# Ch11: Frequency Response Techniques

Contents:

# Frequency Response

# Introduction

Frequency response method, developed by **Nyquist** (/ˈnaɪkwɪst/) and **Bode** (/ˈboʊdi/) in the 1930s, are older than the root locus method, discovered by Evans in 1948. This older method is not as intuitive as the root locus; however, it has distinct advantages in the following situations:

1) When modeling transfer functions of complicated systems from experimental data,
2) When designing lead, lag, and lead-lag compensators to meet a steady-state error requirement and a transient response requirement,
3) When finding the stability of nonlinear systems,
4) In settling ambiguities when sketching a root locus.

In frequency-response methods, we vary the frequency of the input signal over a certain range (say using a function generator) and study the resulting response (say using an oscilloscope).

# A Representation of Sinusoids

Sinusoids

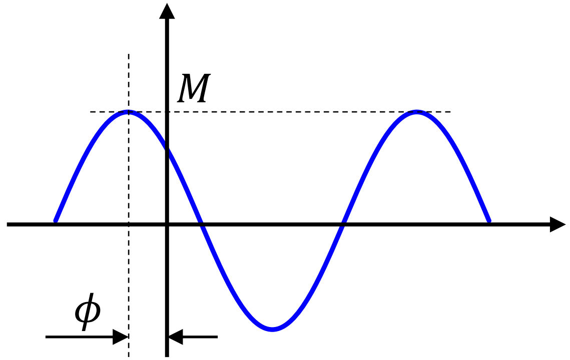
$$A\cos\omega t + B\sin\omega t = \sqrt{A^2+B^2}\cos(\omega t - \tan^{-1} B/A) = M\cos(\omega t + \phi)$$

can be represented as **complex numbers** called **phasors** (phase + vector). The **magnitude** of the complex number is the **amplitude** of the sinusoid $M$, and the **angle** of the complex number is the **phase angle** of the sinusoid $\phi$.

Thus, $M\cos(\omega t + \phi)$ or $A\cos\omega t + B\sin\omega t$ can be represented as

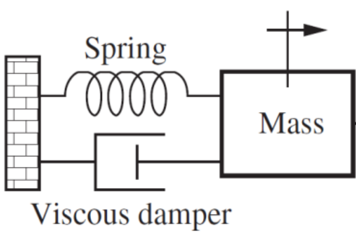
- **Polar Form**: $M\angle\phi$,
- **Euler's Form**: $Me^{j\phi}$
- **Rectangular Form**: $A - jB$

where the frequency $\omega$ is implicit in these forms.



# Concept of Frequency Response

In the steady state, sinusoidal inputs to a **stable**, **linear**, **time-invariant** system generate sinusoidal responses of the **same frequency** but different in amplitude and phase angle from the input. These differences are functions of frequency.



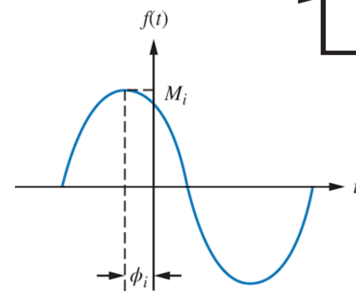

(steady-state output)

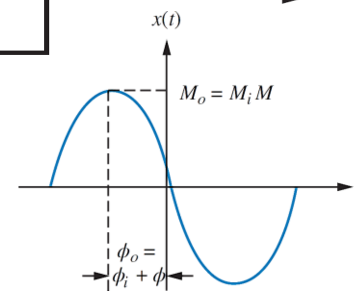
$x(t) = M_o \cos(\omega t + \phi_o)$

$f(t) = M_i \cos(\omega t + \phi_i)$

(input)

$M_i(\omega)\angle\phi_i(\omega)$ → Stable LTI System → $M_o(\omega)\angle\phi_o(\omega)$




Therefore, the system itself can be represented by a **complex number** as $M(\omega)\angle\phi(\omega)$ where

$$M_o(\omega) = M_i(\omega)M(\omega)$$

$$\angle\phi_o(\omega) = \angle\big(\phi_i(\omega) + \phi(\omega)\big)$$

$M_i(\omega)\angle\phi_i(\omega)$ → $M(\omega)\angle\phi(\omega)$ → $M_o(\omega)\angle\phi_o(\omega)$

# Concept of Frequency Response

The combination of the magnitude $M(\omega)$ and phase $\phi(\omega)$ as $M(\omega)\angle\phi(\omega)$ is called the **Frequency Response**.

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)}$$

Ratio of the output sinusoid's magnitude to the input sinusoid's magnitude is called **Magnitude Frequency Response**.

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega)$$

Difference in phase angle between the output and the input sinusoids is called the **Phase Frequency Response**.

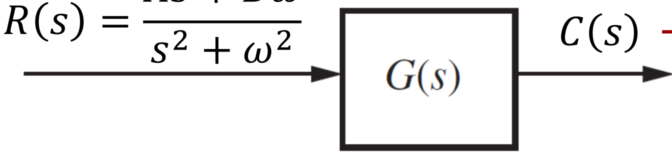
# Relation between Frequency Response and TF

$$r(t) = A\cos\omega t + B\sin\omega t \longrightarrow R(s) = \frac{As + B\omega}{s^2+\omega^2}$$

$$= \sqrt{A^2+B^2}\cos(\omega t - \tan^{-1} B/A)$$

$$= M_i\cos(\omega t + \phi_i) = M_i e^{j\phi_i}$$

$$= A - jB$$




$$C(s) \longrightarrow c(t) = c_f(t) + c_n(t)$$

$$C(s) = \frac{As+B\omega}{(s^2+\omega^2)}G(s) = \frac{As+B\omega}{(s+j\omega)(s-j\omega)}G(s) = \underbrace{\frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega}}_{\text{Force Response}} + \underbrace{\text{Terms from } G(s)}_{\text{Natural Response}}$$

$$K_1 = \frac{As+B\omega}{s-j\omega}G(s)\bigg|_{s\to -j\omega} = \frac{1}{2}(A+jB)G(-j\omega) = \frac{1}{2}M_i e^{-j\phi_i}Me^{-j\phi} = \frac{M_iM}{2}e^{-j(\phi_i+\phi)}$$

$$K_2 = \frac{As+B\omega}{s+j\omega}G(s)\bigg|_{s\to +j\omega} = \frac{1}{2}(A-jB)G(j\omega) = \frac{1}{2}M_i e^{j\phi_i}Me^{j\phi} = \frac{M_iM}{2}e^{j(\phi_i+\phi)} = K_1^*$$

(complex conjugate of $K_1$)

where $M = |G(j\omega)|,\ \phi = \angle G(j\omega)$.

# Relation between Frequency Response and TF

Since the system is stable, $\lim_{t\to\infty} c_n(t) = 0$. Therefore, the sinusoidal steady-state response is determined by the forced response portion of $C(s)$ (first two terms).

$$C_f(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} = \frac{\frac{M_iM}{2}e^{-j(\phi_i+\phi)}}{s+j\omega} + \frac{\frac{M_iM}{2}e^{j(\phi_i+\phi)}}{s-j\omega}$$

$$= M_iM\left(\frac{e^{-j(\omega t+\phi_i+\phi)} + e^{j(\omega t+\phi_i+\phi)}}{2}\right) = M_iM\cos(\omega t + \phi_i + \phi)$$

$$= M_o\cos(\omega t + \phi_o)$$

In phasor form: $M_o\angle\phi_o = (M_i\angle\phi_i)(M\angle\phi)$ where $M = |G(j\omega)|$, $\phi = \angle G(j\omega)$.

Therefore, the **frequency response** of a system whose transfer function is $G(s)$ is

$$G(j\omega) = G(s)\Big|_{s\to j\omega} = |G(j\omega)|\angle G(j\omega) = M(\omega)\angle\phi(\omega)$$

# Plotting Frequency Response

There are 3 commonly used representations of $G(j\omega) = |G(j\omega)|\angle G(j\omega) = M(\omega)\angle\phi(\omega)$:

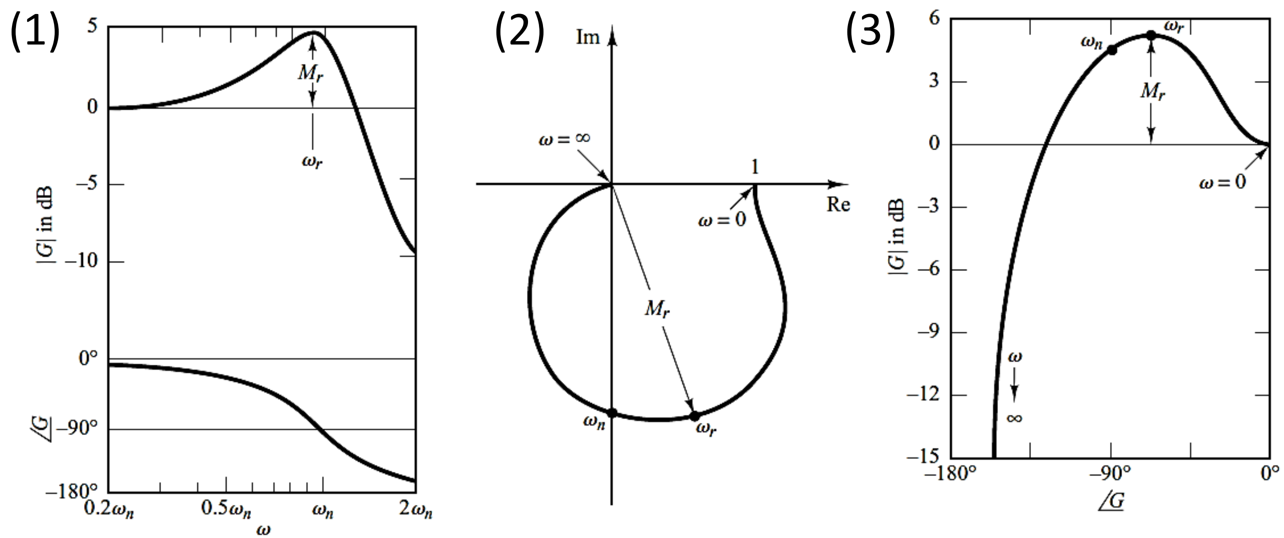
(1) **Bode Plots**: Two separate magnitude and phase plots:

- **Magnitude plot**: Log-magnitude in decibels (dB) (i.e., $20\log|G(j\omega)|$) vs $\omega$.
- **Phase plot**: $\angle G(j\omega)$ vs. $\omega$.

(2) **Nyquist Plot**: As a polar plot, where the phasor length is the magnitude $|G(j\omega)|$ and the phasor angle is the phase $\angle G(j\omega)$.

(3) **Nichols Plot**: Log-magnitude in decibels (dB) (i.e., $20\log|G(j\omega)|$) vs. phase ($\angle G(j\omega)$).

**Note**: Here, log is used to mean $\log_{10}$, or logarithm to the base 10.



# Bode Plots

# Magnitude and Phase Frequency Response

Consider the transfer function $G(s)$:

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$|G(j\omega)| = \left.\frac{K|s+z_1||s+z_2|\cdots|s+z_k|}{|s^m||s+p_1||s+p_2|\cdots|s+p_n|}\right|_{s\to j\omega}$$

$$20\log|G(j\omega)| = \left.\left(20\log K + 20\log|s+z_1| + \cdots + 20\log\left|\frac{1}{s^m}\right| + 20\log\left|\frac{1}{s+p_1}\right| + \cdots\right)\right|_{s\to j\omega}$$

$$\angle|G(j\omega)| = \left.\left(\angle K + \angle(s+z_1) + \cdots + \angle\left(\frac{1}{s^m}\right) + \angle\left(\frac{1}{s+p_1}\right) + \cdots\right)\right|_{s\to j\omega}$$

Therefore, the magnitude $20\log|G(j\omega)|$ and phase $\angle|G(j\omega)|$ frequency response is the **sum** of the magnitude and phase frequency responses of all terms.

# Basic Factors of $G(j\omega)$ for Sketching Bode Plots

1. Gain: $G(s) = K$

2. Integral and derivative factors: $G(s) = s$ , $G(s) = \dfrac{1}{s}$

3. First-order factors: $G(s) = (Ts + 1)$ , $G(s) = \dfrac{1}{Ts + 1}$

4. Quadratic factors: $G(s) = \left(\dfrac{1}{\omega_n^2}s^2 + \dfrac{2\zeta}{\omega_n}s + 1\right)$, $G(s) = \dfrac{1}{\dfrac{1}{\omega_n^2}s^2 + \dfrac{2\zeta}{\omega_n}s + 1}$ $0 \leq \zeta < 1$

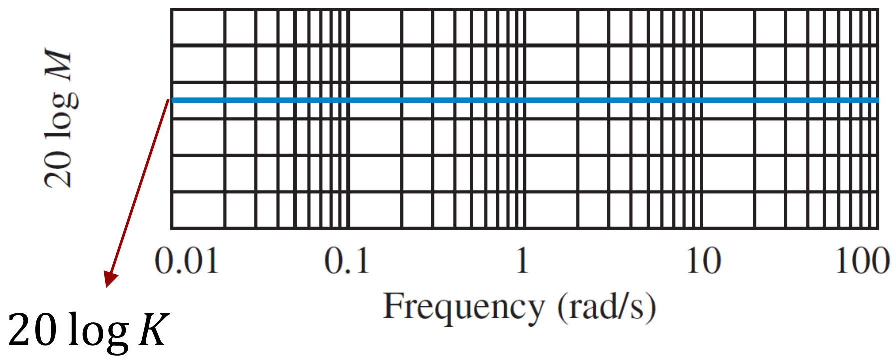
Frequency response of these **basic factors** are **approximated** by **straight-lines** (**asymptotes**). For sketching the frequency response of more complicated transfer functions $G(s)$, these lines are combined.

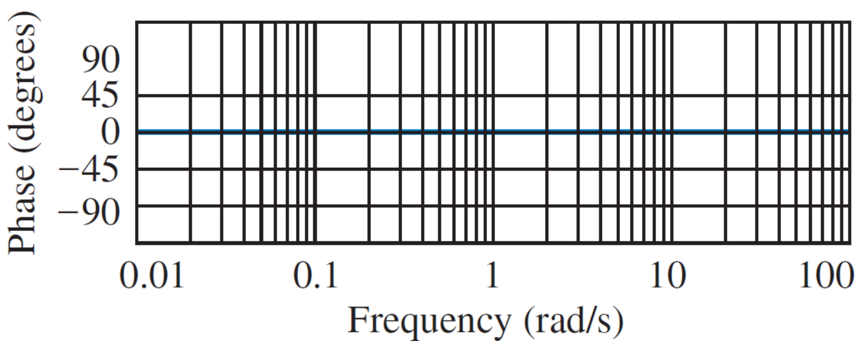
Therefore, sketching Bode plots can be simplified because they can be approximated as a sequence of straight lines (**asymptotes**).

# 1. Bode Plots for *K*

- The log-magnitude curve for a constant gain $K$ is a horizontal straight line at the magnitude of $20 \log K$ dB. A gain $K$ greater than unity has a positive value in decibels, while a number smaller than unity has a negative value

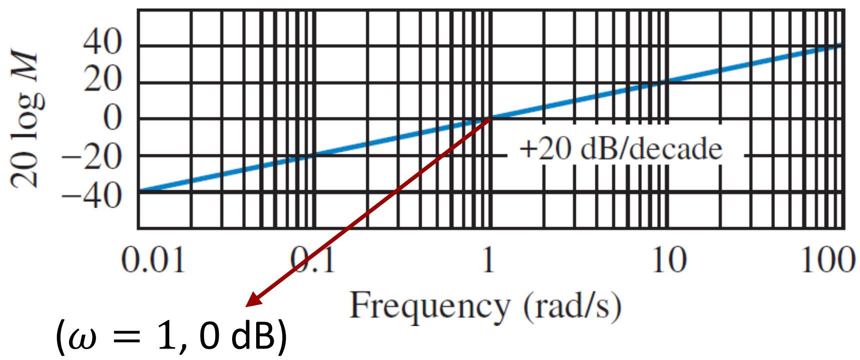
- The phase angle of the gain $K$ is zero.

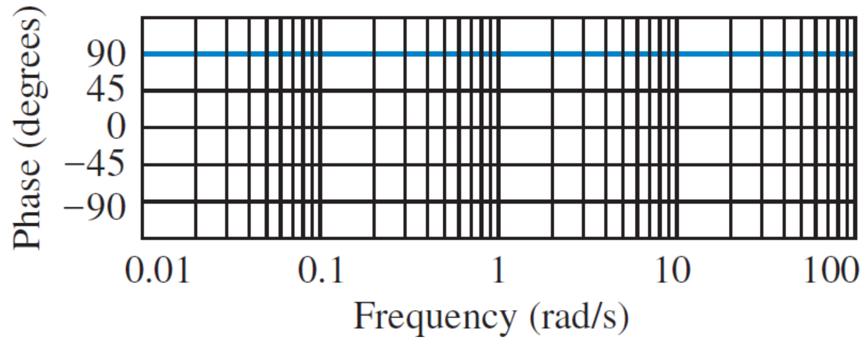


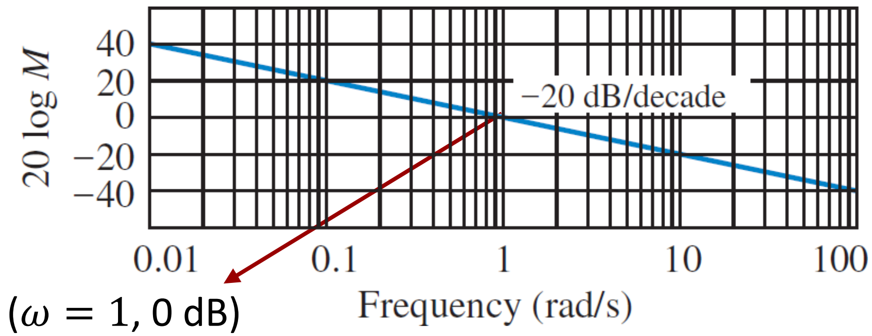
# 2. Bode Plots for $s$ and $\frac{1}{s}$

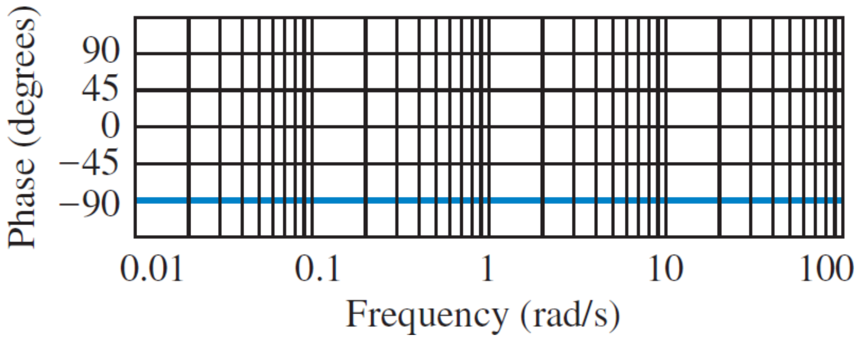
$G(j\omega) = j\omega \quad \rightarrow \quad 20\log|G(j\omega)| = 20\log\omega\,, \qquad \angle G(j\omega) = 90°$




$G(j\omega) = \dfrac{1}{j\omega} \quad \rightarrow \quad 20\log|G(j\omega)| = -20\log\omega\,, \qquad \angle G(j\omega) = -90°$



# 2. Bode Plots for $s^n$ and $\left(\frac{1}{s}\right)^n$

- If the transfer function contains the factor $s^n$ or $(1/s)^n$, the log magnitude becomes:

$$20\log|(j\omega)^n| = n \times 20\log|j\omega| = 20n\log\omega \quad \text{dB}$$

$$20\log\left|\frac{1}{(j\omega)^n}\right| = -n \times 20\log|j\omega| = -20n\log\omega \quad \text{dB}$$

The slopes of the log-magnitude curves for the factors are thus $20n$ dB/decade and – $20n$ dB/decade, respectively.

**Note**: The magnitude curves will pass through the point ($\omega = 1$, 0 dB).

- The phase angle of $s^n$ is equal to $90° \times n$ over the entire frequency range and the phase angle of $(1/s)^n$ is equal to – $90° \times n$ over the entire frequency range.

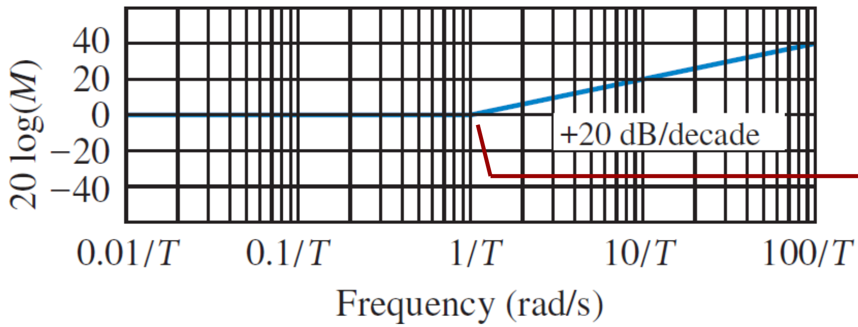
# Bode Plots for $Ts + 1$

$$G(j\omega) = (j\omega T + 1) \quad \Rightarrow \quad \begin{aligned} &20\log(|G(j\omega)|) = 20\log\sqrt{1+\omega^2T^2} \\ &\angle G(j\omega) = \tan^{-1}\omega T \end{aligned}$$

At low frequencies when $0 < \omega < 1/T$: $G(j\omega) \approx 1 \;\rightarrow\; 20\log(|G(j\omega)|) = 20\log 1 = 0$

At high frequencies when $1/T < \omega < \infty$: $G(j\omega) \approx j\omega T \;\rightarrow\; 20\log|G(j\omega)| = 20\log \omega T$

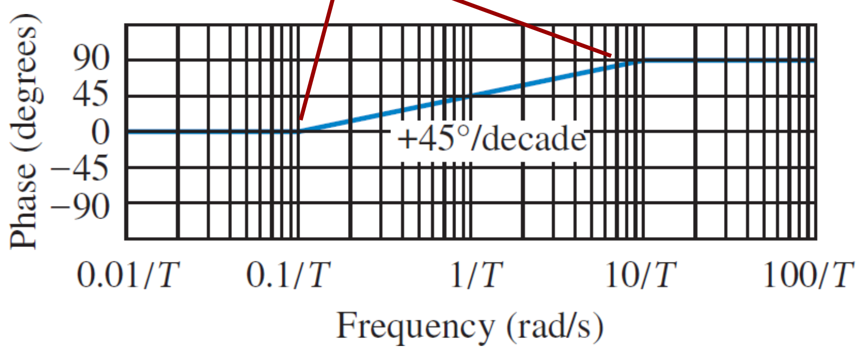
$\Rightarrow$




The frequency at which the two asymptotes meet is called the **corner** or **break** frequency. This frequency is very important in sketching logarithmic frequency-response curves.

At $\omega = 0$, $\angle G(j\omega) = 0°$
At $\omega = 1/\mathrm{T}$, $\angle G(j\omega) = \tan^{-1} 1 = 45°$
At $\omega \rightarrow \infty$, $\angle G(j\omega) = 90$

$\Rightarrow$



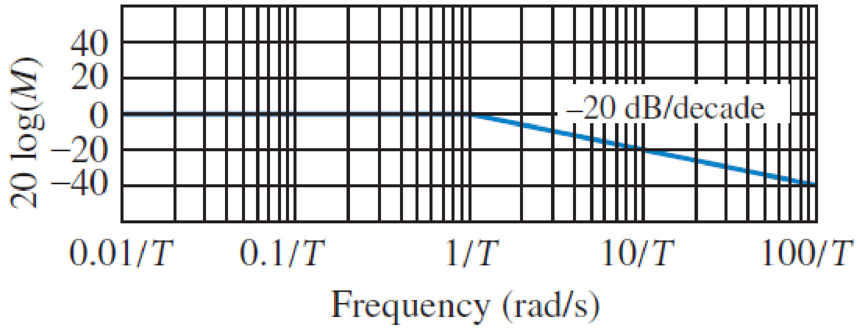
# Bode Plots for $\frac{1}{Ts+1}$

$$G(j\omega) = \frac{1}{(j\omega T + 1)} \quad \Rightarrow \quad \begin{array}{l} 20\log(|G(j\omega)|) = -20\log\sqrt{1+\omega^2T^2} \\ \angle G(j\omega) = -\tan^{-1}\omega T \end{array}$$

At low frequencies when $0 < \omega < 1/T$: $G(j\omega) \approx 1 \rightarrow 20\log(|G(j\omega)|) = 20\log 1 = 0$

At high frequencies when $1/T < \omega < \infty$: $G(j\omega) \approx 1/j\omega T \rightarrow 20\log|G(j\omega)| = -20\log\omega T$

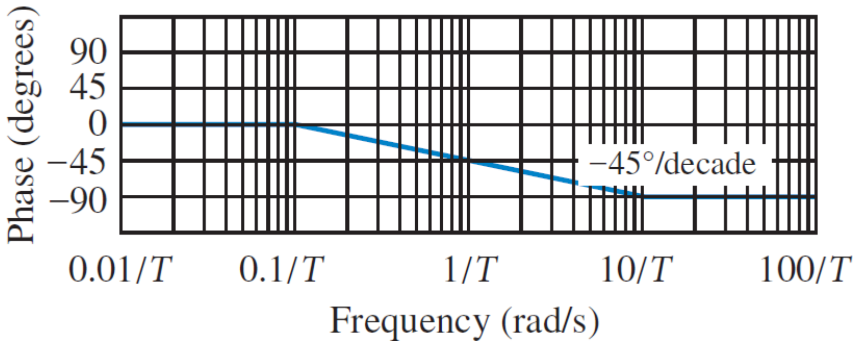
$\Rightarrow$




At $\omega = 0$, $\angle G(j\omega) = 0°$

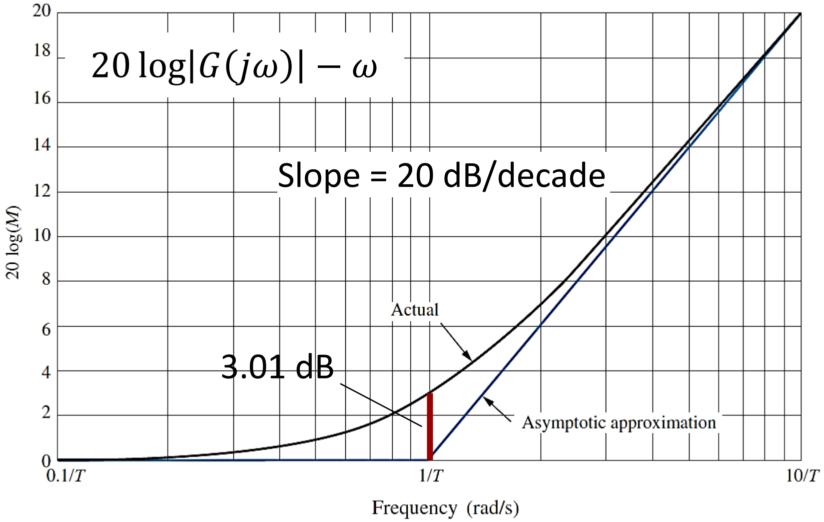
At $\omega = 1/T$, $\angle G(j\omega) = -\tan^{-1}1 = -45°$ $\Rightarrow$

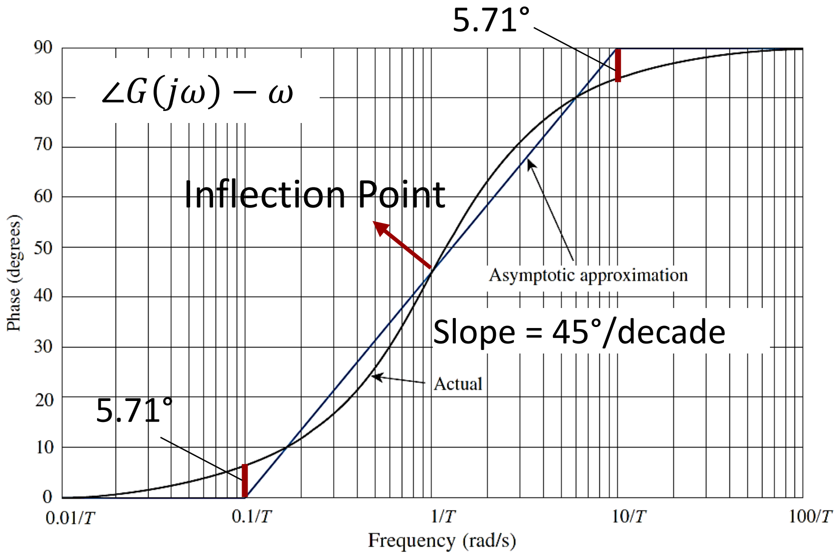
At $\omega \rightarrow \infty$, $\angle G(j\omega) = -90$



# Bode Plots for $Ts + 1$ and $\frac{1}{Ts+1}$

- The maximum difference between the **actual curve** and **asymptotic approximation** for the magnitude curve is 3.01 dB, which occurs at the break frequency and the maximum difference for the phase curve is 5.71°, which occurs at the decades above and below the break frequency.




- For the case where a given transfer function involves terms like $(j\omega T + 1)^{\pm n}$, a similar asymptotic construction may be made, except the high-frequency asymptote has the slope of – $20n$ dB/decade or $20n$ dB/decade; similarly, for the phase angle plots.

# Bode Plots for $\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1$

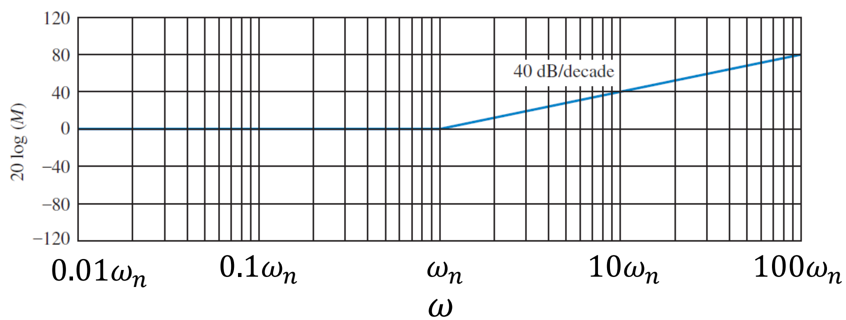
$$G(j\omega) = \frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1$$

$$20\log(|G(j\omega)|) = 20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}, \quad \angle G(j\omega) = \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right)$$

At low frequencies when $0 < \omega < \omega_n$: $G(j\omega) \approx 1 \rightarrow 20\log(|G(j\omega)|) = 20\log 1 = 0$

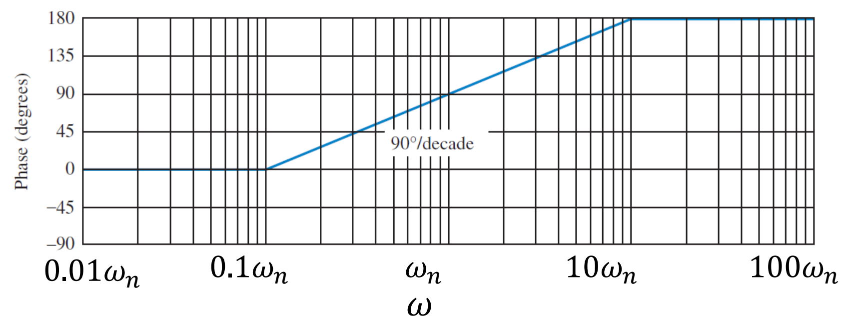
At high frequencies when $\omega_n < \omega < \infty$: $G(j\omega) \approx -\omega^2/\omega_n^2 \rightarrow 20\log|G(j\omega)| = 20\log\frac{\omega^2}{\omega_n^2} = 40\log\frac{\omega}{\omega_n}$

$\Rightarrow$




At $\omega = 0$, $\angle G(j\omega) = 0°$
At $\omega = \omega_n$, $\angle G(j\omega) = 90°$
At $\omega \rightarrow \infty$, $\angle G(j\omega) = 180$

$\Rightarrow$



# Bode Plots for $\dfrac{1}{\frac{1}{\omega_n^2}s^2+\frac{2\zeta}{\omega_n}s+1}$

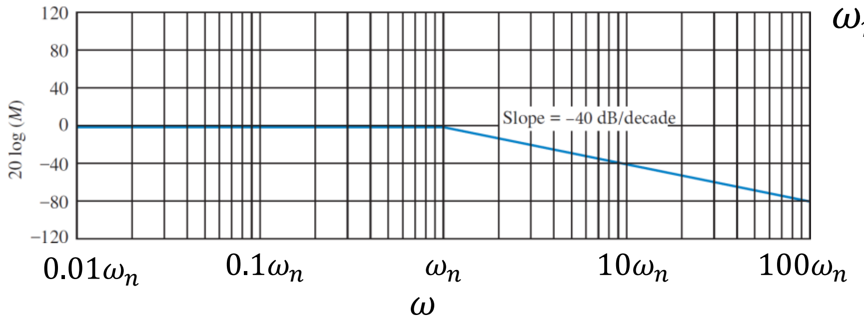
$$G(j\omega)=\frac{1}{\frac{1}{\omega_n^2}(j\omega)^2+\frac{2\zeta}{\omega_n}(j\omega)+1}$$

$$20\log(|G(j\omega)|)=-20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2},\qquad \angle G(j\omega)=-\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right)$$

At low frequencies when $0<\omega<\omega_n$: $\quad G(j\omega)\approx 1 \;\rightarrow 20\log(|G(j\omega)|)=20\log 1=0$

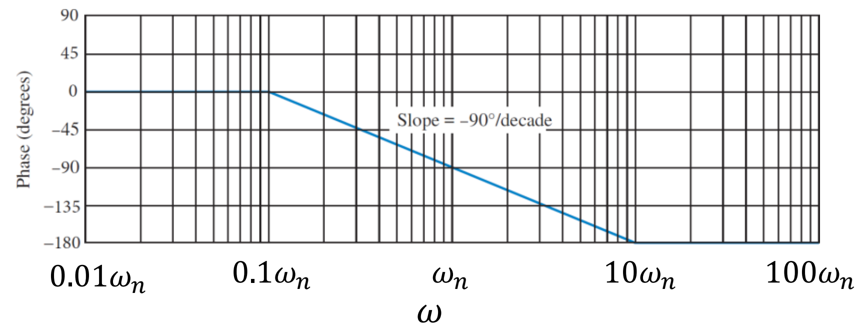
At high frequencies when $\omega_n<\omega<\infty$: $\quad G(j\omega)\approx -\frac{1}{\frac{\omega^2}{\omega_n^2}}\rightarrow 20\log|G(j\omega)|=-20\log\frac{\omega^2}{\omega_n^2}=-40\log\frac{\omega}{\omega_n}$

$\Rightarrow$



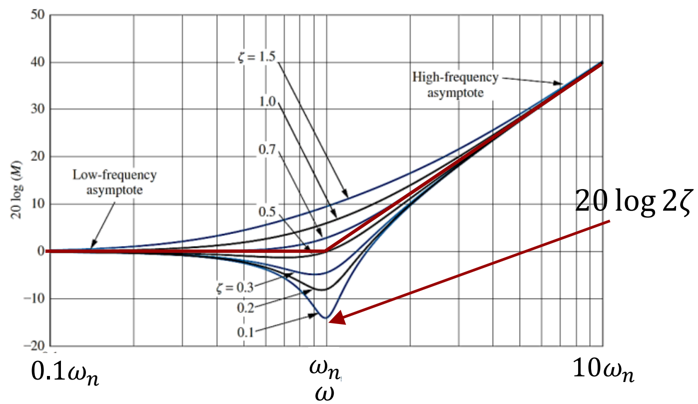

At $\omega=0$, $\quad\angle G(j\omega)=0°$
At $\omega=\omega_n$, $\angle G(j\omega)=-90°$
At $\omega\rightarrow\infty$, $\angle G(j\omega)=-180$

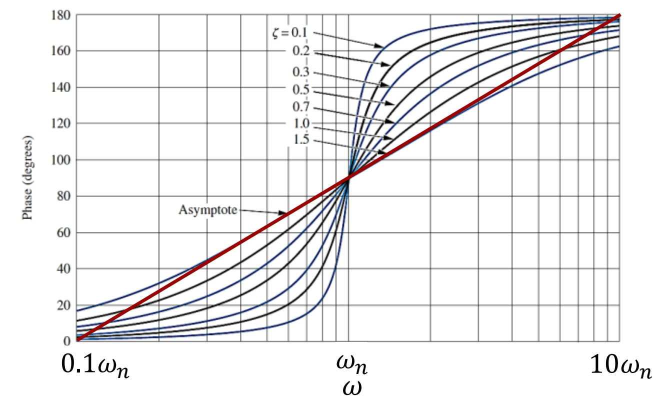
$\Rightarrow$

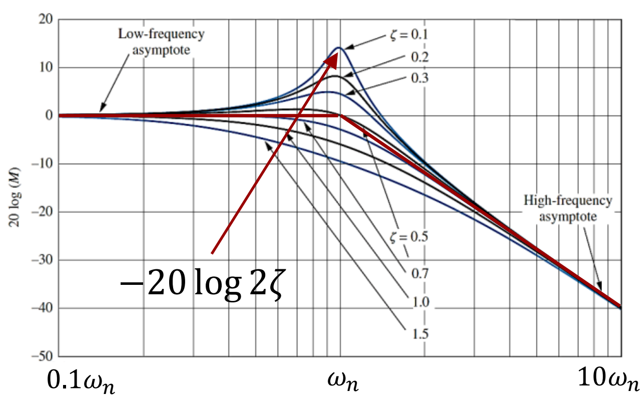


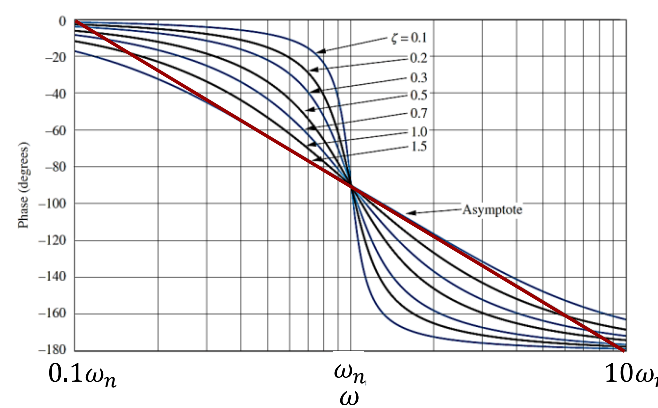
# Actual Bode Plots for $\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1$ & $\frac{1}{\frac{1}{\omega_n^2}s^2+\frac{2\zeta}{\omega_n}s+1}$

Asymptotic approximations to the Bode plots of the quadratic factor are not accurate for small values of $\zeta$ because the magnitude and phase of these factor depend on both the corner frequency $\omega_n$ and the damping ratio $\zeta$. Near $\omega = \omega_n$ a resonant peak occurs. The damping ratio $\zeta$ determines the magnitude of the resonant peak and error. A correction to the Bode plots can be made to improve the accuracy.



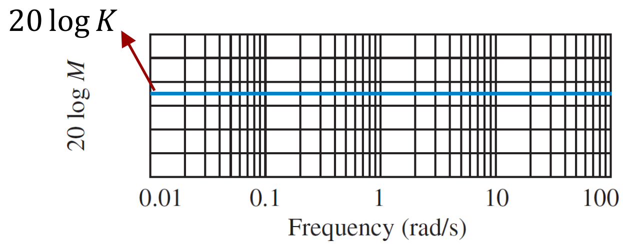



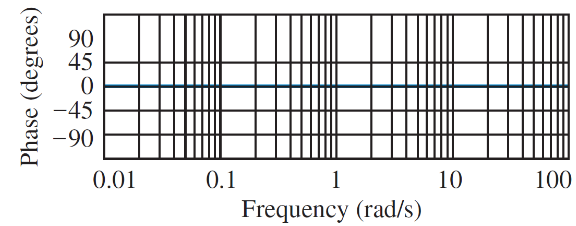



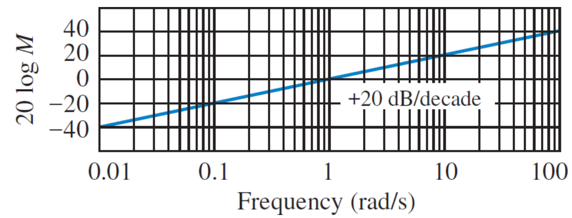
# Bode Plots of Basic Factors: Summery

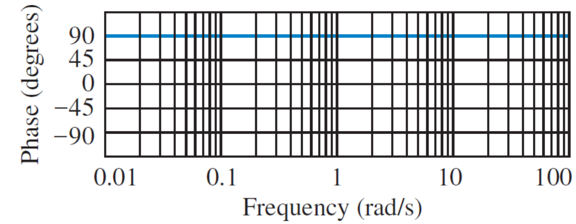
$$G(s) = K$$



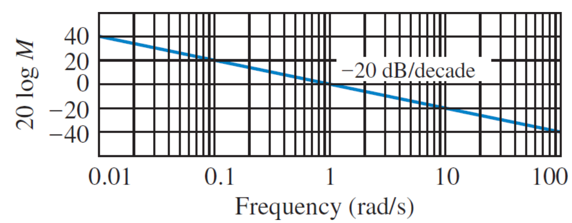



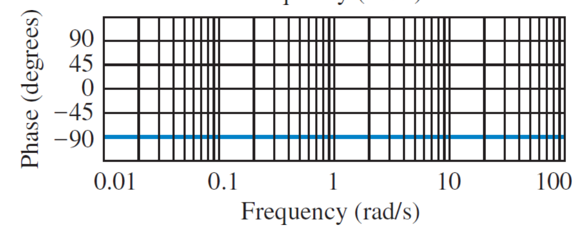

$$G(s) = s$$



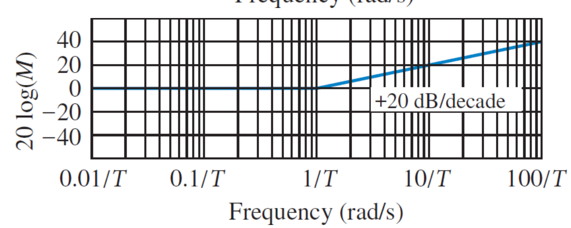



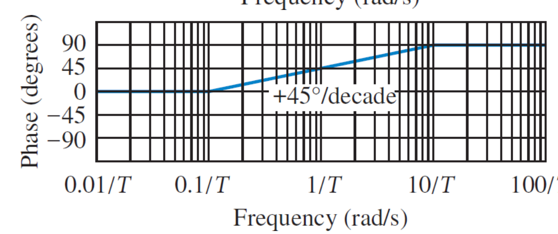

$$G(s) = \frac{1}{s}$$



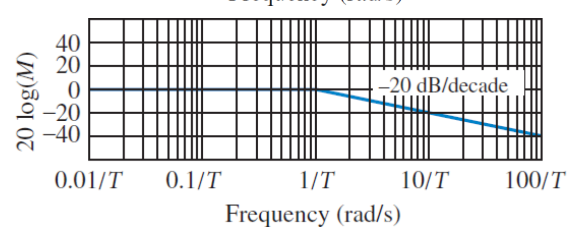



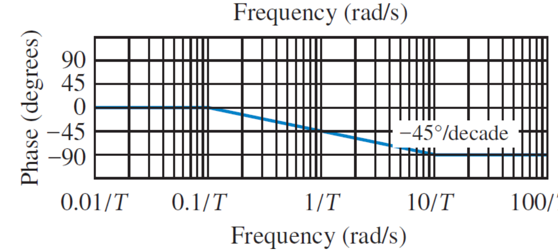

$$G(s) = (Ts + 1)$$



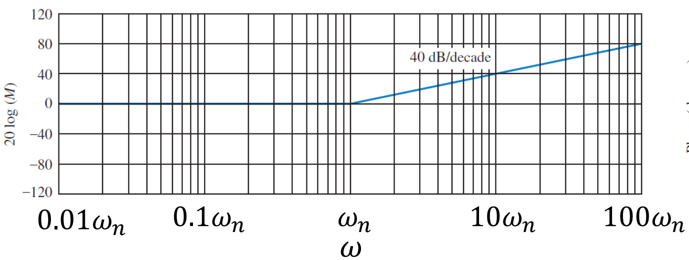

$$G(s) = \frac{1}{Ts + 1}$$

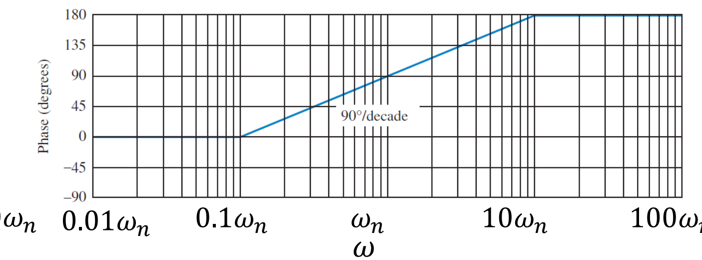


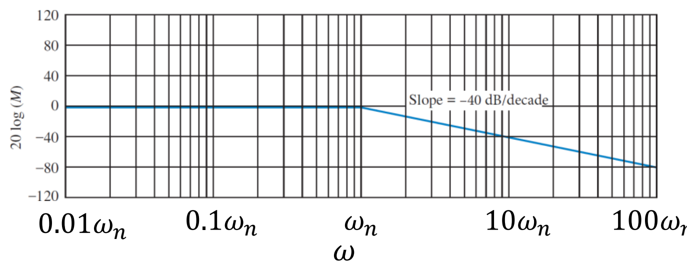
# Bode Plots of Basic Factors: Summery

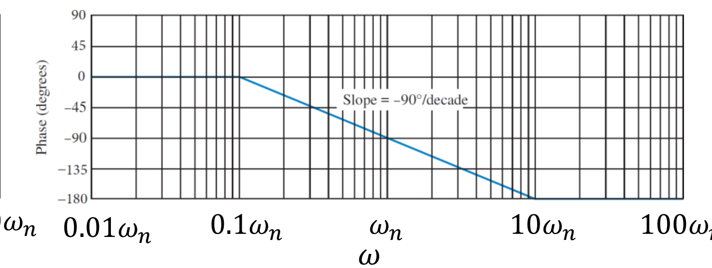
$$G(s) = \left(\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1\right)$$




$$G(s) = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$



# Sketching Bode Plots of Complicated Functions

For sketching the frequency response of more complicated transfer functions,

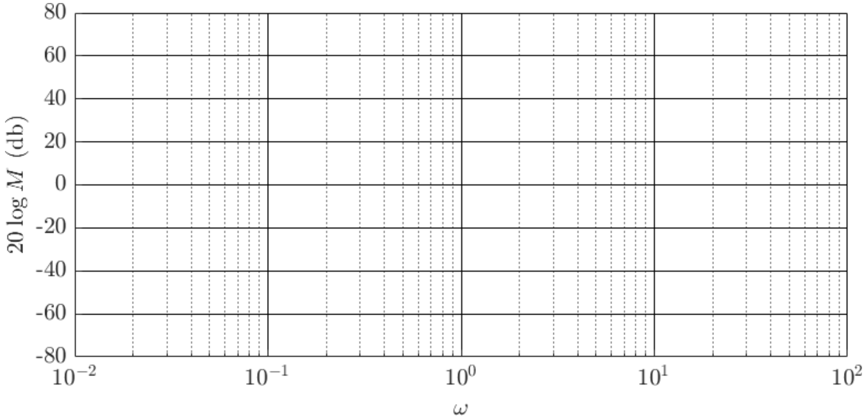
1) First, rewrite $G(j\omega)$ as a product of basic factors.
2) Then, identify all the corner frequencies associated with these basic factors.
3) Finally, draw the asymptotic log-magnitude and phase curves with proper slopes between the corner frequencies. The plots should begin **a decade below the lowest** break frequency and extend **a decade above the highest** break frequency.
4) The exact curve, which lies close to the asymptotic curve, can be obtained by adding proper corrections.

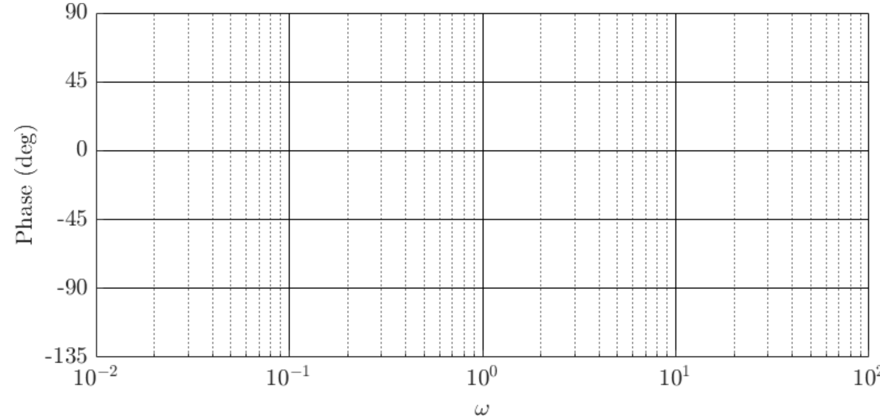
**Note**: The experimental determination of a transfer function $G(s)$ can be made simple if frequency-response data are presented in the form of a **Bode plot**.

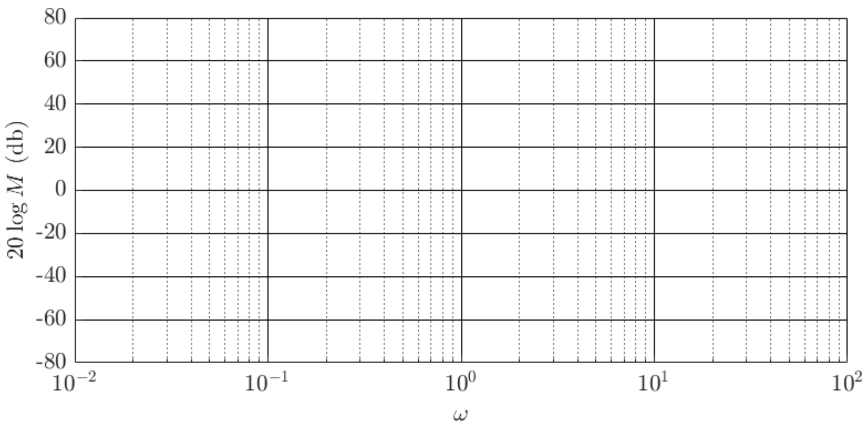
# Example

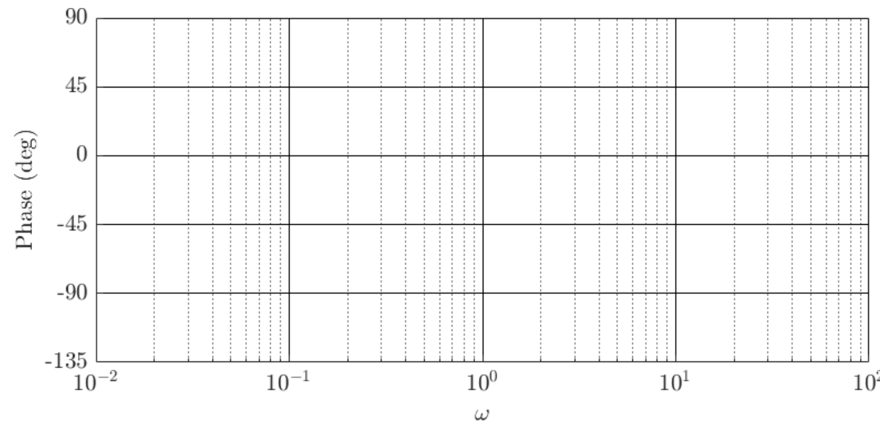
Draw the Bode plots for $G(s)$.

$$G(s) = \frac{10(s+3)}{s(s+5)}$$



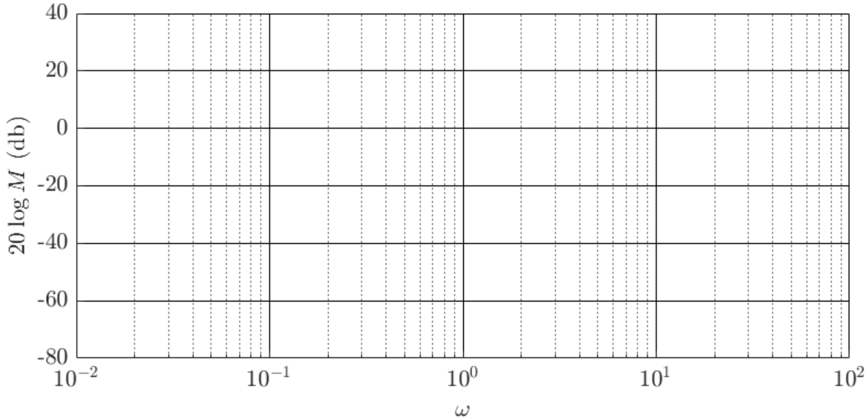



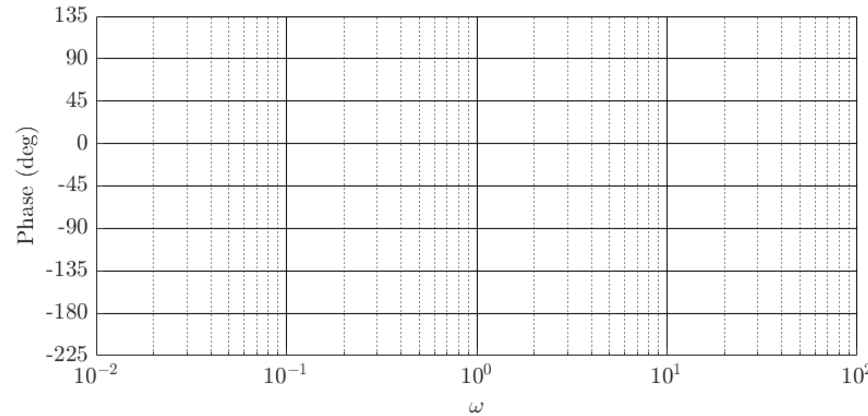



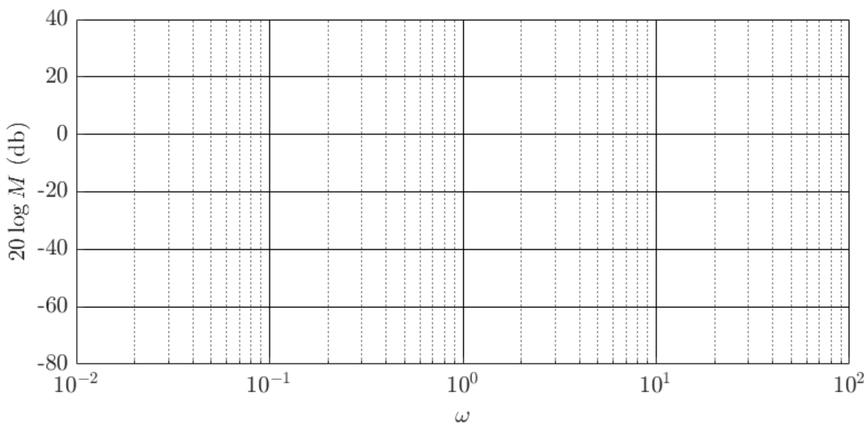
# Example

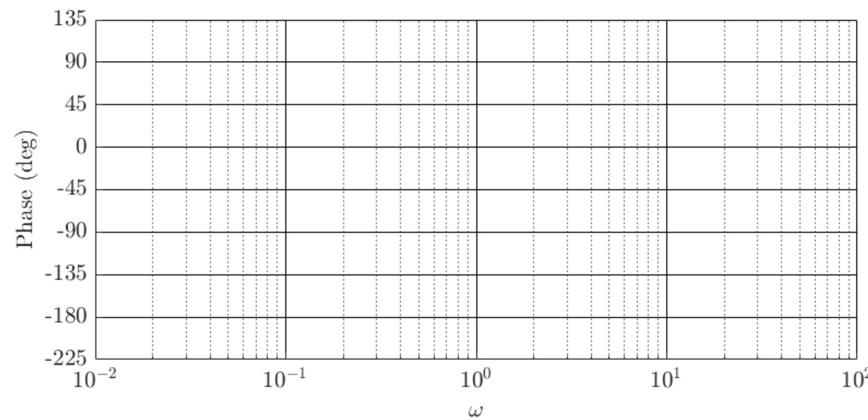
Draw the Bode plots for $G(s)$.

$$G(s) = \frac{s+3}{(s+2)(s^2+2s+25)}$$

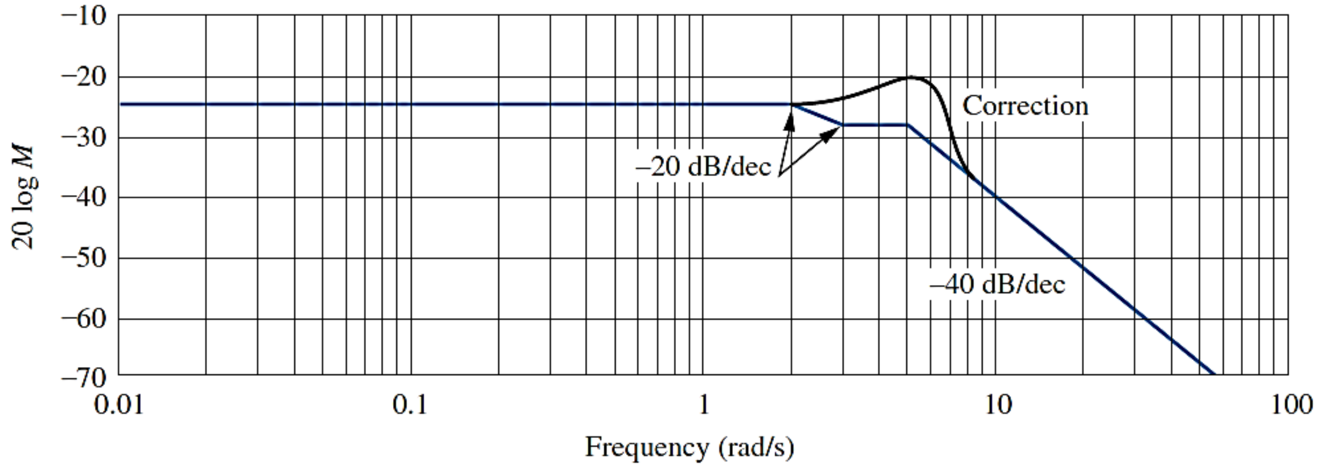


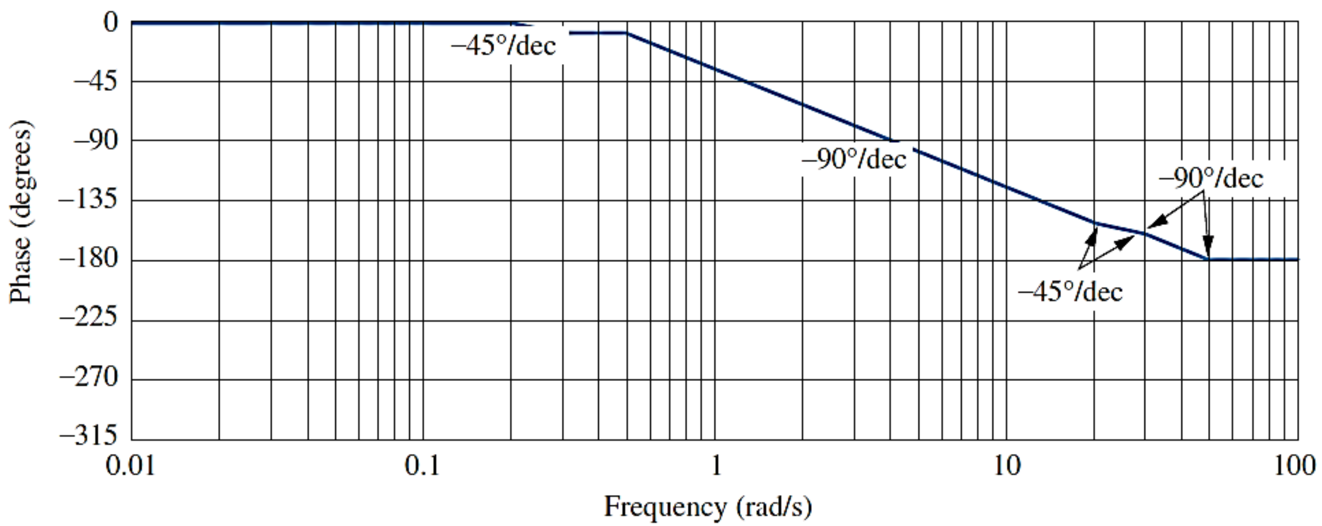



# Final Answer

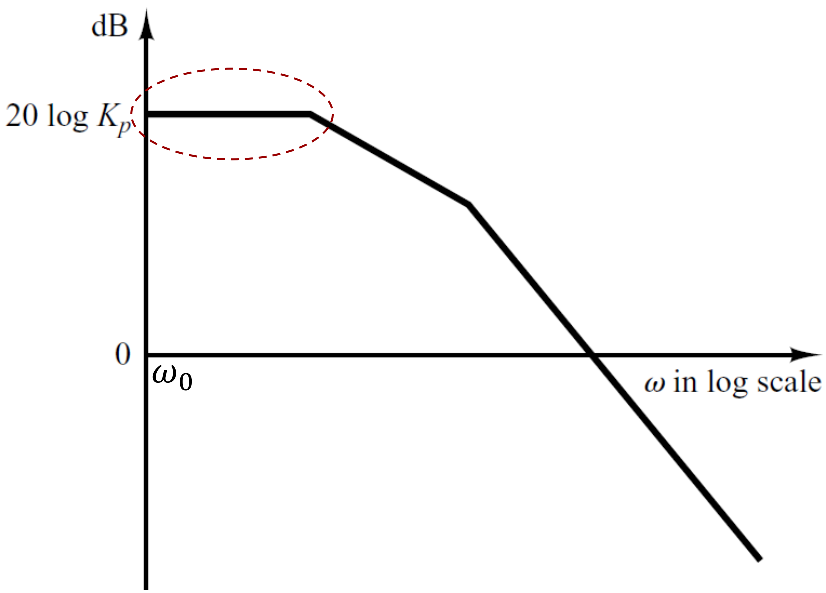


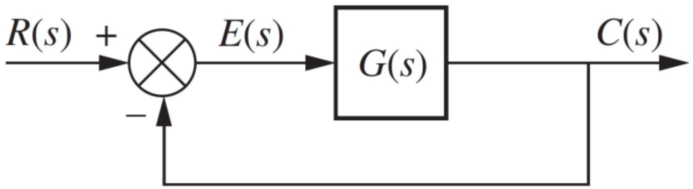
# Bode Plots and Steady-State Error

# Bode Plots and Steady-State Error Characteristics

The **Type** of the system determines the **slope of the log-magnitude curve at low frequencies**. Thus, information concerning the existence and magnitude of the **steady-state error** of a control system to a given input can be determined from the observation of the low-frequency region of the log-magnitude curve.

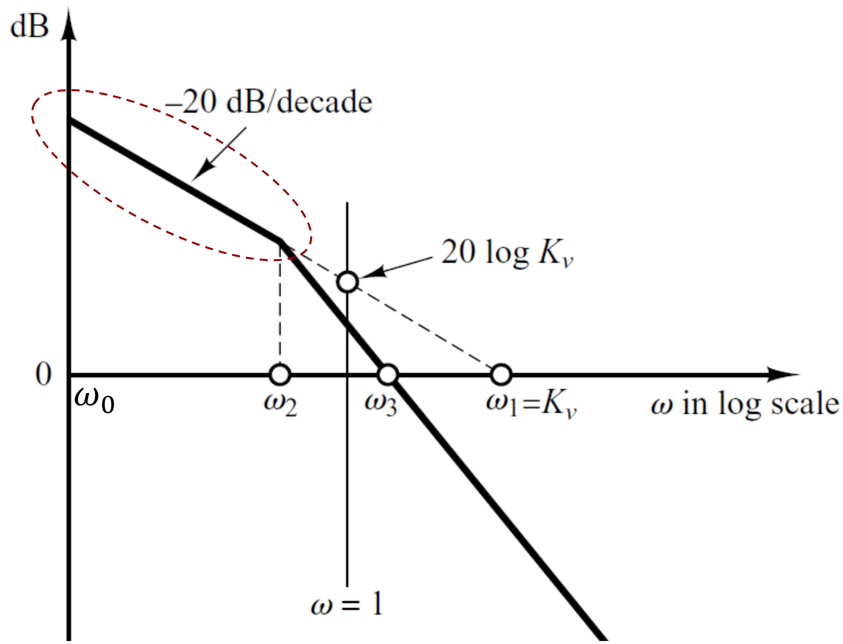



(Log-magnitude curve of a **Type 0** system)

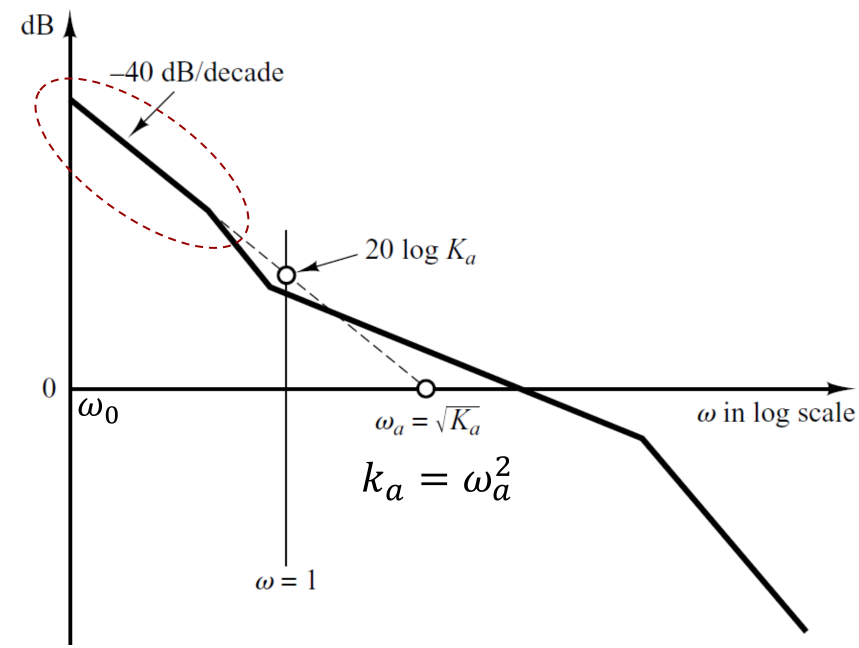



$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^N(s+p_1)(s+p_2)\cdots}$$

# Bode Plots and Steady-State Error Characteristics




(Log-magnitude curve of a **Type 1** system)



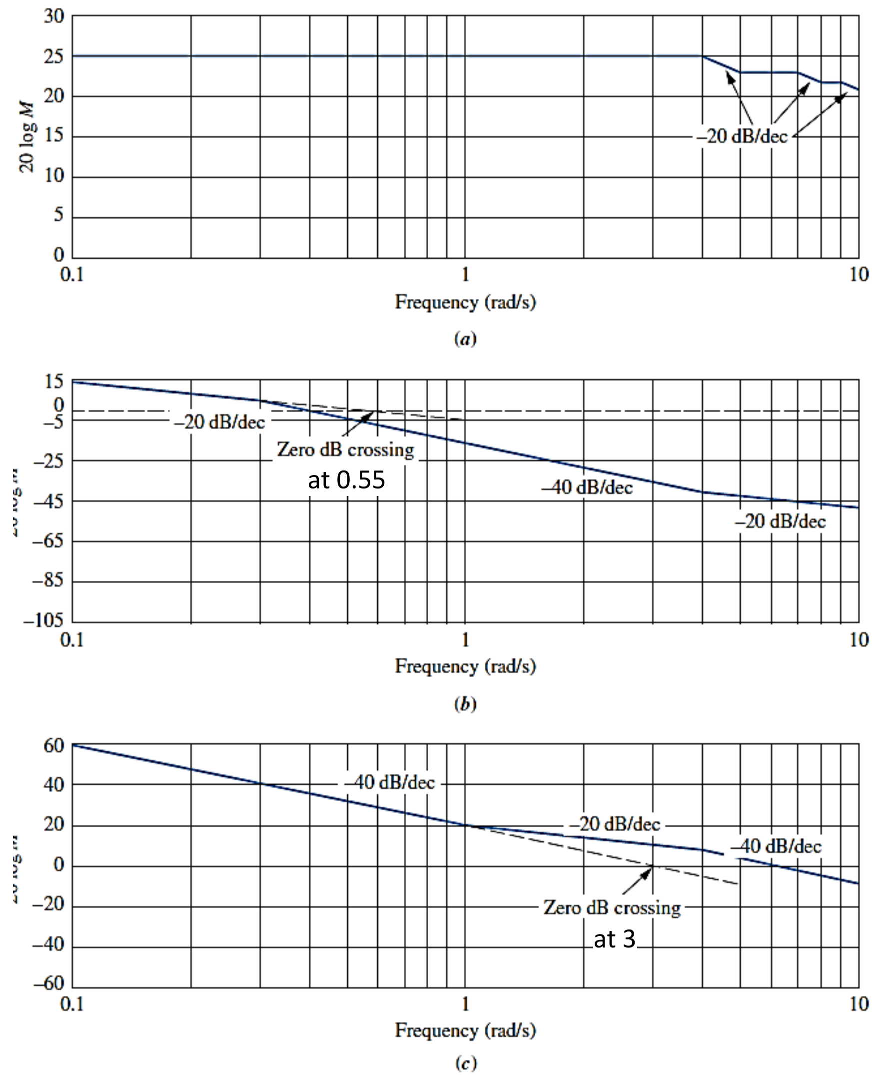

(Log-magnitude curve of a **Type 2** system)

# Example

For each Bode log-magnitude plot,

**a**. Find the system type.

**b**. Find the value of the appropriate static error constant.



# Using MATLAB and Control System Toolbox

# Making Bode Plots Using bode

```
s = tf('s');
G = 10*(s+3)/(s*(s+5));

bode(G,{0.1,100})
grid on

% To store points on the Bode plot
[mag, phase, w]=bode(G);

% List points on Bode plot with magnitude in dB.
points = [20*log10(mag(:,:))', phase(:,:)', w];
```

$$G(s) = \frac{10(s+3)}{s(s+5)}$$