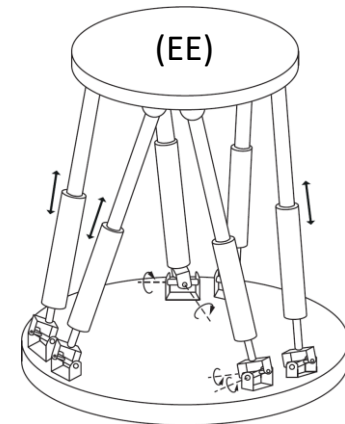


# Ch2: An Introductory Example

# 2R Planar Manipulator

# Robot Mechanical Structure

- A robot is mechanically constructed by connecting a set of bodies, called **Links**, to each other using various types of **Joints**.
  - \* All the robots considered in this course have links that can be modeled as **rigid bodies**.
- **Actuators**, such as electric motors, deliver forces and torques to the joints, thereby causing motion of the robot.
- An **End-Effector** (EE), such as a gripper or hand for grasping and manipulating objects, is attached to a specific link.



# 2R (or RR) Planar Manipulator

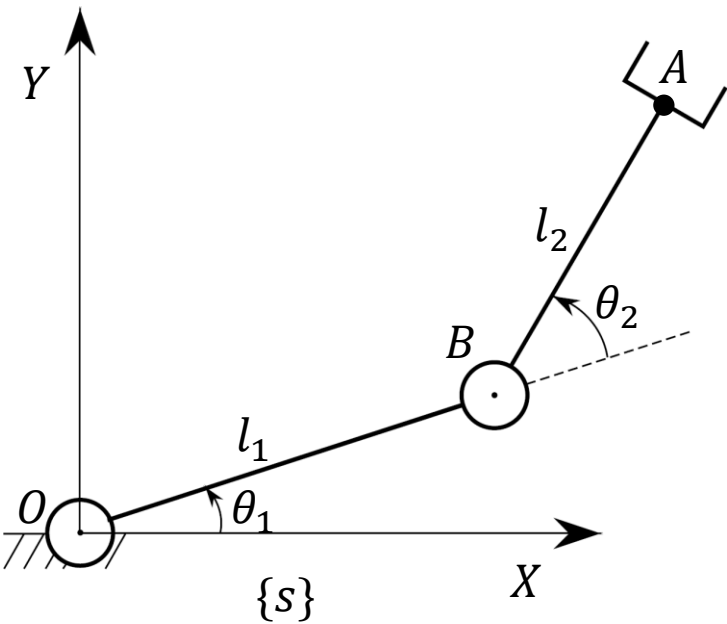
$(\theta_1, \theta_2)$ : Joint angles (or joint positions)

$(x, y)$ : Position of end-effector (point A)

$\{s\}$ : Base frame of manipulator

$l_1$ : Length of link 1

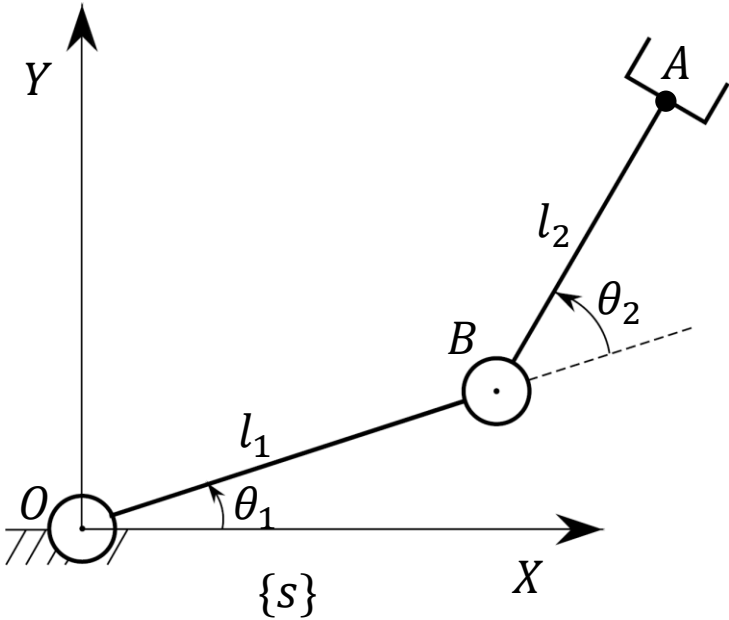
$l_2$ : Length of link 2



# Position Kinematics

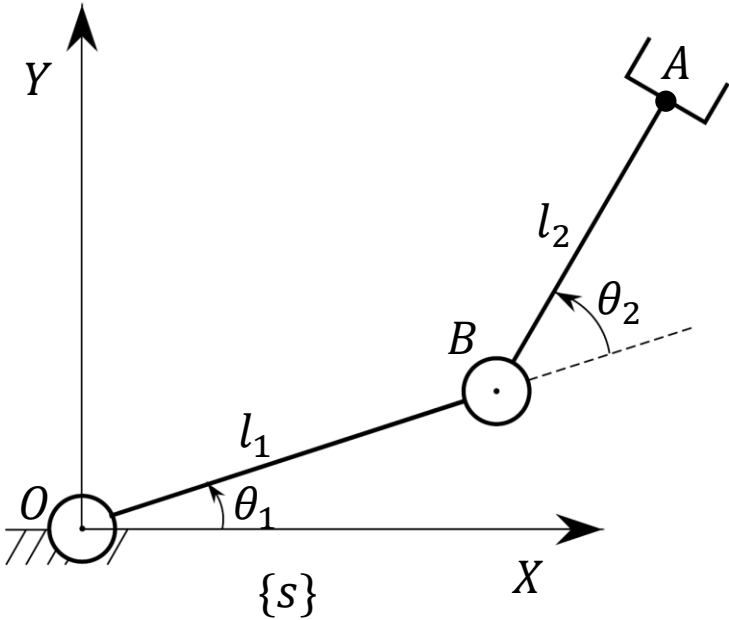
# Relation between Joint Angles and End-Effector Position

What is the relationship between the joint angles,  $(\theta_1, \theta_2)$ , and the position of the end effector point  $A$ ,  $(x, y)$ , in the base frame  $\{s\}$ ?



# Forward (Direct) Position Kinematics

Given the joint angles,  $(\theta_1, \theta_2)$ , of the 2R robot, find the position,  $(x, y)$  of the end-effector point  $A$ , in the base frame  $\{s\}$ .



# Forward (Direct) Position Kinematics

$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ : Vector of joint angles

$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix}$ : Position vector of end-effector point

$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \equiv f_1(\theta_1, \theta_2)$$

$$y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \equiv f_2(\theta_1, \theta_2)$$

$$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$

More abstractly, the forward kinematics map is

$$\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{\theta})$$

where  $\boldsymbol{f}$  is a vector function.

$$\boldsymbol{f}(\boldsymbol{\theta}) = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$



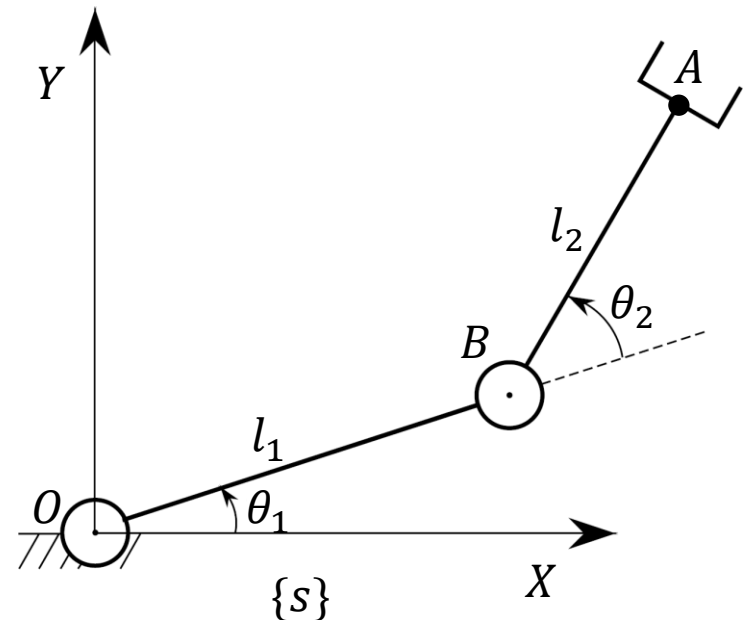
# Inverse Position Kinematics

Given the position,  $(x, y)$ , of the end effector point  $A$ , find the joint angles,  $(\theta_1, \theta_2)$  so that the position  $(x, y)$  is reached.

In other words, from the equations

$$\begin{aligned}x &= l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)\end{aligned}$$

Find  $\theta_1$  and  $\theta_2$  as a function of  $x$  and  $y$ .

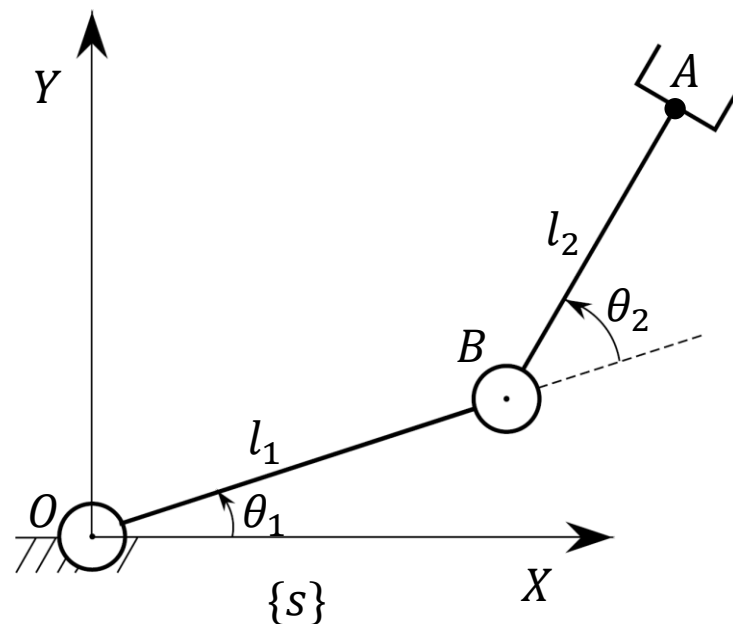


# Numerical Example (Exercise)

Forward and Inverse Position Kinematics:

$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2)$$

$$y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)$$



$$\theta_2 = \text{atan2} \left( \pm \sqrt{1 - u^2}, u \right)$$

$$\theta_1 = \text{atan2}(y, x) - \text{atan2}(l_2 \sin \theta_2, l_1 + l_2 \cos \theta_2)$$

$$u = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

# Velocity Kinematics

# Relation between Joint Angle Rates and End-Effector Velocity

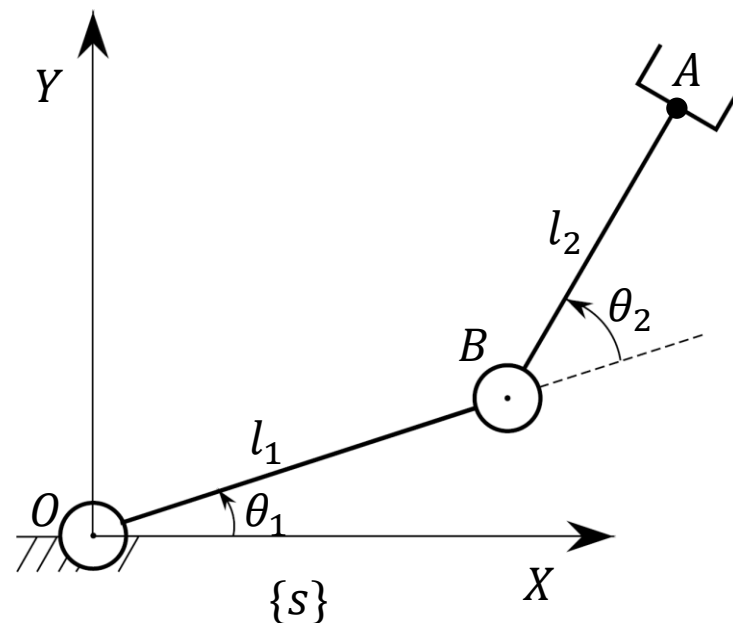
What is the relationship between the joint angle rates of motion (or joint velocities) ( $\dot{\theta}_1, \dot{\theta}_2$ ), and the velocity of the end effector point ( $v_x, v_y$ )?

$\dot{\theta}_1 = \frac{d\theta_1}{dt}$  : Rate of change of angle of joint 1.

$\dot{\theta}_2 = \frac{d\theta_2}{dt}$  : Rate of change of angle of joint 2.

$v_x = \frac{dx}{dt} = \dot{x}$  :  $x$ -component of velocity of point A.

$v_y = \frac{dy}{dt} = \dot{y}$  :  $y$ -component of velocity of point A.



# Relation between Joint Angle Rates and End-Effector Velocity

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The manipulator (analytic) Jacobian is:

$$J(\theta_1, \theta_2) = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$ : Vector of joint angle rates.

$v = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$ : Velocity of end-effector point.

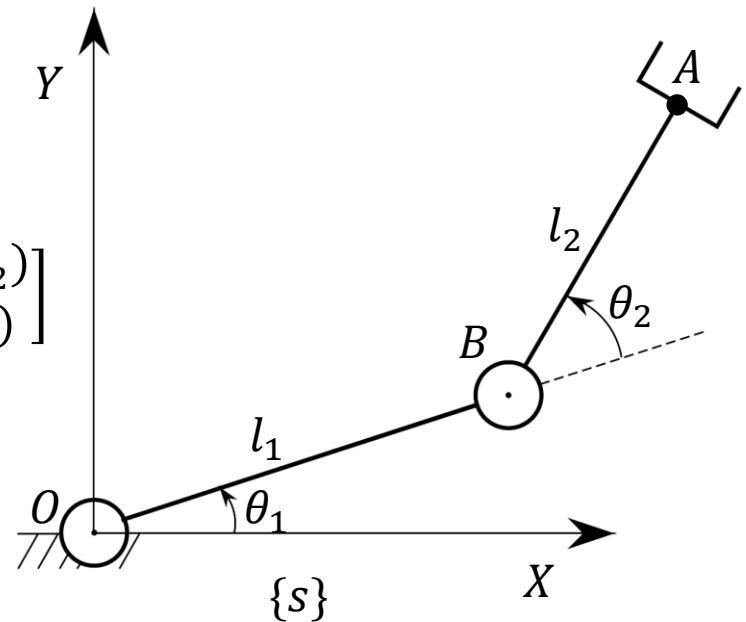
The velocity kinematics equations in vector-matrix form is:  $v = J(\theta)\dot{\theta}$

# Forward (Direct) Velocity Kinematics

Given the configuration of the robot,  $\theta$ , and the joint angle rates,  $\dot{\theta}$ , compute the velocity,  $v$  of the end effector.

$$v = J(\theta)\dot{\theta}$$

$$J(\theta) = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

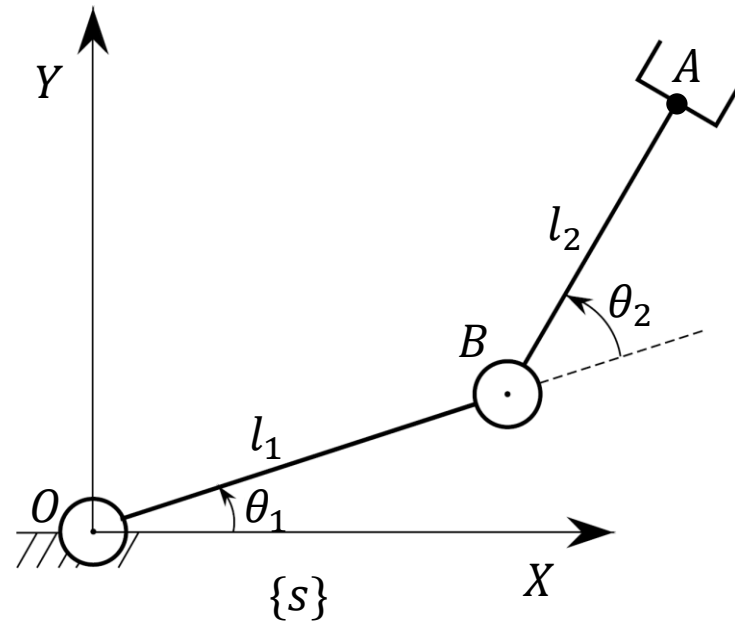


# Inverse Velocity Kinematics

Given the configuration of the robot,  $\theta$ , and the velocity,  $v$ , of the end effector, compute the joint angle rates,  $\dot{\theta}$ .

$$\dot{\theta} = J^{-1}(\theta)v$$

assuming  $J^{-1}(\theta)$  exists or the Jacobian matrix is invertible at the configuration  $\theta$ .

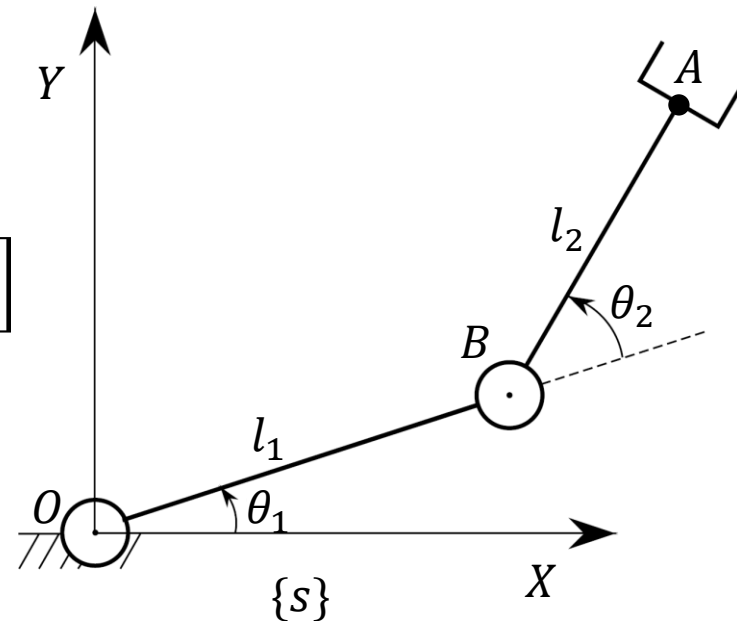


# Kinematic Singularities

The configuration  $\boldsymbol{\theta}$  at which the Jacobian,  $\boldsymbol{J}(\boldsymbol{\theta})$  of a manipulator loses rank is called a **kinematic singularity** or **singular configuration** of the manipulator.

For a 2R manipulator, the Jacobian,  $\boldsymbol{J}(\boldsymbol{\theta})$  losing rank implies  $\det(\boldsymbol{J}(\boldsymbol{\theta})) = 0$ .

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$



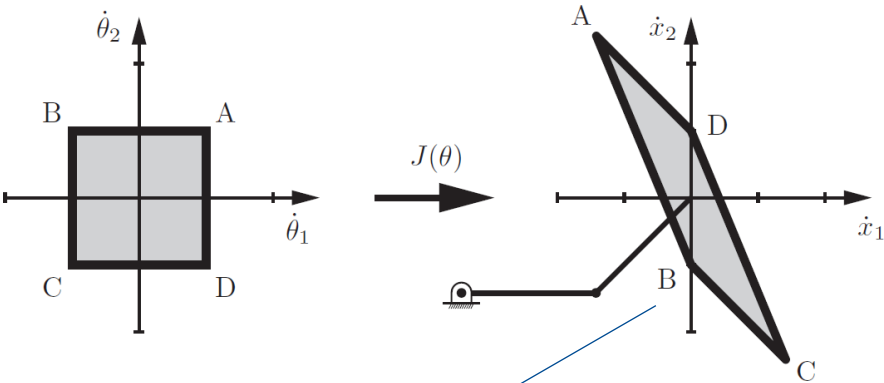


# Physical Implications of Kinematic Singularities

Why should we care about singular (or almost singular) configurations?

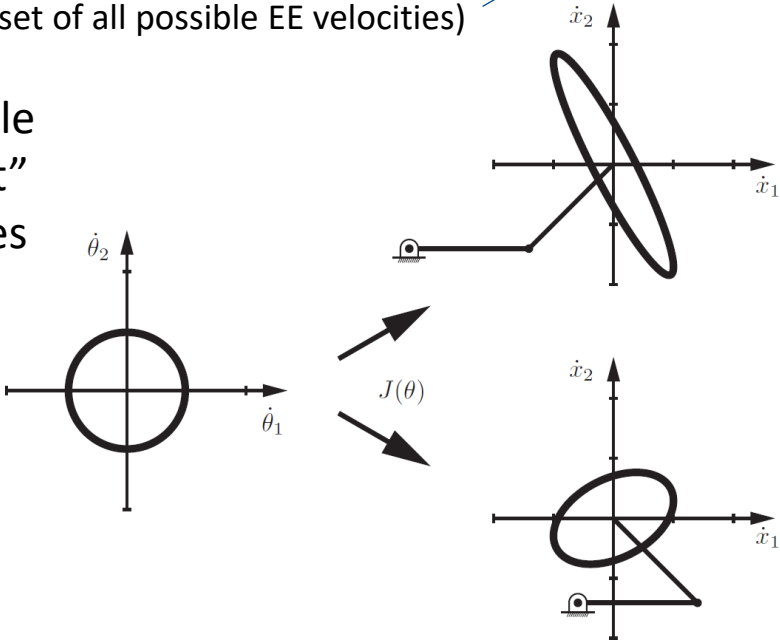
# Velocity Manipulability Ellipsoid

The Jacobian can be used to map bounds on the rotational speed of the joints (which is a polygon) to bounds on  $\mathbf{v}$ .



(set of all possible EE velocities)

The Jacobian can be also used to map a unit circle of joint velocities in the  $\theta_1 - \theta_2$ -plane (“iso-effort” contour) to an ellipse in the space of EE velocities (this ellipse is called the **velocity manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the tip move in arbitrary directions.

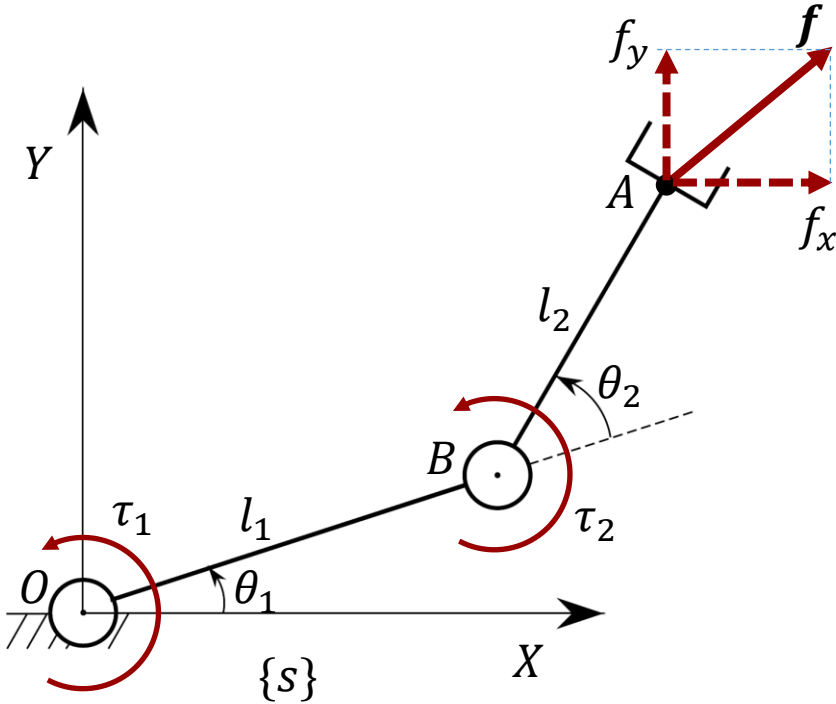
# Statics

# Statics

What is the relationship between the applied force  $\mathbf{f}$  and the joint torques  $\boldsymbol{\tau}$  such that the manipulator is at equilibrium at a given configuration  $\boldsymbol{\theta}$ ?

$\mathbf{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$  : Force acting at end-effector point  $A$

$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$  : Vector of joint torques required to resist  $\mathbf{f}$



(Assume that  $\mathbf{g} = \mathbf{0}$ )

# Statics

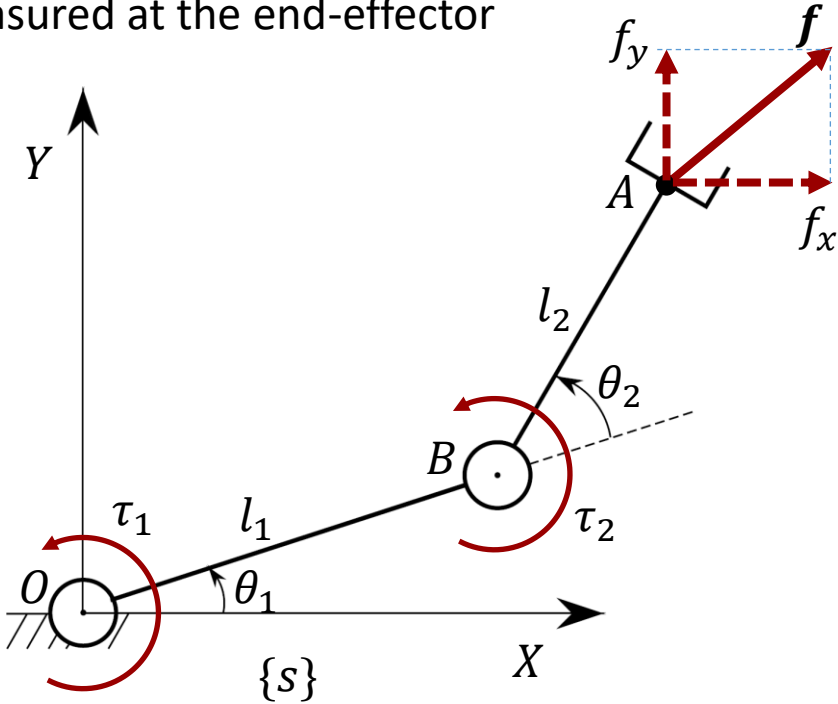
$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ -l_2 \sin(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^T \boldsymbol{f}$$

# Statics

A more general method to derive a relation between  $\mathbf{f}$  and  $\boldsymbol{\tau}$ .

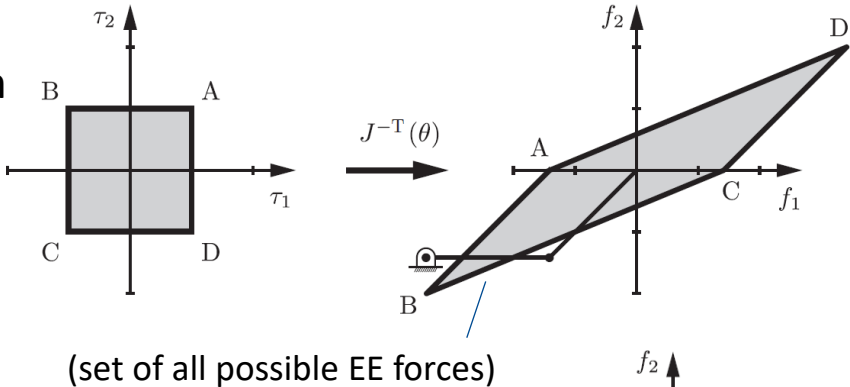
Principle of conservation of power:  
power generated at the joints = power measured at the end-effector



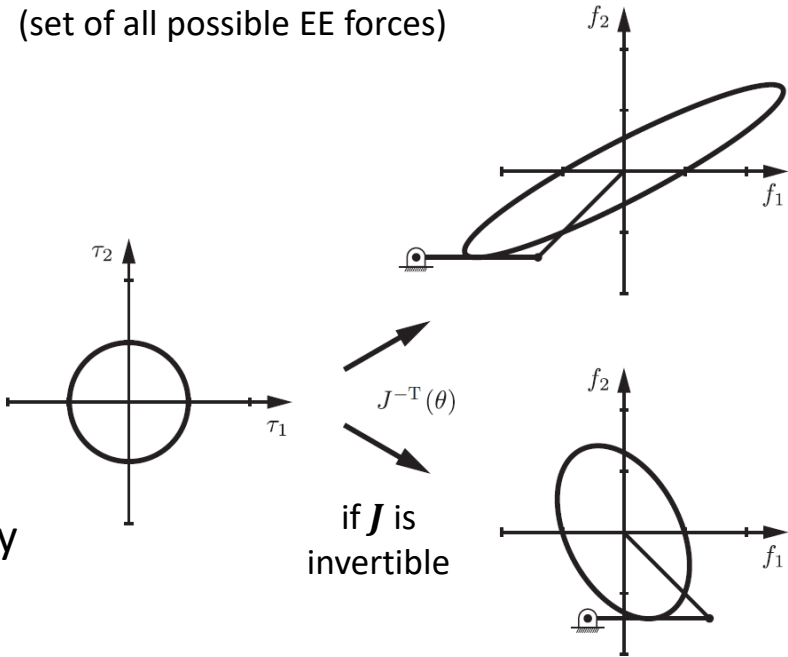
(Assume that  $\mathbf{g} = \mathbf{0}$ )

# Force Manipulability Ellipsoid

Since  $\mathbf{f} = (\mathbf{J}(\boldsymbol{\theta})^T)^{-1}\boldsymbol{\tau} = \mathbf{J}(\boldsymbol{\theta})^{-T}\boldsymbol{\tau}$ , Jacobian transpose inverse can be used to map bounds on the joint torques (which is a polygon) to bounds on end-effector force  $\mathbf{f}$ .



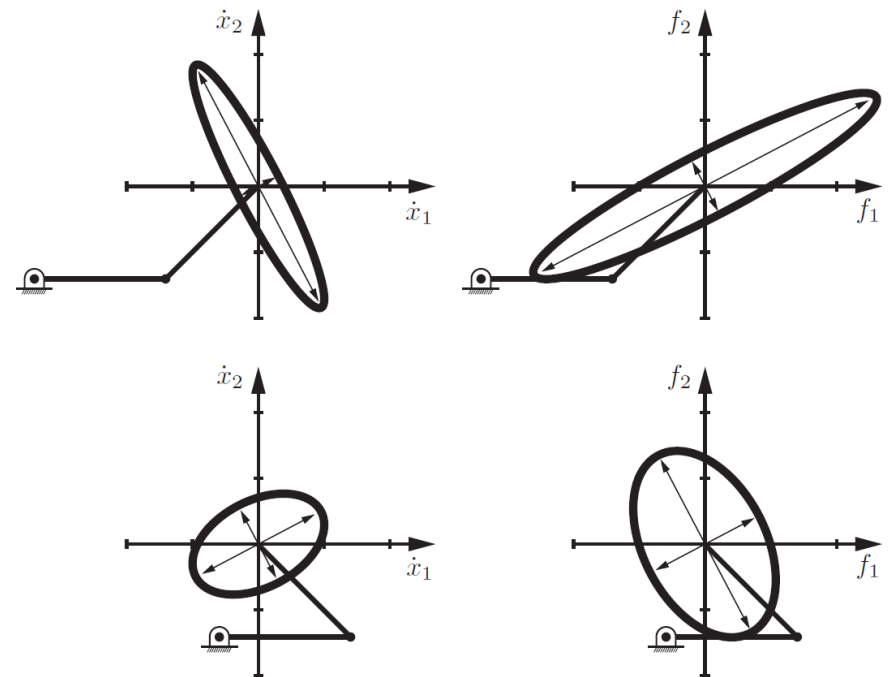
The Jacobian transpose inverse can be also used to map a unit circle of joint torques in the  $\tau_1$ - $\tau_2$ -plane (“iso-effort” contour) to an ellipse in the space of EE forces (this ellipse is called the **force manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the EE generate forces in arbitrary directions.

# Kineto-Statics Duality

If it is easy to generate a tip velocity in a given direction then it is difficult to generate a force in that same direction, and vice versa.



$$\begin{aligned} \mathbf{v} &= \mathbf{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \\ \boldsymbol{\tau} &= \mathbf{J}(\boldsymbol{\theta})^T \mathbf{f} \end{aligned}$$

At a singularity, EE motion capability becomes zero in one or more directions, and it can resist infinite force in one or more directions.