Ch9: Root Locus Techniques

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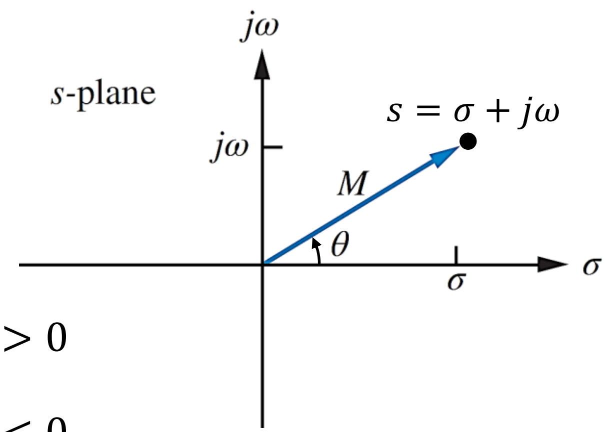
Introduction

Vector Representation of Complex Numbers

Any complex number $s = \sigma + j\omega$ in s-plane can be graphically represented by a **vector**. It also can be described in polar form with magnitude M and angle θ, as $M\angle\theta$.

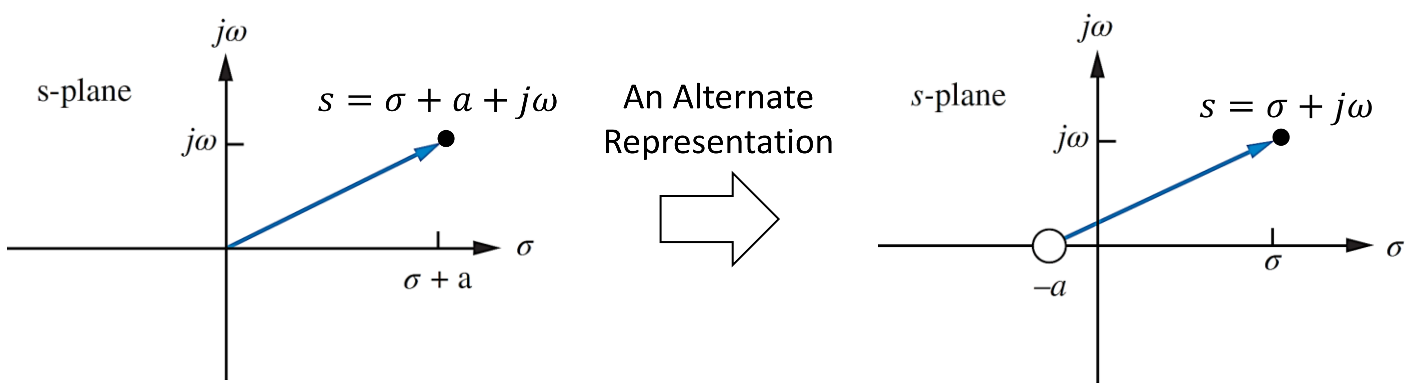


$$s = \sigma + j\omega = M(\cos\theta + j\sin\theta) = M\angle\theta$$

$$M = |(\sigma + j\omega)| = \sqrt{\sigma^2 + \omega^2}, \quad \theta = \angle(\sigma + j\omega) = \begin{cases} \tan^{-1}\dfrac{\omega}{\sigma} & \sigma > 0 \\ \tan^{-1}\dfrac{\omega}{\sigma} + \pi & \sigma < 0 \end{cases}$$

$F(s) = s + a$ (where a is either real or complex) is a complex number which is calculated by substituting the complex number $s = \sigma + j\omega$ into the complex function $F(s)$. $F(s)$ also can be represented by a vector drawn from $-a$ to the point s.

For example, for real a:



Evaluation of a Complex Function via Vectors

Consider a complex function (like a transfer function) as

$$F(s)=\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

(Symbol Π means product)

Since each complex factor in the numerator and denominator can be thought of as a vector, the magnitude M and angle θ of $F(s)$ at any point s are

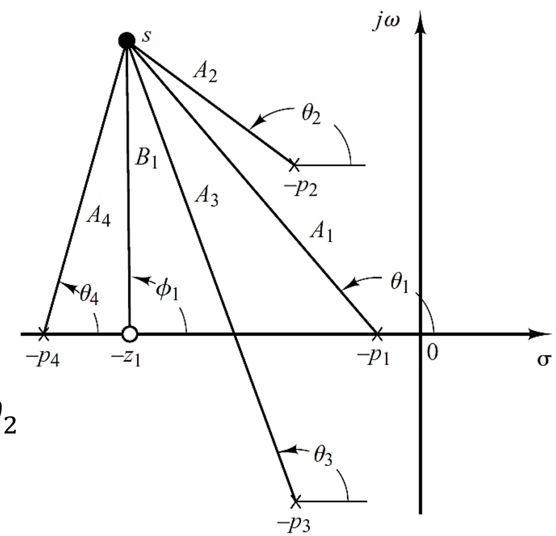
$$M=|F(s)|=\frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{j=1}^{n}|(s+p_j)|}=\frac{\prod \text{zero lengths}}{\prod \text{Pole lengths}}$$

$$\theta=\angle F(s)=\sum_{i=1}^{m}\angle(s+z_i)-\sum_{j=1}^{n}\angle(s+p_j)$$

$$=\sum \text{zero angles}-\sum \text{ploe angles}$$

$$M=\frac{B_1}{A_1A_2A_3A_4}$$

$$\theta=\phi_1-(\theta_1+\theta_2$$



- $|(s+a_i)|$ is the magnitude and $\angle(s+a_i)$ is the angle (measured from the **positive** extension of the real axis σ) of the vector drawn from the zero $-z_i$ or pole $-p_j$ of $F(s)$ to the point s.

Example

Using s-plane, find $F(s)$ at the point $s = -3 + j4$.

$$F(s) = \frac{(s+1)}{s(s+2)}$$

Answer: $M\angle\theta = 0.217\angle - 114.3°$

Infinite Poles and Zeros

A function $G(s)$ can have **poles and zeros at infinity**.

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- If the function $G(s)$ approaches **infinity** as s approaches **infinity**, then the function has a **pole at infinity**. For example,

$$G(s) = s \qquad \text{has a pole at infinity}$$

- If the function $G(s)$ approaches **zero** as s approaches **infinity**, then the function has a **zero at infinity**. For example,

$$G(s) = 1/s \qquad \text{has a zero at infinity}$$

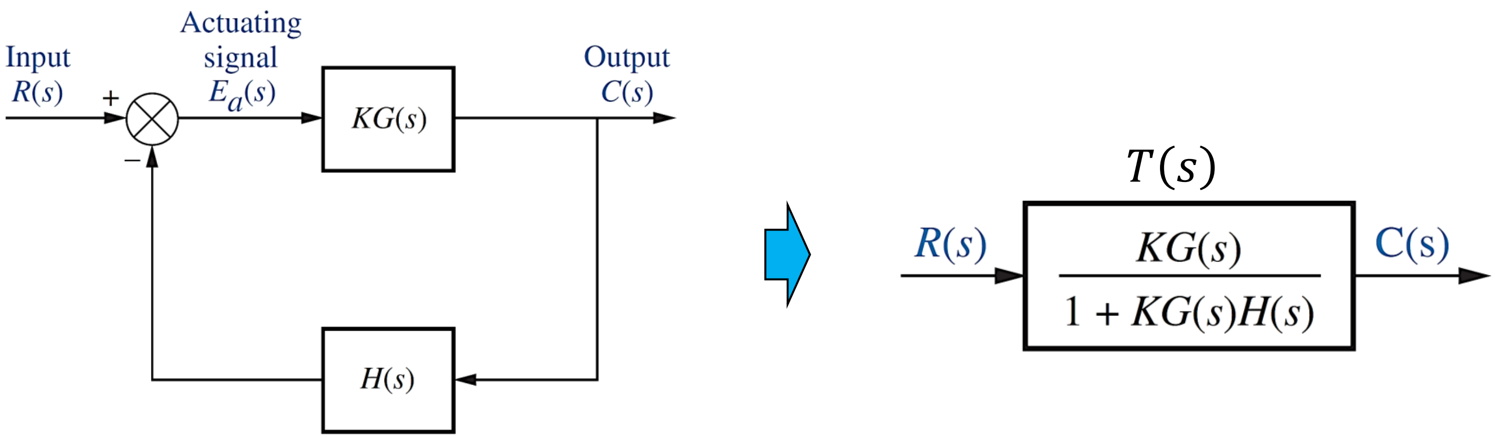
❖ Every function $G(s)$ has always an **equal number of poles and zeros** if we include the **infinite poles and zeros** as well as the **finite poles and zeros**. For example,

$$G(s) = \frac{1}{s(s+1)(s+2)} \qquad \text{has 3 finite poles and 3 infinite zeros.}$$

Defining Root Locus

Defining Root Locus

- The system's <u>transient response</u> and <u>stability</u> are dependent upon the poles of the closed-loop transfer function $T(s)$.

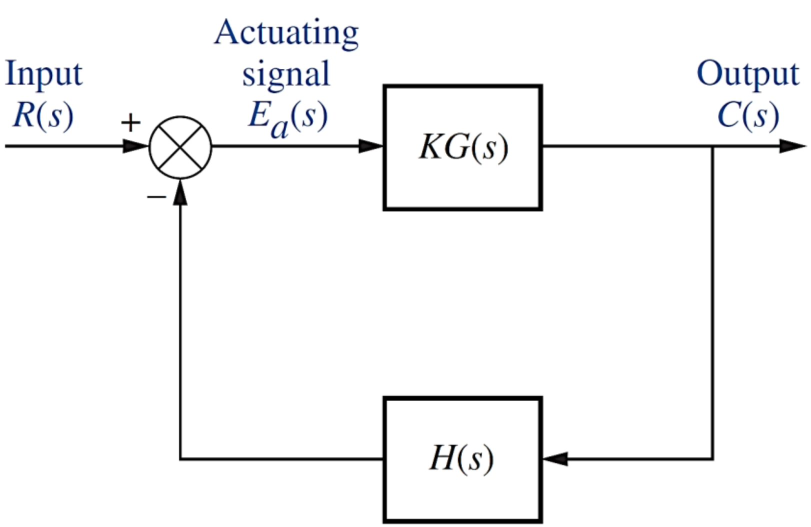


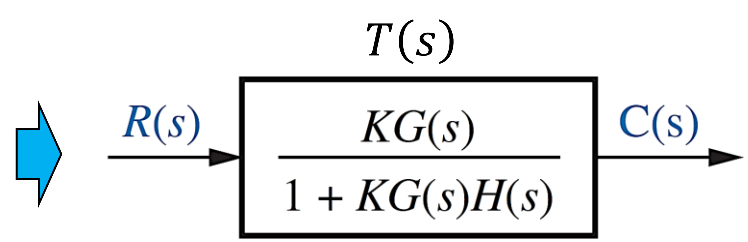
- Typically, the poles of the open-loop transfer function are easily found, but the poles of the closed-loop transfer function are more difficult to find, and they also change with changes in system gain K.
- We have no knowledge of the system's performance unless we find the roots of the denominator for specific values of K.
- The **Root Locus** will be used to give us a vivid picture of the poles of $T(s)$ as K varies.

Defining Root Locus

Root Locus is a graphical presentation of the closed-loop poles as **a system parameter** K is varied.

Root locus is a powerful method of analysis and design for **transient response** and **stability** of systems of order higher than 2 (but it can also be used for first- and second-order systems) without having to factor the denominator of the closed-loop transfer function.

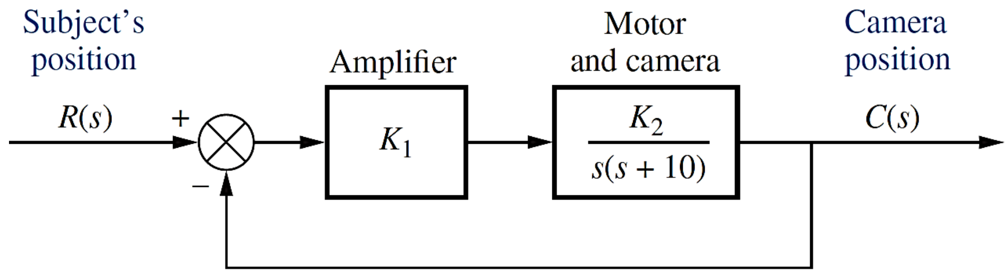


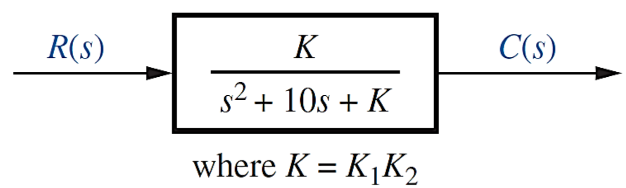


Terminology:
- Forward Transfer Function: $KG(s)$
- Feedback Transfer Function: $H(s)$
- Open-Loop Transfer Function: $KG(s)H(s)$
- Closed-Loop Transfer Function: $T(s)$

Defining Root Locus: An Example

Consider a camera with motor that can be used to follow moving objects automatically.



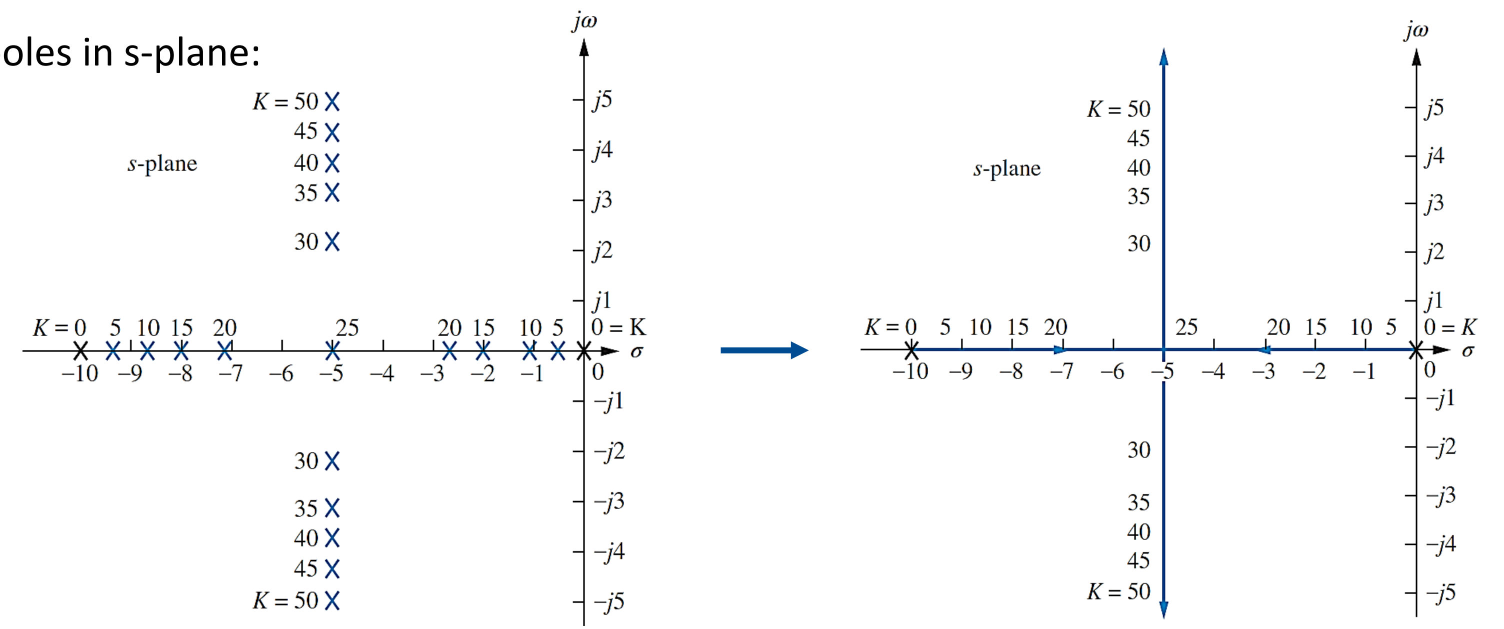


K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

Variation of closed-loop poles location for different values of gain K.

Defining Root Locus: An Example

The closed-loop poles in s-plane:



As the gain K increases, the closed-loop pole which is at -10 for $K = 0$, moves toward the right, and the closed-loop pole which is at 0 for $K = 0$, moves toward the left. They meet at -5, break away from the real axis, and move into the complex plane. One closed-loop pole moves upward while the other moves downward.

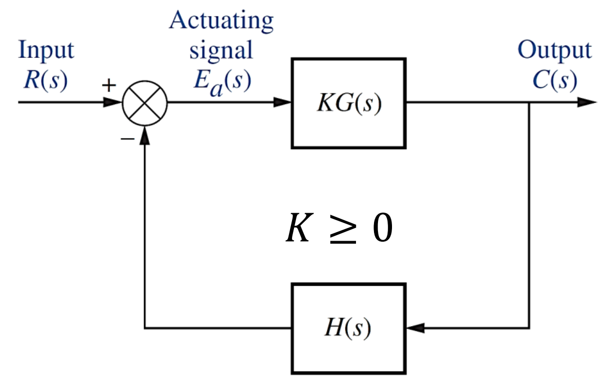
- This representation of the paths (solid lines) of the closed-loop poles as the gain K is varied is called **Root Locus**.

Sketching Root Locus

Properties of the Root Locus

The closed-loop transfer function for the system is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$



$K \geq 0$

For $K \geq 0$, a value of s is a pole of $T(s)$ if

$$KG(s)H(s) = -1 = 1\angle(2k+1)180° \qquad k = 0, \pm 1, \pm 2, \ldots$$

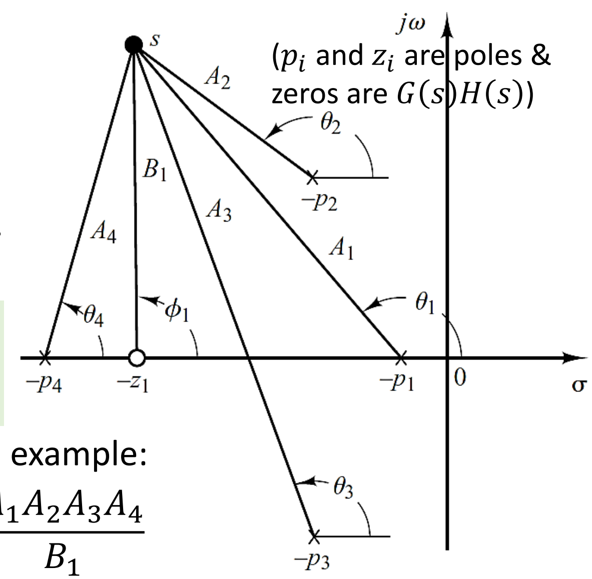
Since K is a positive scalar: $\Rightarrow$ $\angle G(s)H(s) = (2k+1)180°$, $K|G(s)H(s)| = 1$

Hence, point s is on the root locus for a particular value of gain K if

$$\angle G(s)H(s) = \sum \begin{matrix}\text{zero angles of}\\ G(s)H(s)\end{matrix} - \sum \begin{matrix}\text{pole angles of}\\ G(s)H(s)\end{matrix} = (2k+1)180°$$

$$K = \frac{1}{|G(s)H(s)|} = \frac{\prod \text{pole lengths of } G(s)H(s)}{\prod \text{zero lengths of } G(s)H(s)}$$

(Gain at point s)



(p_i and z_i are poles & zeros are $G(s)H(s)$)

In this example:

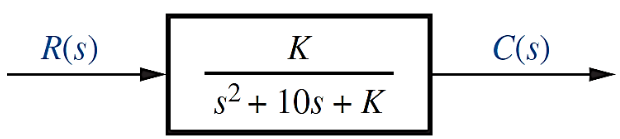
$$K = \frac{A_1 A_2 A_3 A_4}{B_1}$$

$$\phi_1 - (\theta_1 + \theta_2 + \theta_3 + \theta_4) = (2k+1)180°$$

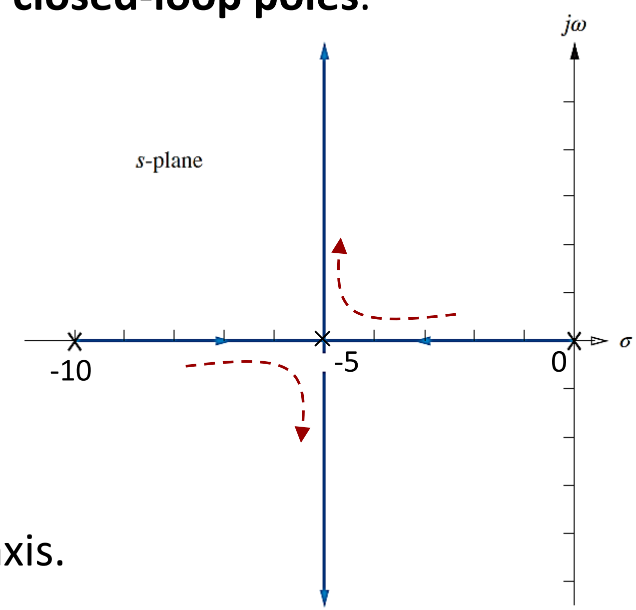
Sketching the Root Locus

There are 5 rules that can be used to rapidly **sketch** the root locus with minimal calculations. The sketch gives intuitive insight into the behavior of a control system. Once a sketch is obtained, it is possible to **refine the sketch** and accurately **plot** it by finding actual points or angles on the root locus, which are required for a particular problem.

1. Number of Branches: If we define a branch as the path that one pole traverses, the number of branches of the root locus equals the number of **closed-loop poles**.



There are **two** branches, one originates at the origin, the other at -10.



2. Symmetry: The root locus is symmetrical about the real axis.

Sketching the Root Locus

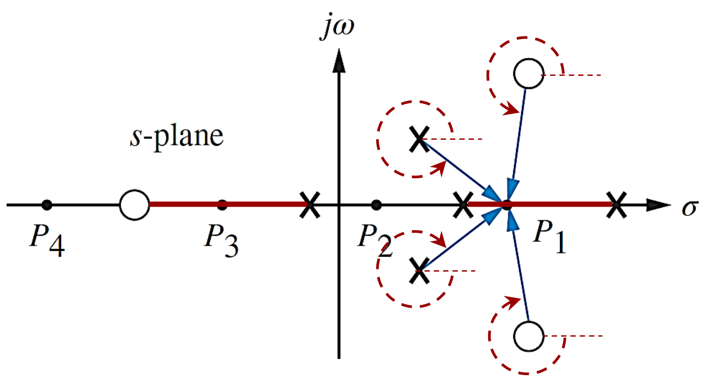
3. Real-Axis Segments: Based on the equation

$$\angle G(s)H(s) = \sum \begin{matrix}\text{zero angles of}\\ G(s)H(s)\end{matrix} - \sum \begin{matrix}\text{pole angles of}\\ G(s)H(s)\end{matrix} = (2k+1)180°$$

at each point P on the real axis σ:

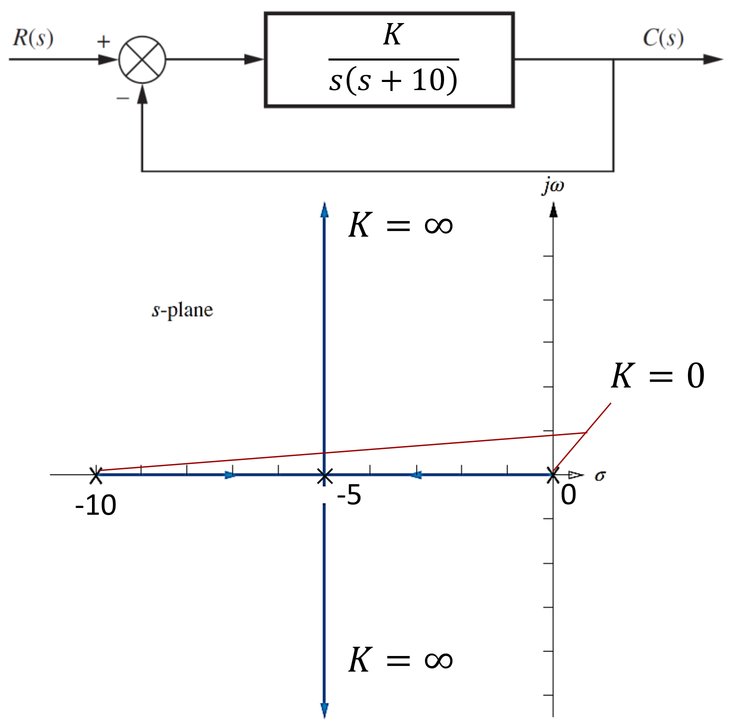
(1) the angular contribution of a pair of open-loop complex poles or zeros is 0,
(2) the angular contribution of the open-loop poles and zeros to the left of the respective point P is 0,
(3) the angular contribution of the even number of open-loop poles and zeros to the right of the respective point P is 0.

❖ Hence, on the real axis, for $K \geq 0$, the root locus exists to the left of an **odd number** of <u>real-axis</u>, <u>finite</u> open-loop ($G(s)H(s)$) poles and/or zeros.

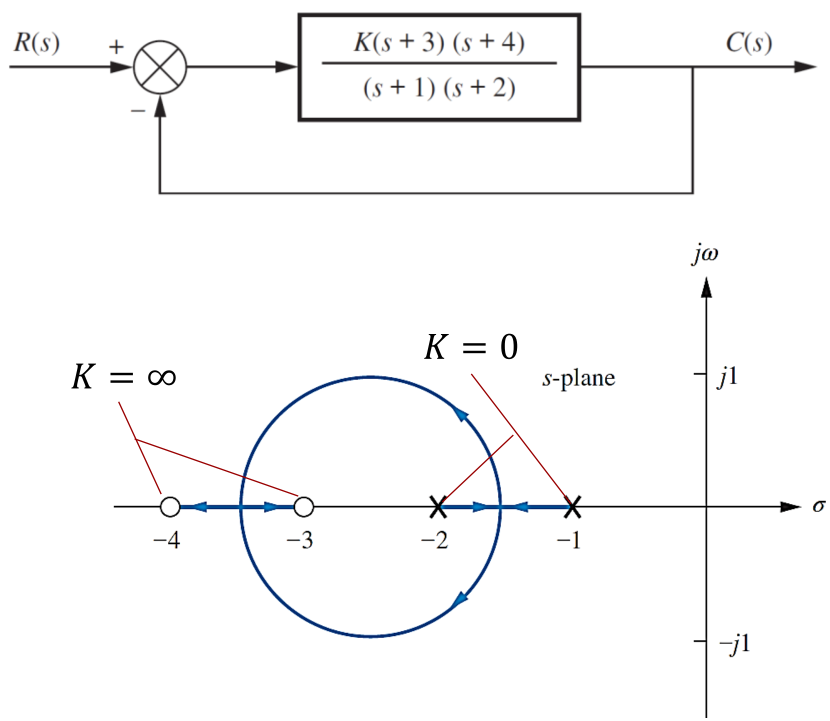


Sketching the Root Locus

4. Starting and Ending Points: The root locus **begins** at the finite and infinite poles of $G(s)H(s)$ where $K = 0$ and **ends** at the finite and infinite zeros of $G(s)H(s)$ where $K = \infty$.



There are two branches that begin at 0 and 10, and end at $\pm\infty$.



There are two branches that begin at -1 and -2, and end at -3 and -4.

Sketching the Root Locus

5. Behavior at Infinity: This rule tell us what the root locus looks like as it approaches the zeros at infinity or as it moves from the poles at infinity.

- The root locus approaches straight lines as **asymptotes** as the locus approaches infinity. The equation of the asymptotes is given by the real-axis intercept σ_a and angle θ_a as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\text{number of finite poles} - \text{number of finite zeros}}$$

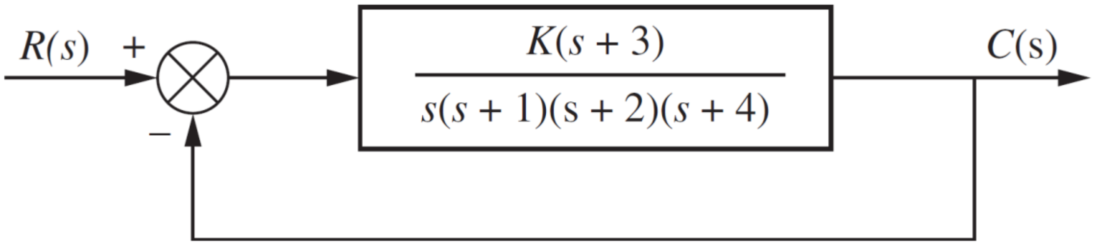
$$\theta_a = \frac{(2k+1)180^\circ}{\text{number of finite poles} - \text{number of finite zeros}}$$

where the angle θ_a is given with respect to the positive extension of the real axis and $k = 0, \pm 1, \pm 2, \ldots$ yields a multiplicity of lines that account for the many branches of a root locus that approach infinity.

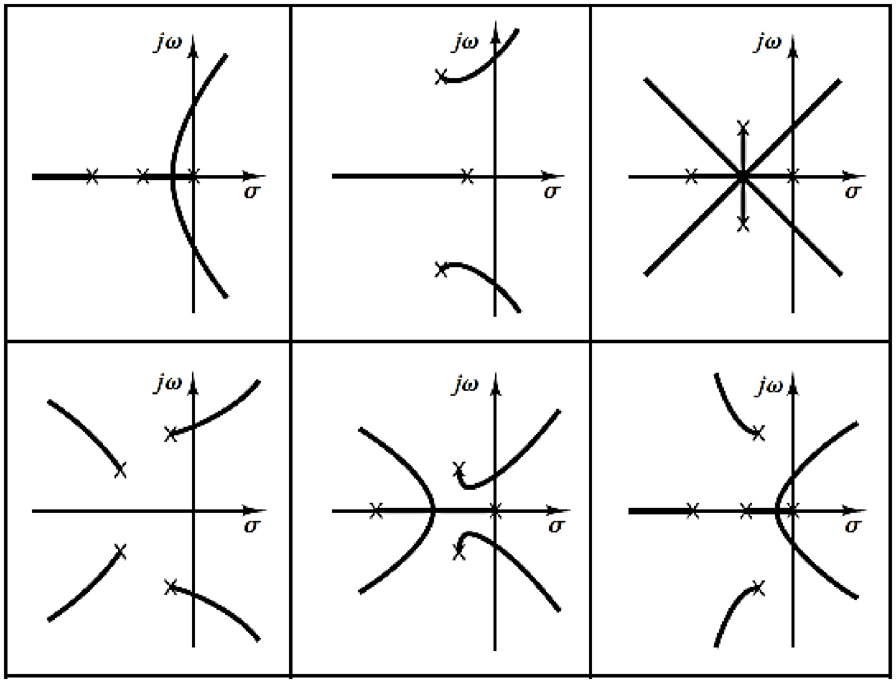
- If the number of poles equals the number of zeros, the root locus does not have any asymptotes.

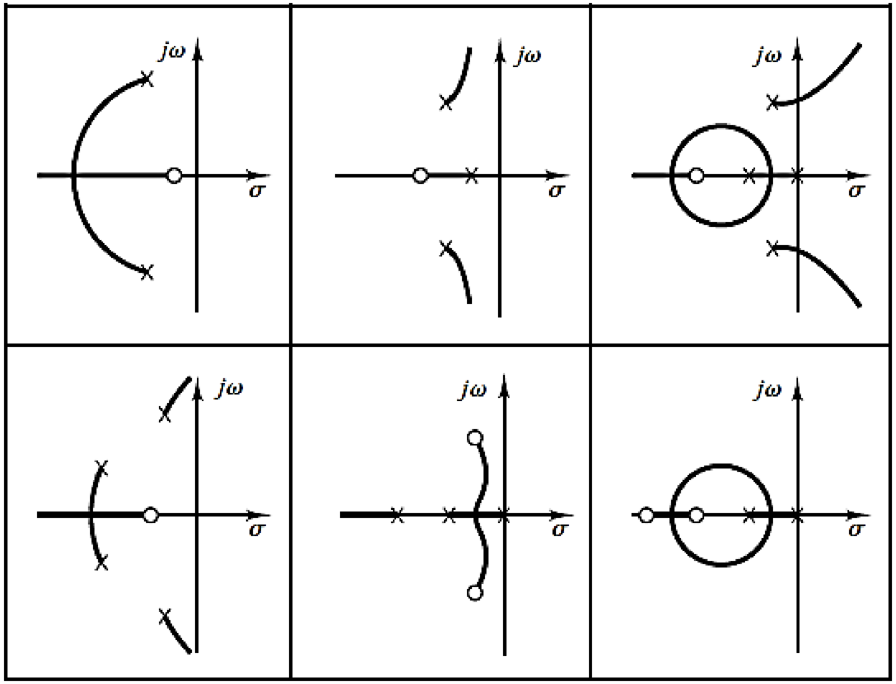
Example

Sketch the root locus for the system.



Typical Pole–Zero Configurations and Corresponding Root Loci





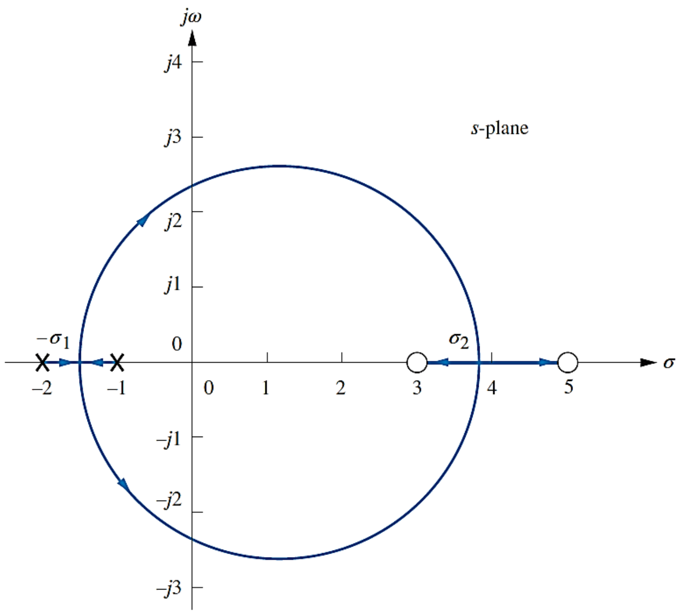
Refining Root Locus

Refining the Sketch

These 5 rules permit us to **sketch** a root locus rapidly. If we want more detail, we must be able to **accurately find important points** on the root locus along with their associated gain K.

1. Real-Axis Breakaway and Break-In Points:

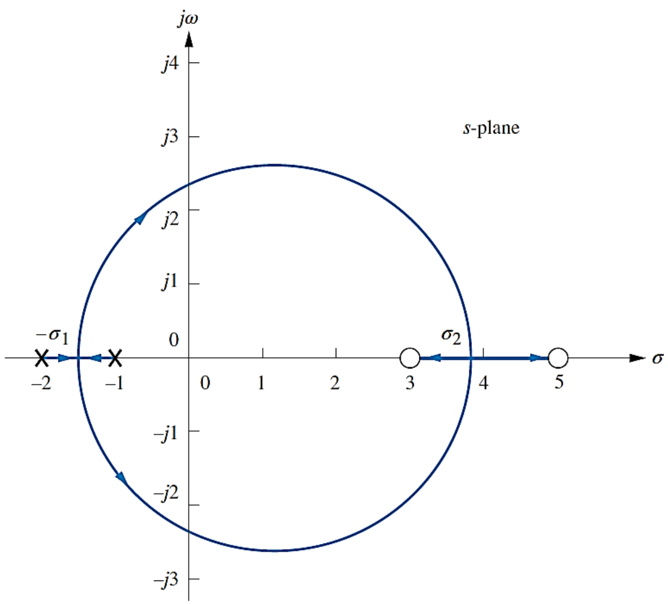
The point where the locus leaves the real axis (e.g., σ_1) is called the **breakaway point**, and the point where the locus returns to the real axis (e.g., σ_2) is called the **break-in point**.

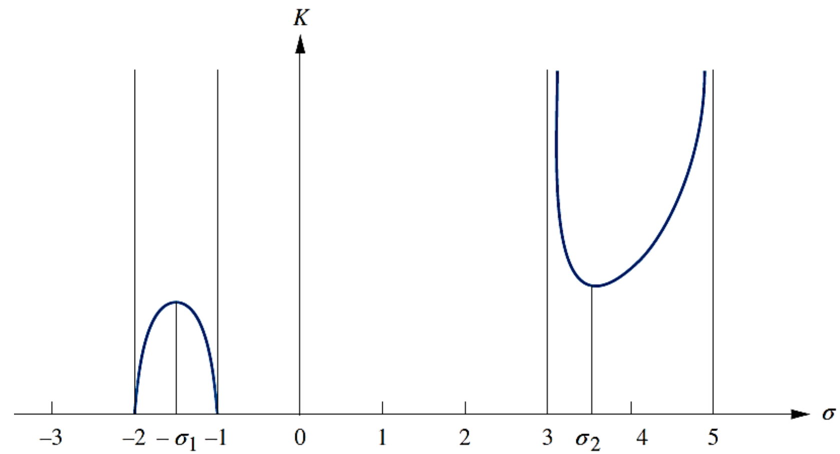


- Angle with Real Axis:
At the breakaway or break-in point, the branches of the root locus form an angle of $180°/n$ with the real axis, where n is the number of closed-loop poles arriving at or departing from the **single** breakaway or break-in point on the real axis.

Refining the Sketch

- Location on Real Axis: The **breakaway** point occurs at a point of **maximum** gain K on the real axis between the open-loop poles. The **break-in** point occurs at a point of **minimum** gain K on the real axis between the open-loop zeros.





(Variation of real-axis gain)

There ate two methods to find the location of these points on the real axis:

Refining the Sketch

Method 1: For points along the real-axis segment of the root locus where breakaway and break-in points could exist, $s = \sigma$. Hence,

$$K = -\frac{1}{G(\sigma)H(\sigma)}$$

By differentiating this equation with respect to σ and setting the derivative equal to zero (i.e., $dK/d\sigma = 0$), the points of maximum and minimum gain K, which are the breakaway and break-in points, can be found.

Method 2: Breakaway and break-in points satisfy the relationship:

$$\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{i=1}^{n} \frac{1}{\sigma + p_i}$$

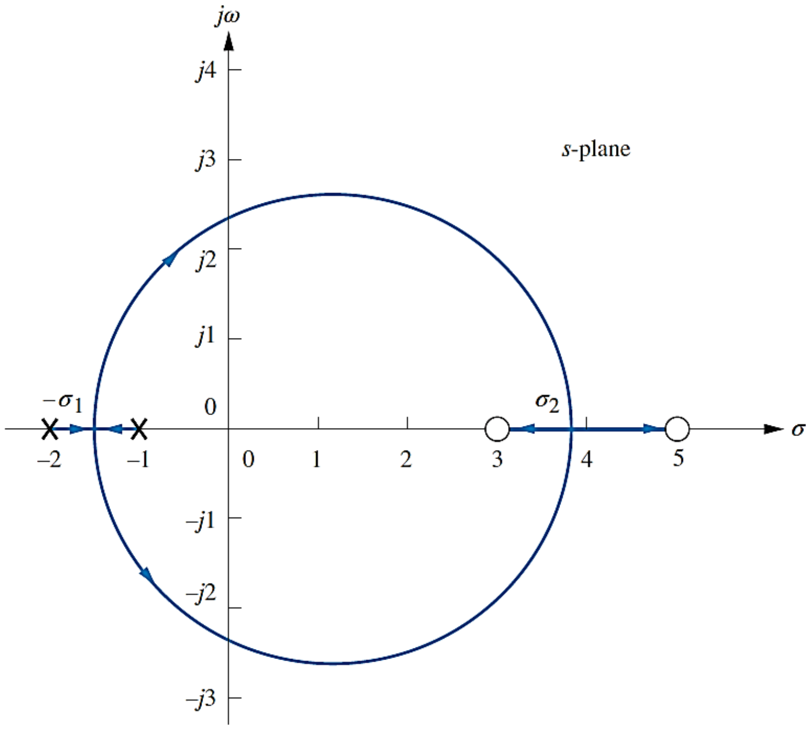
m: is number of zeros
n: is number of poles

$$G(s)H(s) = \frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)}$$

By solving this equation for σ, points of maximum and minimum gain K, which are the breakaway and break-in points, can be found without differentiating.

Example

Find the breakaway and break-in points for the following root locus.



Answer: $\sigma = -1.45, \sigma = 3.82$

Refining the Sketch

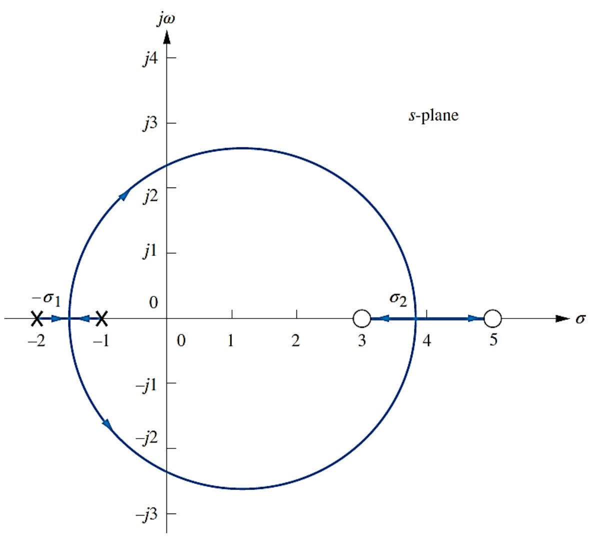
2. Imaginary-Axis ($j\omega$-Axis) Crossings:

The $j\omega$-axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of ω at the $j\omega$-axis crossing yields the frequency of oscillation at marginally stable operation.
There are two methods to find the points where the root loci cross the $j\omega$-axis:

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

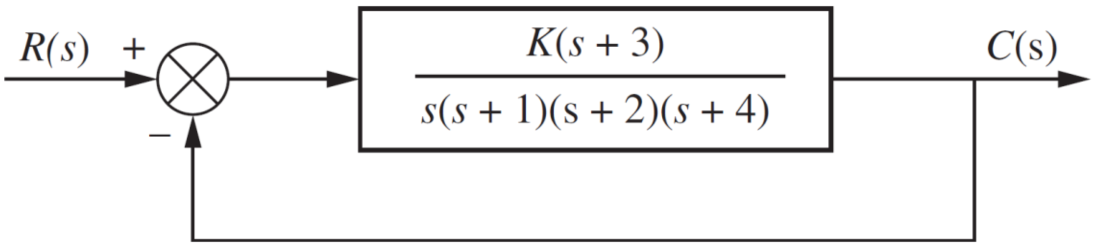
Method 1:
Using **Routh-Hurwitz Criterion** for $1 + KG(s)H(s) = 0$ as follows:
- Forcing a row of zeros in the Routh table will yield the gain K.
- Solving the even auxiliary polynomial for the roots yields the frequency ω at the $j\omega$-axis crossing.

Method 2: Let $s = j\omega$ in the equation $1 + KG(s)H(s) = 0$, equate both the real part and the imaginary part to zero, and then solve for ω and K.



Example

For the following system, find the frequency and gain K for which the root locus crosses the imaginary axis. For what range of K is the system stable?



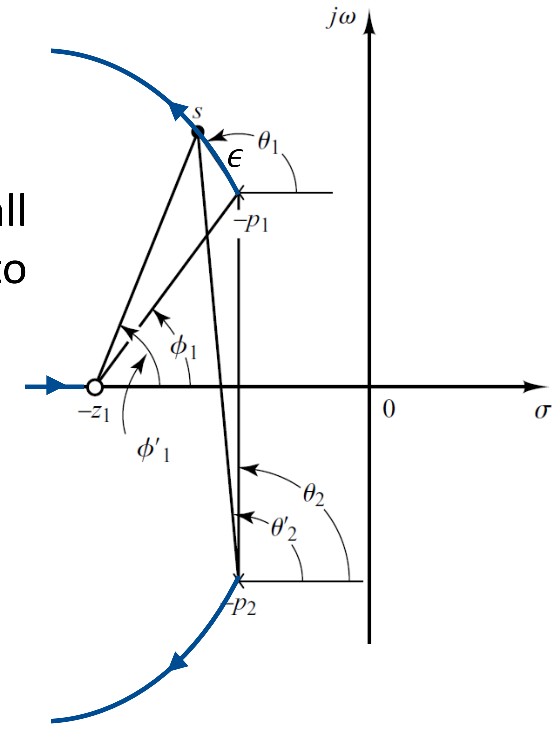
Answer: $K = 9.65$, $\omega = 1.59$, $0 \leq K < 9.65$

Refining the Sketch

3. Angles of Departure and Arrival (departure from complex poles and arrival to complex zeros):

Consider a test point s on the root locus in the very vicinity ϵ of a complex pole/zero. The sum of angles drawn from all finite poles and zeros to this test point must be an odd multiple of 180°.

In this example: $\phi_1' - (\theta_1 + \theta_2') = 180°(2k+1)$



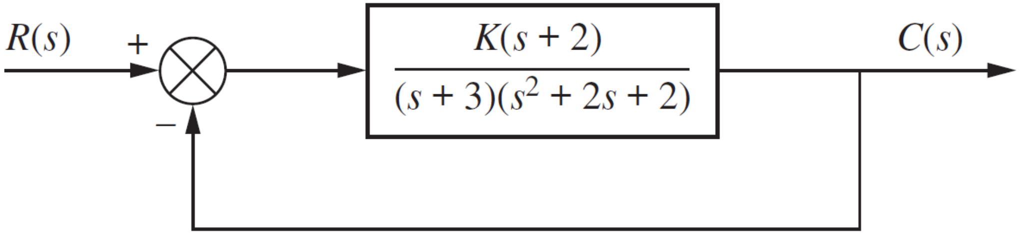
Since $\epsilon \to 0$, to find departure/arrival angle, it is assumed that all angles drawn from all other poles and zeros are drawn directly to the pole/zero that is near the test point s.

In this example, as $\epsilon \to 0$, $\theta_2' \to \theta_2$ and $\phi_1' \to \phi_1$, therefore,

$$\theta_1 = 180°(2k+1) - \theta_2 + \phi_1$$

Example

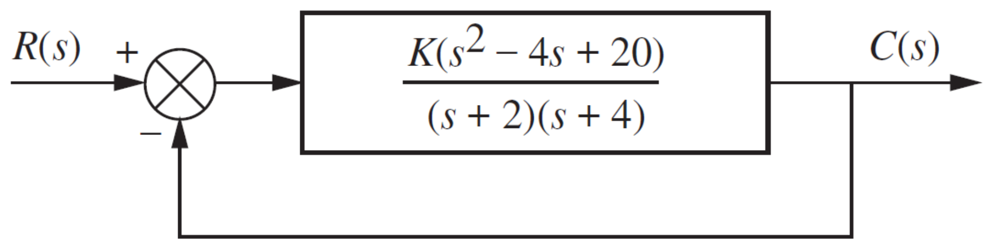
Given the following unity feedback system, find the angle of departure from the complex poles and sketch the root locus.



Answer: $\theta = -251.6^\circ$ or 108.4°

Example

Sketch the root locus for the system and find the following:
a. The exact point and gain where the locus crosses the $j\omega$-axis.
b. The range of K within which the system is stable.
c. The breakaway point on the real axis.

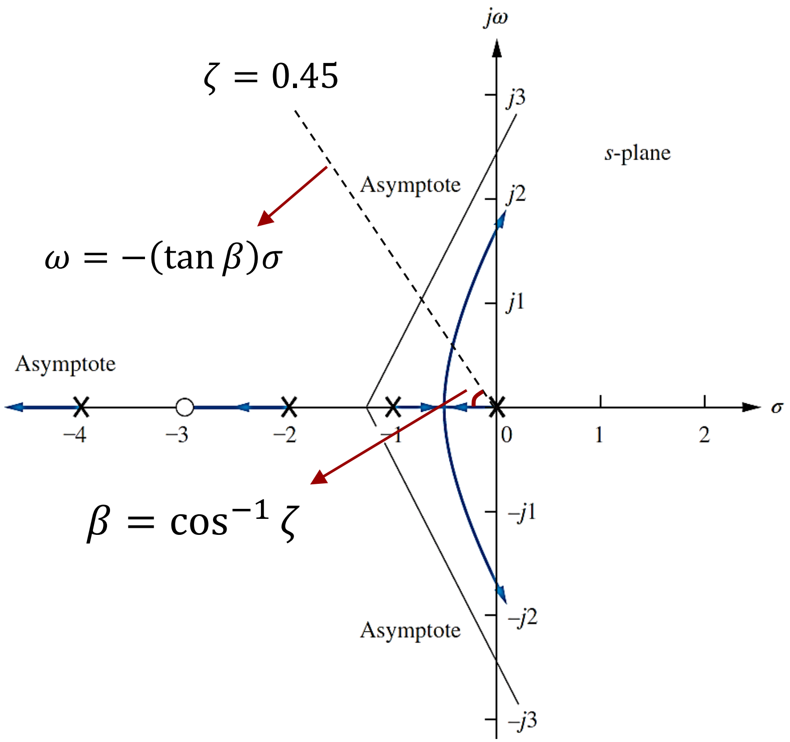


Answer: (a) $K = 1.5$, $\omega = 3.9$, (b) $0 \leq K < 1.5$, (c) $\sigma_a = -2.88$

Refining the Sketch

4. Locating a Point on Root Locus and Finding its Associated Gain K:

We may want to accurately locate some points on the root locus as well as find their associated gain K. For example, finding a pair of dominant complex-conjugate closed-loop poles such that $\zeta = 0.45$.



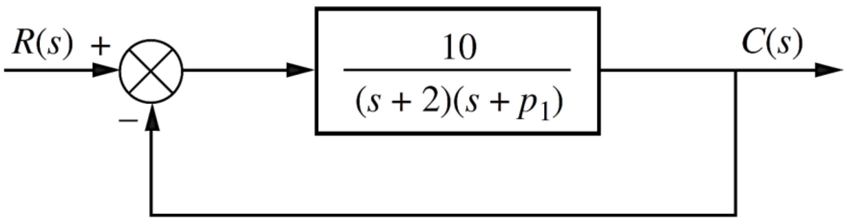
$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

$$1 + KG(s)H(s)\Big|_{s=\sigma+(-\tan\beta)\sigma j} = 0$$

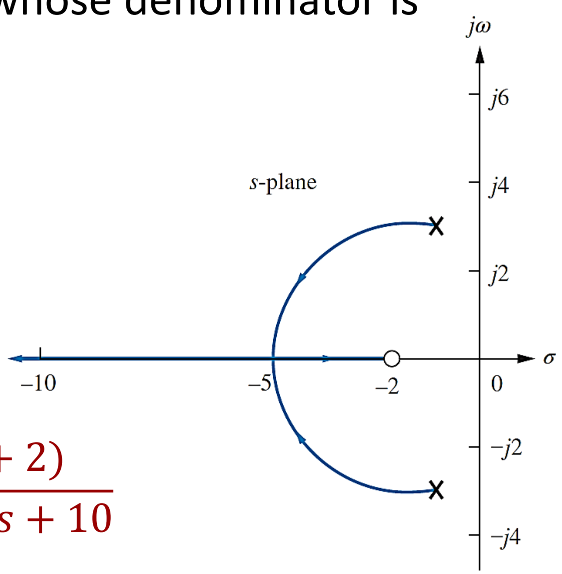
$\Rightarrow$ K, σ is calculated.

Generalized Root Locus

Sometimes we want to know how the closed-loop poles change as a function of <u>another parameter</u> (not the forward-path gain K). For example, in this system the parameter of interest is the open-loop pole at p_1.



The solution to this problem is to create an equivalent system whose denominator is in the form of $1 + p_1 G(s)H(s)$.



$$T(s) = \frac{10}{s^2 + (p_1 + 2)s + 2p_1 + 10} = \frac{10}{s^2 + 2s + 10 + p_1(s+2)}$$

$$T(s) = \frac{\dfrac{10}{s^2 + 2s + 10}}{1 + p_1 \dfrac{(s+2)}{s^2 + 2s + 10}} \equiv \frac{G(s)}{1 + p_1 G(s)H(s)}$$

$$\Rightarrow G(s)H(s) = \frac{(s+2)}{s^2 + 2s + 10}$$

Using MATLAB and Control System Toolbox

Plotting Root Locus Using rlocus

```
num=[1 2];
den=[1 2 3];
GH = tf(num,den);

rlocus(GH)
% or
% K = 0:0.001:10;
% rlocus(GH,K)

zeta = 0.3:0.3:0.9;
wn = 0:1:4;
sgrid(zeta,wn)
```

$$G(s)H(s) = \frac{s+2}{s^2+2s+3}$$

The root locus can be drawn over a grid that shows constant damping ratio (ζ) and constant natural frequency (ω_n) curves.