Ch9: Centralized Control - Position Control

Introduction

Introduction
○●○○

Position Control: PD
○   ○○○○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○
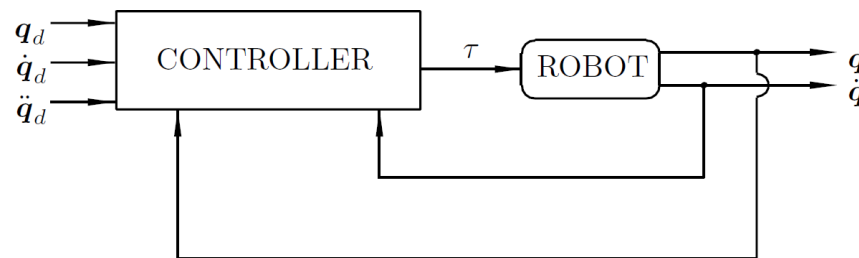
PID
○○○

Stony Brook University

Closed-loop Dynamic Equation

Consider the dynamic model of an n-DOF open-chain manipulator with no friction at the joints and no external force at the end-effector.

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$$

or

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})^{-1}[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})] \end{bmatrix}$$

(state-space form)

In general, a position/motion **Control Law** (**Controller**) with desired joint position $\boldsymbol{q}_d(t) \in \mathbb{R}^n$, velocity $\dot{\boldsymbol{q}}_d(t) \in \mathbb{R}^n$, and acceleration $\ddot{\boldsymbol{q}}_d(t) \in \mathbb{R}^n$ can be expressed as a nonlinear function $\boldsymbol{\tau}_c$ as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_c\big(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d, \boldsymbol{M}(\boldsymbol{q}), \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \boldsymbol{g}(\boldsymbol{q})\big)$$

Note: For practical purposes, it is desirable that the controller $\boldsymbol{\tau}_c$ does not depend on the joint acceleration $\ddot{\boldsymbol{q}}$ since computing or measuring acceleration is usually highly sensitive to noise.

Closed-loop Dynamic Equation (cont.)

Thus, the **closed-loop dynamic equation** is derived as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau_c\big(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d, M(q), C(q, \dot{q}), g(q)\big)$$
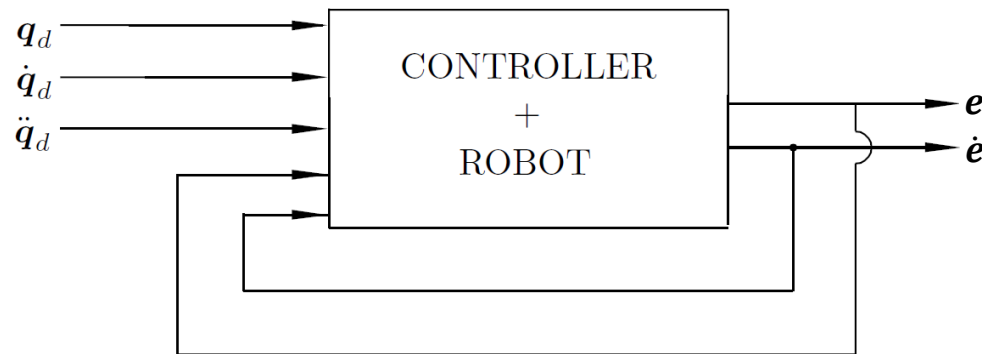
or in the state-space form as
$$\frac{d}{dt}\begin{bmatrix} q_d - q \\ \dot{q}_d - \dot{q} \end{bmatrix} = f\big(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d, M(q), C(q, \dot{q}), g(q)\big)$$

$$e = q_d - q \in \mathbb{R}^n,$$
$$\dot{e} = \dot{q}_d - \dot{q} \in \mathbb{R}^n,$$

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \tilde{f}(t, e, \dot{e})$$

In general, a nonautonomous nonlinear ODE when $q_d = q_d(t)$.

and by replacing q with $q_d(t) - e$ and $\dot{q}$ with $\dot{q}_d(t) - \dot{e}$ in f:

q_d ⟶
$\dot{q}_d$ ⟶
$\ddot{q}_d$ ⟶

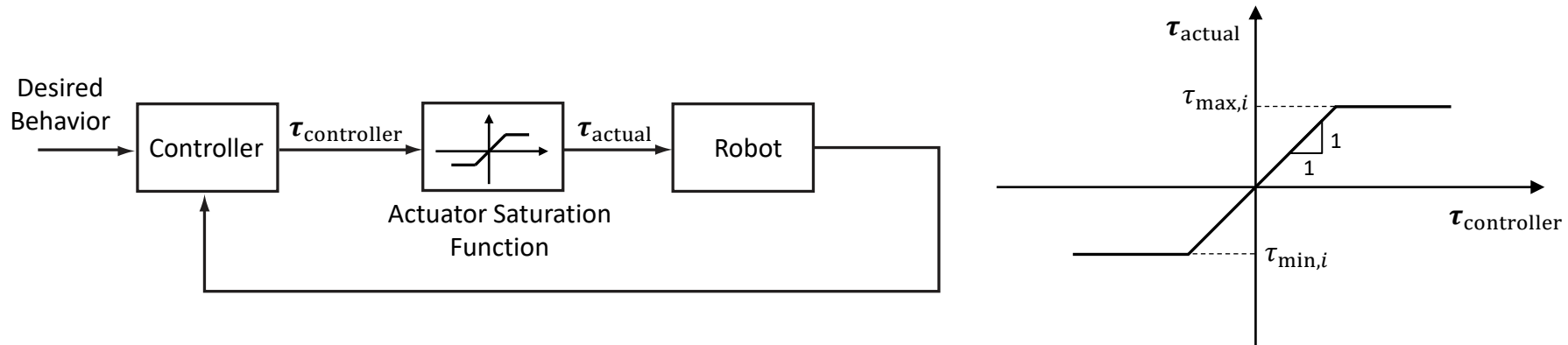CONTROLLER
+
ROBOT

⟶ e
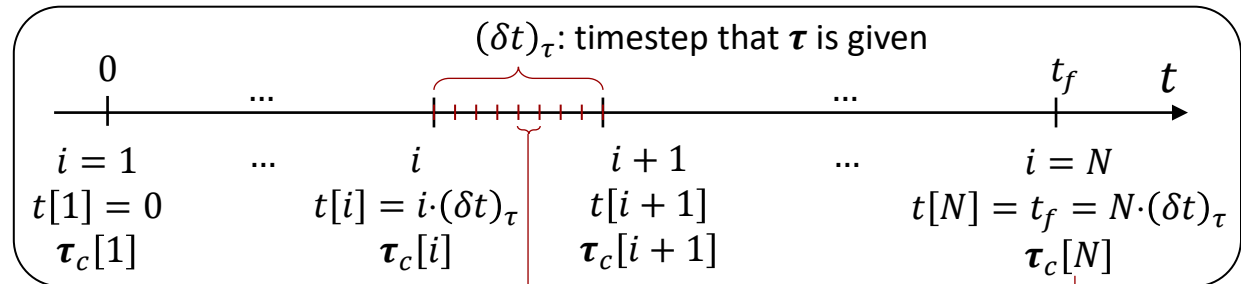⟶ $\dot{e}$

Actuator Saturation

In some controllers, choosing large values for the control parameters causes a large (initial) control torque which is beyond the robot actuators capacity which are limited by maximum and minimum allowable values $\boldsymbol{\tau}_{\max}, \boldsymbol{\tau}_{\min}$. Therefore, the control parameters should be chosen properly.

To consider the actuator saturation limits in the simulation, we add a saturation function as follows:

$$\boldsymbol{\tau}_{\text{actual}} = \text{sat}(\boldsymbol{\tau}_{\text{controller}})$$

Introduction ○○○○● Position Control: ○ PD ○○○○○ PD with Gravity Compensation ○○ PD with Desired Gravity Compensation ○○○ PID ○○○

Stony Brook University

Pseudocode for Controllers

$(\delta t)_\tau$: timestep that $\boldsymbol{\tau}$ is given

0 ... t_f t

$i = 1$ ... i $i + 1$ ... $i = N$
$t[1] = 0$ $t[i] = i\cdot(\delta t)_\tau$ $t[i + 1]$ $t[N] = t_f = N\cdot(\delta t)_\tau$
$\boldsymbol{\tau}_c[1]$ $\boldsymbol{\tau}_c[i]$ $\boldsymbol{\tau}_c[i + 1]$ $\boldsymbol{\tau}_c[N]$

Given $\boldsymbol{\tau}[i], \boldsymbol{\mathcal{F}}_{\text{tip}}[i]$ ($i = 1, \dots, N$)
Set $\boldsymbol{\theta}[1] = \boldsymbol{\theta}(0)$, $\dot{\boldsymbol{\theta}}[1] = \dot{\boldsymbol{\theta}}(0)$, $\boldsymbol{e}_{\text{int}} = \boldsymbol{0}$
Set $\overline{\boldsymbol{\theta}} = \boldsymbol{\theta}[1]$, $\dot{\overline{\boldsymbol{\theta}}} = \dot{\boldsymbol{\theta}}[1]$
For $i = 1$ to $N - 1$

$\left[\boldsymbol{\theta}_d[i], \dot{\boldsymbol{\theta}}_d[i]\right] = \text{trajectory}(t[i] = i\cdot(\delta t)_\tau, \dots)$

$\boldsymbol{\tau}_c[i] = \text{controller}\left(\overline{\boldsymbol{\theta}}, \dot{\overline{\boldsymbol{\theta}}}, \boldsymbol{\theta}_d[i], \dot{\boldsymbol{\theta}}_d[i], \ddot{\boldsymbol{\theta}}_d[i], \boldsymbol{e}_{\text{int}}, \dots\right)$

$\boldsymbol{\tau}_c[i] = \text{sat}(\boldsymbol{\tau}_c[i])$

For $j = 1$ to n_{res}

$\ddot{\boldsymbol{\theta}} = \text{ForwardDynamics}\left(\overline{\boldsymbol{\theta}}, \dot{\overline{\boldsymbol{\theta}}}, \boldsymbol{\tau}_c[i], \boldsymbol{\mathcal{F}}_{\text{tip}}[i]\right)$

$\overline{\boldsymbol{\theta}} = \overline{\boldsymbol{\theta}} + \dot{\overline{\boldsymbol{\theta}}}\cdot(\delta t)_{\ddot{\theta}}$

$\dot{\overline{\boldsymbol{\theta}}} = \dot{\overline{\boldsymbol{\theta}}} + \ddot{\boldsymbol{\theta}}\cdot(\delta t)_{\ddot{\theta}}$

end

$\boldsymbol{e}_{\text{int}} = \boldsymbol{e}_{\text{int}} + (\delta t)_\tau(\boldsymbol{\theta}_d[i] - \overline{\boldsymbol{\theta}})$ % error integral
$\boldsymbol{\theta}[i + 1] = \overline{\boldsymbol{\theta}}$
$\dot{\boldsymbol{\theta}}[i + 1] = \dot{\overline{\boldsymbol{\theta}}}$
end

$(\delta t)_{\ddot{\theta}} = (\delta t)_\tau/n_{\text{res}}$: timestep for motion simulation and computing $\ddot{\boldsymbol{\theta}}$.
n_{res}: Integration resolution.

N: Number of samples

$(\delta t)_\tau, (\delta t)_{\ddot{\theta}} \in \mathbb{R}^+$

First-order Euler Integration
(we can also use any other ODE solver like **ode45** which is based on an explicit **Runge-Kutta** (4,5) formula)

Introduction
○○○○

Position Control:
○

PD
○○○○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○

PID
○○○

Stony Brook
University

Position Control

Introduction
OOOO

Position Control:
●

PD
OOOOO

PD with Gravity Compensation
OO

PD with Desired Gravity Compensation
OOO

PID
OOO

Stony Brook University

Position Control Objective

Given a desired <u>constant</u> joint position (set-point reference) $\boldsymbol{q}_d \in \mathbb{R}^n$, we wish to find joint torques/forces $\boldsymbol{\tau} \in \mathbb{R}^n$ such that the joint position $\boldsymbol{q}(t) \in \mathbb{R}^n$ tend to $\boldsymbol{q}_d$ accurately:

$$\lim_{t\to\infty} \boldsymbol{q}(t) = \boldsymbol{q}_d \quad \Rightarrow \quad \lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{0} \qquad \boldsymbol{e}(t) = \boldsymbol{q}_d - \boldsymbol{q}(t) \in \mathbb{R}^n$$

position error

The most common position controllers:

- PD Control (or P Control Plus Velocity Feedback)
- PD Control with Gravity Compensation
- PD Control with Desired Gravity Compensation
- PID Control

Introduction
○○○○

Position Control:
○

PD
○○○○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○

PID
○○○

PD Control

Introduction
○○○○

Position Control:
○

PD
●○○○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○

PID
○○○

Stony Brook University

PD Control
(or P Control Plus Velocity Feedback)

The PD (Proportional Derivative) control law is given by

$$\boldsymbol{\tau} = \boldsymbol{K}_p \boldsymbol{e} + \boldsymbol{K}_v \dot{\boldsymbol{e}} \quad \xrightarrow[\dot{\boldsymbol{q}}_d = 0]{\text{Since } \boldsymbol{q}_d = \text{constant}} \quad \boldsymbol{\tau} = \boldsymbol{K}_p \boldsymbol{e} - \boldsymbol{K}_v \dot{\boldsymbol{q}}$$

$$\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$$

$\boldsymbol{K}_p, \boldsymbol{K}_v \in \mathbb{R}^{n \times n}$ are symmetric positive definite matrices. If $\boldsymbol{K}_p = \text{diag}\{K_{p,i}\}$, $\boldsymbol{K}_v = \text{diag}\{K_{v,i}\}$, the controller is called PD Independent Joint Control.

This controller is the simplest (linear) controller that may be used to control robot manipulators.

Introduction	Position Control:	PD	PD with Gravity Compensation	PD with Desired Gravity Compensation	PID
○○○○	○	○●○○○○	○○	○○○	○○○

Stony Brook University
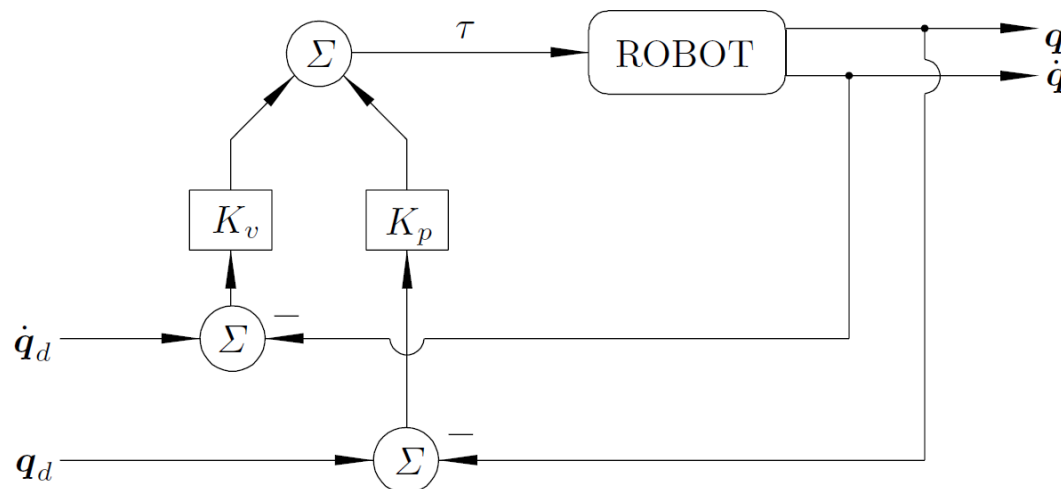
PD Control

The closed-loop dynamic equation is derived as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{K}_p\boldsymbol{e} - \boldsymbol{K}_v\dot{\boldsymbol{q}}$$

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{e} \\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}-\dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})^{-1}\left(\boldsymbol{K}_p\boldsymbol{e} - \boldsymbol{K}_v\dot{\boldsymbol{q}} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right)\end{bmatrix} = \boldsymbol{f}(\boldsymbol{e}, \dot{\boldsymbol{q}}) \quad \overset{(*)}{} \qquad \boldsymbol{q} = \boldsymbol{q}_d - \boldsymbol{e}$$

The system is **autonomous** because $\boldsymbol{q}_d$ is constant.

Note: In general, this system may have several equilibrium points, and the origin $(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \boldsymbol{0} \in \mathbb{R}^{2n}$ is not necessarily an equilibrium point.

$$\boldsymbol{f}(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{q}} = \boldsymbol{0}, \qquad \boldsymbol{K}_p\boldsymbol{e} - \boldsymbol{g}(\boldsymbol{q}_d - \boldsymbol{e}) = \boldsymbol{0}$$

Note: If the manipulator model does not include the gravitational torques term $\boldsymbol{g}(\boldsymbol{q})$ (e.g., those which move only on the horizontal plane), then the only equilibrium is the origin $(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \boldsymbol{0} \in \mathbb{R}^{2n}$.

Introduction
○○○○

Position Control:
○

PD
○○●○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○

PID
○○○

Stony Brook
University

PD Control
(when $g(q) = 0$)

To study the stability of the equilibrium we can use Lyapunov's direct method and LaSalle's Theorem to show asymptotic stability of the origin $(e, \dot{q}) = \mathbf{0}$.

Consider a Lyapunov function candidate as $\quad V(e, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} e^T K_p e > 0 \quad$ (PD)

$$\dot{V}(e, \dot{q}) = \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + e^T K_p \dot{e}$$

Kinetic energy
of the arm

$\forall e, \dot{q} \neq \mathbf{0}$

$$M(q) \ddot{q} = K_p e - K_v \dot{q} - C(q, \dot{q}) \dot{q}, \quad \dot{e} = -\dot{q} \quad (*)$$

$$\dot{q}^T \left[\frac{1}{2} \dot{M} - C \right] \dot{q} = \mathbf{0} \quad \text{(Property of dynamic model)}$$

$$\dot{V}(e, \dot{q}) = -\dot{q}^T K_v \dot{q} \leq 0 \quad \text{(NSD)}$$

Equilibrium Point Theorem $\longrightarrow$ The origin $(e, \dot{q}) = \mathbf{0}$ is (globally) stable and the solutions $e(t)$ and $\dot{q}(t)$ are bounded.

PD Control
(when $g(q) = 0$)

Now, we use LaSalle (invariant set) theorem to analyze the global asymptotic stability of the origin.

$$R = \left\{ (e, \dot{q}) \in \mathbb{R}^{2n} : \dot{V}(e, \dot{q}) = 0 \right\}$$

$(e, \dot{q}) = 0$ is the largest invariant set in R

$\Rightarrow$ The origin $(e, \dot{q}) = 0$ is globally asymptotically stable for any initial condition $q(0), \dot{q}(0) \in \mathbb{R}^n$:

$$\lim_{t \to \infty} e(t) = 0 \qquad \lim_{t \to \infty} \dot{q}(t) = 0$$

$\Rightarrow$ Thus, the control objective is achieved.

Note: Friction at the joints may also affect the position error.

PD Control
(when $g(q) \neq 0$)

The study of unicity of the equilibrium and boundedness of solutions for a control system under PD control when $g(q) \neq 0$ is somewhat more complex than when $g(q) = 0$.

For robots with only revolute joints, we can prove that

- For <u>any</u> $K_p = K_p^T > 0$, $K_v = K_v^T > 0$, it is guaranteed that $e(t)$ and $\dot{q}(t)$ are bounded for all initial conditions. Moreover, $\lim_{t \to \infty} \dot{q}(t) = 0$ (it does not guarantee $\lim_{t \to \infty} q(t) = q_d$ or even $\lim_{t \to \infty} q(t) = \text{constant}$).

- By choosing K_p sufficiently large, e.g., $\lambda_{\min}(K_p) > n \cdot \left(\max_{i,j,q} \left| \frac{\partial g_i(q)}{\partial q_j} \right| \right)$, then the closed-loop equation has a unique equilibrium (but not necessarily at origin).

- The error bound decreases, as $K_{v,i}$ become larger (in case $K_v = \text{diag}\{K_{v,i}\}$), however, large $K_{v,i}$ can saturate the robot actuators.

  $\Rightarrow$ Thus, the control objective <u>cannot</u> be achieved using PD control <u>unless</u> the desired position q_d is such that $g(q_d) = 0$ (i.e., the origin $(e, \dot{q}) = 0$ is an equilibrium).

Note: Friction at the joints may also affect the position error.

Introduction
○○○○

Position Control:
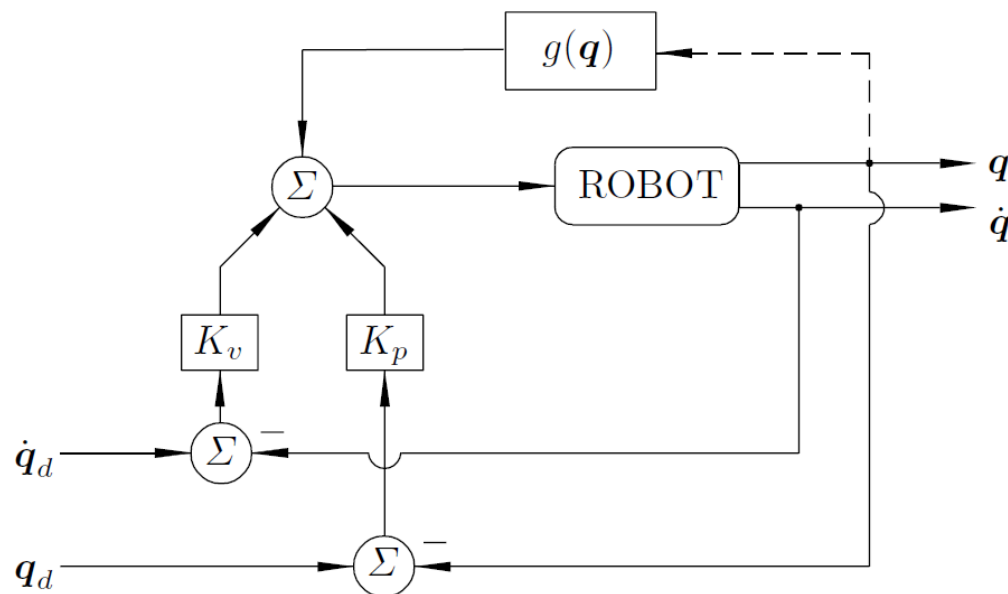○

PD
○○○○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○

PID
○○○

Stony Brook
University

PD Control with Gravity Compensation

Introduction
○○○○

Position Control:
○

PD
○○○○○

PD with Gravity Compensation
●○

PD with Desired Gravity Compensation
○○○

PID
○○○

Stony Brook
University

PD Control with Gravity Compensation

The PD control law with gravity compensation is given by

$$\boldsymbol{\tau} = \boldsymbol{K}_p \boldsymbol{e} + \boldsymbol{K}_v \dot{\boldsymbol{e}} + \boldsymbol{g}(\boldsymbol{q}) \quad \xrightarrow[\dot{\boldsymbol{q}}_d = 0]{\text{Since } \boldsymbol{q}_d = \text{constant}} \quad \boldsymbol{\tau} = \boldsymbol{K}_p \boldsymbol{e} - \boldsymbol{K}_v \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$$

$\boldsymbol{K}_p, \boldsymbol{K}_v \in \mathbb{R}^{n \times n}$ are symmetric positive definite matrices. $\qquad \boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$

Introduction
○○○○

Position Control:
○

PD
○○○○○

PD with Gravity Compensation
○●

PD with Desired Gravity Compensation
○○○

PID
○○○

Stony Brook
University

PD Control with Gravity Compensation

The closed-loop dynamic equation is derived as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{K}_p \boldsymbol{e} - \boldsymbol{K}_v \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$$

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{e} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})^{-1}(\boldsymbol{K}_p \boldsymbol{e} - \boldsymbol{K}_v \dot{\boldsymbol{q}} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}) \end{bmatrix} = \boldsymbol{f}(\boldsymbol{e}, \dot{\boldsymbol{q}}) \qquad \boldsymbol{q} = \boldsymbol{q}_d - \boldsymbol{e}$$

The system is **autonomous**, and the origin $(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \boldsymbol{0} \in \mathbb{R}^{2n}$ is the only equilibrium point.

Note: Using the same proof given for PD Control when $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$, we can show that the origin $(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$ is <u>globally asymptotically stable</u> for any initial condition $\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0) \in \mathbb{R}^n$:

$$\lim_{t \to \infty} \boldsymbol{e}(t) = \boldsymbol{0} \qquad \lim_{t \to \infty} \dot{\boldsymbol{q}}(t) = \boldsymbol{0}$$

$\Rightarrow$ Thus, the position control objective is achieved.

**Note**: Friction at the joints may also affect the position error.

PD Control with Desired Gravity Compensation

Introduction
◯◯◯◯

Position Control:
◯

PD
◯◯◯◯◯

PD with Gravity Compensation
◯◯

PD with Desired Gravity Compensation
●◯◯

PID
◯◯◯

Stony Brook
University

PD Control with Desired Gravity Compensation

Implementation of the PD control with gravity compensation requires **on-line** computation of $\boldsymbol{g}(\boldsymbol{q})$. However, since the elements of $\boldsymbol{g}(\boldsymbol{q})$ involve trigonometric functions of $\boldsymbol{q}$, its real time computation take a longer time than the computation of the 'PD-part' of the control law, especially in high sampling frequency applications. A solution is using **PD Control with Desired Gravity Compensation** which requires only **off-line** computation of $\boldsymbol{g}(\boldsymbol{q}_d)$:

$$\boldsymbol{\tau} = \boldsymbol{K}_p\boldsymbol{e} + \boldsymbol{K}_v\dot{\boldsymbol{e}} + \boldsymbol{g}(\boldsymbol{q}_d) \qquad \boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$$

$\boldsymbol{K}_p, \boldsymbol{K}_v \in \mathbb{R}^{n \times n}$ are symmetric positive definite matrices.
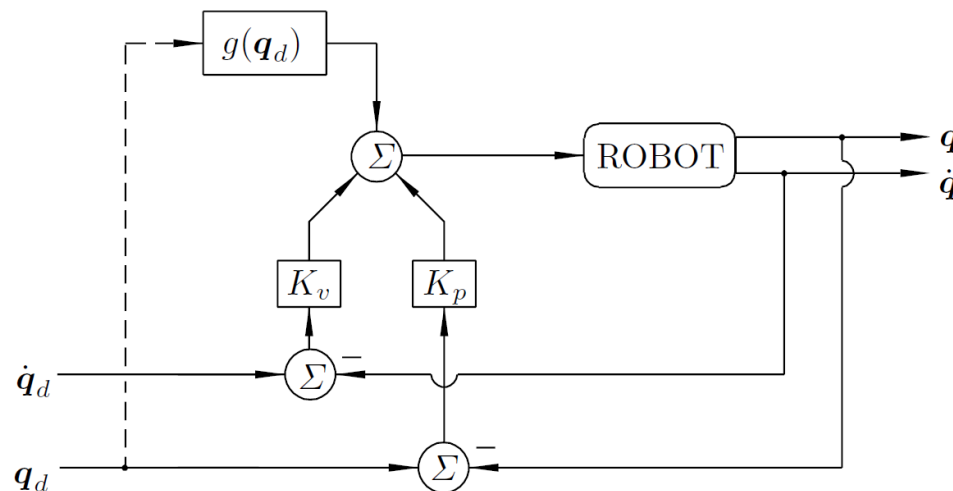
PD Control with Desired Gravity Compensation

The closed-loop dynamic equation is derived as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = K_p e - K_v \dot{q} + g(q_d)$$

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\dot{q} \\ M(q)^{-1}\left(K_p e - K_v \dot{q} - C(q,\dot{q})\dot{q} - g(q) + g(q_d)\right) \end{bmatrix} = f(e,\dot{q})$$

$$q = q_d - e$$

The system is **autonomous** (since q_d is constant), and in general, may have multiple equilibria which the origin $(e,\dot{q}) = 0 \in \mathbb{R}^{2n}$ is always one of them:

$$f(e,\dot{q}) = 0 \quad \Rightarrow \quad \dot{q} = 0, \qquad K_p e - g(q_d - e) + g(q_d) = 0$$

PD Control with Desired Gravity Compensation

For robots with only revolute joints, we can prove that

- For <u>any</u> $\boldsymbol{K}_p = \boldsymbol{K}_p^T > 0$, $\boldsymbol{K}_v = \boldsymbol{K}_v^T > 0$, it is guaranteed that $\boldsymbol{e}(t)$ and $\dot{\boldsymbol{q}}(t)$ are bounded for all initial conditions. Moreover, $\lim_{t\to\infty} \dot{\boldsymbol{q}}(t) = \boldsymbol{0}$ (it does not guarantee $\lim_{t\to\infty} \boldsymbol{q}(t) = \boldsymbol{q}_d$ or even $\lim_{t\to\infty} \boldsymbol{q}(t) = \text{constant}$).

- By choosing $\boldsymbol{K}_p$ sufficiently large, e.g., $\lambda_{\min}(\boldsymbol{K}_p) > n \cdot \left(\max_{i,j,q} \left| \frac{\partial g_i(\boldsymbol{q})}{\partial \boldsymbol{q}_j} \right| \right)$, then the closed-loop equation has a unique equilibrium at origin $(\boldsymbol{e}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$ and it is <u>globally asymptotically stable</u>.

$$\lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{0} \qquad \lim_{t\to\infty} \dot{\boldsymbol{q}}(t) = \boldsymbol{0}$$

$\Rightarrow$ Thus, the position control objective is achieved.

Note: Friction at the joints may also affect the position error.

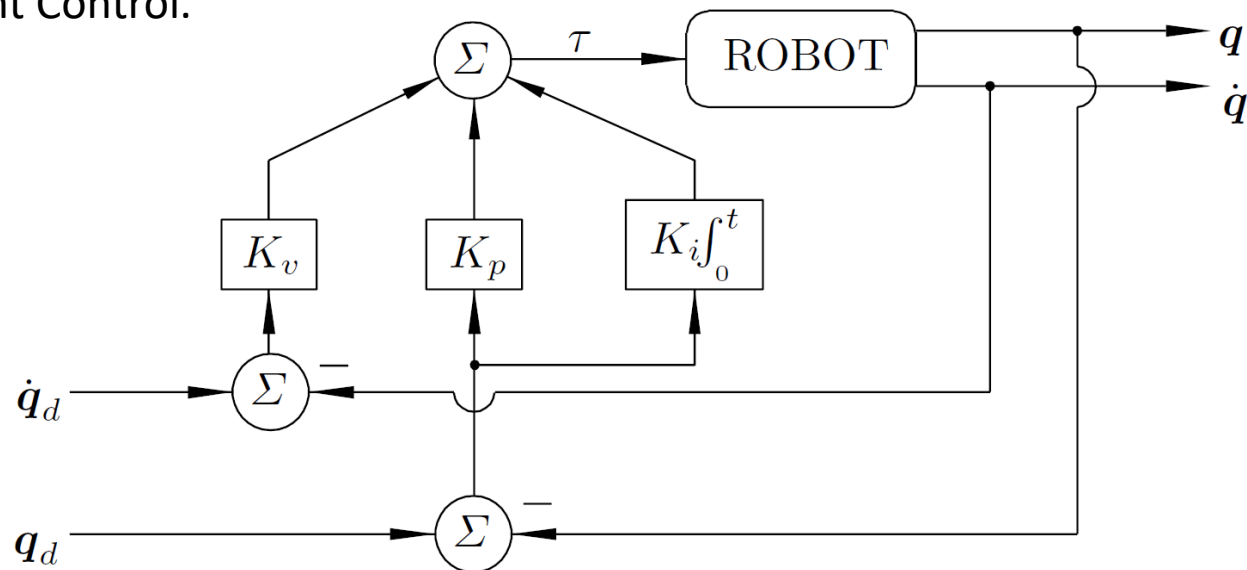Introduction
○○○○

Position Control:
○

PD
○○○○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○

PID
○○○

Stony Brook University

PID Control

PID Control

The PID (Proportional Integral Derivative) control law is given by

$$\boldsymbol{\tau} = \boldsymbol{K}_p \boldsymbol{e} + \boldsymbol{K}_v \dot{\boldsymbol{e}} + \boldsymbol{K}_i \int_0^t \boldsymbol{e}(\tau) d\tau \qquad\qquad \boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$$

$\boldsymbol{K}_p, \boldsymbol{K}_v, \boldsymbol{K}_i \in \mathbb{R}^{n \times n}$ (position, velocity, and integral gains) are symmetric positive definite matrices. If $\boldsymbol{K}_p = \text{diag}\{K_{p,i}\}$, $\boldsymbol{K}_v = \text{diag}\{K_{v,i}\}$, $\boldsymbol{K}_i = \text{diag}\{K_{i,i}\}$, the controller is called PID Independent Joint Control.

Introduction
○○○○

Position Control:
○

PD
○○○○○

PD with Gravity Compensation
○○

PD with Desired Gravity Compensation
○○○

PID
○●○

Stony Brook University

PID Control

The closed-loop dynamic equation is derived as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = K_p e + K_v \dot{e} + K_i \overbrace{\int_0^t e(\tau) d\tau}^{\xi}$$

$$\Rightarrow \begin{cases} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = K_p e + K_v \dot{e} + K_i \xi \\ \dot{\xi} = e \end{cases} \qquad q = q_d - e$$

$$\frac{d}{dt}\begin{bmatrix} \xi \\ e \\ \dot{q} \end{bmatrix} = \begin{bmatrix} e \\ -\dot{q} \\ M(q)^{-1}\left(K_p e - K_v \dot{q} + K_i \xi - C(q,\dot{q})\dot{q} - g(q)\right) \end{bmatrix} \xrightarrow[\text{point}]{\text{equilibrium}} \begin{bmatrix} K_i^{-1} g(q_d) \\ 0 \\ 0 \end{bmatrix}$$

Translating this equilibrium point to the origin via a suitable change of variable:

$$\frac{d}{dt}\begin{bmatrix} z \\ e \\ \dot{q} \end{bmatrix} = \begin{bmatrix} e \\ -\dot{q} \\ M(q)^{-1}\left(K_p e - K_v \dot{q} + K_i z + g(q_d) - C(q,\dot{q})\dot{q} - g(q)\right) \end{bmatrix}$$

$$z = \xi - K_i^{-1} g(q_d)$$

The system is **autonomous,** and its unique equilibrium is the origin $(z, e, \dot{q}) = 0 \in \mathbb{R}^{3n}$.

PID Control: Tuning Method

For robots with only revolute joints, we can prove that the symmetric positive definite matrices $\boldsymbol{K}_p$, $\boldsymbol{K}_i$, $\boldsymbol{K}_v$ which satisfy the following relations can only guarantee achievement of the position control objective by making the origin $(\boldsymbol{z}, \boldsymbol{e}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$ <u>locally asymptotically stable</u> (i.e., if $\boldsymbol{e}(t)$, $\dot{\boldsymbol{q}}(t)$ are "sufficiently small", $\lim\limits_{t \to \infty} \boldsymbol{e}(t) = \boldsymbol{0}$).

$$\lambda_{\max}\{\boldsymbol{K}_i\} \geq \lambda_{\min}\{\boldsymbol{K}_i\} > 0$$
$$\lambda_{\max}\{\boldsymbol{K}_p\} \geq \lambda_{\min}\{\boldsymbol{K}_p\} > n \cdot k_g$$
$$\lambda_{\max}\{\boldsymbol{K}_v\} \geq \lambda_{\min}\{\boldsymbol{K}_v\} > \frac{\lambda_{\max}\{\boldsymbol{K}_i\}}{\lambda_{\min}\{\boldsymbol{K}_p\} - k_g} \cdot \frac{\lambda_{\max}^2(\boldsymbol{M}(\boldsymbol{q}))}{\lambda_{\min}(\boldsymbol{M}(\boldsymbol{q}))}$$

$$k_g = \max_{i,j,q} \left| \frac{\partial g_i(\boldsymbol{q})}{\partial \boldsymbol{q}_j} \right|$$

Note: A system with $\boldsymbol{K}_p$, $\boldsymbol{K}_i$, $\boldsymbol{K}_v$ parameters which satisfy these relations does not necessarily achieve a proper settling time. It is possible to find a set of the symmetric PD matrices $\boldsymbol{K}_p$, $\boldsymbol{K}_i$, $\boldsymbol{K}_v$ which achieve a small settling time, while violating these relations.