# Ch5: Phase Plane Analysis

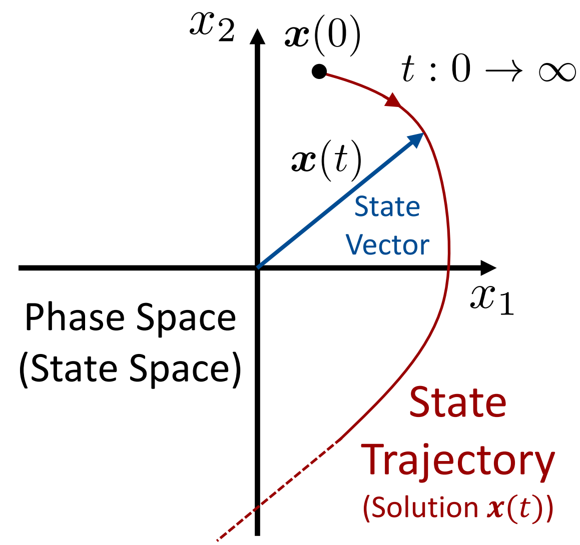
# Phase Plane Concept

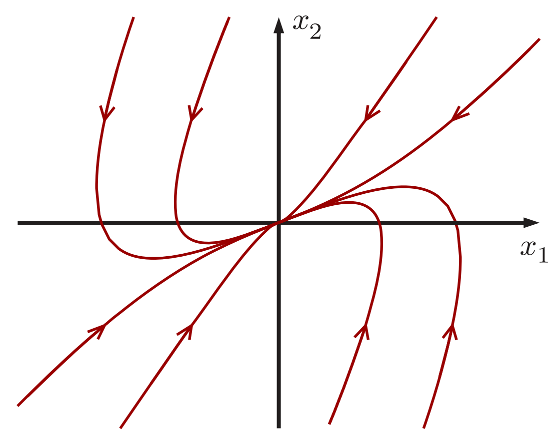
# Phase Plane & Phase Portrait

- A **two-dimensional** state space plane is called the **Phase Plane**.
- Given a set of **initial conditions** $\boldsymbol{x}(0)$, the solution $\boldsymbol{x}(t)$ of a **second-order autonomous** system, when *t* varied from 0 to $\infty$, can be represented geometrically as a curve (**trajectory**) in the phase plane (arrows denote the direction of motion).

$$\dot{\boldsymbol{x}}(t) = \mathbf{f}(\boldsymbol{x}(t)) \quad \Longrightarrow \quad \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

Slope of trajectory: $$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

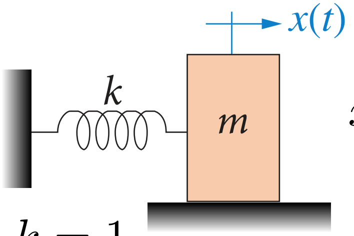



A **family** of phase plane trajectories corresponding to **various initial conditions** is called a **phase portrait** of a system.

# Example: Phase portrait of a linear system

A mass-spring system:




$$\ddot{x} + x = 0$$

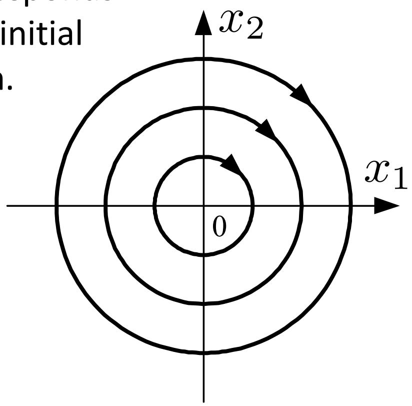
$k = 1$

$m = 1$

$x_0$: Initial position

$\dot{x}_0$: Initial velocity

Each circle corresponds to a different initial condition.



# Singular Point

An **equilibrium point** of a second-order system is called a **Singular Point**.

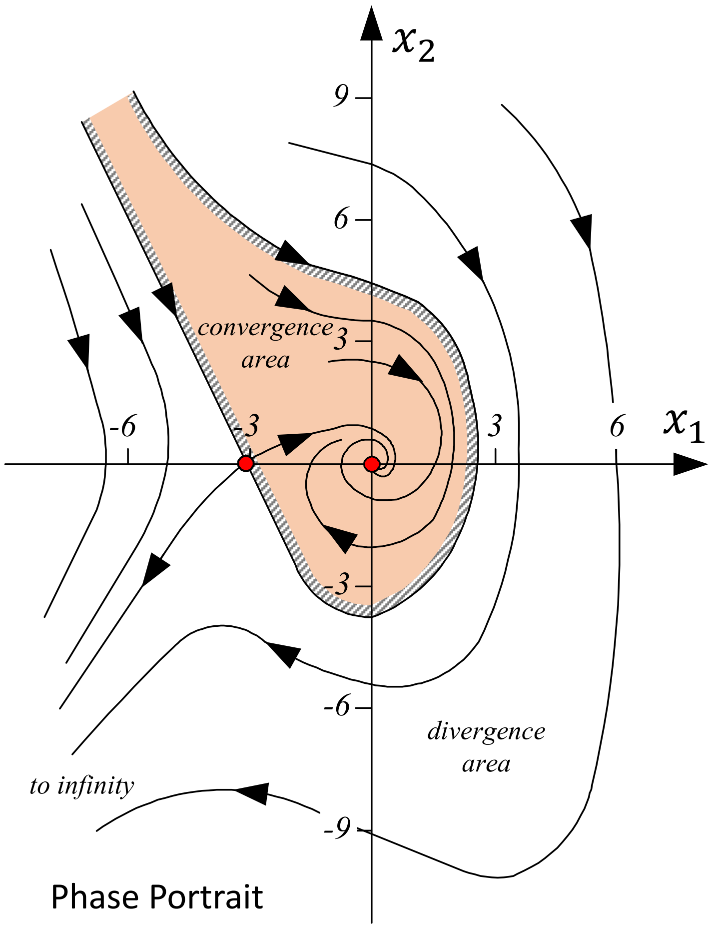
$$\dot{\boldsymbol{x}}(t) = \mathbf{f}(\boldsymbol{x}(t)) = 0 \quad \Longrightarrow \quad \begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}$$

Phase portrait of a nonlinear 2nd order system:

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

The system has two singular points: (0, 0), (-3, 0)

$$x_1 = x$$
$$x_2 = \dot{x}$$

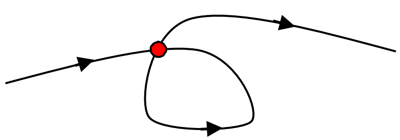


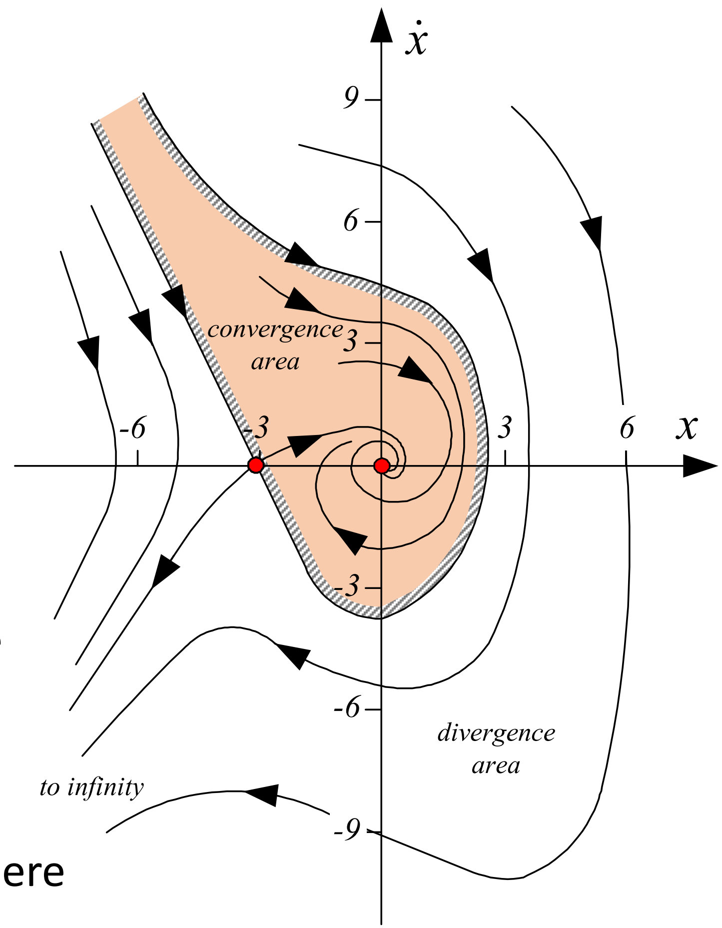

Phase Portrait

# Singular Point

**Note**: The motion patterns of the system trajectories in the vicinity of the two singular points may have different natures!

**Note**: With the functions $f_1$ and $f_2$ assumed to be single valued, a phase trajectory **cannot intersect itself**!






**Note**: Singular points are very important features in the phase plane, e.g., for **linear systems**, the **stability** of the systems is uniquely characterized by the **nature of their singular points**.

**Note**: For **nonlinear systems**, besides singular points, there may be more complex features, such as **limit cycles**.

# Phase Plane for First-order Systems

Although the phase plane method is developed primarily for second-order systems, it can also be applied to the analysis of **first-order** systems of the form

$$\dot{x} + f(x) = 0$$

The difference now is that the phase portrait is composed of a **single trajectory**.

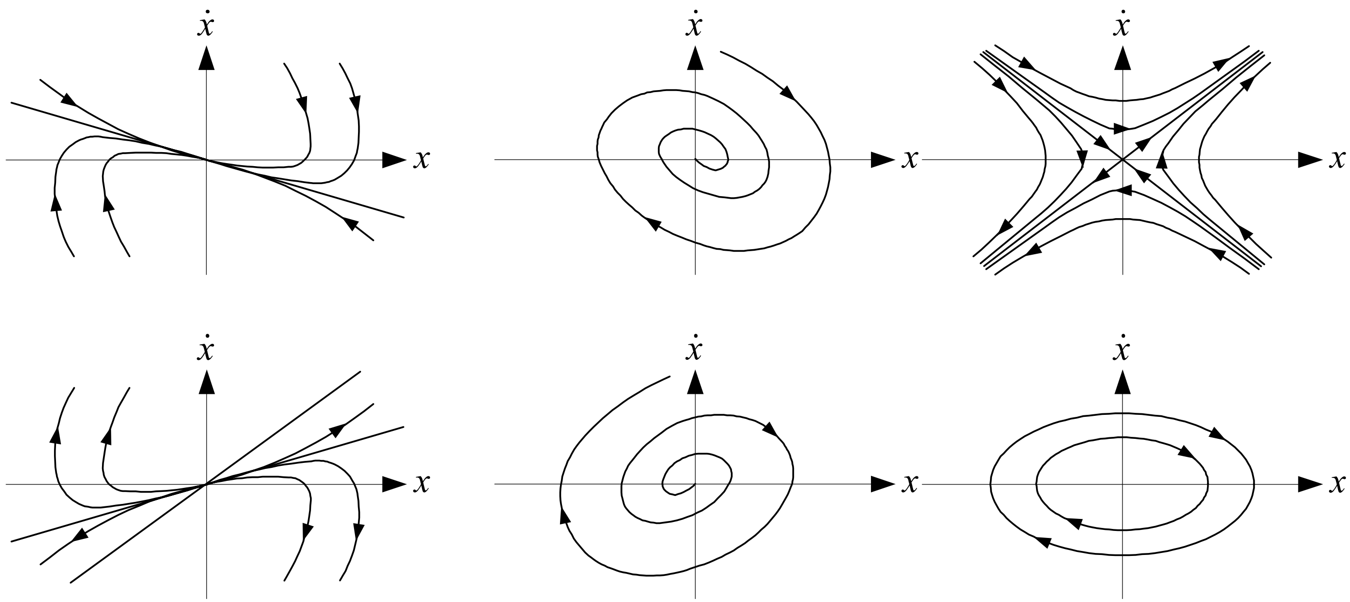
**Example**: Plot the phase portrait for the following first-order system.

$$\dot{x} = -4x + x^3$$

# Phase Plane Analysis: Linear Systems

# Phase Plane Analysis

**Phase plane analysis** is a **graphical method** to visually examine the global behavior of **second-order** autonomous systems, i.e., **stability** and **motion patterns**.




Although the phase plane analysis is applicable only to second-order systems, it can provide **intuitive insights** about **nonlinear effects**.

# Phase Plane Analysis of Linear Systems

General form of a linear second-order system:

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \text{(or)} \qquad \begin{aligned}\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2\end{aligned} \quad \Rightarrow \quad \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$$

Solution:

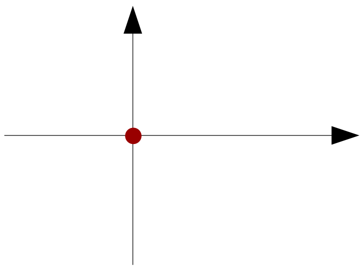
$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \lambda_1 \neq \lambda_2$$
$$x(t) = k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t} \qquad \lambda_1 = \lambda_2$$

$$\lambda_{1,2} = (-a \pm \sqrt{a^2 - 4b})/2$$

(solutions of the characteristic equations $[\lambda^2 + a\lambda + b = 0]$ or eigenvalues of matrix $\mathbf{A}$ $[\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}]$ )

$$\boldsymbol{x}(t) = e^{\mathbf{A}t}\boldsymbol{x}(0)$$

There is only one isolated singular point at origin $\boldsymbol{x} = 0$, assuming $b \neq 0$ or $\mathbf{A}$ is nonsingular ($\det(\boldsymbol{A}) \neq 0$). However, the trajectories in the vicinity of this singularity point can display quite different characteristics, depending on the values of $a$ and $b$.



# Phase Plane Analysis of Linear Systems

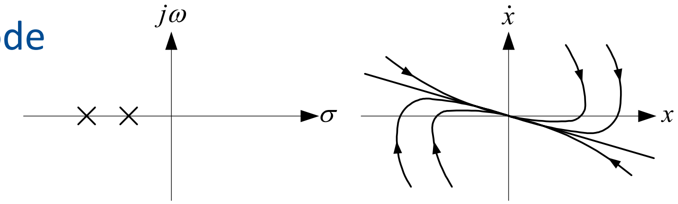
**Stable/Unstable Node**: Both $x(t)$ and $\dot{x}(t)$ converge to/diverge from zero **exponentially**.

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

$\lambda_{1,2} \in \mathbb{R}_{<0}$ Stable Node

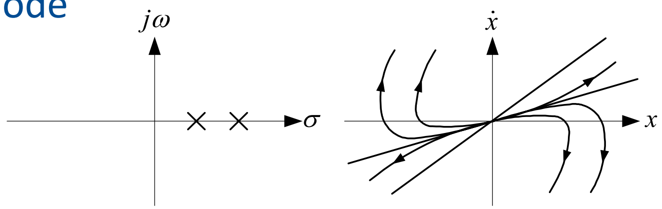
$\lambda_{1,2} \in \mathbb{R}_{>0}$ Unstable Node

Stable Node




$\lambda_{1,2} \in \mathbb{R}_{<0}$

Unstable Node



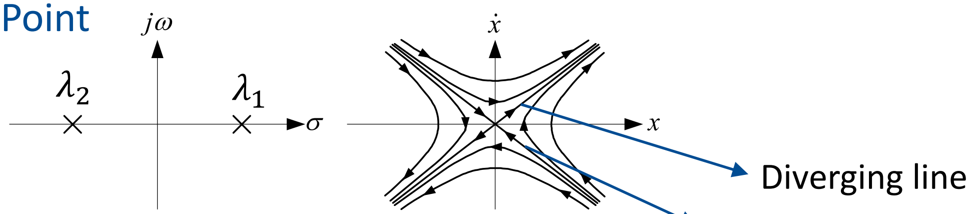

$\lambda_{1,2} \in \mathbb{R}_{>0}$

**Saddle Point**: Because of the unstable pole $\lambda_1$, <u>almost</u> all of the system trajectories diverge to infinity.

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

$\lambda_1 \in \mathbb{R}_{>0}, \lambda_2 \in \mathbb{R}_{<0}$

Saddle Point




$\lambda_1 \in \mathbb{R}_{>0}$
$\lambda_2 \in \mathbb{R}_{<0}$

# Phase Plane Analysis of Linear Systems

**Stable/Unstable Focus**: The trajectories encircle the origin one or more times before converging to it, unlike the situation for a stable node.

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} = K e^{\sigma t} \cos(\omega t - \phi)$$

$$(\lambda_{1,2} = \sigma \pm j\omega)$$

$\sigma \in \mathbb{R}_{<0}$ Stable Focus

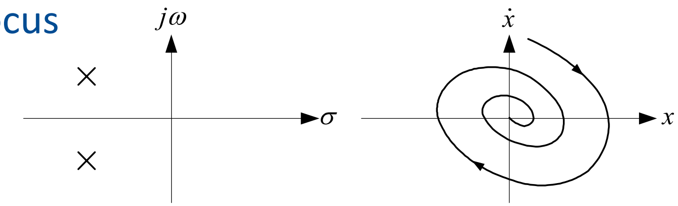
$\sigma \in \mathbb{R}_{>0}$ Unstable Focus

**Center Point**: All trajectories are ellipses, and the singularity point is the center of these ellipses. The system trajectories neither converge to the origin nor diverge to infinity (**marginal stability**).

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} = K \cos(\omega t - \phi)$$

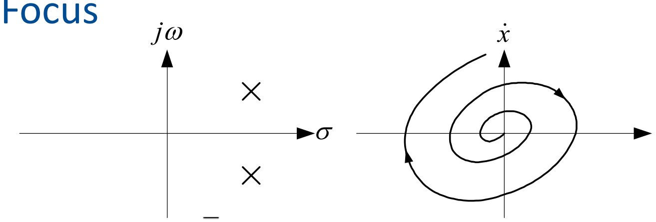
$$(\lambda_{1,2} = \pm j\omega)$$

Stable Focus



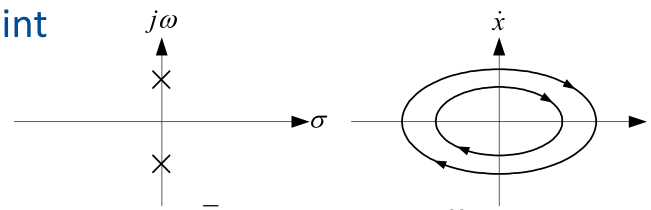

$$\lambda_1, \lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$$
$$\mathrm{Re}(\lambda_1) \in \mathbb{R}_{<0}$$

Unstable Focus




$$\lambda_1, \lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$$
$$\mathrm{Re}(\lambda_1) \in \mathbb{R}_{>0}$$

Center Point

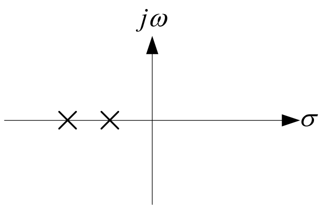


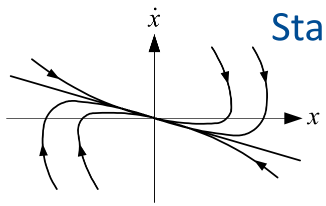

$$\lambda_1, \lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$$
$$\mathrm{Re}(\lambda_1) = 0$$

ellipses

# Phase Plane Analysis of Linear Systems (review)

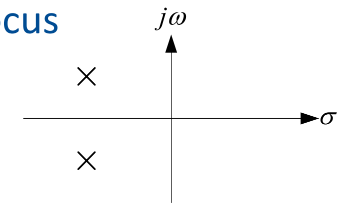
## Stable Node

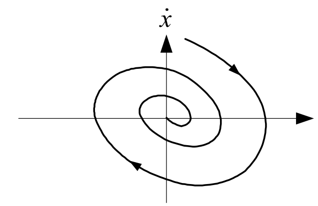



$\lambda_{1,2} \in \mathbb{R}_{<0}$

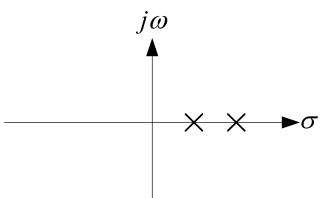
## Stable Focus

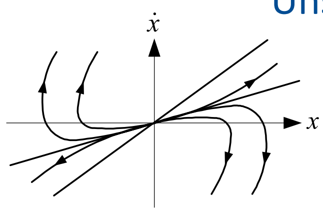



$\lambda_1, \lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$
$\mathrm{Re}(\lambda_1) \in \mathbb{R}_{<0}$

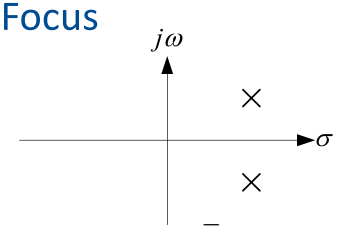
## Unstable Node

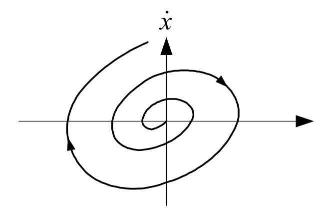



$\lambda_{1,2} \in \mathbb{R}_{>0}$

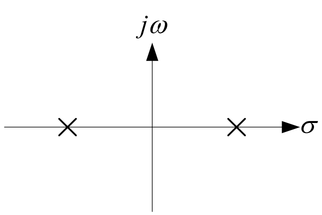
## Unstable Focus

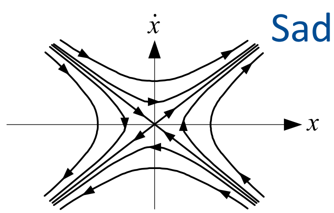



$\lambda_1, \lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$
$\mathrm{Re}(\lambda_1) \in \mathbb{R}_{>0}$

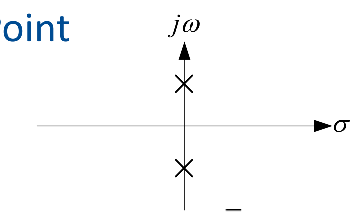
## Saddle Point

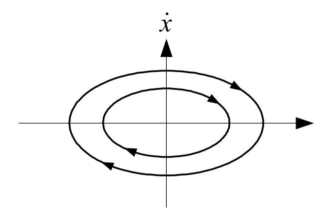



$\lambda_1 \in \mathbb{R}_{>0}$
$\lambda_2 \in \mathbb{R}_{<0}$

## Center Point




ellipses

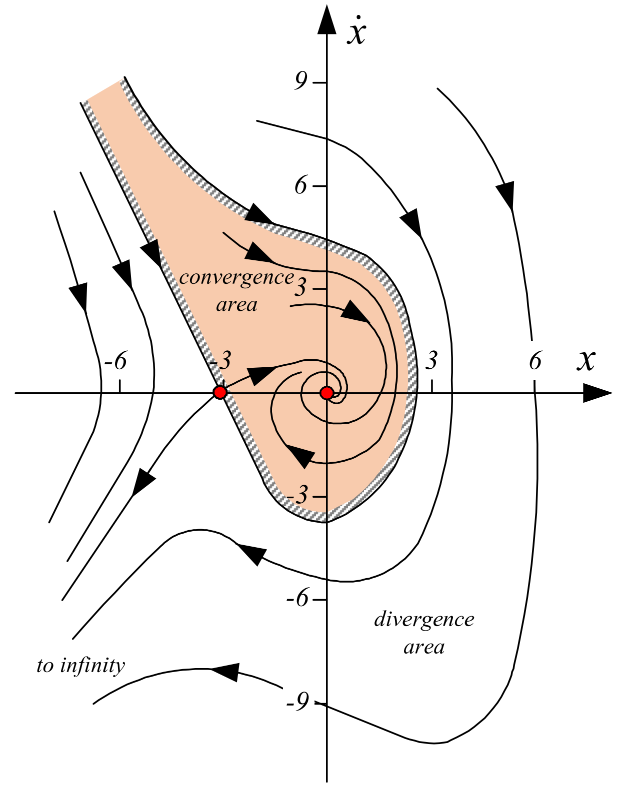
$\lambda_1, \lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$
$\mathrm{Re}(\lambda_1) = 0$

# Phase Plane Analysis: Nonlinear Systems

# Phase Plane Analysis of Nonlinear Systems: Local Behavior

- Nonlinear systems frequently have **more than one equilibrium point**, in contrast to linear systems.
- **Local behavior** of a nonlinear system can be approximated by the behavior of a linear system **in the neighborhood of** each equilibrium point.

(0, 0): Stable Focus
(-3, 0): Saddle Point




This behavior can be determined via **linearization** of the nonlinear system with respect to each equilibrium point.

# Linearization

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}) \longrightarrow \begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

Taylor expansion about $\boldsymbol{x}_e = [x_{e1}, x_{e2}]^{\mathrm{T}}$

$$\left[f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\right]$$

(Higher Order Terms)

$$\dot{x}_1 = \cancel{f_1(x_{e1}, x_{e2})} + a_{11}(x_1 - x_{e1}) + a_{12}(x_2 - x_{e2}) + \cancel{\mathrm{H.O.T}}$$
$$\dot{x}_2 = \cancel{f_2(x_{e1}, x_{e2})} + a_{21}(x_1 - x_{e1}) + a_{22}(x_2 - x_{e2}) + \cancel{\mathrm{H.O.T}}$$

0 0

$\mathbf{f}(\boldsymbol{x}_e) = \mathbf{0}$

Change of variables: $\bar{x}_1 = (x_1 - x_{e2})$, $\bar{x}_2 = (x_2 - x_{e2})$

In the vicinity of $\boldsymbol{x}_e$

**Linearized** state equation:

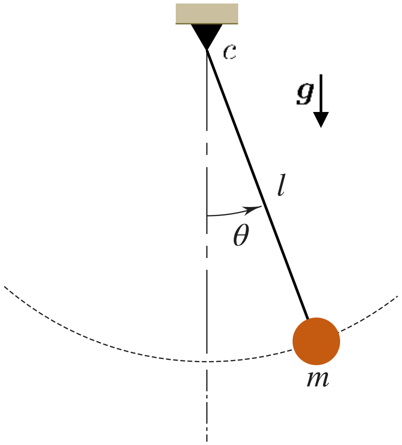
$$\dot{\bar{\boldsymbol{x}}} = \boldsymbol{A}\bar{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\bar{\boldsymbol{x}} = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2}\\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2}\end{bmatrix}\right|_{\boldsymbol{x}=\boldsymbol{x}_e}\bar{\boldsymbol{x}} = \left.\frac{\partial \mathbf{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_e}\bar{\boldsymbol{x}}$$

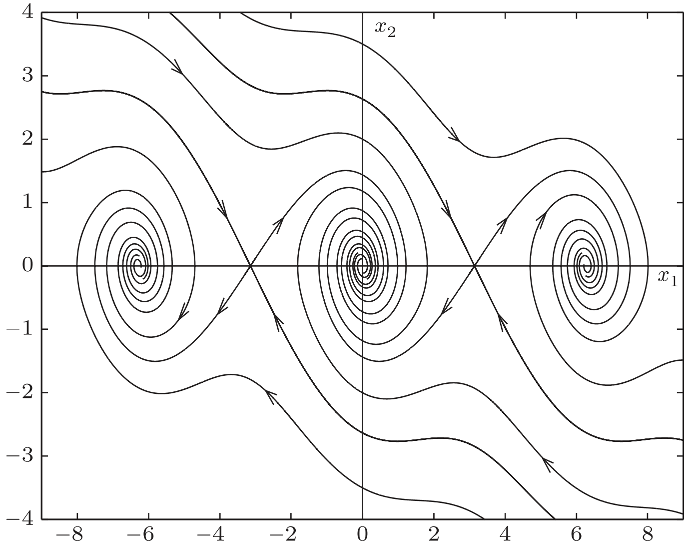
Jacobian of $\mathbf{f}$

# Example: Stability of a Pendulum

$$\ddot{\theta} + \frac{c}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad x_1 = \theta,\ x_2 = \dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ -\dfrac{c}{ml^2}x_2 - \dfrac{g}{l}\sin x_1 \end{bmatrix}$$



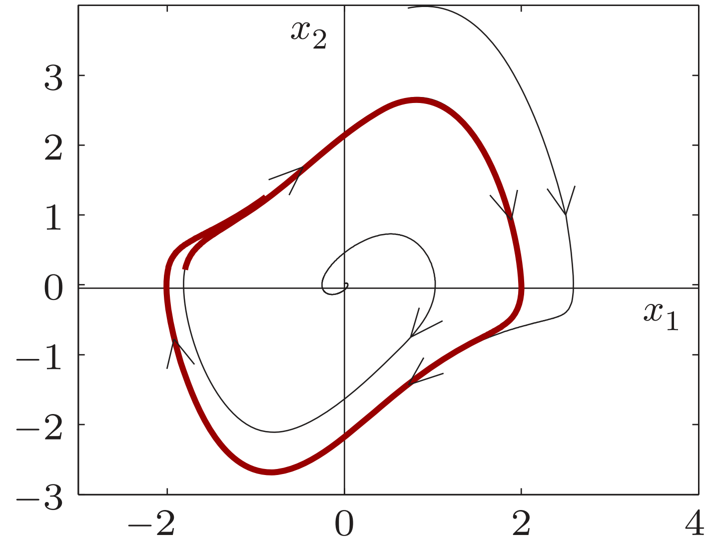
# Limit Cycle

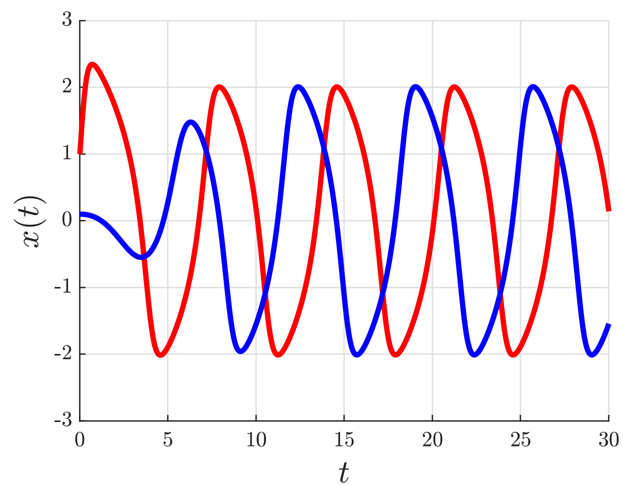
Let's plot phase portrait of the Van der Pol equation:

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0, \quad \mu = 1$$

- An unstable node at the origin.
- A closed curve!

All trajectories inside & outside the curve tend to this curve. A motion started on this curve will stay on it forever, circling periodically around the origin.




This closed curve correspond to oscillations of **fixed amplitude** and **fixed period without external excitation** and **independent of initial conditions**, which is called **Limit Cycle** (LC) or **Self-Excited Oscillations**.

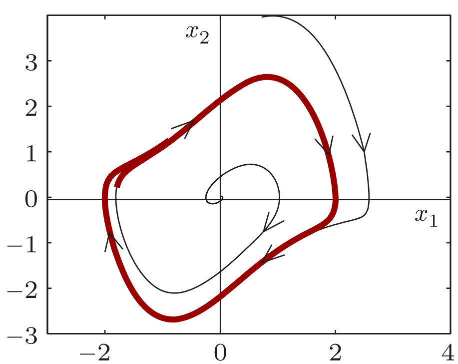
Limit cycles are **unique** features of nonlinear systems.

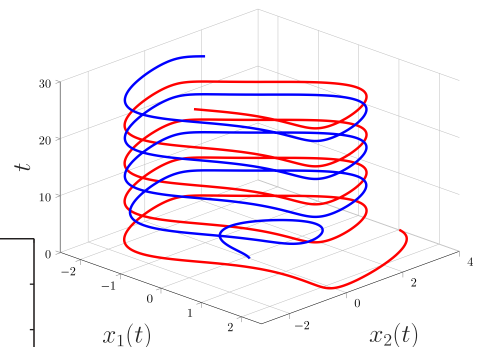
# Limit Cycle

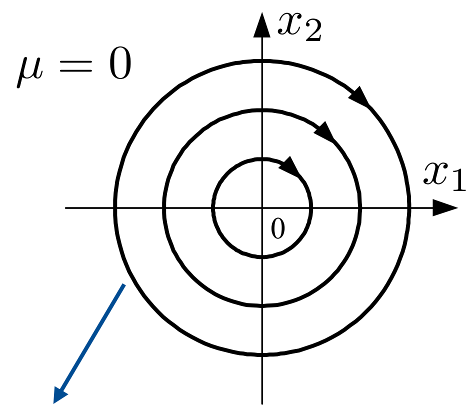
A **Limit Cycle** is defined as an isolated closed curve.

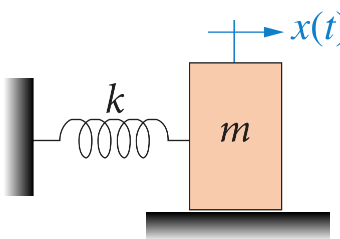
Indicates the limiting nature of the cycle (nearby trajectories converging or diverging from it)

Indicates the periodic nature of the motion.




A marginally stable linear systems.

**Note**: These are not limit cycles, because they are **not isolated**, and the amplitude of the oscillations depends on the initial conditions.

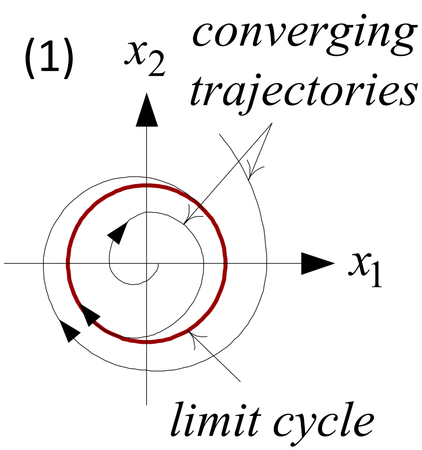
# Limit Cycles

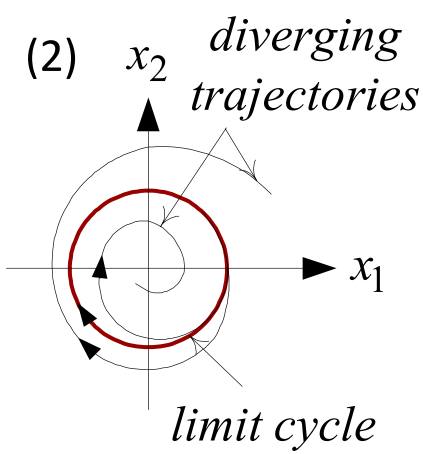
Depending on the motion patterns of the trajectories **in the vicinity of the limit cycle**, there are three kinds of limit cycles:

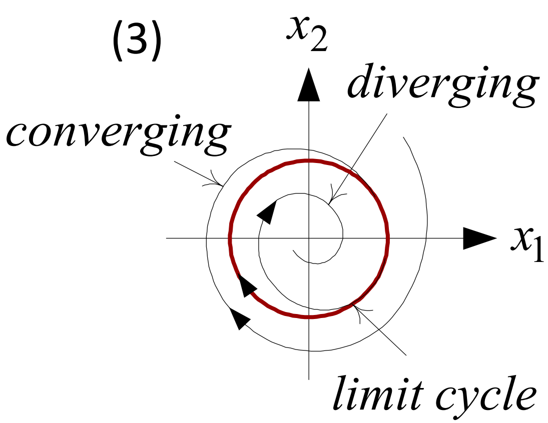
**1) Stable Limit Cycles**: All trajectories in the vicinity of the LC converge to it as $t \to 0$.

**2) Unstable Limit Cycles**: All trajectories in the vicinity of the LC diverge from it as $t \to 0$.

**3) Semi-stable Limit Cycles**: Some of the trajectories in the vicinity of the LC converge to it, while the others diverge from it as $t \to 0$.

(1)



(2)



(3)


# Example: Stability of a Limit Cycle

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{cases}$$

By introducing polar coordinates

$$r^2 = x_1^2 + x_2^2$$
$$\tan\theta = x_2/x_1$$

$$\dot{r} = -r(r^2 - 1)$$
$$\dot{\theta} = -1$$

When the state starts on the unit circle $r = 1$, the $\dot{r} = 0$. This implies that the state will circle around the origin. When $r < 1$, then $\dot{r} > 0$. This implies that the state tends to the circle from inside. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from outside. Therefore, the **unit circle is a stable limit cycle**.

# Constructing Phase Portraits

# Constructing Phase Portraits

Although phase portraits are routinely computer-generated, it is still practically useful to learn how to roughly sketch phase portraits or quickly verify the plausibility of computer outputs.

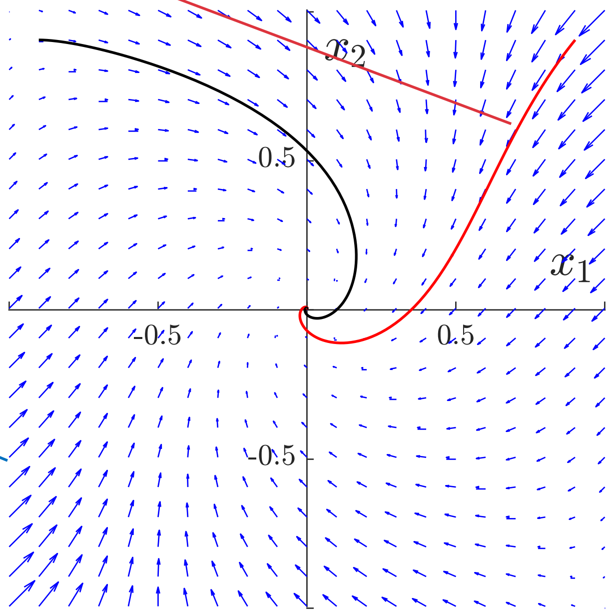
**MATLAB Code**

```
% Phase Trajectory
opts = odeset('RelTol',1e-6,'AbsTol',1e-6);
[t,x] = ode45(@func,[0 10],[0.9; 0.9],opts);

function dxdt = func(t,x)
dxdt = [-x(1) - 2*x(2)*x(1)^2 + x(2); -x(1) - x(2)];
end
```

```
% Phase Portrait
[x1, x2] = meshgrid(-1:0.1:1, -1:0.1:1);
x1dot = -x1 - 2 * x2 .* x1.^2 + x2;
x2dot = -x1 - x2;
quiver(x1,x2,x1dot,x2dot)
```

$$\dot{x}_1 = -x_1 - 2x_2x_1^2 + x_2$$
$$\dot{x}_2 = -x_1 - x_2$$



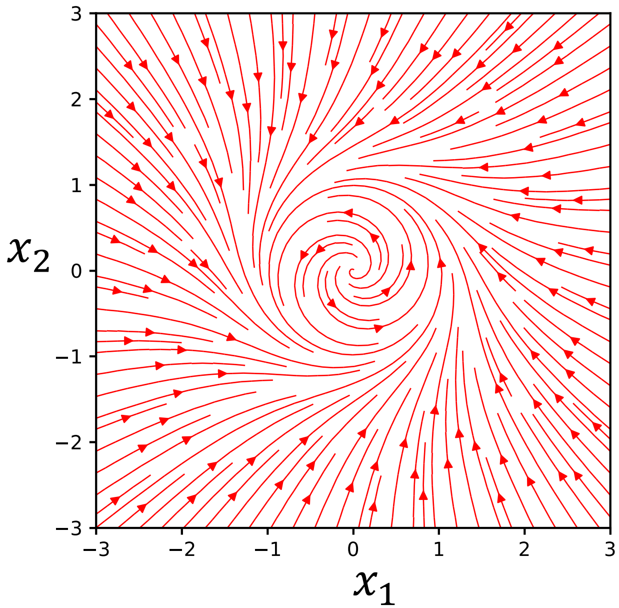

Two simple methods are **Analytical Method** and **Isoclines Method**.

# Method 1: Analytical Method

The method is based on finding a functional relation between the phase variables $x_1$ and $x_2$ of the 2nd-order system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ in the form

$$\underbrace{g(x_1, x_2, c)}_{c} = 0$$

effect of initial conditions




Plotting this relation in the phase plane for **different initial conditions** yields a phase portrait.

**Note**: This method is useful for some **special** nonlinear systems, particularly **piece-wise linear systems**, whose phase portraits can be constructed by piecing together the phase portraits of the related linear systems.

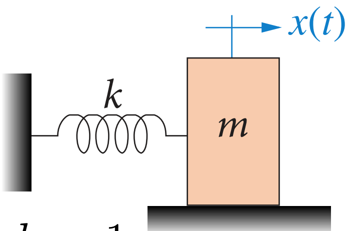
# Method 1: Analytical Method (cont.)

**Technique 1**:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned} \quad \longrightarrow \quad \begin{aligned}x_1 &= g_1(t)\\ x_2 &= g_2(t)\end{aligned} \quad \xrightarrow[\text{from these equations}]{\text{Eliminating time } t} \quad g(x_1, x_2, c) = 0$$

$c$: effect of initial conditions

**Example**: A mass-spring system




$$\ddot{x} + x = 0$$

$k = 1$

$m = 1$

$x_0$: Initial length

$\dot{x}_0$: Initial velocity

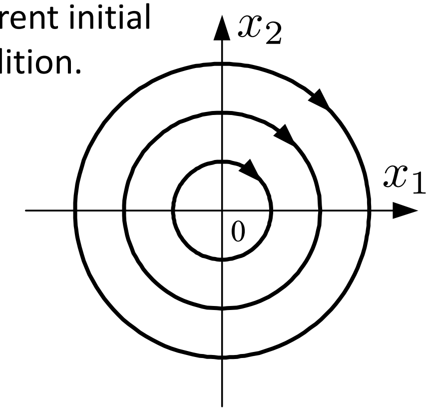
$$x_1 = x \qquad \dot{x}_1 = x_2$$
$$x_2 = \dot{x} \qquad \dot{x}_2 = -x_1$$

$$\begin{aligned}x_1 &= x_0 \cos t + \dot{x}_0 \sin t\\ x_2 &= -x_0 \sin t + \dot{x}_0 \cos t\end{aligned}$$

$$x_1^2 + x_2^2 = x_0^2 + \dot{x}_0^2$$

Equation of the trajectories

Each circle corresponds to a different initial condition.



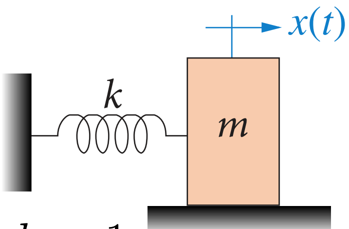
# Method 1: Analytical Method (cont.)

**Technique 2**:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned} \quad \rightarrow \quad \frac{dx_1}{dx_2} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \quad \rightarrow \quad g(x_1, x_2, c) = 0$$

effect of initial conditions

**Example**: A mass-spring system




$$\ddot{x} + x = 0$$

$k = 1$

$m = 1$

$x_0$: Initial length

$\dot{x}_0$: Initial velocity

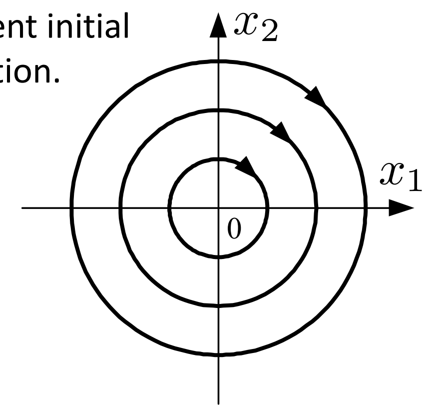
$$\begin{aligned}x_1 &= x\\ x_2 &= \dot{x}\end{aligned} \qquad \begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1\end{aligned}$$

$$\frac{dx_1}{dx_2} = \frac{x_2}{-x_1} \quad \rightarrow \quad -x_1 dx_1 = x_2 dx_2$$

$$x_1^2 + x_2^2 = x_0^2 + \dot{x}_0^2$$

Equation of the trajectories

Each circle corresponds to a different initial condition.



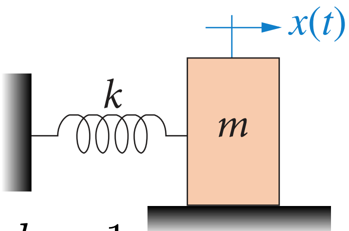
# Method 2: Isoclines Method

An **isocline** is defined to be the locus of the points with a given tangent slope $\alpha$.

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)} = \alpha \quad \longrightarrow \quad f_2(x_1,x_2) = \alpha f_1(x_1,x_2)$$ (isocline equation)

All points on this curve have the same tangent slope $\alpha$.

**Example 1**: A mass-spring system




$$\ddot{x} + x = 0$$

$k = 1$

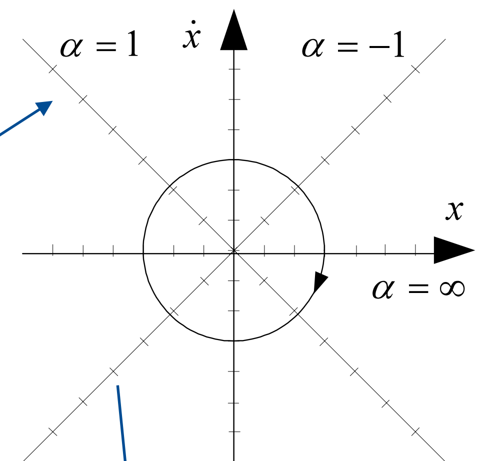
$m = 1$

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned} \quad \longrightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha$$

$$\alpha x_2 = -x_1$$

isoclines




We assume that the tangent slopes are locally constant. Therefore, a trajectory starting from any point in the field of directions can be found by connecting a sequence of line segments.

Short line segments with slope $\alpha$ to generate a field of directions (same scales should be used for the $x_1$, $x_2$ axes)

# Method 2: Isoclines Method (cont.)

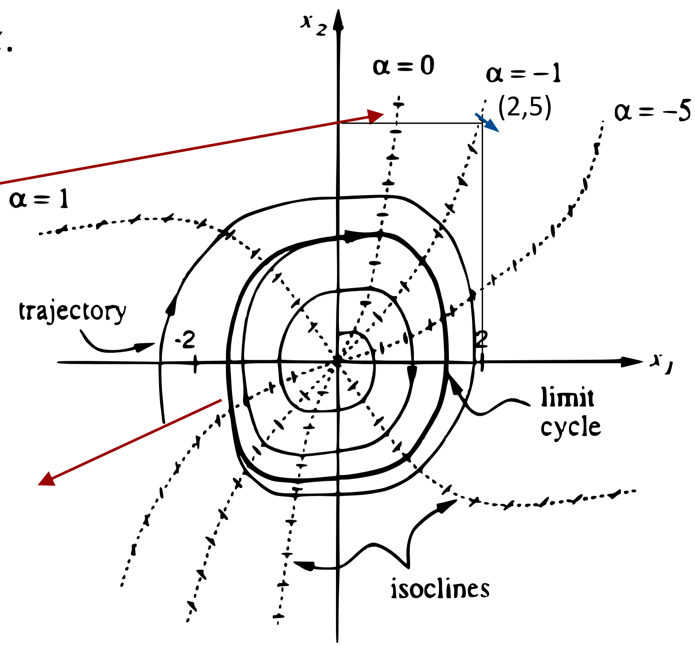
**Example 2**: Van der Pol Equation

$$\ddot{x} + 0.2(x^2 - 1)\dot{x} + x = 0 \longrightarrow \frac{dx_2}{dx_1} = -\frac{0.2(x_1^2 - 1)x_2 + x_1}{x_2} = \alpha$$

$$0.2(x_1^2 - 1)x_2 + x_1 + \alpha x_2 = 0 \qquad \text{(isocline equation)}$$

All points on this curve have the same tangent slope $\alpha$.

By taking $\alpha$ of different values, different isoclines can be obtained.




* For connecting the segments, we can first determine the type of the equilibrium points and check if there is a limit cycle.

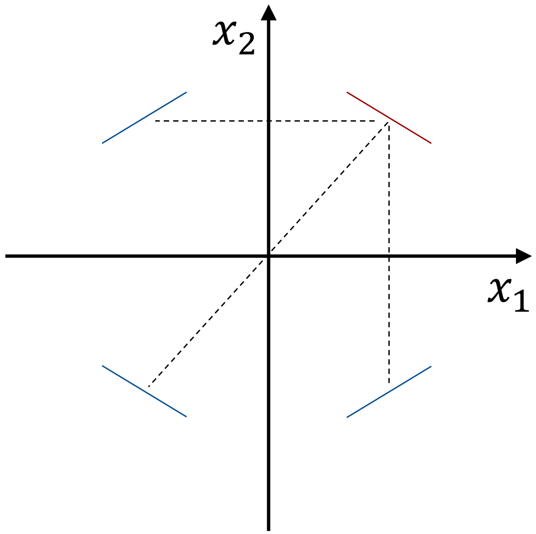
The trajectories starting from both outside and inside converge to the limit cycle.

# Symmetry in Phase Plane Portraits

A phase portrait may have a priori known symmetry properties, which can simplify its generation and study (e.g., studying one half or one quarter of it).

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = g(x_1, x_2)$$




Symmetry of the phase portraits implies symmetry of the slope:

$g(x_1, x_2) = -g(x_1, -x_2) \Rightarrow$ symmetry about the $x_1$ axis

$g(x_1, x_2) = -g(-x_1, x_2) \Rightarrow$ symmetry about the $x_2$ axis

$g(x_1, x_2) = g(-x_1, -x_2) \Rightarrow$ symmetry about the origin

Mass-spring system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$