 1, \ldots, c \\ \dfrac{\lambda_s - \lambda_r}{\lambda_s} \displaystyle\sum_{j=1}^{c} f_x(x \mid \omega_j) P(\omega_j) & i = c+1 \end{cases}$$

Plot these discriminant functions and the decision regions for the two-category one dimensional case having

$$f_x(x \mid \omega_1) \sim N(1,1)$$
$$f_x(x \mid \omega_2) \sim N(-1,1)$$
$$P(\omega_1) = P(\omega_2) = \frac{1}{2}$$
$$\frac{\lambda_r}{\lambda_s} = \frac{1}{4}$$

Describe quantitatively what happens as $\frac{\lambda_r}{\lambda_s}$ increased from 0 to 1.

PROBLEM 9

Consider minimax criterion for the zero-one loss function, i.e. $\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

a) Prove that in this case the decision regions will satisfy

$$\int_{R_2} P(x \mid \omega_1) dx = \int_{R_1} P(x \mid \omega_2) dx$$

b) Is this solution always unique? If not, construct a simple counterexample.

PROBLEM 10 (Bonus)

Consider a simple communication system shown in figure 1. The source module sends message m with value equal to $m=0$ and $m=1$ with probabilities equal to $P(m=0)=3/4$, and $P(m=1)=1/4$. The receiver, however, receives the messages that contain noise n that is independent of m with values of -1 ‹0 ‹1 with probabilities equal to $P(n=-1)=1/8$, $P(n=0)=5/8$ and $P(n=1)=2/8$. The receiver module receives signals equal to $r=m+n$. The receiver module should decide (m') about every incoming signal r, whether the original message was equal to $m=1$ or $m=0$. So if $m'=m$, the decision was correct. Design a classifier that maximizes the performance of the receiver module.

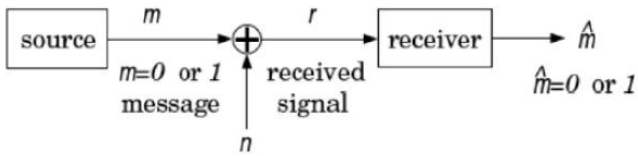

Figure 2 .Simple communication system

NOTES

1. Please make sure you reach the deadline because there would be no extra time available.
2. Late policy would be as bellow:
   - Every student has a budget for late submission during the semester. This budget is two weeks for all the assignments.
   - Late submission more than two weeks may cause lost in your scores.
3. Analytical problems can be solved on papers and there is no need to type the answers. The only thing matters is the quality of your pictures. Scanning your answer sheets is recommended. If you are using your smartphones you may use scanner apps such as CamScanner or google drive application.
4. Simulation problems need report as well as source codes and results. This report must be prepared as a standard scientific report.
5. You have to prepare your final report including the analytical problems answer sheets and your simulation report in a single pdf file.
6. Finalized report and your source codes must be uploaded to the course page as a ".zip" file (not ".rar") with the file name format as bellow:

   PR_Assignment #[Assignment Number]_Surname_Name_StudentID.zip

7. Plagiarisms would be strictly penalized.
8. You may ask your questions from corresponding TAs.