Ch6: Stability for Autonomous Systems

Concepts of Stability

Introduction

Given a control system, the first and most important question about its various properties is whether it is **Stable**.

The most useful and general approach for studying the stability of nonlinear control systems is the theory introduced in 1892 by the Russian mathematician Alexandr Mikhailovich **Lyapunov**.

1857-1918

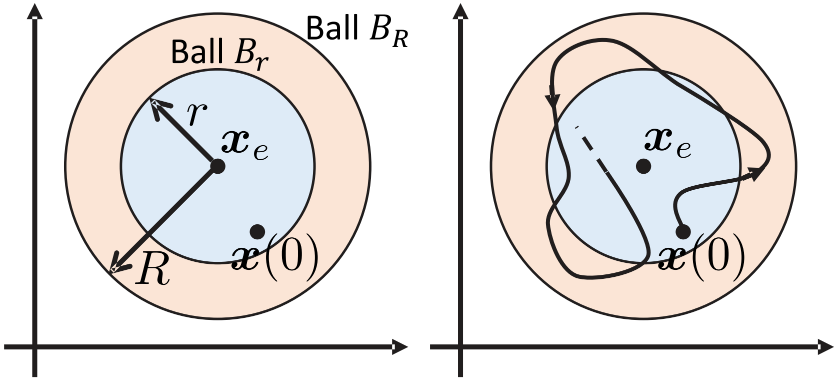
$$\left\{\begin{array}{ll} \forall & \text{for any} \\ \exists & \text{there exists (at least one)} \\ \in & \text{in the set} \\ \Rightarrow & \text{implies that} \\ \Leftrightarrow & \text{equivalent to} \end{array}\right.$$

Lyapunov Stability and Instability

The equilibrium point $\boldsymbol{x}_e$ is said to be **Stable** if for any (arbitrary) $R > 0$, there exists $r = r(R) > 0$, such that if $\|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < r$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < R$ for all $t \geq 0$. Otherwise, the equilibrium point is **Unstable**.

$$\boxed{\forall R > 0, \exists r > 0 : \|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < r \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < R, \ \forall t \geq 0}$$

An equilibrium point is **stable** if starting the system somewhere (sufficiently) near the point (i.e., anywhere in the ball B_r) implies that the system trajectory will stay (arbitrarily) around the point (i.e., in the ball B_R) ever after.

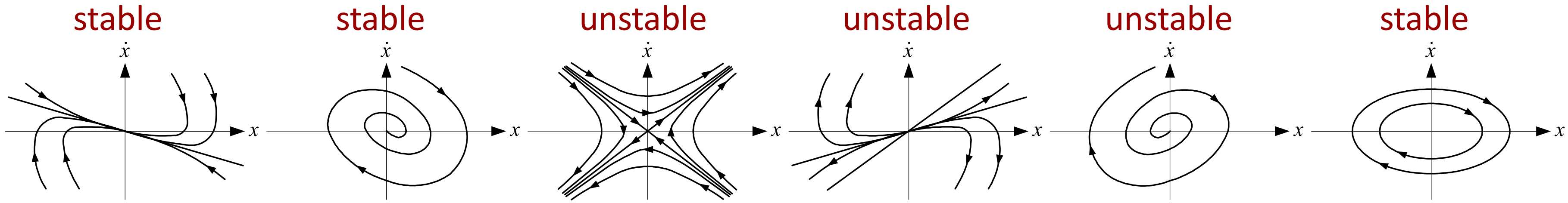


An equilibrium point is **unstable** if there exists at least one ball B_R, such that for every $r > 0$, no matter how small, it is always possible for the system trajectory to start somewhere within the ball B_r, and eventually leave the ball B_R.

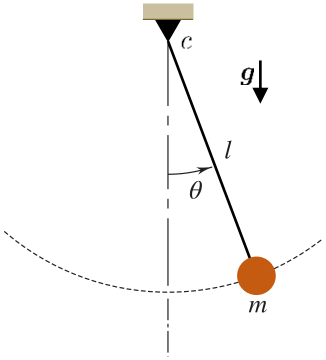
This is also called **Stability in the Sense of Lyapunov**.

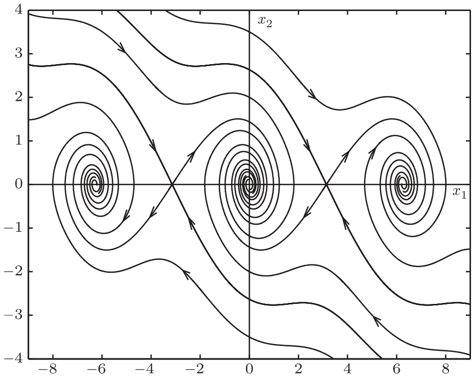
Lyapunov Stability and Instability (cont.)

Example: Linear systems or Local linearization of nonlinear systems.


stable
stable
unstable
unstable
unstable
stable
$\dot{x}$
x

Example: In a pendulum, the vertical up and down positions, are unstable and stable, respectively.





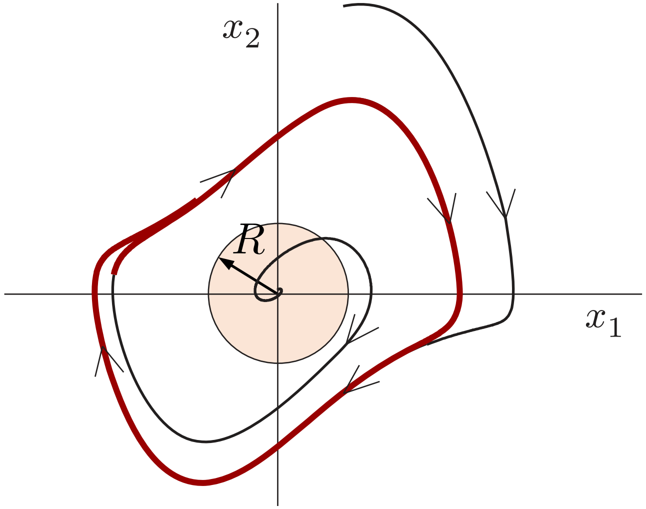
Instability of an equilibrium point is typically undesirable, because it often leads the system into limit cycles or results in damage to the involved mechanical or electrical components.

Instability in Linear and Nonlinear Systems

- In linear systems, instability is equivalent to **blowing up** (moving all trajectories close to equilibrium point to infinity).
- In nonlinear systems, blowing up is **only one way of instability**.

For example, consider Van der Pol Oscillator:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \left(1 - x_1^2\right) x_2 \end{aligned}$$

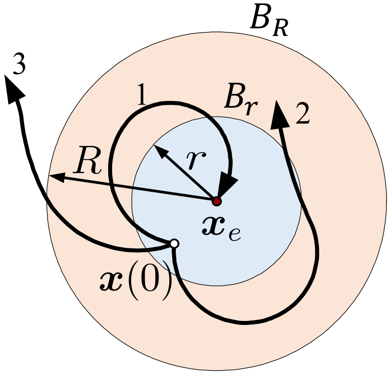


- If we choose the circle of radius R to fall completely within the limit cycle, then system trajectories starting near the origin will eventually get out of this circle. This implies **instability** of the origin.
- Thus, even though the state of the system does remain around the equilibrium point in a certain sense, it cannot stay arbitrarily close to it.

Asymptotic and Marginal Stability

In many applications, Lyapunov stability is not enough. For example, (1) and (2) are stable, but their behavior is not the same.

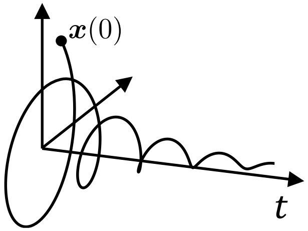
1) **Stable** (asymptotically)
2) **Stable** (marginally)
3) **Unstable**



▶ The equilibrium point $\boldsymbol{x}_e$ is said to be **Asymptotically Stable** if it is **Lyapunov Stable** and there exists $r > 0$ such that if $\|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < r$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \to 0$ as $t \to \infty$.

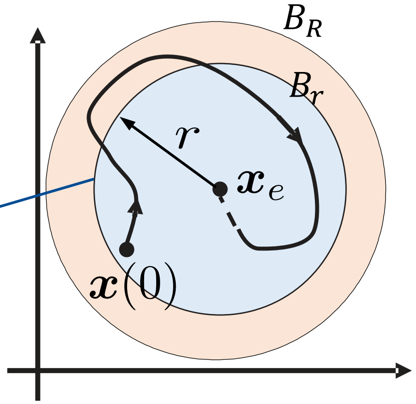
$$\boxed{\exists r > 0 : \|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < r \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \to 0, \text{ as } t \to \infty}$$

The states started close to $\boldsymbol{x}_e$ converge to $\boldsymbol{x}_e$ as $t \to \infty$.



- The region with the **largest** r is called **Domain of Attraction** of $\boldsymbol{x}_e$.

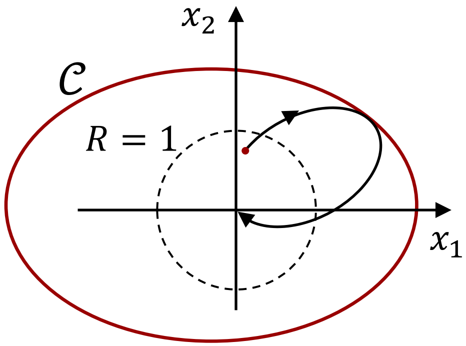
- An equilibrium point which is **Lyapunov Stable** but not asymptotically stable is called **Marginally Stable**.



Asymptotic and Marginal Stability (cont.)

$*$ State convergence does not necessarily imply stability.

Example 1: In the system studied by Vinograd, all the trajectories starting from non-zero initial points within the unit disk first reach the curve $\mathcal{C}$ before converging to the origin.



The **origin** is **unstable** in the sense of Lyapunov, despite the state convergence.

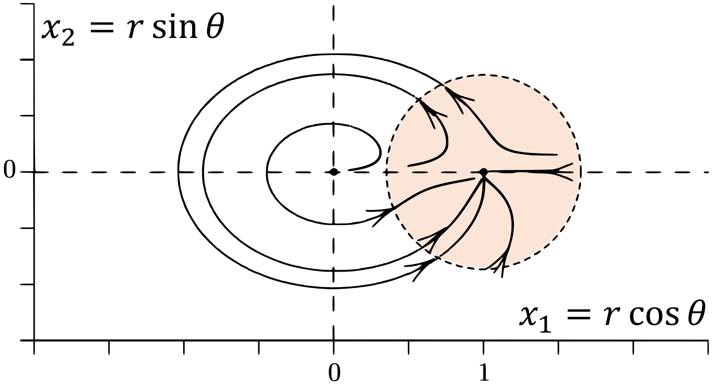
Example 2: Consider the system expressed in polar coordinates.

$$\dot{r} = 0.05r(1-r)$$
$$\dot{\theta} = \sin^2(\theta/2) \qquad \theta \in [0,2\pi).$$

- Equilibrium points: [0, 0], [1, 0].

- All the solutions of the system tend asymptotically to [1, 0].

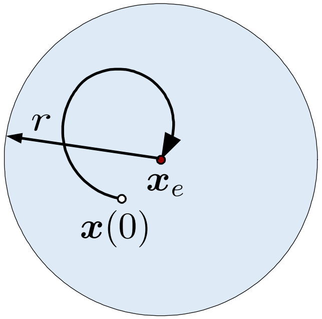
- For each initial condition inside the dashed disk the generated trajectory goes asymptotically to [1, 0]. However, this equilibrium is **unstable** in the sense of Lyapunov, because there are always solutions that leave the disk before coming back towards the equilibrium.



Exponential Stability

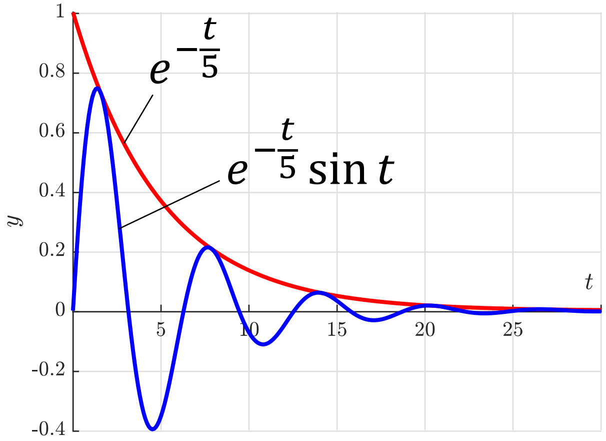
How fast the system trajectory approaches $\boldsymbol{x}_e$?

▶ The equilibrium point $\boldsymbol{x}_e$ is said to be **Exponentially Stable** if there exist $\alpha, \lambda, r > 0$ such that if $\|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < r$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \alpha \|\boldsymbol{x}(0) - \boldsymbol{x}_e\| e^{-\lambda t}$.



$$\boxed{\exists \alpha, \lambda, r > 0 : \|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < r \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \alpha \|\boldsymbol{x}(0) - \boldsymbol{x}_e\| e^{-\lambda t}}$$

λ: exponential convergence rate



Note: **Exponential stability** itself implies **asymptotic stability**. Thus, in this definition, there is no need to explicitly mention “if the system is asymptotically stable”.

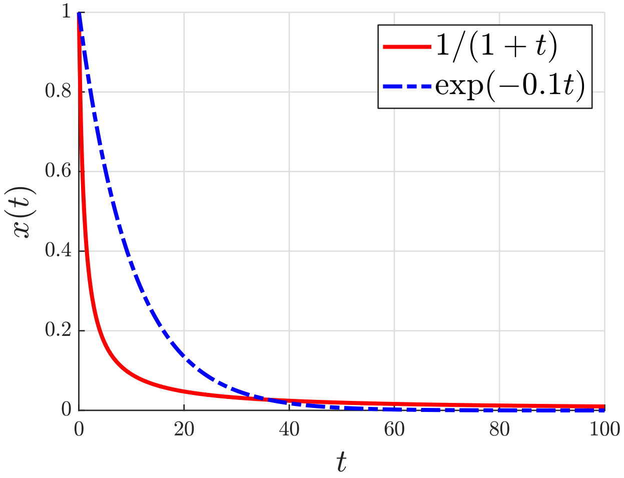
Exponential Stability (cont.)

But asymptotic stability does not guarantee exponential stability.

Example:

$$\dot{x} = -x^2, \quad x(0) = 1 \quad \Rightarrow \quad x = \frac{1}{1+t}$$

The function converges to 0 slower than any exponential function with $\lambda > 0$.



Local and Global Stability

The above definitions are formulated to characterize the **local behavior** of systems, i.e., how the state evolves after starting near $\boldsymbol{x}_e$. What will be the behavior of systems when the initial state is some distance away from $\boldsymbol{x}_e$?

► If asymptotic (or exponential) stability holds for **any initial states**, i.e., $r = +\infty$, the equilibrium point $\boldsymbol{x}_e$ is said to be **Globally Asymptotically (or Exponentially) Stable**.

$\Downarrow$

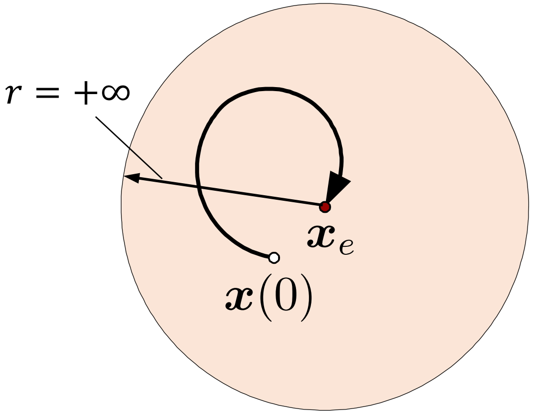
Starting the system from anywhere,
it ends up the equilibrium point $\boldsymbol{x}_e$.

$\Downarrow$

There is only 1 equilibrium points.

$\Downarrow$

Stability of the equilibrium point $\boldsymbol{x}_e \equiv$ Stability of the system.

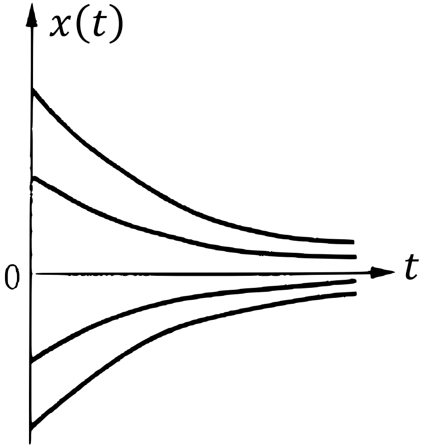


Local and Global Stability (cont.)

Examples:

$$\dot{x} = -x, \quad x(0) = x_0 \quad \Rightarrow \quad x(t) = x_0 e^{-t}$$

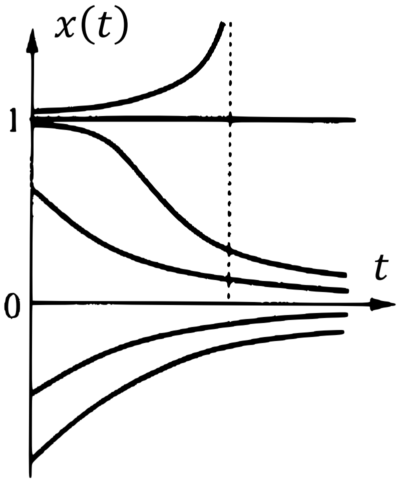
Globally Exponentially Stable



- **LTI** systems are either **globally exponentially stable**, **marginally stable**, or **unstable**.

$$\dot{x} = -x + x^2, \quad x(0) = x_0 \quad \Rightarrow \quad x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

(Locally) Exponentially Stable @ 0, Unstable @ 1



Stability of a Motion

In some problems, we are not concerned with stability around an equilibrium point, but rather with the **stability of a motion**, i.e., whether a system will remain close to its original motion trajectory if slightly perturbed away from it.

These problems can be **transformed** into an equivalent stability problem around an equilibrium point, although the equivalent system may be now non-autonomous.

Consider $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$

(nominal motion trajectory)

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \xrightarrow{\text{solution}} \boldsymbol{x}^*(t), \qquad \dot{\boldsymbol{x}}^* = \mathbf{f}(\boldsymbol{x}^*)$$

Perturbing the initial condition $\Downarrow$

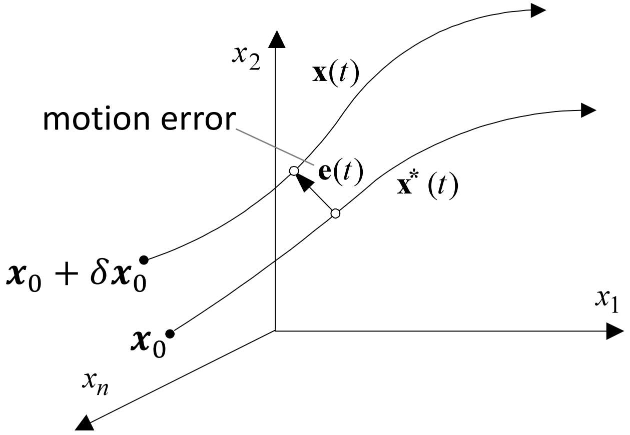
$$\boldsymbol{x}(0) = \boldsymbol{x}_0 + \delta\boldsymbol{x}_0 \xrightarrow{\text{solution}} \boldsymbol{x}(t), \qquad \dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$$

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*(t) \;\Rightarrow\; \dot{\boldsymbol{e}}(t) = \mathbf{f}(\boldsymbol{x}) - \mathbf{f}(\boldsymbol{x}^*)$$

$$\Rightarrow \dot{\boldsymbol{e}}(t) = \mathbf{f}(\boldsymbol{e} + \boldsymbol{x}^*) - \mathbf{f}(\boldsymbol{x}^*) = \mathbf{g}(\boldsymbol{e}, t)$$

$$\boldsymbol{e}(0) = \delta\boldsymbol{x}_0$$

(due to the presence of $\boldsymbol{x}^*(t)$)



Stability of a Motion (cont.)

$$\Rightarrow \qquad \dot{\boldsymbol{e}}(t) = \mathbf{g}(\boldsymbol{e}, t)$$ (a non-autonomous system)

Since $\mathbf{g}(\mathbf{0}, t) = \mathbf{0}$, the new dynamic system $\dot{\boldsymbol{e}}(t) = \mathbf{g}(\boldsymbol{e}, t)$ with $\boldsymbol{e}$ as state has an equilibrium point $\mathbf{0}$. Therefore, instead of studying the deviation of $\boldsymbol{x}(t)$ from $\boldsymbol{x}^*(t)$ for the original system, we can simply study the stability of $\dot{\boldsymbol{e}}(t) = \mathbf{g}(\boldsymbol{e}, t)$ with respect to the equilibrium point $\mathbf{0}$.

Results:

- Each particular nominal motion of an **autonomous system** corresponds to an equivalent **non-autonomous system**.
- For **non-autonomous** nonlinear systems, the stability problem around a nominal motion can also be transformed as a stability problem around the origin for an equivalent **non-autonomous** system.
- If the original system is **autonomous** and **linear** as $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, then the equivalent system is still **autonomous**, since it can be written as

$$\dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e}$$ (Prove it!)

Stability of a Motion: Example

Consider the autonomous mass-spring system

$$m\ddot{x} + k_1 x + k_2 x^3 = 0$$

Study the stability of the motion $x^*(t)$ which starts from initial position x_0.

Slightly Perturbing the initial condition $\quad x(0) = x_0 + \delta x_0 \xrightarrow{\text{solution}} x(t)$

$$\begin{array}{rl} & m\ddot{x} + k_1 x + k_2 x^3 = 0 \\ - & m\ddot{x}^* + k_1 x^* + k_2 {x^*}^3 = 0 \\ \hline \end{array}$$

$$e(t) = x(t) - x^*(t)$$

$$m\ddot{e} + k_1 e + k_2[e^3 + 3e^2 x^*(t) + 3e{x^*}^2(t)] = 0$$

(a non-autonomous system)

Stability Theories

Two techniques are typically used in the study of the stability of nonlinear systems:

- ☐ **Input-Output Stability**: Stability of the system from an input-output perspective.
- ☐ **Lyapunov Stability**: Stability of the system using state variables description.

Lyapunov Stability Theory includes two methods:

1) **Indirect Method** or **Linearization Method**: It is restricted to **local** stability around an equilibrium point.
2) **Direct Method** or **Second Method**: This is a powerful tool for nonlinear system analysis and design.
    - **Equilibrium Point Theorem**
    - **Invariant Set Theorem (LaSalle Theorem)**

Lyapunov's Linearization Method

Lyapunov's Linearization Method

Lyapunov's linearization method (or **indirect method**) is concerned with the **local stability** of a nonlinear system.

- It states that a nonlinear system should behave similarly to its **linearized approximation** for small range motions in the close vicinity of an equilibrium point. Thus, the **local stability of a nonlinear system** around an equilibrium point is the same as the stability properties of its linear approximation.
- The method serves as the theoretical justification for using **linear control** for physical systems. It shows that stable design by linear control guarantees the local stability of the physical system, which are always inherently nonlinear.

Linearization

- Dynamic of a nonlinear autonomous system $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, \boldsymbol{u})$ when $\boldsymbol{u} = \mathbf{0}$ can be represented as

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$$

- Moreover, the closed-loop dynamics of a feedback control system when $\boldsymbol{u} = \mathbf{k}(\boldsymbol{x})$ can be also represented as

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, \boldsymbol{u}) \longrightarrow \dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, \mathbf{k}(\boldsymbol{x})) \longrightarrow \dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$$

$$\xrightarrow{\text{Taylor Expansion}} \quad \dot{\boldsymbol{x}} = \cancelto{\mathbf{0}}{\mathbf{f}(\boldsymbol{x}_{eq})} + \left(\frac{\partial \mathbf{f}}{\partial \boldsymbol{x}}\right)_{\boldsymbol{x}=\boldsymbol{x}_{eq}} \left(\boldsymbol{x} - \boldsymbol{x}_{eq}\right) + \underbrace{\mathbf{f}_{\text{h.o.t.}}(\boldsymbol{x})}_{\text{(higher-order terms)}}$$

Assumptions:
- $\mathbf{f}(\boldsymbol{x})$ is continuously differentiable.
- $\boldsymbol{x}_{eq}$ is an equilibrium point, i.e., $\mathbf{f}\left(\boldsymbol{x}_{eq}\right) = \mathbf{0}$.

$\downarrow$

$\mathbf{A}$: $n \times n$ Jacobian matrix of $\mathbf{f}$ with respect to $\boldsymbol{x}$ $\quad A_{ij} = \dfrac{\partial f_i}{\partial x_j}$

$$\Rightarrow \quad \boxed{\dot{\bar{\boldsymbol{x}}} = \mathbf{A}\bar{\boldsymbol{x}}}$$

$$\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_{eq}$$

Linearization (or linear approximation) of the nonlinear system $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$ at the equilibrium point $\boldsymbol{x}_{eq}$.

Linearization: Example

Example: Linearization the nonlinear system at the equilibrium point $\boldsymbol{x}_{eq} = \mathbf{0}$.

$$\begin{aligned}\dot{x}_1 &= x_2^2 + x_1 \cos x_2 \\ \dot{x}_2 &= x_2 + (x_1+1)x_1 + x_1 \sin x_2\end{aligned} \quad \Rightarrow \quad \begin{aligned}\dot{x}_1 &= 0 + x_1.1 \\ \dot{x}_2 &= x_2 + 0 + x_1 + x_1 x_2\end{aligned} \quad \Rightarrow \quad \begin{aligned}\dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 + x_1\end{aligned}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{x}$$

(unstable)

Example: Linearization the nonlinear system $\ddot{x} + 4\dot{x}^5 + (x^2+1)u = 0$ about $x = 0$ when $u = \sin x + x^3 + \dot{x}\cos^2 x$

$$\ddot{x} + 4\dot{x}^5 + (x^2+1)u = 0 \quad \Rightarrow \quad \ddot{x} + u = 0$$

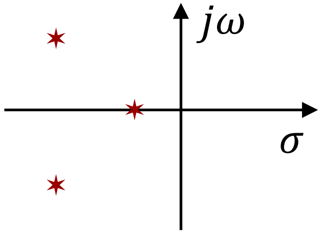
$$u = \sin x + x^3 + \dot{x}\cos^2 x \quad \Rightarrow \quad u = x + \dot{x}$$

$$\Rightarrow \quad \ddot{x} + \dot{x} + x = 0$$

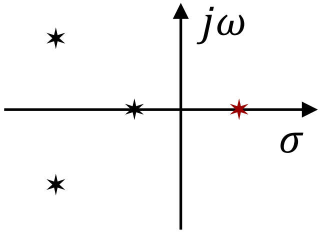
(stable)

Lyapunov's Linearization Method: Stability

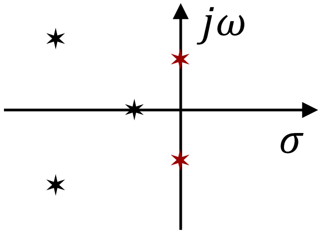
The relationship between the **local stability of a nonlinear system** $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$ around an equilibrium point $\boldsymbol{x}_{eq}$ and that of the its linear approximation $\dot{\bar{\boldsymbol{x}}} = \mathbf{A}\bar{\boldsymbol{x}}$:



1) If the linearized system is **strictly stable** (i.e., if all eigenvalues of $\mathbf{A}$ are strictly in the left-half complex plane), then the equilibrium point is (locally) **asymptotically stable** for the nonlinear system.



2) If the linearized system is **unstable** (i.e., if at least one eigenvalue of $\mathbf{A}$ is strictly in the right-half complex plane), then the equilibrium point is (locally) **unstable** for the nonlinear system.



3) If the linearized system is **marginally stable** (i.e., all eigenvalues of $\mathbf{A}$ are in the left-half complex plane, but at least one of them is on the $j\omega$ axis), then one **cannot conclude anything** from the linear approximation (and $\mathbf{f}_{\text{h.o.t.}}(\boldsymbol{x})$ have a decisive effect on whether the equilibrium point is **stable**, **asymptotically stable**, or **unstable** for the nonlinear system).

Example

Consider the first order system $\dot{x} = ax + bx^5$

The origin 0 is one of the equilibrium points of this system. The linearization of this system around the origin is

$$\dot{x} = ax$$

$\Downarrow$ Lyapunov's linearization method

$a < 0$: asymptotically stable

$a > 0$: unstable

$a = 0$: cannot tell from linearization

But

How large is the linear range?

What is the extent of stability ?

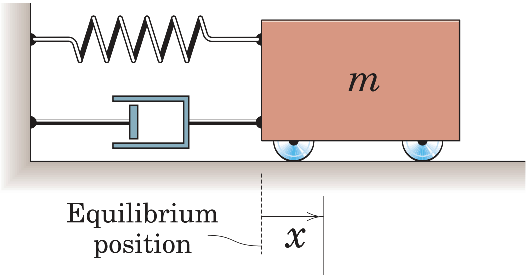
The **Lyapunov's Direct Method** can answer these questions.

Equilibrium Point Theorem

Motivation

Consider a nonlinear mass-damper-spring system. Will the system be stable if the mass is released from a large $x(0) = x_0$?

$$m\ddot{x} + \underbrace{b\dot{x}|\dot{x}|}_{\text{Nonlinear Damper}} + \underbrace{k_0 x + k_1 x^3}_{\text{Nonlinear Spring}} = 0$$



1) Using the definitions of stability?

It is very difficult, because the general solution of this nonlinear equation is unavailable.

2) Using the Lyapunov’s linearization method?

It cannot be used, because the motion starts outside the linear range. If it is used, the system's linear approximation is only marginally stable.

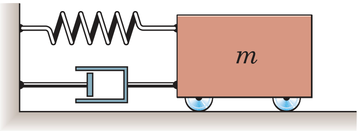
$$m\ddot{x} + k_0 x = 0$$

Motivation: Lyapunov's Direct Method

The basic philosophy of **Lyapunov's Direct Method** is the mathematical extension of a fundamental physical observation:

If the **total energy** of a mechanical/electrical system is continuously **dissipated**, the system must eventually settle down to an **equilibrium point**.

The total mechanical energy of this nonlinear mass-damper-spring system is



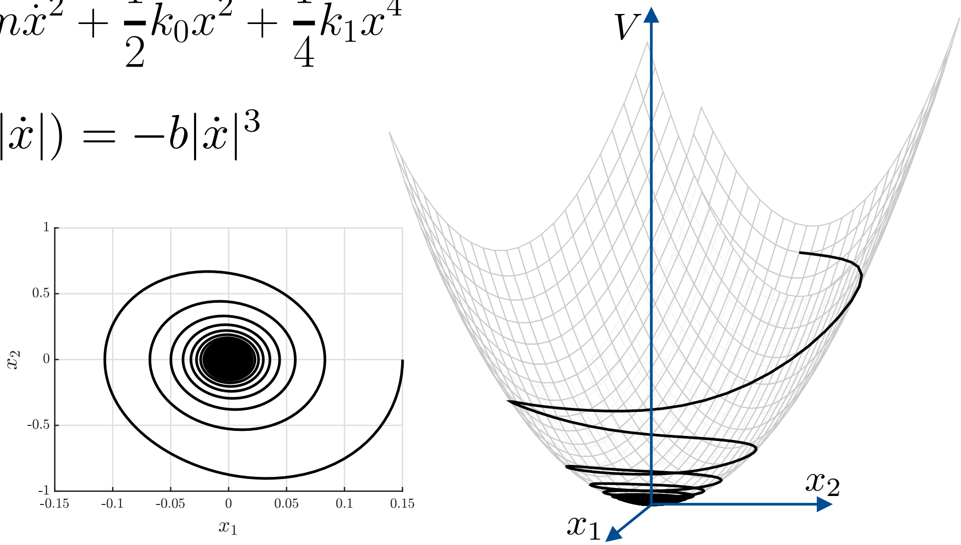
$$V(\boldsymbol{x}) = \frac{1}{2}m\dot{x}^2 + \int_0^x \left(k_o\bar{x} + k_1\bar{x}^3\right)d\bar{x} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4$$

$$\dot{V}(\boldsymbol{x}) = m\dot{x}\ddot{x} + \left(k_o x + k_1 x^3\right)\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

↓

Energy of the system is dissipated by the damper until the mass settles down at the natural length of the spring.

$$x_1 = x$$
$$x_2 = \dot{x}$$



Motivation: Lyapunov’s Direct Method (cont.)

Thus, we can conclude that value of V indirectly reflects the magnitude of the state vector $\boldsymbol{x}$, consequently, the stability of a system can be examined by the variation of a single scalar function V.

- Zero energy (or V) corresponds to the **equilibrium point** ($\boldsymbol{x} = \boldsymbol{x}_{eq}$).
- **Asymptotic stability** corresponds to the convergence of energy (or V) to zero.
- **Instability** corresponds to the growth of energy (or V).

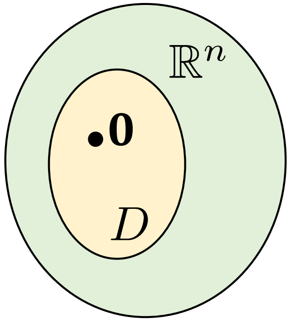
$*$ In using the **Lyapunov's direct method** to analyze the stability of a nonlinear system, the idea is to generate a **scalar "energy-like" function** (a **Lyapunov function**) V for the system and examine the time variation of the function to see whether it decreases (without using the difficult stability definitions or requiring explicit knowledge of solutions).

Positive Definite Functions

A scalar, continuous function $V(\boldsymbol{x})$ ($V\colon D\to\mathbb{R}$, $D\subset\mathbb{R}^n$, $\mathbf{0}\in D$) is said to be **Locally Positive Definite** if

1) $V(\mathbf{0})=0$,

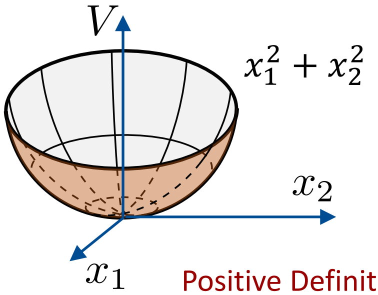
2) $V(\boldsymbol{x})>0\quad\forall\boldsymbol{x}\in D$ with $\boldsymbol{x}\neq\mathbf{0}$.



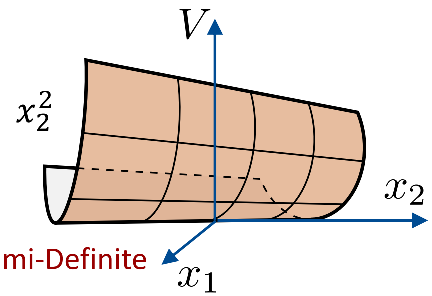
$V(\boldsymbol{x})$ is said to be **Globally Positive Definite** if $D=\mathbb{R}^n$.

$\therefore V(\boldsymbol{x})$ has a unique minimum at $\mathbf{0}$.

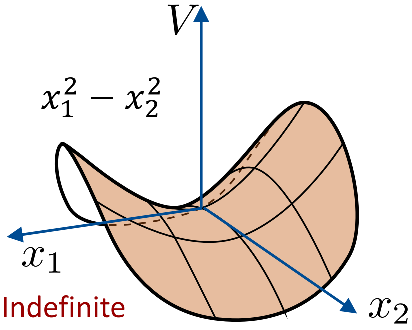
- A function $V(\boldsymbol{x})$ is **positive semi-definite** if $V(\mathbf{0})=0$ and $V(\boldsymbol{x})\geq 0$, $\forall\boldsymbol{x}\in D$ with $\boldsymbol{x}\neq\mathbf{0}$.
- A function $V(\boldsymbol{x})$ is **negative (semi-)definite** if $-V(\boldsymbol{x})$ is positive (semi-)definite.



Positive Definite



Positive Semi-Definite

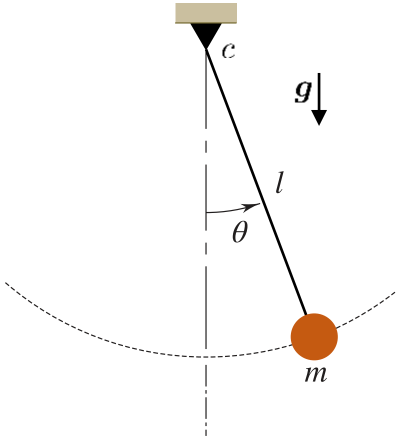


Indefinite

Examples

$$V(\boldsymbol{x}) = \frac{1}{2}ml^2x_2^2 + mlg\left(1 - \cos x_1\right)$$

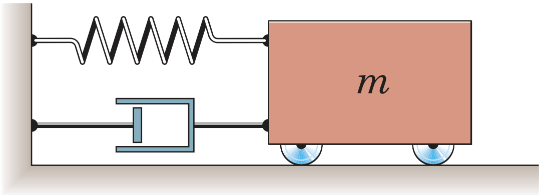
(locally positive definite)



D:
$-\pi < x_1 < \pi$
$x_2 \in \mathbb{R}$

$$V\left(\boldsymbol{x}\right) = \underbrace{\frac{1}{2}m\dot{x}^2}_{} + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4$$

(globally positive definite)

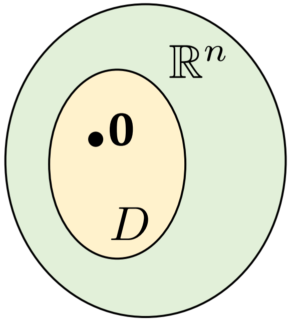


Note: This term is not positive definite by itself, because it can equal zero for non-zero values of x.

Note: All the quadratic functions $f(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$ ($f\colon \mathbb{R}^n \to \mathbb{R}$) with positive definite matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ are globally positive definite.

Lyapunov Functions

Consider an autonomous system, $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$, with an equilibrium point at origin, $\boldsymbol{x} = \mathbf{0}$. A scalar, continuously differentiable function $V(\boldsymbol{x})$ ($V: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, $\mathbf{0} \in D$) is said to be **Lyapunov Function** for the system if



1) $V(\boldsymbol{x})$ is **positive definite** (locally in D), i.e.,
   1.1) $V(\mathbf{0}) = 0$,
   1.2) $V(\boldsymbol{x}) > 0 \quad \forall \boldsymbol{x} \in D$ with $\boldsymbol{x} \neq \mathbf{0}$.

2) $\dot{V}(\boldsymbol{x})$ is **negative semi-definite** (locally in D), i.e.,
   2.1) $\dot{V}(\mathbf{0}) = 0$
   2.2) $\dot{V}(\boldsymbol{x}) \leq 0 \quad \forall \boldsymbol{x} \in D$ with $\boldsymbol{x} \neq \mathbf{0}$.

Note: $V(\boldsymbol{x})$ is an implicit function of time t.

$$\dot{V}(\boldsymbol{x}) = \frac{dV}{dt} = \frac{\partial V}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = \frac{\partial V}{\partial \boldsymbol{x}} \mathbf{f}(\boldsymbol{x})$$

(chain rule)

Equilibrium Point Theorem:

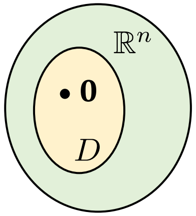
(The relation between Lyapunov Functions & Stability)

Consider an autonomous system, $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$, with an equilibrium point at origin, $\boldsymbol{x} = \mathbf{0}$.

Local Stability (in the vicinity of equilibrium point $\mathbf{0}$):

If there exists a scalar, continuously differentiable function $V(\boldsymbol{x})$ ($V: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, $\mathbf{0} \in D$) such that

1) $V(\boldsymbol{x}) > 0$ (locally in D),
2) $\dot{V}(\boldsymbol{x}) \leq 0$ (locally in D),



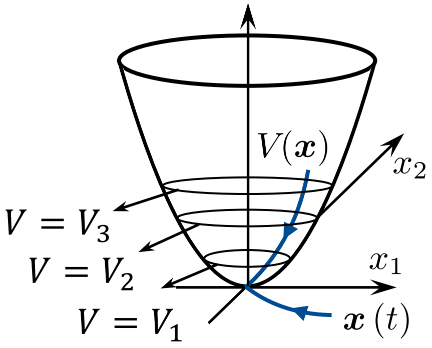
the equilibrium point $\mathbf{0}$ is **Locally Stable**. If $\dot{V}(\boldsymbol{x})$ is **negative definite** ($\dot{V}(\boldsymbol{x}) < 0$, locally in D), the equilibrium point $\mathbf{0}$ is **Locally Asymptotically Stable**.

Global Stability: $D = \mathbb{R}^n$

If there exists a scalar, continuously differentiable function $V(\boldsymbol{x})$ ($V: \mathbb{R}^n \to \mathbb{R}$) such that

1) $V(\boldsymbol{x}) > 0$ (globally positive definite),
2) $\dot{V}(\boldsymbol{x}) < 0$ (globally negative definite),
3) $V(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$ (i.e., $V(\boldsymbol{x})$ is radially unbounded),

the equilibrium point $\mathbf{0}$ is **Globally Asymptotically Stable**.

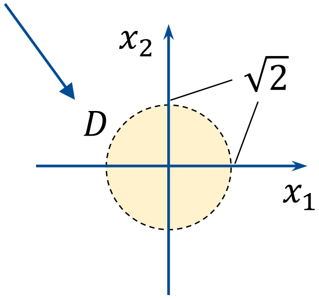


Examples

Example: $$\begin{aligned}\dot{x}_1 &= x_1\left(x_1^2+x_2^2-2\right)-4x_1x_2^2\\ \dot{x}_2 &= 4x_1^2x_2+x_2\left(x_1^2+x_2^2-2\right)\end{aligned}$$

Consider a Lyapunov Function as $V(\boldsymbol{x}) = x_1^2+x_2^2$

$$\dot{V} = 2\left(x_1^2+x_2^2\right)\left(x_1^2+x_2^2-2\right)$$ Locally Negative Definite $\left(x_1^2+x_2^2<2\right)$



The system is **Locally Asymptotically Stable**.

Example: $$\begin{aligned}\dot{x}_1 &= x_2-x_1\left(x_1^2+x_2^2\right)\\ \dot{x}_2 &= -x_1-x_2\left(x_1^2+x_2^2\right)\end{aligned}$$

Consider a Lyapunov Function as $V(\boldsymbol{x}) = x_1^2+x_2^2$

$$\dot{V}(\boldsymbol{x}) = 2x_1\dot{x}_1+2x_2\dot{x}_2 = -2\left(x_1^2+x_2^2\right)^2$$ Negative Definite

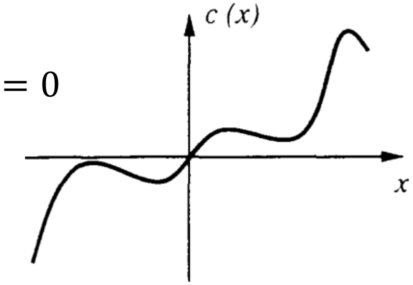
The origin is **Globally Asymptotically Stable**.

Example

A Class of First-Order Nonlinear Systems

Consider the nonlinear first-order system $\dot{x} + c(x) = 0$, where c is any continuous function of the same sign as x, i.e., $xc(x) > 0$ for $x \neq 0$.

Since c is continuous, $c(0) = 0$



Consider as the Lyapunov function candidate: $V = x^2$

$$V > 0$$
$$\dot{V} = 2x\dot{x} = -2xc(x) < 0 \quad \Rightarrow \quad \text{The origin is } \textbf{Globally Asymptotically Stable}.$$
V is radially unbounded

For instance,

- $\dot{x} + x - \sin^2 x = 0$ Since $\sin^2 x \leq |\sin x| < |x|$, $x - \sin^2 x$ has the same sign as x.

  $\Rightarrow$ The origin is **Globally Asymptotically Stable**.

- $\dot{x} + x^3 = 0$ $\Rightarrow$ The origin is **Globally Asymptotically Stable**.

Notice that the system's linear approximation ($\dot{x} \approx 0$) is inconclusive, even about local stability.

Remarks

- Lyapunov function **is not unique** for a system. Many Lyapunov functions may exist for the same system.

For instance, if V is a Lyapunov function for a given system, so is $V_1 = \rho V^{\alpha}$

$$\rho, \alpha \in \mathbb{R}, \ \ \rho > 0, \alpha > 1$$

(The positive definiteness of V implies that of V_1, the negative (semi-)definiteness of $\dot{V}$ implies that of $\dot{V}_1$, and the radial unboundedness of V implies that of V_1.)

- The theorems in Lyapunov analysis are all **sufficiency theorems**. If for a particular choice of Lyapunov function candidate V, the conditions on $\dot{V}$ are not met, one cannot draw any conclusions on the stability or instability of the system, the only conclusion one should draw is that a different Lyapunov function candidate should be tried.

- For a given system, specific choices of Lyapunov functions may yield **more precise results** on the stability of the system than others (see the next example).

Example

A Pendulum with Viscous Damping

Consider a simple pendulum with viscous damping: $\ddot{\theta} + \dot{\theta} + \sin\theta = 0$

Let's consider pendulum total energy as Lyapunov Function: $V(\boldsymbol{x}) = (1 - \cos\theta) + \dfrac{\dot{\theta}^2}{2}$

$$\dot{V}(\boldsymbol{x}) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = -\dot{\theta}^2 \leq 0$$

Positive definite locally in $D = \{(\theta, \dot{\theta}) : \theta \in (-\pi, \pi)\}$

The origin is a **Locally Stable** equilibrium point. However, with this Lyapunov function, one cannot draw conclusions on the asymptotic stability of the system.

Now, let's consider a Lyapunov Function (without obvious physical meaning) as

$$V(\boldsymbol{x}) = 2(1 - \cos\theta) + \frac{\dot{\theta}^2}{2} + \frac{1}{2}(\dot{\theta} + \theta)^2$$

$$\dot{V}(\boldsymbol{x}) = -(\dot{\theta}^2 + \theta\sin\theta) < 0$$

$(\forall \boldsymbol{x} \in D$ with $\boldsymbol{x} \neq \boldsymbol{0})$ $\Rightarrow$ The origin is **Locally Asymptotically Stable**.

$D = \{(\theta, \dot{\theta}) : \theta \in (-\pi, \pi)\}$

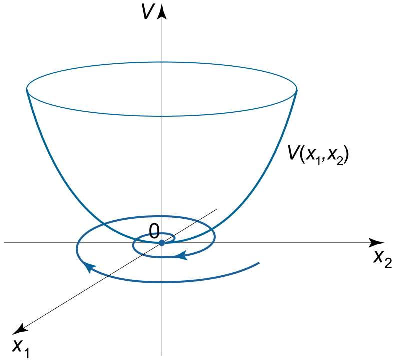
Invariant Set Theorem

Determining the Asymptotic Stability of Systems

Asymptotic stability of a control system is usually a very important property to be determined. Using **Equilibrium Point Theorem** for determining the **asymptotic stability** is often difficult, because it often happens that $\dot{V}(x)$ is only negative semi-definite.

↓

In these situations, **Invariant Set Theorem (LaSalle Theorem)** can be used to conclude the **asymptotic stability** of the system. It can also determine the **domain of attraction** and describe convergence to a **limit cycle**.



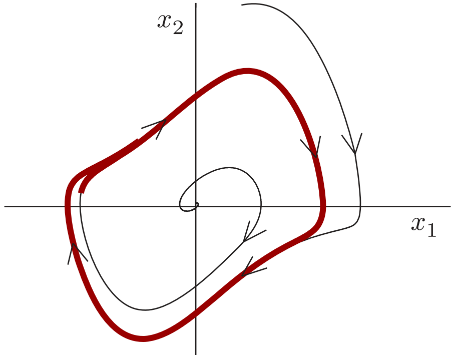
Invariant Set
(A generalization of the concept of equilibrium point)

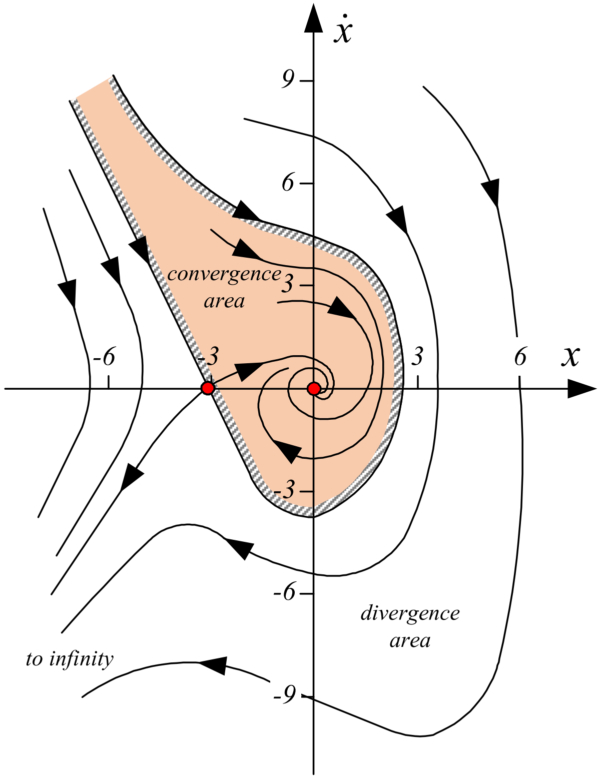
A set M is an invariant set for a dynamic system $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$ if every system trajectory which starts from a point in M remains in M for all future time.

$$\boldsymbol{x}(0) \in M \quad \Rightarrow \quad \boldsymbol{x}(t) \in M, \qquad \forall t \in \mathbb{R}$$

Examples of invariant set for an autonomous system:

- Any equilibrium point,
- Domain of attraction of an equilibrium point,
- Any of the trajectories in state-space,
- Limit cycles,
- Whole state-space (a trivial example).





Local Invariant Set Theorem (LaSalle Theorem)

Consider an autonomous system $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$. Let $V(\boldsymbol{x})$ ($V: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$) be a scalar function with continuous first partial derivatives. Assume that

- $\exists l > 0$, the region Ω_l defined by $V(\boldsymbol{x}) < l$ is bounded.
- $\dot{V}(\boldsymbol{x}) \leq 0, \;\; \forall \boldsymbol{x} \in \Omega_l$.

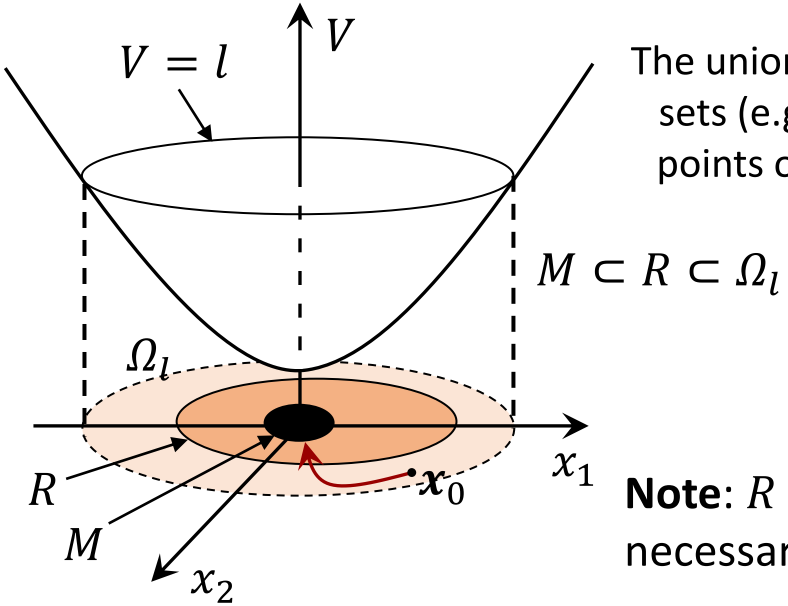
Let R be the set of all points within Ω_l where $\dot{V}(\boldsymbol{x}) = 0$, and M be the largest invariant set in R. Then, every solution $\boldsymbol{x}(t)$ originating in Ω_l tends to M as $t \to \infty$.

$$R = \left\{\boldsymbol{x} \in D \subset \mathbb{R}^n : \dot{V}(\boldsymbol{x}) = 0\right\}$$

The union of all invariant sets (e.g., equilibrium points or limit cycles).

- A special case of the invariant set theorem: When M consists only of the origin, it results in the **local asymptotic stability** of the **origin**.

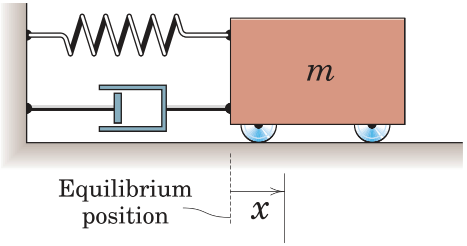
- Note the relaxation of the **positive definiteness requirement** on the function V, as compared with the **Equilibrium Point Theorem**.



Note: R and M are not necessarily connected.

Example: Asymptotic Stability

Consider the system $m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$

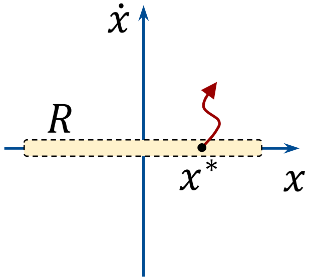


with a Lyapunov function chosen as

$$V(\boldsymbol{x}) = \frac{1}{2}m\dot{x}^2 + \int_0^x \left(k_o\bar{x} + k_1\bar{x}^3\right) d\bar{x} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4$$

$$\dot{V}(\boldsymbol{x}) = m\dot{x}\ddot{x} + \left(k_o x + k_1 x^3\right)\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

- Using Lyapunov's linearization method: **Marginally Stable** (inconclusive)
- Using equilibrium point theorem: **Stable**
- Using invariant set theorem:



$R = \{(x,\dot{x}):\ \dot{x} = 0\}$ (the whole horizontal axis in the phase plane)

Assume that the largest invariant set $M \subset R$ contains a point with a nonzero position x^*. $\Rightarrow \ddot{x} = -k_0/mx^* - k_1/mx^{*3} \neq 0 \Rightarrow$ The Trajectory will move out of R.

$\Rightarrow$ M contains only the origin. $\Rightarrow$ **(Globally) Asymptotically Stable**

Example: Domain of Attraction

Consider the system $\begin{aligned}\dot{x}_1 &= x_1\left(x_1^2+x_2^2-2\right)-4x_1x_2^2\\ \dot{x}_2 &= 4x_1^2x_2+x_2\left(x_1^2+x_2^2-2\right)\end{aligned}$

with a Lyapunov function chosen as $V(\boldsymbol{x})=x_1^2+x_2^2$

$$\dot{V}=2\left(x_1^2+x_2^2\right)\left(x_1^2+x_2^2-2\right)$$

For $l=2$, the region Ω_2 defined by $V(\boldsymbol{x})<2$ is bounded, and $\dot{V}(\boldsymbol{x})\le 0,\ \forall \boldsymbol{x}\in\Omega_2$.

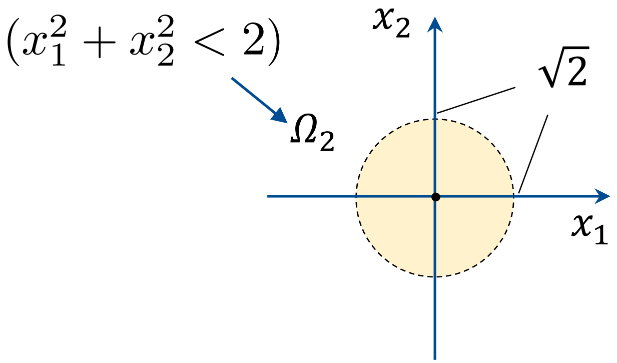
The set R is simply the origin $\mathbf{0}$, which is an invariant set (since it is an equilibrium point), thus, $M=R$.

$$\Downarrow$$

every solution $\boldsymbol{x}(t)$ starting within the circle Ω_2 converges to the origin.

$$\Downarrow$$

Ω_2 is the **domain of attraction**.



Global Invariant Set Theorem (LaSalle Theorem)

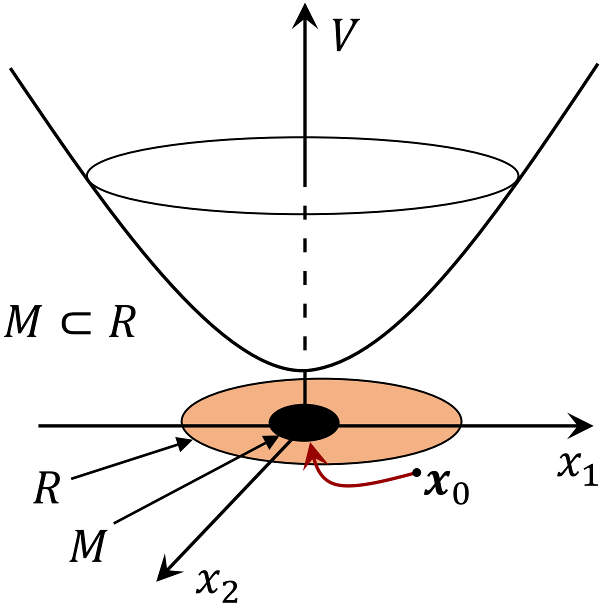
Consider an autonomous system $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$. Let $V(\boldsymbol{x})$ ($V: \mathbb{R}^n \to \mathbb{R}$) be a scalar function with continuous first partial derivatives. Assume that

- $\dot{V}(\boldsymbol{x}) \leq 0,\ \forall \boldsymbol{x} \in \mathbb{R}^n$,
- $V(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$ (i.e., $V(\boldsymbol{x})$ is radially unbounded).

Let R be the set of all points within $\mathbb{R}^n$ where $\dot{V}(\boldsymbol{x}) = 0$, and M be the largest invariant set in R. Then, every solution $\boldsymbol{x}(t)$ **globally** converge to M as $t \to \infty$.

The union of all invariant sets (e.g., equilibrium points or limit cycles).

$$R = \{\boldsymbol{x} \in \mathbb{R}^n : \dot{V}(\boldsymbol{x}) = 0\}$$



- A special case of the invariant set theorem: When M consists only of the origin, it results in the **global asymptotic stability** of the **origin**.

- Note the relaxation of the **positive definiteness requirement** on the function V, as compared with the **Equilibrium Point Theorem**.

Note: R and M are not necessarily connected.

Example:
A Class of Second-Order Nonlinear Systems

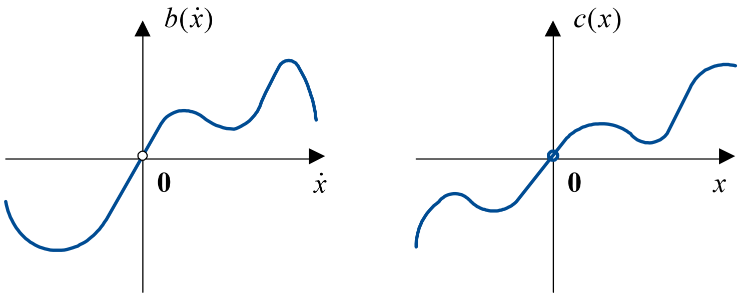
Consider the second-order system $\ddot{x} + b(\dot{x}) + c(x) = 0$ where b and c are continuous functions verifying the sign conditions as:

$$\begin{aligned} \dot{x}b(\dot{x}) &> 0 \quad \text{for } \dot{x} \neq 0 \\ xc(x) &> 0 \quad \text{for } x \neq 0 \end{aligned}$$

The continuity assumptions and the sign conditions imply that $b(0) = 0$ and $c(0) = 0$.

Consider a function V as the sum of the kinetic and potential energy of the system:

$$V = \frac{1}{2}\dot{x}^2 + \int_0^x c(y)dy$$



★ If $\int_0^x c(y)dy$ is unbounded as $\|\boldsymbol{x}\| \to \infty$, then $V(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$.

$$\dot{V} = \dot{x}\ddot{x} + c(x)\dot{x} = -\dot{x}b(\dot{x}) - \dot{x}c(x) + c(x)\dot{x} = -\dot{x}b(\dot{x}) \leq 0$$

(A representation of the power dissipation in the system)

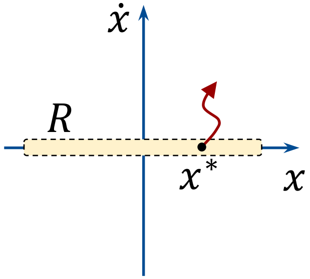
Example:

A Class of Second-Order Nonlinear Systems (cont.)

R: $\dot{V} = 0 \;\Rightarrow\; \dot{x} = 0$

$R = \{(x, \dot{x}) \colon \dot{x} = 0\}$

(the whole horizontal axis in the phase plane)



Assume that the largest invariant set $M \subset R$ contains a point with a nonzero position x^*. $\Rightarrow$ $\ddot{x} = -c(x^*) \neq 0$ $\Rightarrow$ The Trajectory will move out of R.

$\Rightarrow$ M contains only the origin. $\Rightarrow$ The origin is **Globally Asymptotically Stable**.

► For instance, the system $\ddot{x} + \dot{x}^3 + x^5 = x^4 \sin^2 x$ is globally asymptotically convergent to the origin, while its linear approximation $\ddot{x} = 0$ would be inconclusive, even about its local stability.

Example:

Multimodal Lyapunov Function

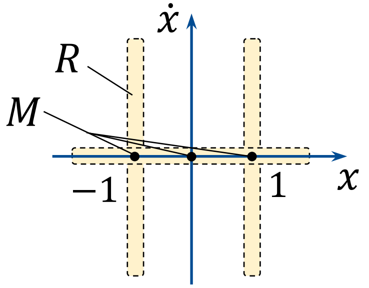
Consider the system $\ddot{x} + |x^2 - 1|\dot{x}^3 + x = \sin\dfrac{\pi x}{2}$

Consider a function V as the sum of the kinetic and potential energy of the system:

$$V = \frac{1}{2}\dot{x}^2 + \int_0^x \left(y - \sin\frac{\pi y}{2}\right) dy$$

$$\dot{V} = |x^2 - 1|\dot{x}^4 \leq 0, \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

$$V \to \infty \text{ as } \|\boldsymbol{x}\| \to \infty$$



$$R = \left\{(x, \dot{x}) \colon \dot{V}(\boldsymbol{x}) = 0\right\} \;\Rightarrow\; \dot{V} = 0 \;\Rightarrow\; \dot{x} = 0 \text{ or } x = \pm 1$$

$$\dot{x} = 0 \;\Rightarrow\; \ddot{x} = \sin\frac{\pi x}{2} - x \neq 0$$ Except for $x = 0$ or $x = \pm 1$

$$x = \pm 1 \;\Rightarrow\; \dot{x} = 0 \;\Rightarrow\; \ddot{x} = 0$$

$$\Rightarrow \; M = \{(0,0), (1,0), (-1,0)\}$$

The invariant set theorem indicates that the system converges globally to M.

Example:

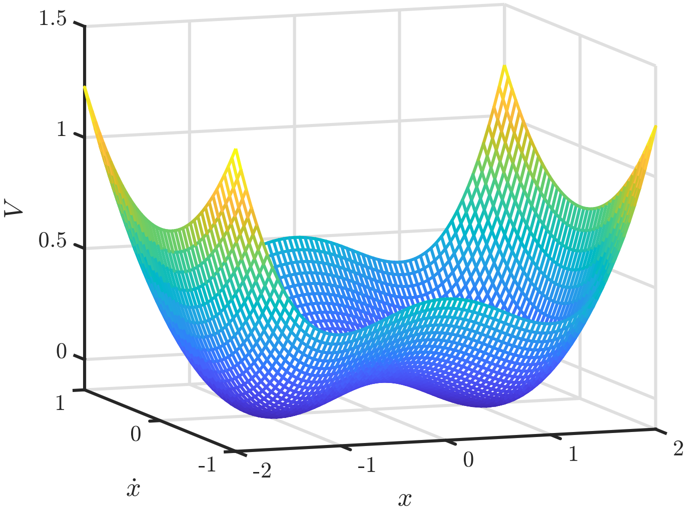
Multimodal Lyapunov Function (cont.)

Linearization about $(0,0)$: $\ddot{x} = \left(\frac{\pi}{2} - 1\right) x \quad \Rightarrow$ **Unstable**

Linearization about $(\pm 1,0)$: $\ddot{z} = -z \quad \Rightarrow$ Inconclusive (marginally stable)

$(z = x \mp 1)$

$$V = \frac{1}{2}\dot{x}^2 + \frac{2}{\pi}\cos\frac{\pi x}{2} + \frac{x^2}{2} - \frac{2}{\pi}$$



Function V has two minima at $(\pm 1,0)$ and a saddle point at $(0,0)$. Thus, $(\pm 1,0)$ are **Stable**.

Note: Since several Lyapunov functions may exist for a given system, several associated invariant sets M_i may be derived. The system converges to the (necessarily non-empty) intersection of the invariant sets, which may give a more precise result than that obtained from any of the Lyapunov functions taken separately.

A Corollary of Local Invariant Set Theorem (LaSalle Theorem)

Consider an autonomous system, $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x})$, with an equilibrium point at origin, $\boldsymbol{x} = \mathbf{0}$.

Local Stability (in the vicinity of equilibrium point $\mathbf{0}$):

If there exists a scalar, continuously differentiable function $V(\boldsymbol{x})$ ($V: \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $\mathbf{0} \in \Omega$) such that

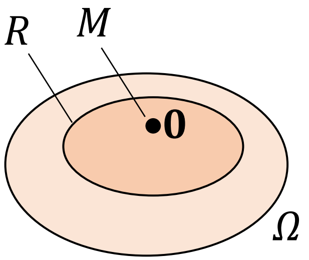
1) $V(\boldsymbol{x}) > 0$ (locally in Ω),
2) $\dot{V}(\boldsymbol{x}) \leq 0$ (locally in Ω),
3) $\boldsymbol{x} = \mathbf{0}$ is the only invariant set in $R = \{\boldsymbol{x}\text{: } \dot{V}(\boldsymbol{x}) = 0\}$,

Then, the equilibrium point $\mathbf{0}$ is **Locally Asymptotically Stable**.

Global Stability:

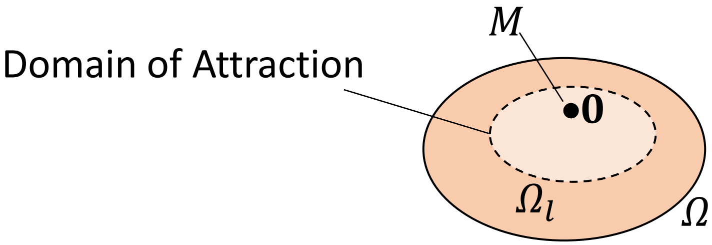
4) $\Omega = \mathbb{R}^n$,
5) $V(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$, (i.e., $V(\boldsymbol{x})$ is radially unbounded),

Then, the equilibrium point $\mathbf{0}$ is **Globally Asymptotically Stable**.



Remarks

- This corollary is used for **asymptotic stability** of an equilibrium point.
- This corollary replaces the **negative definiteness condition** on $\dot{V}$ in Equilibrium Point Theorem by a **negative semi-definiteness condition** on $\dot{V}$, combined with a condition ($\boldsymbol{x} = \mathbf{0}$ is the only invariant set in R), for **Local/Global Asymptotic Stability**.
- The largest connected region of the form Ω_l (defined by $V(\boldsymbol{x}) < l$) within Ω is a **domain of attraction** of the equilibrium point, but not necessarily the whole domain of attraction, because the function V is not unique.



Example:
A Pendulum with Viscous Damping

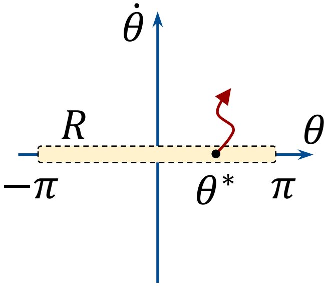
Consider a simple pendulum with viscous damping: $\ddot{\theta} + \dot{\theta} + \sin\theta = 0$

Let's consider pendulum total energy as Lyapunov Function: $V(\boldsymbol{x}) = (1 - \cos\theta) + \dfrac{\dot{\theta}^2}{2}$

Positive definite locally in $\Omega = \{(\theta, \dot{\theta}) : \theta \in (-\pi, \pi)\}$

$$\dot{V}(\boldsymbol{x}) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = -\dot{\theta}^2 \leq 0$$

The set R results in: $R = \{(\theta, \dot{\theta}) : \dot{\theta} = 0\}$



Assume that R contains a point with a nonzero position θ^*. $\Rightarrow$ $\ddot{\theta} = -\sin\theta^* \neq 0$ (except in $\theta^* = 0$)

$\Rightarrow$ The trajectory will move out of R. $\Rightarrow$ (0,0) is the only invariant set in R.

$\Rightarrow$ The origin is **Locally Asymptotically Stable**.