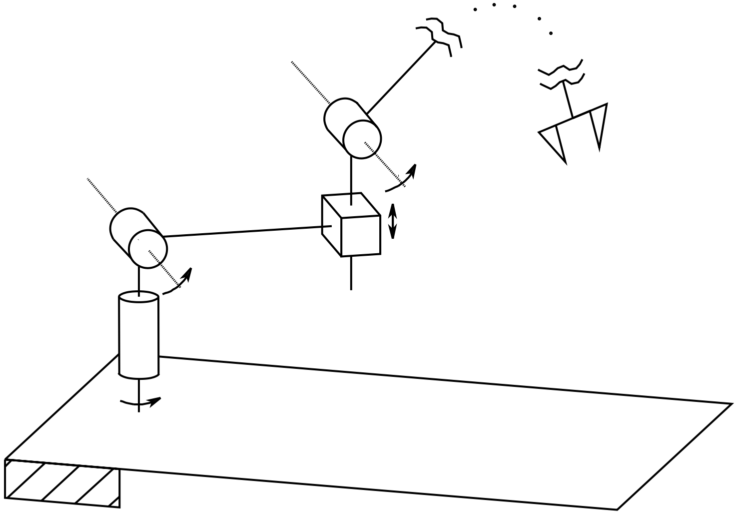
Ch2: Robot Dynamics – Part 1

Dynamic Equations

Dynamic Equations

The dynamic equations (equations of motion) of an open-chain manipulator are a set of 2nd-order ordinary differential equations of the form



$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$$

$\boldsymbol{\theta} \in \mathbb{R}^n$: Joint Variables (or joint coordinates or joint positions)

$\boldsymbol{\tau} \in \mathbb{R}^n$: Joint Torques/Forces (applied at the joints by the actuators)

$\boldsymbol{M}(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$: Mass Matrix:

$\boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$: Coriolis, Centripetal, Gravitational, and Frictional Terms

Forward & Inverse Dynamics

Forward Dynamics:
Finding the acceleration $\ddot{\boldsymbol{\theta}}$ given the state $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and the joint torques/forces $\boldsymbol{\tau}$:

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}(\boldsymbol{\theta})(\boldsymbol{\tau} - \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}))$$

Inverse Dynamics:
Finding the joint torques/forces $\boldsymbol{\tau}$ given the state $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and acceleration $\ddot{\boldsymbol{\theta}}$.

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$$

Two equivalent approaches to derive dynamic equations:

1) **Lagrangian Formulation** (variational, based on energy)
2) **Newton–Euler Formulation**

Lagrangian Formulation

Lagrangian Formulation

- Lagrangian function: $\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \mathcal{P}(\boldsymbol{q})$

Kinetic Energy   Potential Energy

$\mathcal{L}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$

$\boldsymbol{q} \in \mathbb{R}^n$: Generalized Coordinates

(Due only to conservative forces such as gravitational energy and energy stored in springs.)

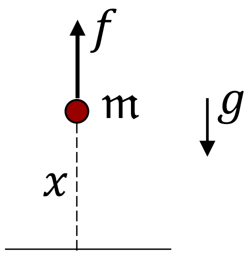
- Equations of Motion: $$\boldsymbol{f} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}$$

$\boldsymbol{f} \in \mathbb{R}^n$: Generalized (Nonconservative) Forces (e.g., external forces/torques or friction forces) such that $\boldsymbol{f}$ and $\dot{\boldsymbol{q}}$ are dual to each other, i.e., the $\boldsymbol{f}^T\dot{\boldsymbol{q}}$ corresponds to power.

In components: $$f_i = \frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial q_i} \qquad i = 1,\cdots,n$$

Example 1

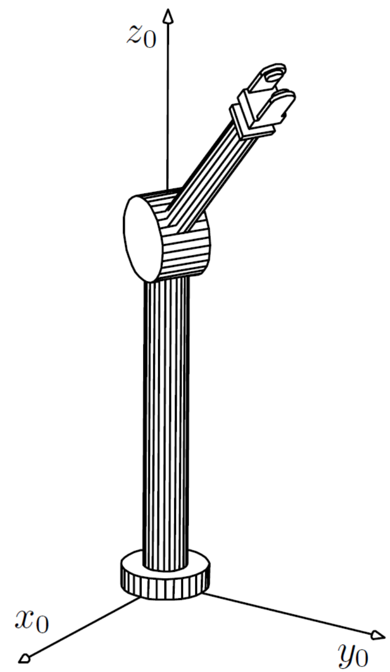
Derive equations of motion for a particle of mass $\mathfrak{m}$ constrained to move on a vertical line.

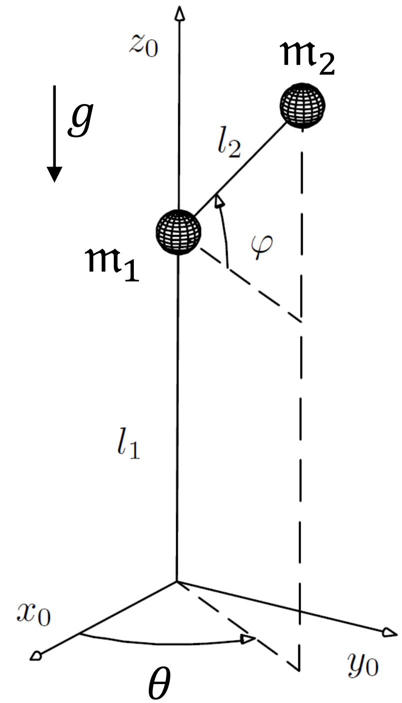


$$f = \mathfrak{m}\ddot{x} + \mathfrak{m}g$$

Example 2

Consider the 1-DOF mechanism. It consists of a rigid link formed by two parts, of lengths l_1 and l_2, whose masses $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are, for simplicity, considered to be concentrated at their respective centers of mass, located at the ends. The angle φ is constant. The mechanism possesses only revolute motion about the z_0 axis, the angle of which is represented by θ. Derive equations of motion for the mechanism moving in the presence of gravity.

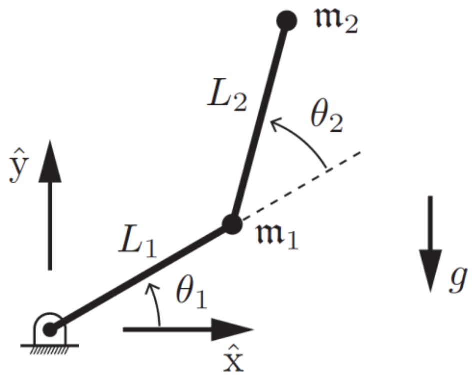




Example 3

Derive equations of motion for a planar 2R open chain moving in the presence of gravity.

For the sake of simplicity, model the links as point masses $\mathfrak{m}_1$, $\mathfrak{m}_2$ concentrated at the ends of each link.

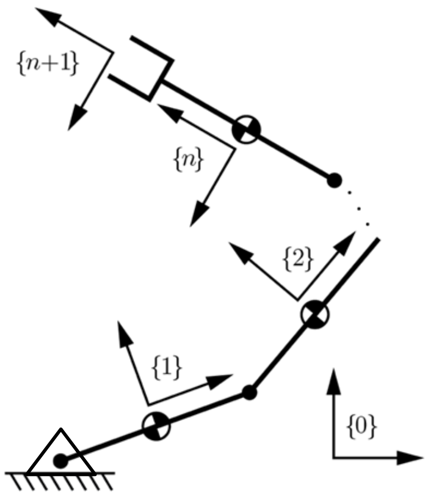


Note:
For an n-link open-chain manipulator:

Generalized coordinates: $\boldsymbol{\theta} \in \mathbb{R}^n$

Generalized forces: $\boldsymbol{\tau} \in \mathbb{R}^n$

(for a revolute/prismatic joint θ_i, τ_i is a torque/force)



Example 3 (cont.)

$$\tau_1 = \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right)\ddot{\theta}_1$$
$$+\mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_2 - \mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)$$
$$+(\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2),$$

$$\tau_2 = \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_1 + \mathfrak{m}_2L_2^2\ddot{\theta}_2 + \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2$$
$$+\mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2).$$

We can gather terms together into an equation of the form: $\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) + \boldsymbol{g}(\boldsymbol{\theta})$

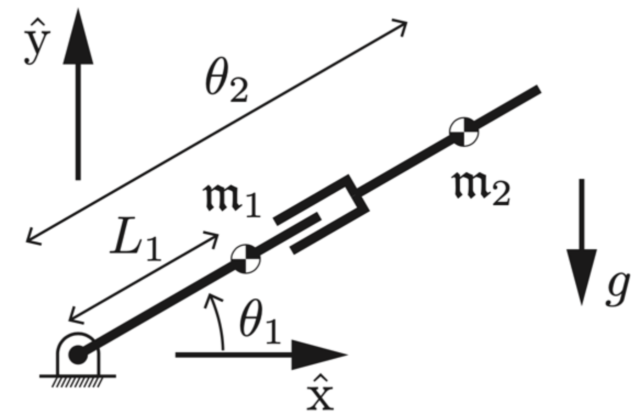
$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2L_2^2 \end{bmatrix}$$

$$\boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) = \begin{bmatrix} -\mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2 \end{bmatrix}$$

$$\boldsymbol{g}(\boldsymbol{\theta}) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) \\ \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) \end{bmatrix}$$

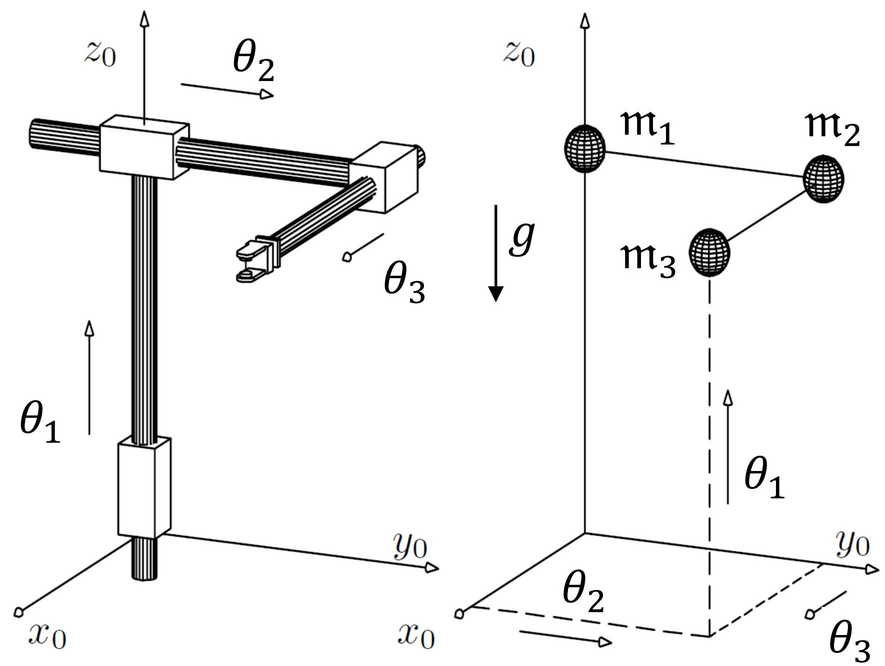
Example 4

Derive equations of motion for a planar RP open chain moving in the presence of gravity.



Example 5

Consider the 3-DOF Cartesian robot manipulator. The manipulator consists of three rigid links mutually orthogonal. The three joints of the robot are prismatic. Derive equations of motion for the robot manipulator moving in the presence of gravity.

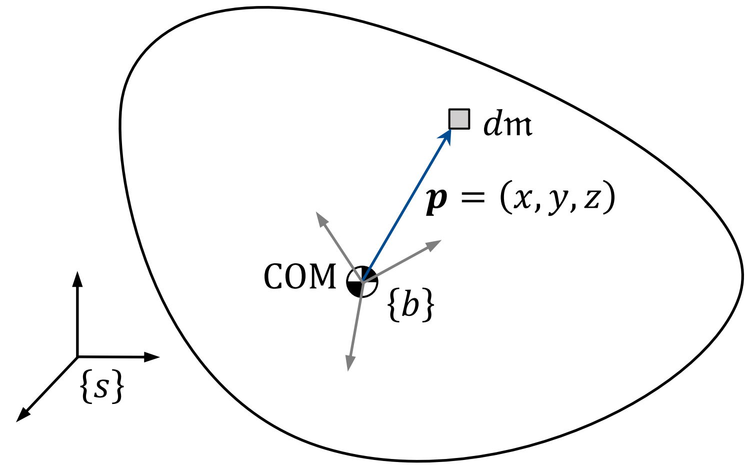


Newton–Euler Formulation

Dynamics of a Single Rigid Body

$$\dot{\boldsymbol{p}} = \boldsymbol{v}_b + \boldsymbol{\omega}_b \times \boldsymbol{p}$$

$$\begin{aligned}\ddot{\boldsymbol{p}} &= \dot{\boldsymbol{v}}_b + \frac{d}{dt}\boldsymbol{\omega}_b \times \boldsymbol{p} + \boldsymbol{\omega}_b \times \frac{d}{dt}\boldsymbol{p} \\ &= \dot{\boldsymbol{v}}_b + \dot{\boldsymbol{\omega}}_b \times \boldsymbol{p} + \boldsymbol{\omega}_b \times (\boldsymbol{v}_b + \boldsymbol{\omega}_b \times \boldsymbol{p}) \\ &= \dot{\boldsymbol{v}}_b + [\dot{\boldsymbol{\omega}}_b]\boldsymbol{p} + [\boldsymbol{\omega}_b]\boldsymbol{v}_b + [\boldsymbol{\omega}_b]^2\boldsymbol{p}\end{aligned}$$



$$\begin{aligned}d\boldsymbol{f} &= d\mathfrak{m}\,\ddot{\boldsymbol{p}} \\ &= d\mathfrak{m}\,(\dot{\boldsymbol{v}}_b + [\dot{\boldsymbol{\omega}}_b]\boldsymbol{p} + [\boldsymbol{\omega}_b]\boldsymbol{v}_b + [\boldsymbol{\omega}_b]^2\boldsymbol{p})\end{aligned}$$

at COM: $\int \boldsymbol{p}\, d\mathfrak{m} = \boldsymbol{0}$

$$d\boldsymbol{m} = \boldsymbol{p} \times d\boldsymbol{f} = [\boldsymbol{p}]d\boldsymbol{f}$$

$$\begin{bmatrix}\boldsymbol{m}_b \\ \boldsymbol{f}_b\end{bmatrix} = \begin{bmatrix}\int d\boldsymbol{m} \\ \int d\boldsymbol{f}\end{bmatrix}$$

Rotational dynamics

Translational dynamics

$$= \begin{bmatrix}\boldsymbol{I}_b\dot{\boldsymbol{\omega}}_b + [\boldsymbol{\omega}_b]\boldsymbol{I}_b\boldsymbol{\omega}_b \\ \mathfrak{m}(\dot{\boldsymbol{v}}_b + [\boldsymbol{\omega}_b]\boldsymbol{v}_b)\end{bmatrix}$$

← Euler's Equation

$$\boldsymbol{I}_b = -\int [\boldsymbol{p}]^2 d\mathfrak{m} \in \mathbb{R}^{3\times 3}$$

Inertia Matrix

(symmetric and positive definite)

Inertia Matrix

$$\boldsymbol{I}_b = -\int [\boldsymbol{p}]^2 d\mathfrak{m} \qquad \boldsymbol{p} = (x, y, z)$$

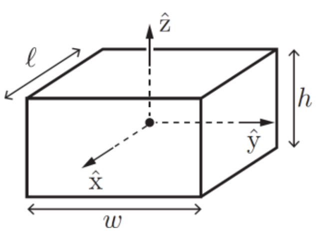
$$= \begin{bmatrix} \int (y^2 + z^2) d\mathfrak{m} & -\int xy d\mathfrak{m} & -\int xz d\mathfrak{m} \\ -\int xy d\mathfrak{m} & \int (x^2 + z^2) d\mathfrak{m} & -\int yz d\mathfrak{m} \\ -\int xz d\mathfrak{m} & -\int yz d\mathfrak{m} & \int (x^2 + y^2) d\mathfrak{m} \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

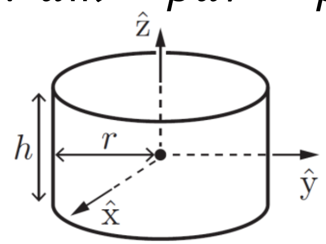
Rotational Kinetic Energy:

$$\mathcal{K} = \frac{1}{2} \boldsymbol{\omega}_b^{\mathrm{T}} \boldsymbol{I}_b \boldsymbol{\omega}_b$$

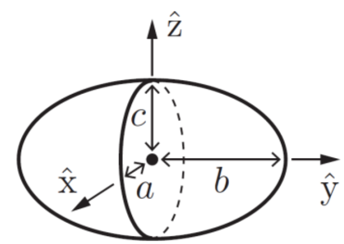
- If the body has uniform density: $d\mathfrak{m} = \rho dV = \rho dxdydz$



$$I_{xx} = \mathfrak{m}(w^2 + h^2)/12$$
$$I_{yy} = \mathfrak{m}(\ell^2 + h^2)/12$$
$$I_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$$



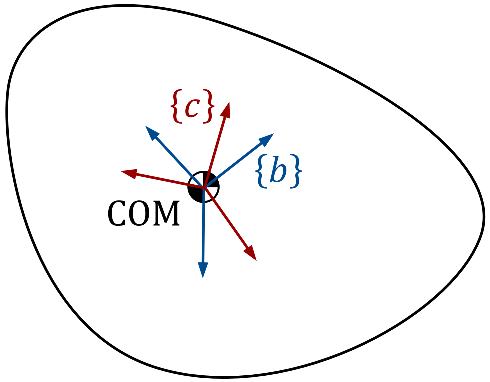
$$I_{xx} = \mathfrak{m}(3r^2 + h^2)/12$$
$$I_{yy} = \mathfrak{m}(3r^2 + h^2)/12$$
$$I_{zz} = \mathfrak{m}r^2/2$$



$$I_{xx} = \mathfrak{m}(b^2 + c^2)/5$$
$$I_{yy} = \mathfrak{m}(a^2 + c^2)/5$$
$$I_{zz} = \mathfrak{m}(a^2 + b^2)/5$$

Expressing Inertia Matrix $\boldsymbol{I}_b$ in a Rotated Frame

Let $\boldsymbol{I}_c$ be inertia matrix in a rotated frame $\{c\}$ described by $\boldsymbol{R}_{bc}$:



Rotational kinetic energy of the rotating body is independent of the chosen frame:

$$\begin{aligned}\frac{1}{2}\boldsymbol{\omega}_c^{\mathrm{T}}\boldsymbol{I}_c\boldsymbol{\omega}_c &= \frac{1}{2}\boldsymbol{\omega}_b^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{\omega}_b \\ &= \frac{1}{2}(\boldsymbol{R}_{bc}\boldsymbol{\omega}_c)^{\mathrm{T}}\boldsymbol{I}_b(\boldsymbol{R}_{bc}\boldsymbol{\omega}_c) \\ &= \frac{1}{2}\boldsymbol{\omega}_c^{\mathrm{T}}\left(\boldsymbol{R}_{bc}^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{R}_{bc}\right)\boldsymbol{\omega}_c\end{aligned}$$

$\Rightarrow$

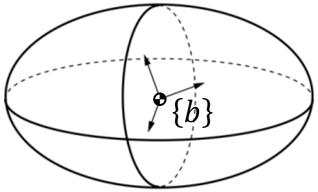
$$\boldsymbol{I}_c = \boldsymbol{R}_{bc}^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{R}_{bc}$$

Diagonalizing Inertia Matrix $\boldsymbol{I}_b$

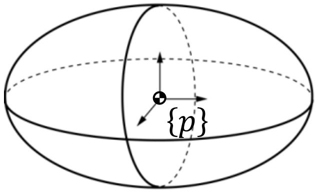
Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be the eigenvectors of $\boldsymbol{I}_b$ and $\lambda_1, \lambda_2, \lambda_3$ be the corresponding eigenvalues.

- **Principal Axes of Inertia** are in the directions of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ (expressed in $\{b\}$).
- **Principal Moments of Inertia** (about $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$), are $\lambda_1, \lambda_2, \lambda_3 > 0$.

$$\boldsymbol{R}_{bp} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$$



$$\boldsymbol{I}_b = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$



$$\boldsymbol{I}_c = \boldsymbol{R}_{bp}^{\mathrm{T}} \boldsymbol{I}_b \boldsymbol{R}_{bp} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\boldsymbol{\omega}_b = (\omega_x, \omega_y, \omega_z)$$

- If $\{b\}$ is aligned with the principal axes of inertia :

$$\boldsymbol{m}_b = \boldsymbol{I}_b \dot{\boldsymbol{\omega}}_b + [\boldsymbol{\omega}_b] \boldsymbol{I}_b \boldsymbol{\omega}_b = \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}$$

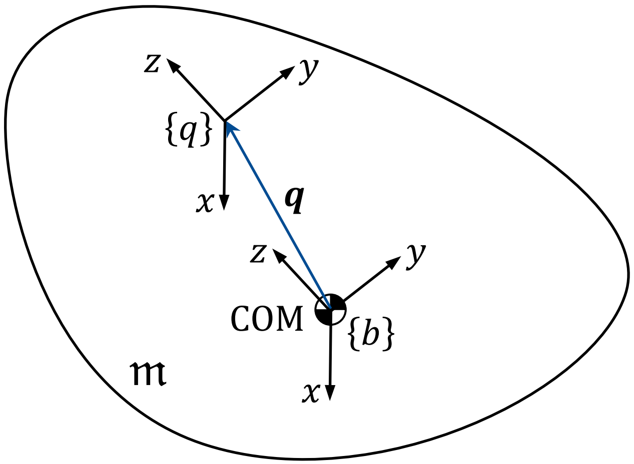
Inertia Matrix: Steiner's Theorem

Inertia matrix $\boldsymbol{I}_q$ about a frame $\{q\}$ aligned with $\{b\}$ (at the center of mass), but at a point $\boldsymbol{q} = \left(q_x, q_y, q_z\right)$ in $\{b\}$:

$$\boldsymbol{I}_q = \boldsymbol{I}_b + \mathfrak{m}\left(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q}\boldsymbol{I}_3 - \boldsymbol{q}\boldsymbol{q}^{\mathrm{T}}\right)$$

$$= \boldsymbol{I}_b + \mathfrak{m}[\boldsymbol{q}]^{\mathrm{T}}[\boldsymbol{q}]$$

$\boldsymbol{I}_3 = \mathrm{diag}(1) \in \mathbb{R}^3$
(identity matrix)



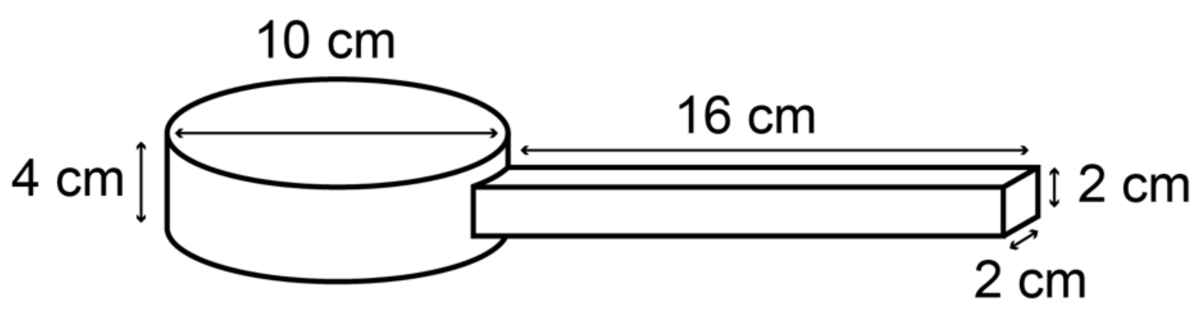
Note: The inertia matrix of a compound/composite body is the sum of their inertias when expressed in a common frame.

Example

A compound object consists of a uniform-density cylinder and a uniform-density rectangular prism. The mass of the cylinder is 2 kg and the mass of the prism is 1 kg. A frame $\{a\}$ is defined at the center of the cylinder, with the x-axis along the prism and the z-axis vertical.

(I) Where is the CM of the compound object in $\{a\}$?

(II) In a frame $\{b\}$ at the CM, aligned with $\{a\}$, what is the inertia of the compound object?



Twist–Wrench Formulation

$$\underbrace{\mathcal{F}_b}_{\text{Body Wrench}} = \begin{bmatrix} \boldsymbol{m}_b \\ \boldsymbol{f}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_b\dot{\boldsymbol{\omega}}_b + [\boldsymbol{\omega}_b]\boldsymbol{I}_b\boldsymbol{\omega}_b \\ \mathfrak{m}(\dot{\boldsymbol{v}}_b + [\boldsymbol{\omega}_b]\boldsymbol{v}_b) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_b & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{\omega}}_b \\ \dot{\boldsymbol{v}}_b \end{bmatrix} + \begin{bmatrix} [\boldsymbol{\omega}_b] & \mathbf{0} \\ \mathbf{0} & [\boldsymbol{\omega}_b] \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_b & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{I}_b & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{\omega}}_b \\ \dot{\boldsymbol{v}}_b \end{bmatrix} + \begin{bmatrix} [\boldsymbol{\omega}_b] & [\boldsymbol{v}_b] \\ \mathbf{0} & [\boldsymbol{\omega}_b] \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_b & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \boldsymbol{I}_b & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix}}_{\substack{\mathcal{G}_b \in \mathbb{R}^{6\times 6}: \\ \text{Spatial Inertia Matrix} \\ \text{(symmetric \& positive definite)}}}\begin{bmatrix} \dot{\boldsymbol{\omega}}_b \\ \dot{\boldsymbol{v}}_b \end{bmatrix} - \underbrace{\begin{bmatrix} [\boldsymbol{\omega}_b] & \mathbf{0} \\ [\boldsymbol{v}_b] & [\boldsymbol{\omega}_b] \end{bmatrix}}_{[\mathrm{ad}_{\mathcal{V}_b}]}{}^{\mathrm{T}}\begin{bmatrix} \boldsymbol{I}_b & \mathbf{0} \\ \mathbf{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix}\underbrace{\begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix}}_{\mathcal{V}_b:\ \text{Body Twist}}$$

$$\mathcal{F}_b = \mathcal{G}_b\dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b$$ (Inverse Dynamics of Rigid Body)

$$\dot{\mathcal{V}}_b = \mathcal{G}_b^{-1}\left(\mathcal{F}_b + [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b\right)$$ (Forward Dynamics of Rigid Body)

$$\text{Total Kinetic Energy} = \frac{1}{2}\boldsymbol{\omega}_b^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{\omega}_b + \frac{1}{2}\mathfrak{m}\boldsymbol{v}_b^{\mathrm{T}}\boldsymbol{v}_b = \frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b$$

Lie Bracket of Two Twists

Given two twists $\mathcal{V}_1 = (\boldsymbol{\omega}_1, \boldsymbol{v}_1) \in \mathbb{R}^6$ and $\mathcal{V}_2 = (\boldsymbol{\omega}_2, \boldsymbol{v}_2) \in \mathbb{R}^6$, the Lie Bracket of $\mathcal{V}_1$ and $\mathcal{V}_2$ is defined as $[\mathrm{ad}_{\mathcal{V}_1}]\mathcal{V}_2 \in \mathbb{R}^6$ where

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\boldsymbol{\omega}] & \mathbf{0} \\ [\boldsymbol{v}] & [\boldsymbol{\omega}] \end{bmatrix} \in \mathbb{R}^{6\times 6}, \qquad \mathcal{V} = (\boldsymbol{\omega}, \boldsymbol{v})$$

This is generalization of the cross product to two twists $\mathcal{V}_1$ and $\mathcal{V}_2$.

Dynamics in Other Frames

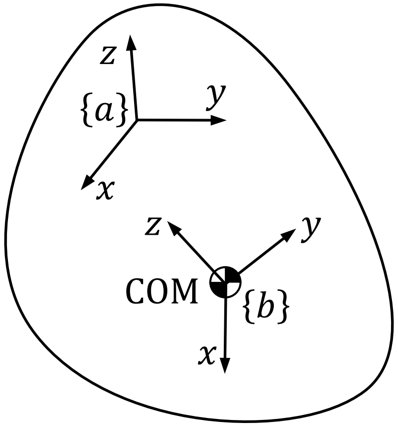
Kinetic energy of the rigid body is independent of the frame of representation:

$$
\begin{aligned}
\frac{1}{2}\boldsymbol{\mathcal{V}}_a^{\mathrm{T}}\boldsymbol{\mathcal{G}}_a\boldsymbol{\mathcal{V}}_a &= \frac{1}{2}\boldsymbol{\mathcal{V}}_b^{\mathrm{T}}\boldsymbol{\mathcal{G}}_b\boldsymbol{\mathcal{V}}_b \\
&= \frac{1}{2}\left([\mathrm{Ad}_{\boldsymbol{T}_{ba}}]\boldsymbol{\mathcal{V}}_a\right)^{\mathrm{T}}\boldsymbol{\mathcal{G}}_b[\mathrm{Ad}_{\boldsymbol{T}_{ba}}]\boldsymbol{\mathcal{V}}_a \\
&= \frac{1}{2}\boldsymbol{\mathcal{V}}_a^{\mathrm{T}}\underbrace{[\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{\mathrm{T}}\boldsymbol{\mathcal{G}}_b[\mathrm{Ad}_{\boldsymbol{T}_{ba}}]}_{\boldsymbol{\mathcal{G}}_a}\boldsymbol{\mathcal{V}}_a
\end{aligned}
$$

$\Rightarrow$

$$\boldsymbol{\mathcal{G}}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{\mathrm{T}}\boldsymbol{\mathcal{G}}_b[\mathrm{Ad}_{\boldsymbol{T}_{ba}}]$$

This is a generalization of Steiner's theorem.



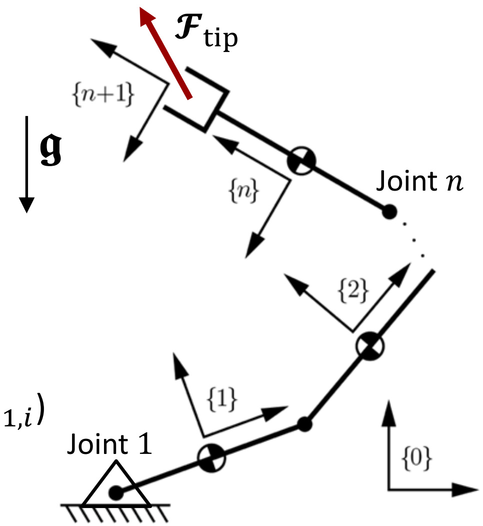
$$
\begin{aligned}
\boldsymbol{\mathcal{F}}_a &= [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{T}\boldsymbol{\mathcal{F}}_b \\
&= [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{T}\boldsymbol{\mathcal{G}}_b\dot{\boldsymbol{\mathcal{V}}}_b - [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{T}[\mathrm{ad}_{\boldsymbol{\mathcal{V}}_b}]^{T}\boldsymbol{\mathcal{G}}_b\boldsymbol{\mathcal{V}}_b \\
&= [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{T}\boldsymbol{\mathcal{G}}_b[\mathrm{Ad}_{\boldsymbol{T}_{ba}}]\dot{\boldsymbol{\mathcal{V}}}_a - [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{T}[\mathrm{ad}_{\boldsymbol{\mathcal{V}}_b}]^{T}\boldsymbol{\mathcal{G}}_b[\mathrm{Ad}_{\boldsymbol{T}_{ba}}]\boldsymbol{\mathcal{V}}_a \\
&= \boldsymbol{\mathcal{G}}_a\dot{\boldsymbol{\mathcal{V}}}_a - [\mathrm{ad}_{\boldsymbol{\mathcal{V}}_a}]^{T}\boldsymbol{\mathcal{G}}_a\boldsymbol{\mathcal{V}}_a
\end{aligned}
$$

$\Rightarrow$ $\boldsymbol{\mathcal{F}}_a = \boldsymbol{\mathcal{G}}_a\dot{\boldsymbol{\mathcal{V}}}_a - [\mathrm{ad}_{\boldsymbol{\mathcal{V}}_a}]^{T}\boldsymbol{\mathcal{G}}_a\boldsymbol{\mathcal{V}}_a$

The form of the equations of motion is independent of the frame of representation.

Dynamics of an Open Chain Manipulator

Consider an n-link open chain manipulator connected by 1 DOF joints. Attach a frame $\{0\}$ to the base, frames $\{1\}$ to $\{n\}$ to the centers of mass of links $\{1\}$ to $\{n\}$, and a frame $\{n+1\}$ at the end-effector, fixed in the frame $\{n\}$.



$\mathcal{G}_i \in \mathbb{R}^{6\times6}$: spatial inertia matrix of link i in $\{i\}$: $\mathcal{G}_i = \begin{bmatrix} \boldsymbol{I}_i & 0 \\ 0 & \mathfrak{m}_i \boldsymbol{I} \end{bmatrix}$

$\boldsymbol{M}_{i,i-1} \in SE(3)$: $\{i-1\}$ in $\{i\}$ at home configuration ($\boldsymbol{\theta} = \boldsymbol{0}$).

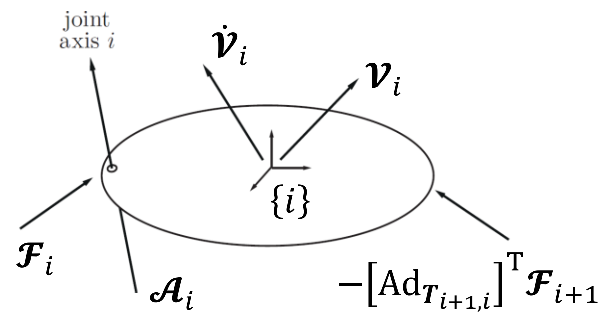
For given $\boldsymbol{M}_{i-1,i}$: $\boldsymbol{M}_{i,i-1} = \left(\boldsymbol{M}_{i-1,i}\right)^{-1}$. $\quad(\boldsymbol{M}_{0,i} = \boldsymbol{M}_{0,1}\boldsymbol{M}_{1,2} \dots \boldsymbol{M}_{i-1,i})$

$\boldsymbol{\mathcal{A}}_i \in \mathbb{R}^6$: screw axis of joint i in $\{i\}$. $\quad \boldsymbol{\mathcal{A}}_i = \left[\mathrm{Ad}_{\boldsymbol{M}_{0,i}^{-1}}\right]\boldsymbol{\mathcal{S}}_i = \left[\mathrm{Ad}_{\boldsymbol{M}_{0,i}}\right]^{-1}\boldsymbol{\mathcal{S}}_i$

$\boldsymbol{\mathcal{V}}_i = (\boldsymbol{\omega}_i, \boldsymbol{v}_i) \in \mathbb{R}^6$: twist of link i in $\{i\}$.

$\boldsymbol{\mathcal{V}}_0 = (\boldsymbol{\omega}_0, \boldsymbol{v}_0) = \boldsymbol{0}$.

$\dot{\boldsymbol{\mathcal{V}}}_0 = (\dot{\boldsymbol{\omega}}_0, \dot{\boldsymbol{v}}_0) = (\boldsymbol{0}, -\mathfrak{g})$. $\quad \mathfrak{g} \in \mathbb{R}^3$ gravity vector in $\{0\}$

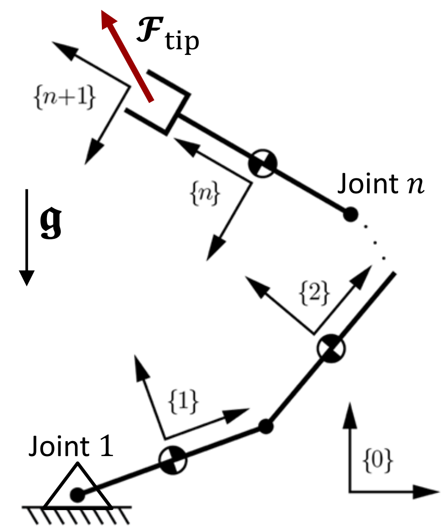


$\boldsymbol{\mathcal{F}}_i = (\boldsymbol{m}_i, \boldsymbol{f}_i) \in \mathbb{R}^6$: wrench at joint i in $\{i\}$.

$\boldsymbol{\mathcal{F}}_{n+1} = \boldsymbol{\mathcal{F}}_{\text{tip}} \in \mathbb{R}^6$: wrench applied to the environment by end-effector in $\{i+1\}$.

Recursive Newton-Euler Inverse Dynamics Algorithm

Forward Iterations: Determining twists $\mathcal{V}_i$ and accelerations $\dot{\mathcal{V}}_i$ of each link by moving outward from the base to the tip. Given $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\ddot{\boldsymbol{\theta}}$, for $i = 1$ to n, find



$$T_{i,i-1} = e^{-[\mathcal{A}_i]\theta_i} M_{i,i-1} \in SE(3) \qquad (T_{n+1,n} = M_{n+1,n})$$

$$\mathcal{V}_i = [\mathrm{Ad}_{T_{i,i-1}}]\mathcal{V}_{i-1} + \mathcal{A}_i\dot{\theta}_i$$

$$\dot{\mathcal{V}}_i = [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + \frac{d}{dt}([\mathrm{Ad}_{T_{i,i-1}}])\mathcal{V}_{i-1} + \mathcal{A}_i\ddot{\theta}_i$$
$$= [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + [\mathrm{ad}_{\mathcal{V}_i}]\mathcal{A}_i\dot{\theta}_i + \mathcal{A}_i\ddot{\theta}_i$$

Backward Iterations: Determining wrenches $\mathcal{F}_i$ experienced by each link, and then, the joint torques/forces τ_i by moving inward from the tip to the base. For $i = n$ to 1, find

$$\mathcal{F}_i = [\mathrm{Ad}_{T_{i+1,i}}]^{\mathrm{T}}\mathcal{F}_{i+1} + \mathcal{G}_i\dot{\mathcal{V}}_i - [\mathrm{ad}_{\mathcal{V}_i}]^{\mathrm{T}}\mathcal{G}_i\mathcal{V}_i$$
$$\tau_i = \mathcal{F}_i^{\mathrm{T}}\mathcal{A}_i$$