Ch3: Configuration Space

Introduction

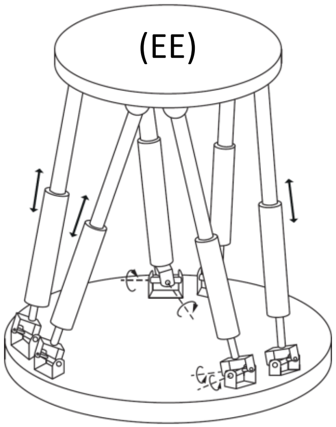
Robot Mechanical Structure

- A robot is mechanically constructed by connecting a set of bodies, called **Links**, to each other using various types of **Joints**.

  * All the robots considered in this course have links that can be modeled as **rigid bodies**.

- **Actuators**, such as electric motors, deliver forces and torques to the joints, thereby causing motion of the robot.

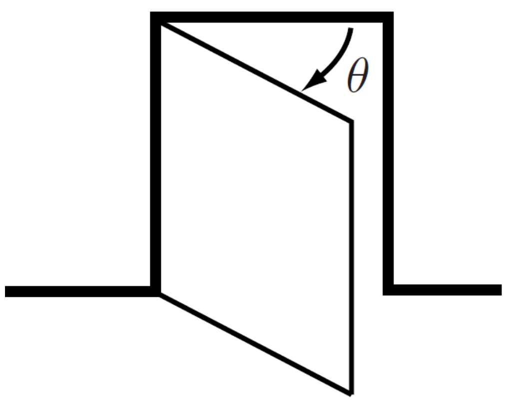
- An **End-Effector** (EE), such as a gripper or hand for grasping and manipulating objects, is attached to a specific link.

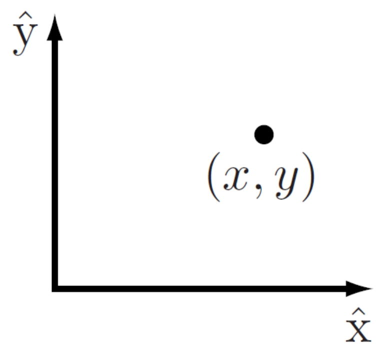


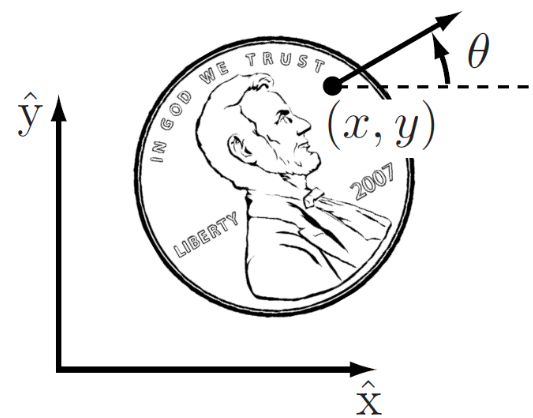
Configuration, DOF, and C-Space of a Robot

Configuration: A complete specification of the positions of all points of a robot/mechanism.

Since the robot's links are rigid and of a known shape/geometry, only a few numbers are needed to represent its configuration.

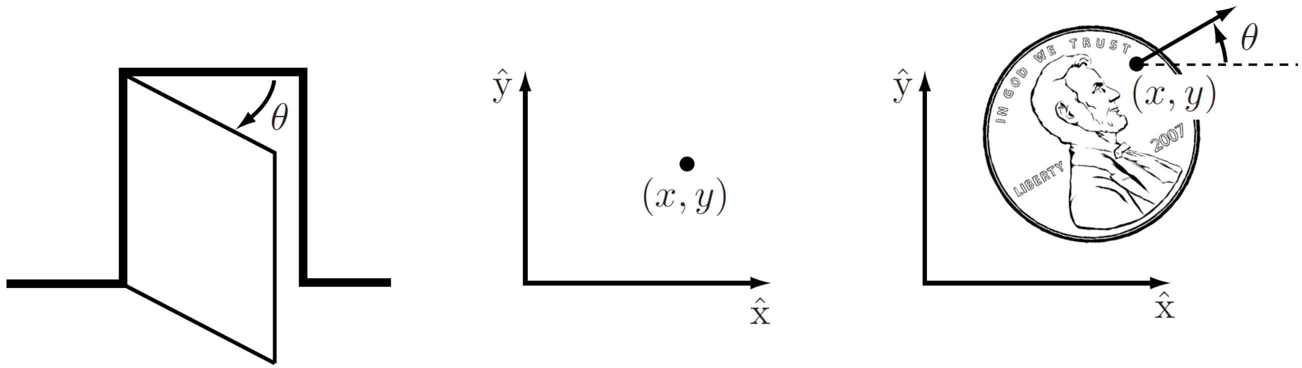




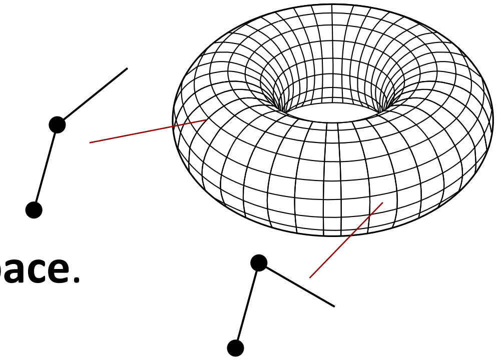


Configuration, DOF, and C-Space of a Robot

Degrees of Freedom (DOF): The minimum number n of real-valued coordinates needed to represent the **configuration** of a robot/mechanism.



Configuration Space (C-Space): The n-dimensional space containing all possible configurations of the robot/mechanism.

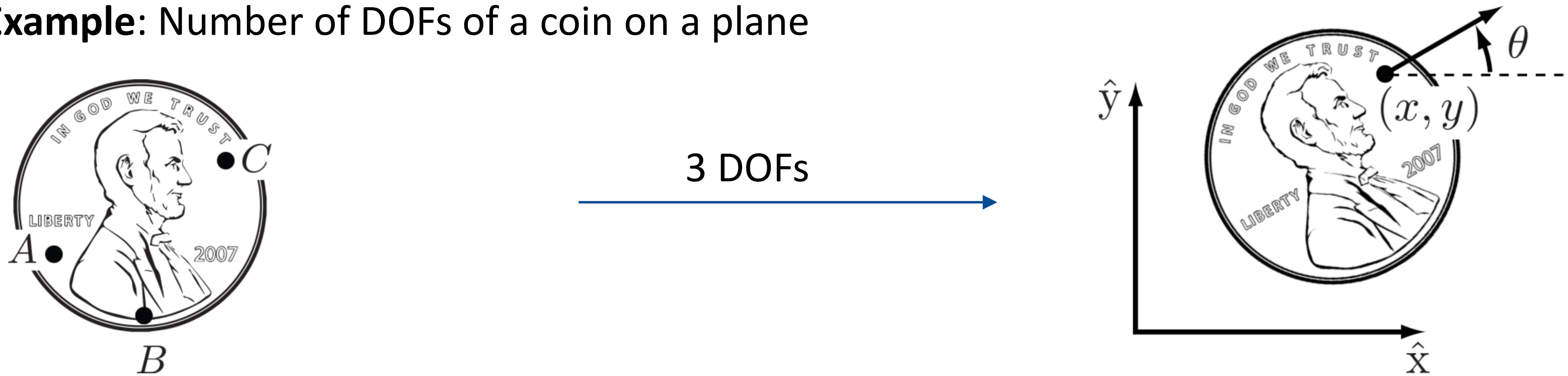


* The **configuration** of a robot is represented by a point in its **C-space**.

Degrees of Freedom (DOF)

DOFs of a Rigid Body in 2D Space

Example: Number of DOFs of a coin on a plane



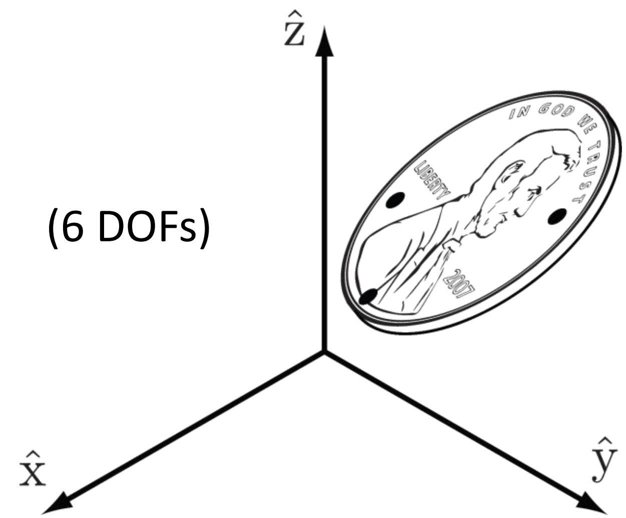
A general rule for determining the number of DOFs of rigid bodies :

DOF = (number of variables) – (number of independent equations/constraints)

DOFs of a Rigid Body in 3D Space

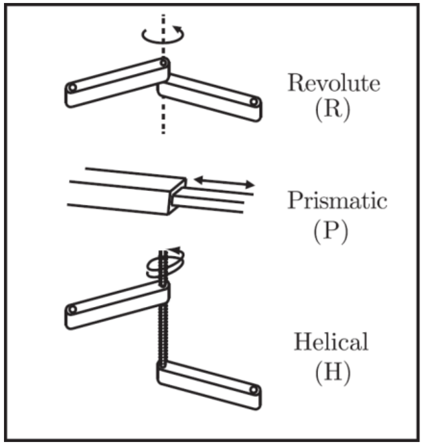
Example: Number of DOFs of a coin in space

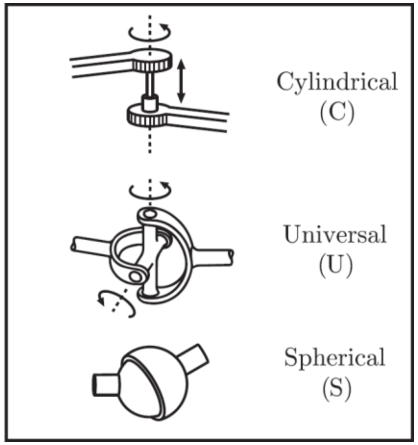
(6 DOFs)



In summary, a **spatial rigid body**, has **six** degrees of freedom and a **planar rigid body** has **three** degrees of freedom.

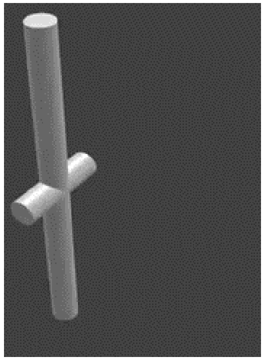
DOFs of Robots: Typical Robot Joints

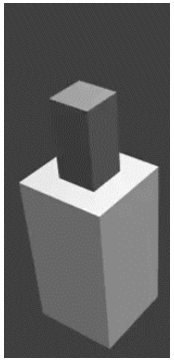


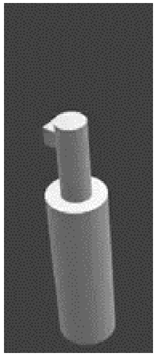


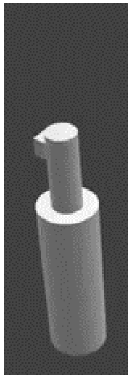
Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

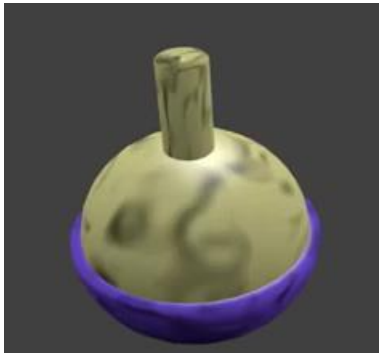
Note: Every joint connects exactly two links.











# of DOFs of a joint = (# of DOFs of a rigid body) − (# of constraints provided by a joint)

DOFs of Robots: Grübler’s Formula

A general rule for determining the number of DOFs of mechanisms consist of rigid bodies:

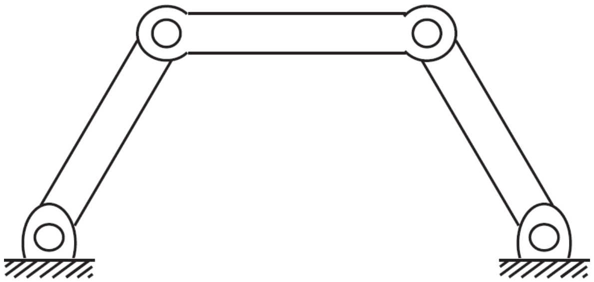
DOF = (sum of freedoms of the bodies) – (number of independent constraints)

Grübler’s Formula for the number of degrees of freedom of mechanisms/robots:

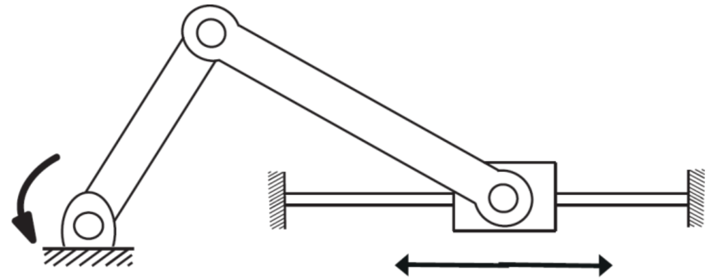
$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

Note: This formula fails when the joint constraints are not independent!

Examples: Number of DOFs

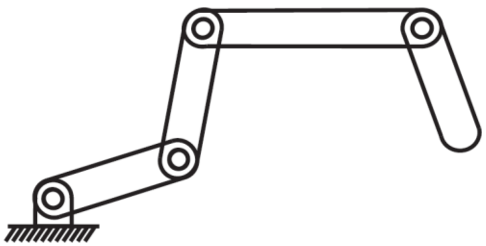


(Four-bar linkage)

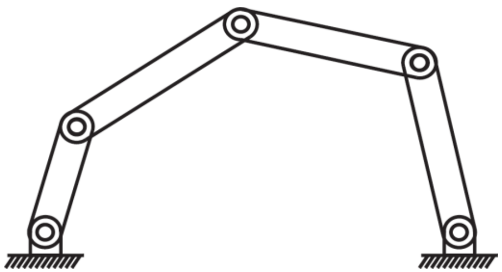


(Slider-crank mechanism)

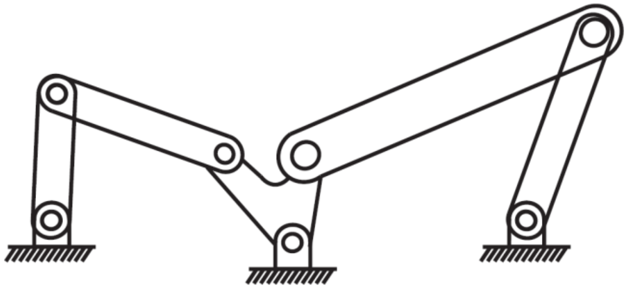
Examples: Number of DOFs



*k*R robot

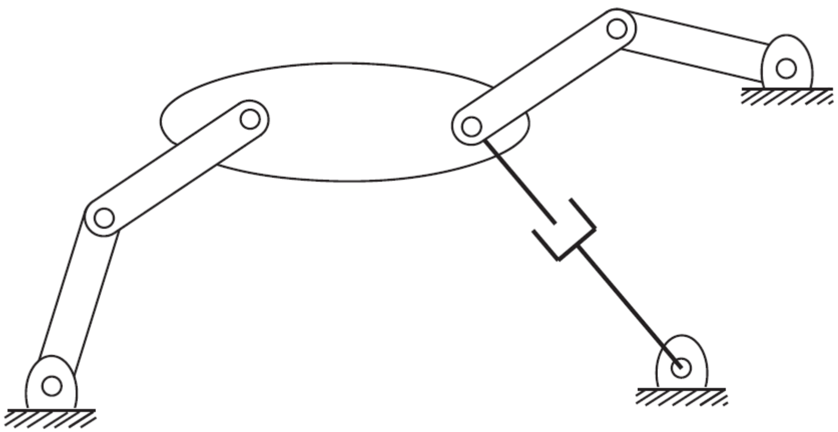


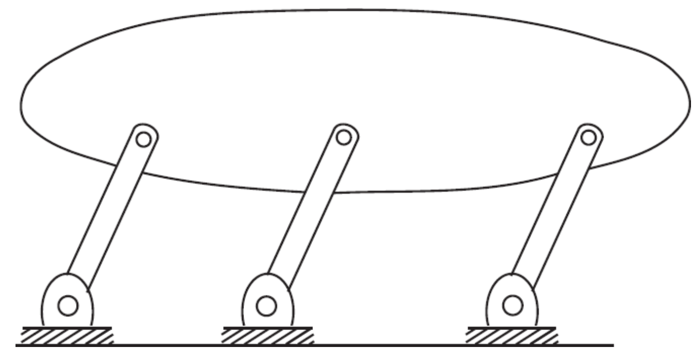
Five-bar linkage



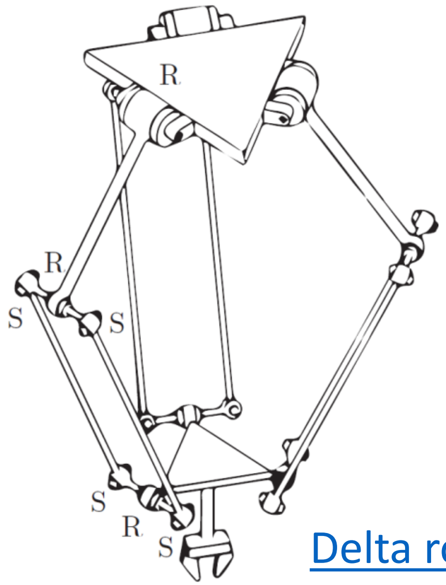
Watt six-bar linkage

Examples: Number of DOFs

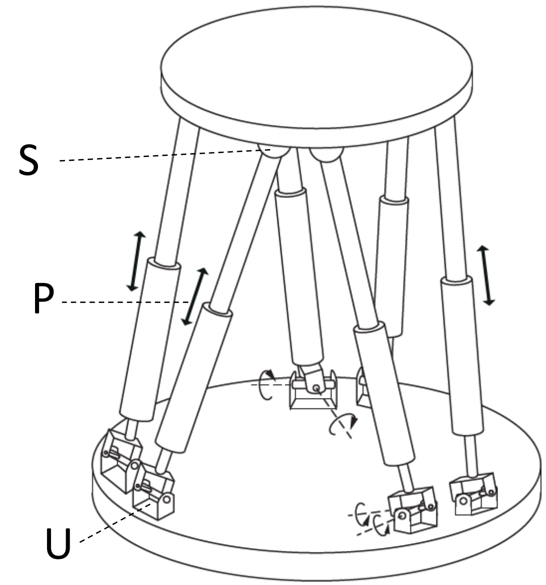




Examples: Number of DOFs



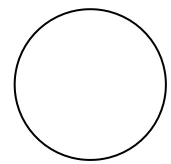
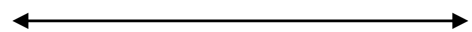
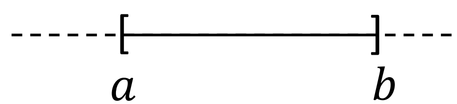
Delta robot



Stewart–Gough platform

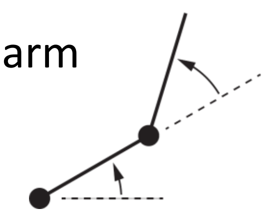
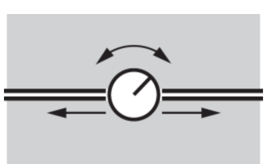
Configuration Space Topology and Representation

Topologies of 1D C-Space

System		C-Space Topology
(a) A point moving on a Circle (or any closed loop):		S^1
(b) A point moving on a Line:		$\mathbb{E}^1$ or $\mathbb{R}^1$
(c) A point moving on a Closed Interval of Line:		$[a, b] \subset \mathbb{R}^1$

Two spaces are **topologically equivalent** if one can be continuously deformed into the other without cutting or gluing.

Topologies of 2D C-Space

System		**C-Space Topology**
A point moving on a plane		$\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$ (or $\mathbb{E}^1 \times \mathbb{E}^1 = \mathbb{E}^2$)
Spherical pendulum		S^2
2R robot arm 		$S^1 \times S^1 = T^2$
Rotating sliding robot 		$\mathbb{R}^1 \times S^1$ (or $\mathbb{E}^1 \times S^1$)

C-Space: More Examples

- A rigid body in the plane

- A PR robot arm

- A mobile robot with a 2R robot arm

- A rigid body in three dimensions

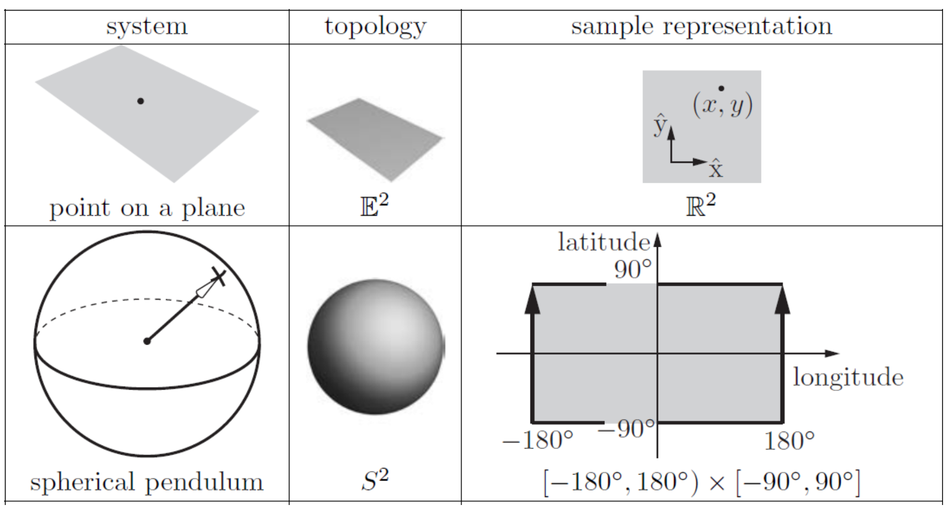
C-Space: More Examples

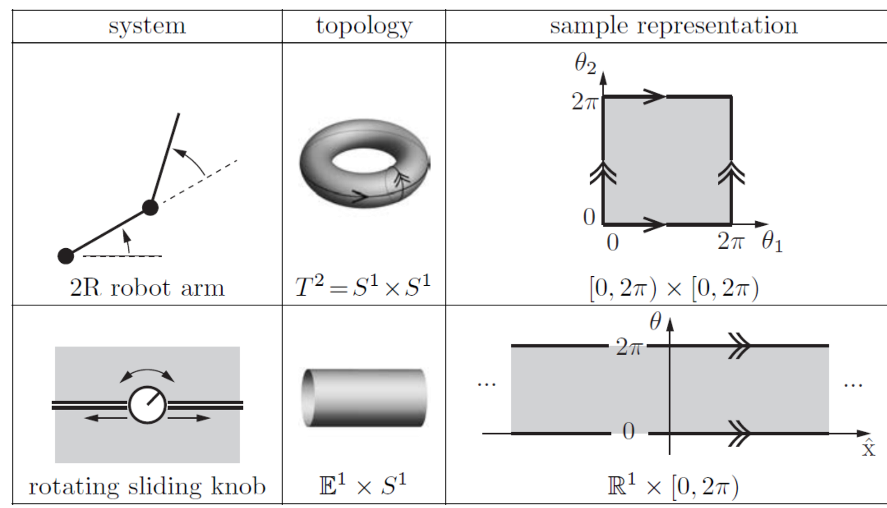
- Hexrotor UAV with two 5-DOF arms (without and with arm joint limits)

C-Space Representation

To perform computations, we must have a numerical representation of the space, consisting of a set of real numbers.

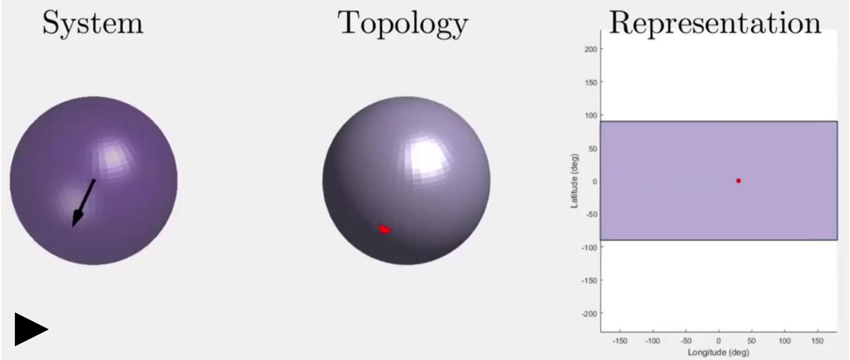
Note that the topology of a space is a fundamental property of the space itself and is **independent of how we choose coordinates to represent points in the space**.

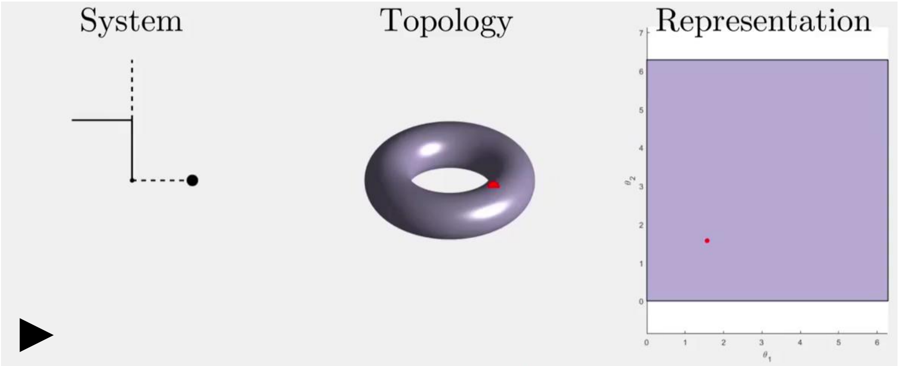


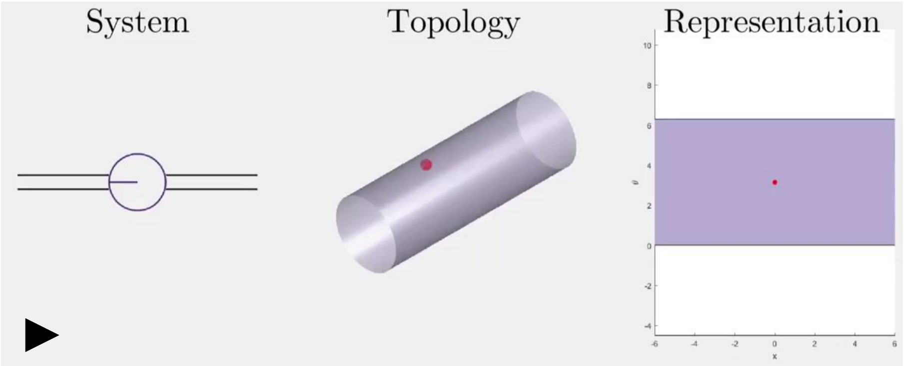


Explicit & Implicit Representations

A choice of n coordinates or parameters (in Euclidian space) to represent an n-dimensional C-space is called an **explicit representation** of the C-space.







Disadvantage: Singularities of Representation

Explicit & Implicit Representations

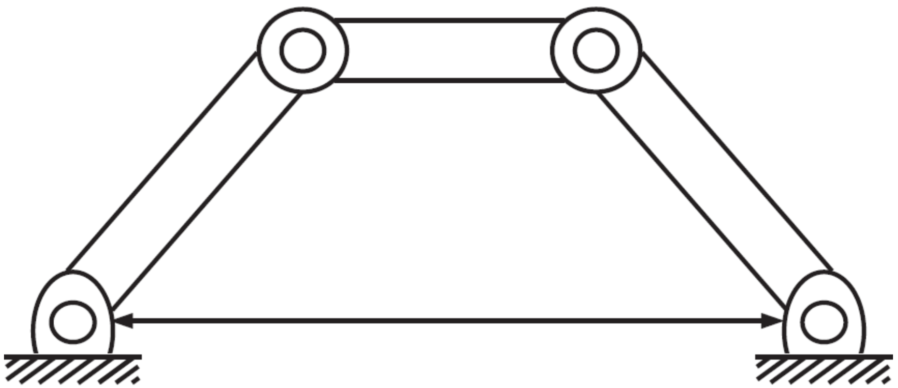
To overcome the Singularities of Representation:

Use an implicit representation which views the n-dimensional space as embedded in a Euclidean space of more than n dimensions.

Configuration and Velocity Constraints

Configuration and Velocity Constraints

For robots containing one or more closed loops, usually an **implicit representation** is more easily obtained than an explicit parametrization.



C-Space: one-dimensional space
Joint Space (J-Space): three-dimensional space

Holonomic Constraints

For general robots containing one or more closed loops:

- Implicit representation of C-space: $\boldsymbol{\theta} = [\theta_1, \dots, \theta_n]^T \in \mathbb{R}^n$
  (or J-Space)

- Constraint (loop-closure equations): $$\boldsymbol{g}(\boldsymbol{\theta}) = \begin{bmatrix} g_1(\theta_1, \dots, \theta_n) \\ \vdots \\ g_k(\theta_1, \dots, \theta_n) \end{bmatrix} = \mathbf{0}$$

(a set of k independent equations, with $k \leq n$)

- Such constraints are known as **holonomic constraints**.
- These constraints reduce the dimension of implicit representation of C-space.

$$\Rightarrow \text{ DOF} = n - k$$

Pfaffian Constraints

Let's suppose that a closed-chain robot is in motion. $\dfrac{d\boldsymbol{g}(\boldsymbol{\theta})}{dt} = \mathbf{0}$

$$\begin{bmatrix} \dfrac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_1} & \cdots & \dfrac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_k(\boldsymbol{\theta})}{\partial \theta_1} & \cdots & \dfrac{\partial g_k(\boldsymbol{\theta})}{\partial \theta_n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad \frac{\partial \boldsymbol{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \mathbf{0} \quad \Rightarrow \quad \mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0}$$

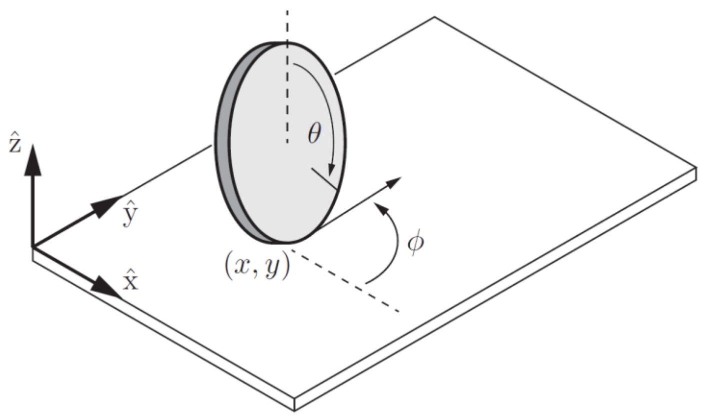
Velocity constraints of this form are called **Pfaffian Constraints**.

* If a Pfaffian constraint is integrable, the equivalent configuration constraints are **Holonomic Constraints**.

Nonholonomic Constraints

* If a Pfaffian constraint of the form $\mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0}$ is nonintegrable to equivalent configuration constraints, it is called a **Nonholonomic Constraint**.

Example: An upright coin of radius r rolling (without slipping) on a plane.



$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = r\dot{\theta} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

Nonholonomic constraints reduce the dimension of the feasible velocities of the system but do not reduce the dimension of the C-space.

Workspace and Task Space

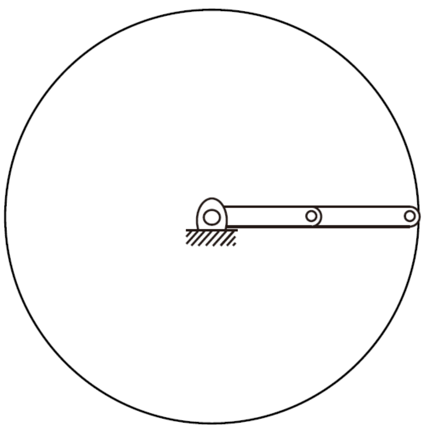
Workspace and Task Space

Workspace: The configuration space of the **end-effector** that the robot can **reach** (by at least one configuration of the robot), which is primarily determined by the robot's structure and independent of the task.

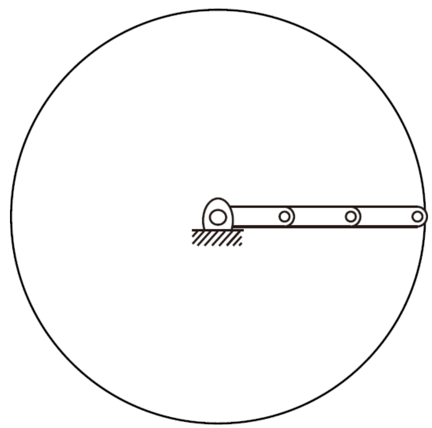
Task Space (T-Space): The space of configurations as specified by the robot's task itself and independent of the robot.

Examples:
- Task space of a robot plotting with a pen on a piece of paper:
- Task space of a robot manipulating a rigid body:
- Task space for operating a laser pointer:
- Task space for carrying a tray of glasses to keep them vertical:

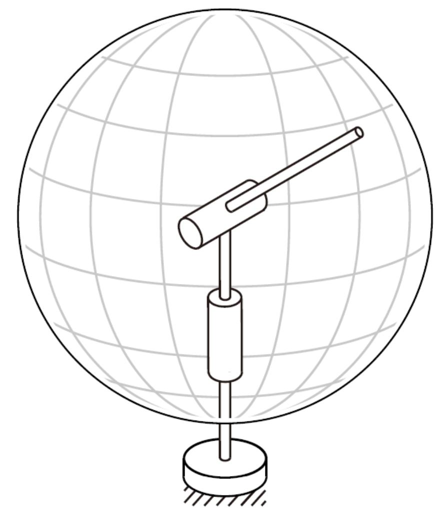
Workspace and Task Space



(2R)



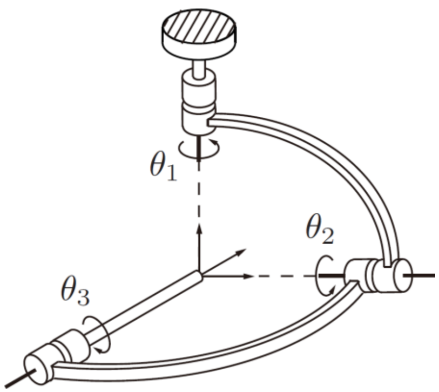
(3R)



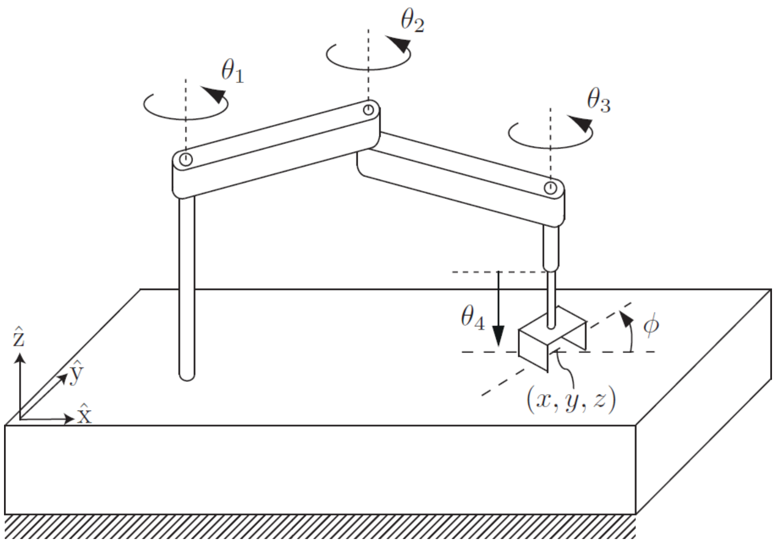
(2R)

- Two mechanisms with different C-spaces may have the same workspace.
- Two mechanisms with the same C-space may also have different workspaces.

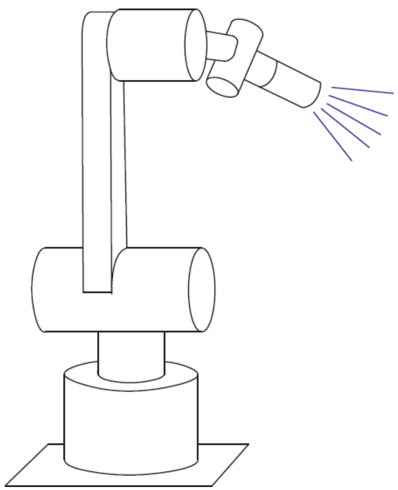
Workspace and Task Space: Some Examples



(3R wrist mechanism)
(for pointing a laser)



(SCARA Robot)
(RRRP)



(A spray-painting robot)
(6R)