

Ch4: Equilibrium of Rigid Bodies

Contents:

Equilibrium and Free-Body Diagrams

Equilibrium in Two Dimensions

Two Special Cases

Equilibrium in Three Dimensions

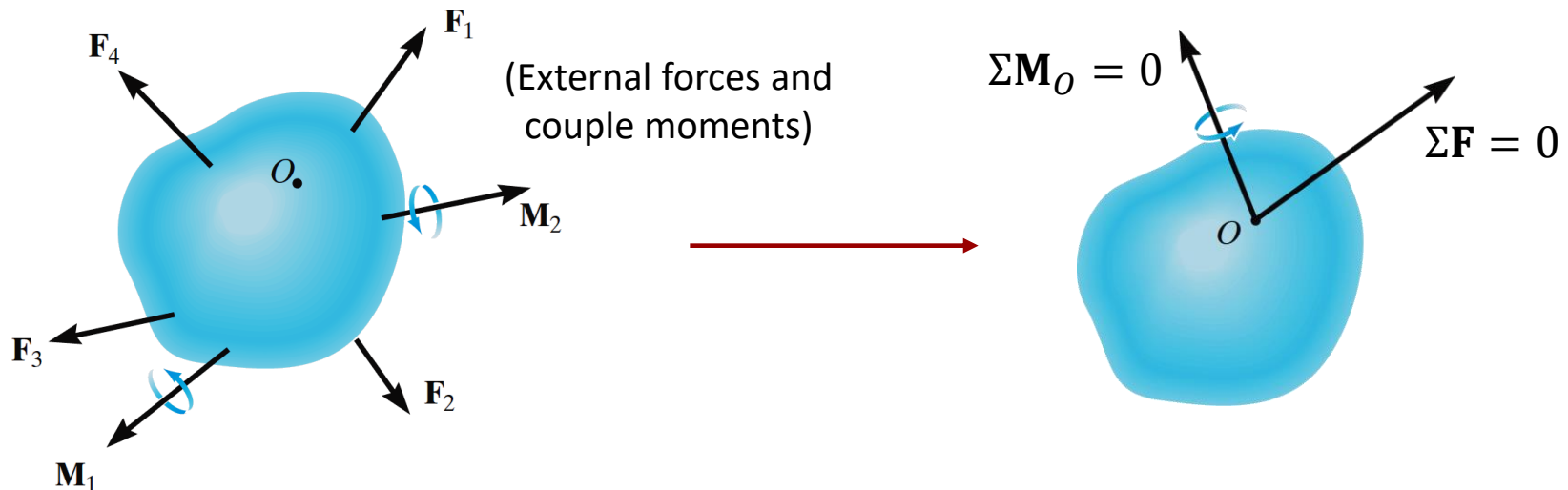
Statical Determinacy and Constraints

Equilibrium and Free-Body Diagrams

Equilibrium Conditions

If the **resultant force and couple moment** (at any arbitrary point O on or off the body) are both equal to **zero**, then the body is said to be in **Equilibrium**.

$$\begin{cases} \Sigma \mathbf{F} = \mathbf{0} \\ \Sigma \mathbf{M}_O = \mathbf{0} \end{cases}$$

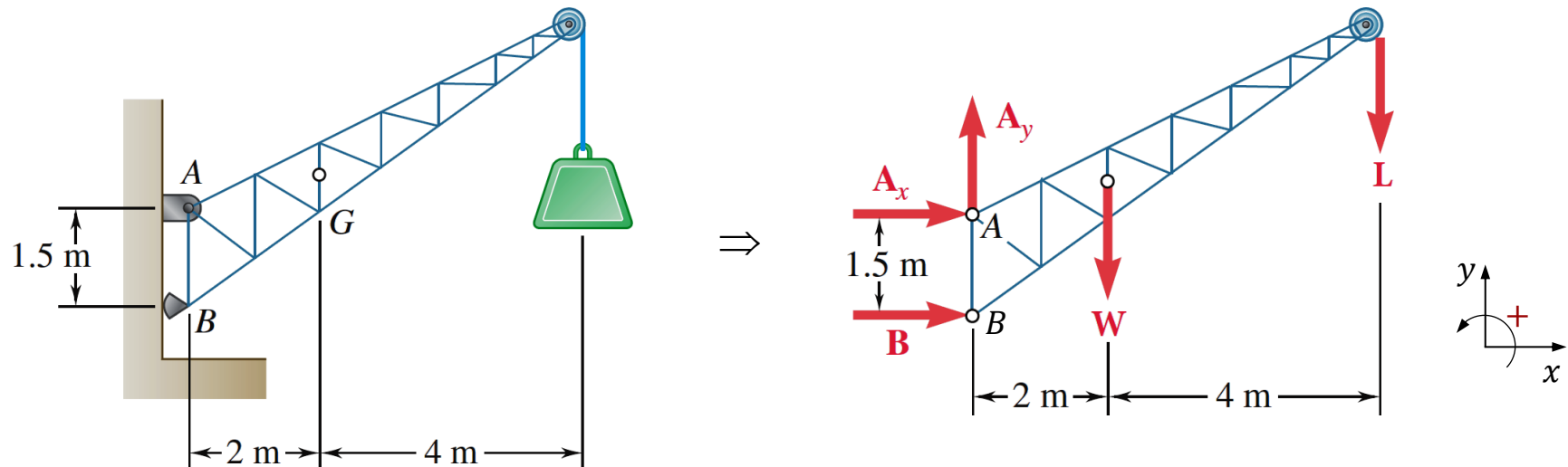


We can use these equations to determine unknown forces **applied** to the rigid body or unknown **reactions** exerted on it by its supports.

Free-Body Diagrams

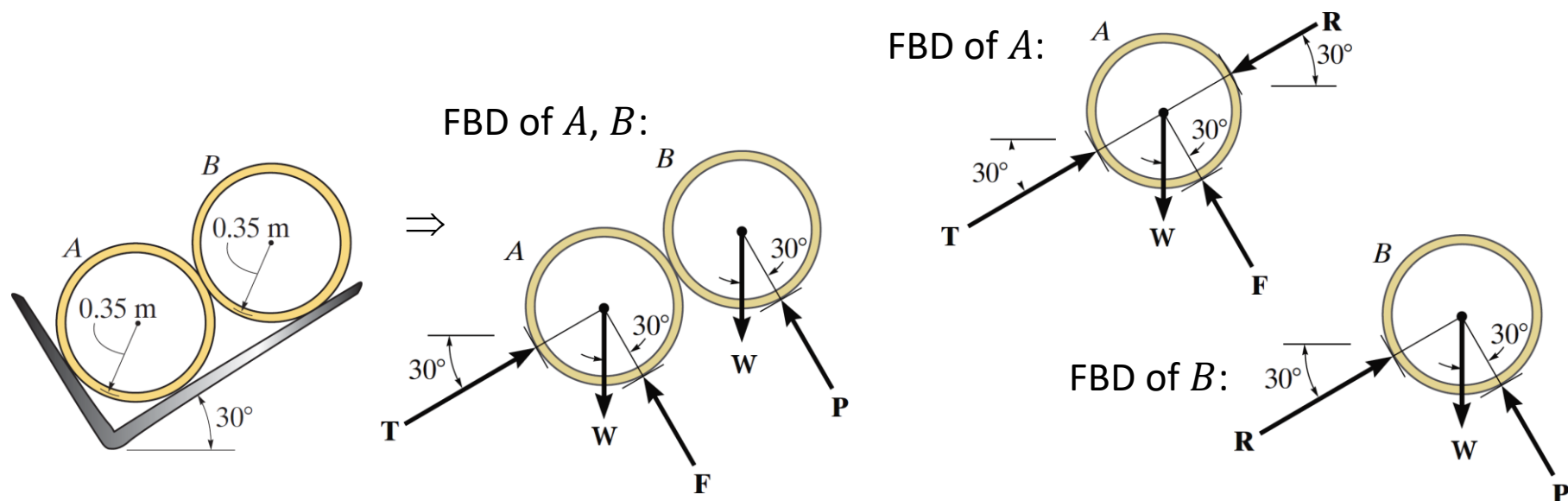
In order to write the equations of equilibrium for a rigid body, we must first draw a **free-body diagram (FBD)** of the rigid body under consideration.

This diagram is an **isolated** sketch of the body from its surroundings. The free-body diagram should include **all the external forces and couple moments** that the surroundings exert on the body (including the reactions exerted on it by its supports, and the weight of the body) and **dimensions** (for computing moments of forces).



Free-Body Diagrams

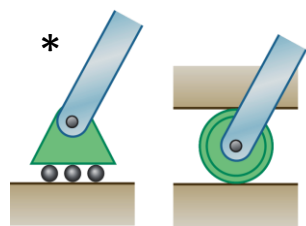
- If the body under consideration is made of several parts, do not include the **internal forces** the various parts exert on each other among the external forces, since they occur in equal but opposite collinear pairs (Newton's third law) and therefore cancel out.
- Include the weight **W** of the body, acting at the **center of gravity** G of the body, among the external forces. When the body is **uniform**, the center of gravity will be located at the body's **geometric center** or **centroid**.



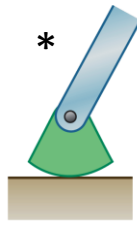
Equilibrium in Two Dimensions

Support Reactions

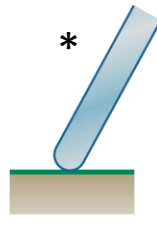
- Reactions with **one unknown** (the line of action is known, but the magnitude is unknown):



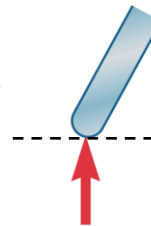
Rollers



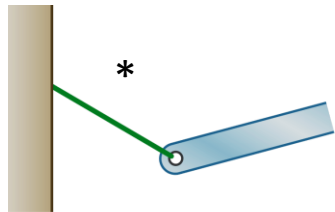
Rocker



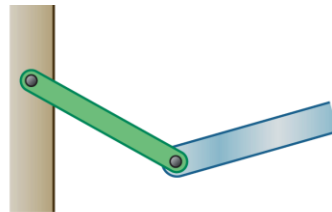
Frictionless Surface



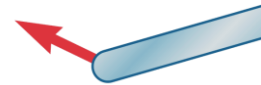
Force is perpendicular to surface.



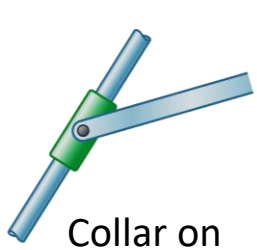
Weightless Cable



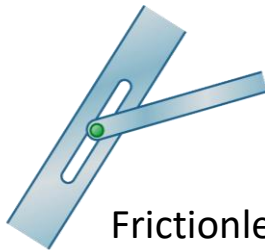
Weightless Link



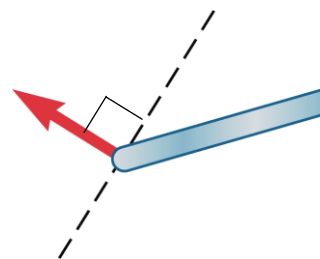
Force is along cable or link.



Collar on frictionless rod



Frictionless pin in slot

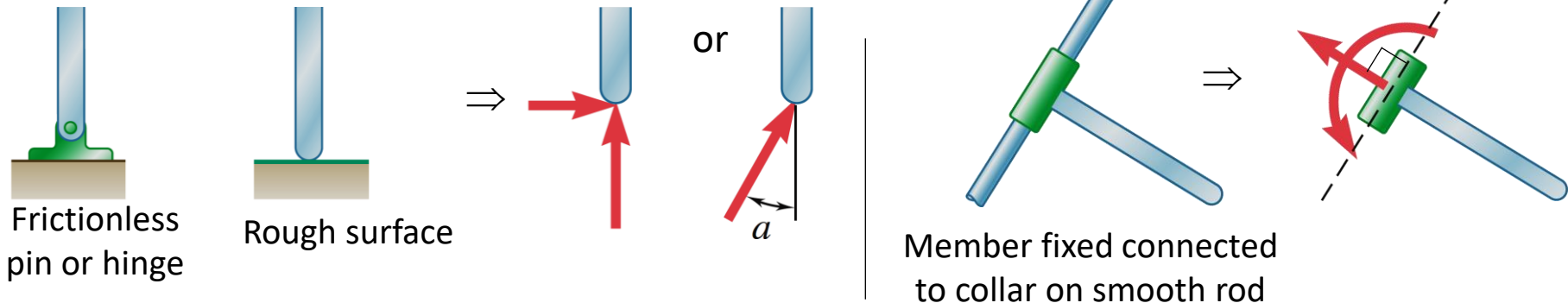


Force is perpendicular to rod or slot.

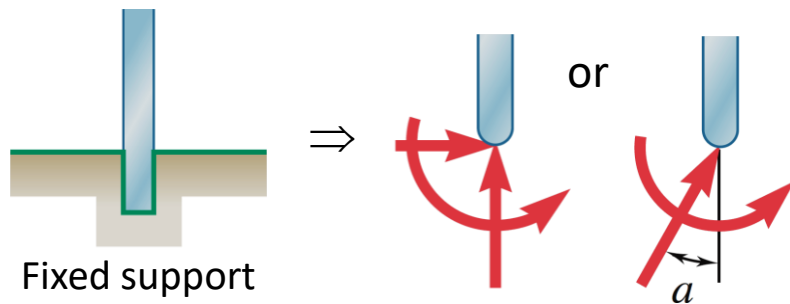
(* the sense is also known)

Support Reactions

- Reactions with **two unknowns**:



- Reactions with **three unknowns**:



A support prevents the translation/rotation of a body in a given direction by exerting a force/couple moment on the body in the opposite direction.

Note: When the sense of an unknown force or couple is unknown, arbitrarily assume the sense; the sign of the answer will indicate whether the assumption is correct or not.

A **positive** [**negative**] answer means the assumption is **correct** [**incorrect**].

Equations of Equilibrium

If the force system acting on a rigid body lies in a single plane (say $x - y$) and any couple moments acting on the body are directed perpendicular to this plane, the force and couple system is **two-dimensional (2D)**. Thus, Equations of equilibrium are

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M_O = 0$$



(O is any point in the plane of the body)

- Two alternative sets of three independent equilibrium equations:

$$\Sigma F_x = 0, \quad \Sigma M_A = 0, \quad \Sigma M_B = 0$$

or y

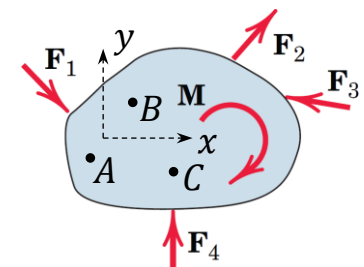
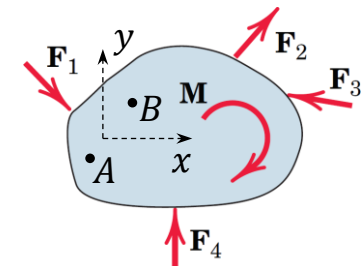
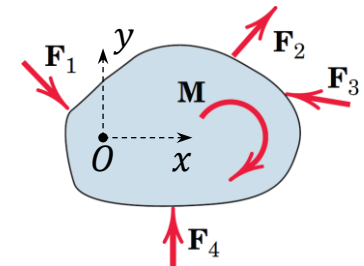
Only when the line AB is not perpendicular to the x axis.

or y

$$\Sigma M_A = 0, \quad \Sigma M_B = 0, \quad \Sigma M_C = 0$$

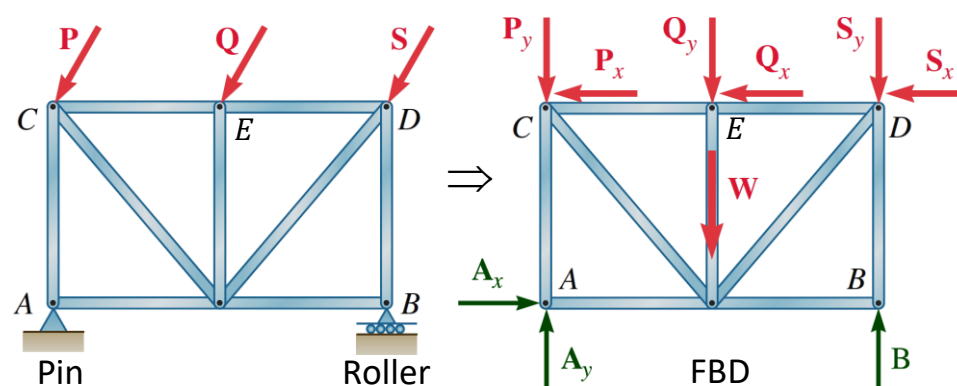
Only when the points A , B , and C do not lie along a straight line.

These three equations can be solved for **up to three unknowns**.



Equations of Equilibrium

- ◆ It is desirable to choose equations of equilibrium containing **only one unknown** in each equation, because this eliminates the necessity of solving simultaneous equations. To do so,
- Sum moments** about the point of intersection of the lines of action of two unknown forces (for example, point E is not a proper choice in the following figure).
 - Sum force components** in a direction perpendicular to two unknown parallel forces.



For given \mathbf{P} , \mathbf{Q} , \mathbf{S} , and \mathbf{W} :

$$\Sigma M_A = 0 \rightarrow B \checkmark \text{ (It doesn't contain } A_x, A_y \text{)}$$

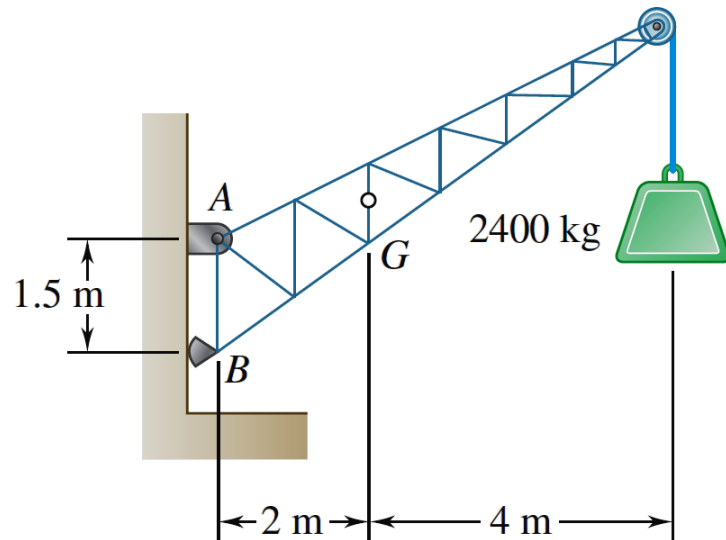
$$\Sigma F_x = 0 \rightarrow A_x \checkmark \text{ (It doesn't contain } A_y, B \text{)}$$

$$\Sigma F_y = 0 \rightarrow A_y \checkmark$$

- ◆ Any additional equation (e.g., $\Sigma M_B = 0$) is not independent (does not contain any new information) and cannot be used to determine a fourth unknown (however, they can be useful for checking the solution obtained from the original three equations of equilibrium!).

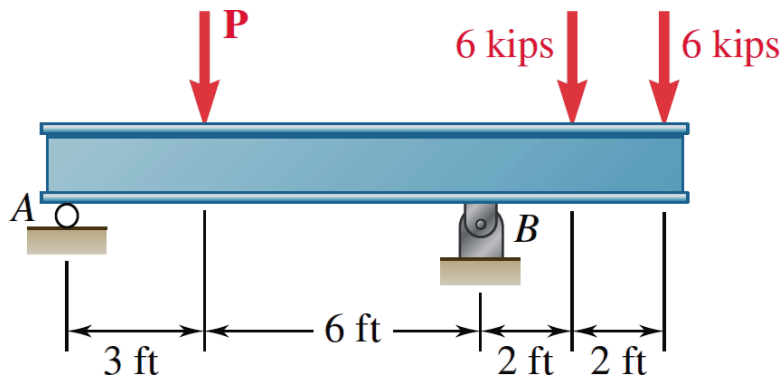
Sample Problem 4.1

A fixed crane has a mass of 1000 kg and is used to lift a 2400-kg crate. It is held in place by a pin at A and a rocker at B . The center of gravity of the crane is located at G . Determine the components of the reactions at A and B .



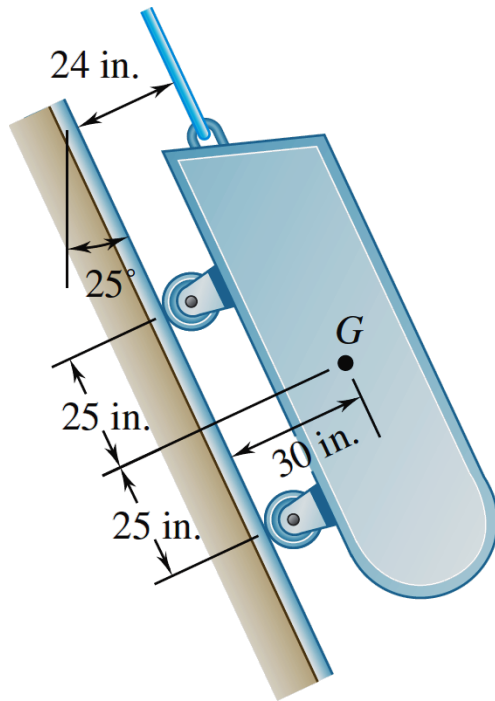
Sample Problem 4.2

Three loads are applied to a beam as shown. The beam is supported by a roller at A and by a pin at B . Neglecting the weight of the beam, determine the reactions at A and B when $P = 15$ kips.



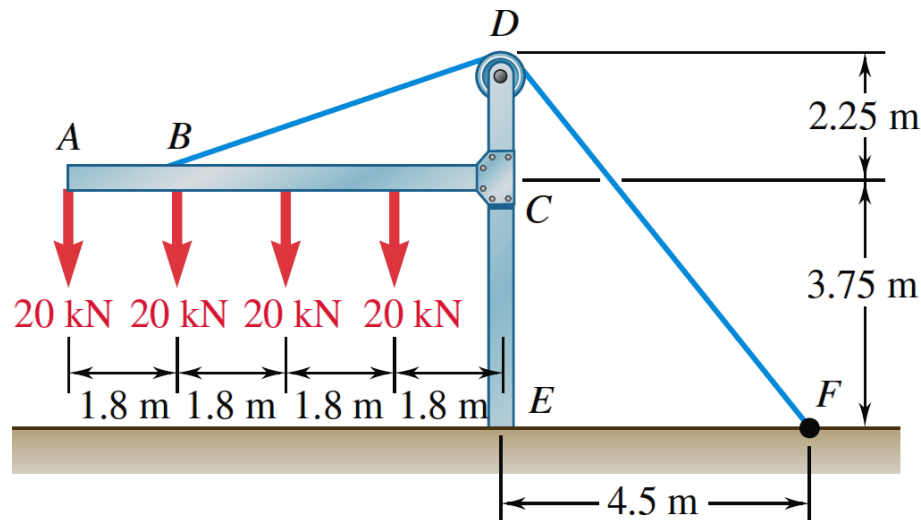
Sample Problem 4.3

A loading car is at rest on a track forming an angle of 25° with the vertical. The gross weight of the car and its load is 5500 lb, and it acts at point G , 30 in. from the track, halfway between the two axles. The car is held by a cable attached 24 in. from the track. Determine the tension in the cable and the reaction at each pair of wheels.



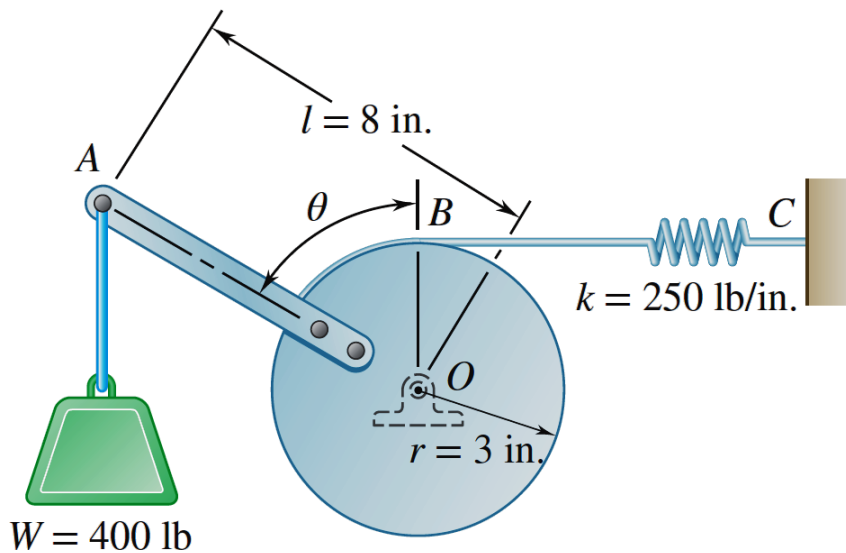
Sample Problem 4.4

The frame shown supports part of the roof of a small building. Knowing that the tension in the cable is 150 kN, determine the reaction at the fixed end E .



Sample Problem 4.5

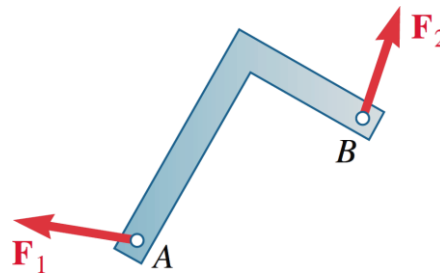
A 400-lb weight is attached at A to the lever shown. The constant of the spring BC is $k = 250$ lb/in., and the spring is unstretched when $\theta = 0$. Determine the position of equilibrium.



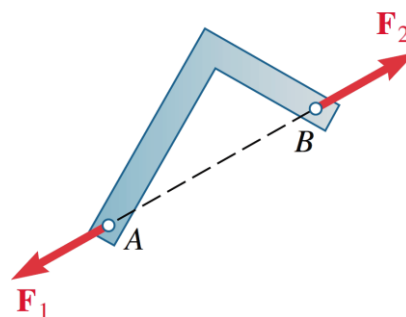
Two Special Cases

Equilibrium of a Two-Force Body

A **two-force body** is defined as a rigid body subjected to forces acting at only two points.

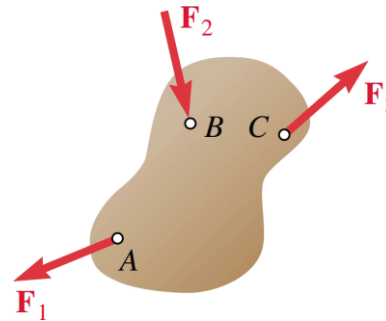


If a **two-force body is in equilibrium**, the two forces must have the **same magnitude**, **opposite sense**, and the **same line of action** (directed along the line joining the two points where these forces act).

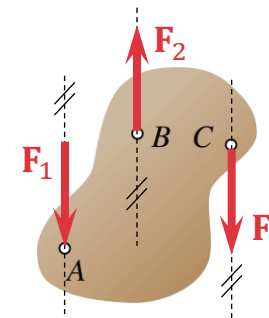
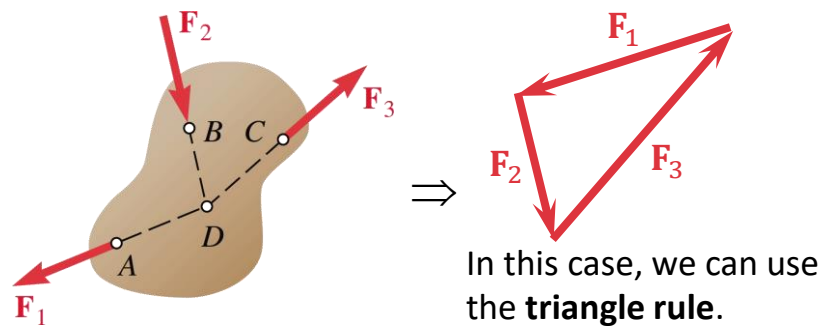


Equilibrium of a Three-Force Body

A **three-force body** is defined as a rigid body subjected to forces acting at only three points.

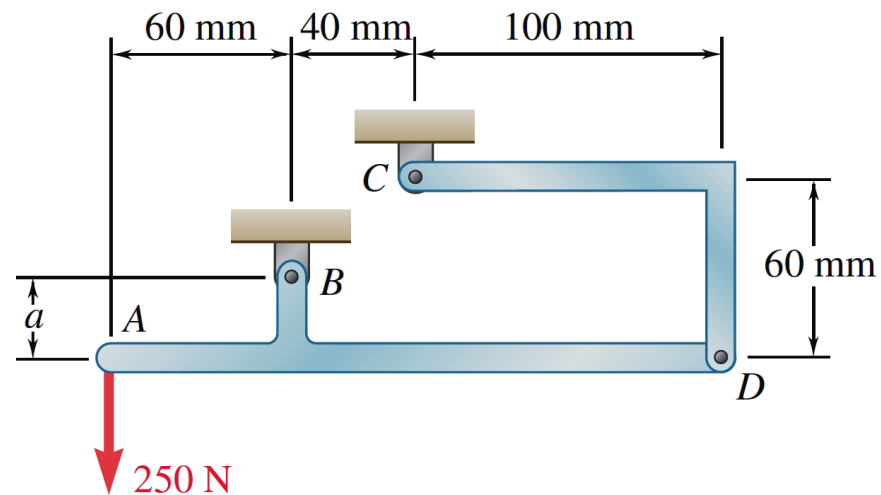


If a **three-force body is in equilibrium**, the lines of action of the three forces must be either **concurrent** or **parallel**.



Problem 4.65

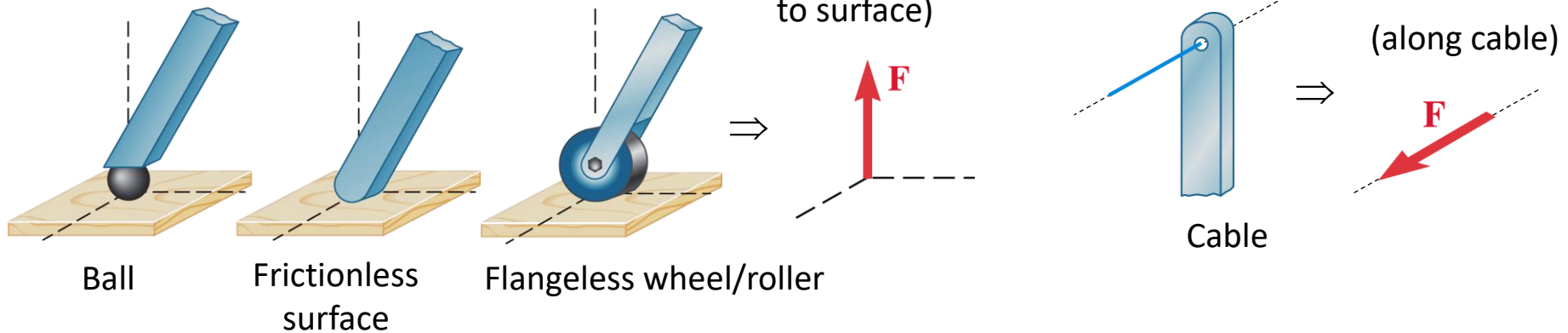
Determine the reactions at B and C when $a = 30$ mm.



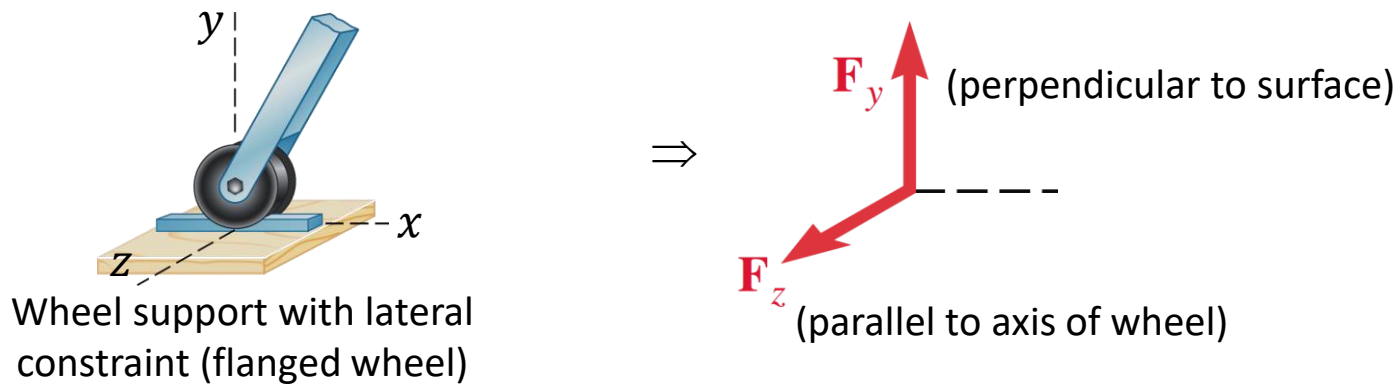
Equilibrium in Three Dimensions

Support Reactions

Reactions with **one unknown**:

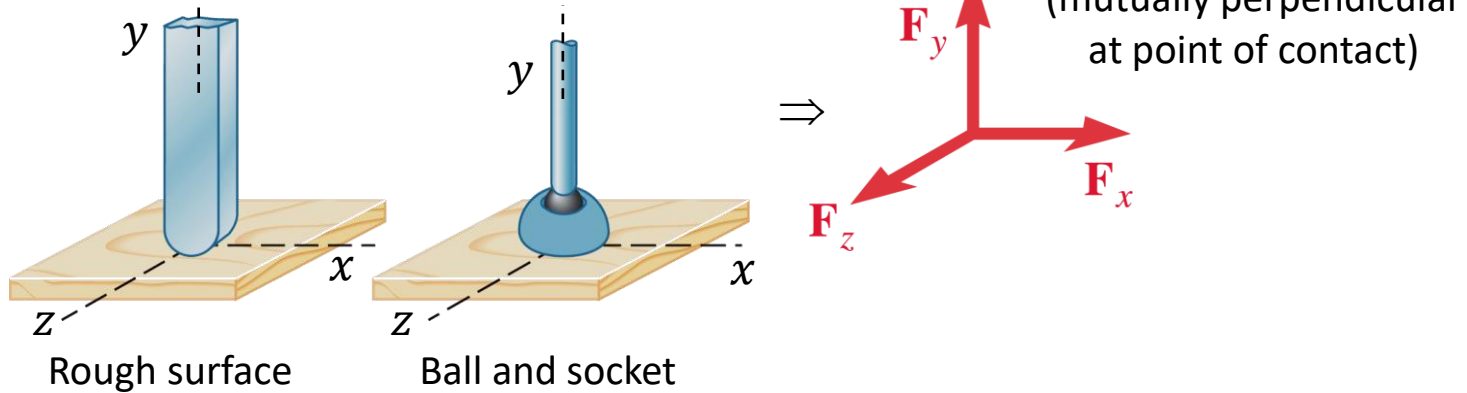


Reactions with **two unknowns**:

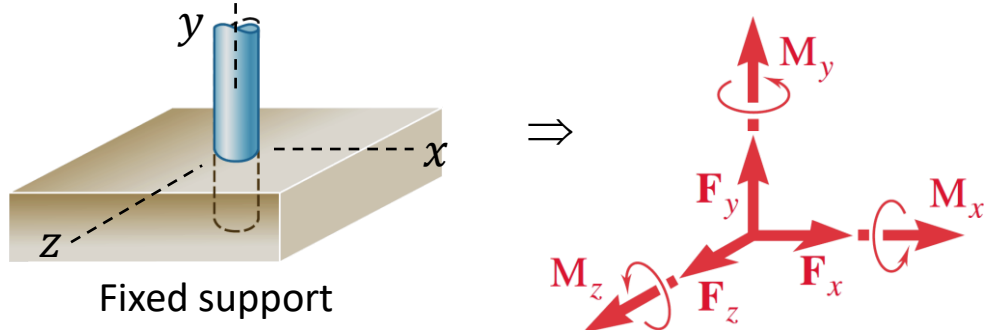


Support Reactions

Reactions with **three unknowns**:

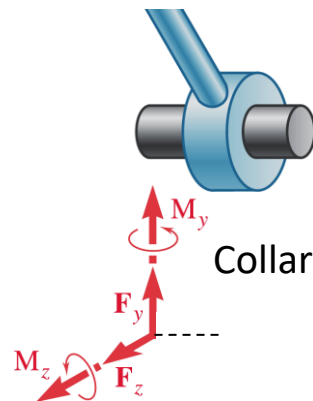


Reactions with **six unknowns**:

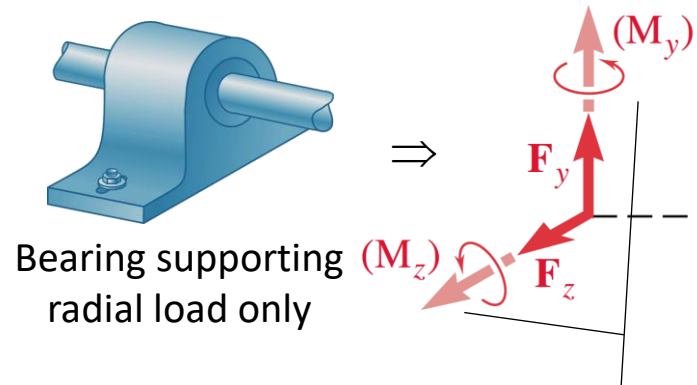
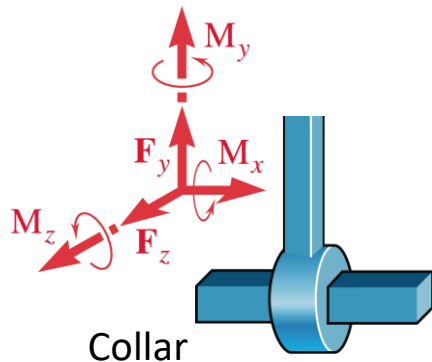


Support Reactions

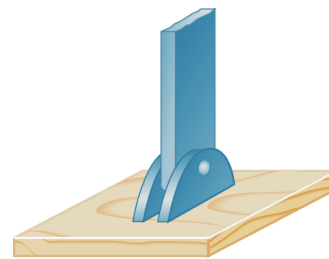
Reactions with **four unknowns**:



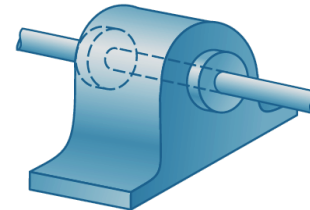
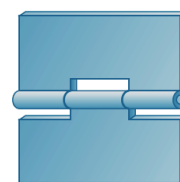
Reactions with **five unknowns**:



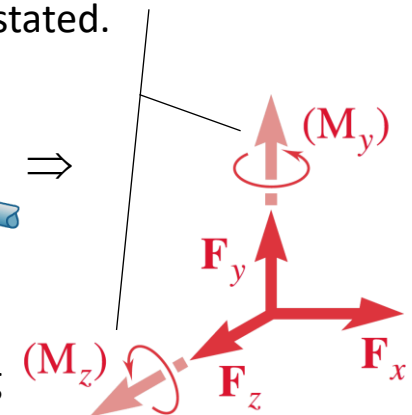
◆ Bearings are primarily designed to prevent only translation. Thus, only force components should be included **unless** otherwise stated.



Pin and bracket



Hinge and Bearing supporting axial thrust and radial load



Equations of Equilibrium

The most general situation of rigid-body equilibrium occurs in three dimensions.

$$\begin{cases} \Sigma \mathbf{F} = \mathbf{0} \\ \Sigma \mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) + \Sigma \mathbf{M}_i = \mathbf{0} \end{cases}$$

(O is any point located
either on or off the body)

(or)

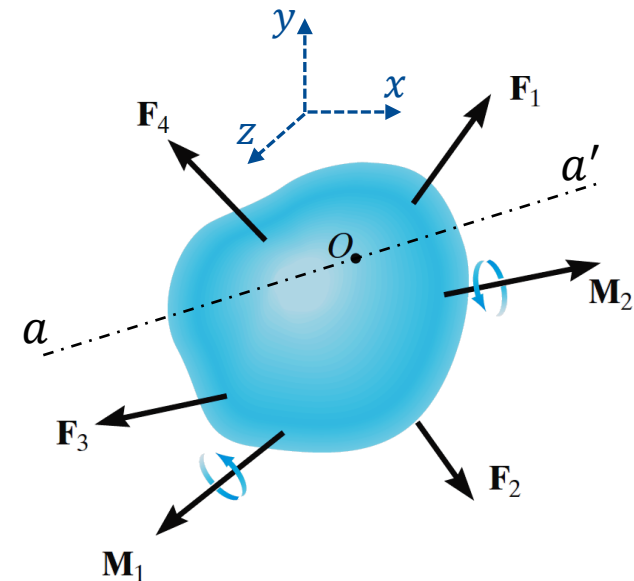
$$\begin{cases} \Sigma F_x = 0 & \Sigma F_y = 0 & \Sigma F_z = 0 \\ \Sigma M_x = 0 & \Sigma M_y = 0 & \Sigma M_z = 0 \end{cases}$$

$$\Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k}$$

$$\Sigma \mathbf{M}_O = \Sigma M_x \mathbf{i} + \Sigma M_y \mathbf{j} + \Sigma M_z \mathbf{k}$$

◆ Any arbitrary **axis** aa' can be also chosen for summing moments:

$$\Sigma \mathbf{M}_{aa'} = \mathbf{0}$$

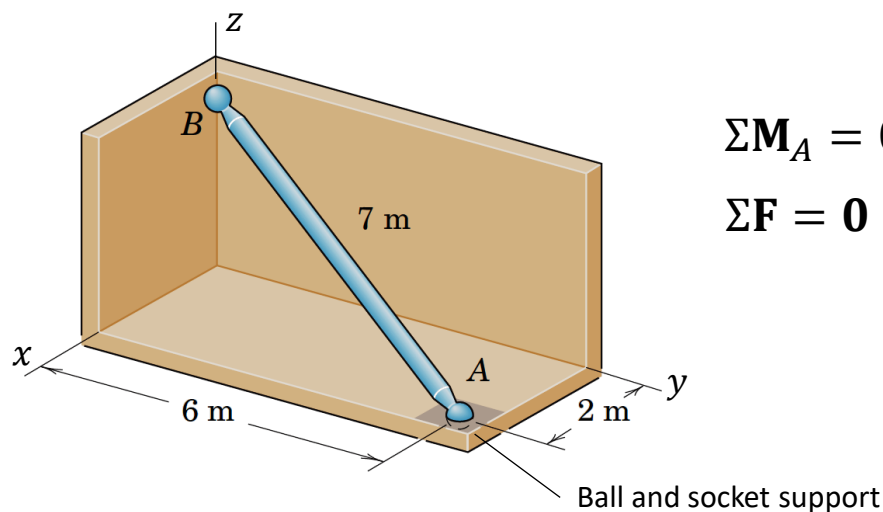


These equations can be solved for **up to six unknowns**.

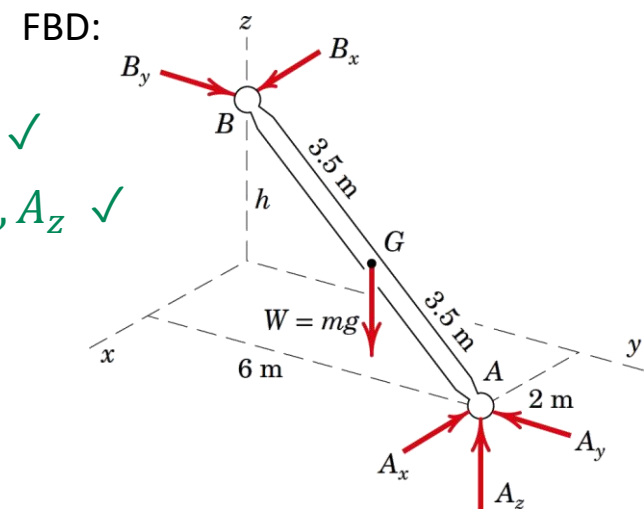
Equations of Equilibrium

◆ It is desirable to choose equations of equilibrium containing as few unknowns as possible. Two helpful strategies:

(1) Sum moments **about a support** to eliminate three unknown reaction components.



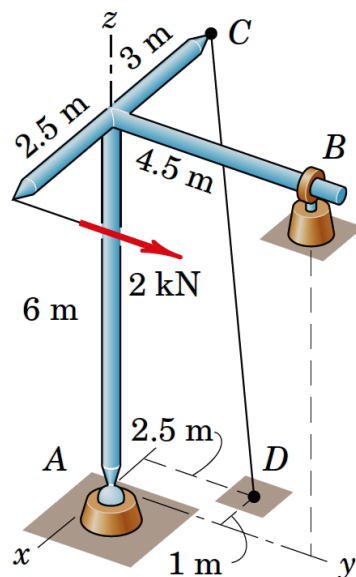
$$\begin{aligned}\Sigma \mathbf{M}_A &= \mathbf{0} \rightarrow B_x, B_y \checkmark \\ \Sigma \mathbf{F} &= \mathbf{0} \rightarrow A_x, A_y, A_z \checkmark\end{aligned}$$



Note: When the sense of an unknown force or couple is unknown, arbitrarily assume the sense. If the sign of the answer yields a **negative** scalar for a force or couple moment magnitude, it indicates that the correct sense is **opposite** to that assumed on the free-body diagram.

Equations of Equilibrium

(2) Sum moments **about an axis** that it intersects the lines of action of as many unknown forces as possible (say all but one of the unknown reactions).

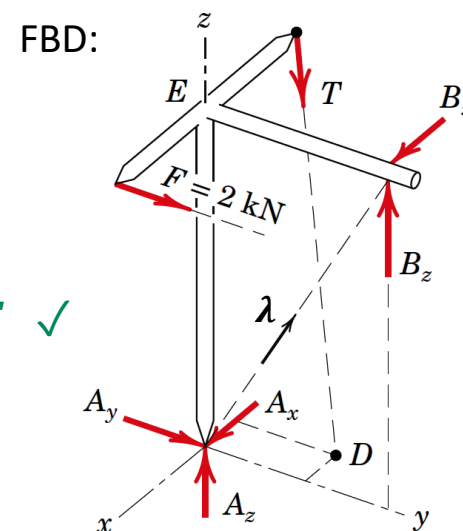


Finding tension T :

$$\Sigma \mathbf{M}_{AB} = \mathbf{0} \rightarrow T \quad \checkmark$$

VS

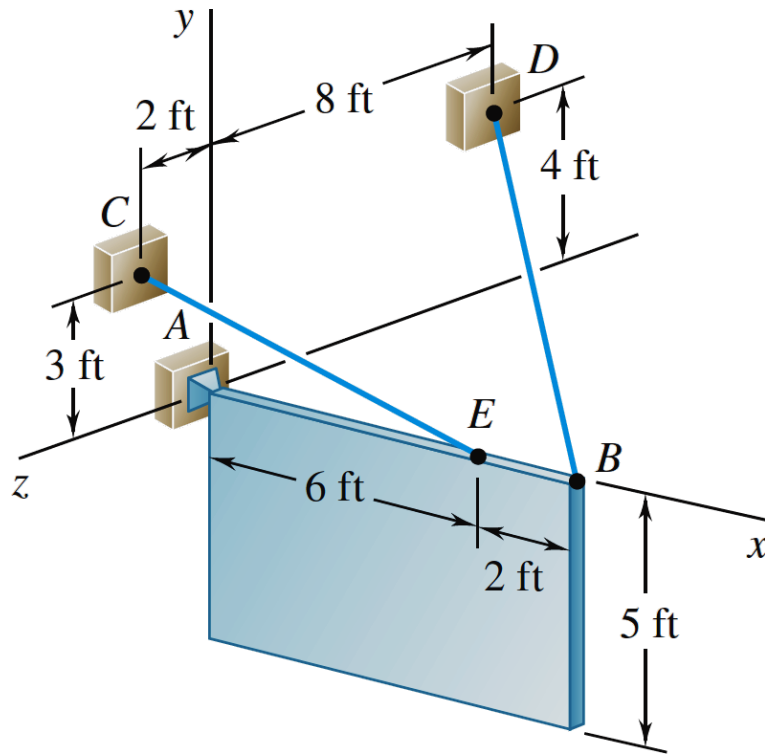
$$\Sigma \mathbf{M}_A = \mathbf{0} \rightarrow B_x, B_z, T \quad \checkmark$$



In this way, we can eliminate the reactions at two points A and B by writing $\Sigma \mathbf{M}_{AB} = \mathbf{0}$, which involves the computation of the moments of the forces about an axis AB joining points A and B .

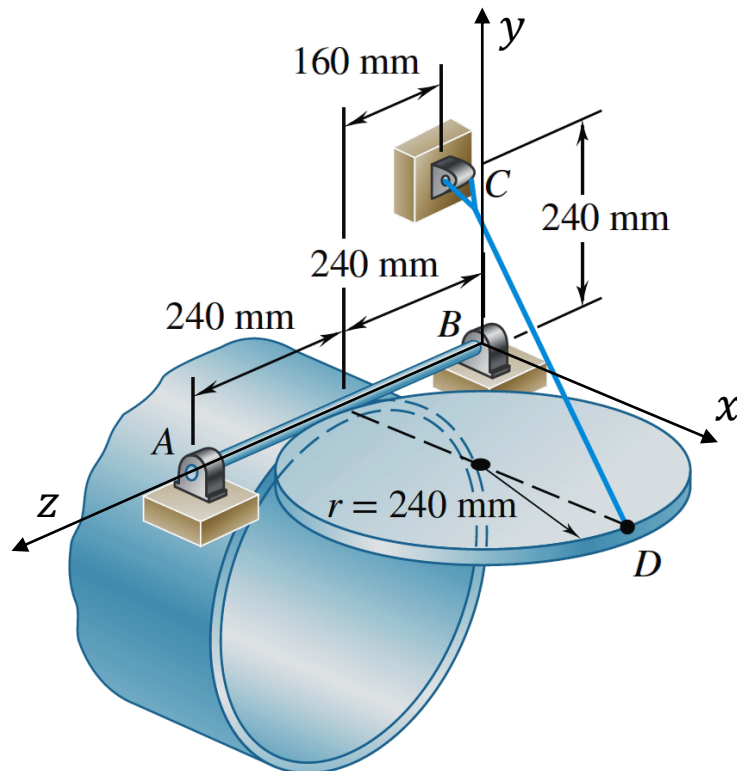
Sample Problem 4.8

A 5×8 -ft sign of uniform density weighs 270 lb and is supported by a ball-and-socket joint at A and by two cables. Determine the tension in each cable and the reaction at A .



Sample Problem 4.9

A uniform pipe cover of radius $r = 240$ mm and mass $m = 30$ kg is held in a horizontal position by the cable CD . Assuming that the bearing at B does not exert any axial force and the bearing at A exerts an axial force, determine the tension in the cable and the reactions at A and B ($g = 9.81$ m/s²).

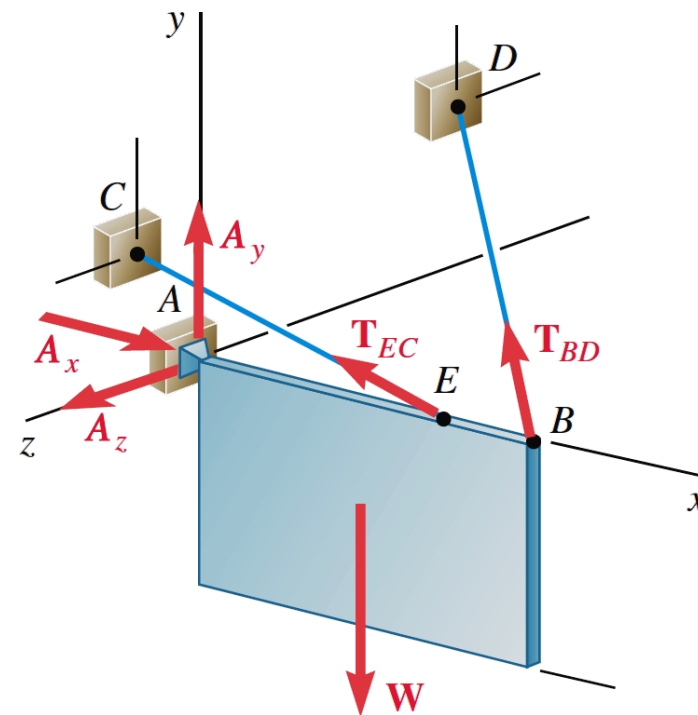
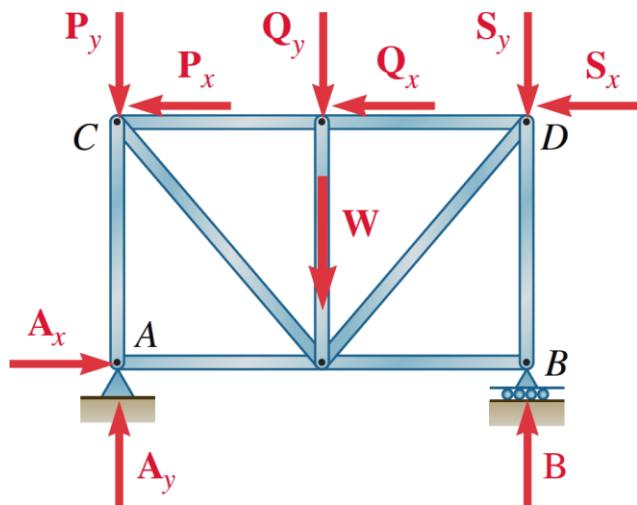


Static Determinacy and Constraints

Static Determinacy

A rigid body which has **the minimum number of supports/constraints** necessary to maintain an equilibrium configuration under given loads is called **statically determinate**. For such bodies,

- The equilibrium equations are **sufficient** to determine all the unknown reactions (up to three unknowns in 2D or six unknowns in 3D).



Static Determinacy

A rigid body which has **more supports/constraints than are necessary** to maintain an equilibrium configuration under given loads is called **statically indeterminate**.

For such bodies,

- The equilibrium equations are **insufficient** to determine all the unknown reactions (e.g., **more than** three unknowns in 2D or six unknowns in 3D).
- You may be able to calculate **a few of reactions**. The remaining reactions can be determined by additional equations obtained from the deformation conditions at the points of support (MEC 363 [Mechanics of Solids]).

