Ch7: Inverse Kinematics

Inverse Kinematics

Inverse Kinematics

The inverse kinematics of a robot refers to the calculation of the joint coordinates $\boldsymbol{\theta}$ from the position and orientation (**pose**) of its end-effector frame.

- "Geometric" inverse kinematics:

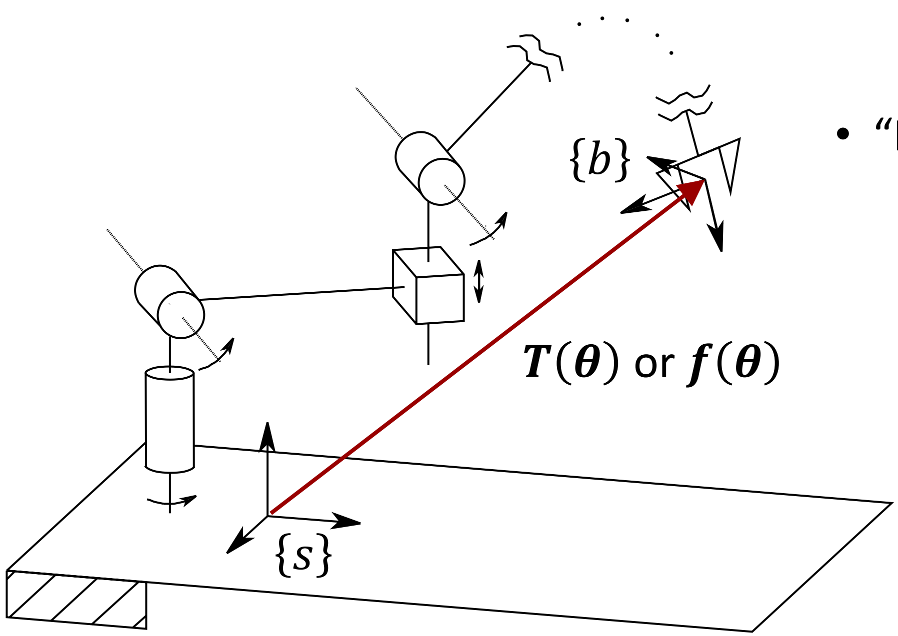
$$\text{Given } \boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3), \text{ Find } \boldsymbol{\theta} \in \mathbb{R}^n$$

$$\boldsymbol{T}: \mathbb{R}^n \to SE(3)$$

- "Minimum-Coordinate" inverse kinematics:

$$\text{Given } \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m, \text{ Find } \boldsymbol{\theta} \in \mathbb{R}^n$$

$$(m \leq n) \qquad \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m$$



Complexities of Inverse Kinematics

- The equations to solve are in general nonlinear. Thus, it is not always possible to find a closed-form solution.
- Multiple (finite) solutions may exist.
- Infinite solutions may exist (e.g., in the case of a kinematically redundant manipulator).
- There might be no admissible solutions (e.g., when the given EE pose does not belong to the manipulator dexterous workspace.).

► Solving Inverse Kinematics Problems:

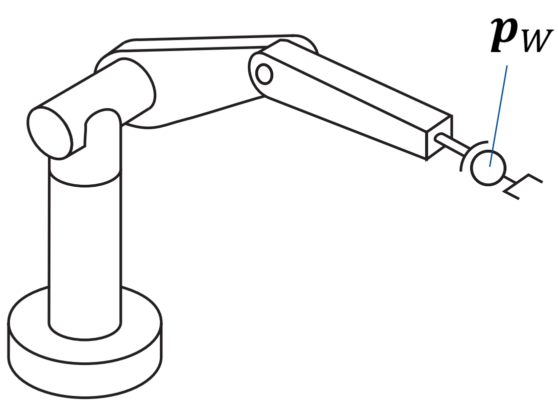
- **Analytic Methods**: Finding closed-form solutions using algebraic intuition or geometric intuition.

- **Iterative Numerical Methods**: When there are no (or it is difficult to find) closed-form solutions.

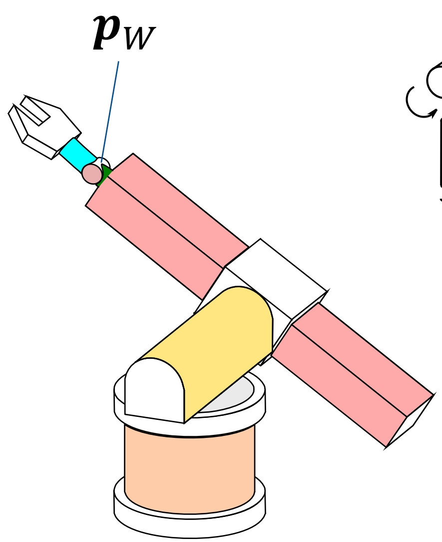
Analytic Inverse Kinematics

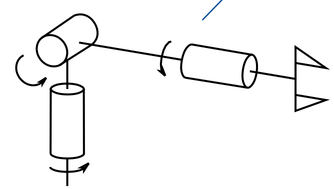
Analytic Inverse Kinematics

Most of the existing manipulators are typically formed by an **arm** and a **spherical wrist** (i.e., three consecutive revolute joint axes intersect at a common point). Thus, we can <u>decouple</u> the solution for the position (e.g., point $\boldsymbol{p}_W$ at the intersection of the three revolute axes) from that for the orientation.

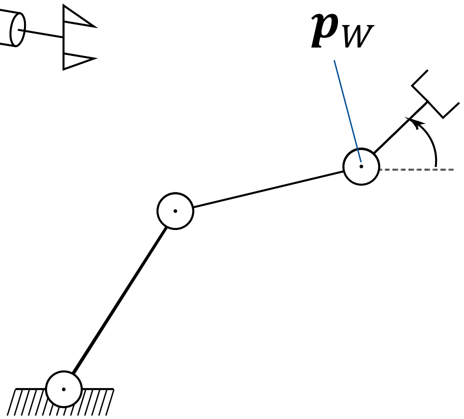


PUMA Arm (6R)





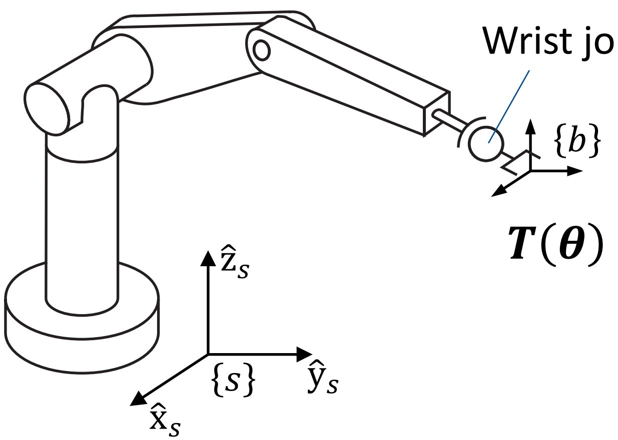
Stanford Arm (RRPRRR)



3R Planar Arm

∗ Therefore, it is possible to solve the inverse kinematics for the arm separately from the inverse kinematics for the spherical wrist.

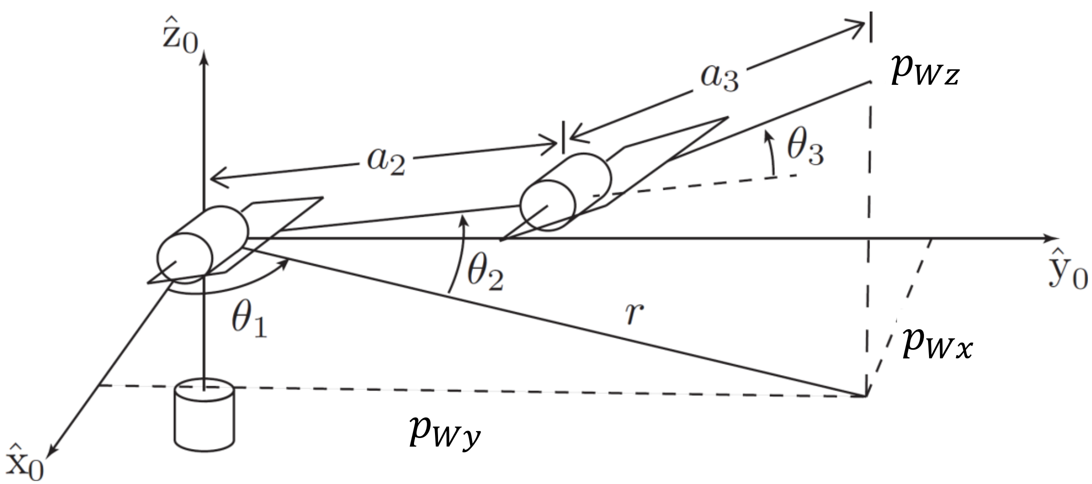
6R PUMA-Type Arms



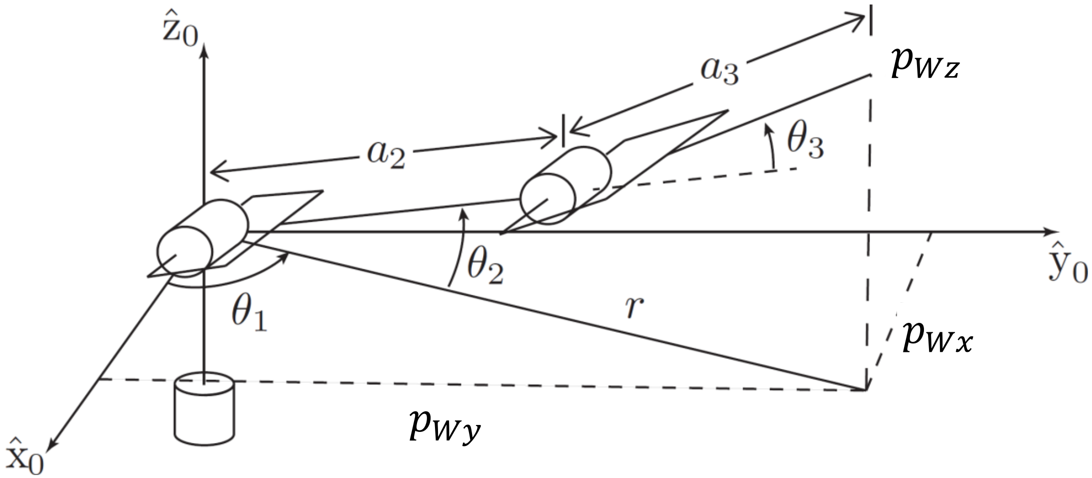
Wrist joints intersect orthogonally at a common point $\boldsymbol{p}_W$

$$T(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{R}_{sb} & \boldsymbol{p}_b \\ \boldsymbol{0} & 1 \end{bmatrix} = e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2} e^{[\boldsymbol{S}_3]\theta_3} e^{[\boldsymbol{S}_4]\theta_4} e^{[\boldsymbol{S}_5]\theta_5} e^{[\boldsymbol{S}_6]\theta_6} \boldsymbol{M}$$

By having $\boldsymbol{p}_b$, we can find $\boldsymbol{p}_W = (p_{Wx}, p_{Wy}, p_{Wz})$



6R PUMA-Type Arms (cont.)



$$p_{Wx} = c_1(a_2c_2 + a_3c_{23}) = c_1 r$$
$$p_{Wy} = s_1(a_2c_2 + a_3c_{23}) = s_1 r$$
$$p_{Wz} = a_2s_2 + a_3s_{23}$$

❖ Inverse position problem of finding $(\theta_1, \theta_2, \theta_3)$ using <u>algebraic intuition</u>:

$$p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 = a_2^2 + a_3^2 + 2a_2a_3c_3$$

$$c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2a_3}$$
$$s_3 = \pm\sqrt{1 - c_3^2}$$

$\Rightarrow$ $\theta_3 = \operatorname{atan2}(s_3, c_3)$ $\Rightarrow$ $\theta_{3,\mathrm{I}} \in [-\pi, \pi]$, $\theta_{3,\mathrm{II}} = -\theta_{3,I}$

6R PUMA-Type Arms (cont.)

$$p_{Wx}^2 + p_{Wy}^2 = (a_2 c_2 + a_3 c_{23})^2 \longrightarrow \left.\begin{aligned} a_2 c_2 + a_3 c_{23} &= \pm\sqrt{p_{Wx}^2 + p_{Wy}^2} = \pm r \\ p_{Wz} &= a_2 s_2 + a_3 s_{23} \\ s_{23} &= s_2 c_3 + s_3 c_2 \\ c_{23} &= c_2 c_3 - s_2 s_3 \end{aligned}\right\} \Rightarrow$$

$$c_2 = \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3 c_3) + p_{Wz} a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3}$$

$$s_2 = \frac{p_{Wz}(a_2 + a_3 c_3) - \left(\pm\sqrt{p_{Wx}^2 + p_{Wy}^2} a_3 s_3\right)}{a_2^2 + a_3^2 + 2a_2 a_3 c_3} \Rightarrow \theta_2 = \operatorname{atan2}(s_2, c_2)$$

For each θ_3, we have two solutions for θ_2:

$$\begin{cases} \theta_{3,\mathrm{I}} \begin{cases} \theta_{2,\mathrm{I}} & (+r) \\ \theta_{2,\mathrm{II}} & (-r) \end{cases} \\ \theta_{3,\mathrm{II}} \begin{cases} \theta_{2,\mathrm{III}} & (+r) \\ \theta_{2,\mathrm{IV}} & (-r) \end{cases} \end{cases}$$

$$\theta_{3,\mathrm{I}} \to (\theta_{2,\mathrm{I}}, \theta_{2,\mathrm{II}})$$
$$\theta_{3,\mathrm{II}} \to (\theta_{2,\mathrm{III}}, \theta_{2,\mathrm{IV}})$$

6R PUMA-Type Arms (cont.)

$$p_{Wx} = c_1(a_2c_2 + a_3c_{23})$$
$$p_{Wy} = s_1(a_2c_2 + a_3c_{23})$$
$$a_2c_2 + a_3c_{23} = \pm\sqrt{p_{Wx}^2 + p_{Wy}^2} = \pm r$$

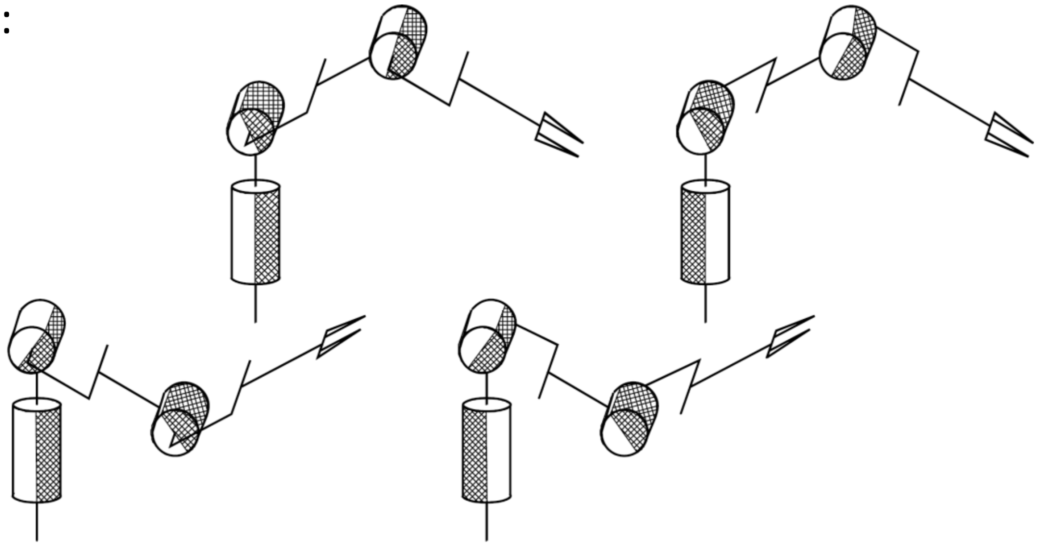
$$\Rightarrow \quad p_{Wx} = \pm c_1\sqrt{p_{Wx}^2 + p_{Wy}^2}, \quad p_{Wy} = \pm s_1\sqrt{p_{Wx}^2 + p_{Wy}^2}$$

$$\Rightarrow \quad \theta_{1,\mathrm{I}} = \operatorname{atan2}(p_{Wy}, p_{Wx}), \quad \theta_{1,\mathrm{II}} = \operatorname{atan2}(-p_{Wy}, -p_{Wx})$$

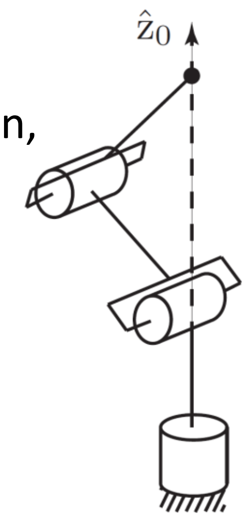
Thus, in total, there exist four solutions:

$$\begin{cases} \theta_{3,\mathrm{I}} \begin{cases} \theta_{2,\mathrm{I}} \quad (+r) \longrightarrow \theta_{1,\mathrm{I}} \\ \theta_{2,\mathrm{II}} \quad (-r) \longrightarrow \theta_{1,\mathrm{II}} \end{cases} \\ \theta_{3,\mathrm{II}} \begin{cases} \theta_{2,\mathrm{III}} \quad (+r) \longrightarrow \theta_{1,\mathrm{I}} \\ \theta_{2,\mathrm{IV}} \quad (-r) \longrightarrow \theta_{1,\mathrm{II}} \end{cases} \end{cases}$$

$$(\theta_{1,\mathrm{I}}, \theta_{2,\mathrm{I}}, \theta_{3,\mathrm{I}})$$
$$(\theta_{1,\mathrm{I}}, \theta_{2,\mathrm{III}}, \theta_{3,\mathrm{II}})$$
$$(\theta_{1,\mathrm{II}}, \theta_{2,\mathrm{II}}, \theta_{3,\mathrm{I}})$$
$$(\theta_{1,\mathrm{II}}, \theta_{2,\mathrm{IV}}, \theta_{3,\mathrm{II}})$$



6R PUMA-Type Arms (cont.)



Note: When $p_{Wx} = p_{Wy} = 0$, the arm is in a kinematically singular configuration, and there are infinitely many possible solutions for θ_1.

❖ Inverse orientation problem of finding $(\theta_4, \theta_5, \theta_6)$:

$$e^{[\boldsymbol{S}_4]\theta_4} e^{[\boldsymbol{S}_5]\theta_5} e^{[\boldsymbol{S}_6]\theta_6} = \underbrace{e^{-[\boldsymbol{S}_3]\theta_3} e^{-[\boldsymbol{S}_2]\theta_2} e^{-[\boldsymbol{S}_1]\theta_1} \boldsymbol{T}(\boldsymbol{\theta}) \boldsymbol{M}^{-1} = \boldsymbol{T}'}_{\text{known}}$$

Assume that the joint axes $(\boldsymbol{S}_4, \boldsymbol{S}_5, \boldsymbol{S}_6)$ of the spherical wrist are aligned in the $(\hat{z}_s, \hat{y}_s, \hat{x}_s)$ directions, respectively:

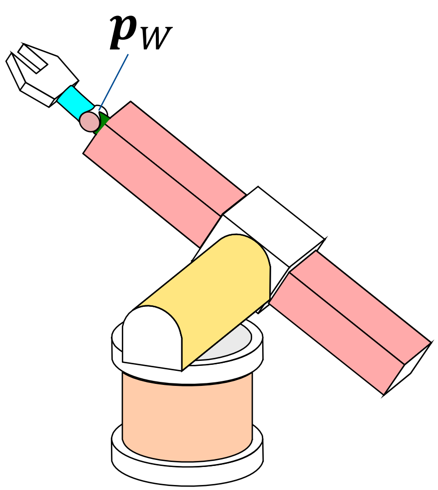
$$\boldsymbol{S}_{\omega_4} = (0,0,1)$$
$$\boldsymbol{S}_{\omega_5} = (0,1,0)$$
$$\boldsymbol{S}_{\omega_6} = (1,0,0)$$

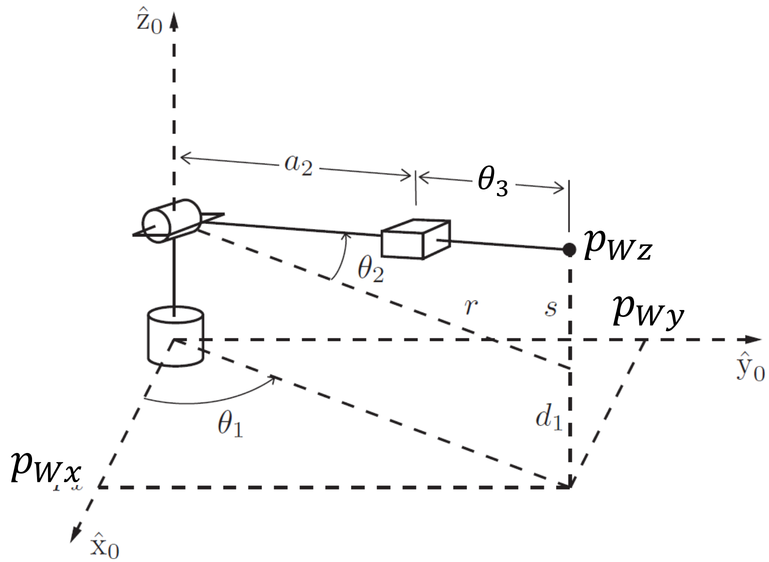
$\Rightarrow$ $\mathrm{Rot}(\hat{z}, \theta_4)\mathrm{Rot}(\hat{y}, \theta_5)\mathrm{Rot}(\hat{x}, \theta_6) = \boldsymbol{R}'$ $\Rightarrow$ This corresponds to the ZYX Euler angles.

$$\boldsymbol{T}' = (\boldsymbol{R}', \boldsymbol{p}')$$

$\Rightarrow (\theta_4, \theta_5, \theta_6)$

Stanford-Type Arms





$$r^2 = p_{Wx}^2 + p_{Wy}^2$$
$$s = p_{Wz} - d_1$$

❖ Inverse position problem of finding $(\theta_1, \theta_2, \theta_3)$ using <u>geometric intuition</u>:

If $p_{Wx}, p_{Wy} \neq 0$: $\begin{cases} \theta_1 = \text{atan2}(p_{Wy}, p_{Wx}) \\ \theta_2 = \text{atan2}(s, r) \end{cases}$, $\quad \begin{cases} \theta_1 = \pi + \text{atan 2}(p_{Wy}, p_{Wx}) \\ \theta_2 = \pi - \text{atan 2}(s, r) \end{cases}$

$$(\theta_3 + a_2)^2 = r^2 + s^2 \longrightarrow \theta_3 = \sqrt{r^2 + s^2} - a_2 = \sqrt{p_{Wx}^2 + p_{Wy}^2 + (p_{Wz} - d_1)^2} - a_2$$

$\Rightarrow$ Thus, there are 2 solutions to the inverse kinematics problem.

❖ Inverse orientation problem of finding $(\theta_4, \theta_5, \theta_6)$ is similar to PUMA.

Numerical Inverse Kinematics

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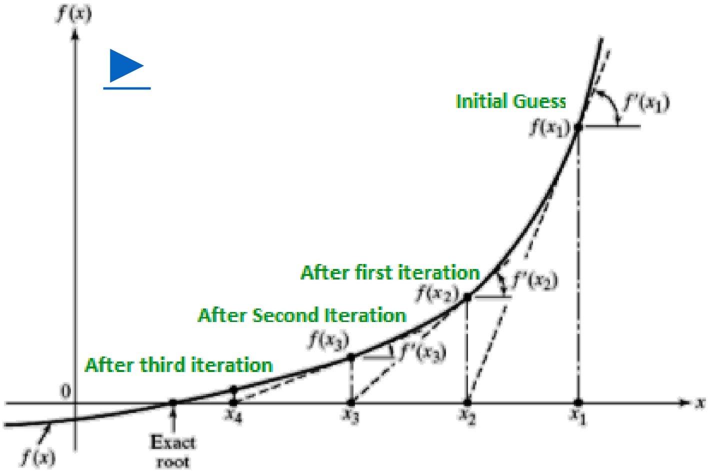
Newton–Raphson Method

Newton–Raphson Method is an iterative method for numerically finding the roots of a nonlinear equation $f(\theta) = 0$ where $f: \mathbb{R} \to \mathbb{R}$ is differentiable.



If θ^0 is an initial guess for the solution, Taylor expansion of $f(\theta)$ at θ^0 is

$$f(\theta) = f(\theta^0) + \frac{\partial f}{\partial \theta}(\theta^0)(\theta - \theta^0) + \underbrace{\cancel{\text{higher-order terms (h.o.t)}}}_{\cong 0} \xrightarrow{f(\theta)=0} \theta = \theta^0 - \left(\frac{\partial f}{\partial \theta}(\theta^0)\right)^{-1} f(\theta^0)$$

Using θ as the new guess for the solution and repeating: $\theta^{k+1} = \theta^k - \left(\frac{\partial f}{\partial \theta}(\theta^k)\right)^{-1} f(\theta^k)$

The iteration is repeated until some stopping criterion is satisfied, e.g., $\frac{|f(\theta^k) - f(\theta^{k+1})|}{|f(\theta^k)|} \le \epsilon$

ϵ: a given threshold value

Inverse Kinematics Based on Newton–Raphson Method (Minimum-Coordinate IK)

Assume that the EE pose is represented by the minimum number of coordinates, i.e., $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$, $\boldsymbol{\theta} \in \mathbb{R}^n$. Thus, given a desired EE pose $\boldsymbol{x}_d$, the goal is to find joint coordinates $\boldsymbol{\theta} = \boldsymbol{\theta}_d$ such that

$$\boldsymbol{x}_d = \boldsymbol{f}(\boldsymbol{\theta}_d)$$

(Assumption: $\boldsymbol{f}$ is differentiable)

- We use a method similar to the Newton–Raphson method for nonlinear root-finding:

Given an initial guess $\boldsymbol{\theta}^0$ which is “close to” a solution $\boldsymbol{\theta}_d$, and using the Taylor expansion:

$$\boldsymbol{x}_d = \boldsymbol{f}(\boldsymbol{\theta}) = \boldsymbol{f}(\boldsymbol{\theta}^0) + \underbrace{\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}^0}}_{\boldsymbol{J}(\boldsymbol{\theta}^0) \in \mathbb{R}^{m \times n}} \underbrace{(\boldsymbol{\theta} - \boldsymbol{\theta}^0)}_{\Delta\boldsymbol{\theta}} + \text{h.o.t.}$$

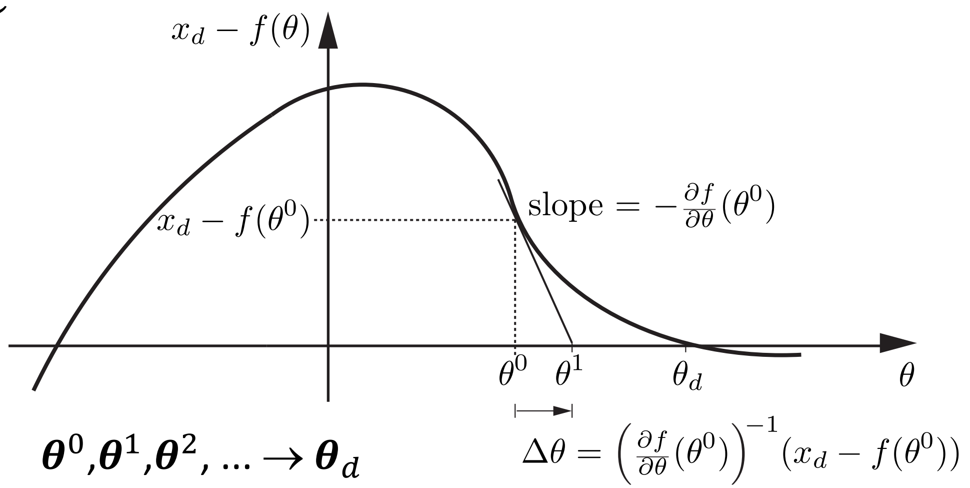
Analytical Jacobian at $\boldsymbol{\theta}^0$

Approximately:

$$\xrightarrow{(\text{h.o.t.} = \mathbf{0})} \quad \boldsymbol{J}(\boldsymbol{\theta}^0)\Delta\boldsymbol{\theta} = \boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^0)$$

* If $\boldsymbol{J}$ is square ($m = n$) and invertible:

$$\Delta\boldsymbol{\theta} = \boldsymbol{J}^{-1}(\boldsymbol{\theta}^0)\big(\boldsymbol{x}_d - \boldsymbol{f}(\theta^0)\big)$$



$$\boldsymbol{\theta}^0, \boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \ldots \rightarrow \boldsymbol{\theta}_d$$

Inverse Kinematics Based on Newton–Raphson Method (Minimum Coordinate IK) (cont.)

∗ If $\boldsymbol{J}$ is not invertible, either because it is not square or because it is singular,:

$$\Delta\boldsymbol{\theta} = \boldsymbol{J}^{\dagger}(\boldsymbol{\theta}^0)\big(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^0)\big)$$

$\boldsymbol{J}^{\dagger}$: Moore–Penrose pseudoinverse

In this case, if $\boldsymbol{J}$ is full rank ($\mathrm{rank}(\boldsymbol{J}) = \min(m, n)$), i.e., the robot is not at a singularity:

- If $n > m$ (the robot has more joints n than end-effector coordinates m):

(Fat Matrix)

$$\boldsymbol{J}^{\dagger} = \boldsymbol{J}^{\mathrm{T}}\left(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\right)^{-1}$$

Note: If there are multiple inverse kinematics solutions, the iterative process tends to converge to the solution that is “closest” to the initial guess $\boldsymbol{\theta}^0$.

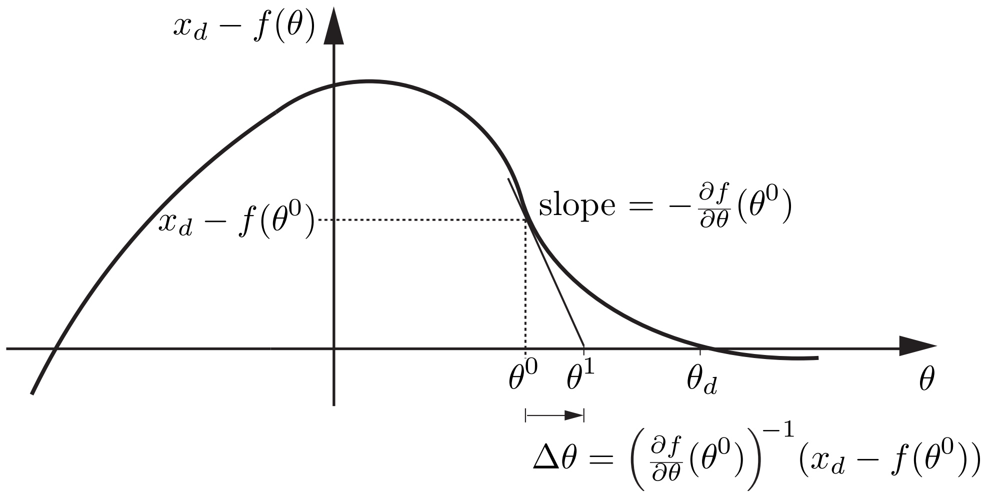
Note: **Methods of optimization** are needed in situations where an exact solution may not exist and we seek the closest approximate solution; or, conversely, an infinity of inverse kinematics solutions exists (i.e., if the robot is kinematically redundant) and we seek a solution that is optimal with respect to some criterion.

Algorithm for Minimum Coordinate Representation

a) Initialization: Given $\boldsymbol{x}_d \in \mathbb{R}^m$ and an initial guess $\boldsymbol{\theta}^0 \in \mathbb{R}^n$, set $i = 0$.

b) Iteration: Set $\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^i)$. While $\|\boldsymbol{e}\| > \epsilon$ for some small $\epsilon \in \mathbb{R}$:

- Set $\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \boldsymbol{J}^\dagger(\boldsymbol{\theta}^i)\boldsymbol{e}$.
- Increment i.



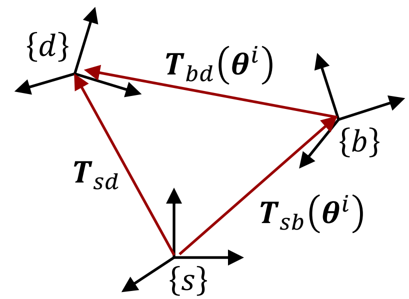
Algorithm in MATLAB:

```
max_iterations = 20;
i = 0;
Theta = Theta_0;
e = X_d - FK(Theta);
while norm(e) > epsilon && i < max_iterations
    Theta = Theta + pinv(J(Theta)) * e;
    i = i + 1;
    e = X_d - FK(Theta);
end
```

Inverse Kinematics Based on Newton–Raphson Method (Geometric IK)

Assume that the EE pose is represented by a Transformation Matrix, i.e., $\boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$, $\boldsymbol{\theta} \in \mathbb{R}^n$. Thus, given a desired EE pose $\boldsymbol{T}_{sd}$, the goal is to find joint coordinates $\boldsymbol{\theta} = \boldsymbol{\theta}_d$ such that

$$\boldsymbol{T}_{sd} = \boldsymbol{T}(\boldsymbol{\theta}_d)$$



Algorithm for Matrix Transformation Representation:

Algorithm in Body Frame:

a) Initialization: Given $\boldsymbol{T}_{sd} \in SE(3)$ and an initial guess $\boldsymbol{\theta}^0 \in \mathbb{R}^n$, set $i = 0$.

b) Iteration: Set $[\boldsymbol{\mathcal{V}}_b] = \log\left(\boldsymbol{T}_{bd}(\boldsymbol{\theta}^i)\right) = \log(\boldsymbol{T}_{sb}^{-1}(\boldsymbol{\theta}^i)\boldsymbol{T}_{sd})$. While $\|\boldsymbol{\omega}_b\| > \epsilon_\omega$ or $\|\boldsymbol{v}_b\| > \epsilon_v$ for some small $\epsilon_\omega, \epsilon_v \in \mathbb{R}$:

- Set $\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \boldsymbol{J}_b^\dagger(\boldsymbol{\theta}^i)\boldsymbol{\mathcal{V}}_b$. ($\boldsymbol{\mathcal{V}}_b$ is the twist that takes $\boldsymbol{T}_{sb}$ to $\boldsymbol{T}_{sd}$ in 1s)
- Increment i. (ϵ_ω has the unit of radian and the dimension of ϵ_v is length)

Algorithm in Space Frame:

a) Initialization: Given $\boldsymbol{T}_{sd} \in SE(3)$ and an initial guess $\boldsymbol{\theta}^0 \in \mathbb{R}^n$, set $i = 0$.

b) Iteration: Set $[\boldsymbol{\mathcal{V}}_s] = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\log\left(\boldsymbol{T}_{bd}(\boldsymbol{\theta}^i)\right) = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\log(\boldsymbol{T}_{sb}^{-1}(\boldsymbol{\theta}^i)\boldsymbol{T}_{sd})$. While $\|\boldsymbol{\omega}_s\| > \epsilon_\omega$ or $\|\boldsymbol{v}_s\| > \epsilon_v$ for some small $\epsilon_\omega, \epsilon_v \in \mathbb{R}$:

- Set $\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \boldsymbol{J}_s^\dagger(\boldsymbol{\theta}^i)\boldsymbol{\mathcal{V}}_s$. ($\boldsymbol{\mathcal{V}}_s$ is the twist that takes $\boldsymbol{T}_{sb}$ to $\boldsymbol{T}_{sd}$ in 1s)
- Increment i.