

MEC549: Robot Dynamics and Control

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Ch1: Summary of Linear Algebra & Robot Kinematics

Linear Algebra

Basic Notation

\forall

\exists

\in

\Rightarrow

\Leftrightarrow

\dot{x}

$:=$

$f: \mathcal{D} \rightarrow \mathcal{R}$

\mathbb{R}

$\mathbb{R}_+, \mathbb{R}_{++}$

$|x|$

Vector

$\mathbf{x} \in \mathbb{R}^n$: (an n -dimensional real vector in the column format)

\mathbb{R}^n : n -dimensional real space
(Euclidian Space)

Vector Norm

General Definition: Given $\boldsymbol{x} \in \mathbb{R}^n$, vector norm $\|\boldsymbol{x}\| \in \mathbb{R}_+$ is defined such that

- $\|\boldsymbol{x}\| > 0$ when $\boldsymbol{x} \neq \mathbf{0}$ and $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \mathbf{0}$.
- $\|k\boldsymbol{x}\| = |k|\|\boldsymbol{x}\|$, $\forall k \in \mathbb{R}$.
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$, $\forall \boldsymbol{y} \in \mathbb{R}^n$.

❖ The p -norm (or ℓ_p -norm) of \boldsymbol{x} for $p \in \mathbb{R}, p \geq 1$ is defined as $\|\boldsymbol{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$

e.g. $\|\boldsymbol{x}\|_2 = \|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$ (Euclidean Norm)

Special case: $\|\boldsymbol{x}\|_\infty := \max_i |x_i|$

Schwartz Inequality: $|\boldsymbol{x}^T \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

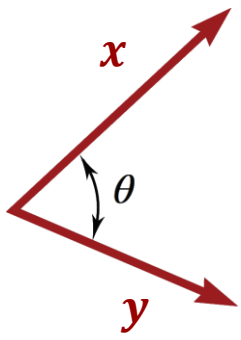
Unit Vector: $\|\hat{\boldsymbol{x}}\|_2 = \hat{\boldsymbol{x}}^T \hat{\boldsymbol{x}} = 1, \quad \hat{\boldsymbol{x}} = \boldsymbol{x} / \|\boldsymbol{x}\|_2$

Dot Product or Scalar Product or Inner Product

Dot Product or Scalar Product or Inner Product of two vectors $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^n$ is a scalar defined as

(Algebraic Definition) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x}$

(Geometric Definition) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos \theta$
 $(0 \leq \theta \leq 180^\circ)$



Orthogonal Vectors:

Matrix

$\mathbf{A} \in \mathbb{R}^{m \times n}$ (an m by n dimensional real matrix)

\mathbf{A}^T

Matrix-vector multiplication $\mathbf{A}\mathbf{x}$ as linear combination of columns of \mathbf{A} :

Particular Matrices

Square Matrix:

- Upper Triangular
- Lower Triangular
- Diagonal
 - Identity Matrix
- Null Matrix

Symmetric Matrix:

Skew-symmetric Matrix:

Partitioned Matrix: A matrix whose elements are matrices (blocks) of proper dimensions.

Matrix Operations

Trace of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$: $\text{tr}(\mathbf{A})$

Sum of matrices: $\mathbf{C} = \mathbf{A} + \mathbf{B}$

Symmetric and skew-symmetric part of a square matrix \mathbf{A} :

Product of matrices: $\mathbf{C} = \mathbf{AB}$

Determinant of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$: $\det(\mathbf{A})$

Singular and Nonsingular Matrices:

Matrix Operations

Rank of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$: $\text{rank}(\mathbf{A})$

Inverse of \mathbf{A} : \mathbf{A}^{-1}

Orthogonal Matrix:

Linearly Independent Vectors $\mathbf{x}_i \in \mathbb{R}^m, i = 1, \dots, n$

Derivative of $\mathbf{A}(t) \in \mathbb{R}^{m \times n}$: $\frac{d}{dt} \mathbf{A}(t) = \dot{\mathbf{A}}(t)$

Derivative of $\mathbf{A}^{-1}(t) \in \mathbb{R}^{n \times n}$:

Jacobian

For a **vector function** $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose elements f_i are differentiable with respect to the elements x_i of $\mathbf{x} \in \mathbb{R}^n$, its **Jacobian** with respect to \mathbf{x} is matrix $\mathbf{J}_f \in \mathbb{R}^{m \times n}$ as:

$$\mathbf{J}_f(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial \mathbf{x}} \\ \frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial f_m(\mathbf{x})}{\partial \mathbf{x}} \end{bmatrix}$$

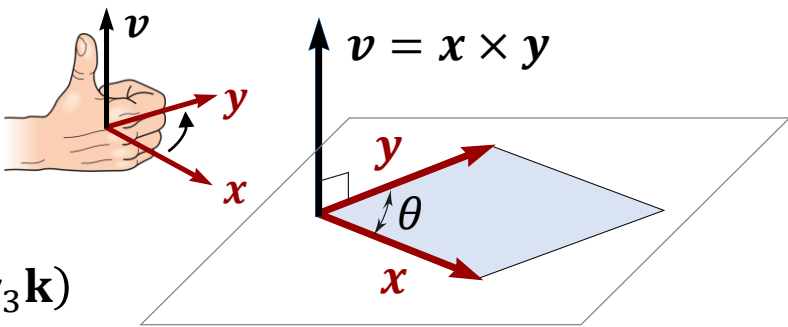
If $\mathbf{x}(t)$ is a differentiable function with respect to t :

$$\dot{\mathbf{f}}(\mathbf{x}) = \frac{d}{dt} \mathbf{f}(\mathbf{x}(t)) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \dot{\mathbf{x}} = \mathbf{J}_f(\mathbf{x}) \dot{\mathbf{x}} \quad (\text{Chain Rule})$$

Cross Product or Vector Product

Cross product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ (in the Euclidean space) is defined as a vector $\mathbf{v} = \mathbf{x} \times \mathbf{y} \in \mathbb{R}^3$ that is orthogonal to both \mathbf{x} and \mathbf{y} ($\mathbf{v} \perp \mathbf{x}, \mathbf{v} \perp \mathbf{y}$), with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

$$\|\mathbf{v}\|_2 = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \sin \theta \quad (0 \leq \theta \leq 180^\circ)$$



$$\mathbf{v} = \mathbf{x} \times \mathbf{y} = (x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}) \times (y_1 \mathbf{i} + y_2 \mathbf{j} + y_3 \mathbf{k})$$

Coordinate notation

$$\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3)$$

$$\mathbf{v} = \mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Matrix notation

Cross Product as a Matrix-Vector Multiplication

Cross product $\mathbf{x} \times \mathbf{y}$ ($\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$) can be thought of as a multiplication of a vector by a 3×3 skew-symmetric matrix as

$$\mathbf{x} \times \mathbf{y} = \underbrace{\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}}_{[\mathbf{x}]} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [\mathbf{x}]\mathbf{y} = -[\mathbf{y}]\mathbf{x}$$

The matrix $[\mathbf{x}]$ is a 3x3 skew-symmetric matrix representation of \mathbf{x} . $[\mathbf{x}] = -[\mathbf{x}]^T$

Eigenvalues and Eigenvectors

If the vector resulting from the linear transformation $A \in \mathbb{R}^{n \times n}$ on a vector u has the same direction of u (with $u \neq 0$), then $Au = \lambda u$.

For each square matrix $A \in \mathbb{R}^{n \times n}$ there exist n **eigenvalues** (in general, complex numbers) denoted by $\lambda_i(A)$, $i = 1, \dots, n$ that satisfy

(characteristic equation)

$\det(A - \lambda_i(A)I) = 0$

$I = \text{diag}(1) \in \mathbb{R}^{n \times n}$

- If $A = A^T$, then $\lambda_i(A) \in \mathbb{R}$, $i = 1, \dots, n$.

Eigenvectors u_i associated with the eigenvalues λ_i satisfy $(A - \lambda_i I)u_i = 0 \quad i = 1, \dots, n$

- If the eigenvectors u_i of A are linearly independent, matrix U formed by the column vectors u_i is invertible and $\Lambda = U^{-1}AU$ where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. If A is symmetric, U is orthogonal ($UU^T = U^T U = I$) and $\Lambda = U^T AU$.
 \Rightarrow Eigendecomposition: $A = U\Lambda U^{-1}$ and if A is symmetric $A = U\Lambda U^T$.

- $\det(A) = \prod_{i=1}^n \lambda_i$
- $\lambda(A^T) = \lambda(A)$

Matrix Norm

General Definition: Given $A \in \mathbb{R}^{n \times n}$, vector norm $\|A\| \in \mathbb{R}_+$ is defined such that

- $\|A\| > 0$ when $A \neq \mathbf{0}$ and $\|A\| = 0$ iff $A = \mathbf{0}$.
- $\|kA\| = |k|\|A\|$, $\forall k \in \mathbb{R}$.
- $\|A + B\| \leq \|A\| + \|B\|$, $\forall B \in \mathbb{R}^{n \times n}$.
- $\|AB\| \leq \|A\|\|B\|$, $\forall B \in \mathbb{R}^{n \times n}$.

The p -norm of A (induced by vector p -norms) for $0 \leq p \leq \infty$ is defined as

$$\|A\|_p = \sup_{\|x\|_p=1} \|Ax\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \quad \forall x \in \mathbb{R}^n$$

Matrix Norm (cont.)

In the special cases of $p = 1, 2, \infty$, these norms can be computed/estimated by:

- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$ (the max. absolute column sum of A)
- $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} = \sigma_{\max}(A)$ (Spectral Norm)

If $A = A^T$

$\left\{ \begin{aligned} \|A\|_2 &= \max_i |\lambda_i(A)| \\ \|A^{-1}\|_2 &= 1/\min_i |\lambda_i(A)| \end{aligned} \right.$

(the square root of the maximum eigenvalue of $A^T A$, or the largest singular value of A)
- $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$ (the max. absolute row sum of A)
- Frobenius Norm:** $\|A\|_F = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2} = \sqrt{\text{tr}(A^T A)}$

$\|Ax\|_2 \leq \|A\|_2 \|x\|_2 , \quad \|A\|_2 \leq \|A\|_F$

Quadratic Form

A **Quadratic Form** is a polynomial with terms all of degree two:

$$\begin{aligned}
 Q(x) &= ax^2 \\
 Q(x_1, x_2) &= ax_1^2 + bx_1x_2 + cx_2^2 \\
 Q(x_1, x_2, x_3) &= ax_1^2 + bx_1x_2 + cx_2^2 + dx_2x_3 + ex_3^2 + fx_1x_3
 \end{aligned}$$

The quadratic form associated with a $n \times n$ matrix $A \in \mathbb{R}^{n \times n}$ is the function $Q: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $Q(x) = x^T A x$ for all x .

- The quadratic function associated with a skew-symmetric matrix A_{SS} is always **zero**.

$$A_{SS} \text{ is skew-symmetric} \quad \Leftrightarrow \quad x^T A_{SS} x = 0 \quad (\forall x)$$

- Each quadratic function $x^T A x$ is always equal to a quadratic function with the symmetric part of matrix.

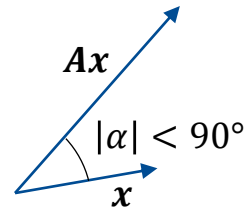
$$Q(x) = x^T A x = x^T (A_S + A_{SS}) x = x^T A_S x$$

- If $A = A^T$: $\nabla_x Q(x) = \left(\frac{\partial Q(x)}{\partial x} \right)^T = 2Ax$, $\dot{Q}(x) = \frac{d}{dt} Q(x(t)) = 2x^T A \dot{x} + x^T \dot{A} x$

Definite and Semi-Definite Matrices

A square not necessarily symmetric matrix $A \in \mathbb{R}^{n \times n}$ is

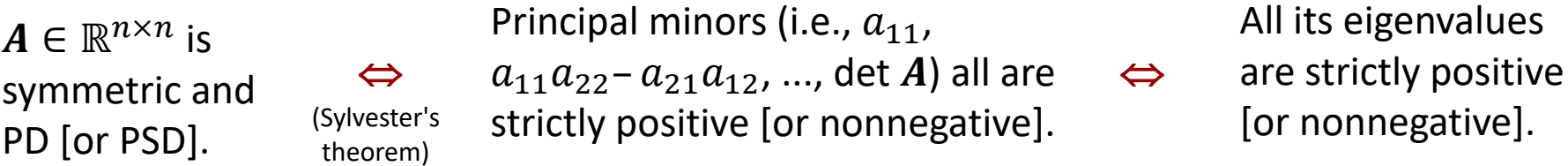
- **Positive Definite** (PD or $A > 0$) if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$.
- **Positive Semi-Definite** (PSD or $A \geq 0$) if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$.
- **Negative Definite** (ND or $A < 0$) if $x^T A x < 0$ for all nonzero $x \in \mathbb{R}^n$.
- **Negative Semi-Definite** (NSD or $A \leq 0$) if $x^T A x \leq 0$ for all $x \in \mathbb{R}^n$.
- **Indefinite** if A neither positive semi-definite nor negative semi-definite.



Geometric Interpretation of the Positive Definiteness of A .

- A square matrix $A \in \mathbb{R}^{n \times n}$ is negative definite if $-A$ is positive definite and it is negative semidefinite if $-A$ is positive semidefinite.
- A **necessary** condition for $A \in \mathbb{R}^{n \times n}$ to be PD [or PSD] is that its diagonal elements be strictly positive [or nonnegative].
- Since $x^T A_{ss} x = 0$, the test for the definiteness of A may be done by considering only its symmetric part.

Definite and Semi-Definite Matrices (cont.)



- Any symmetric PD matrix $A = A^T > 0$ is always full-rank and nonsingular.
- Let $A \in \mathbb{R}^{n \times n}$ be a symmetric PD matrix and λ_{\min} , λ_{\max} be the minimum and maximum eigenvalues of A . For any $x \in \mathbb{R}^n$,

$$\lambda_{\min}(A)\|x\|_2^2 \leq x^T A x \leq \lambda_{\max}(A)\|x\|_2^2$$

(Rayleigh–Ritz Theorem)

- Semi-definiteness implies that $\text{rank}(A) = r < n$, and thus r eigenvalues of A are positive/negative and $n - r$ are 0.
- A matrix inequality of the form $A_1 > A_2$, where $A_1, A_2 \in \mathbb{R}^{n \times n}$ means that $A_1 - A_2 > 0$, i.e., $A_1 - A_2$ is PD. Similar notations apply to the concepts of PSD, ND, NSD.

Rigid-Body Motions

Rigid-Body Motions

Rotations	Rigid-Body Motions
$R \in SO(3)$: 3×3 matrices $R^T R = I, \det(R) = 1$	$T \in SE(3)$: 4×4 matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$ where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^T$	$R^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
Change of coordinate frame: $R_{ab} R_{bc} = R_{ac}, R_{ab} p_b = p_a$	Change of coordinate frame: $T_{ab} T_{bc} = T_{ac}, T_{ab} p_b = p_a$

Rigid-Body Motions

Rotations	Rigid-Body Motions
<p>Rotating a frame {b}:</p> $\mathbf{R} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$ <p>$\mathbf{R}_{sb'} = \mathbf{R}\mathbf{R}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$</p> <p>$\mathbf{R}_{sb''} = \mathbf{R}_{sb}\mathbf{R}$: rotate θ about $\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}$</p>	<p>Displacing a frame {b}:</p> $\mathbf{T} = \begin{bmatrix} \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$ <p>$\mathbf{T}_{sb'} = \mathbf{T}\mathbf{T}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ (moves {b} origin), translate \mathbf{p} in {s}</p> <p>$\mathbf{T}_{sb''} = \mathbf{T}_{sb}\mathbf{T}$: translate \mathbf{p} in {b}, rotate θ about $\hat{\boldsymbol{\omega}}$ in new body frame</p>
<p>Unit rotation axis is $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$, where $\ \hat{\boldsymbol{\omega}}\ = 1$</p>	<p>“Unit” screw axis is $\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\ \mathbf{S}_\omega\ = 1$ or (ii) $\ \mathbf{S}_\omega\ = 0, \ \mathbf{S}_v\ = 1$</p>
	<p>For a screw axis $\{\mathbf{q}, \hat{\mathbf{s}}, h\}$ with finite h,</p> $\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \mathbf{q} + h\hat{\mathbf{s}} \end{bmatrix}$
<p>Angular velocity is $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}$</p>	<p>Twist is $\boldsymbol{\mathcal{V}} = \mathbf{S}\dot{\theta}$</p>

Rigid-Body Motions

Rotations	Rigid-Body Motions
<p>For any $\omega \in \mathbb{R}^3$,</p> $[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ <p>Properties: For any $\omega, x \in \mathbb{R}^3, R \in SO(3)$:</p> $[\omega] = -[\omega]^T, [\omega]x = -[x]\omega,$ $[\omega][x] = ([x][\omega])^T, R[\omega]R^T = [R\omega]$	<p>For any $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$ or $S = \begin{bmatrix} S_\omega \\ S_v \end{bmatrix} \in \mathbb{R}^6$,</p> $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3),$ $[S] = \begin{bmatrix} [S_\omega] & S_v \\ 0 & 0 \end{bmatrix} \in se(3)$
$\dot{R}R^{-1} = [\omega_s], \quad R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s], \quad T^{-1}\dot{T} = [\mathcal{V}_b]$
	$[Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6 \times 6}$ <p>Properties: $[Ad_T]^{-1} = [Ad_{T^{-1}}],$</p> $[Ad_{T_1}][Ad_{T_2}] = [Ad_{T_1T_2}]$
<p>Change of coordinate frame:</p> $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \quad \omega_a = R_{ab}\omega_b$	<p>Change of coordinate frame:</p> $S_a = [Ad_{T_{ab}}]S_b, \quad \mathcal{V}_a = [Ad_{T_{ab}}]\mathcal{V}_b$

Rigid-Body Motions

Rotations	Rigid-Body Motions
Exponential coordinate for $\boldsymbol{R} \in SO(3)$: $\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$	Exponential coordinate for $\boldsymbol{T} \in SE(3)$: $\boldsymbol{S}\theta \in \mathbb{R}^6$
$\exp: [\hat{\boldsymbol{\omega}}]\theta \in so(3) \rightarrow \boldsymbol{R} \in SO(3)$ $\boldsymbol{R} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = e^{[\hat{\boldsymbol{\omega}}]\theta}$ $\boldsymbol{R} = \boldsymbol{I} + \sin \theta [\hat{\boldsymbol{\omega}}] + (1 - \cos \theta) [\hat{\boldsymbol{\omega}}]^2$ (Rodrigues' formula for rotations)	$\exp: [\boldsymbol{S}]\theta \in se(3) \rightarrow \boldsymbol{T} \in SE(3)$ $\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$ $\boldsymbol{T} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix}$ $\boldsymbol{G}(\theta)$ $= \boldsymbol{I}\theta + (1 - \cos \theta)[\boldsymbol{S}_\omega] + (\theta - \sin \theta)[\boldsymbol{S}_\omega]^2$
$\log: \boldsymbol{R} \in SO(3) \rightarrow [\hat{\boldsymbol{\omega}}]\theta \in so(3)$ $\log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta$	$\log: \boldsymbol{T} \in SE(3) \rightarrow [\boldsymbol{S}]\theta \in se(3)$ $\log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$
Moment change of coordinate frame: $\boldsymbol{m}_a = \boldsymbol{R}_{ab}\boldsymbol{m}_b$	Wrench change of coordinate frame: $\boldsymbol{\mathcal{F}}_a = (\boldsymbol{m}_a, \boldsymbol{f}_a) = [\text{Ad}_{\boldsymbol{T}_{ba}}]^T \boldsymbol{\mathcal{F}}_b$

Forward/Velocity/Inverse Kinematics

Forward Kinematics

The forward kinematics of a robot refers to the calculation of the position and orientation (**pose**) of its end-effector frame from its joint positions θ .

- “Geometric” forward kinematics:

Given $\theta \in \mathbb{R}^n$, Find $T_{sb} = T(\theta) \in SE(3)$

$$T: \mathbb{R}^n \rightarrow SE(3)$$

(Using PoE or D-H Method)



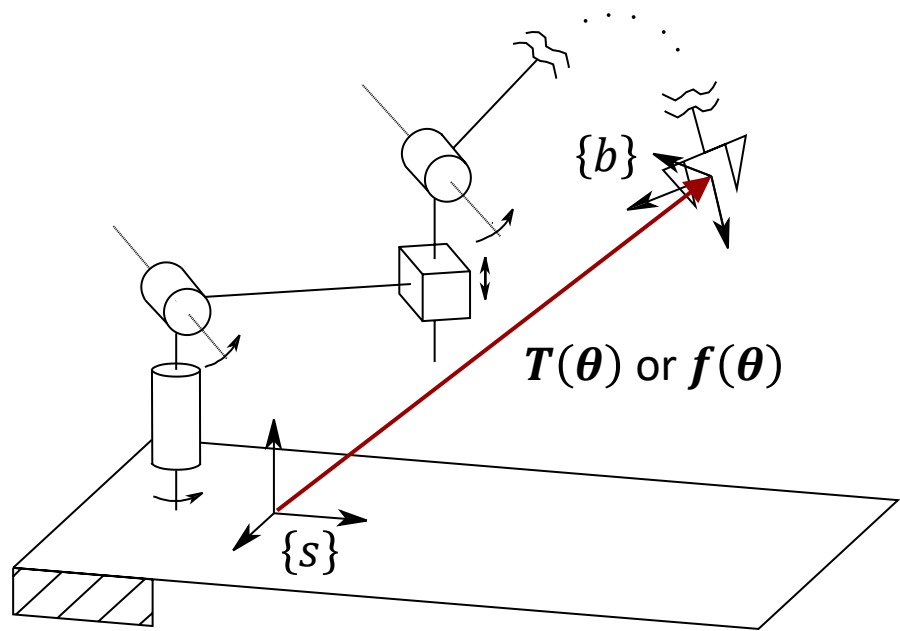
$$T(\theta) = e^{[S_1]\theta_1} \dots e^{[S_{n-1}]\theta_{n-1}} e^{[S_n]\theta_n} M$$

where $M = T_{sb}(0) \in SE(3)$ and S_1, \dots, S_n are screw axes expressed in $\{s\}$ when $\theta = 0$.

- “Minimum-Coordinate” forward kinematics:

Given $\theta \in \mathbb{R}^n$, Find $x = f(\theta) \in \mathbb{R}^m$

$$(m \leq n) \quad f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$



Velocity Kinematics

- $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ \mathcal{V}_s \end{bmatrix} = J_s(\theta)\dot{\theta}$
 - $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ \mathcal{V}_b \end{bmatrix} = J_b(\theta)\dot{\theta}$
 - $\begin{bmatrix} \omega_s \\ \dot{p} \end{bmatrix} = J_g(\theta)\dot{\theta}$

Geometric Jacobian
- $\begin{bmatrix} \dot{\phi} \\ \dot{p} \end{bmatrix} = J_{a,\phi}(\theta)\dot{\theta} \quad \phi = (\alpha, \beta, \gamma)$
 - $\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = J_{a,q}(\theta)\dot{\theta} \quad q = (q_0, q_1, q_2, q_3)$
 - $\begin{bmatrix} \dot{r} \\ \dot{p} \end{bmatrix} = J_{a,r}(\theta)\dot{\theta} \quad r = \hat{\omega}\theta$

Analytic Jacobian

- $J_s(\theta) = [J_{s1} \quad J_{s2}(\theta) \quad \cdots \quad J_{sn}(\theta)]$, $J_{si}(\theta) = \left[\text{Ad}_{e^{[s_1]\theta_1} \dots e^{[s_{i-1}]\theta_{i-1}}} \right] S_i \quad \begin{matrix} i = 2, \dots, n, \\ J_{s1} = S_1 \end{matrix}$
- $J_b(\theta) = [\text{Ad}_{T_{bs}}] J_s(\theta)$
- Statics: $\tau = J_b^T(\theta) \mathcal{F}_b$, $\tau = J_s^T(\theta) \mathcal{F}_s$
- In singular configuration θ^* , $J(\theta^*) \in \mathbb{R}^{r \times n}$ is rank-deficient, i.e., $\text{rank}(J(\theta^*)) < r$.

Inverse Kinematics

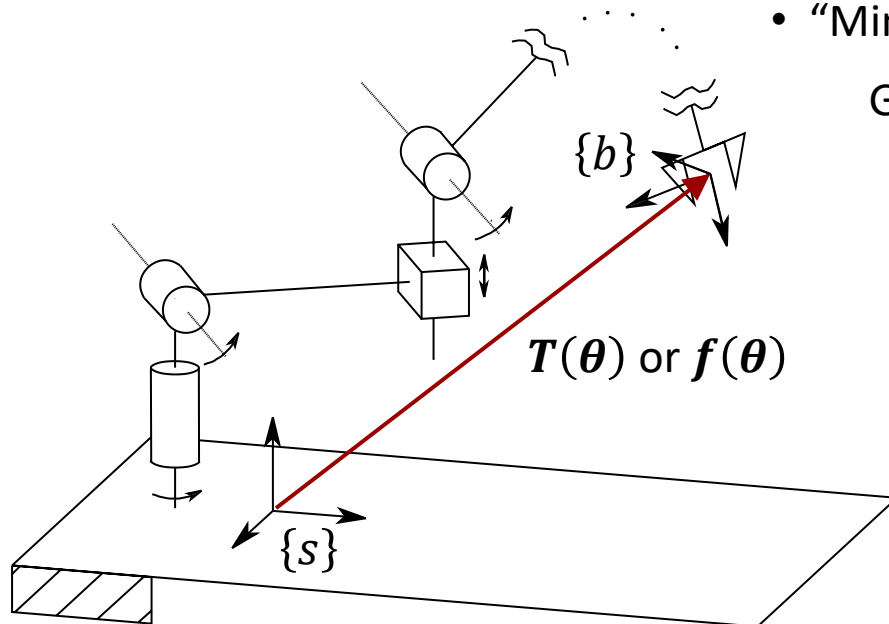
The inverse kinematics of a robot refers to the calculation of the joint coordinates θ from the position and orientation (**pose**) of its end-effector frame.

- “Geometric” inverse kinematics:

Given $T_{sb} = T(\theta) \in SE(3)$, Find $\theta \in \mathbb{R}^n$

- “Minimum-Coordinate” inverse kinematics:

Given $x = f(\theta) \in \mathbb{R}^m$, Find $\theta \in \mathbb{R}^n$



- **Analytic Methods:** Finding closed-form solutions using algebraic or geometric intuition intuitions.
- **Iterative Numerical Methods:** For instance, using Newton–Raphson method:

$$\theta^{i+1} = \theta^i + J^\dagger(\theta^i)e = \theta^i + J^\dagger(\theta^i)(x_d - f(\theta^i))$$

Trajectory Generation

Trajectory Generation: Path & Time Scaling

Trajectory $\mathcal{C}(s(t))$ or $\mathcal{C}(t)$ specifies the robot configuration as a function of time, i.e., the combination of a **path** $\mathcal{C}(s)$ and a **time scaling** $s(t)$.

$$\mathcal{C}: [0,1] \rightarrow \mathbb{C}$$

$$s: [0, T] \rightarrow [0,1]$$

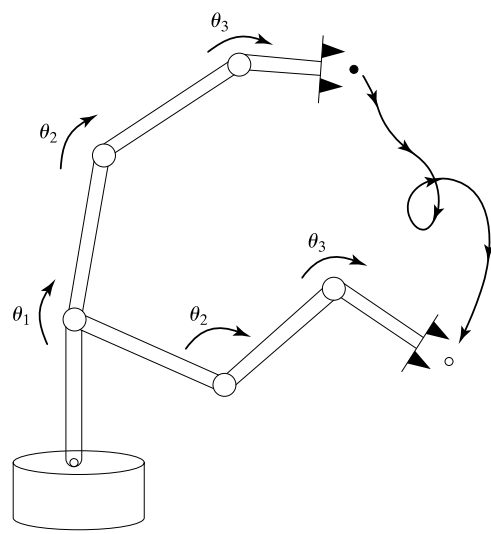
- Straight-Line Path in Joint Space: $\boldsymbol{\theta}(s) = \boldsymbol{\theta}_{\text{start}} + s(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$
- Straight-Line Path in Task Space:

$$(1) \mathbf{x}(s) = \mathbf{x}_{\text{start}} + s(\mathbf{x}_{\text{end}} - \mathbf{x}_{\text{start}}) \in \mathbb{R}^m$$

$$(2) \mathbf{p}(s) = \mathbf{p}_{\text{start}} + s(\mathbf{p}_{\text{end}} - \mathbf{p}_{\text{start}}) \in \mathbb{R}^3$$

$$\mathbf{R}(s) = \mathbf{R}_{\text{start}} \exp(\log(\mathbf{R}_{\text{start}}^T \mathbf{R}_{\text{end}}) s) \in SO(3)$$

$$(3) \mathbf{T}(s) = \mathbf{T}_{\text{start}} \exp(\log(\mathbf{T}_{\text{start}}^{-1} \mathbf{T}_{\text{end}}) s) \in SE(3)$$



Examples of Time Scaling:

- 3rd-Order, 5th-Order Polynomial Position Profile
- Trapezoidal/S-Curve Velocity Profile
- Polynomial Via Point Trajectories

$$\begin{cases} s(t) = a_0 + a_1t + a_2t^2 + a_3t^3 \\ s(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 \end{cases}$$