Ch6: Inverse Kinematics

Inverse Kinematics

Inverse Kinematics

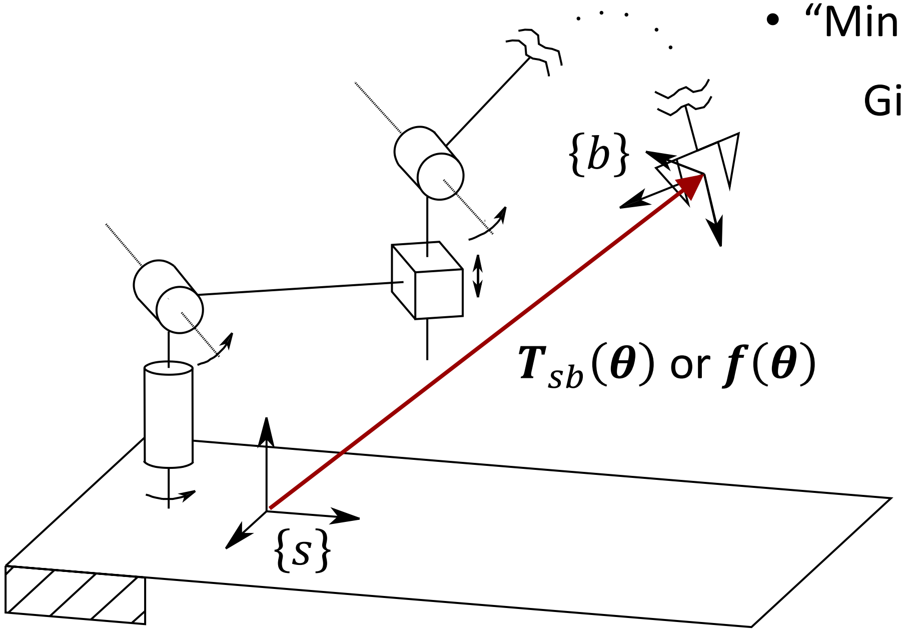
The inverse kinematics of a robot refers to the calculation of the joint coordinates $\boldsymbol{\theta}$ from the position and orientation (**pose**) of its end-effector frame.

- "Geometric" inverse kinematics:

  Given $\boldsymbol{T}_{sb}(\boldsymbol{\theta}) = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$, Find $\boldsymbol{\theta} \in \mathbb{R}^n$

- "Minimum-Coordinate" inverse kinematics:

  Given $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$, Find $\boldsymbol{\theta} \in \mathbb{R}^n$


$\{b\}$
$\boldsymbol{T}_{sb}(\boldsymbol{\theta})$ or $\boldsymbol{f}(\boldsymbol{\theta})$
$\{s\}$

Complexities of Inverse Kinematics

• The equations to solve are in general nonlinear, and thus it is not always possible to find a closed-form solution.
• Multiple solutions may exist.
• Infinite solutions may exist (e.g., in the case of a kinematically redundant manipulator).
• There might be no admissible solutions (e.g., when the given EE pose does not belong to the manipulator dexterous workspace.).

► Solving Inverse Kinematics Problems:

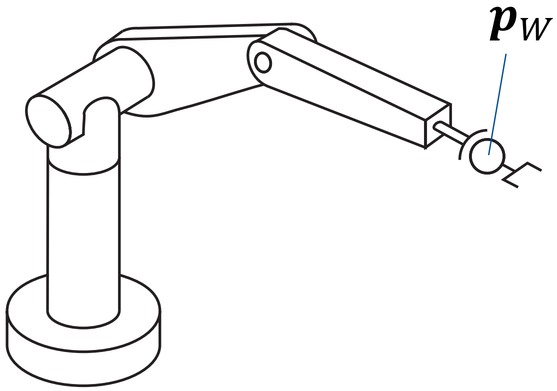
- **Analytic Methods**: Finding closed-form solutions using algebraic intuition or geometric intuition.

- **Iterative Numerical Methods**: When there are no (or it is difficult to find) closed-form solutions.

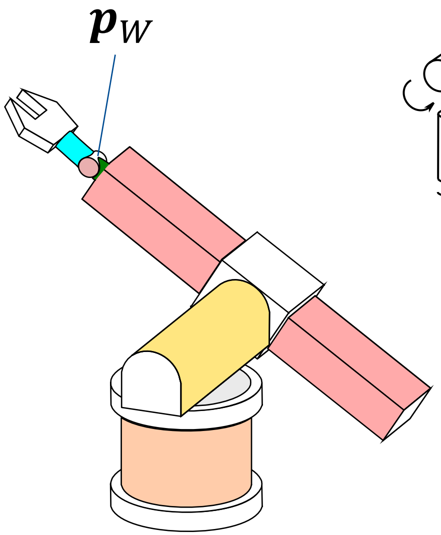
Analytic Inverse Kinematics

Analytic Inverse Kinematics

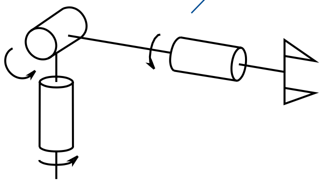
Most of the existing manipulators are typically formed by an **arm** and a **spherical wrist** (i.e., three consecutive revolute joint axes intersect at a common point). Thus, we can decouple the solution for the position (e.g., point $\boldsymbol{p}_W$ at the intersection of the three revolute axes) from that for the orientation.

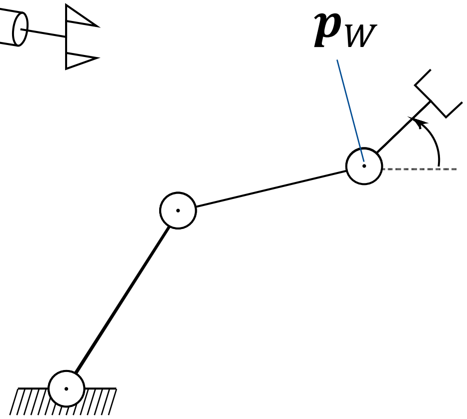


PUMA Arm (6R)



Stanford Arm (RRPRRR)

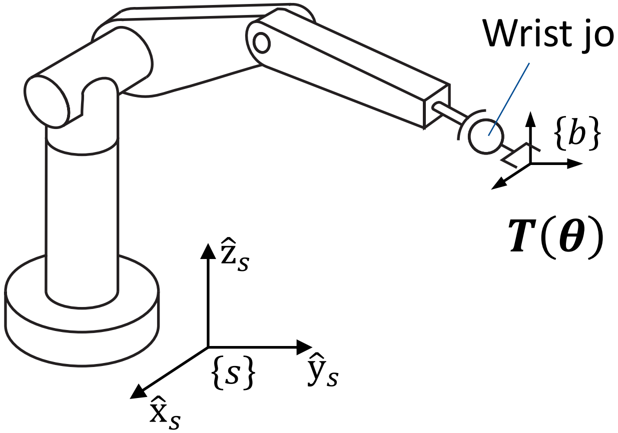




3R Planar Arm

∗ Therefore, it is possible to solve the inverse kinematics for the arm separately from the inverse kinematics for the spherical wrist.

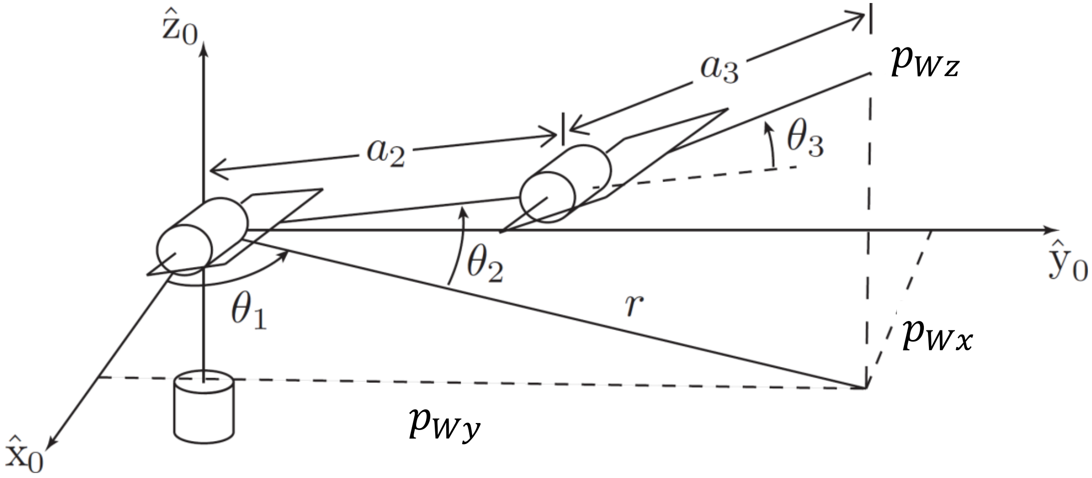
6R PUMA-Type Arms



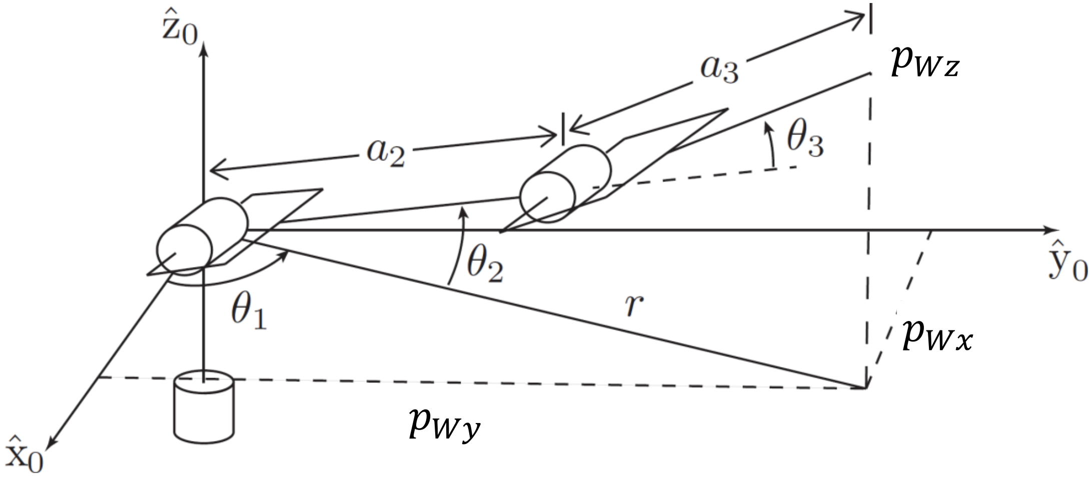
Wrist joints intersect orthogonally at a common point $\boldsymbol{p}_W$

$$\boldsymbol{T}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{R}_{sb} & \boldsymbol{p}_b \\ \boldsymbol{0} & 1 \end{bmatrix} = e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2} e^{[\boldsymbol{S}_3]\theta_3} e^{[\boldsymbol{S}_4]\theta_4} e^{[\boldsymbol{S}_5]\theta_5} e^{[\boldsymbol{S}_6]\theta_6} \boldsymbol{M}$$

By having $\boldsymbol{p}_b$, we can find $\boldsymbol{p}_W = (p_{Wx}, p_{Wy}, p_{Wz})$



6R PUMA-Type Arms (cont.)



$$p_{Wx} = c_1(a_2c_2 + a_3c_{23})$$
$$p_{Wy} = s_1(a_2c_2 + a_3c_{23})$$
$$p_{Wz} = a_2s_2 + a_3s_{23}$$

- Inverse position problem of finding $(\theta_1, \theta_2, \theta_3)$ using algebraic intuition:

$$p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 = a_2^2 + a_3^2 + 2a_2a_3c_3$$

$$c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2a_3} \quad \Rightarrow \quad \theta_3 = \text{atan2}(s_3, c_3) \quad \Rightarrow \quad \begin{array}{c} \theta_{3,\text{I}} \in [-\pi, \pi] \\ \theta_{3,\text{II}} = -\theta_{3,I} \end{array}$$

$$s_3 = \pm\sqrt{1 - c_3^2}$$

6R PUMA-Type Arms (cont.)

$$p_{Wx}^2 + p_{Wy}^2 = (a_2 c_2 + a_3 c_{23})^2 \longrightarrow \left.\begin{aligned} a_2 c_2 + a_3 c_{23} &= \pm\sqrt{p_{Wx}^2 + p_{Wy}^2} \\ p_{Wz} &= a_2 s_2 + a_3 s_{23} \\ s_{23} &= s_2 c_3 + s_3 c_2 \\ c_{23} &= c_2 c_3 - s_2 s_3 \end{aligned}\right\} \Rightarrow$$

$$c_2 = \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3 c_3) + p_{Wz} a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3}$$

$$s_2 = \frac{p_{Wz}(a_2 + a_3 c_3) \mp \sqrt{p_{Wx}^2 + p_{Wy}^2} a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3} \Rightarrow \qquad \theta_2 = \operatorname{atan2}(s_2, c_2)$$

For each θ_3, we have two solutions for θ_2:

$$\theta_{3,\mathrm{I}} \rightarrow \left(\theta_{2,\mathrm{I}}, \theta_{2,\mathrm{II}}\right)$$

$$\theta_{3,\mathrm{II}} \rightarrow \left(\theta_{2,\mathrm{III}}, \theta_{2,\mathrm{IV}}\right)$$

6R PUMA-Type Arms (cont.)

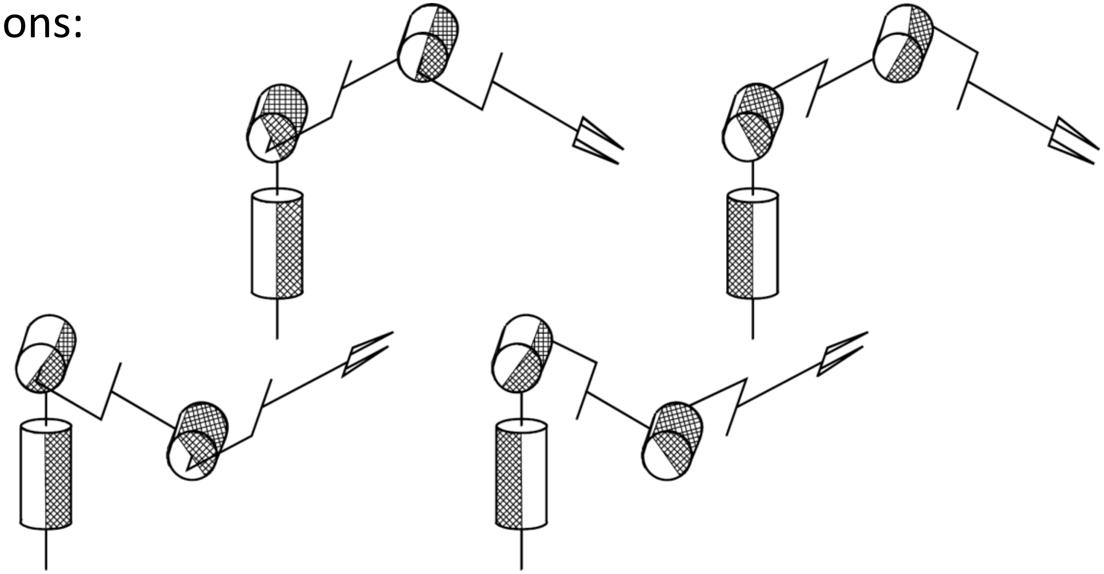
$$p_{Wx} = c_1(a_2c_2 + a_3c_{23})$$
$$p_{Wy} = s_1(a_2c_2 + a_3c_{23})$$
$$a_2c_2 + a_3c_{23} = \pm\sqrt{p_{Wx}^2 + p_{Wy}^2}$$

$$\Rightarrow \quad p_{Wx} = \pm c_1\sqrt{p_{Wx}^2 + p_{Wy}^2}$$
$$p_{Wy} = \pm s_1\sqrt{p_{Wx}^2 + p_{Wy}^2}$$

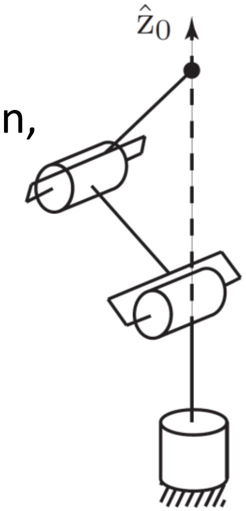
$$\Rightarrow \quad \theta_{1,\mathrm{I}} = \mathrm{atan2}(p_{Wy}, p_{Wx})$$
$$\theta_{1,\mathrm{II}} = \mathrm{atan2}(-p_{Wy}, -p_{Wx})$$

Thus, in total, there exist four solutions:

$$(\theta_{1,\mathrm{I}}, \theta_{2,\mathrm{I}}, \theta_{3,\mathrm{I}})$$
$$(\theta_{1,\mathrm{I}}, \theta_{2,\mathrm{III}}, \theta_{3,\mathrm{II}})$$
$$(\theta_{1,\mathrm{II}}, \theta_{2,\mathrm{II}}, \theta_{3,\mathrm{I}})$$
$$(\theta_{1,\mathrm{II}}, \theta_{2,\mathrm{IV}}, \theta_{3,\mathrm{II}})$$



6R PUMA-Type Arms (cont.)



Note: When $p_{Wx} = p_{Wy} = 0$, the arm is in a kinematically singular configuration, and there are infinitely many possible solutions for θ_1.

❖ Inverse orientation problem of finding $(\theta_4, \theta_5, \theta_6)$:

$$e^{[\boldsymbol{S}_4]\theta_4} e^{[\boldsymbol{S}_5]\theta_5} e^{[\boldsymbol{S}_6]\theta_6} = \underbrace{e^{-[\boldsymbol{S}_3]\theta_3} e^{-[\boldsymbol{S}_2]\theta_2} e^{-[\boldsymbol{S}_1]\theta_1} \boldsymbol{T}(\boldsymbol{\theta}) \boldsymbol{M}^{-1}}_{\text{known}} = \boldsymbol{T}'$$

Assume that the joint axes $(\boldsymbol{S}_4, \boldsymbol{S}_5, \boldsymbol{S}_6)$ of the spherical wrist are aligned in the $(\hat{z}_s, \hat{y}_s, \hat{x}_s)$ directions, respectively:

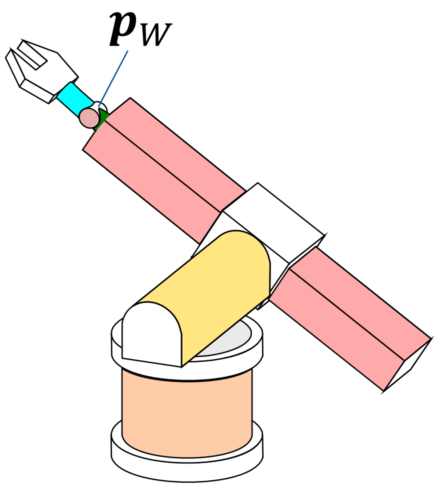
$$\boldsymbol{S}_{\omega_4} = (0,0,1) \quad \boldsymbol{S}_{\omega_5} = (0,1,0) \quad \boldsymbol{S}_{\omega_6} = (1,0,0)$$

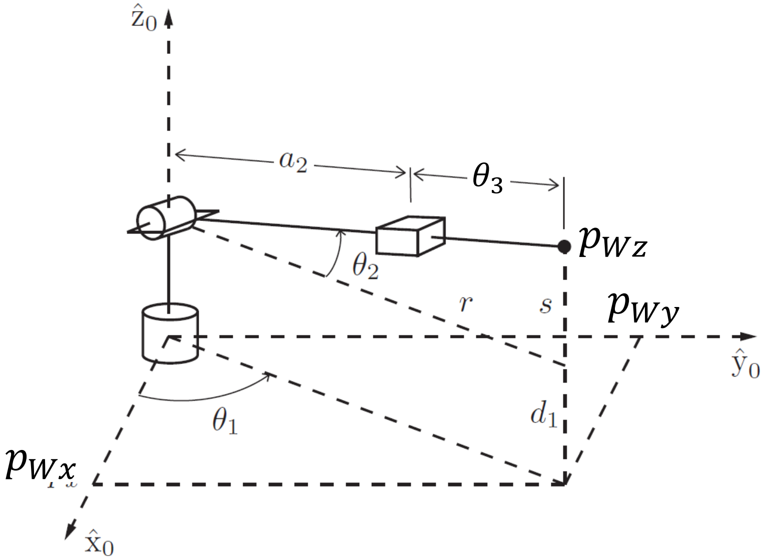
⇨ $\text{Rot}(\hat{z}, \theta_4)\text{Rot}(\hat{y}, \theta_5)\text{Rot}(\hat{x}, \theta_6) = \boldsymbol{R}'$ ⇨ This corresponds to the ZYX Euler angles.

$$\boldsymbol{T}' = (\boldsymbol{R}', \boldsymbol{p}')$$

⇨ $(\theta_4, \theta_5, \theta_6)$

Stanford-Type Arms





$$r^2 = p_{Wx}^2 + p_{Wy}^2$$
$$s = p_{Wz} - d_1$$

- ❖ Inverse position problem of finding $(\theta_1, \theta_2, \theta_3)$ using geometric intuition:

If $p_{Wx}, p_{Wy} \neq 0$: $\begin{cases} \theta_1 = \text{atan2}(p_{Wy}, p_{Wx}) \\ \theta_2 = \text{atan2}(s, r) \end{cases}$, $\begin{cases} \theta_1 = \pi + \text{atan}\,2(p_{Wy}, p_{Wx}) \\ \theta_2 = \pi - \text{atan}\,2(s, r) \end{cases}$

$$(\theta_3 + a_2)^2 = r^2 + s^2 \quad \longrightarrow \quad \theta_3 = \sqrt{r^2 + s^2} = \sqrt{p_{Wx}^2 + p_{Wy}^2 + (p_{Wz} - d_1)^2} - a_2$$

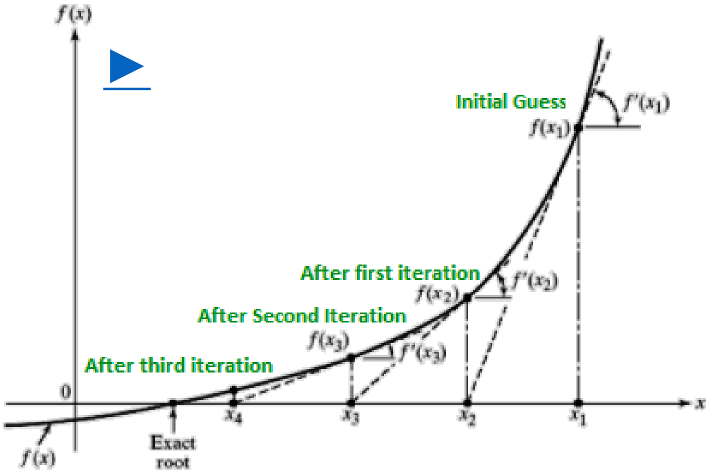
$\Rightarrow$ Thus, there are 2 solutions to the inverse kinematics problem.

- ❖ Inverse orientation problem of finding $(\theta_4, \theta_5, \theta_6)$ is similar to PUMA.

Numerical Inverse Kinematics

Newton–Raphson Method

Newton–Raphson Method is an iterative method for numerically finding the roots of a nonlinear equation $f(\theta) = 0$ where $f: \mathbb{R} \to \mathbb{R}$ is differentiable.



If θ^0 is an initial guess for the solution, Taylor expansion of $f(\theta)$ at θ^0 is

$$f(\theta) = f(\theta^0) + \frac{\partial f}{\partial \theta}(\theta^0)(\theta - \theta^0) + \overset{\cong 0}{\cancel{\text{higher-order terms (h.o.t)}}} \xrightarrow{f(\theta)=0} \theta = \theta^0 - \left(\frac{\partial f}{\partial \theta}(\theta^0)\right)^{-1} f(\theta^0)$$

Using θ as the new guess for the solution and repeating: $$\theta^{k+1} = \theta^k - \left(\frac{\partial f}{\partial \theta}(\theta^k)\right)^{-1} f(\theta^k)$$

The iteration is repeated until some stopping criterion is satisfied: $$\frac{\left|f(\theta^k) - f(\theta^{k+1})\right|}{\left|f(\theta^k)\right|} \leq \epsilon$$

ϵ: a given threshold value

Inverse Kinematics Based on Newton–Raphson Method (Minimum-Coordinate IK)

❖ If the desired end-effector configuration represented by the minimum number of coordinates, i.e., $\boldsymbol{x}_d = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$, $\boldsymbol{\theta} \in \mathbb{R}^n$, then the goal is to find joint coordinates $\boldsymbol{\theta}_d$ such that

$$\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}_d) = \boldsymbol{0}$$

(Assumption: $\boldsymbol{f}$ is differentiable)

- We use a method similar to the Newton–Raphson method for nonlinear root-finding:

Given an initial guess $\boldsymbol{\theta}^0$ which is "close to" a solution $\boldsymbol{\theta}_d$, and using the Taylor expansion:

$$\boldsymbol{x}_d = \boldsymbol{f}(\boldsymbol{\theta}_d) = \boldsymbol{f}(\boldsymbol{\theta}^0) + \underbrace{\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}^0}}_{\boldsymbol{J}(\boldsymbol{\theta}^0) \in \mathbb{R}^{m \times n}} \underbrace{(\boldsymbol{\theta}_d - \boldsymbol{\theta}^0)}_{\Delta\boldsymbol{\theta}} + \text{h.o.t.}$$

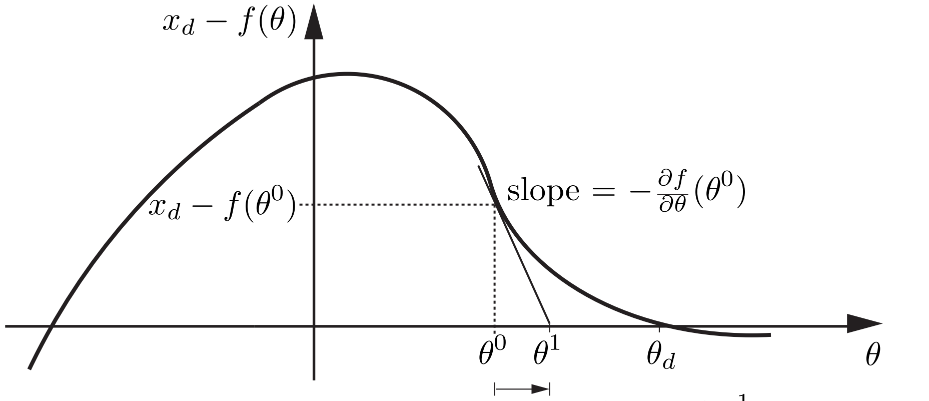
Analytical Jacobian at $\boldsymbol{\theta}^0$

Approximately: $\boldsymbol{J}(\boldsymbol{\theta}^0)\Delta\boldsymbol{\theta} = \boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^0)$

(h.o.t. = **0**)

∗ If $\boldsymbol{J}$ is square ($m = n$) and invertible:

$$\Delta\boldsymbol{\theta} = \boldsymbol{J}^{-1}(\boldsymbol{\theta}^0)\big(\boldsymbol{x}_d - \boldsymbol{f}(\theta^0)\big)$$



$$\Delta\theta = \left(\frac{\partial f}{\partial \theta}(\theta^0)\right)^{-1}(x_d - f(\theta^0))$$

$\boldsymbol{\theta}^0, \boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \ldots \rightarrow \boldsymbol{\theta}_d$

Inverse Kinematics Based on Newton–Raphson Method (Minimum Coordinate IK)

∗ If $\boldsymbol{J}$ is not invertible, either because it is not square or because it is singular,:

$$\Delta\boldsymbol{\theta} = \boldsymbol{J}^{\dagger}(\boldsymbol{\theta}^0)\left(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^0)\right)$$

$\boldsymbol{J}^{\dagger}$: Moore–Penrose pseudoinverse

∗ If $\boldsymbol{J}$ is full rank ($\mathrm{rank}(\boldsymbol{A}) = \min(m, n)$), i.e., the robot is not at a singularity:

- If $n > m$ (the robot has more joints n than end-effector coordinates m):

$$\boldsymbol{J}^{\dagger} = \boldsymbol{J}^{\mathrm{T}}\left(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\right)^{-1}$$

- If $n < m$ (the robot has fewer joints n than end-effector coordinates m or it is at a singularity).

$$\boldsymbol{J}^{\dagger} = \left(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J}\right)^{-1}\boldsymbol{J}^{\mathrm{T}}$$

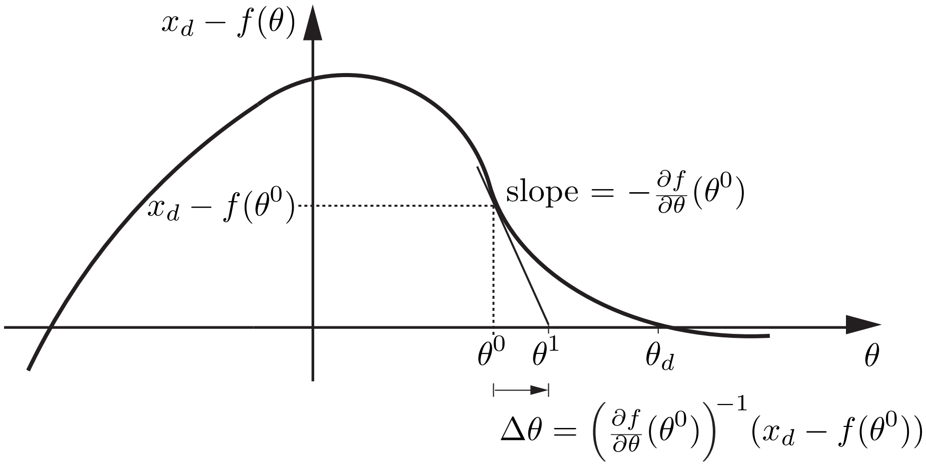
Note: If there are multiple inverse kinematics solutions, the iterative process tends to converge to the solution that is “closest” to the initial guess $\boldsymbol{\theta}^0$.

Algorithm for Minimum Coordinate Representation

a) Initialization: Given $\boldsymbol{x}_d \in \mathbb{R}^m$ and an initial guess $\boldsymbol{\theta}^0 \in \mathbb{R}^n$, set $i = 0$.

b) Iteration: Set $\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^i)$. While $\|\boldsymbol{e}\| > \epsilon$ for some small $\epsilon \in \mathbb{R}$:

- Set $\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \boldsymbol{J}^{\dagger}(\boldsymbol{\theta}^i)\boldsymbol{e}$.
- Increment i.


$x_d - f(\theta)$
$x_d - f(\theta^0)$
slope $= -\frac{\partial f}{\partial \theta}(\theta^0)$
θ^0
θ^1
θ_d
θ
$\Delta\theta = \left(\frac{\partial f}{\partial \theta}(\theta^0)\right)^{-1}(x_d - f(\theta^0))$

```
max_iterations = 20;
i = 0;
Theta = Theta_0;
e = X_d - FK(Theta);
while norm(e) > epsilon && i < max_iterations
    Theta = Theta + pinv(J(Theta)) * e;
    i = i + 1;
    e = X_d - FK(Theta);
end
```

Inverse Kinematics Based on Newton–Raphson Method (Geometric IK)

❖ If the desired end-effector configuration represented as the Transformation Matrix, i.e., $\boldsymbol{T}_{sd} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$, $\boldsymbol{\theta} \in \mathbb{R}^n$, then the goal is to find joint coordinates $\boldsymbol{\theta}_d$ such that

$$\boldsymbol{T}_{sd} - \boldsymbol{T}(\boldsymbol{\theta}_d) = \boldsymbol{0}$$

Algorithm for Matrix Transformation Representation:

a) **Initialization**: Given $\boldsymbol{T}_{sd} \in SE(3)$ and an initial guess $\boldsymbol{\theta}^0 \in \mathbb{R}^n$, set $i = 0$.

b) **Iteration**: Set $[\boldsymbol{\mathcal{V}}_b] = \log\left(\boldsymbol{T}_{bd}(\theta^i)\right) = \log\left(\boldsymbol{T}_{sb}^{-1}(\theta^i)\boldsymbol{T}_{sd}\right)$. While $\|\boldsymbol{\omega}_b\| > \epsilon_\omega$ or $\|\boldsymbol{v}_b\| > \epsilon_v$ for some small $\epsilon_\omega, \epsilon_v \in \mathbb{R}$:

(in Body Frame)

- Set $\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \boldsymbol{J}_b^{\dagger}(\boldsymbol{\theta}^i)\boldsymbol{\mathcal{V}}_b$.
- Increment i.

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a) **Initialization**: Given $\boldsymbol{T}_{sd} \in SE(3)$ and an initial guess $\boldsymbol{\theta}^0 \in \mathbb{R}^n$, set $i = 0$.

b) **Iteration**: Set $[\boldsymbol{\mathcal{V}}_s] = \left[\mathrm{Ad}_{\boldsymbol{T}_{sb}}\right]\log\left(\boldsymbol{T}_{bd}(\theta^i)\right) = \left[\mathrm{Ad}_{\boldsymbol{T}_{sb}}\right]\log\left(\boldsymbol{T}_{sb}^{-1}(\theta^i)\boldsymbol{T}_{sd}\right)$. While $\|\boldsymbol{\omega}_s\| > \epsilon_\omega$ or $\|\boldsymbol{v}_s\| > \epsilon_v$ for some small $\epsilon_\omega, \epsilon_v \in \mathbb{R}$:

(in Space Frame)

- Set $\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \boldsymbol{J}_s^{\dagger}(\boldsymbol{\theta}^i)\boldsymbol{\mathcal{V}}_s$.
- Increment i.

($\boldsymbol{\mathcal{V}}_s$ or $\boldsymbol{\mathcal{V}}_b$ is the twist that takes $\boldsymbol{T}_{sb}$ to $\boldsymbol{T}_{sd}$ in 1s)