Ch9: Sinusoids and Phasors

Sinusoids

Alternating Current (AC)

Thus far our analysis has been limited for the most part to **DC Circuits**: those circuits excited by constant or time-invariant sources. We now begin the analysis of **Alternating Current (AC) Circuits** in which the source voltage or current is time-varying.

In the late 1800's, there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission from the power generating plant to the consumer.

A sinusoidal current has alternately positive and negative values at regular time intervals.

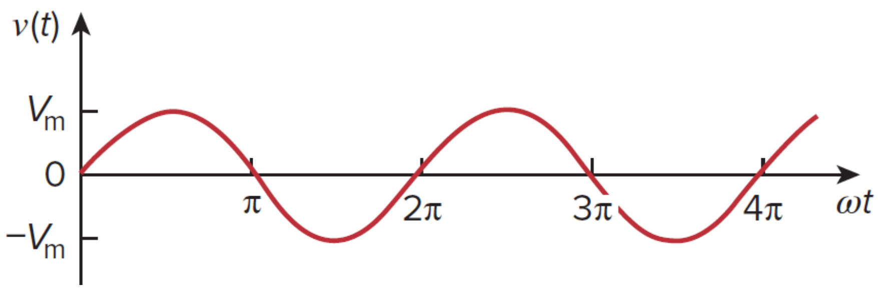
A sinusoidal signal is easy to generate and transmit. Thus, it is the dominant form of electrical power that is delivered to homes and industry. Moreover, they are very easy to handle mathematically.

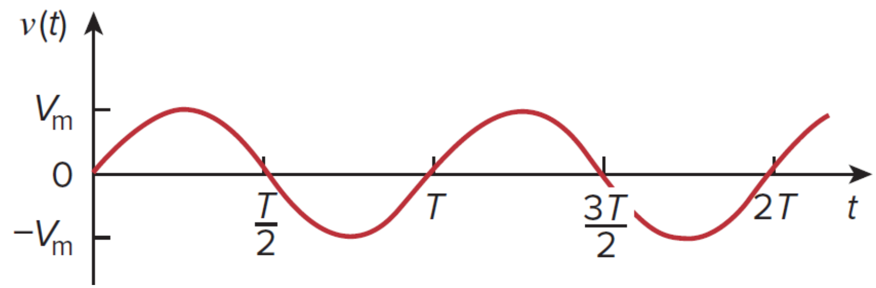
Sinusoids

A sinusoidal voltage may be represented as:

$$v(t) = V_m \sin \omega t$$

V_m: Amplitude of the sinusoid
ω: Angular frequency in radians/s
ωt: Argument of the sinusoid





- It is evident that the sinusoid $v(t)$ repeats itself every T seconds; thus, T is called the **period** of the sinusoid.

$$T = \frac{2\pi}{\omega}$$

Period T is
the time of one complete cycle or
the number of seconds per cycle.

Sinusoids

- The fact that $v(t)$ repeats itself every T seconds can be shown:

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t+\frac{2\pi}{\omega}\right)$$
$$= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t)$$

Hence, $v(t+T) = v(t)$ $\Rightarrow$ v has the same value at $t+T$ as it does at t and $v(t)$ is said to be **periodic**.

- The reciprocal of period T is the number of cycles per second, which is known as the **cyclic frequency** f of the sinusoid. Thus,

$$f = \frac{1}{T}$$ Unit: Hertz (Hz)

- It is often useful to refer to frequency f in angular terms: $\omega = 2\pi f$

  Here the angular frequency ω is in radians per second (rad/s).

Sinusoids

In general, we need to account for relative timing of one wave versus another. This can be done by including a **phase** shift ϕ. Therefore, a more general expression for the sinusoid is

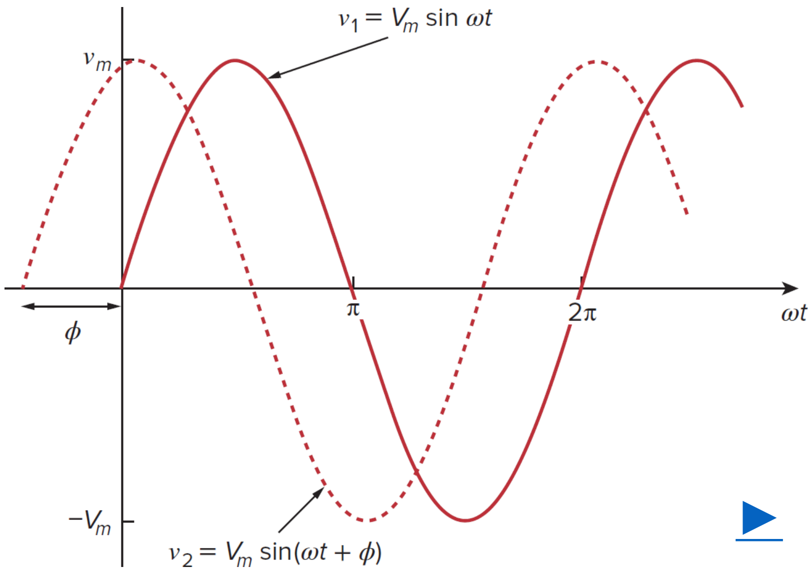
$$v(t) = V_m \sin(\omega t + \phi)$$

Both argument $(\omega t + \phi)$ and phase ϕ can be in radians or degrees.

- Consider the two sinusoids:

  $$v_1(t) = V_m \sin \omega t \qquad v_2(t) = V_m \sin(\omega t + \phi)$$

  - If $\phi \neq 0$, we say that v_1 and v_2 are **out of phase**. If $\phi = 0$, then v_1 and v_2 are said to be **in phase**.
  - If starting point of v_2 occurs first in time ($\phi > 0$), we say that v_2 **leads** v_1 by ϕ or that v_1 **lags** v_2 by ϕ.



Sinusoids

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes using the following trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\longrightarrow$$

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos \omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin \omega t$$

$$A\cos \omega t + B\sin \omega t = C\cos(\omega t - \theta)$$
$$C = \sqrt{A^2 + B^2}, \theta = \tan^{-1}\frac{B}{A}$$

Example

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10°)\text{V}$$

Example

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50°)$ and $v_2 = 12\sin(\omega t - 10°)$. State which sinusoid is leading.

Phasors

Introduction: Complex Numbers

- A complex number z can be represented in **rectangular form** as

$$z = x + jy$$

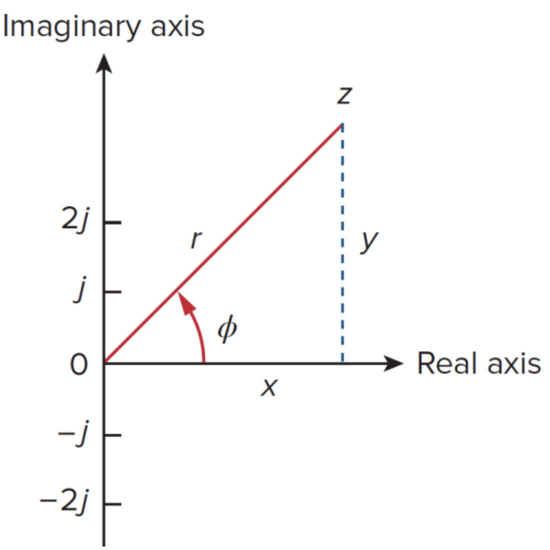
where $j = \sqrt{-1}$, x is the real part of z, and y is the imaginary part of z.

- The complex number z can also be written in **polar form** or **exponential form** as

$$z = r\angle\phi = re^{j\phi}$$

where r is the magnitude of z, and ϕ is the phase of z.

$z = x + jy$	Rectangular form
$z = r\angle\phi$	Polar form
$z = re^{j\phi}$	Exponential form



Introduction: Complex Numbers

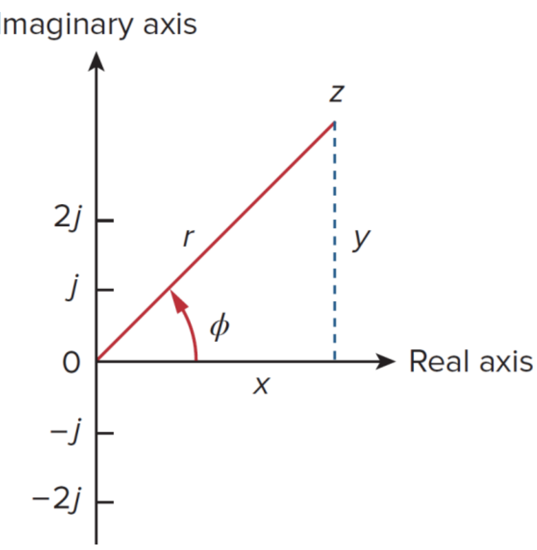
The different forms can be interconverted.

- Given x and y, we can get r and ϕ as

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

- Given r and ϕ, we can get x and y as

$$x = r\cos\phi\,, \qquad y = r\sin\phi$$



$$\boxed{z = x + jy = r\angle\phi = re^{j\phi} = r(\cos\phi + j\sin\phi)}$$

Introduction: Complex Numbers

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication, division, reciprocal, and square root are better done in polar form. Given the complex numbers $z_1 = x_1 + jy_1 = r_1\angle\phi_1$ and $z_2 = x_2 + jy_2 = r_2\angle\phi_2$:

- Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

- Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

- Multiplication

$$z_1 z_2 = r_1 r_2\angle(\phi_1 + \phi_2)$$

- Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

- Reciprocal

$$\frac{1}{z} = \frac{1}{r}\angle(-\phi) \qquad \frac{1}{j} = -j$$

- Square Root

$$\sqrt{z} = \sqrt{r}\angle(\phi/2)$$

Complex Conjugate of complex number z:

$$z^* = x - jy = r\angle-\phi = re^{-j\phi}$$

Example

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Phasors

- Sinusoids are easily expressed in terms of **Phasors**, which are more convenient to work with than sine and cosine functions. A **Phasor** is a complex number that represents the **amplitude** (or magnitude) and **phase** of a **sinusoid**.
- **Phasors** provide a simple means of analyzing linear circuits excited by sinusoidal sources.
- The idea of a phasor representation is based on Euler's identity:

$$e^{j\phi} = \cos\phi + j\sin\phi$$

$$\cos\phi = \mathrm{Re}\left(e^{j\phi}\right)$$
$$\sin\phi = \mathrm{Im}\left(e^{j\phi}\right)$$

Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we can represent it as the real component of a vector in the complex plane:

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}\left(V_m e^{j(\omega t + \phi)}\right) = \mathrm{Re}\left(V_m e^{j\phi} e^{j\omega t}\right)$$

$$\Rightarrow \quad v(t) = \mathrm{Re}\left(\mathbf{V} e^{j\omega t}\right), \qquad \boxed{\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi}$$

$\mathbf{V}$ is the phasor representation of the sinusoid $v(t) = V_m \cos(\omega t + \phi)$.

Phasors

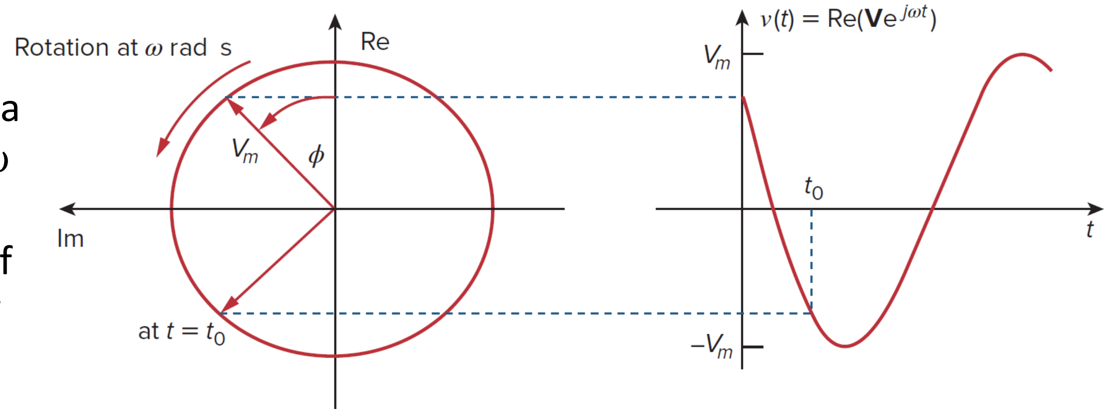
Thus, to get the phasor corresponding to a sinusoid, we first express the sinusoid in the **cosine form** so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$, and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the **sinusoid from the time domain** to the **phasor domain**.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(time-domain representation) (phasor-domain representation)

(term $e^{j\omega t}$ is implicitly present)

As time increases, $V_m e^{j(\omega t+\phi)}$ rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction. $v(t)$ is its projection on the real axis. The value of $V_m e^{j(\omega t+\phi)}$ at time $t = 0$ is the phasor of the sinusoid $v(t)$.



Sinusoid-Phasor Transformation

$v(t)$ Time domain representation	$\mathbf{V}$ Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \underline{/\phi}$
$V_m \sin(\omega t + \phi)$	$V_m \underline{/\phi - 90^\circ}$
$I_m \cos(\omega t + \theta)$	$I_m \underline{/\theta}$
$I_m \sin(\omega t + \theta)$	$I_m \underline{/\theta - 90^\circ}$

Note: The frequency ω of the phasor is not explicitly shown in the phasor diagram. For this reason, phasor domain is also known as **frequency domain**.

Summary:
1. $v(t)$ is the instantaneous or time domain representation, while $\mathbf{V}$ is the frequency or phasor domain representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not.
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

Sinusoid-Phasor Transformation

- Applying a derivative to a phasor yields:

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90°)$$
$$= \mathrm{Re}\left(\omega V_m e^{j\omega t} e^{j\phi} e^{j90°}\right) = \mathrm{Re}\left(j\omega \mathbf{V} e^{j\omega t}\right)$$

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi$$

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega \mathbf{V}$$

(time-domain) (phasor-domain)

- Similarly, applying an integral to a phasor yields:

$$\int v dt \quad \Leftrightarrow \quad \frac{\mathbf{V}}{j\omega}$$

(time-domain) (phasor-domain)

Example

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40°)$ A
(b) $v = -4\sin(30t + 50°)$ V

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20°}$ V

Example

Given $i_1(t) = 4\cos(\omega t + 30°)\mathrm{A}$ and $i_2(t) = 5\sin(\omega t - 20°)\mathrm{A}$, find their sum using the phasor approach.

Example

Using the phasor approach, determine the steady-state current $i(t)$ in a circuit described by the following integrodifferential equation

$$4i(t) + 8\int i(t)dt - 3\frac{di(t)}{dt} = 50\cos(2t + 75°)$$

Assume that $\omega = 2$.

Phasor and Circuit Elements

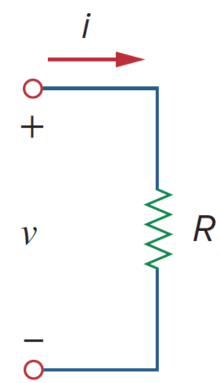
Phasor Relationships for Resistors

Each circuit element (e.g., R, L, and C) has a relationship between its current and voltage. These can be mapped into phasor relationships.

$$i = I_m \cos(\omega t + \phi)$$
$$v = iR = RI_m \cos(\omega t + \phi)$$

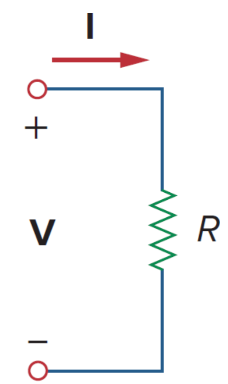
$$\mathbf{I} = I_m \angle \phi$$
$$\mathbf{V} = RI_m \angle \phi$$

$$\boxed{\mathbf{V} = R\mathbf{I}}$$



$v = iR$

time domain

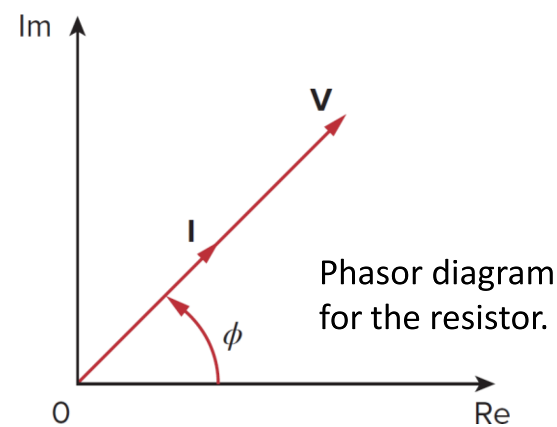


$\mathbf{V} = \mathbf{I}R$

frequency domain

Notes:

- The voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain.
- The voltage and current are in phase with each other.



Phasor diagram for the resistor.

Phasor Relationships for Inductors

$$i = I_m \cos(\omega t + \phi)$$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

$$= \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

$$\mathbf{I} = I_m \angle \phi$$

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)}$$

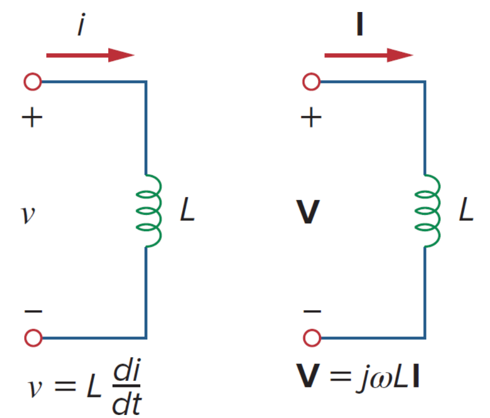
$$= \omega L I_m e^{j\phi} e^{j90^\circ}$$

$$= \omega L I_m \angle(\phi + 90^\circ)$$

$$= j\omega L I_m e^{j\phi}$$

$$= j\omega L \mathbf{I}$$

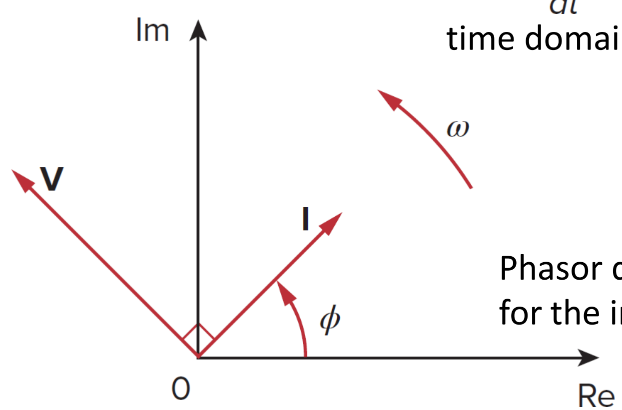
$$\boxed{\mathbf{V} = j\omega L \mathbf{I}}$$



time domain

frequency domain

Notes: The voltage and current are 90° out of phase. Specifically, the <u>current</u> **lags** the <u>voltage</u> by 90° (or the <u>voltage</u> **leads** the <u>current</u> by 90°).



Phasor diagram for the inductor.

Phasor Relationships for Capacitors

$$v = V_m \cos(\omega t + \phi)$$

$$i = C\frac{dv}{dt}$$

$$= -\omega C V_m \sin(\omega t + \phi)$$

$$= \omega C V_m \cos(\omega t + \phi + 90^\circ)$$

⇨

$$\boldsymbol{V} = V_m \angle \phi$$

$$\boldsymbol{I} = \omega C V_m e^{j(\phi + 90^\circ)}$$

$$= \omega C V_m e^{j\phi} e^{j90^\circ}$$

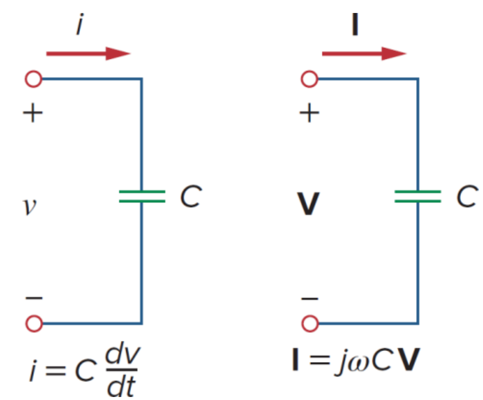
$$= \omega C V_m \angle(\phi + 90^\circ)$$

$$= j\omega C V_m e^{j\phi}$$

$$= j\omega C \mathbf{V}$$

⇨

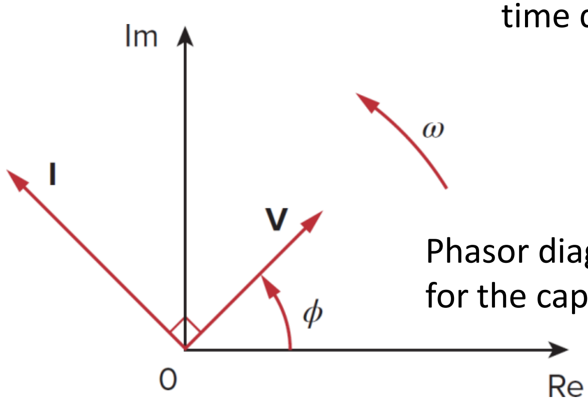
$$\boxed{\mathbf{V} = \frac{\mathbf{I}}{j\omega C}}$$



time domain

frequency domain

Notes: The voltage and current are 90° out of phase. Specifically, the <u>current</u> **leads** the <u>voltage</u> by 90° (or the <u>voltage</u> **lags** the <u>current</u> by 90°).



Phasor diagram for the capacitor.

Summary: Time-Domain and Phasor-Domain Voltage-Current Relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Example

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Impedance and Admittance

Impedance and Admittance

It is possible to expand Ohm's law to capacitors and inductors. In frequency domain, it is straightforward.

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

The **Impedance** $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\boldsymbol{V}$ to the phasor current $\boldsymbol{I}$, measured in ohms (Ω).

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I}$$

- $\mathbf{Z}$ is a frequency-dependent quantity and represents the opposition that the circuit exhibits to the flow of sinusoidal current.

The **Admittance** $\boldsymbol{Y}$ is the reciprocal of impedance, measured in siemens (S).

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$$

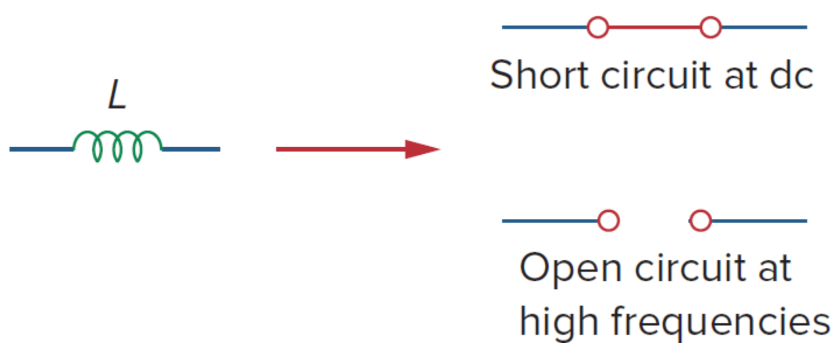
Note: The values obtained for Impedance and Admittance are only valid at that frequency ω.

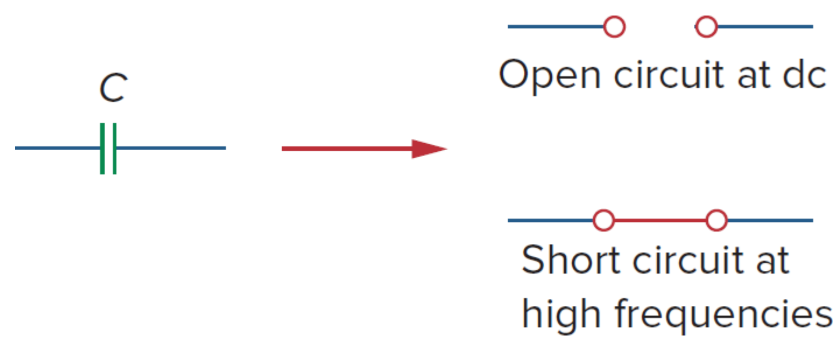
Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Impedance and Admittance

Consider two extreme cases of angular frequency ω.

- When $\omega = 0$ (i.e., for DC sources), $\boldsymbol{Z}_L = 0$ and $\boldsymbol{Z}_C \rightarrow \infty$, confirming that the inductor acts like a short circuit, while the capacitor acts like an open circuit.
- When $\omega \rightarrow \infty$ (i.e., for high frequencies), $\boldsymbol{Z}_L \rightarrow \infty$ and $\boldsymbol{Z}_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit.





Impedance and Admittance

As a complex quantity, the impedance may be expressed in rectangular or polar form.

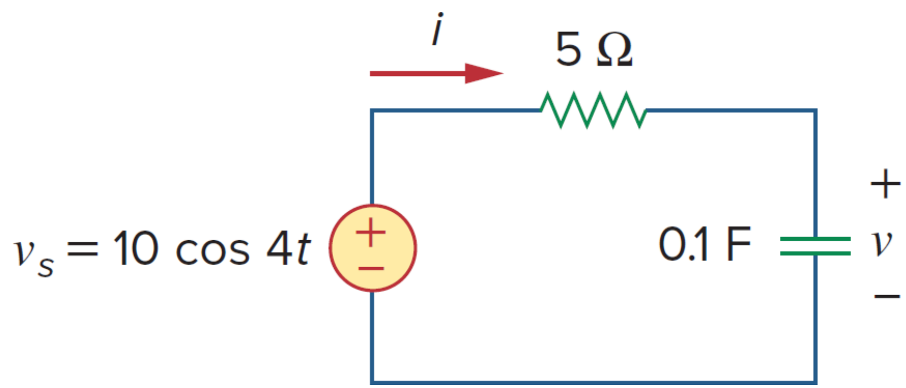
$$\mathbf{Z} = R \pm jX \qquad \text{or} \qquad \mathbf{Z} = |\mathbf{Z}| \angle \theta$$

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{\pm X}{R} \qquad\qquad R = |\mathbf{Z}| \cos\theta\,, \qquad X = |\mathbf{Z}| \sin\theta$$

- The real part R is called ***resistance*** (in ohms) and the imaginary part X is called the ***reactance*** (in ohms).
- The impedance $Z = R + jX$ is said to be inductive or lagging since current lags voltage, while impedance $Z = R - jX$ is capacitive or leading because current leads voltage.
- As a complex quantity, the admittance may be expressed in rectangular form as $\mathbf{Y} = G + jB$, where G is called the ***conductance*** (in siemens) and B is called the ***susceptance*** (in siemens).

Example

Find $v(t)$ and $i(t)$ in the circuit.



Kirchoff's Laws

Kirchoff's Laws in Frequency Domain

A powerful aspect of phasors is that Kirchoff's laws apply to them as well. This means that a circuit transformed to frequency domain can be evaluated by the same methodology developed for KVL and KCL.

KVL: Let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop at time t, then,

$$v_1 + v_2 + \cdots + v_n = 0$$

If $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$ are the phasor forms of the sinusoids $v_1, v_2, \ldots, v_n$, then,

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

KCL: Let $i_1, i_2, \ldots, i_n$ be the current leaving or entering a closed surface in a network at time t, then,

$$i_1 + i_2 + \cdots + i_n = 0$$

If $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ are the phasor forms of the sinusoids $i_1, i_2, \ldots, i_n$, then,

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0$$

Impedance Combinations

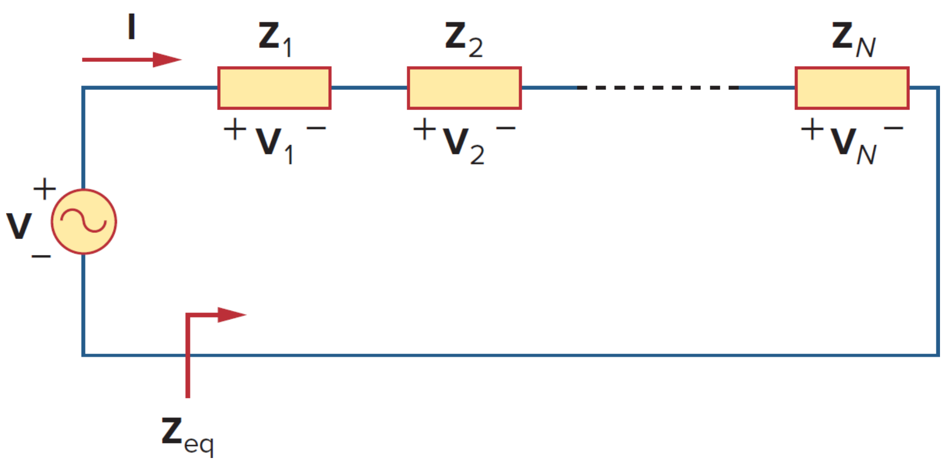
Impedance Series Combinations

Consider the N series-connected impedances shown. The same current $\mathbf{I}$ flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{\mathrm{eq}} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$



Note: The total or equivalent impedance of **series-connected impedances** is the **sum of the individual impedances**. This is similar to the series connection of **resistances**.

Impedance Series Combinations

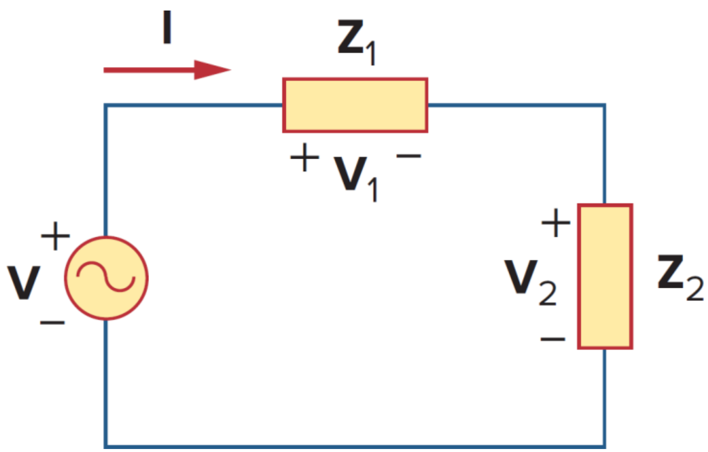
For example, if $N = 2$,

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

Because $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

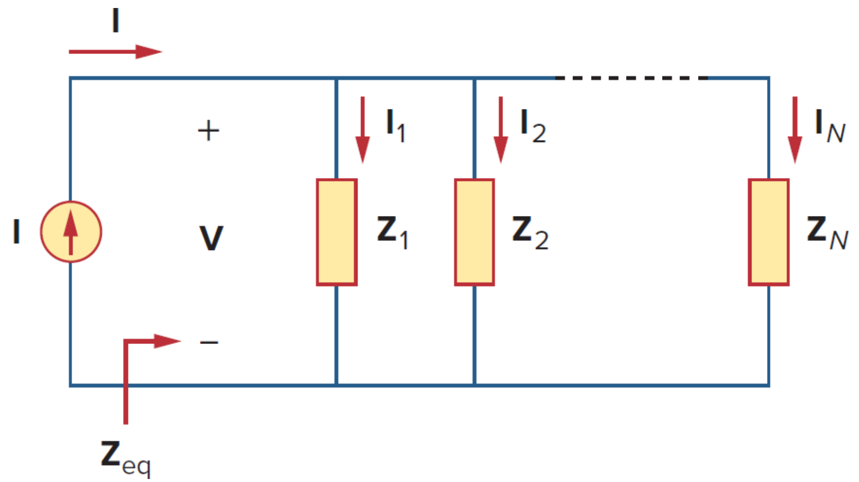
which is the voltage-division relationship.



Impedance Parallel Combination

Consider the N parallel-connected impedances shown. The voltage $\mathbf{V}$ across each impedance is the same. Applying KCL at the top node gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$



The equivalent impedance at the input terminals is

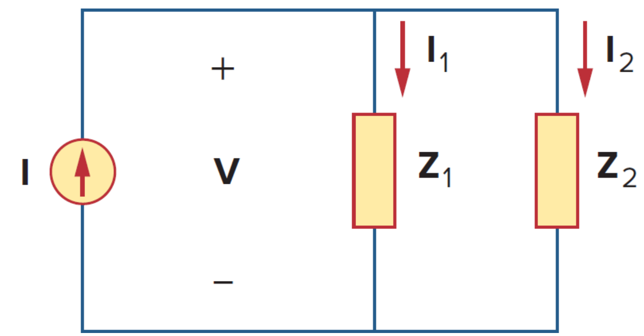
$$\boxed{\frac{1}{\mathbf{Z}_{\text{eq}}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}}$$

$$\mathbf{Y}_{\text{eq}} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N$$

Impedance Parallel Combination

For example, if $N = 2$,

$$\mathbf{Z}_{\text{eq}} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$



Since

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{\text{eq}} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

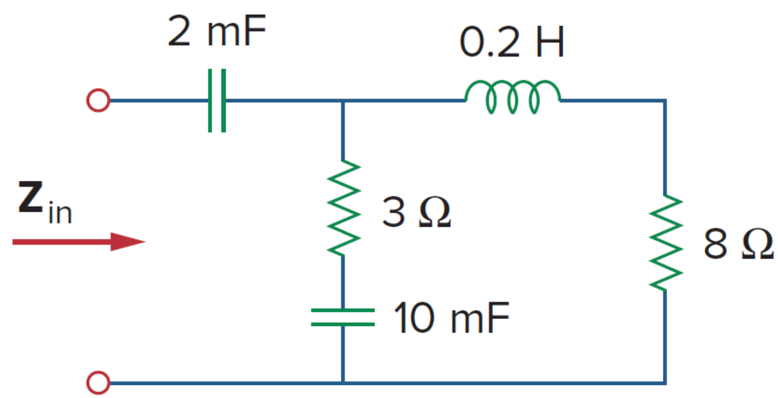
the currents in the impedances are

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

which is the current-division relationship.

Example

Find the input impedance of the circuit. Assume that the circuit operates at $\omega = 50$ rad/s.



Example

Determine $v_o(t)$ in the circuit.