

# **MEC549: Robot Dynamics and Control**

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# **Ch1: Summary of Linear Algebra & Robot Kinematics**

# Linear Algebra

# Basic Notation

$\forall$

$\exists$

$\in$

$\Rightarrow$

$\Leftrightarrow$

$:=$

$\dot{x}$

$f: \mathcal{D} \rightarrow \mathcal{R}$

$\mathbb{R}$

$\mathbb{R}_+, \mathbb{R}_{++}$

$|x|$

# Vector

$\mathbf{x} \in \mathbb{R}^n$ : (an  $n$ -dimensional real vector in the column format)

$\mathbb{R}^n$ :  $n$ -dimensional real space  
(Euclidian Space)

$\mathbf{x}^T$ :

# Vector Norm

**General Definition:** Given  $\boldsymbol{x} \in \mathbb{R}^n$ , vector norm  $\|\boldsymbol{x}\| \in \mathbb{R}_+$  is defined such that

- $\|\boldsymbol{x}\| > 0$  when  $\boldsymbol{x} \neq \mathbf{0}$  and  $\|\boldsymbol{x}\| = 0$  iff  $\boldsymbol{x} = \mathbf{0}$ .
- $\|k\boldsymbol{x}\| = |k|\|\boldsymbol{x}\|$ ,  $\forall k \in \mathbb{R}$ .
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ ,  $\forall \boldsymbol{y} \in \mathbb{R}^n$ .

❖ The  $p$ -norm (or  $\ell_p$ -norm) of  $\boldsymbol{x}$  for  $p \in \mathbb{R}, p \geq 1$  is defined as  $\|\boldsymbol{x}\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$

e.g.  $\|\boldsymbol{x}\|_2 = \|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$  (Euclidean Norm)

Special case:  $\|\boldsymbol{x}\|_\infty := \max_i |x_i|$

**Schwartz Inequality:**  $|\boldsymbol{x}^T \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

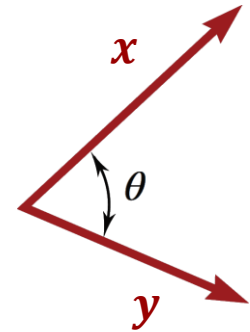
**Unit Vector:**  $\|\hat{\boldsymbol{x}}\|_2 = \hat{\boldsymbol{x}}^T \hat{\boldsymbol{x}} = 1, \quad \hat{\boldsymbol{x}} = \boldsymbol{x} / \|\boldsymbol{x}\|_2$

# Dot Product or Scalar Product or Inner Product

Dot Product or Scalar Product or Inner Product of two vectors  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{y} \in \mathbb{R}^n$  is a scalar defined as

(Algebraic Definition)  $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x}$

(Geometric Definition)  $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos \theta$   
( $0 \leq \theta \leq 180^\circ$ )



Orthogonal Vectors:

# Matrix

$\mathbf{A} \in \mathbb{R}^{m \times n}$  (an  $m$  by  $n$  dimensional real matrix)

$\mathbf{A}^T$

Matrix-vector multiplication  $\mathbf{A}\mathbf{x}$  as linear combination of columns of  $\mathbf{A}$ :



# Particular Matrices

Square Matrix:

- Upper Triangular
- Lower Triangular
- Diagonal
  - Identity Matrix
- Null Matrix

Symmetric Matrix:

Skew-symmetric Matrix:

Partitioned Matrix: A matrix whose elements are matrices (blocks) of proper dimensions.

# Matrix Operations

Trace of a square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ :  $\text{tr}(\mathbf{A})$

Sum of matrices:  $\mathbf{C} = \mathbf{A} + \mathbf{B}$

Symmetric and skew-symmetric part of a square matrix  $\mathbf{A}$ :

Product of matrices:  $\mathbf{C} = \mathbf{AB}$

Determinant of a square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ :  $\det(\mathbf{A})$

Singular and Nonsingular Matrices:

# Matrix Operations

Rank of a matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ :  $\text{rank}(\mathbf{A})$

Inverse of  $\mathbf{A}$ :  $\mathbf{A}^{-1}$

Orthogonal Matrix:

Linearly Independent Vectors  $\mathbf{x}_i \in \mathbb{R}^m, i = 1, \dots, n$

Derivative of  $\mathbf{A}(t) \in \mathbb{R}^{m \times n}$ :  $\frac{d}{dt} \mathbf{A}(t) = \dot{\mathbf{A}}(t)$

Derivative of  $\mathbf{A}^{-1}(t) \in \mathbb{R}^{n \times n}$ :

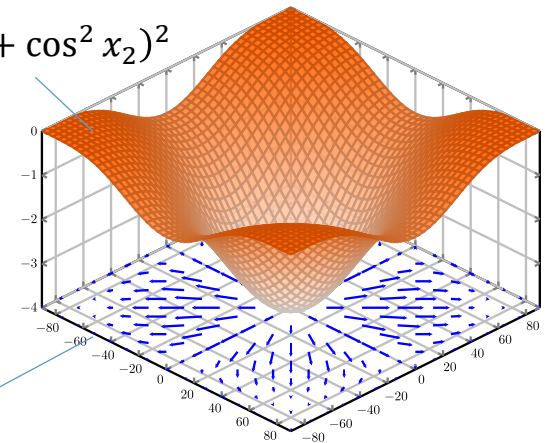
# Gradient

For a **scalar function**  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  which is differentiable with respect to the elements  $x_i$  of  $\mathbf{x} \in \mathbb{R}^n$ , its **gradient** with respect to  $\mathbf{x}$  is an  $n$ -dimensional column vector  $\nabla_{\mathbf{x}} f \in \mathbb{R}^n$  as:

(nabla symbol and pronounced "del")

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^T = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

$$f(x_1, x_2) = -(\cos^2 x_1 + \cos^2 x_2)^2$$



The gradient depicted as a projected vector field.

If  $\mathbf{x}(t)$  is a differentiable function with respect to  $t$ :

$$\dot{f}(\mathbf{x}) =$$

# Jacobian

For a **vector function**  $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$  whose elements  $f_i$  are differentiable with respect to the elements  $x_i$  of  $\mathbf{x} \in \mathbb{R}^n$ , its **Jacobian** with respect to  $\mathbf{x}$  is matrix  $\mathbf{J}_f \in \mathbb{R}^{m \times n}$  as:

$$\mathbf{J}_f(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial \mathbf{x}} \\ \frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial f_m(\mathbf{x})}{\partial \mathbf{x}} \end{bmatrix}$$

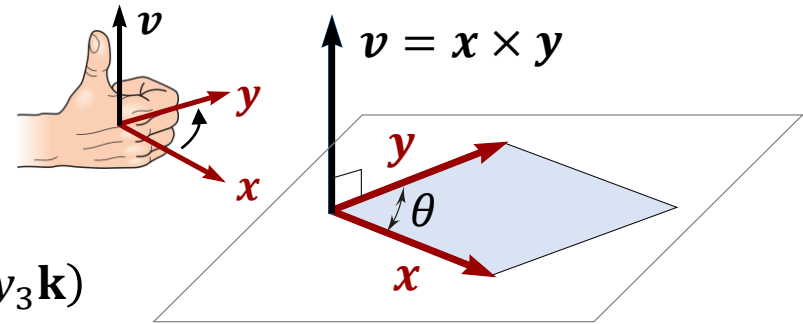
If  $\mathbf{x}(t)$  is a differentiable function with respect to  $t$ :

$$\dot{\mathbf{f}}(\mathbf{x}) =$$

# Cross Product or Vector Product

Cross product of  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$  (in the Euclidean space) is defined as a vector  $\mathbf{v} = \mathbf{x} \times \mathbf{y} \in \mathbb{R}^3$  that is orthogonal to both  $\mathbf{x}$  and  $\mathbf{y}$  ( $\mathbf{v} \perp \mathbf{x}, \mathbf{v} \perp \mathbf{y}$ ), with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

$$\|\mathbf{v}\|_2 = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \sin \theta \quad (0 \leq \theta \leq 180^\circ)$$



$$\mathbf{v} = \mathbf{x} \times \mathbf{y} = (x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}) \times (y_1 \mathbf{i} + y_2 \mathbf{j} + y_3 \mathbf{k})$$

Coordinate notation

$$\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3)$$

$$\mathbf{v} = \mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Matrix notation

# Cross Product as a Matrix-Vector Multiplication

Cross product  $\mathbf{x} \times \mathbf{y}$  ( $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ ) can be thought of as a multiplication of a vector by a  $3 \times 3$  skew-symmetric matrix as

$$\mathbf{x} \times \mathbf{y} = \underbrace{\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}}_{[\mathbf{x}]} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [\mathbf{x}]\mathbf{y} = -[\mathbf{y}]\mathbf{x}$$

The matrix  $[\mathbf{x}]$  is a 3x3 skew-symmetric matrix representation of  $\mathbf{x}$ .  $[\mathbf{x}] = -[\mathbf{x}]^T$

# Eigenvalues and Eigenvectors

If the vector resulting from the linear transformation  $A \in \mathbb{R}^{n \times n}$  on a vector  $u$  has the same direction of  $u$  (with  $u \neq 0$ ), then  $Au = \lambda u$ .

For each square matrix  $A \in \mathbb{R}^{n \times n}$  there exist  $n$  **eigenvalues** (in general, complex numbers) denoted by  $\lambda_i(A)$ ,  $i = 1, \dots, n$  that satisfy

(characteristic equation)

$\det(A - \lambda_i(A)I) = 0$

$I = \text{diag}(1) \in \mathbb{R}^{n \times n}$

- If  $A = A^T$ , then  $\lambda_i(A) \in \mathbb{R}$ ,  $i = 1, \dots, n$ .

**Eigenvectors**  $u_i$  associated with the eigenvalues  $\lambda_i$  satisfy  $(A - \lambda_i I)u_i = 0 \quad i = 1, \dots, n$

- If the eigenvectors  $u_i$  of  $A$  are linearly independent, matrix  $U$  formed by the column vectors  $u_i$  is invertible and  $\Lambda = U^{-1}AU$  where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ . If  $A$  is symmetric,  $U$  is orthogonal ( $UU^T = U^T U = I$ ) and  $\Lambda = U^T AU$ .  
 $\Rightarrow$  Eigendecomposition:  $A = U\Lambda U^{-1}$  and if  $A$  is symmetric  $A = U\Lambda U^T$ .

•  $\det(A) = \prod_{i=1}^n \lambda_i$

•  $\lambda(A^T) = \lambda(A)$

•  $\lambda(A^{-1}) = 1/\lambda(A)$



# Matrix Norm

**General Definition:** Given  $A \in \mathbb{R}^{n \times n}$ , matrix norm  $\|A\| \in \mathbb{R}_+$  is defined such that

- $\|A\| > 0$  when  $A \neq \mathbf{0}$  and  $\|A\| = 0$  iff  $A = \mathbf{0}$ .
- $\|kA\| = |k|\|A\|$ ,  $\forall k \in \mathbb{R}$ .
- $\|A + B\| \leq \|A\| + \|B\|$ ,  $\forall B \in \mathbb{R}^{n \times n}$ .
- $\|AB\| \leq \|A\|\|B\|$ ,  $\forall B \in \mathbb{R}^{n \times n}$ .

The  $p$ -norm of  $A$  (induced by vector  $p$ -norms) for  $0 \leq p \leq \infty$  is defined as

$$\|A\|_p = \sup_{\|x\|_p=1} \|Ax\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \quad \forall x \in \mathbb{R}^n$$

# Matrix Norm (cont.)

In the special cases of  $p = 1, 2, \infty$ , these norms can be computed/estimated by:

- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$  (the max. absolute column sum of  $A$ )
- $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} = \sigma_{\max}(A)$  (Spectral Norm)
 

(the square root of the maximum eigenvalue of  $A^T A$ , or the largest singular value of  $A$ )
 

If  $A = A^T$ 
 $\left\{ \begin{array}{l} \|A\|_2 = \max_i |\lambda_i(A)| \\ \|A^{-1}\|_2 = 1/\min_i |\lambda_i(A)| \end{array} \right.$
- $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$  (the max. absolute row sum of  $A$ )
- Frobenius Norm:**  $\|A\|_F = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2} = \sqrt{\text{tr}(A^T A)}$

$\|Ax\|_2 \leq \|A\|_2 \|x\|_2 , \quad \|A\|_2 \leq \|A\|_F$

# Quadratic Form

A **Quadratic Form** is a polynomial with terms all of degree two:

$$\begin{aligned}
 Q(x) &= ax^2 \\
 Q(x_1, x_2) &= ax_1^2 + bx_1x_2 + cx_2^2 \\
 Q(x_1, x_2, x_3) &= ax_1^2 + bx_1x_2 + cx_2^2 + dx_2x_3 + ex_3^2 + fx_1x_3
 \end{aligned}$$

The quadratic form associated with a  $n \times n$  matrix  $A \in \mathbb{R}^{n \times n}$  is the function  $Q: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $Q(x) = x^T A x$  for all  $x$ .

- The quadratic function associated with a skew-symmetric matrix  $A_{ss}$  is always **zero**.

$$A_{ss} \text{ is skew-symmetric} \quad \Leftrightarrow \quad x^T A_{ss} x = 0 \quad (\forall x)$$

- Each quadratic function  $x^T A x$  is always equal to a quadratic function with the symmetric part of matrix.

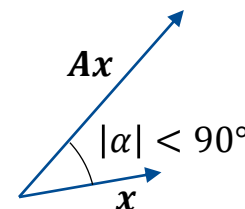
$$Q(x) = x^T A x = x^T (A_s + A_{ss}) x = x^T A_s x$$

- If  $A = A^T$ :  $\nabla_x Q(x) = \left( \frac{\partial Q(x)}{\partial x} \right)^T = 2Ax$ ,  $\dot{Q}(x) = \frac{d}{dt} Q(x(t)) = 2x^T A \dot{x} + x^T \dot{A} x$

# Definite and Semi-Definite Matrices

A square not necessarily symmetric matrix  $A \in \mathbb{R}^{n \times n}$  is

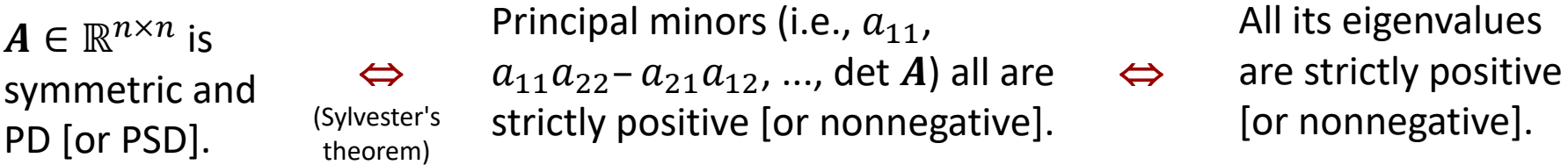
- **Positive Definite** (PD or  $A > 0$ ) if  $x^T A x > 0$  for all nonzero  $x \in \mathbb{R}^n$ .
- **Positive Semi-Definite** (PSD or  $A \geq 0$ ) if  $x^T A x \geq 0$  for all  $x \in \mathbb{R}^n$ .
- **Negative Definite** (ND or  $A < 0$ ) if  $x^T A x < 0$  for all nonzero  $x \in \mathbb{R}^n$ .
- **Negative Semi-Definite** (NSD or  $A \leq 0$ ) if  $x^T A x \leq 0$  for all  $x \in \mathbb{R}^n$ .
- **Indefinite** if  $A$  neither positive semi-definite nor negative semi-definite.



Geometric Interpretation of the Positive Definiteness of  $A$ .

- A square matrix  $A \in \mathbb{R}^{n \times n}$  is negative definite if  $-A$  is positive definite and it is negative semidefinite if  $-A$  is positive semidefinite.
- A **necessary** condition for  $A \in \mathbb{R}^{n \times n}$  to be PD [or PSD] is that its diagonal elements be strictly positive [or nonnegative].
- Since  $x^T A_{ss} x = 0$ , the test for the definiteness of  $A$  may be done by considering only its symmetric part.

# Definite and Semi-Definite Matrices (cont.)



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- Any symmetric PD matrix  $A = A^T > 0$  is always full-rank and nonsingular.
  - Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric PD matrix and  $\lambda_{\min}$ ,  $\lambda_{\max}$  be the minimum and maximum eigenvalues of  $A$ . For any  $x \in \mathbb{R}^n$ ,

$(x^T x = \|x\|_2^2)$

$$\lambda_{\min}(A)x^T x \leq x^T A x \leq \lambda_{\max}(A)x^T x$$

(Rayleigh–Ritz Theorem)

- Semi-definiteness of  $A \in \mathbb{R}^{n \times n}$  implies that  $\text{rank}(A) = r < n$ , and thus  $r$  eigenvalues of  $A$  are positive/negative and  $n - r$  are 0.
- A matrix inequality of the form  $A_1 > A_2$ , where  $A_1, A_2 \in \mathbb{R}^{n \times n}$  means that  $A_1 - A_2 > 0$ , i.e.,  $A_1 - A_2$  is PD. Similar notations apply to the concepts of PSD, ND, NSD.

# Rigid-Body Motions

# Rigid-Body Motions

Rotations	Rigid-Body Motions
$R \in SO(3)$ : $3 \times 3$ matrices $R^T R = R R^T = I, \det(R) = 1$	$T \in SE(3)$ : $4 \times 4$ matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$ where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^T$	$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
Change of coordinate frame: $R_{ab} R_{bc} = R_{ac}, R_{ab} p_b = p_a$ $(R_{ab} = R_{ba}^{-1})$	Change of coordinate frame: $T_{ab} T_{bc} = T_{ac}, T_{ab} p_b = p_a$ $(T_{ab} = T_{ba}^{-1})$

# Rigid-Body Motions

Rotations	Rigid-Body Motions
<p>Rotating a frame {b}:</p> $\mathbf{R} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$ <p><math>\mathbf{R}_{sb'} = \mathbf{R}\mathbf{R}_{sb}</math>: rotate <math>\theta</math> about <math>\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}</math></p> <p><math>\mathbf{R}_{sb''} = \mathbf{R}_{sb}\mathbf{R}</math>: rotate <math>\theta</math> about <math>\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}</math></p>	<p>Displacing a frame {b}:</p> $\mathbf{T} = \begin{bmatrix} \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$ <p><math>\mathbf{T}_{sb'} = \mathbf{T}\mathbf{T}_{sb}</math>: rotate <math>\theta</math> about <math>\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}</math> (moves {b} origin), translate <math>\mathbf{p}</math> in {s}</p> <p><math>\mathbf{T}_{sb''} = \mathbf{T}_{sb}\mathbf{T}</math>: translate <math>\mathbf{p}</math> in {b}, rotate <math>\theta</math> about <math>\hat{\boldsymbol{\omega}}</math> in new body frame</p>
<p>Unit rotation axis is <math>\hat{\boldsymbol{\omega}} \in \mathbb{R}^3</math>, where <math>\ \hat{\boldsymbol{\omega}}\  = 1</math></p>	<p>“Unit” screw axis is <math>\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} \in \mathbb{R}^6</math>, where either (i) <math>\ \mathbf{S}_\omega\  = 1</math> or (ii) <math>\ \mathbf{S}_\omega\  = 0, \ \mathbf{S}_v\  = 1</math></p>
	<p>For a screw axis <math>\{\mathbf{q}, \hat{\mathbf{s}}, h\}</math> with finite <math>h</math>,</p> $\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \mathbf{q} + h\hat{\mathbf{s}} \end{bmatrix}$
<p>Angular velocity is <math>\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}</math></p>	<p>Twist is <math>\boldsymbol{\mathcal{V}} = \mathbf{S}\dot{\theta}</math></p>



# Rigid-Body Motions

Rotations	Rigid-Body Motions
<p>For any <math>\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3</math>,</p> $[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ <p><b>Properties:</b> For any <math>\boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)</math>:</p> $[\boldsymbol{\omega}] = -[\boldsymbol{\omega}]^T, [\boldsymbol{\omega}]\boldsymbol{x} = -[\boldsymbol{x}]\boldsymbol{\omega},$ $[\boldsymbol{\omega}][\boldsymbol{x}] = ([\boldsymbol{x}][\boldsymbol{\omega}])^T, \boldsymbol{R}[\boldsymbol{\omega}]\boldsymbol{R}^T = [\boldsymbol{R}\boldsymbol{\omega}]$	<p>For any <math>\boldsymbol{v} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} \in \mathbb{R}^6</math> or <math>\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6</math>,</p> $[\boldsymbol{v}] = \begin{bmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3),$ $[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$
$\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s], \quad \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b] \quad (\boldsymbol{R} := \boldsymbol{R}_{sb})$	$\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\boldsymbol{v}_s], \quad \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{v}_b] \quad (\boldsymbol{T} := \boldsymbol{T}_{sb})$
	$[\text{Ad}_{\boldsymbol{T}}] = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ [\boldsymbol{p}]\boldsymbol{R} & \boldsymbol{R} \end{bmatrix} \in \mathbb{R}^{6 \times 6}$ <p><b>Properties:</b> <math>[\text{Ad}_{\boldsymbol{T}}]^{-1} = [\text{Ad}_{\boldsymbol{T}^{-1}}],</math></p> $[\text{Ad}_{\boldsymbol{T}_1}][\text{Ad}_{\boldsymbol{T}_2}] = [\text{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]$
<p>Change of coordinate frame:</p> $\hat{\boldsymbol{\omega}}_a = \boldsymbol{R}_{ab}\hat{\boldsymbol{\omega}}_b, \quad \boldsymbol{\omega}_a = \boldsymbol{R}_{ab}\boldsymbol{\omega}_b$	<p>Change of coordinate frame:</p> $\boldsymbol{S}_a = [\text{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b, \quad \boldsymbol{v}_a = [\text{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{v}_b$

# Rigid-Body Motions

Rotations	Rigid-Body Motions
$\hat{\omega}_s = R_{sb} \hat{\omega}_b$	$S_s = [\text{Ad}_{T_{sb}}] S_b, \mathcal{V}_s = [\text{Ad}_{T_{sb}}] \mathcal{V}_b, [\text{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$
Exponential coordinate for $R \in SO(3)$ : $\hat{\omega}\theta \in \mathbb{R}^3$	Exponential coordinate for $T \in SE(3)$ : $S\theta \in \mathbb{R}^6$
$\text{exp}: [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$ $R = \text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta}$ $R = I + \sin \theta [\hat{\omega}] + (1 - \cos \theta) [\hat{\omega}]^2$ (Rodrigues' formula for rotations)	$\text{exp}: [S]\theta \in se(3) \rightarrow T \in SE(3)$ $T = e^{[S]\theta}$ $T = \begin{bmatrix} e^{[S_\omega]\theta} & G(\theta)S_v \\ 0 & 1 \end{bmatrix}$ $G(\theta) = I\theta + (1 - \cos \theta)[S_\omega] + (\theta - \sin \theta)[S_\omega]^2$
$\text{log}: R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$ $\text{log}(R) = [\hat{\omega}]\theta$	$\text{log}: T \in SE(3) \rightarrow [S]\theta \in se(3)$ $\text{log}(T) = [S]\theta$
Moment change of coordinate frame: $m_a = R_{ab} m_b$	Wrench change of coordinate frame: $\mathcal{F}_a = (m_a, f_a) = [\text{Ad}_{T_{ba}}]^T \mathcal{F}_b$

# Forward/Velocity/Inverse Kinematics

# Forward Kinematics

The forward kinematics of a robot refers to the calculation of the position and orientation (**pose**) of its end-effector frame from its joint positions  $\theta$ .

- “Geometric” forward kinematics:

Given  $\theta \in \mathbb{R}^n$ , Find  $T_{sb} = T(\theta) \in SE(3)$

$$T: \mathbb{R}^n \rightarrow SE(3)$$

(Using PoE or D-H Method)

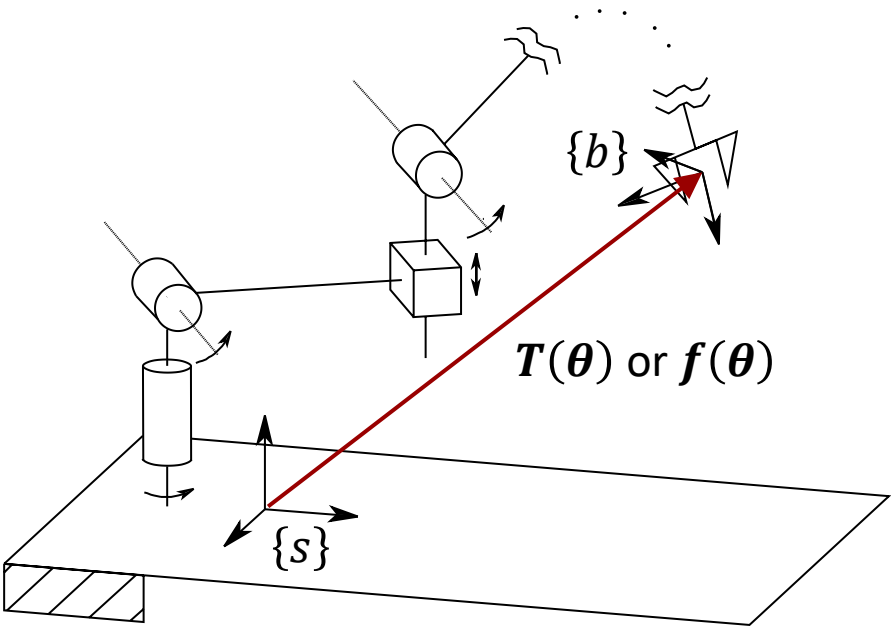
$$T(\theta) = e^{[S_1]\theta_1} \dots e^{[S_{n-1}]\theta_{n-1}} e^{[S_n]\theta_n} M$$

where  $M = T_{sb}(0) \in SE(3)$  and  $S_1, \dots, S_n$  are screw axes expressed in  $\{s\}$  when  $\theta = 0$ .

- “Minimum-Coordinate” forward kinematics:

Given  $\theta \in \mathbb{R}^n$ , Find  $x = f(\theta) \in \mathbb{R}^m$

$$(m \leq n) \quad f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$



# Velocity Kinematics

- $\mathbf{v}_s = \begin{bmatrix} \boldsymbol{\omega}_s \\ \mathbf{v}_s \end{bmatrix} = J_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
  - $\mathbf{v}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \mathbf{v}_b \end{bmatrix} = J_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
  - $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\mathbf{p}} \end{bmatrix} = J_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

**Geometric Jacobian**
- $\begin{bmatrix} \dot{\boldsymbol{\phi}} \\ \dot{\mathbf{p}} \end{bmatrix} = J_{a,\boldsymbol{\phi}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \quad \boldsymbol{\phi} = (\alpha, \beta, \gamma)$
  - $\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = J_{a,q}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \quad \mathbf{q} = (q_0, q_1, q_2, q_3)$
  - $\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \end{bmatrix} = J_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \quad \mathbf{r} = \hat{\boldsymbol{\omega}}\boldsymbol{\theta}$

**Analytic Jacobian**

- $J_s(\boldsymbol{\theta}) = [J_{s1} \quad J_{s2}(\boldsymbol{\theta}) \quad \cdots \quad J_{sn}(\boldsymbol{\theta})], J_{si}(\boldsymbol{\theta}) = \left[ \text{Ad}_{e^{[s_1]\theta_1} \dots e^{[s_{i-1}]\theta_{i-1}}} \right] \mathbf{s}_i \quad \begin{matrix} i = 2, \dots, n, \\ J_{s1} = \mathbf{s}_1 \end{matrix}$
- $J_b(\boldsymbol{\theta}) = [\text{Ad}_{T_{bs}}] J_s(\boldsymbol{\theta})$
- Statics:  $\boldsymbol{\tau} = J_b^T(\boldsymbol{\theta}) \mathcal{F}_b, \quad \boldsymbol{\tau} = J_s^T(\boldsymbol{\theta}) \mathcal{F}_s$
- In singular configuration  $\boldsymbol{\theta}^*, J(\boldsymbol{\theta}^*) \in \mathbb{R}^{r \times n}$  is rank-deficient, i.e.,  $\text{rank}(J(\boldsymbol{\theta}^*)) < r$ .

# Inverse Kinematics

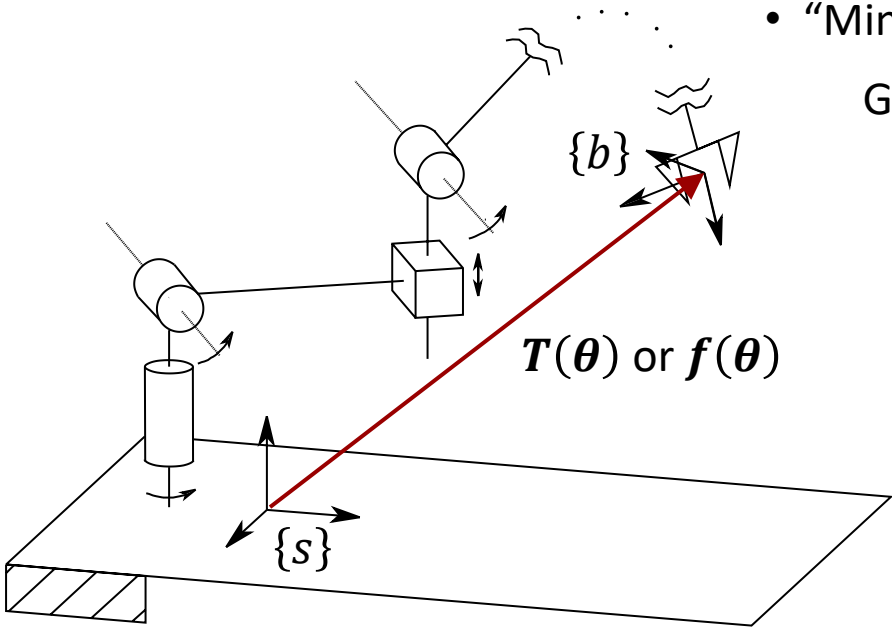
The inverse kinematics of a robot refers to the calculation of the joint coordinates  $\theta$  from the position and orientation (**pose**) of its end-effector frame.

- “Geometric” inverse kinematics:

Given  $T_{sb} = T(\theta) \in SE(3)$ , Find  $\theta \in \mathbb{R}^n$

- “Minimum-Coordinate” inverse kinematics:

Given  $x = f(\theta) \in \mathbb{R}^m$ , Find  $\theta \in \mathbb{R}^n$



- **Analytic Methods:** Finding closed-form solutions using algebraic or geometric intuition intuitions.
- **Iterative Numerical Methods:** For instance, using Newton–Raphson method:  
$$\theta^{i+1} = \theta^i + J^\dagger(\theta^i)e = \theta^i + J^\dagger(\theta^i)(x_d - f(\theta^i))$$

# Trajectory Generation

# Trajectory Generation: Path & Time Scaling

**Trajectory**  $\mathcal{C}(s(t))$  or  $\mathcal{C}(t)$  specifies the robot configuration as a function of time, i.e., the combination of a **path**  $\mathcal{C}(s)$  and a **time scaling**  $s(t)$ .

$$\mathcal{C}: [0,1] \rightarrow \mathbb{C}$$

$$s: [0, t_f] \rightarrow [0,1]$$

- Straight-Line Path in Joint Space:  $\boldsymbol{\theta}(s) = \boldsymbol{\theta}_{\text{start}} + s(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$
- Straight-Line Path in Task Space:

$$(1) \mathbf{x}(s) = \mathbf{x}_{\text{start}} + s(\mathbf{x}_{\text{end}} - \mathbf{x}_{\text{start}}) \in \mathbb{R}^m$$

$$(2) \mathbf{p}(s) = \mathbf{p}_{\text{start}} + s(\mathbf{p}_{\text{end}} - \mathbf{p}_{\text{start}}) \in \mathbb{R}^3$$

$$\mathbf{R}(s) = \mathbf{R}_{\text{start}} \exp(\log(\mathbf{R}_{\text{start}}^T \mathbf{R}_{\text{end}}) s) \in SO(3)$$

$$(3) \mathbf{T}(s) = \mathbf{T}_{\text{start}} \exp(\log(\mathbf{T}_{\text{start}}^{-1} \mathbf{T}_{\text{end}}) s) \in SE(3)$$

Examples of Time Scaling:

- 3<sup>rd</sup>-Order, 5<sup>th</sup>-Order Polynomial Position Profile
- Trapezoidal/S-Curve Velocity Profile
- Polynomial Via Point Trajectories

$$\begin{cases} s(t) = a_0 + a_1t + a_2t^2 + a_3t^3 \\ s(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 \end{cases}$$

