Ch3: Rigid-Body Motions – Part 2

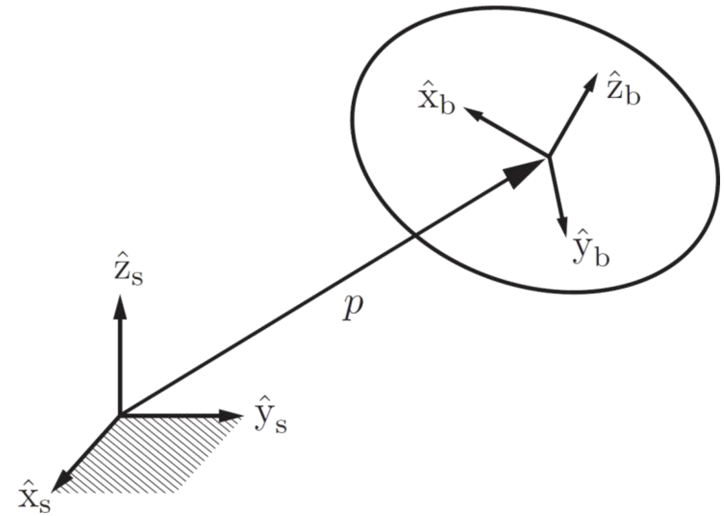
Rigid-Body Motions

Homogeneous Transformation Matrices

Rigid-body configuration can be represented by the pair $(\boldsymbol{R}, \boldsymbol{p})$ ($\boldsymbol{R} \in SO(3)$, $\boldsymbol{p} \in \mathbb{R}^3$). We can package $(\boldsymbol{R}, \boldsymbol{p})$ into a single 4×4 matrix as

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

(an implicit representation of the C-space)



Special Euclidean Group $SE(n)$

The **Special Euclidean Group** $SE(3)$, also known as the **group of rigid-body motions** or **homogeneous transformation matrices** in $\mathbb{R}^3$, is the set of all 4×4 real matrices $\boldsymbol{T}$ of the form

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T} \in SE(3)$$
$$\boldsymbol{R} \in SO(3)$$
$$\boldsymbol{p} \in \mathbb{R}^3$$

The **special Euclidean group** $SE(2)$ is the set of all 3×3 real matrices $\boldsymbol{T}$ of the form

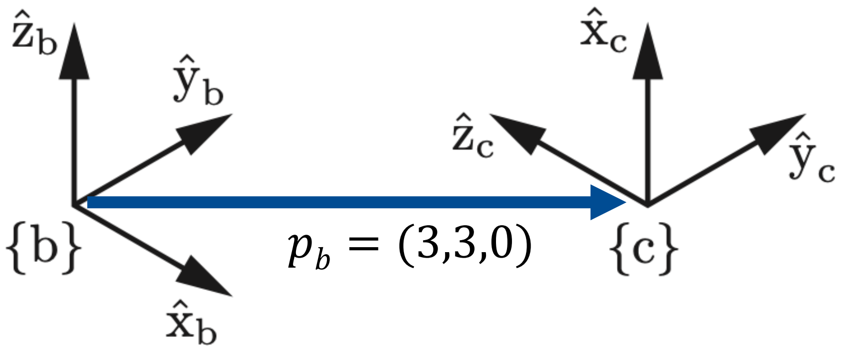
$$T = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T} \in SE(2)$$
$$\boldsymbol{R} \in SO(2)$$
$$\boldsymbol{p} \in \mathbb{R}^2$$
$$\theta \in [0,2\pi)$$

$SE(2)$ is a subgroup of $SE(3)$: $\quad SE(2) \subset SE(3)$

$$\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) \in SE(3) \qquad SE(3) = \{(\boldsymbol{R}, \boldsymbol{p}) \mid \boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3\}$$

Example



T_{bc}?

Properties of Transformation Matrices

$SE(3)$ (or $SE(2)$) is a **matrix (Lie) group** (and the group operation • is matrix multiplication).

Closure: $\boldsymbol{T}_1\boldsymbol{T}_2 \in SE(3)$

Associative: $(\boldsymbol{T}_1\boldsymbol{T}_2)\boldsymbol{T}_3 = \boldsymbol{T}_1(\boldsymbol{T}_2\boldsymbol{T}_3)$ (but generally not commutative, $\boldsymbol{T}_1\boldsymbol{T}_2 \neq \boldsymbol{T}_2\boldsymbol{T}_1$)

Identity: $\exists\, \boldsymbol{I} \in SE(3)$ such that $\boldsymbol{T}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{T} = \boldsymbol{T}$

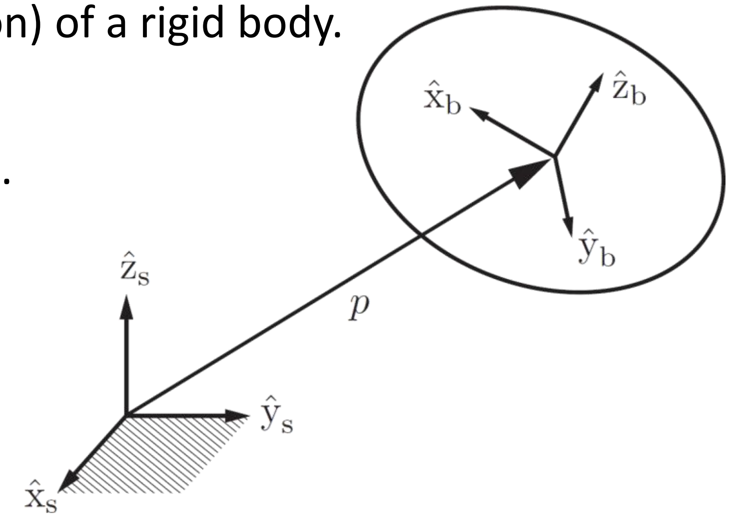
Inverse: $\exists\, \boldsymbol{T}^{-1} \in SE(3)$ such that $\boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{I}$

$$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Note: $\boldsymbol{T}$ preserves both distances and angles.

Uses of Transformation Matrices (1)

(1) Representing configuration (position and orientation) of a rigid body.



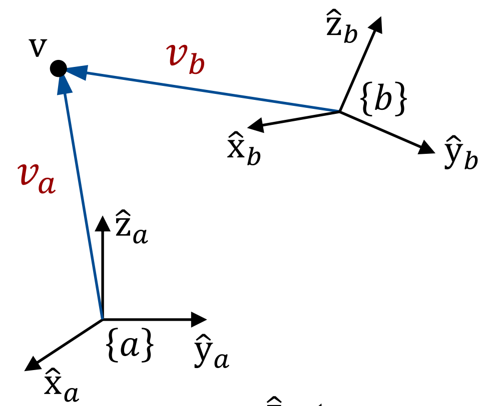
Notation: $\boldsymbol{T}_{sb}$ is the configuration of $\{b\}$ relative to $\{s\}$.

$$\boldsymbol{T}_{sb} = \begin{bmatrix} \boldsymbol{R}_{sb} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

$$\boldsymbol{T}_{sb}\boldsymbol{T}_{bs} = \boldsymbol{I} \quad \text{or} \quad \boldsymbol{T}_{bs} = \boldsymbol{T}_{sb}^{-1} = \begin{bmatrix} \boldsymbol{R}_{sb}^T & -\boldsymbol{R}_{sb}^T\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Uses of Transformation Matrices (2)

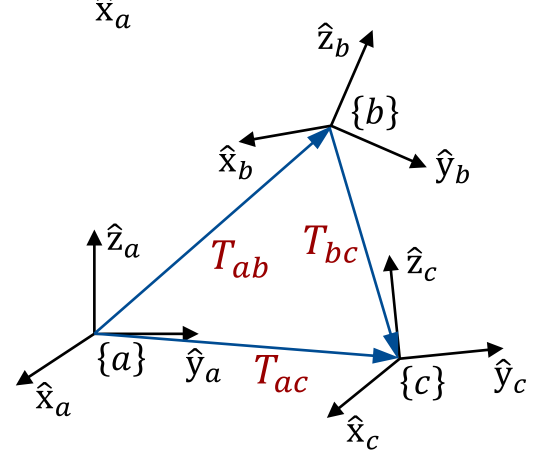
(2) Changing the reference frame of a <u>vector</u> or <u>frame</u>.



Subscript Cancellation Rule:

$$\boldsymbol{T}_{ab}\boldsymbol{v}_b = \boldsymbol{T}_{a\cancel{b}}\boldsymbol{v}_{\cancel{b}} = \boldsymbol{v}_a$$

$$\boldsymbol{T}_{ab}\boldsymbol{T}_{bc} = \boldsymbol{T}_{a\cancel{b}}\boldsymbol{T}_{\cancel{b}c} = \boldsymbol{T}_{ac}$$



$\boldsymbol{T}_{ab}$ can be viewed as a <u>**mathematical operator**</u> that changes the reference frame from $\{b\}$ to $\{a\}$.

Note: To calculate $\boldsymbol{Tv}$, we append a “1” to $\boldsymbol{v}$ and it is called **homogeneous coordinates** representation of $\boldsymbol{v}$.

$$\boldsymbol{v} = [v_1 \;\; v_2 \;\; v_3 \;\; 1]^T$$

Uses of Transformation Matrices (3)

(3) Displacing (rotating and translating) a vector or frame.

$$\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) = (\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta), \boldsymbol{p}) = \mathrm{Trans}(\boldsymbol{p})\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

$$\mathrm{Trans}(\boldsymbol{p}) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{p} \\ \mathbf{0} & 1 \end{bmatrix} \qquad \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} \boldsymbol{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$\boldsymbol{T}$ can be viewed as a **mathematical operator** that rotates a frame or vector about a unit axis $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$ by an amount θ + translating it by $\boldsymbol{p}$.

Uses of Transformation Matrices (3) (cont.)

- Rotation of vector $\boldsymbol{v}$ about a unit axis $\hat{\boldsymbol{\omega}}$ (expressed in the same frame) by an amount θ and translation of it by $\boldsymbol{p}$ (expressed in the same frame) is vector $\boldsymbol{v}'$ expressed in the same frame:

$$\boldsymbol{v}' = \boldsymbol{T}\boldsymbol{v} = \text{Trans}(\boldsymbol{p})\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v}$$

Interpretation

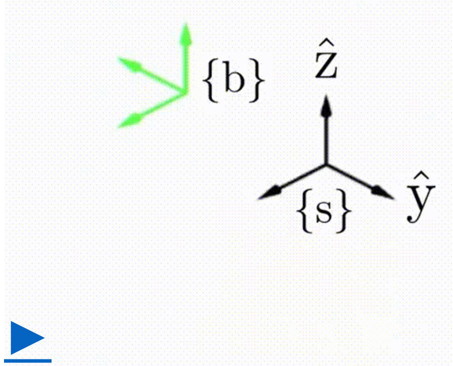
- **Fixed-frame Transformation**:

2. Translating it by $\boldsymbol{p}$ in $\{s\}$ to get $\{b'\}$

1. Rotating $\{b\}$ by θ about $\hat{\boldsymbol{\omega}}$ in $\{s\}$ (this moves $\{b\}$ origin)

$$\boldsymbol{T}_{sb'} = \boldsymbol{T}\boldsymbol{T}_{sb} = \text{Trans}(\boldsymbol{p})\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{T}_{sb}$$

Interpretation



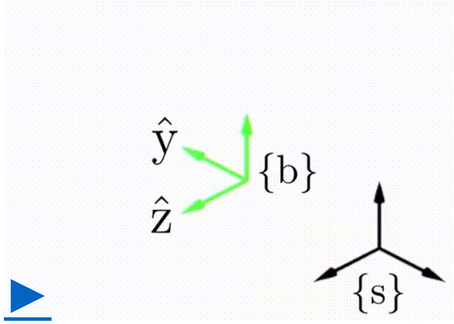
- **Body-frame Transformation**:

1. Translating $\{b\}$ by $\boldsymbol{p}$ in $\{b\}$

2. Rotating it by θ about $\hat{\boldsymbol{\omega}}$ in the new body frame to get $\{b''\}$

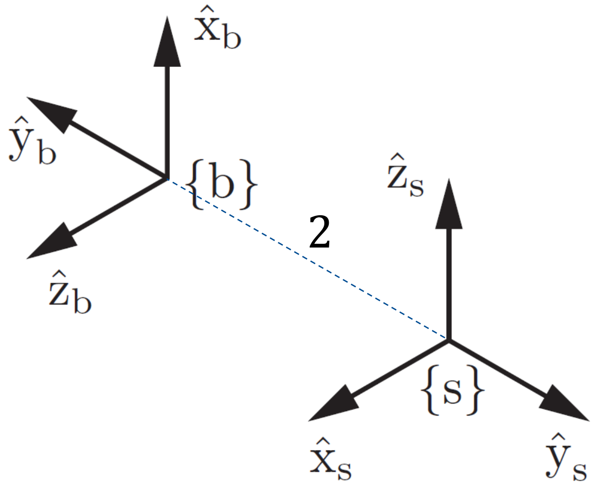
$$\boldsymbol{T}_{sb''} = \boldsymbol{T}_{sb}\boldsymbol{T} = \boldsymbol{T}_{sb}\text{Trans}(\boldsymbol{p})\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

Interpretation



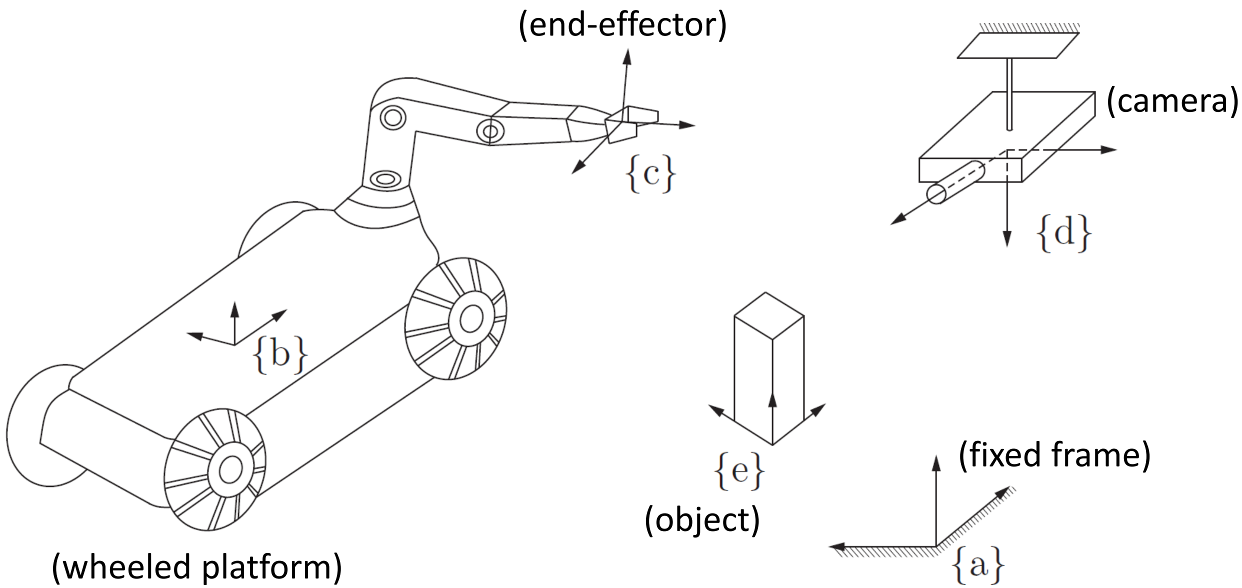
Example

Fixed-frame and body-frame transformations corresponding to $\hat{\boldsymbol{\omega}} = (0,0,1)$, $\theta = 90°$, and $\boldsymbol{p} = (0,2,0)$.



Example

A robot arm mounted on a wheeled mobile platform moving in a room, and a camera fixed to the ceiling. The robot must pick up an object with body frame $\{e\}$. What is the configuration of the object relative to the robot hand, $\boldsymbol{T}_{ce}$, given $\boldsymbol{T}_{db}$, $\boldsymbol{T}_{de}$, $\boldsymbol{T}_{bc}$, and $\boldsymbol{T}_{ad}$?

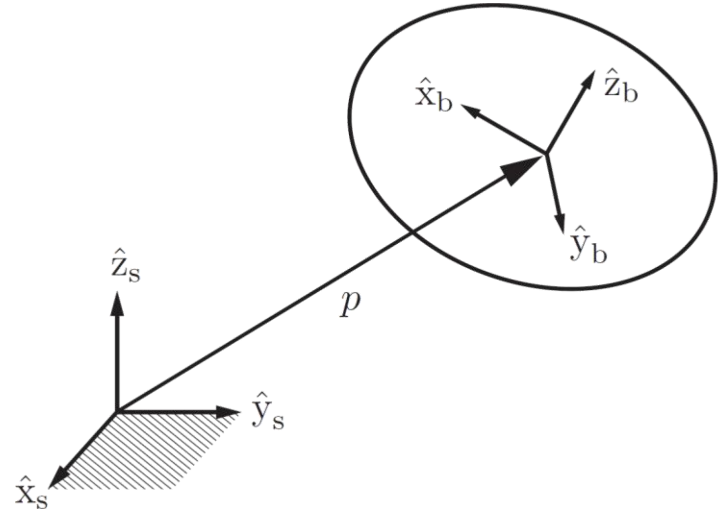


Twists

Spatial Velocity or Twist

Finding both the linear and angular velocity of frame $\{b\}$ attached to a moving body.

$$\boldsymbol{T}(t) = \begin{bmatrix} \boldsymbol{R}(t) & \boldsymbol{p}(t) \\ \boldsymbol{0} & 1 \end{bmatrix} \; : \; \boldsymbol{T}_{sb} \text{ at time } t$$



Body Twist

Similar to $\boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b]$, lets compute $\boldsymbol{T}^{-1}\dot{\boldsymbol{T}}$:

$$\boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{R}} & \dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}} & \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{v}_b \\ \boldsymbol{0} & 0 \end{bmatrix} \xrightarrow[{[\boldsymbol{\omega}_b] \in so(3)}]{\boldsymbol{v}_b \in \mathbb{R}^3} \boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} \in \mathbb{R}^6$$

$\boldsymbol{\mathcal{V}}_b$ is defined as **Body Twist**
(or spatial velocity in the body frame)

$$\boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{\mathcal{V}}_b] = \begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{v}_b \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$$

The set of all 4×4 matrices of this form is called $se(3)$ and comprises the 4×4 matrix representations of the **body twists** associated with the rigid-body configurations $SE(3)$.

($se(3)$ is called the Lie algebra of the Lie group $SE(3)$)

Spatial Twist

Similar to $\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s]$, lets compute $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$:

$$\begin{aligned}\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} &= \begin{bmatrix}\dot{\boldsymbol{R}} & \dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1\end{bmatrix} \\ &= \begin{bmatrix}\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}} & \dot{\boldsymbol{p}} - \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 0\end{bmatrix} \\ &= \begin{bmatrix}[\boldsymbol{\omega}_s] & \boldsymbol{v}_s \\ \boldsymbol{0} & 0\end{bmatrix}\end{aligned}$$

$$\xrightarrow[]{\boldsymbol{v}_s \in \mathbb{R}^3,\ [\boldsymbol{\omega}_s] \in so(3)} \quad \boldsymbol{\mathcal{V}}_s = \begin{bmatrix}\boldsymbol{\omega}_s \\ \boldsymbol{v}_s\end{bmatrix} \in \mathbb{R}^6$$

$\boldsymbol{\mathcal{V}}_s$ is defined as **Spatial Twist**
(or spatial velocity in the space frame)

$$\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\boldsymbol{\mathcal{V}}_s] = \begin{bmatrix}[\boldsymbol{\omega}_s] & \boldsymbol{v}_s \\ \boldsymbol{0} & 0\end{bmatrix} \in se(3)$$

The set of all 4×4 matrices of this form is called $se(3)$ and comprises the 4×4 <u>matrix representations</u> of the **spatial twists** associated with the rigid-body configurations $SE(3)$.

Adjoint Map

$$\cancel{\mathcal{V}_s = T_{sb}\mathcal{V}_b}$$

T_{sb}: 4×4 → 6×6

$$[\mathcal{V}_b] = T^{-1}\dot{T}$$
$$[\mathcal{V}_s] = \dot{T}T^{-1}$$

$$\longrightarrow [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1} \longrightarrow$$

$$[\mathcal{V}_s] = \begin{bmatrix} R[\omega_b]R^{\mathrm{T}} & -R[\omega_b]R^{\mathrm{T}}p + Rv_b \\ \mathbf{0} & 0 \end{bmatrix}$$

$$R[\omega]R^{\mathrm{T}} = [R\omega]$$
$$[\omega]p = -[p]\omega \quad p, \omega \in \mathbb{R}^3$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Adjoint Map associated with T
or Adjoint Representation of T

$$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$$

Similarly, $\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s = \mathrm{Ad}_{T_{bs}}(\mathcal{V}_s)$

Adjoint Map Properties

- Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in SE(3)$ and $\boldsymbol{\mathcal{V}} = (\boldsymbol{\omega}, \boldsymbol{v})$. Then,

$$[\mathrm{Ad}_{\boldsymbol{T}_1}][\mathrm{Ad}_{\boldsymbol{T}_2}]\boldsymbol{\mathcal{V}} = [\mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]\boldsymbol{\mathcal{V}} \qquad \text{or} \qquad \mathrm{Ad}_{\boldsymbol{T}_1}\big(\mathrm{Ad}_{\boldsymbol{T}_2}(\boldsymbol{\mathcal{V}})\big) = \mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}(\boldsymbol{\mathcal{V}})$$

- For any $\boldsymbol{T} \in SE(3)$, $[\mathrm{Ad}_{\boldsymbol{T}}]^{-1} = [\mathrm{Ad}_{\boldsymbol{T}^{-1}}]$

- For any two frames $\{c\}$ and $\{d\}$, a twist represented as $\boldsymbol{\mathcal{V}}_c$ in $\{c\}$ is related to its representation $\boldsymbol{\mathcal{V}}_d$ in $\{d\}$ by

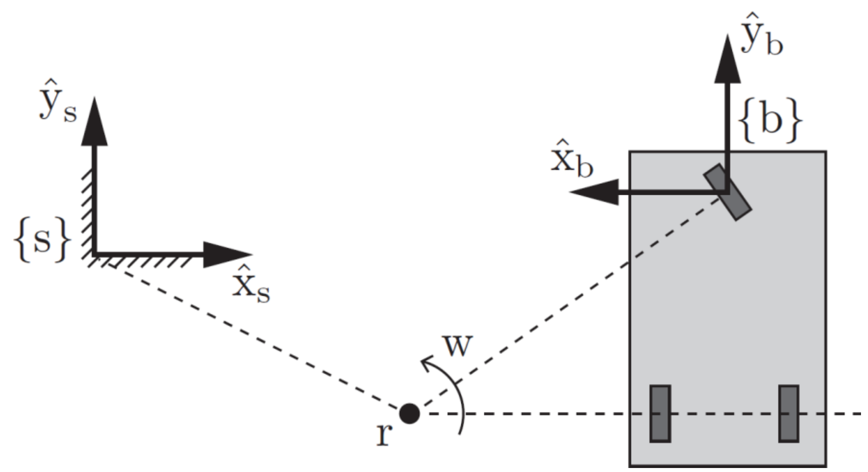
$$\boldsymbol{\mathcal{V}}_c = [\mathrm{Ad}_{\boldsymbol{T}_{cd}}]\boldsymbol{\mathcal{V}}_d \qquad\qquad \boldsymbol{\mathcal{V}}_d = [\mathrm{Ad}_{\boldsymbol{T}_{dc}}]\boldsymbol{\mathcal{V}}_c$$

(changing the reference frame of a twist)

Example

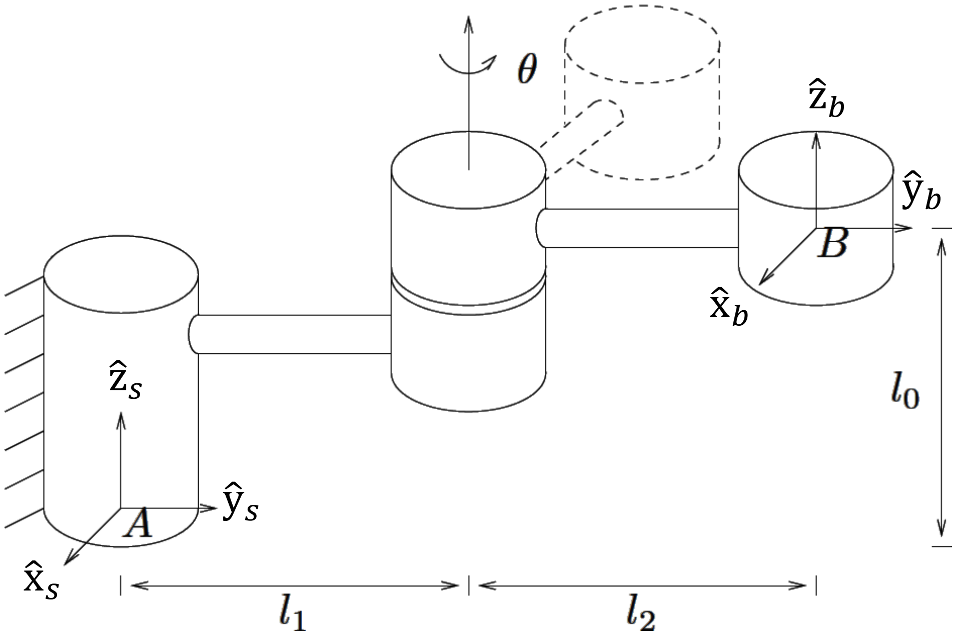
Consider a three-wheeled car with a single steerable front wheel, driving on a plane. The angle of the front wheel of the car causes the car's motion to be a pure angular velocity 2 rad/s about an axis out of the page at the point r in the plane. Find $\mathcal{V}_s$ and $\mathcal{V}_b$ when

$$r_s = (2, -1, 0)$$
$$r_b = (2, -1.4, 0)$$



Example

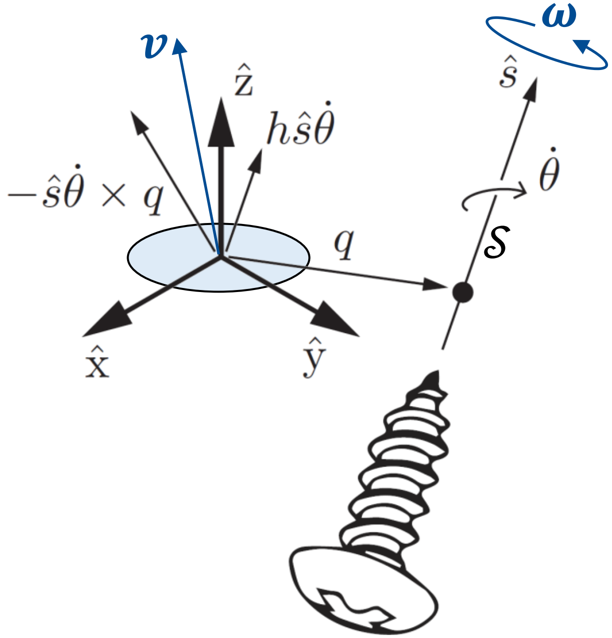
Find $\mathcal{V}_s$ and $\mathcal{V}_b$ for the shown one degree of freedom manipulator.



Screw Interpretation of a Twist

Any rigid-body velocity or twist $\boldsymbol{\mathcal{V}}$ is equivalent to the instantaneous velocity $\dot{\theta}$ about some screw axis $\mathcal{S}$ (i.e., rotating about the axis while also translating along the axis).

A screw axis $\mathcal{S}$ represented by a point $\boldsymbol{q} \in \mathbb{R}^3$ on the axis, a unit vector $\hat{\boldsymbol{s}} \in S^2$ in the direction of the axis, and a pitch $h \in \mathbb{R}$ (linear velocity along the axis / angular velocity $\dot{\theta}$ about the axis) as $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$.



Thus, twist $\boldsymbol{\mathcal{V}}$ can be represented as

$$\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}}\dot{\theta} \\ -\hat{\boldsymbol{s}}\dot{\theta} \times \boldsymbol{q} + h\dot{\theta}\hat{\boldsymbol{s}} \end{bmatrix}$$

Due to rotation about $\mathcal{S}$
(which is in the plane orthogonal to $\hat{\boldsymbol{s}}$)

Due to translation along $\mathcal{S}$
(which is in the direction of $\hat{\boldsymbol{s}}$)

Screw Interpretation of a Twist

Instead of representing the screw axis $\mathcal{S}$ as $\{\boldsymbol{q}, \hat{\mathbf{s}}, h\}$ (with the non-uniqueness of $\boldsymbol{q}$), we represent a "unit" screw axis as a vector as

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6 \quad \text{where} \quad \boldsymbol{\mathcal{V}} = \boldsymbol{S}\dot{\theta} \in \mathbb{R}^6 \qquad \boldsymbol{S}_\omega, \boldsymbol{S}_v \in \mathbb{R}^3$$

Finding $\boldsymbol{S}$ by having $\boldsymbol{\mathcal{V}}$:

(a) If $\|\boldsymbol{\omega}\| \neq 0$ ($\equiv$ rotation with/without translation along $\hat{\mathbf{s}}$):

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \boldsymbol{\mathcal{V}}/\|\boldsymbol{\omega}\| = \begin{bmatrix} \boldsymbol{\omega}/\|\boldsymbol{\omega}\| \\ \boldsymbol{v}/\|\boldsymbol{\omega}\| \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \boldsymbol{q} + h\hat{\mathbf{s}} \end{bmatrix}$$

$$= \begin{bmatrix} \text{angular velocity when } \dot{\theta} = 1 \\ \text{linear velocity of origin when } \dot{\theta} = 1 \end{bmatrix}$$

Pitch h is finite, $h = \boldsymbol{\omega}^T \boldsymbol{v}/\|\boldsymbol{\omega}\|^2$

$\hat{\mathbf{s}} = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|$, $\quad \|\boldsymbol{S}_\omega\| = 1$

$\dot{\theta} = \|\boldsymbol{\omega}\|$ is interpreted as angular velocity about $\hat{\mathbf{s}}$

(b) If $\|\boldsymbol{\omega}\| = 0$ ($\equiv$ pure translation along $\hat{\mathbf{s}}$):

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \boldsymbol{\mathcal{V}}/\|\boldsymbol{v}\| = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v}/\|\boldsymbol{v}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{s}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} \\ \text{normalized linear velocity of origin} \end{bmatrix}$$

Pitch h is infinite, $\quad \|\boldsymbol{S}_\omega\| = 0$

$\hat{\mathbf{s}} = \boldsymbol{v}/\|\boldsymbol{v}\|$, $\quad \|\boldsymbol{S}_v\| = 1$

$\dot{\theta} = \|\boldsymbol{v}\|$ is interpreted as linear velocity along $\hat{\mathbf{s}}$

Screw Interpretation of a Twist

❖ Since a screw axis $\boldsymbol{S}$ is just a normalized twist, the 4×4 matrix representation $[\boldsymbol{S}]$ of $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ is

$$[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \mathbf{0} & 0 \end{bmatrix} \in se(3) \qquad\qquad [\boldsymbol{\mathcal{V}}] = [\boldsymbol{S}]\dot{\theta} \in se(3)$$

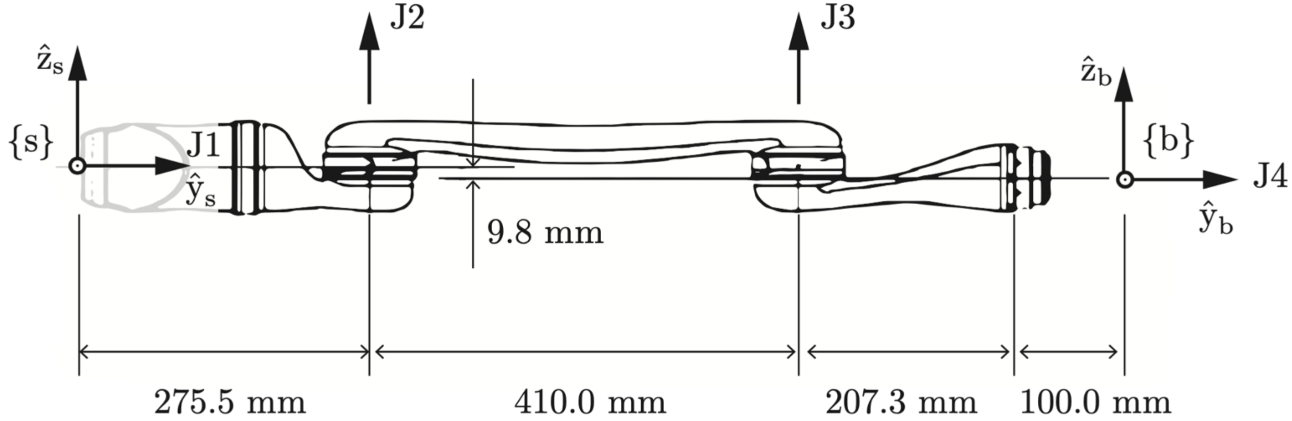
❖ Relation between a screw axis represented as $\boldsymbol{S}_a$ in a frame $\{a\}$ and $\boldsymbol{S}_b$ in a frame $\{b\}$:

$$\boldsymbol{S}_a = \left[\mathrm{Ad}_{\boldsymbol{T}_{ab}}\right]\boldsymbol{S}_b \qquad\qquad \boldsymbol{S}_b = \left[\mathrm{Ad}_{\boldsymbol{T}_{ba}}\right]\boldsymbol{S}_a$$

(changing the reference frame of a screw axis)

Example

Kinova lightweight 4-dof arm:



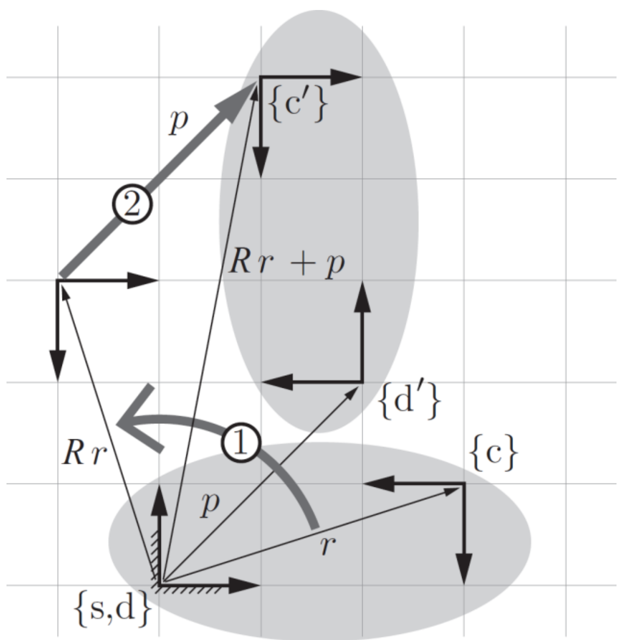
What are the screw axis $\boldsymbol{S}_b$ and $\boldsymbol{S}_s$ for J4 and J2?

Exponential Coordinate Representation of Rigid-Body Motion

Screw Motion

Instead of viewing a displacement as a rotation followed by a translation, both rotation and translation can be performed simultaneously.

Planar example of a screw motion:
The displacement can be viewed as a rotation of $\beta = 90°$ about a fixed point s.



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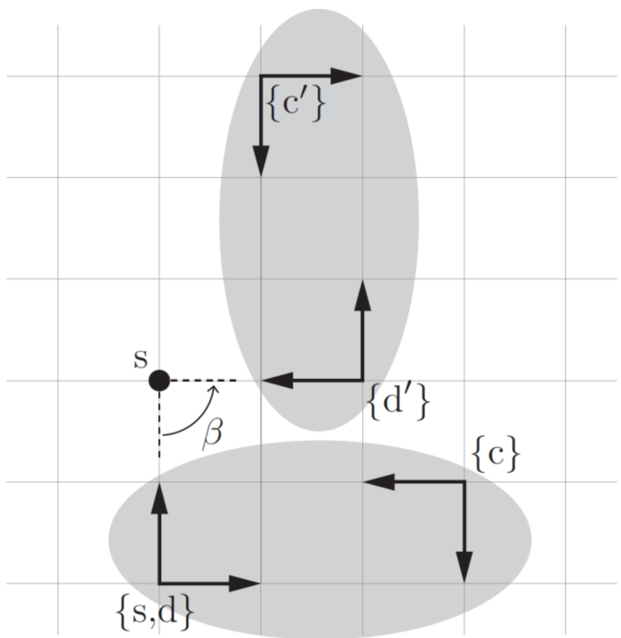


Exponential Coordinates of Rigid-Body Motions

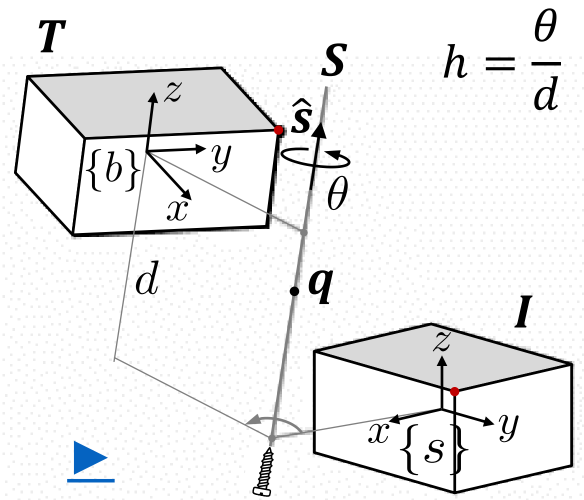
Chasles–Mozzi theorem states that every rigid-body displacement can be expressed as a finite rotation and translation about a fixed screw axis in space.

This theorem motivates a six-parameter representation of a configuration (or a homogeneous transformation $\boldsymbol{T} \in SE(3)$) called the **exponential coordinates** as $\boldsymbol{S}\theta \in \mathbb{R}^6$, where $\boldsymbol{S}$ is the screw axis and θ is the distance that must be traveled along the screw axis to take a frame from the origin $\boldsymbol{I}$ to $\boldsymbol{T}$.

Note: $\boldsymbol{T}$ is equivalent to the displacement obtained by rotating a frame from $\boldsymbol{I}$ about $\boldsymbol{S}$

- by an angle θ, or
- at a speed $\dot{\theta} = 1$ rad/s for θs, or
- at a speed $\dot{\theta} = \theta$ for unit time , or
- by twist $\boldsymbol{\mathcal{V}}$ for unit time.

Constant Screw Motion: A rotation θ + a translation d about/along a fixed screw axis $\boldsymbol{S}$.



$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \boldsymbol{q} + h\hat{\mathbf{s}} \end{bmatrix}$$ (for rotation with/without translation along $\hat{\mathbf{s}}$)

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{s}} \end{bmatrix}$$ (for pure translation along $\hat{\mathbf{s}}$)

Exponential Coordinates of Rigid-Body Motions

As with rotations, we can define a matrix exponential (exp) and matrix logarithm (log). For any transformation matrix $\boldsymbol{T} \in SE(3)$, we can always find a screw axis $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v) \in \mathbb{R}^6$ ($\|\boldsymbol{S}_\omega\| = 1$ or $\boldsymbol{S}_\omega = \boldsymbol{0}$, $\|\boldsymbol{S}_v\| = 1$) and scalar $\theta \in \mathbb{R}$ such that $\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$.

$$\begin{array}{llcll} \text{exp:} & [\boldsymbol{S}]\theta \in se(3) & \rightarrow & \boldsymbol{T} \in SE(3) & : \quad e^{[\boldsymbol{S}]\theta} = \boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) \\ \text{log:} & \boldsymbol{T} \in SE(3) & \rightarrow & [\boldsymbol{S}]\theta \in se(3) & : \quad \log(\boldsymbol{T}) = [\boldsymbol{S}]\theta \end{array}$$

$\boldsymbol{S}\theta \in \mathbb{R}^6$: Exponential coordinates of $\boldsymbol{T} \in SE(3)$

$[\boldsymbol{S}]\theta = [\boldsymbol{S}\theta] \in se(3)$: Matrix logarithm of $\boldsymbol{T}$ (inverse of the matrix exponential)

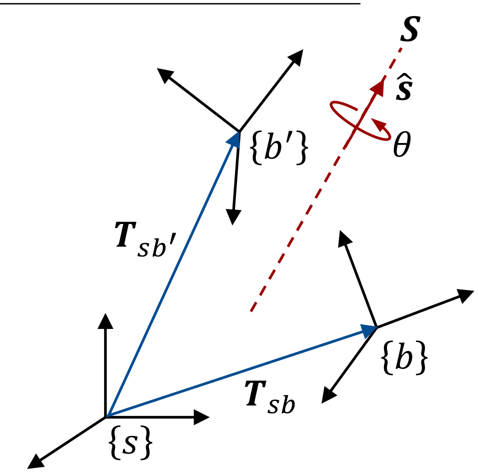
For a given $\boldsymbol{S}$: $(\boldsymbol{S}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b)$

$\boldsymbol{S}$ is expressed in $\{b\}$

Body-frame displacement: $\boldsymbol{T}_{sb'} = \boldsymbol{T}_{sb} e^{[\boldsymbol{S}_b]\theta}$

Fixed-frame displacement: $\boldsymbol{T}_{sb'} = e^{[\boldsymbol{S}_s]\theta} \boldsymbol{T}_{sb}$

$\boldsymbol{S}$ is expressed in $\{s\}$



Matrix Exponential

$$\text{exp:} \quad [\boldsymbol{S}]\theta \in se(3) \ \rightarrow \ \boldsymbol{T} \in SE(3) \quad : \quad e^{[\boldsymbol{S}]\theta} = \boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$$

❖ Finding $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$ by having $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ and θ:

(a) If $\|\boldsymbol{S}_\omega\| = 1$:

$$e^{[\boldsymbol{S}]\theta} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

$$\boldsymbol{G}(\theta) = \boldsymbol{I}\theta + (1 - \cos\theta)[\boldsymbol{S}_\omega] + (\theta - \sin\theta)[\boldsymbol{S}_\omega]^2 \in \mathbb{R}^{3\times 3}$$

(b) If $\boldsymbol{S}_\omega = \boldsymbol{0}$ ($\|\boldsymbol{S}_v\| = 1$):

$$e^{[\boldsymbol{S}]\theta} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{S}_v\theta \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Matrix Logarithm

$$\log: \quad \boldsymbol{T} \in SE(3) \quad \rightarrow [\boldsymbol{S}]\theta \in se(3) \quad : \quad \log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$$

❖ Finding $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ and $\theta \in [0, \pi]$ by having $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$:

(a) If $\boldsymbol{R} = \boldsymbol{I}$, then set $\boldsymbol{S}_\omega = \boldsymbol{0}$, $\boldsymbol{S}_v = \boldsymbol{p}/\|\boldsymbol{p}\|$, and $\theta = \|\boldsymbol{p}\|$.

(b) Otherwise, use the matrix logarithm $\log(\boldsymbol{R}) = [\boldsymbol{S}_\omega]\theta$ to determine $\boldsymbol{S}_\omega$ ($\hat{\boldsymbol{\omega}}$ in the $SO(3)$ algorithm) and $\theta \in [0, \pi]$. Then, $\boldsymbol{S}_v$ is calculated as

$$\boldsymbol{S}_v = \boldsymbol{G}^{-1}(\theta)\boldsymbol{p}$$

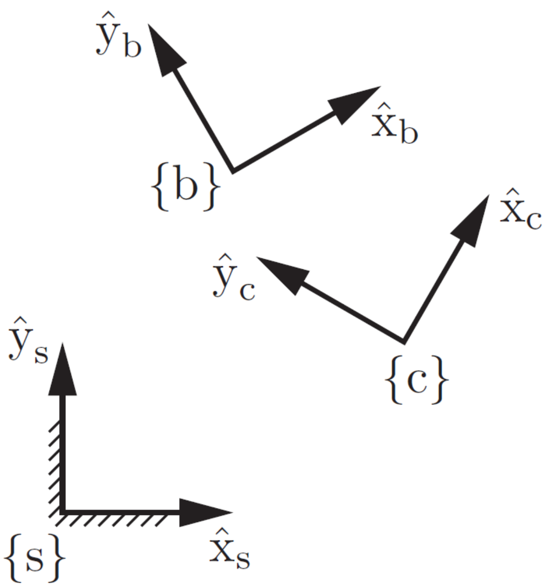
$$\boldsymbol{G}^{-1}(\theta) = \frac{1}{\theta}\boldsymbol{I} - \frac{1}{2}[\boldsymbol{S}_\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\boldsymbol{S}_\omega]^2 \in \mathbb{R}^{3\times3}$$

Example

The initial frame $\{b\}$ and final frame $\{c\}$ are given. Find the screw motion that displaces the frame at T_{sb} to T_{sc}.

$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Wrenches

wrench

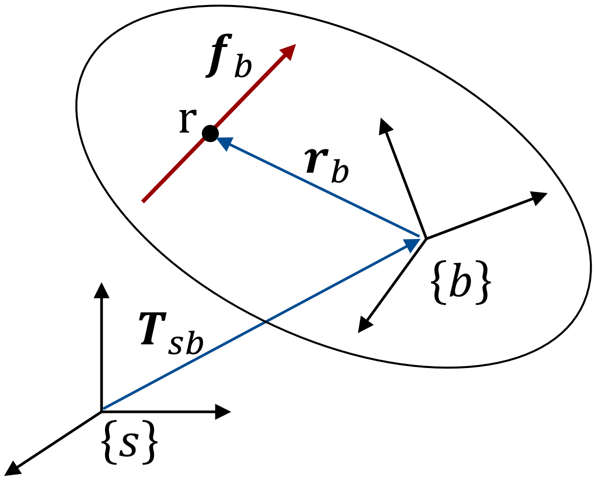
Consider a linear force $\mathbf{f}$ acting on a rigid body at a point r. Both $\boldsymbol{f}_b \in \mathbb{R}^3$ and $\boldsymbol{r}_b \in \mathbb{R}^3$ are represented in $\{b\}$. This force creates a torque or moment $\boldsymbol{m}_b \in \mathbb{R}^3$ in $\{b\}$ as

$$\boldsymbol{m}_b = \boldsymbol{r}_b \times \boldsymbol{f}_b$$

We can package the moment and force together in a single six-dimensional vector called **wrench** (or **spatial force**) in $\{b\}$ as

$$\boldsymbol{\mathcal{F}}_b = \begin{bmatrix} \boldsymbol{m}_b \\ \boldsymbol{f}_b \end{bmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{\mathcal{F}}_s = ?$$



wrench

The **power** is a coordinate-independent quantity, i.e., the power generated (or dissipated) by an $(\mathcal{F},\mathcal{V})$ pair (wrench and twist) must be the same regardless of the frame in which it is represented:

$$(\mathcal{V} \cdot \mathcal{F} = \text{power}) \qquad \mathcal{V}_s^{\mathrm{T}}\mathcal{F}_s = \mathcal{V}_b^{\mathrm{T}}\mathcal{F}_b = \text{power} \qquad (\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s)$$

$$\mathcal{V}_s^{\mathrm{T}}\mathcal{F}_s = \left([\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s\right)^{\mathrm{T}}\mathcal{F}_b$$

$$= \mathcal{V}_s^{\mathrm{T}}[\mathrm{Ad}_{T_{bs}}]^{\mathrm{T}}\mathcal{F}_b$$

Since this must hold for all $\mathcal{V}_s$

$$\mathcal{F}_s = [\mathrm{Ad}_{T_{bs}}]^{\mathrm{T}}\mathcal{F}_b$$

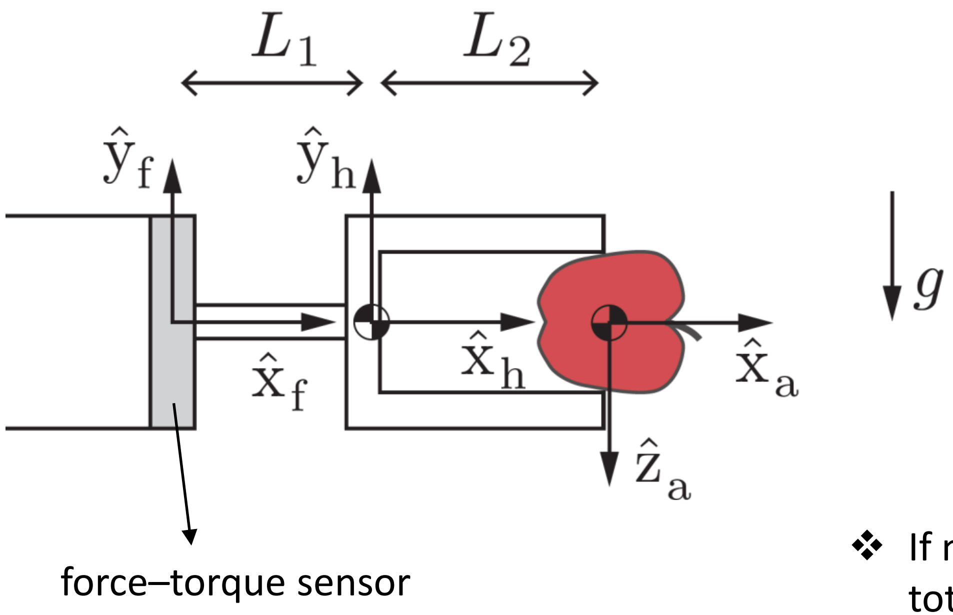
spatial wrench body wrench

$$\mathcal{F}_a = [\mathrm{Ad}_{T_{ba}}]^T\mathcal{F}_b \qquad \mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^T\mathcal{F}_a$$

(changing the reference frame of a twist)

Example

The robot hand shown is holding an apple with a mass of 0.1 kg in a gravitational field g=10 m/s^2. The mass of the hand is 0.5 kg, L_1=10 cm, and L_2=15 cm. What is the force and torque measured by the six-axis force–torque sensor between the hand and the robot arm?



- If more than one wrench acts on a rigid body, the total wrench on the body is simply the vector sum of the individual wrenches, provided that the wrenches are expressed in the same frame.

Review

Rotations	Rigid-Body Motions
$R \in SO(3)$: 3×3 matrices $R^{\mathrm{T}}R = I, \det R = 1$	$T \in SE(3)$: 4×4 matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$ where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^{\mathrm{T}}$	$T^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$
change of coordinate frame: $R_{ab}R_{bc} = R_{ac}, \quad R_{ab}p_b = p_a$	change of coordinate frame: $T_{ab}T_{bc} = T_{ac}, \quad T_{ab}p_b = p_a$

Review

rotating a frame {b}:	displacing a frame {b}:
$R = \mathrm{Rot}(\hat{\omega}, \theta)$	$T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$
$R_{sb'} = RR_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb}R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	$T_{sb'} = TT_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb}T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\lVert\hat{\omega}\rVert = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\lVert\omega\rVert = 1$ or (ii) $\omega = 0$ and $\lVert v\rVert = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity is $\omega = \hat{\omega}\dot{\theta}$	twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$

Review

for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$ identities, $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}, [\omega]x = -[x]\omega,$ $[\omega][x] = ([x][\omega])^{\mathrm{T}}, R[\omega]R^{\mathrm{T}} = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)
$\dot{R}R^{-1} = [\omega_s], \ R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s], \ T^{-1}\dot{T} = [\mathcal{V}_b]$
	$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1 T_2}]$
change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \ \omega_a = R_{ab}\omega_b$	change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \ \mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$

Review

exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$
$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$	$\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$	$\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$
moment change of coord frame: $m_a = R_{ab} m_b$	wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{F}_b$