Ch4: Rigid-Body Motions – Rotation

Reference Frames

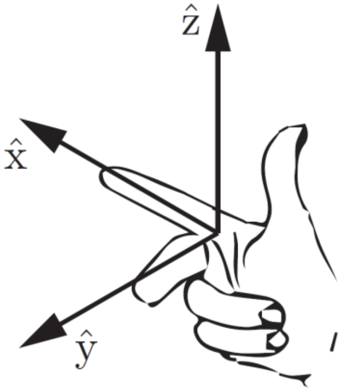
Reference Frames

- **Fixed Space Frame** $\{s\}$: A stationary, inertial frame and there is only one.
- **Body-attached Frame**: A frame fixed to a body and moves with it.
- **Body Frame** $\{b\}$: A stationary, inertial frame that is instantaneously coincident with the body-attached frame.

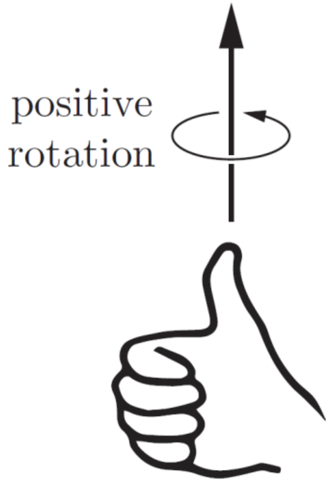
In this course, all frames are instantaneously stationary.

Reference Frames

All reference frames are **right-handed**.



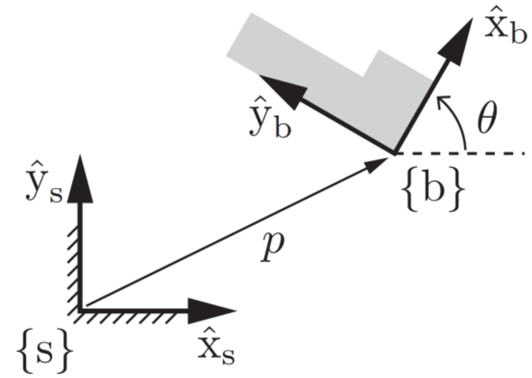
A **positive rotation** about an axis is defined as the direction in which the fingers of the right hand curl when the thumb is pointed along the axis.



Rotation Matrices

Rotation in 2D Space

In 2D, the simplest way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by specifying the angle θ.



Another way is to specify the directions of the unit axes $\hat{x}_b$ and $\hat{y}_b$ of $\{b\}$ relative to $\{s\}$.

$$\Rightarrow \quad \boldsymbol{R} = [\hat{x}_b \quad \hat{y}_b] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \in [0, 2\pi)$$

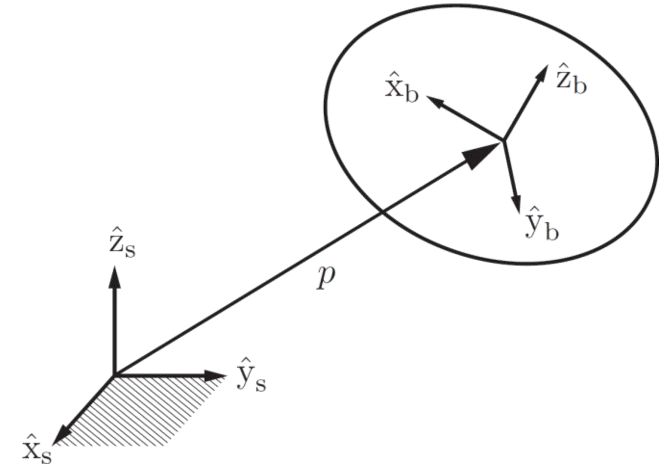
Rotation Matrix

Rotation in 3D Space

In 3D, a way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by specifying the directions of the unit axes $\hat{x}_b$, $\hat{y}_b$, and $\hat{z}_b$ of $\{b\}$ relative to $\{s\}$.

$$\boldsymbol{R} = [\hat{x}_b \quad \hat{y}_b \quad \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \boldsymbol{R} \in \mathbb{R}^{3\times 3}$$

Rotation Matrix



This is as Implicit representation.

$$\det(\boldsymbol{R}) = 1.$$

Constraints on Rotation Matrix

1- The unit norm condition: $\hat{x}_b$, $\hat{y}_b$, and $\hat{z}_b$ are all unit vectors.

2- The orthogonality condition: $\hat{x}_b . \hat{y}_b = \hat{x}_b . \hat{z}_b = \hat{y}_b . \hat{z}_b = 0$

Compact form: $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I}_3$

For right-handed frames: $\det(\boldsymbol{R}) = 1$

Definition of a Group

A **group** is a set of elements $G = \{a, b, c, \dots\}$ and a binary operation $\bullet$ on any two elements satisfying

- **Closure**: $a \bullet b \in G \quad \forall a, b \in G$
- **Associativity**: $(a \bullet b) \bullet c = a \bullet (b \bullet c) \qquad \forall a, b, c \in G$
- **Identity Element Existence**: $\exists I \in G$ such that $a \bullet I = I \bullet a = a \qquad \forall a \in G$
- **Inverse Element Existence**: $\forall a \in G, \exists a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = I$

Special Orthogonal Group $SO(n)$

The **special orthogonal group** $SO(n)$, $n = 2,3$, also known as the (Lie) group of rotation matrices, is the set of all $n \times n$ real matrices $\boldsymbol{R}$ that satisfy **(i)** $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I}_3$ and **(ii)** $\det(\boldsymbol{R}) = 1$.

orthogonal

special

$SO(2)$ is a subgroup of $SO(3)$: $SO(2) \subset SO(3)$

$\boldsymbol{R} \in SO(3)$

$$SO(3) = \left\{\boldsymbol{R} \in \mathbb{R}^{3\times 3} \middle| \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}_3, \det(\boldsymbol{R}) = 1\right\}$$

Properties of Rotation Matrices

$SO(3)$ (or $SO(2)$) is a **matrix (Lie) group** (and the group operation • is matrix multiplication).

- **Closure**: $\boldsymbol{R}_1\boldsymbol{R}_2 \in \mathrm{SO}(3)$
- **Associative**: $(\boldsymbol{R}_1\boldsymbol{R}_2)\boldsymbol{R}_3 = \boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{R}_3)$ (but generally not commutative, $\boldsymbol{R}_1\boldsymbol{R}_2 \neq \boldsymbol{R}_2\boldsymbol{R}_1$)
- **Identity**: $\exists\ \boldsymbol{I} \in SO(3)$ such that $\boldsymbol{RI} = \boldsymbol{IR} = \boldsymbol{R}$
- **Inverse**: $\exists\ \boldsymbol{R}^{-1} \in SO(3)$ such that $\boldsymbol{RR}^{-\mathbf{1}} = \boldsymbol{R}^{-\mathbf{1}}\boldsymbol{R} = \boldsymbol{I}$ $\quad (\Rightarrow \boldsymbol{R}^{-1} = \boldsymbol{R}^T)$

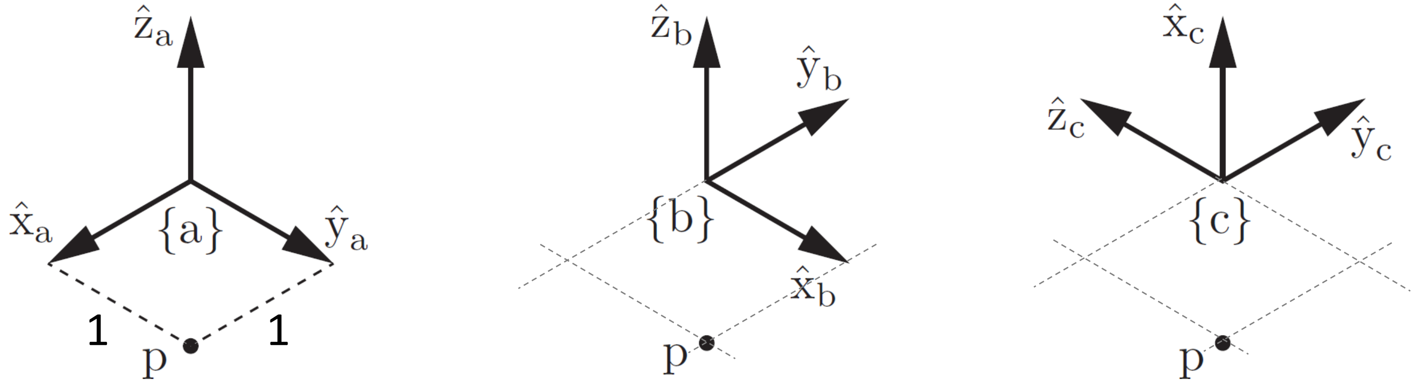
* For any vector $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{R} \in \mathrm{SO}(3)$, the vector $\boldsymbol{y} = \boldsymbol{Rx}$ has the same length as $\boldsymbol{x}$ ($\|\boldsymbol{x}\| = \|\boldsymbol{Rx}\|$).

Uses of Rotation Matrices (1)

(1) Representing orientation of a frame relative to another frame.

Notation: $\boldsymbol{R}_{bc}$ is the orientation of $\{c\}$ relative to $\{b\}$.

Example:
Find $\boldsymbol{R}_{ab}$ and $\boldsymbol{R}_{ac}$.
(All frames have the same origin)



Uses of Rotation Matrices (2)

(2) Changing the reference frame of a vector or frame.

Subscript Cancellation Rule:

$$\boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{p}_{\cancel{b}} = \boldsymbol{p}_a$$

$$\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{R}_{\cancel{b}c} = \boldsymbol{R}_{ac}$$

$\boldsymbol{R}_{ab}$ can be viewed as a **mathematical operator** that changes the reference frame from $\{b\}$ to $\{a\}$.

Note: $\boldsymbol{R}_{bc}\boldsymbol{R}_{cb} = \boldsymbol{I}$ or $\boldsymbol{R}_{bc} = \boldsymbol{R}_{cb}^T = \boldsymbol{R}_{cb}^{-1}$

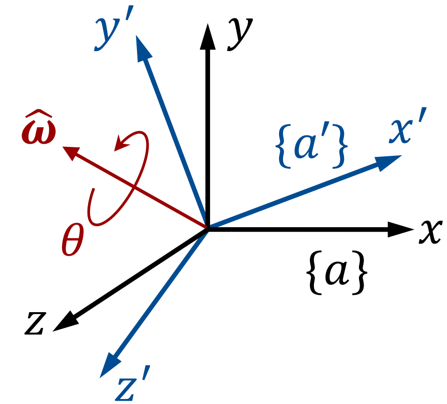
Example

Given $\boldsymbol{R}_1 = \boldsymbol{R}_{ab}$, $\boldsymbol{R}_2 = \boldsymbol{R}_{bc}$, and $\boldsymbol{R}_3 = \boldsymbol{R}_{ad}$, write $\boldsymbol{R}_{dc}$ in terms of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{R}_3$.

Given $\boldsymbol{p}_b$, what is $\boldsymbol{p}_d$ in terms of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{R}_3$?

Uses of Rotation Matrices (3)

(3) Rotating a <u>vector</u> or <u>frame</u> (about a unit axis $\hat{\boldsymbol{\omega}}$ by an amount θ).



$$\boldsymbol{R} = \boldsymbol{R}_{aa'} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

$\boldsymbol{R}$ can be viewed as a **<u>mathematical operator</u>** that rotates $\{a\}$ about a unit axis $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$ (expressed in $\{a\}$) by an amount θ to obtain $\{a'\}$.

$$\mathrm{Rot}(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \mathrm{Rot}(\hat{y}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \mathrm{Rot}(\hat{z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta,\ c_\theta = \cos\theta$$

- $\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \mathrm{Rot}(-\hat{\boldsymbol{\omega}}, -\theta)$

Uses of Rotation Matrices (3) (cont.)

- Rotation of vector $\boldsymbol{v}$ about a unit axis $\hat{\boldsymbol{\omega}}$ (expressed in the same frame) by an amount θ is vector $\boldsymbol{v}'$ expressed in the same frame:

$$\boldsymbol{v}' = \boldsymbol{R}\boldsymbol{v} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v}$$

- **Fixed-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{s\}$ by an amount θ is frame $\{b'\}$:

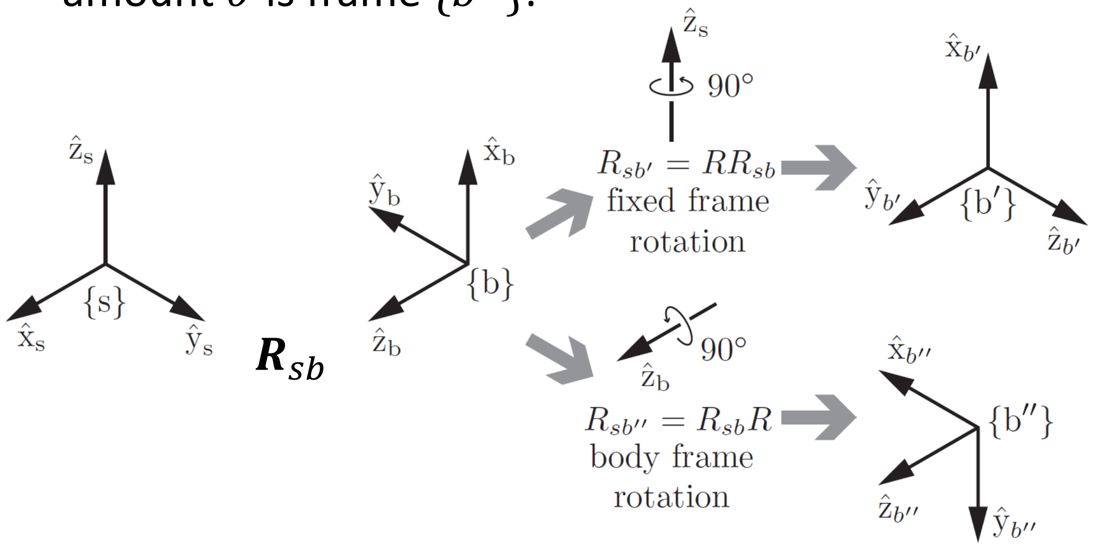
$$\boldsymbol{R}_{sb'} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{R}_{sb}$$

Interpretation

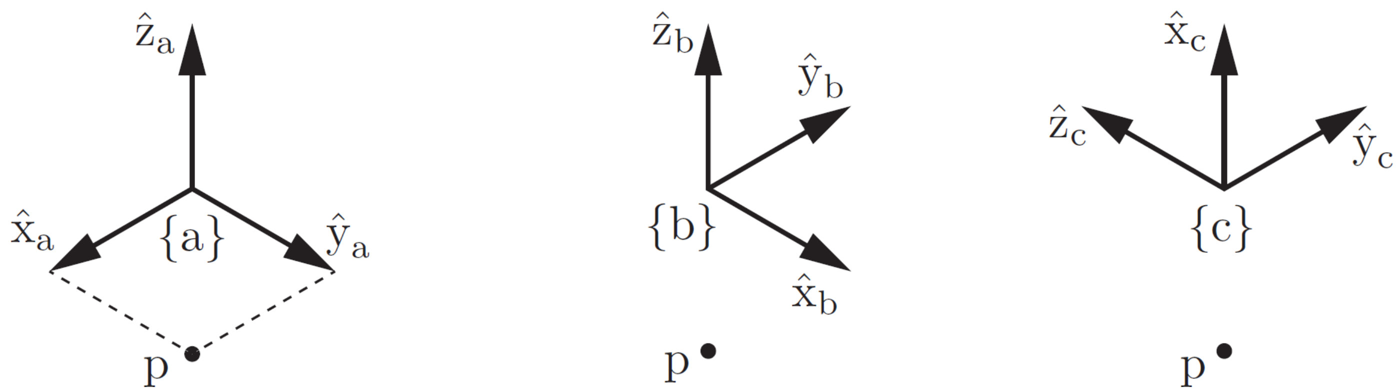
- **Body-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{b\}$ by an amount θ is frame $\{b''\}$:

$$\boldsymbol{R}_{sb''} = \boldsymbol{R}_{sb}\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

Interpretation



Examples



$\boldsymbol{R} = \boldsymbol{R}_{ba} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$: $\quad \theta = \dfrac{\pi}{2}$, $\quad \hat{\boldsymbol{\omega}} =?$

$\boldsymbol{R}_{bc'} = \boldsymbol{R}\boldsymbol{R}_{bc} =?$ $\qquad$ $\boldsymbol{R}_{bc''} = \boldsymbol{R}_{bc}\boldsymbol{R} =?$

Angular Velocities

Set of Skew-Symmetric Matrices $so(3)$

The set of all 3×3 real skew-symmetric matrices is called $so(3)$ (which is the Lie algebra of the Lie group $SO(3)$).

$$so(3) = \{\mathbf{S} \in \mathbb{R}^{3\times3} | \mathbf{S}^T = -\mathbf{S}\} \qquad so(3) \subset \mathbb{R}^{3\times3}$$

$$\boldsymbol{x} \in \mathbb{R}^3 \qquad [\boldsymbol{x}] \in so(3)$$

- Given any $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{R} \in SO(3)$, $\boldsymbol{R}[\boldsymbol{x}]\boldsymbol{R}^T = [\boldsymbol{Rx}]$.

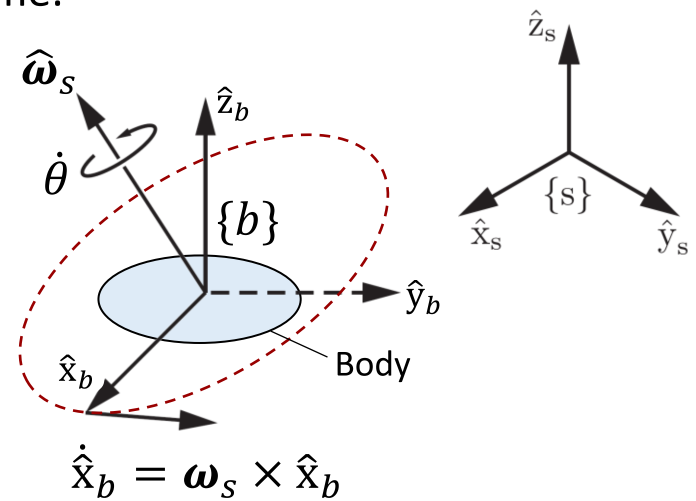
- Given $[\boldsymbol{x}] \in so(3)$, $[\boldsymbol{x}]^2 = \boldsymbol{x}\boldsymbol{x}^T - \|\boldsymbol{x}\|^2\boldsymbol{I}$ and $[\boldsymbol{x}]^3 = -\|\boldsymbol{x}\|^2[\boldsymbol{x}]$

and higher powers of $[\boldsymbol{x}]$ can be calculated recursively.

Angular Velocities

Let's find the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of a frame fixed to a rotating body. Body Frame $\{b\}$ is instantaneously coincident with this body-attached frame.

- If $\boldsymbol{\omega}$ is expressed in $\{s\}$: $\quad \boldsymbol{\omega} = \boldsymbol{\omega}_s = \dot{\theta}\hat{\boldsymbol{\omega}}_s$



$\boldsymbol{R}(t) = [\hat{\mathrm{x}}_b \quad \hat{\mathrm{y}}_b \quad \hat{\mathrm{z}}_b]$: $\boldsymbol{R}_{sb}$ at time t

$\dot{\boldsymbol{R}}(t) = [\dot{\hat{\mathrm{x}}}_b \quad \dot{\hat{\mathrm{y}}}_b \quad \dot{\hat{\mathrm{z}}}_b]$: Time rate of change of $\boldsymbol{R}_{sb}$ at time t

$$\dot{\hat{\mathrm{x}}}_b = \boldsymbol{\omega}_s \times \hat{\mathrm{x}}_b$$
$$\dot{\hat{\mathrm{y}}}_b = \boldsymbol{\omega}_s \times \hat{\mathrm{y}}_b$$
$$\dot{\hat{\mathrm{z}}}_b = \boldsymbol{\omega}_s \times \hat{\mathrm{z}}_b$$

$\boldsymbol{\omega}_s$: Fixed-frame angular velocity

$$\dot{\boldsymbol{R}} = [[\boldsymbol{\omega}_s]\hat{\mathrm{x}}_b \quad [\boldsymbol{\omega}_s]\hat{\mathrm{y}}_b \quad [\boldsymbol{\omega}_s]\hat{\mathrm{z}}_b] = [\boldsymbol{\omega}_s]\boldsymbol{R} \quad \Rightarrow \quad \boxed{[\boldsymbol{\omega}_s] = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1}}$$

Note: $\boldsymbol{\omega}_s$ does not depend on the choice of body frame $\{b\}$.

Angular Velocities

- If $\boldsymbol{\omega}$ is expressed in $\{b\}$: $\boldsymbol{\omega} = \boldsymbol{\omega}_b = \dot{\theta}\hat{\boldsymbol{\omega}}_b$

$\boldsymbol{\omega}_b$: Body-frame angular velocity

$$\boldsymbol{\omega}_s = \boldsymbol{R}\boldsymbol{\omega}_b$$

$$\boldsymbol{\omega}_b = \boldsymbol{R}^{-1}\boldsymbol{\omega}_s = \boldsymbol{R}^T\boldsymbol{\omega}_s$$

$$\begin{aligned}[\boldsymbol{\omega}_b] &= [\boldsymbol{R}^{\mathrm{T}}\boldsymbol{\omega}_s] \\ &= \boldsymbol{R}^{\mathrm{T}}[\boldsymbol{\omega}_s]\boldsymbol{R} \\ &= \boldsymbol{R}^{\mathrm{T}}(\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}})\boldsymbol{R} \\ &= \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}} = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\end{aligned}$$

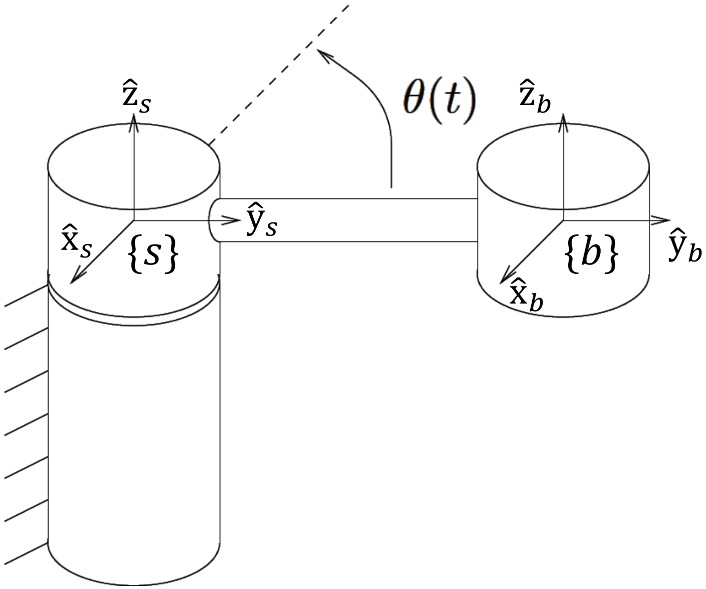
Recall: $\boldsymbol{R}[\boldsymbol{x}]\boldsymbol{R}^{\mathrm{T}} = [\boldsymbol{R}\boldsymbol{x}]$.

$$\Rightarrow \quad \boxed{[\boldsymbol{\omega}_b] = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}}$$

Note: $\boldsymbol{\omega}_b$ does not depend on the choice of fixed frame $\{s\}$.

Example

Find $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_b$ for rotational motion of a one degree of freedom manipulator.



Exponential Coordinate Representation of Rotation

Matrix Exponential

Scalar Linear ODE:

$$\dot{x}(t) = ax(t) \xrightarrow[x(0) = x_0]{x(t) \in \mathbb{R},\, a \in \mathbb{R} \text{ is constant}} x(t) = e^{at}x_0$$

Vector Linear ODE:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \xrightarrow[\boldsymbol{x}(0) = \boldsymbol{x}_0]{\boldsymbol{x}(t) \in \mathbb{R}^n,\, \boldsymbol{A} \in \mathbb{R}^{n\times n} \text{ is constant}} \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$$

Properties of Matrix Exponential $e^{\boldsymbol{A}t}$

$\forall \boldsymbol{A} \in \mathbb{R}^{n\times n}, t \in \mathbb{R}$: $$d\left(e^{\boldsymbol{A}t}\right)/dt = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$$

If $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ for some $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ and invertible $\boldsymbol{P} \in \mathbb{R}^{n\times n}$: $$e^{\boldsymbol{A}t} = \boldsymbol{P}e^{\boldsymbol{D}t}\boldsymbol{P}^{-1}$$

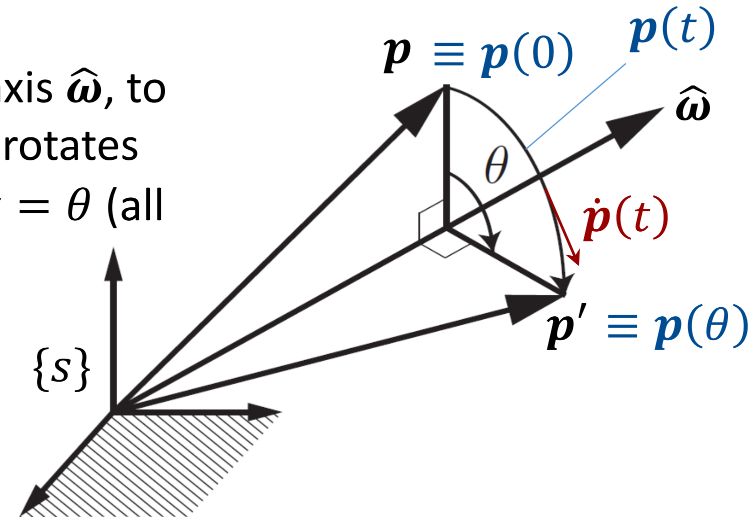
If $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ is diagonal, i.e., $\boldsymbol{D} = \mathrm{diag}\{d_1, d_2, \dots, d_n\}$: $$e^{\boldsymbol{D}t} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}$$

If $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$, then $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}}$.

$$\left(e^{\boldsymbol{A}}\right)^{-1} = e^{-\boldsymbol{A}}$$

Exponential Coordinates of Rotations

The vector $\boldsymbol{p}$ is rotated by an angle θ about the unit axis $\hat{\boldsymbol{\omega}}$, to $\boldsymbol{p}'$. This rotation can be achieved by imagining that $\boldsymbol{p}$ rotates at a constant rate of $\dot{\theta} = 1$ rad/s from time $t = 0$ to $t = \theta$ (all vectors are expressed in $\{s\}$).



$$\dot{\boldsymbol{p}} = \dot{\theta}\hat{\boldsymbol{\omega}} \times \boldsymbol{p}(t) = [\hat{\boldsymbol{\omega}}]\boldsymbol{p}(t) \qquad (\|\hat{\boldsymbol{\omega}}\| = 1, \dot{\theta} = 1 \text{ rad/s})$$

$$\boldsymbol{p}(t) = e^{[\hat{\boldsymbol{\omega}}]t}\boldsymbol{p}(0)$$

at $t = \theta$

$$\boldsymbol{p}(\theta) = e^{[\hat{\boldsymbol{\omega}}]\theta}\boldsymbol{p}(0) \xrightarrow{\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}} \boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) \in SO(3) \qquad [\hat{\boldsymbol{\omega}}]\theta = [\hat{\boldsymbol{\omega}}\theta] \in so(3)$$

$\Rightarrow$ Any rotation matrix $\boldsymbol{R} \in SO(3)$ can be obtained by rotating from the identity matrix $\boldsymbol{I}$ about a unit rotation axis $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{\omega}}\| = 1$) by an angle of rotation $\theta \in \mathbb{R}$ about that axis. This motivates a three-parameter representation of a rotation $\boldsymbol{R}$ called the **exponential coordinates** as $\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ (equivalently, $\hat{\boldsymbol{\omega}}$ and θ can be written individually as the **axis-angle representation** of a rotation).

Exponential Coordinates of Rotations

For any rotation matrix $\boldsymbol{R} \in SO(3)$, we can always find a unit rotation axis $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{\omega}}\| = 1$) and scalar $\theta \in \mathbb{R}$ such that $\boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta}$.

$$
\begin{array}{llcllcl}
\text{exp:} & [\hat{\boldsymbol{\omega}}]\theta \in so(3) & \rightarrow & \boldsymbol{R} \in SO(3) & : & e^{[\hat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R} \\
\text{log:} & \boldsymbol{R} \in SO(3) & \rightarrow & [\hat{\boldsymbol{\omega}}]\theta \in so(3) & : & \log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta
\end{array}
$$

$\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$: Exponential coordinates of $\boldsymbol{R} \in SO(3)$

$[\hat{\boldsymbol{\omega}}]\theta = [\hat{\boldsymbol{\omega}}\theta] \in so(3)$: Matrix logarithm of $\boldsymbol{R}$ (inverse of the matrix exponential)

Note: $\boldsymbol{R}$ and $\hat{\boldsymbol{\omega}}$ have the same base.

Matrix Exponential

$$\boxed{\exp: \quad [\hat{\boldsymbol{\omega}}]\theta \in so(3) \ \rightarrow \ \boldsymbol{R} \in SO(3) \qquad : e^{[\hat{\boldsymbol{\omega}}]\theta} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R}}$$

❖ Finding $\boldsymbol{R}$ by having $\hat{\boldsymbol{\omega}}$ and θ:

$$e^{[\hat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + [\hat{\boldsymbol{\omega}}]\theta + [\hat{\boldsymbol{\omega}}]^2 \frac{\theta^2}{2!} + [\hat{\boldsymbol{\omega}}]^3 \frac{\theta^3}{3!} + \cdots$$

$$= \boldsymbol{I} + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta} [\hat{\boldsymbol{\omega}}] + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}_{1-\cos\theta} [\hat{\boldsymbol{\omega}}]^2$$

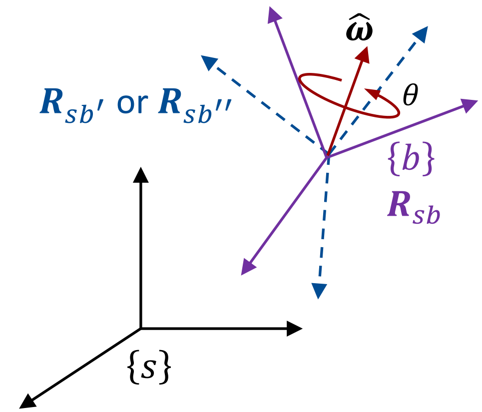
$$\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = e^{[\hat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + \sin\theta [\hat{\boldsymbol{\omega}}] + (1 - \cos\theta)[\hat{\boldsymbol{\omega}}]^2$$ (Rodrigues' formula for rotations)

$$\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$s_\theta = \sin\theta, c_\theta = \cos\theta, \quad \hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$

Matrix Exponential: Remark

For a given $\hat{\boldsymbol{\omega}}_s$ or $\hat{\boldsymbol{\omega}}_b$: $(\hat{\boldsymbol{\omega}}_s = \boldsymbol{R}_{sb}\hat{\boldsymbol{\omega}}_b)$



$\hat{\boldsymbol{\omega}}$ is expressed in $\{s\}$

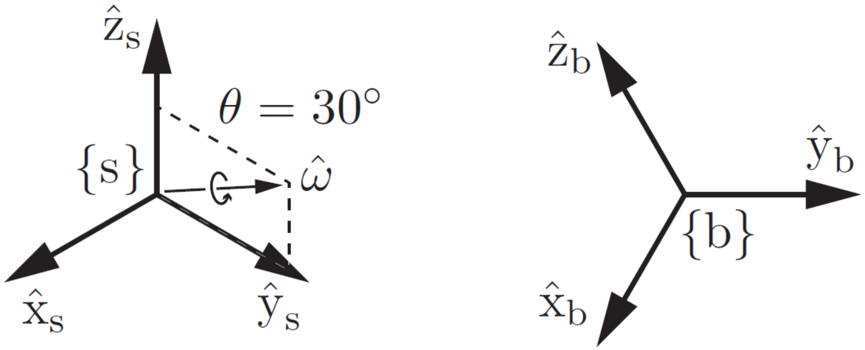
Fixed-frame Rotation: $\boldsymbol{R}_{sb'} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}_s, \theta)\boldsymbol{R}_{sb} = e^{[\hat{\boldsymbol{\omega}}_s]\theta}\boldsymbol{R}_{sb}$

Body-frame Rotation: $\boldsymbol{R}_{sb''} = \boldsymbol{R}_{sb}\mathrm{Rot}(\hat{\boldsymbol{\omega}}_b, \theta) = \boldsymbol{R}_{sb}e^{[\hat{\boldsymbol{\omega}}_b]\theta}$

$\hat{\boldsymbol{\omega}}$ is expressed in $\{b\}$

Example

The frame $\{b\}$ is obtained by a rotation from $\{s\}$ by $\theta_1 = 30°$ about $\hat{\boldsymbol{\omega}}_1 = (0,0.866,0.5)$. Find the rotation matrix representation of $\{b\}$.



Find the new rotation matrix if $\{b\}$ is then rotated by θ_2 about
(a) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{s\}$.
(b) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{b\}$.

Matrix Logarithm

$$\log: \quad \boldsymbol{R} \in SO(3) \quad \rightarrow \quad [\hat{\boldsymbol{\omega}}]\theta \in so(3) \quad : \quad \log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta$$

❖ Finding $\hat{\boldsymbol{\omega}}$ and $\theta \in [0, \pi]$ by having $\boldsymbol{R}$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$\operatorname{tr}\boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$

$$\frac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = [\hat{\boldsymbol{\omega}}]$$

$$\boldsymbol{R}\Big|_{\theta=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{R}\Big|_{\theta=\pi} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -1 + 2\hat{\omega}_1^2 & 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_1\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & -1 + 2\hat{\omega}_2^2 & 2\hat{\omega}_2\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_3 & 2\hat{\omega}_2\hat{\omega}_3 & -1 + 2\hat{\omega}_3^2 \end{bmatrix}$$

Matrix Logarithm: Algorithm

(a) If $\mathrm{tr}\boldsymbol{R} = 3$ (or $\boldsymbol{R} = \boldsymbol{I}$), then $\theta = 0$ and $\hat{\boldsymbol{\omega}}$ is undefined.

(b) If $\mathrm{tr}\boldsymbol{R} = -1$, then $\theta = \pi$ and $\hat{\boldsymbol{\omega}}$ is equal to any of the three vectors that is a feasible solution:

$$\hat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \quad \text{or} \quad \hat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix} \quad \text{or} \quad \hat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}$$

(Note that if $\hat{\boldsymbol{\omega}}$ is a solution, then so is $-\hat{\boldsymbol{\omega}}$)

(c) Otherwise,

$$\theta = \cos^{-1}\left(\frac{1}{2}(\mathrm{tr}\,\boldsymbol{R} - 1)\right) \in (0, \pi)$$

$$[\hat{\boldsymbol{\omega}}] = \frac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right)$$

Other Representations of Rotations

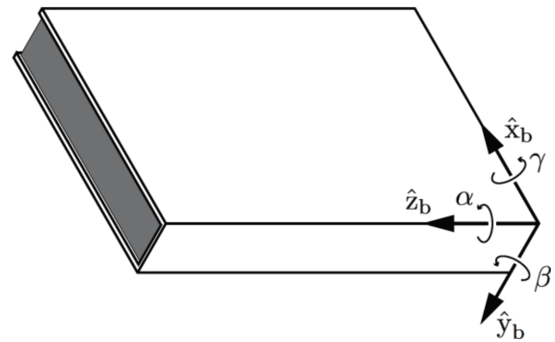
Euler Angles

Another minimal representation of orientation can be obtained by using a set of three angles (α, β, γ), i.e., by composing a suitable sequence of three elementary rotations about the (fixed frame $\{s\}$ or body/current frame $\{b\}$) coordinate axes.

Two Examples:

- ZYX Euler angles (with rotations about the body/current frame $\{b\}$).
- XYZ Euler angles (with rotations about the fixed frame $\{s\}$). This is also called **roll–pitch–yaw** angles.

Euler Angles ZYX
(about the body/current frame)



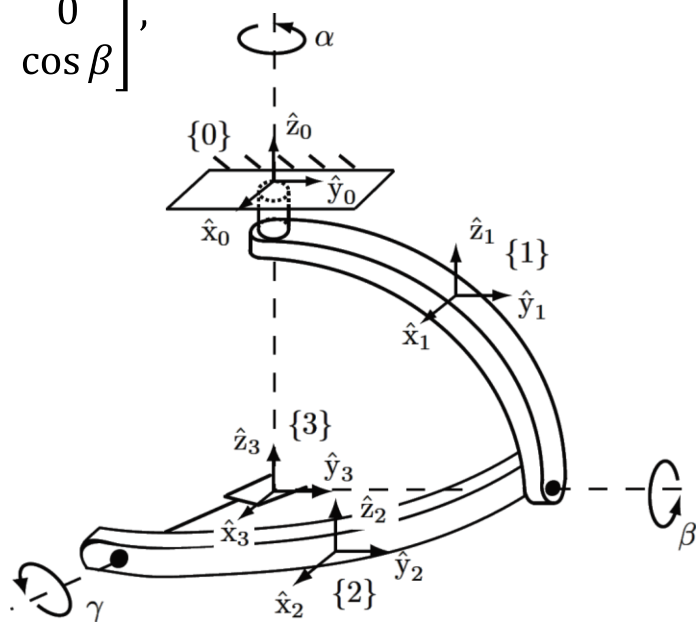
ZYX Euler angles (with rotations about the body/current frame $\{b\}$):

$$\boldsymbol{R}(\alpha,\beta,\gamma) = \boldsymbol{I}\,\mathrm{Rot}(\hat{\boldsymbol{z}},\alpha)\mathrm{Rot}(\hat{\boldsymbol{y}},\beta)\mathrm{Rot}(\hat{\boldsymbol{x}},\gamma)$$

$$\mathrm{Rot}(\hat{\boldsymbol{x}},\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}, \mathrm{Rot}(\hat{\boldsymbol{y}},\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\boldsymbol{z}},\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{R}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$



Euler Angles ZYX
(about the body/current frame)

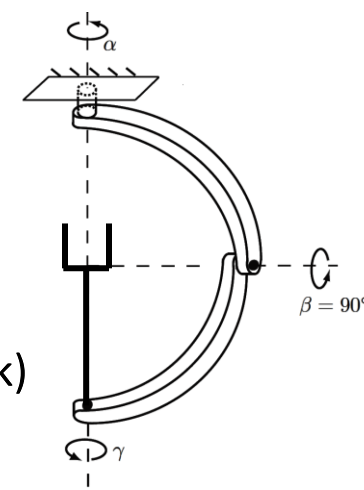
Finding (α, β, γ) for any given rotation matrix $\boldsymbol{R} \in SO(3)$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

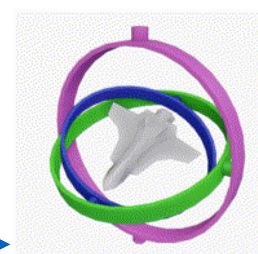
- If $r_{31} \neq \pm 1$ (i.e., when $\beta \in (-\pi/2, \pi/2)$):

$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$
$$\alpha = \operatorname{atan2}(r_{21}, r_{11})$$
$$\gamma = \operatorname{atan2}(r_{32}, r_{33})$$

- If $r_{31} = -1$, then $\beta = \pi/2$, and if $r_{31} = 1$, then $\beta = -\pi/2$. In these cases, it is possible to determine only the sum or difference of α and γ.



Singularity of the Euler angles:

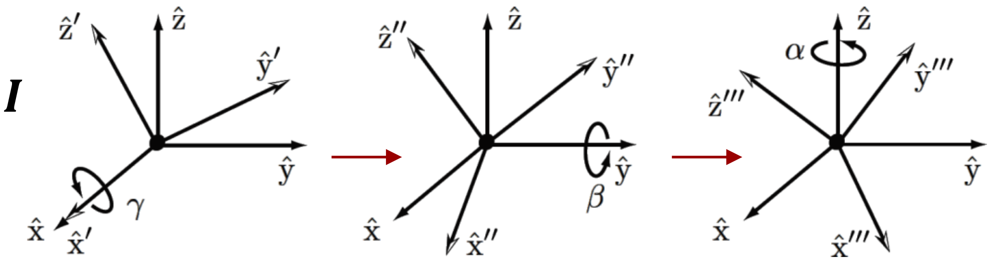


(Gimbal lock)

Roll–Pitch–Yaw Angles (XYZ)
(about the fixed frame)

XYZ Euler angles (with rotations about the fixed frame $\{s\}$):

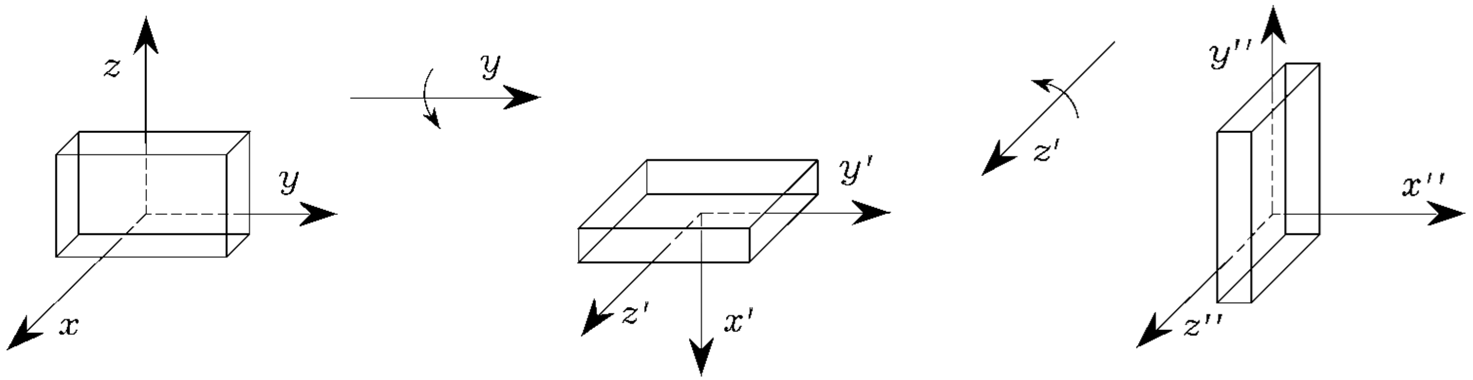
$$\boldsymbol{R}(\alpha,\beta,\gamma) = \mathrm{Rot}(\hat{\boldsymbol{z}},\alpha)\,\mathrm{Rot}(\hat{\boldsymbol{y}},\beta)\,\mathrm{Rot}(\hat{\boldsymbol{x}},\gamma)\,\boldsymbol{I}$$

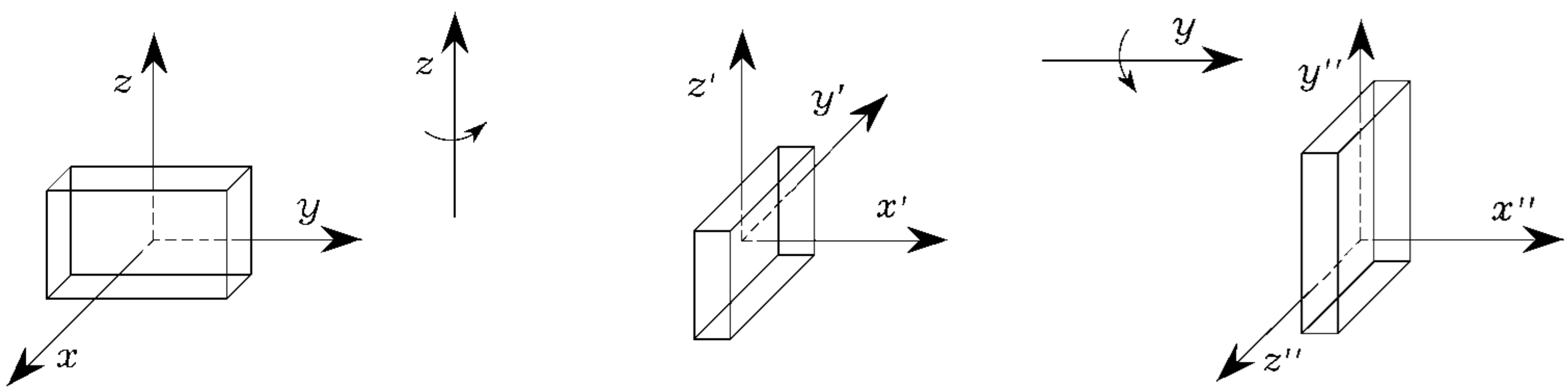


$$\boldsymbol{R}(\alpha,\beta,\gamma) = \begin{bmatrix} \mathrm{c}_\alpha \mathrm{c}_\beta & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma - \mathrm{s}_\alpha \mathrm{c}_\gamma & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma + \mathrm{s}_\alpha \mathrm{s}_\gamma \\ \mathrm{s}_\alpha \mathrm{c}_\beta & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma + \mathrm{c}_\alpha \mathrm{c}_\gamma & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma - \mathrm{c}_\alpha \mathrm{s}_\gamma \\ -\mathrm{s}_\beta & \mathrm{c}_\beta \mathrm{s}_\gamma & \mathrm{c}_\beta \mathrm{c}_\gamma \end{bmatrix}$$

This product of three rotations is the same as that for the ZYX Euler angles with rotations about the body/current frame $\{b\}$, i.e., the same product of three rotations admits two different physical interpretations.

Successive Rotations about Axes of Fixed & Body/Current Frames





Unit Quaternions

The unit quaternions are an alternative representation of rotations that alleviates the singularity of the division by $\sin\theta$ in the logarithm formula of the exponential coordinates, but at the cost of four variables subject to one constraint in the representation.

Let $\boldsymbol{R} \in SO(3)$ have the exponential coordinate representation $\hat{\boldsymbol{\omega}}\theta$, i.e., $\boldsymbol{R} = e^{[\hat{\omega}]\theta}$, where $\|\hat{\boldsymbol{\omega}}\| = 1$ and $\theta = [0, \pi]$.

$$\boldsymbol{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \hat{\boldsymbol{\omega}}\sin(\theta/2) \end{bmatrix} \in \mathbb{R}^4$$

Clearly $\|\boldsymbol{q}\| = 1$

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$\operatorname{tr}\boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$

$$\Rightarrow \quad q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} \quad , \quad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix}$$

Unit Quaternions

$$\boldsymbol{q} = (q_0, q_1, q_2, q_3) \quad \Rightarrow \quad \boldsymbol{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

It is interpreted as a rotation about the unit axis, in the direction of (q_1, q_2, q_3) by an angle $2\cos^{-1} q_0$.

- If $\boldsymbol{p} = (p_0, p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_0, q_1, q_2, q_3)$, then $\boldsymbol{n} = \boldsymbol{pq}$ is computed by:

$$\boldsymbol{R}_n = \boldsymbol{R}_p\boldsymbol{R}_q \quad \leftrightarrow \quad \boldsymbol{n} = \boldsymbol{pq}$$

$$\begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + p_0q_1 - q_2p_3 + q_3p_2 \\ q_0p_2 + p_0q_2 + q_1p_3 - q_3p_1 \\ q_0p_3 + p_0q_3 - q_1p_2 + q_2p_1 \end{bmatrix}$$

- The rotation of a point or vector $\boldsymbol{v} \in \mathbb{R}^3$ by the angle θ about an axis in the direction $\hat{\boldsymbol{\omega}}$ through the origin is determined as $\boldsymbol{q}_{v'} = \boldsymbol{q}\boldsymbol{q}_v\boldsymbol{q}^*$ where $\boldsymbol{q}$ is quaternion representation of $\hat{\boldsymbol{\omega}}\theta$, $\boldsymbol{q}^* = (q_0, -q_1, -q_2, -q_3)$ is conjugate of $\boldsymbol{q}$, $\boldsymbol{q}_v = (0, \boldsymbol{v})$, and $\boldsymbol{q}_{v'} = (0, \boldsymbol{v}')$.

$$\boldsymbol{v}' = \boldsymbol{R}_q\boldsymbol{v} \quad \leftrightarrow \quad \boldsymbol{q}_{v'} = \boldsymbol{q}\boldsymbol{q}_v\boldsymbol{q}^*$$