

Ch2: Laplace Transform Review

Contents:

Laplace Transform

Partial-Fraction Expansion

Using MATLAB and Control System Toolbox

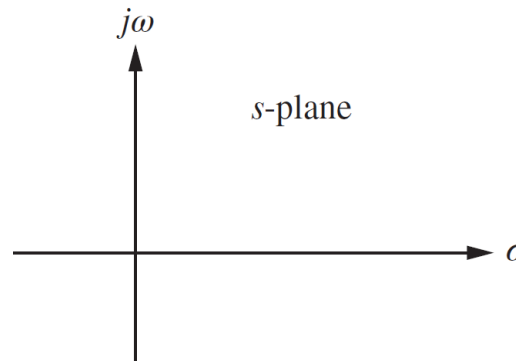
Laplace Transform

Laplace Transform

The definition of the unilateral (or one-sided) **Laplace Transform** is:

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

where $s = \sigma + j\omega$ is a **complex variable** (with real numbers σ and ω) and 0^- is a value just before $t = 0$ (which is applicable for discontinuous functions like impulse function or discontinuous initial conditions of differential equations at $t = 0$).



Note: The Laplace Transform exists if there exists a real number σ_1 such that:

$$\lim_{t \rightarrow \infty} |f(t)e^{-\sigma_1 t}| = 0$$

Example

Find the Laplace transform of $f(t) = Ae^{-at}$ ($t \geq 0$).

Answer: $F(s) = \frac{A}{s + a}$

Inverse Laplace Transform

Finding $f(t)$ from $F(s)$:

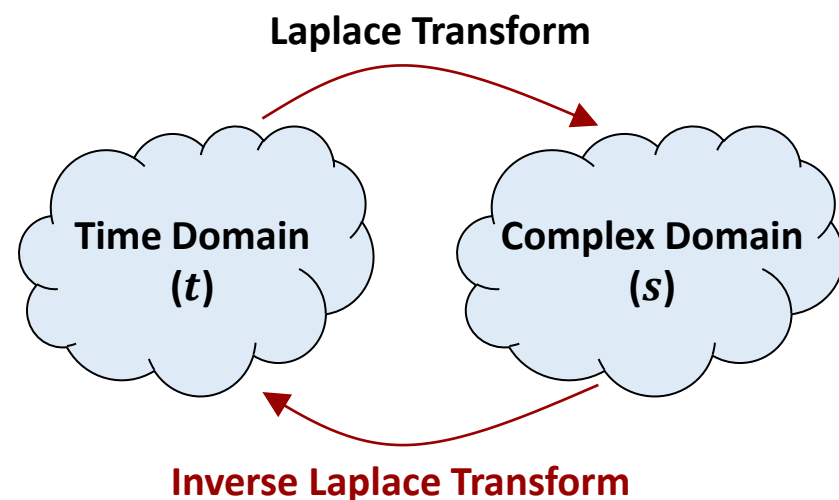
$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds = f(t) \quad (t \geq 0)$$

where σ , the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount σ from it. This path of integration is to the right of all singular points.

Evaluating the inversion integral appears complicated. In practice, we frequently use the **Laplace Transform Theorems** and **Partial-Fraction Expansion Method** for transforming between $f(t)$ and $F(s)$.

Laplace Transform Pairs

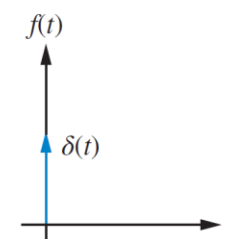
$f(t), t > 0$	$F(s)$
(impulse) $\delta(t)$	1
(step) $u(t)$	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$



$$\delta(t) = \infty \text{ for } 0- < t < 0+$$

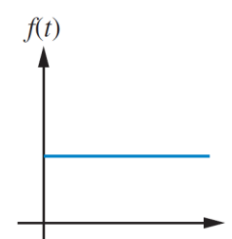
$$= 0 \text{ elsewhere}$$

$$\int_{0-}^{0+} \delta(t) dt = 1$$



$$u(t) = 1 \text{ for } t > 0$$

$$= 0 \text{ for } t < 0$$



Laplace Transform Theorems

No.	Theorem	Name
1.	$\mathcal{L}[k_1 f_1(t) + k_2 f_2(t)] = k_1 F_1(s) + k_2 F_2(s)$	Linearity theorem
2.	$\mathcal{L}[e^{-at} f(t)] = F(s + a)$	Frequency shift theorem
3.	$\mathcal{L}[f(t - T)] = e^{-sT} F(s)$	Time shift theorem
4.	$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
5.	$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^-)$	Differentiation theorem
6.	$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - sf(0^-) - f'(0^-)$	Differentiation theorem
7.	$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{k-1}(0^-)$	Differentiation theorem
8.	$\mathcal{L}\left[\int_{0^-}^t f(\tau) d\tau\right] = \frac{F(s)}{s}$	Integration theorem

Laplace Transform Theorems

No.	Theorem	Name
9.	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$	Multiplication by time
10.	$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}, (n = 1, 2, \dots)$	Multiplication by time
11.	$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem ¹
12.	$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem ²
13.	$\mathcal{L}^{-1}[F_1(s)F_2(s)] = f_1(t) * f_2(t)$	Convolution Integral ³

¹ For this theorem to yield correct finite results, all **roots** of the denominator of $F(s)$ must have **negative real parts**, and no more than one can be at the origin.

² For this theorem to be valid, $f(t)$ must be **continuous** or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

³ $f_1(t) * f_2(t) = \int_0^t f_1(t - \tau)f_2(\tau)d\tau = \int_0^t f_1(\tau)f_2(t - \tau)d\tau$ and $f_1(t)$ and $f_2(t)$ are 0 for $t < 0$.

Example

$$f(t) = 1 + 2 \sin \omega t \quad \mathcal{L}\{f(t)\} = ?$$

$$f(t) = A \sin(t - t_d) \quad \mathcal{L}\{f(t)\} = ?$$

$$f(t) = Ae^{-at} \sin \omega t \quad \mathcal{L}\{f(t)\} = ?$$

$$F(s) = \frac{1}{(s+3)^2} \quad \mathcal{L}^{-1}\{F(s)\} = ?$$

$$F(s) = \frac{1}{s^2(s-a)} \quad \mathcal{L}^{-1}\{F(s)\} = ?$$

Answers:

$$F(s) = \frac{s^2 + 2\omega s + \omega^2}{s^3 + \omega^2 s}$$

$$F(s) = \frac{A}{s^2 + 1} e^{-st_d}$$

$$F(s) = \frac{A\omega}{(s+a)^2 + \omega^2}$$

$$f(t) = e^{-3t} t$$

$$f(t) = \frac{1}{a^2} (e^{at} - at - 1)$$

Partial-Fraction Expansion

Partial-Fraction Expansion

To find the **inverse Laplace transform** of a complicated function $F(s) = N(s)/D(s)$, we can **convert** the function to a sum of **simpler terms** for which we know the Laplace transform of each term using the Tables and Theorems.

If the order of $N(s)$ is less than the order of $D(s)$, then a Partial-Fraction Expansion can be made. If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then first $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator (i.e., $F(s) = R(s) + N(s)/D(s)$).

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} \longrightarrow F(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Based on roots of $D(s)$ there are three cases:

Case 1: Roots of the Denominator of $F(s)$ Are Real and Distinct

Case 2: Roots of the Denominator of $F(s)$ Are Real and Repeated

Case 3: Roots of the Denominator of $F(s)$ Are Complex or Imaginary

Case 1: Real and Distinct Roots

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_i) \cdots (s + p_n)}$$
$$= \frac{K_1}{(s + p_1)} + \frac{K_2}{(s + p_2)} + \cdots + \frac{K_i}{(s + p_i)} + \cdots + \frac{K_n}{(s + p_n)}$$

Note: Order of $N(s)$ is less than the order of $D(s)$.

K_i is constant and called Residue.

$$K_i = (s + p_i)F(s) \Big|_{s \rightarrow -p_i} = \frac{(s + p_i)N(s)}{(s + p_1)(s + p_2) \cdots (s + p_i) \cdots (s + p_n)} \Big|_{s \rightarrow -p_i}, i = 1, \dots, n$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = K_1 e^{-p_1 t} + \cdots + K_i e^{-p_i t} + \cdots + K_n e^{-p_n t} \quad \text{for } t \geq 0$$

Example

$$F(s) = \frac{2}{(s+1)(s+2)}$$

$$\mathcal{L}^{-1}\{F(s)\} = ?$$

Answer:

$$f(t) = 2e^{-t} - 2e^{-2t} \quad t \geq 0$$

Case 2: Real and Repeated Roots

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)^r (s + p_2) \cdots (s + p_n)}$$

Note: Order of $N(s)$ is less than the order of $D(s)$.

$$= \frac{K_1}{(s + p_1)^r} + \underbrace{\frac{K_2}{(s + p_1)^{r-1}} + \cdots + \frac{K_r}{(s + p_1)}}_{\text{(Each multiple root generates additional terms consisting of denominator factors of reduced multiplicity)}} + \frac{K_{r+1}}{(s + p_2)} + \cdots + \frac{K_n}{(s + p_n)}$$

(Each multiple root generates additional terms consisting of denominator factors of reduced multiplicity)

$$\left\{ \begin{array}{l} K_1, K_{r+1}, \dots, K_n \text{ can be found using the method explained in Case 1.} \\ K_2, \dots, K_r \text{ can be found using: } K_i = \frac{1}{(i-1)!} \frac{d^{i-1} \{(s + p_1)^r F(s)\}}{ds^{i-1}} \bigg|_{s \rightarrow -p_1}, i = 2, \dots, r \end{array} \right.$$

Note: For finding $f(t)$, we know $\mathcal{L}^{-1} \left\{ \frac{1}{(s + a)^n} \right\} = e^{-at} \frac{t^{n-1}}{(n-1)!}$

Example

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = ?$$

Answer:

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t} \quad t \geq 0$$

Case 3: Complex or Imaginary Roots

Method 1

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s^2 + as + b) \dots}$$

$$= \frac{K_1}{(s + p_1)} + \frac{(K_2s + K_3)}{(s^2 + as + b)} + \dots \quad (*)$$

Note: Order of $N(s)$ is less than the order of $D(s)$.

K_1 can be found using the method explained in **Case 1**.

K_2 and K_3 can be found by multiplying both sides of equation (*) by $D(s)$ and balancing coefficients of both sides of equation:

$$\frac{N(s)}{D(s)} = \frac{K_1}{(s + p_1)} + \frac{(K_2s + K_3)}{(s^2 + as + b)} + \dots \Rightarrow N(s) = K_1(s^2 + as + b) + (K_2s + K_3)(s + p_1) + \dots$$

Note: For finding $f(t)$, we know

$$\left\{ \begin{aligned} s^2 + as + b &= \left(s + \frac{a}{2}\right)^2 + b - \left(\frac{a}{2}\right)^2 = (s + \sigma)^2 + \omega^2, \\ \mathcal{L}^{-1} \left\{ \frac{A(s + \sigma) + B\omega}{(s + \sigma)^2 + \omega^2} \right\} &= Ae^{-\sigma t} \cos \omega t + Be^{-\sigma t} \sin \omega t \end{aligned} \right.$$

Example

$$F(s) = \frac{3}{s(s^2 + 2s + 5)}$$

$$\mathcal{L}^{-1}\{F(s)\} = ?$$

Answer:

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) \quad t \geq 0$$

Case 3: Complex or Imaginary Roots

Method 2 (optional)

The techniques described for real roots, i.e., **Case 1** and **Case 2**, can be also used for complex and imaginary roots.

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s^2 + as + b) \dots} = \frac{K_1}{(s + p_1)} + \frac{K_2}{(s + \sigma + j\omega)} + \frac{K_3}{(s + \sigma - j\omega)} + \dots$$

- K_1 and K_2 can be found using the method explained in **Case 1**, and K_3 will be the **complex conjugate** of K_2 .
- Using this general method, inverse Laplace transform of a function with Repeated Complex or Imaginary Roots can be also found using the method explained in **Case 2**.

Note: For finding $f(t)$, we know

$$\left\{ \begin{array}{l} \frac{e^{j\omega t} + e^{-j\omega t}}{2} = \cos \omega t \\ \frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \sin \omega t \end{array} \right.$$

Example

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} \quad \mathcal{L}^{-1}\{F(s)\} = ?$$

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2}{s + 1 + j2} + \frac{K_3}{s + 1 - j2}$$

K_1 , K_2 , and K_3 can be found using method explained in Case 1:

$$K_1 = \frac{3}{5}, \quad K_2 = \frac{3}{s(s + 1 - j2)} \Big|_{s \rightarrow -1 - j2} = -\frac{3}{20}(2 + j1), \quad K_3 \text{ is complex conjugate of } K_2: K_3 = -\frac{3}{20}(2 - j1)$$

$$F(s) = \frac{3}{5} \frac{1}{s} - \frac{3}{20} \left(\frac{2 + j1}{s + 1 + j2} + \frac{2 - j1}{s + 1 - j2} \right)$$

$$f(t) = \frac{3}{5} - \frac{3}{20} [(2 + j1)e^{-(1+j2)t} + (2 - j1)e^{-(1-j2)t}]$$

$$f(t) = \frac{3}{5} - \frac{3}{20} e^{-t} \left[4 \left(\frac{e^{j2t} + e^{-j2t}}{2} \right) + 2 \left(\frac{e^{j2t} - e^{-j2t}}{2j} \right) \right] \Rightarrow f(t) = \frac{3}{5} - \frac{3}{5} e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) \quad t \geq 0$$

Example

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s + 1)(s + 2)}$$

$$\mathcal{L}^{-1}\{G(s)\} = ?$$

Answer:

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t} \quad t \geq 0$$

Example

$$\mathcal{L}^{-1} \left\{ \frac{s^2 + 2s + 3}{(s + 1)^3} \right\} = ?$$

Answer:

$$(1 + t^2)e^{-t} \quad t \geq 0$$

Example: Solving an ODE Using Laplace Transform

$$\ddot{x} + 2\dot{x} + 5x = 3, \quad \dot{x}(0) = 0, \quad x(0) = 0$$

Answer:

$$x(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\cos 2t - \frac{3}{10}e^{-t}\sin 2t$$

Using MATLAB and Control System Toolbox

Laplace and Inverse Laplace Transforms Using laplace, ilaplace

```
syms s t w A a
```

```
F = laplace(A*exp(-a*t) * sin(w*t));  
f = ilaplace(1/(s + 3)^2);
```

$$f(t) = Ae^{-at} \sin \omega t \quad \mathcal{L}\{f(t)\} = ?$$

$$F(s) = \frac{1}{(s + 3)^2} \quad \mathcal{L}^{-1}\{F(s)\} = ?$$

Partial Fraction Expansion/Decomposition Using residue

`[r,p,k] = residue(b,a)` finds the residues, poles, and direct term of a Partial Fraction Expansion of the ratio of two polynomials:

$$\frac{b(s)}{a(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{a_ns^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} = \frac{r_n}{s - p_n} + \dots + \frac{r_2}{s - p_2} + \frac{r_1}{s - p_1} + k(s)$$

where $b = [b_m \dots b_1 b_0]$ and $a = [a_n \dots a_1 a_0]$ are coefficients of the polynomials, $r = [r_n \dots r_2 r_1]$ are the residues, $p = [p_n \dots p_2 p_1]$ are the poles, and k is a polynomial.

`[b,a] = residue(r,p,k)` converts the partial fraction expansion back to the ratio of two polynomials and returns the coefficients in b and a .

Example: $F(s) = \frac{-4s + 8}{s^2 + 6s + 8} = \frac{-12}{s + 4} + \frac{8}{s + 2}$

