Ch2: Configuration Space

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Introduction

# Robot Mechanical Structure

- A robot is mechanically constructed by connecting a set of bodies, called **Links**, to each other using various types of **Joints**.

- **Actuators**, such as electric motors, deliver forces and torques to the joints, thereby causing motion of the robot

- An **End-Effector** (EE), such as a gripper or hand for grasping and manipulating objects, is attached to a specific link.

* All the robots considered in this course have links that can be modeled as **rigid bodies**.

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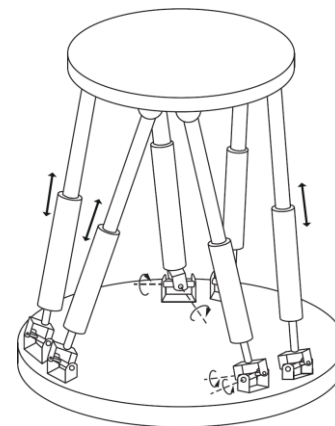
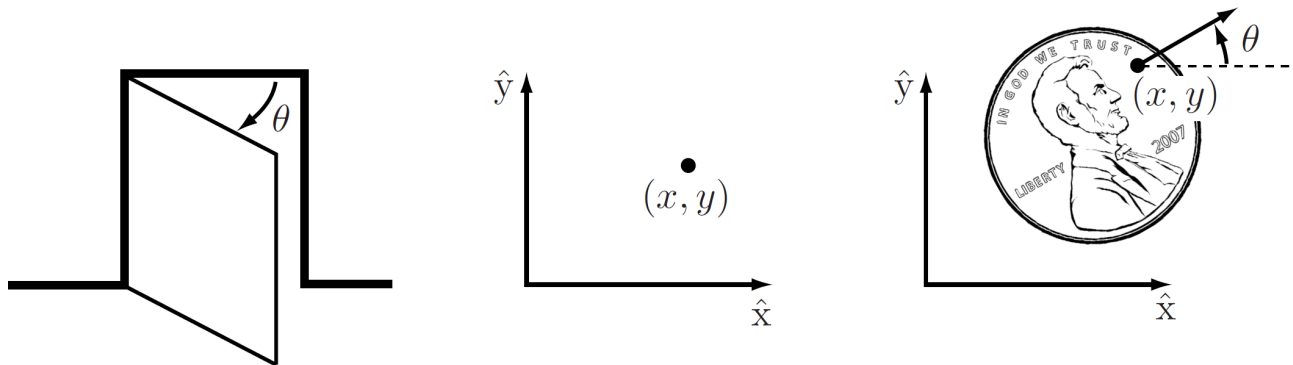# Configuration, DOF, and C-Space of a Robot

Configuration: A complete specification of the positions of all points of a robot/mechanism.

Since the robot's links are rigid and of a <u>known shape/geometry</u>, only a few numbers are needed to represent its configuration.
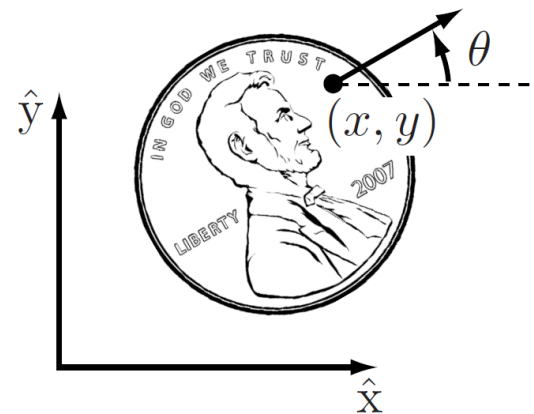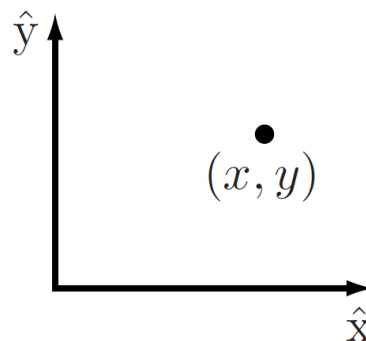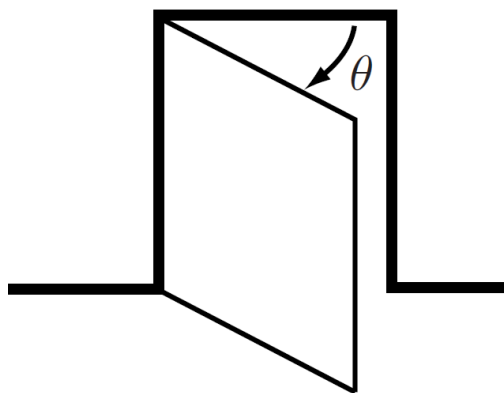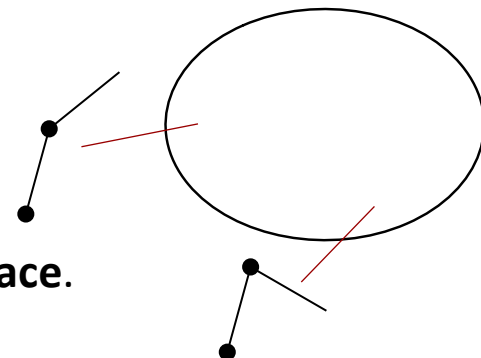
Configuration, DOF, and C-Space of a Robot

Degrees of Freedom (DOF): The minimum number n of real-valued coordinates needed to represent the **configuration** of a robot/mechanism.

Configuration Space (C-Space): The n-dimensional space containing all possible configurations of the robot/mechanism.

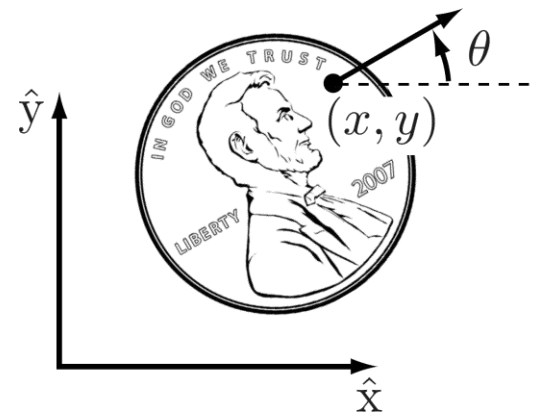* The **configuration** of a robot is represented by a <u>point</u> in its **C-space**.

Degrees of Freedom (DOF)

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DOFs of a Rigid Body in 2D Space

Example: Number of DOFs of a coin on a plane

3 DOFs

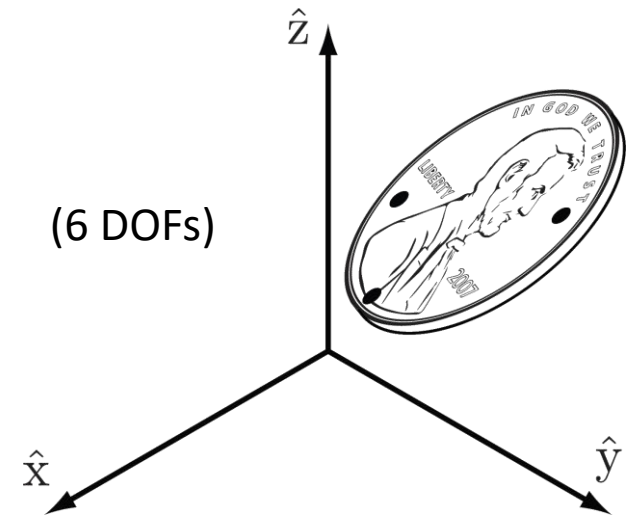A general rule for determining the number of DOFs of rigid bodies :

**DOF = (number of variables) − (number of independent equations/constraints)**
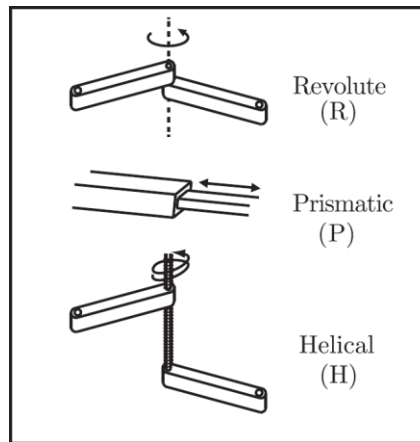
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DOFs of a Rigid Body in 3D Space

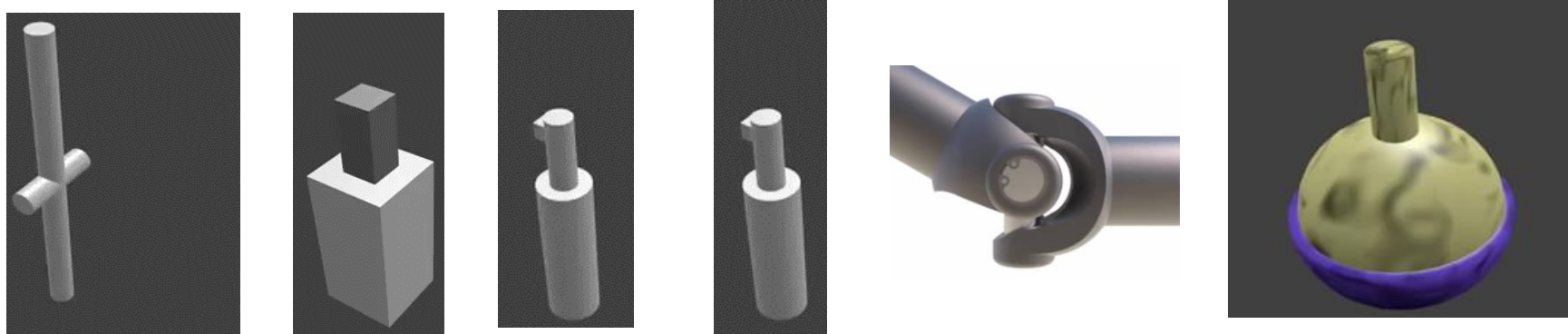**Example**: Number of DOFs of a coin in space

(6 DOFs)

In summary, a **spatial rigid body**, has **six** degrees of freedom and a **planar rigid body** has **three** degrees of freedom.

DOFs of Robots: Typical Robot Joints
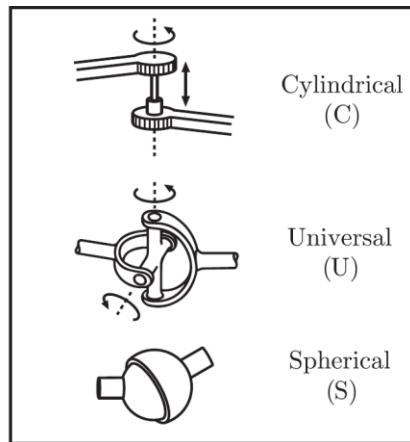
Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Note: Every joint connects exactly two links.

# of DOFs of a joint = (# of DOFs of a rigid body) − (# of constraints provided by a joint)

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DOFs of Robots: Grübler's Formula

A general rule for determining the number of DOFs of mechanisms consist of rigid bodies:
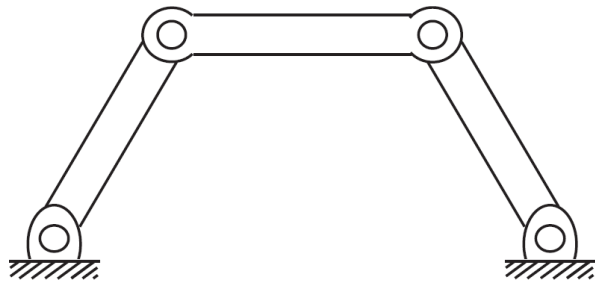   **DOF = (sum of freedoms of the bodies) − (number of <u>independent</u> constraints)**

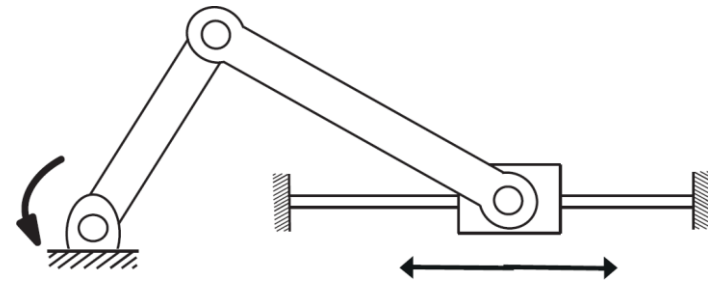Grübler's Formula for the number of degrees of freedom of mechanisms/robots:

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

Note: This formula fails when the joint constraints are not independent!

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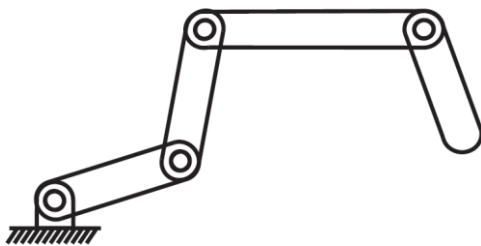Stony Brook University

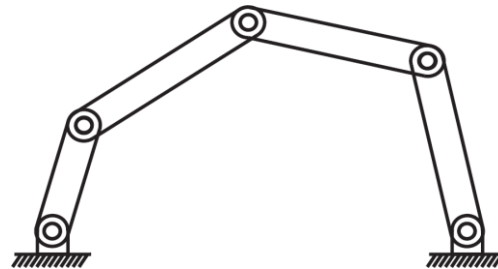# Examples: Number of DOFs

(Four-bar linkage)
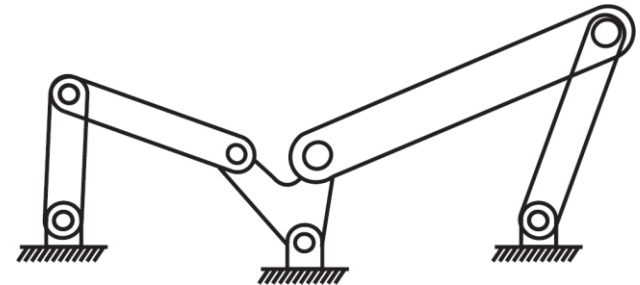
(Slider-crank mechanism)
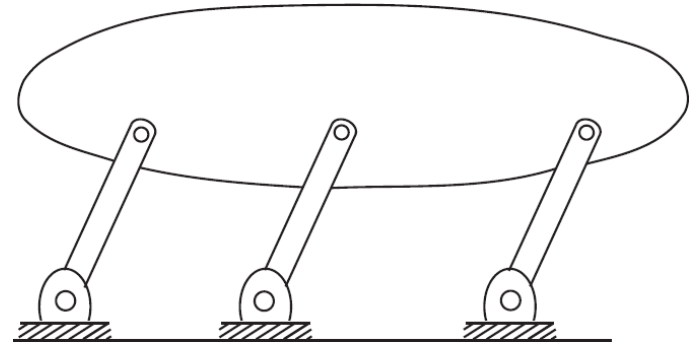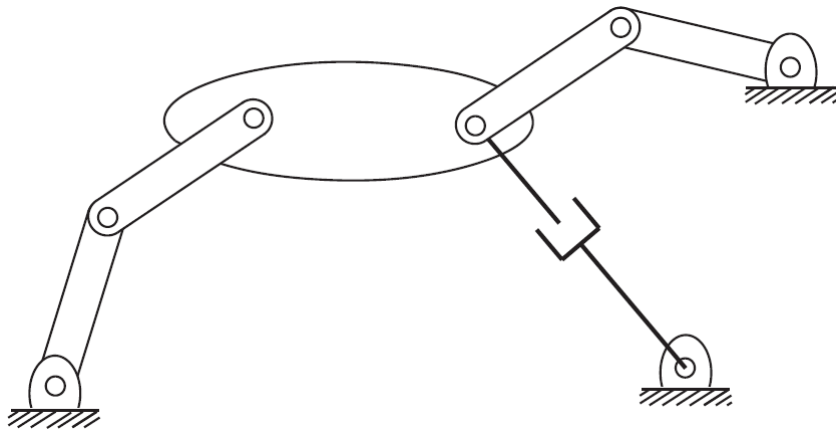
Examples: Number of DOFs

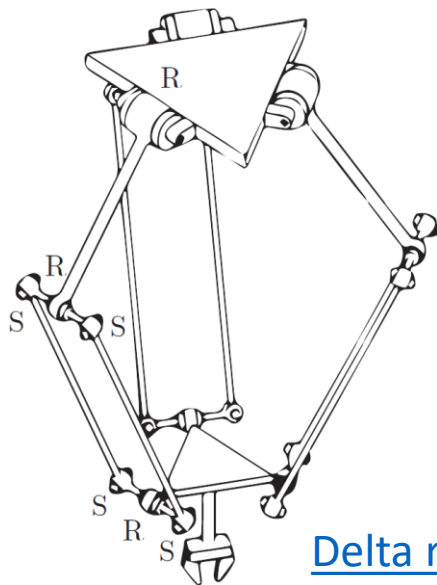
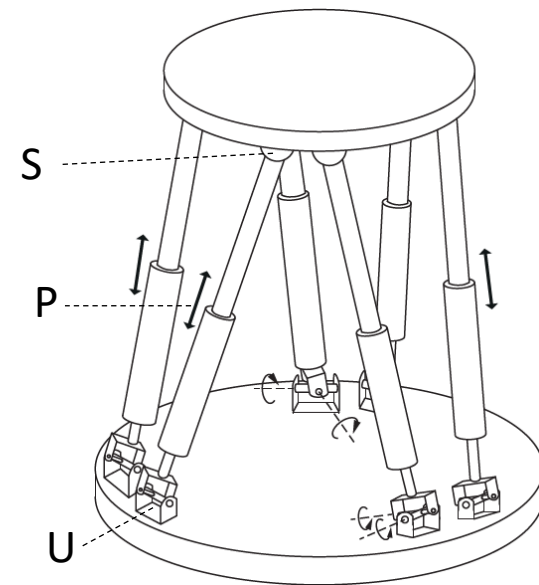*k*R robot   Five-bar linkage   Watt six-bar linkage

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Examples: Number of DOFs

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Examples: Number of DOFs

Delta robot

S

P

U

Stewart–Gough platform

Configuration Space Topology and Representation

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Topologies of 1D C-Space

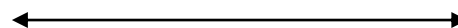**System**   **C-Space Topology**

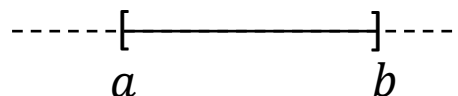(a) A point moving on a Circle
(or any closed loop):
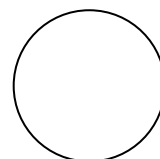
S^1

(b) A point moving on a Line:

$\mathbb{E}^1$ or $\mathbb{R}^1$

(c) A point moving on a Closed
Interval of Line:

$[a, b] \subset \mathbb{R}^1$

$a \qquad b$

Two spaces are **topologically equivalent** if one can be continuously
deformed into the other without cutting or gluing.

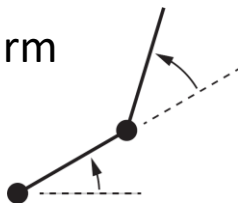https://en.wikipedi
a.org/wiki/Topology
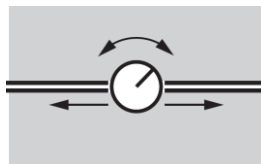
Topologies of 2D C-Space

System

A point moving on a plane

Spherical pendulum

2R robot arm

Rotating sliding robot

C-Space Topology

$$\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$$
(or $\mathbb{E}^1 \times \mathbb{E}^1 = \mathbb{E}^2$)

$$S^2$$

$$S^1 \times S^1 = T^2$$

$$\mathbb{R}^1 \times S^1$$
(or $\mathbb{E}^1 \times S^1$)

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C-Space: More Examples

- A rigid body in the plane

- A PR robot arm

- A mobile robot with a 2R robot arm

- A rigid body in three dimensions
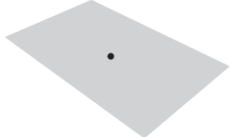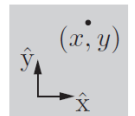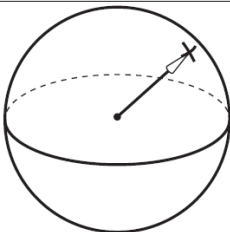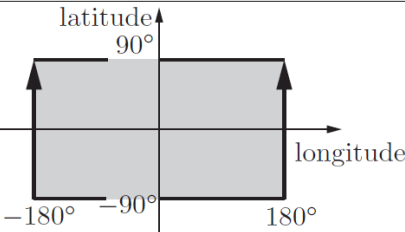
C-Space: More Examples

- Hexrotor UAV with two 5-DOF arms (without and with arm joint limits)
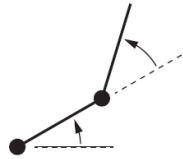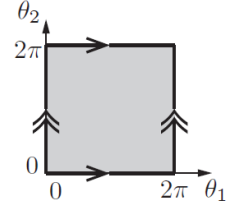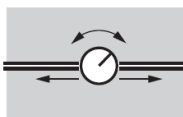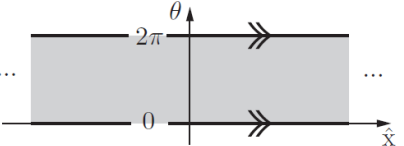
C-Space Representation

To perform computations, we must have a numerical representation of the space, consisting of a set of real numbers.

Note that the topology of a space is a fundamental property of the space itself and is **independent of how we choose coordinates to represent points in the space**.

system	topology	sample representation
point on a plane	$\mathbb{E}^2$	$\mathbb{R}^2$
spherical pendulum	S^2	$[-180°, 180°) \times [-90°, 90°]$

system	topology	sample representation
2R robot arm	$T^2 = S^1 \times S^1$	$[0, 2\pi) \times [0, 2\pi)$
rotating sliding knob	$\mathbb{E}^1 \times S^1$	$\mathbb{R}^1 \times [0, 2\pi)$

Explicit & Implicit Representations

A choice of $\underline{n}$ $\underline{\text{coordinates}}$, or parameters, to represent an $\underline{n}$-$\underline{\text{dimensional}}$ space is called an **explicit representation** of the space.
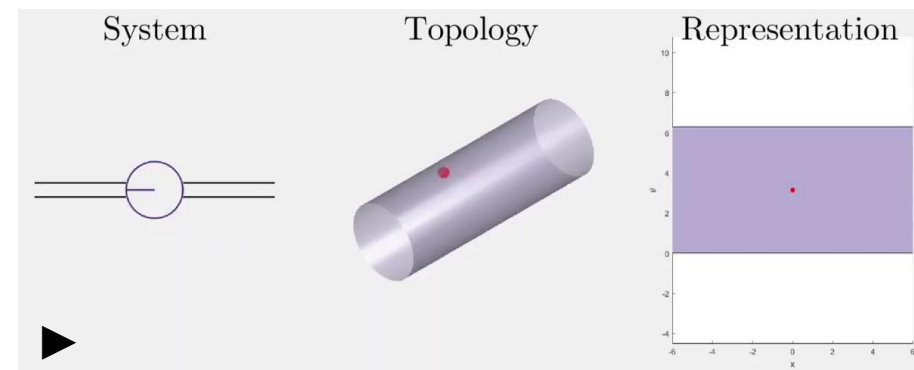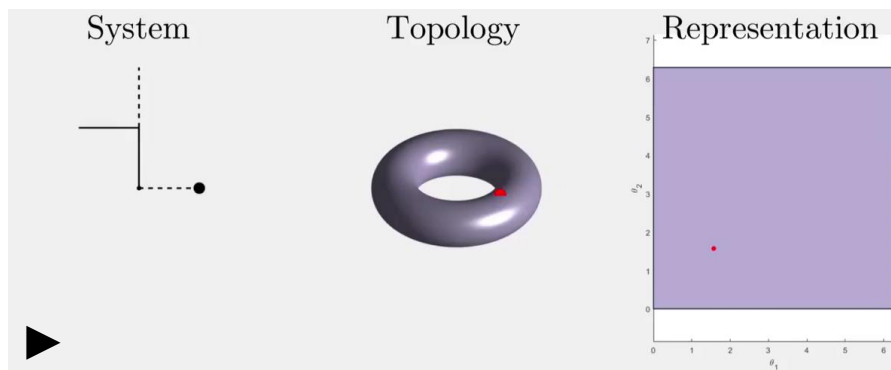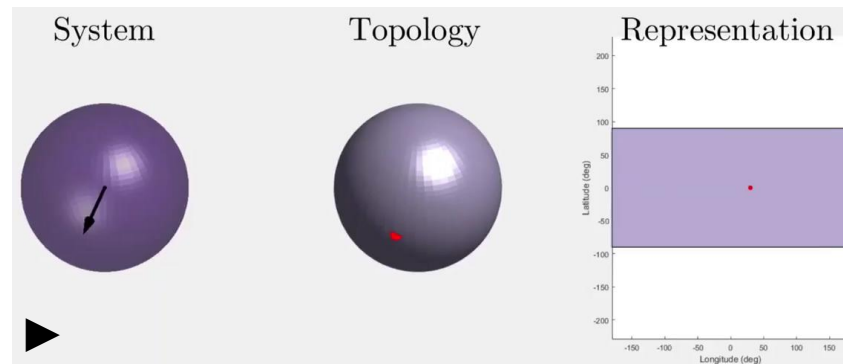
Disadvantage: Singularities of Representation

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Explicit & Implicit Representations

To overcome the Singularities of Representation:

Use an implicit representation which views the n-dimensional space as embedded in a Euclidean space of more than n dimensions.

Configuration and Velocity Constraints

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Configuration and Velocity Constraints

For robots containing one or more closed loops, usually an **implicit representation** is more easily obtained than an explicit parametrization.
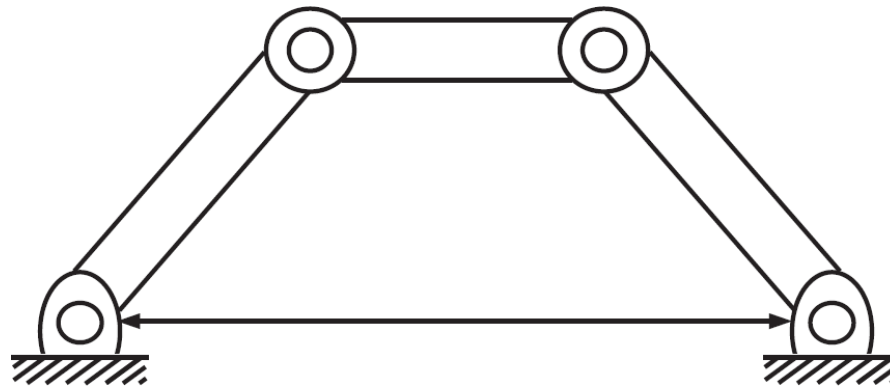
C-Space: one-dimensional space
Joint Space (J-Space): three-dimensional space

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Holonomic Constraints

For general robots containing one or more closed loops:

- Implicit representation of C-space: $\boldsymbol{\theta} = [\theta_1, \dots, \theta_n]^T \in \mathbb{R}^n$

- Constraint (loop-closure equations):
$$\boldsymbol{g}(\boldsymbol{\theta}) = \begin{bmatrix} g_1(\theta_1, \dots, \theta_n) \\ \vdots \\ g_k(\theta_1, \dots, \theta_n) \end{bmatrix} = \boldsymbol{0}$$

(a set of k independent equations, with $k \leq n$)

o Such constraints are known as **holonomic constraints**.
o These constraints reduce the dimension of C-space.

$$\Rightarrow \mathrm{DOF} = n - k$$

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Pfaffian Constraints

Let's suppose that a closed-chain robot is in motion. $\dfrac{d\boldsymbol{g}(\boldsymbol{\theta})}{dt} = \boldsymbol{0}$

$$\begin{bmatrix} \dfrac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_1} & \cdots & \dfrac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_k(\boldsymbol{\theta})}{\partial \theta_1} & \cdots & \dfrac{\partial g_k(\boldsymbol{\theta})}{\partial \theta_n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = 0 \qquad \Rightarrow \qquad \dfrac{\partial \boldsymbol{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{0} \qquad \Rightarrow \qquad \mathbf{A}(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}} = \boldsymbol{0}$$

Velocity constraints of this form are called **Pfaffian Constraints**.

* If a Pfaffian constraint is integrable, the equivalent configuration constraints are **Holonomic Constraints**.

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Nonholonomic Constraints

* If a Pfaffian constraint of the form $\mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0}$ is <u>nonintegrable</u> to equivalent configuration constraints, it is called a **Nonholonomic Constraint**.

Example: An upright coin of radius r rolling (without slipping) on a plane.
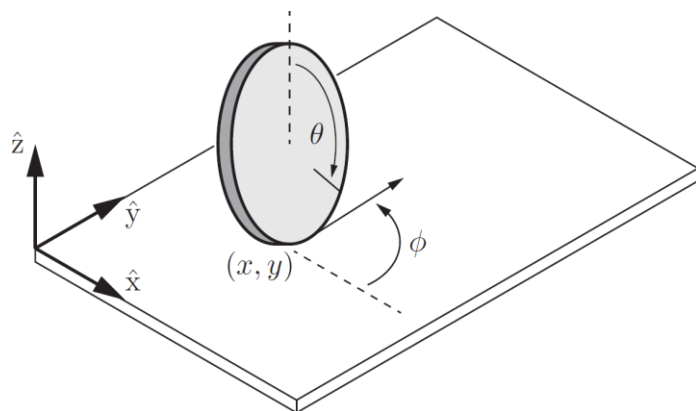
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = r\dot{\theta} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

Nonholonomic constraints reduce the dimension of the <u>feasible velocities</u> of the system but do not reduce the dimension of the C-space.

Task Space and Workspace

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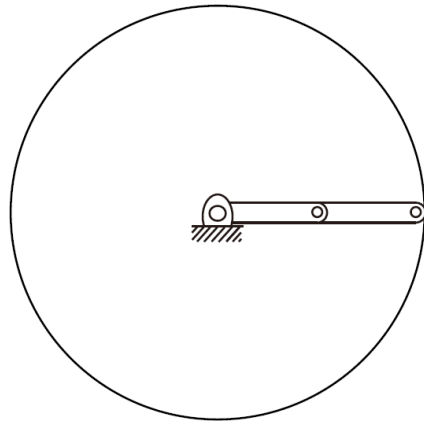
Task Space and Workspace

Task Space: The space of configurations as specified by the robot's task itself and independent of the robot.
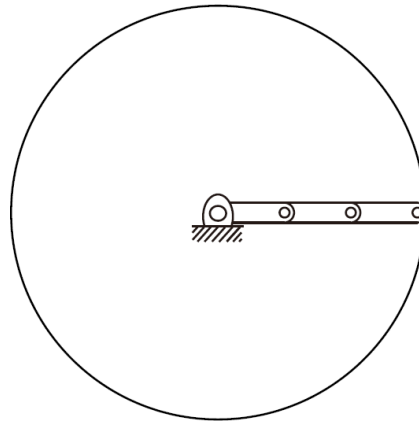
Examples:
- Task space of a robot plotting with a pen on a piece of paper:
- Task space of a robot manipulating a rigid body:
- Task space for operating a laser pointer:
- Task space for carrying a tray of glasses to keep them vertical:

Workspace: The configuration space of the end-effector that the robot can **reach** (by at least one configuration of the robot), which is primarily determined by the robot's structure and independent of the task.
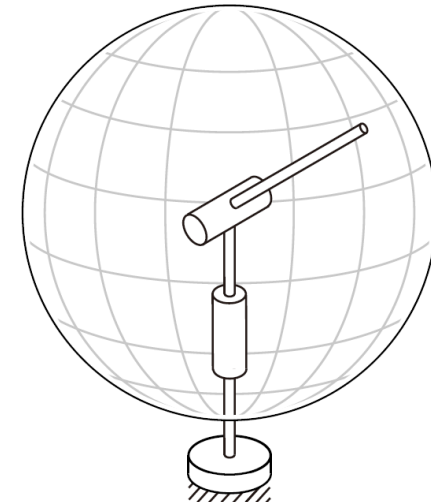
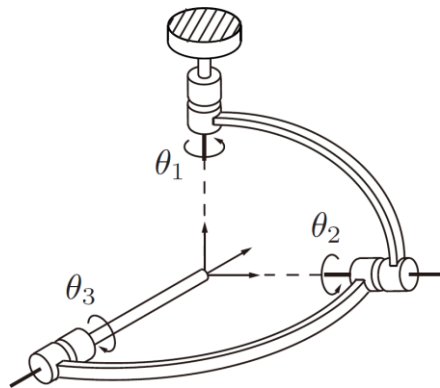# Task Space and Workspace

(2R)

(3R)

(2R)

- Two mechanisms with different C-spaces may have the same workspace.

- Two mechanisms with the same C-space may also have different workspaces.

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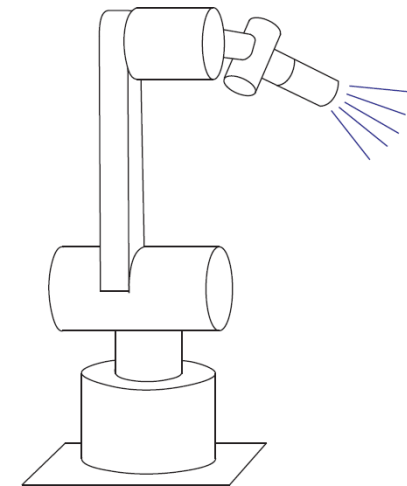Task Space and Workspace
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Task Space and Workspace: Some Examples

(3R wrist mechanism)
(for pointing a laser)
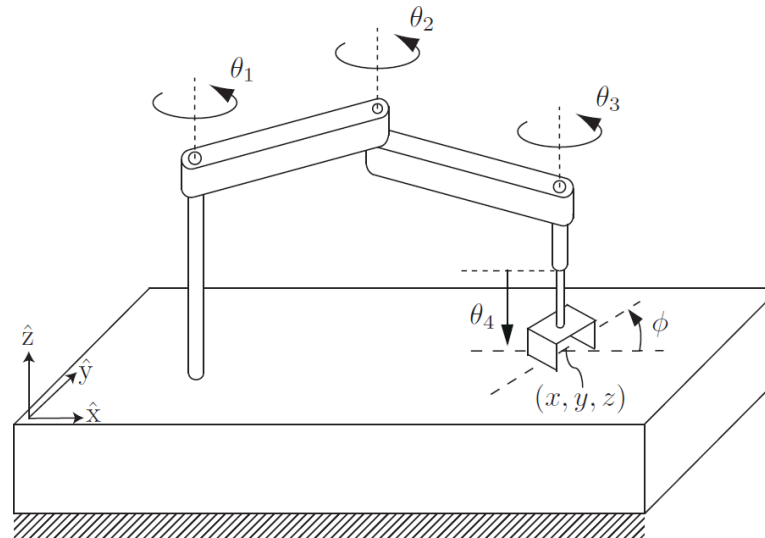
(SCARA Robot)
(RRRP)

(A spray-painting robot)
(6R)