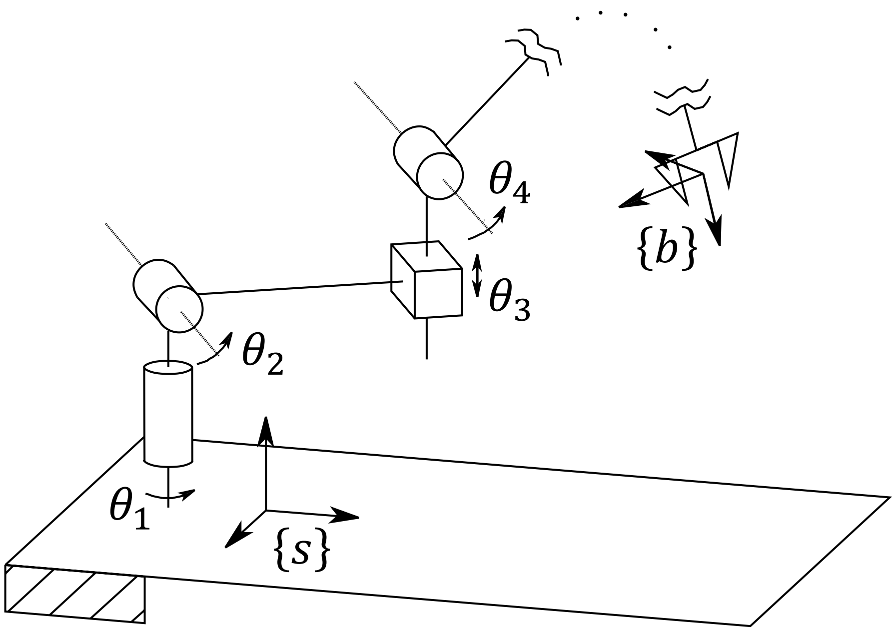
Ch4: Forward Kinematics

Forward Kinematics

Open-Chain Robot Manipulators

Robot manipulators are articulated mechanical systems composed of links connected by joints. In this course, we consider only n-DOF open-chain (serial) robot manipulators with revolute and/or prismatic joints.



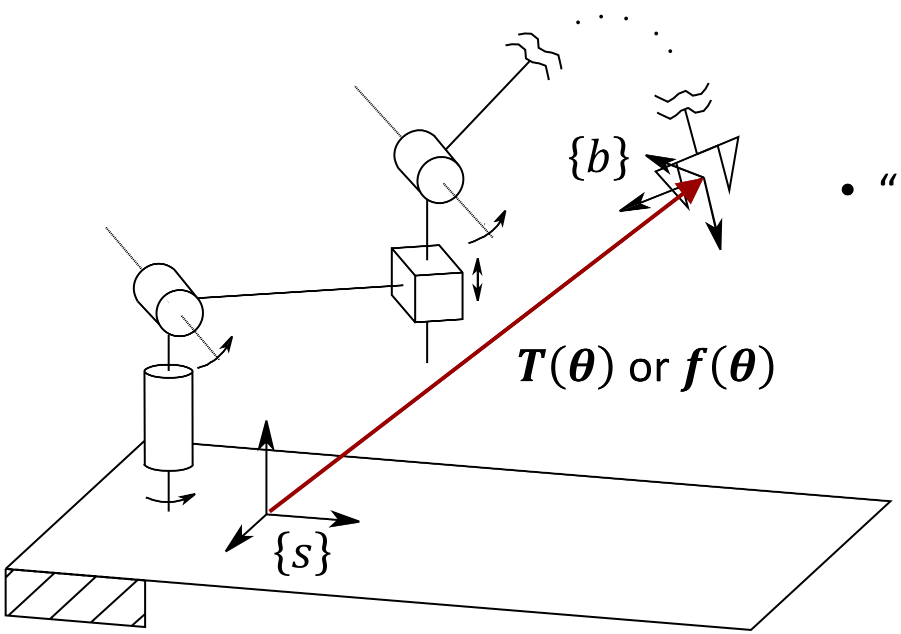
- The generalized joint coordinate (joint position) denoted by θ_i corresponds to the angular displacement of a revolute joint or the linear displacement of a prismatic joint.

Vector of joint positions: $\boldsymbol{\theta} \in \mathbb{R}^n$

- Each joint is independently controlled through an actuator.
- The joint positions are measured by sensors placed at the actuators, that are usually located at the joints.

Forward Kinematics

The forward kinematics of a robot refers to the calculation of the position and orientation (**pose**) of its end-effector frame from its joint positions $\boldsymbol{\theta}$.



- "Geometric" forward kinematics:

  Given $\boldsymbol{\theta} \in \mathbb{R}^n$, Find $\boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$

  $$\boldsymbol{T}: \mathbb{R}^n \to SE(3)$$

  (Using PoE or D-H Method)

- "Minimum-Coordinate" forward kinematics:

  Given $\boldsymbol{\theta} \in \mathbb{R}^n$, Find $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$

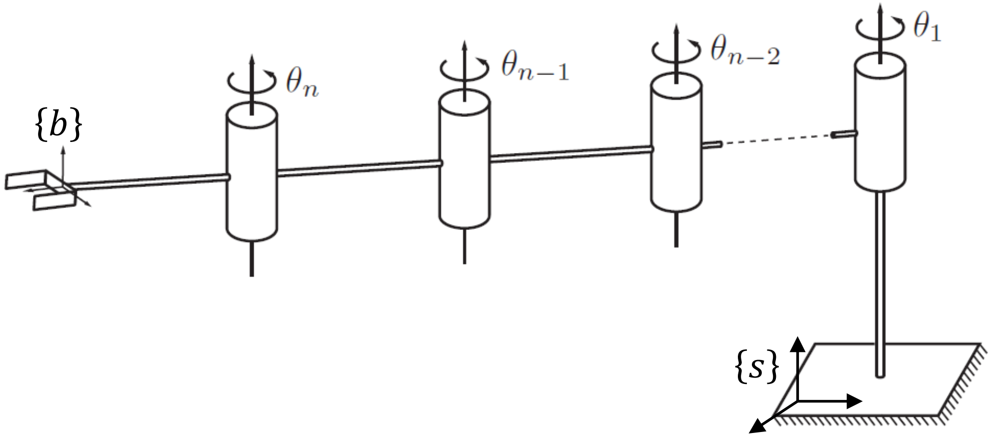
  $$(m \leq n) \qquad \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m$$

Product of Exponentials (PoE) Formulation in the Base Frame

Forward Kinematics Using Product of Exponentials (PoE) Formulation

Calculating the forward kinematics of an open chain using the **space form** of the PoE formula:

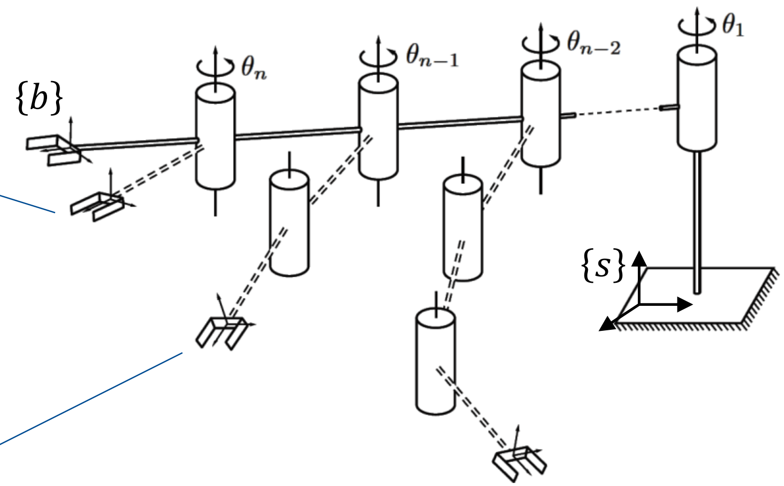
- Assign a fixed base frame $\{s\}$.
- Assign an end-effector frame $\{b\}$.
- Let $\boldsymbol{M} = \boldsymbol{T}_{sb}(\mathbf{0}) \in SE(3)$ be the configuration of $\{b\}$ relative to $\{s\}$ when the robot is in its home or zero configuration ($\boldsymbol{\theta} = \mathbf{0}$).
- Specify the direction of positive displacement (rotation for revolute joints, translation for prismatic joints) for each joint.



Forward Kinematics Using Product of Exponentials Formulation

- Suppose that joint n is displaced by θ_n (for $\theta_{1,\dots,n-1} = 0$). Then, the new configuration of $\{b\}$ is

$$\boldsymbol{T} = e^{[\boldsymbol{S}_n]\theta_n}\boldsymbol{M}$$



$$\boldsymbol{S}_n = \begin{bmatrix} \boldsymbol{S}_{\omega,n} \\ \boldsymbol{S}_{v,n} \end{bmatrix} \in \mathbb{R}^6 \qquad [\boldsymbol{S}_n] = \begin{bmatrix} [\boldsymbol{S}_{\omega,n}] & \boldsymbol{S}_{v,n} \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$$

$\boldsymbol{S}_n$: Screw axis of joint n as expressed in $\{s\}$.

- Suppose that joint $n-1$ is displaced by θ_{n-1} (for $\theta_{1,\dots,n-2} = 0$ and any fixed, but arbitrary, θ_n). Then, the new configuration of $\{b\}$ is

$$\boldsymbol{T} = e^{[\boldsymbol{S}_{n-1}]\theta_{n-1}}\left(e^{[\boldsymbol{S}_n]\theta_n}\boldsymbol{M}\right)$$

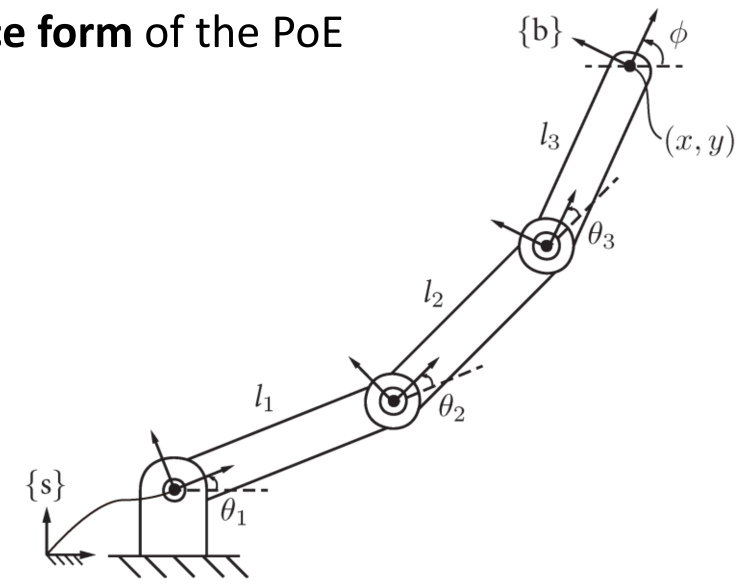
Continuing this for all the joints:

$$\boxed{\boldsymbol{T}(\boldsymbol{\theta}) = e^{[\boldsymbol{S}_1]\theta_1} \dots e^{[\boldsymbol{S}_{n-1}]\theta_{n-1}} e^{[\boldsymbol{S}_n]\theta_n}\boldsymbol{M}}$$

The screw axes $\boldsymbol{S}_1, \dots, \boldsymbol{S}_n$ expressed in $\{s\}$, corresponding to the joint motions when the robot is at its home configuration ($\boldsymbol{\theta} = \boldsymbol{0}$).

Example: 3R Planar Robot

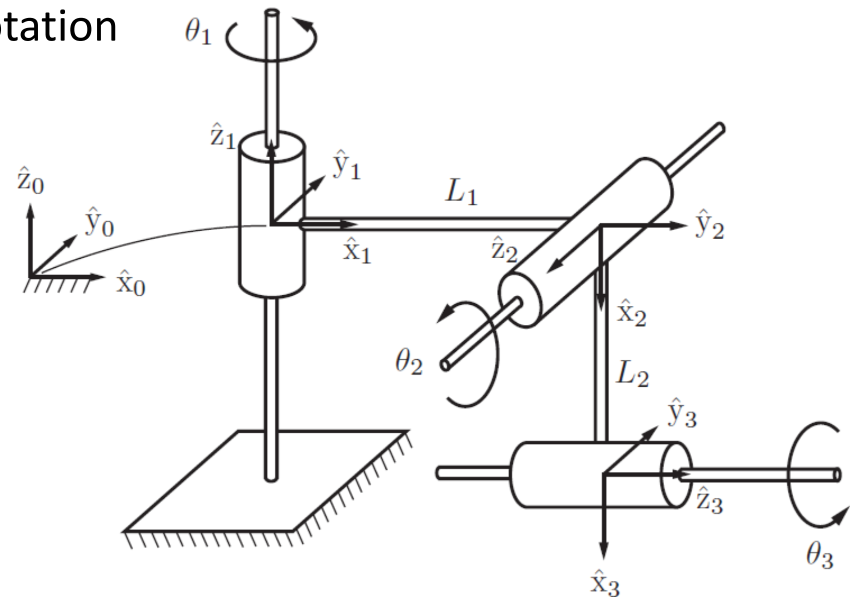
Find the Geometric forward kinematics using the **space form** of the PoE formulation.



Example: 3R Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

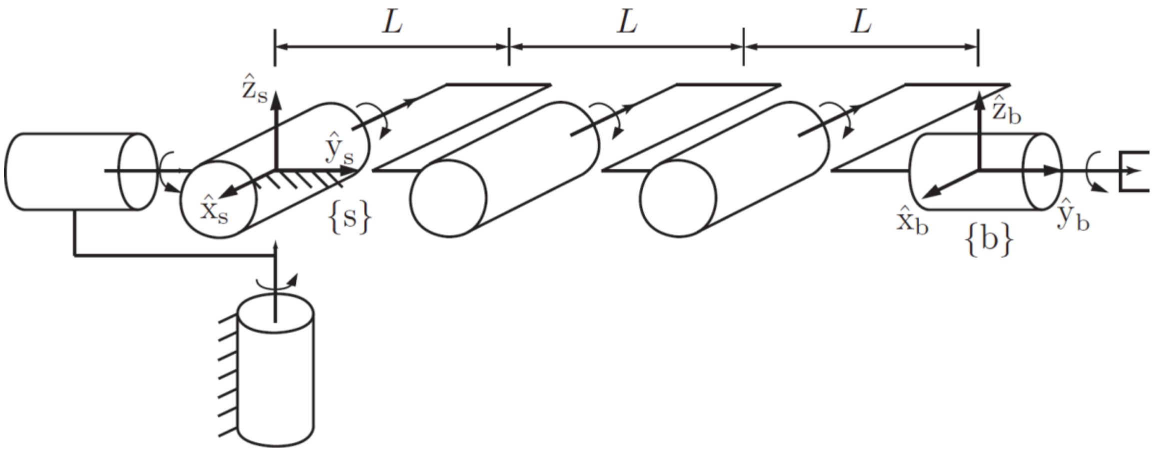
Find the Geometric forward kinematics using the **space form** of the PoE formulation.



Example: 6R Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

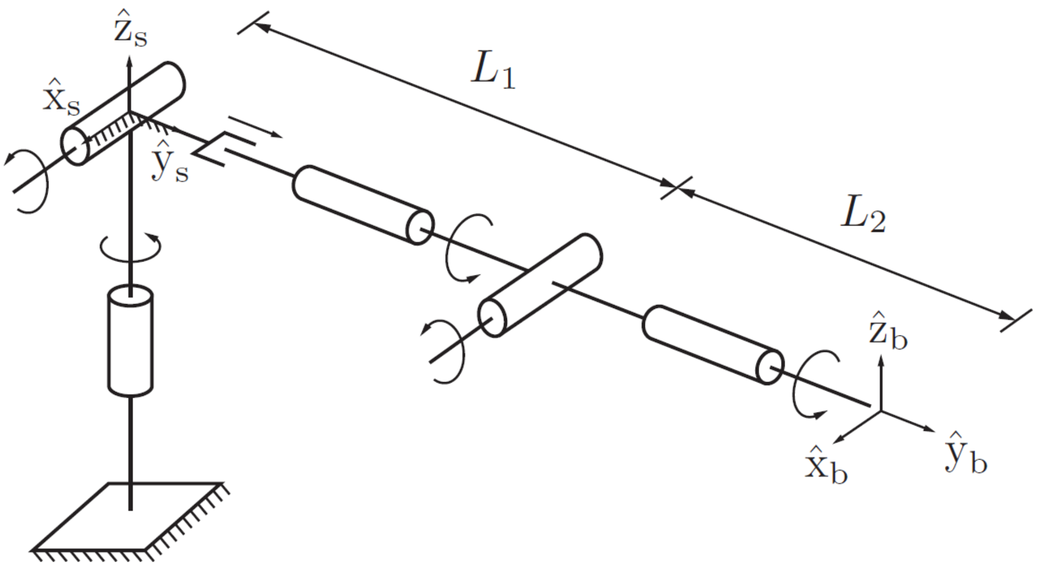
Find the Geometric forward kinematics using the **space form** of the PoE formulation.



Example: RRPRRR Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

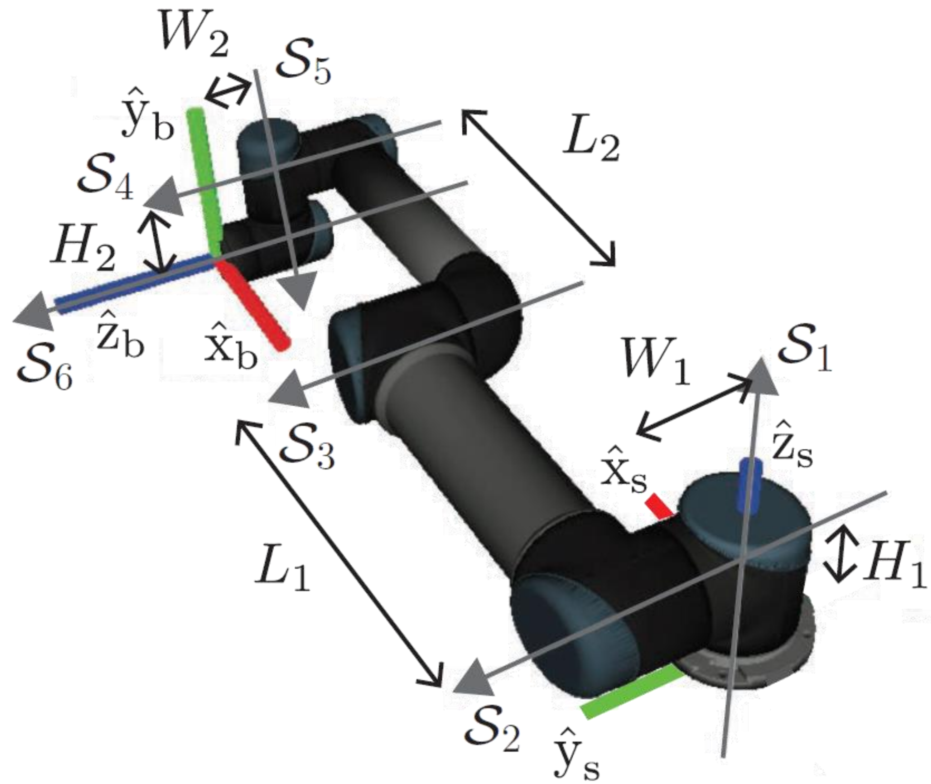
Find the Geometric forward kinematics using the **space form** of the PoE formulation.



Example: 6R Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

Find the Geometric forward kinematics using the **space form** of the PoE formulation.

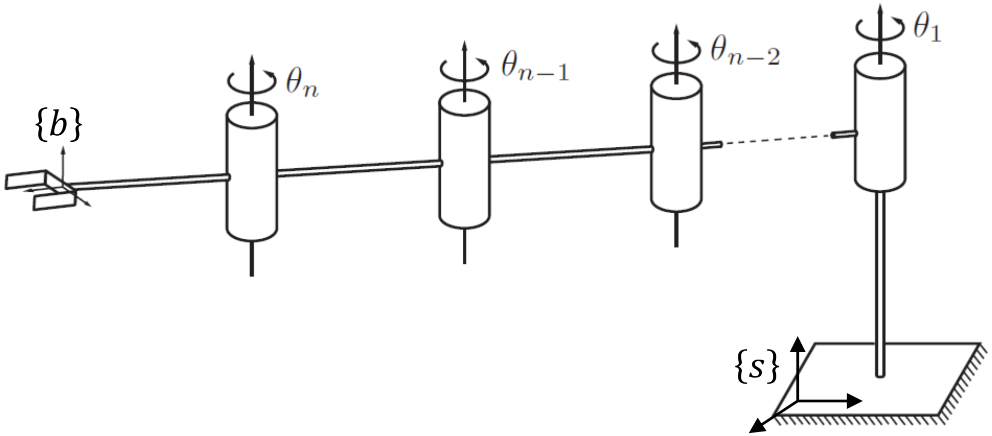


Product of Exponentials (PoE) Formulation in the End-Effector Frame

Forward Kinematics Using Product of Exponentials (PoE) Formulation

An alternative method to calculate the forward kinematics of an open chain is using the **body form** of the PoE formula.

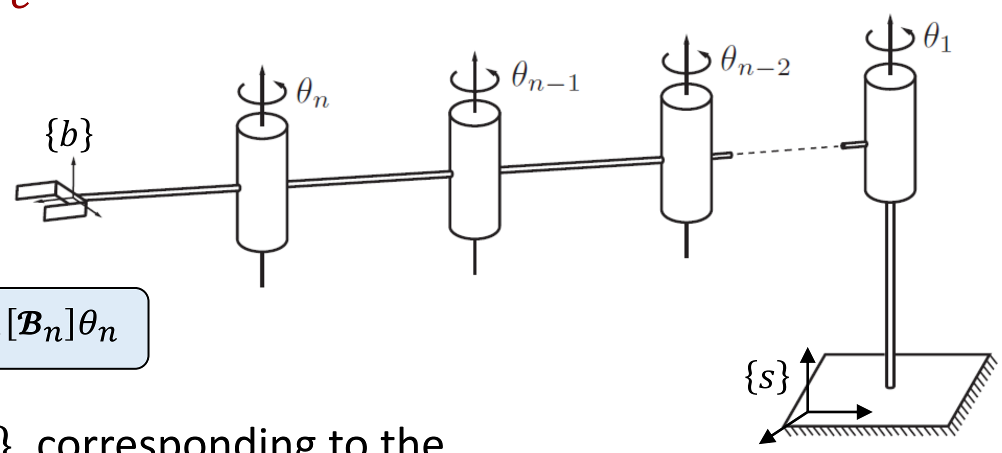
- Assign a fixed base frame $\{s\}$.
- Assign an end-effector frame $\{b\}$.
- Let $\boldsymbol{M} \in SE(3)$ be the configuration of $\{b\}$ relative to $\{s\}$ when the robot is in its <u>home</u> or <u>zero</u> configuration.
- Specify the direction of positive displacement (rotation for revolute joints, translation for prismatic joints) for each joint.



Forward Kinematics Using Product of Exponentials (PoE) Formulation

We know that $e^{\boldsymbol{M}^{-1}\boldsymbol{P}\boldsymbol{M}} = \boldsymbol{M}^{-1}e^{\boldsymbol{P}}\boldsymbol{M}$, similarly $\boldsymbol{M}e^{\boldsymbol{M}^{-1}\boldsymbol{P}\boldsymbol{M}} = e^{\boldsymbol{P}}\boldsymbol{M}$.

$$
\begin{aligned}
\boldsymbol{T}(\boldsymbol{\theta}) &= e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} \boldsymbol{M} \\
&= e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} \boldsymbol{M} e^{\boldsymbol{M}^{-1}[\mathcal{S}_n]\boldsymbol{M}\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} \dots \boldsymbol{M} e^{\boldsymbol{M}^{-1}[\mathcal{S}_{n-1}]\boldsymbol{M}\theta_{n-1}} e^{\boldsymbol{M}^{-1}[\mathcal{S}_n]\boldsymbol{M}\theta_n} \\
&= \boldsymbol{M} e^{\boldsymbol{M}^{-1}[\mathcal{S}_1]\boldsymbol{M}\theta_1} \dots e^{\boldsymbol{M}^{-1}[\mathcal{S}_{n-1}]\boldsymbol{M}\theta_{n-1}} e^{\boldsymbol{M}^{-1}[\mathcal{S}_n]\boldsymbol{M}\theta_n} \\
&= \boldsymbol{M} e^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}
\end{aligned}
$$



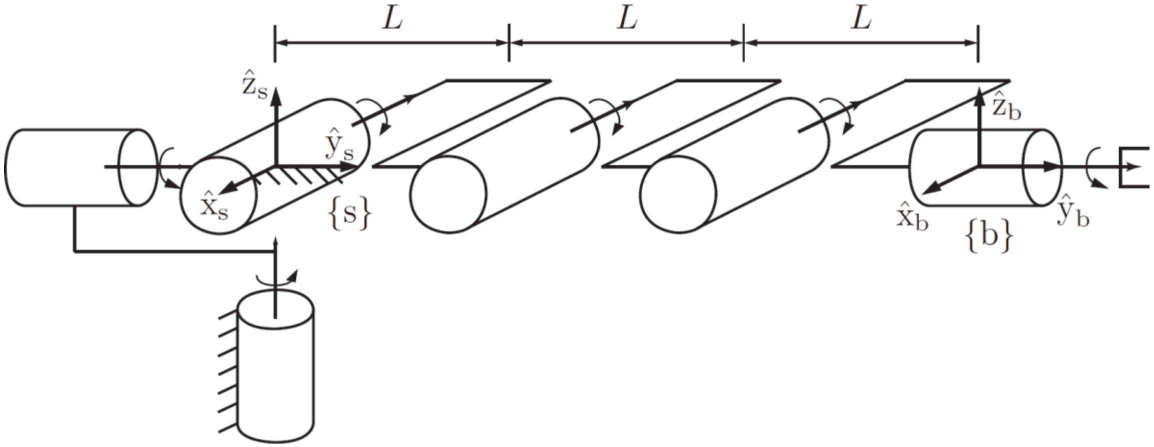
$$
\boxed{\boldsymbol{T}(\boldsymbol{\theta}) = \boldsymbol{M} e^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}}
$$

The screw axes $\mathcal{B}_1, \dots, \mathcal{B}_n$ expressed in $\{b\}$, corresponding to the joint motions when the robot is at its home configuration ($\boldsymbol{\theta} = \boldsymbol{0}$).

Example: 6R Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

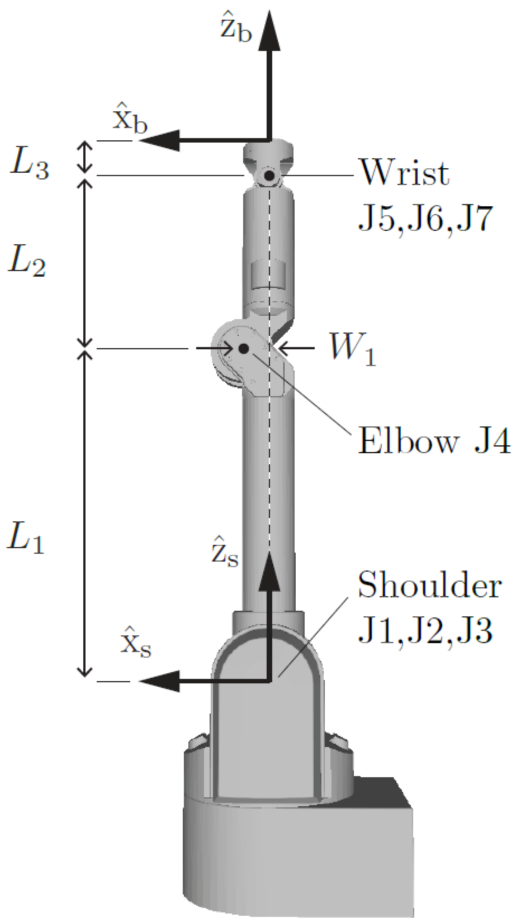
Find the Geometric forward kinematics using the **body form** of the PoE formulation.



Example: 7R Spatial Robot

At the zero configuration, axes 1, 3, 5, and 7 are along $\hat{z}_s$ and axes 2, 4, and 6 are aligned with $\hat{y}_s$. Positive rotations are given by the right-hand rule. Axes 1, 2, and 3 intersect at the origin of $\{s\}$ and axes 5, 6, and 7 intersect at a point 60mm from $\{b\}$. The zero configuration is singular.

Find the Geometric forward kinematics using the **body form** of the PoE formulation.



Redundancy

Intrinsic and Kinematic Redundancy

$\dim(\text{C–Space}) = \dim(\text{J–Space}) = n$ (Configuration Space or Joint space)

$\dim(\text{O–Space}) = m$ (Operational space)

$\dim(\text{T–Space}) = r$ (Task space)

A manipulator is **intrinsically redundant** when the dimension of the joint space is greater than the dimension of the operational space (i.e., $n > m$).

A manipulator is **kinematically redundant** when the dimension of the joint space is greater than the dimension of the task space (i.e, $n > r$) and there exist $n - r$ redundant DOFs.

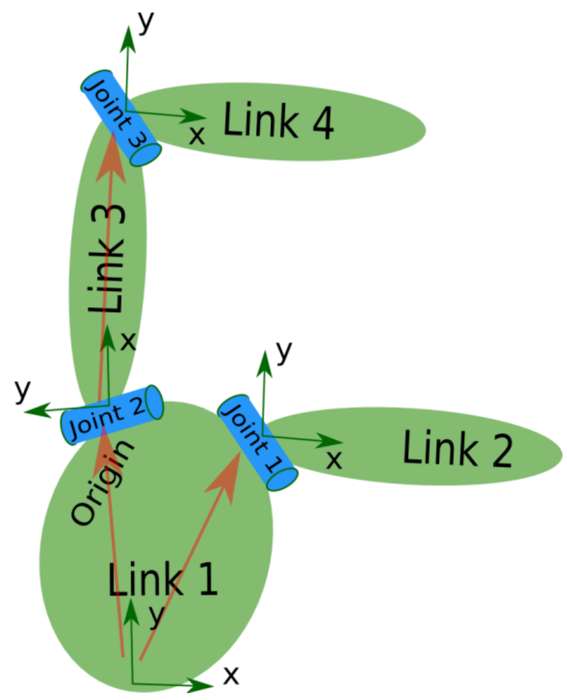
- A manipulator can be redundant with respect to a task and nonredundant with respect to another.
- Even in the case of $m = n$, a manipulator can be functionally redundant when only r components of operational space are of concern for the specific task, with $r < m$.

Universal Robot Description Format (URDF)

Universal Robot Description Format

The Universal Robot Description Format (URDF) is an XML (eXtensible Markup Language) file format used by the Robot Operating System (ROS) to describe the kinematics (in defining joints), inertial properties, and link geometry of robots (in defining links) of open-chain robots.

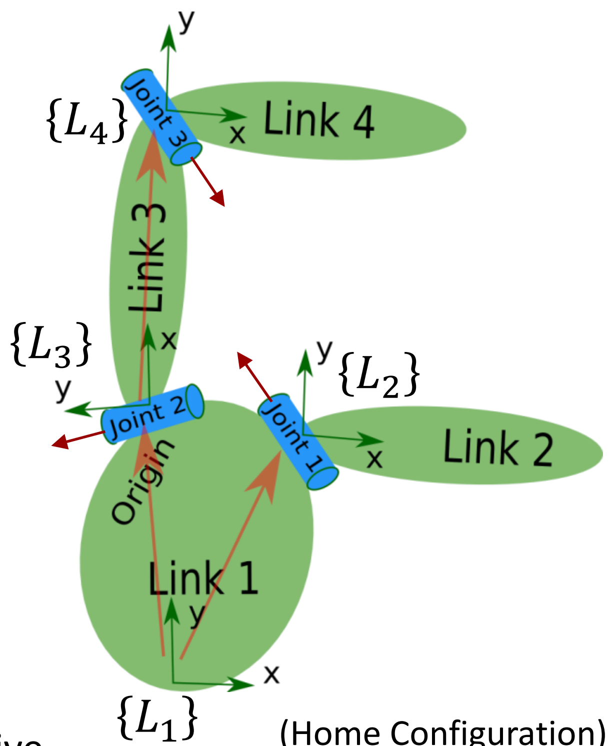
```
<joint name="joint1" type="continuous">
        <parent link="link1"/>
        <child link="link2"/>
        <origin xyz="0.5 0.3 0" rpy="0 0 0" />
        <axis xyz="-0.9 0.15 0" />
</joint>

<link name="link1">
        <inertial>
                <mass value="1"/>
                <origin rpy="0.1 0 0" xyz="0 0 0"/>
                <inertia ixx="0.004" ixy="0" ixz="0"
                        iyy="0.004" iyz="0" izz="0.007"/>
        </inertial>
</link>
```



[Link](#)

URDF: Defining Joints

Joints connect two links: a parent link and a child link. The reference frame of each (child) link $\{L_i\}$ is located (at the bottom of the link) on the joint's axis.




```
<joint name="joint3" type="continuous">
        <parent link="link3"/>
        <child link="link4"/>
        <origin xyz="0.5 0 0" rpy="0 0 -1.57" />
        <axis xyz="0.707 -0.707 0" />
</joint>
```

: $\{L_4\}$ w.r.t. $\{L_3\}$

: in $\{L_4\}$

"origin" frame defines the pose of the child link frame relative to the parent link frame when the joint variable is zero.

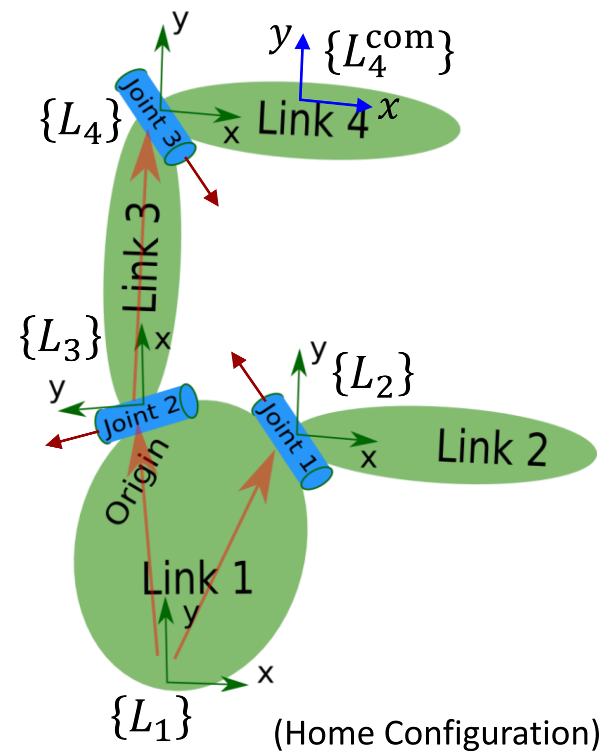
"axis" defines the joint's axis, a unit vector expressed in the child link's frame, in the direction of positive rotation for a revolute joint or positive translation for a prismatic joint.

URDF: Defining Links

```
<link name=" link4">
    <inertial>
        <mass value="1"/>
        <origin xyz="0.1 0 0" rpy="0 0 0"/>
        <inertia ixx="0.004" ixy="0" ixz="0"
            iyy="0.004" iyz="0" izz="0.007"/>
    </inertial>
    <visual>
        <geometry>
            <mesh filename=".../link1.stl" />
        </geometry>
        <material name="DarkGrey">
            <color rgba="0.3 0.3 0.3 1.0"/>
        </material>
    </visual>
</link>
```

$\boxed{:\ \{L_4^{\mathrm{com}}\}\ \text{w.r.t.}\ \{L_4\}}$

$\boxed{:\ \text{in}\ \{L_4^{\mathrm{com}}\}}$



"inertia" six elements of inertia matrix relative to the link's center of mass.

"origin" frame defines the position and orientation of a frame at the link's center of mass relative to the link's frame at its joint.