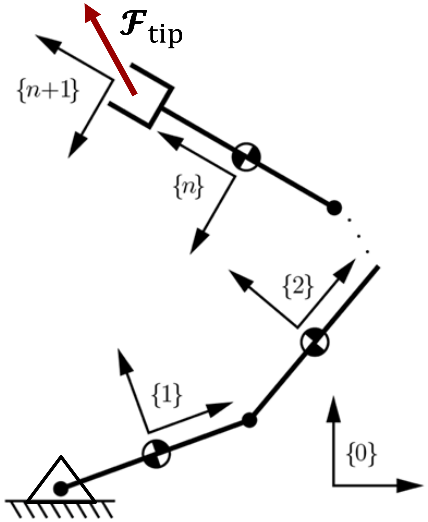
Ch2: Robot Dynamics – Part 2

Inverse Dynamics

Inverse Dynamic Equations in Closed Form

Inverse dynamic equations of an open-chain manipulator (finding $\boldsymbol{\tau}$ given $\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}, \boldsymbol{\mathcal{F}}_{\text{tip}}$) can be organized into a closed-form as

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}\end{aligned}$$



$\boldsymbol{\theta} \in \mathbb{R}^n$: Joint Variables

$\boldsymbol{\tau} \in \mathbb{R}^n$: Joint Torques/Forces

$\boldsymbol{M}(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$: Mass Matrix

$\boldsymbol{g}(\boldsymbol{\theta}) \in \mathbb{R}^n$: Gravitational Terms

$\boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$: Coriolis and Centripetal, and Gravitational Terms

$\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$: Coriolis and Centripetal Terms (velocity-product term or quadratic velocity term)

$\boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^{n\times n}$: Coriolis Matrix

$\boldsymbol{\Gamma}(\boldsymbol{\theta})$: $n \times n \times n$ matrix of Christoffel symbols of the first kind

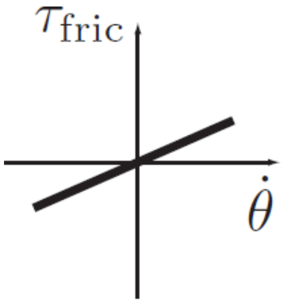
$\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{n\times 6}$: Jacobian in the same frame as $\boldsymbol{\mathcal{F}}_{\text{tip}}$

$\boldsymbol{\mathcal{F}}_{\text{tip}} \in \mathbb{R}^6$: Wrench applied to the environment by end-effector

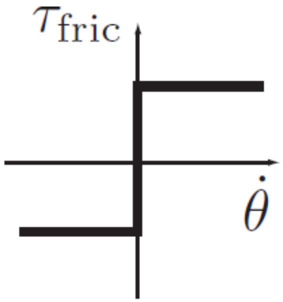
Friction Torques/Forces at Joints

The Lagrangian and Newton–Euler dynamics do not account for friction at the joints. However, the friction torques/forces in gearheads and bearings may be significant.

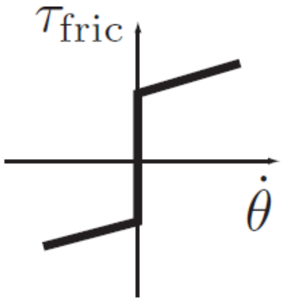
Friction models often include a static friction term and a velocity-dependent viscous friction term.



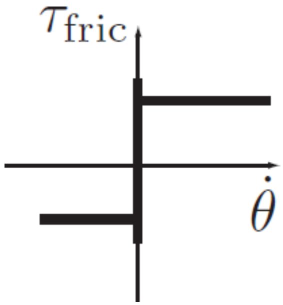
Viscous friction



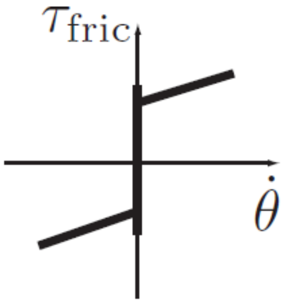
Coulomb friction



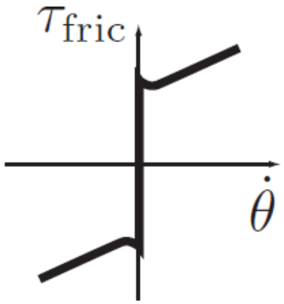
Static and viscous friction



Static and kinetic friction



Static, kinetic, and viscous friction

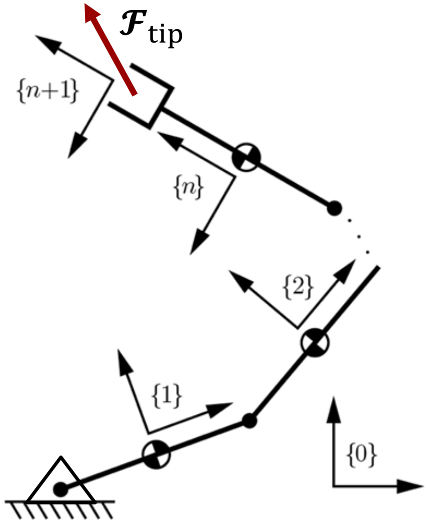


A friction law exhibiting the Stribeck effect

Inverse Dynamic Equations in Closed Form

In the presence of the viscous and static friction torques/forces at the joints:

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{f}_v(\dot{\boldsymbol{\theta}}) + \boldsymbol{f}_s(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \underbrace{\boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s\,\mathbf{sgn}(\dot{\boldsymbol{\theta}})}_{\text{simplified models}} + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}}\end{aligned}$$



$\boldsymbol{F}_v \in \mathbb{R}^{n\times n}$: Diagonal matrix of viscous friction coefficients

$\boldsymbol{F}_s \in \mathbb{R}^{n\times n}$: Diagonal matrix of Coulomb friction coefficients

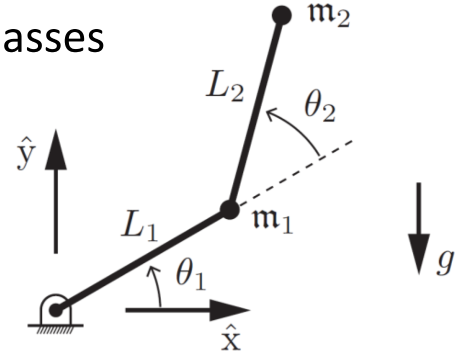
$\mathbf{sgn}(\dot{\boldsymbol{\theta}}) \in \mathbb{R}^{n\times 1}$: A vector whose components are the sign functions of $\dot{\theta}_i$

We can also add a disturbance $\boldsymbol{\tau}_{\mathrm{dist}}$ to represent inaccurately modeled dynamics, etc.

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s\mathbf{sg\,n}(\dot{\boldsymbol{\theta}}) + \boldsymbol{\tau}_{\mathrm{dist}} + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

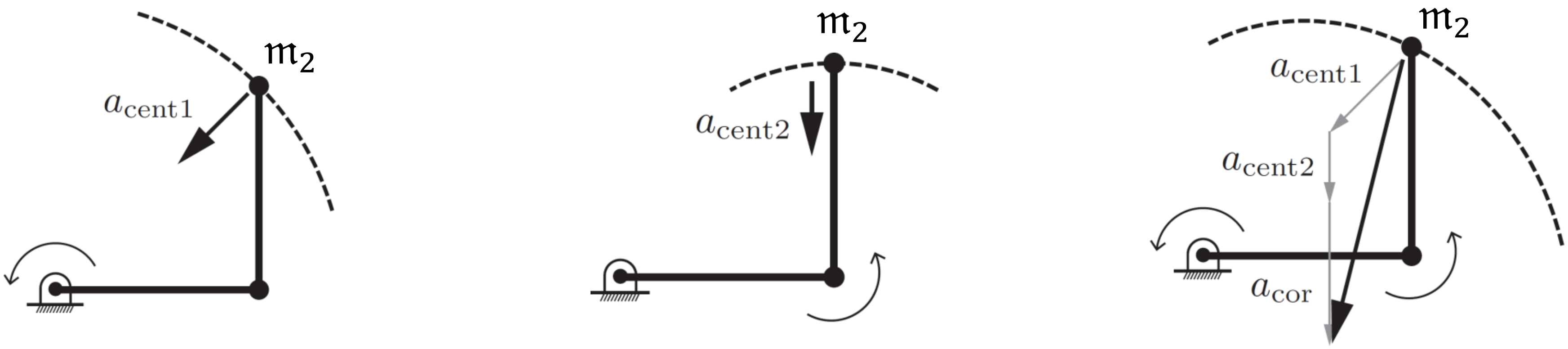
Understanding Centripetal and Coriolis Terms

Consider a planar 2R open chain whose links are modeled as point masses concentrated at the ends of each link:


$\mathfrak{m}_2$
L_2
θ_2
ŷ
L_1
$\mathfrak{m}_1$
g
θ_1
x̂

Accelerations of $\mathfrak{m}_2$ when $\boldsymbol{\theta} = (0, \pi/2)$ and $\ddot{\boldsymbol{\theta}} = \mathbf{0}$:

$$\begin{bmatrix}\ddot{x}_2 \\ \ddot{y}_2\end{bmatrix} = \underbrace{\begin{bmatrix} -L_1\dot{\theta}_1^2 \\ -L_2\dot{\theta}_1^2 - L_2\dot{\theta}_2^2 \end{bmatrix}}_{\text{centripetal terms}} + \underbrace{\begin{bmatrix} 0 \\ -2L_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}}_{\text{Coriolis terms}}$$


$\mathfrak{m}_2$
a_{cent1}
$\mathfrak{m}_2$
a_{cent2}
$\mathfrak{m}_2$
a_{cent1}
a_{cent2}
a_{cor}

$$\boldsymbol{a}_{\text{cent1}} = (-L_1\dot{\theta}_1^2, -L_2\dot{\theta}_1^2)$$

$$\dot{\theta}_1 > 0, \qquad \dot{\theta}_2 = 0$$

$$\boldsymbol{a}_{\text{cent2}} = (0, -L_2\dot{\theta}_2^2)$$

$$\dot{\theta}_1 = 0, \qquad \dot{\theta}_2 > 0$$

$$\boldsymbol{a}_{\text{cor}} = (0, -2L_2\dot{\theta}_1\dot{\theta}_2)$$

$$\dot{\theta}_1, \dot{\theta}_2 > 0$$

Understanding Mass Matrix

The total kinetic energy $\mathcal{K}$ of a robot can be expressed as the sum of the kinetic energies of each link:

$$\mathcal{K} = \frac{1}{2}\sum_{i=1}^{n} \boldsymbol{\mathcal{V}}_i^{\mathrm{T}} \boldsymbol{\mathcal{G}}_i \boldsymbol{\mathcal{V}}_i$$

twist of link frame $\{i\}$ in $\{i\}$ spatial inertia matrix of link i in $\{i\}$

Let define $\boldsymbol{J}_{ib}(\boldsymbol{\theta})$ as body Jacobian of link frame $\{i\}$ such that $\boldsymbol{\mathcal{V}}_i = \boldsymbol{J}_{ib}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}, i = 1, \dots, n$, thus:

$$\mathcal{K} = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}} \underbrace{\left(\sum_{i=1}^{n} \boldsymbol{J}_{ib}^{\mathrm{T}}(\boldsymbol{\theta}) \boldsymbol{\mathcal{G}}_i \boldsymbol{J}_{ib}(\boldsymbol{\theta})\right)}_{\text{This is the mass matrix}} \dot{\boldsymbol{\theta}}$$

$$\boldsymbol{M}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{J}_{ib}^{\mathrm{T}}(\boldsymbol{\theta}) \boldsymbol{\mathcal{G}}_i \boldsymbol{J}_{ib}(\boldsymbol{\theta})$$

$$\Rightarrow \quad \mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}} \boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

kinetic energy of an open-chain robot

Mass matrix $\boldsymbol{M}(\boldsymbol{\theta})$ is always **symmetric** and **positive-definite** ($\boldsymbol{x}^T \boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \neq \boldsymbol{0}$), and **depends** only on $\boldsymbol{\theta}$. Moreover, $\boldsymbol{M}^{-1}(\boldsymbol{\theta})$ always exist.

Understanding Mass Matrix (cont.)

A mass matrix $\boldsymbol{M}(\boldsymbol{\theta})$ presents a different effective mass in different acceleration directions. For better understanding, let represent $\boldsymbol{M}(\boldsymbol{\theta})$ as an effective (or apparent) mass of the end-effector $\boldsymbol{M}_C(\boldsymbol{\theta})$, because it is possible to feel this mass directly by grabbing and moving the end-effector.

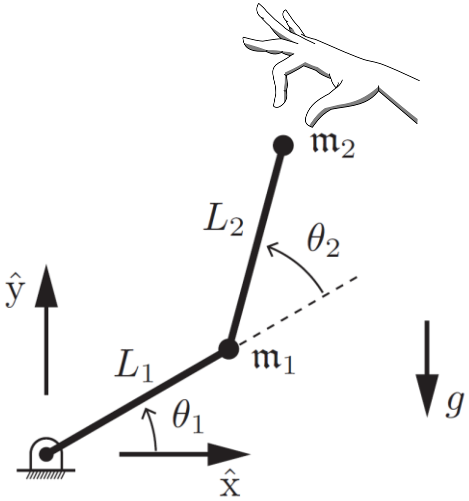
If $\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is the end-effector twist and $\boldsymbol{J}(\boldsymbol{\theta})$ is invertible,

$$\mathcal{K} = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \frac{1}{2}\boldsymbol{\mathcal{V}}^{\mathrm{T}}\boldsymbol{M}_C(\boldsymbol{\theta})\boldsymbol{\mathcal{V}}$$

Kinetic energy of the robot regardless of the coordinates.

$$\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{M}_C(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\boldsymbol{M}_C(\boldsymbol{\theta}) = \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{J}^{-1}(\boldsymbol{\theta})$$



A general expression that applies to both redundant and nonredundant manipulators:

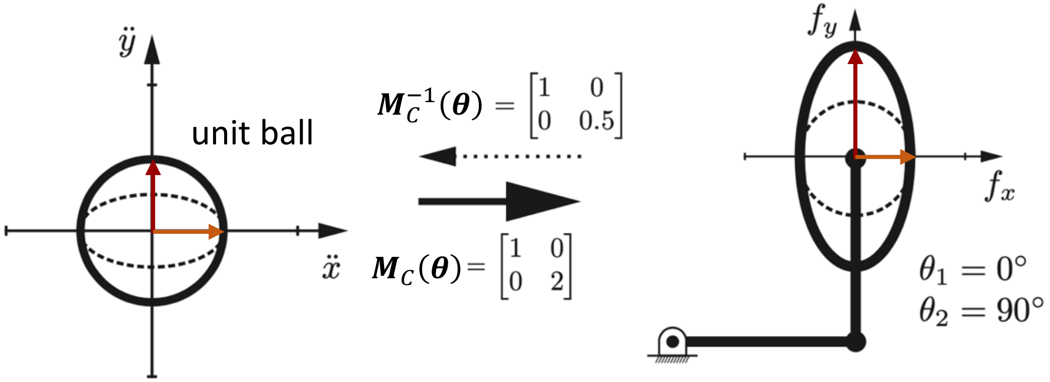
$$\boldsymbol{M}_C(\boldsymbol{\theta}) = \left(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\right)^{-1}$$

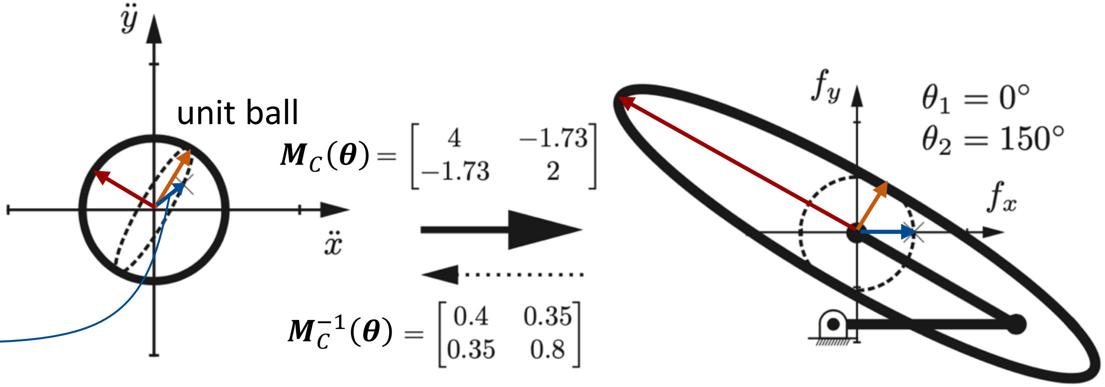
Understanding Mass Matrix (cont.)

Consider the 2R robot with $L_1 = L_2 = \mathfrak{m}_1 = \mathfrak{m}_2 = 1$. When the robot is at rest ($\dot{\boldsymbol{\theta}} = \mathbf{0}$) and $g = 0$, $\boldsymbol{M}_C(\boldsymbol{\theta})$ maps the endpoint acceleration $(\ddot{x}, \ddot{y})$ to (f_x, f_y), i.e., $(f_x, f_y) = \boldsymbol{M}_C(\boldsymbol{\theta})(\ddot{x}, \ddot{y})$.

Force and acceleration are only parallel along principal axes.

(Principal-axis directions given by the eigenvectors of $\boldsymbol{M}_C(\boldsymbol{\theta})$ and principal s axis lengths given by its eigenvalues.)


$\ddot{y}$
unit ball
$\ddot{x}$
$\boldsymbol{M}_C^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$
$\boldsymbol{M}_C(\boldsymbol{\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
f_y
f_x
$\theta_1 = 0°$
$\theta_2 = 90°$


$\ddot{y}$
unit ball
$\ddot{x}$
$\boldsymbol{M}_C(\boldsymbol{\theta}) = \begin{bmatrix} 4 & -1.73 \\ -1.73 & 2 \end{bmatrix}$
$\boldsymbol{M}_C^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} 0.4 & 0.35 \\ 0.35 & 0.8 \end{bmatrix}$
f_y
f_x
$\theta_1 = 0°$
$\theta_2 = 150°$

An example where force and acceleration are not parallel.

Finding Dynamic Terms Using Lagrangian Formulation

$$\boldsymbol{\tau} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}{\partial \dot{\boldsymbol{\theta}}}\right] - \frac{\partial \mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}}, \qquad \mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \mathcal{P}(\boldsymbol{\theta}), \qquad \mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\theta}}} = \frac{\partial \mathcal{K}}{\partial \dot{\boldsymbol{\theta}}} = \boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\theta}}} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}, \qquad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\theta}}\left(\dot{\boldsymbol{\theta}}^{T}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right) - \frac{\partial \mathcal{P}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$

$$\Rightarrow \quad \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\theta}}\left[\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right] + \frac{\partial \mathcal{P}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \quad \xrightarrow[\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta})]{\text{Comparing with}}$$

$$\Rightarrow \quad \begin{cases} \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} - \dfrac{1}{2}\dfrac{\partial}{\partial \boldsymbol{\theta}}\left[\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right] = \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} - \dfrac{\partial \mathcal{K}}{\partial \boldsymbol{\theta}}, \\[2ex] \boldsymbol{g}(\boldsymbol{\theta}) = \dfrac{\partial \mathcal{P}}{\partial \boldsymbol{\theta}} \end{cases}$$

Finding Dynamic Terms Using Lagrangian Formulation

(cont.)

Componentwise Analysis: $\tau_k = \frac{d}{dt}\frac{\partial \mathcal{L}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}{\partial \dot{\theta}_k} - \frac{\partial \mathcal{L}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}{\partial \theta_k} \qquad k = 1, \dots, n$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_k} = \sum_{j=1}^{n} m_{kj}(\theta)\dot{\theta}_j$$

$$\frac{\partial \mathcal{L}}{\partial \theta_k} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial m_{ij}}{\partial \theta_k}\dot{\theta}_j\dot{\theta}_i - \frac{\partial P}{\partial \theta_k}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_k} = \sum_{j=1}^{n}\left(m_{kj}(\theta)\ddot{\theta}_j + \left[\frac{d}{dt}m_{kj}(\theta)\right]\dot{\theta}_j\right)$$

$$= \sum_{j=1}^{n} m_{kj}\ddot{\theta}_j + \underbrace{\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{kj}}{\partial \theta_i}\dot{\theta}_i\dot{\theta}_j}$$

(due to symmetry)

$$\boxed{\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{kj}}{\partial \theta_i}\dot{\theta}_i\dot{\theta}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j}\right]\dot{\theta}_j\dot{\theta}_i}$$

$$\Rightarrow \quad \tau_k = \sum_{j=1}^{n} m_{kj}\ddot{\theta}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{\frac{1}{2}\left[\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right]}_{\Gamma_{ijk}(\boldsymbol{\theta})}\dot{\theta}_j\dot{\theta}_i + \frac{\partial P}{\partial \theta_k}, \qquad k = 1, \dots, n$$

$\Gamma_{ijk}(\boldsymbol{\theta})$is a $n \times n \times n$ matrix known as Christoffel symbols of the first kind.

Finding Dynamic Terms Using Lagrangian Formulation

(cont.)

Thus, we can write the components of $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ as

$$c_k(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{ijk}(\boldsymbol{\theta})\dot{\theta}_j\dot{\theta}_i$$

⇨ $$\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \underbrace{\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}\,\dot{\boldsymbol{\theta}} = \underbrace{\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}}$$

$$C_{kj}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \sum_{i=1}^{n}\Gamma_{ijk}(\boldsymbol{\theta})\dot{\theta}_i$$

$$= \sum_{i=1}^{n}\frac{1}{2}\left[\frac{\partial m_{kj}}{\partial\theta_i}+\frac{\partial m_{ki}}{\partial\theta_j}-\frac{\partial m_{ij}}{\partial\theta_k}\right]\dot{\theta}_i$$

$$\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \equiv \begin{bmatrix}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}_1(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\\ \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}_2(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\\ \vdots\\ \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}_n(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\end{bmatrix}$$

$\boldsymbol{\Gamma}_i(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{\Gamma}_i(\boldsymbol{\theta}) = \boldsymbol{\Gamma}_i^T(\boldsymbol{\theta})$

(j,k)th element of $\Gamma_{ijk}(\boldsymbol{\theta})$

Finding Dynamic Terms Using Newton–Euler Formulation

We know that using the recursive Newton-Euler inverse dynamics algorithm we can find $\boldsymbol{\tau}$. Thus,

Term $\boldsymbol{b}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\mathcal{F}}_{\text{tip}}) = \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\text{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$ is computed by finding $\boldsymbol{\tau}|_{\ddot{\boldsymbol{\theta}}=\mathbf{0}}$

Term $\boldsymbol{g}(\boldsymbol{\theta})$ is computed by finding $\boldsymbol{\tau}|_{\dot{\boldsymbol{\theta}}=\ddot{\boldsymbol{\theta}}=\mathbf{0},\, \boldsymbol{\mathcal{F}}_{\text{tip}}=\mathbf{0}}$

Term $\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is computed by finding $\boldsymbol{\tau}|_{\ddot{\boldsymbol{\theta}}=\mathbf{0},\, \boldsymbol{\mathcal{F}}_{\text{tip}}=\mathbf{0},\, \mathfrak{g}=\mathbf{0}}$

Term $\boldsymbol{J}^{\text{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$ is computed by finding $\boldsymbol{\tau}|_{\dot{\boldsymbol{\theta}}=\ddot{\boldsymbol{\theta}}=\mathbf{0},\, \mathfrak{g}=\mathbf{0}}$

Term $\boldsymbol{M}(\boldsymbol{\theta}) = [\boldsymbol{M}_1(\boldsymbol{\theta}), \dots, \boldsymbol{M}_n(\boldsymbol{\theta})]$ is computed by

$$\boldsymbol{M}_i(\boldsymbol{\theta}) = \boldsymbol{\tau}\Big|_{\dot{\boldsymbol{\theta}}=\mathbf{0},\, \boldsymbol{\mathcal{F}}_{\text{tip}}=\mathbf{0},\, \mathfrak{g}=\mathbf{0},\, \ddot{\theta}_i=1,\, \ddot{\theta}_j=0,\, \forall j\neq i}$$

(Alternatively, we can use: $\boldsymbol{M}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{J}_{ib}^{\text{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}_i \boldsymbol{J}_{ib}(\boldsymbol{\theta})$)

Forward Dynamics

Forward Dynamics

Finding $\ddot{\boldsymbol{\theta}}$ given the $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$, $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

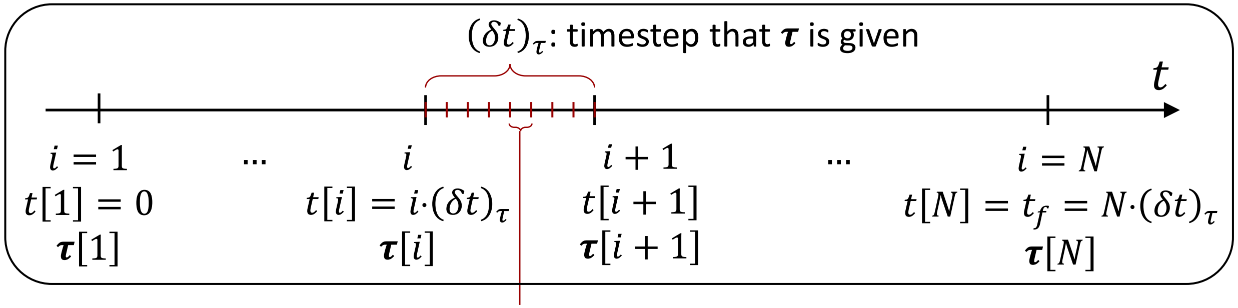
After computing $\boldsymbol{b}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\mathcal{F}}_{\mathrm{tip}}) = \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$ and $\boldsymbol{M}(\boldsymbol{\theta})$, we can use any efficient algorithm to solve $\boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \boldsymbol{\tau} - \boldsymbol{b}$ for $\ddot{\boldsymbol{\theta}}$.

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}(\boldsymbol{\theta})\left(\boldsymbol{\tau} - \boldsymbol{b}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\mathcal{F}}_{\mathrm{tip}})\right)$$

or $\ddot{\boldsymbol{\theta}} = \boldsymbol{M}(\boldsymbol{\theta})\backslash\left(\boldsymbol{\tau} - \boldsymbol{b}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\mathcal{F}}_{\mathrm{tip}})\right)$ in MATLAB

Numerical Simulation of Robot Motion

Forward dynamics can be used to **simulate the motion of the robot** for $t \in [0, t_f]$ given $\boldsymbol{\tau}(t), \boldsymbol{\mathcal{F}}_{\text{tip}}(t)$, its initial state $\boldsymbol{\theta}(0), \dot{\boldsymbol{\theta}}(0)$, and the number of integration steps N:


$(\delta t)_\tau$: timestep that $\boldsymbol{\tau}$ is given
t
$i = 1$
$t[1] = 0$
$\boldsymbol{\tau}[1]$
…
i
$t[i] = i \cdot (\delta t)_\tau$
$\boldsymbol{\tau}[i]$
$i + 1$
$t[i+1]$
$\boldsymbol{\tau}[i+1]$
…
$i = N$
$t[N] = t_f = N \cdot (\delta t)_\tau$
$\boldsymbol{\tau}[N]$

```
Given $\boldsymbol{\tau}[i], \boldsymbol{\mathcal{F}}_{\text{tip}}[i]$ $(i = 1, \ldots, N)$
Set $\boldsymbol{\theta}[1] = \boldsymbol{\theta}(0)$,  $\dot{\boldsymbol{\theta}}[1] = \dot{\boldsymbol{\theta}}(0)$
For $i = 1$ to $N - 1$
    For $j = 1$ to $n_{\text{res}}$
        $\ddot{\boldsymbol{\theta}} = \text{ForwardDynamics}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\tau}[i], \boldsymbol{\mathcal{F}}_{\text{tip}}[i])$
        $\boldsymbol{\theta} = \boldsymbol{\theta} + \dot{\boldsymbol{\theta}} \cdot (\delta t)_{\ddot{\theta}}$
        $\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}} + \ddot{\boldsymbol{\theta}} \cdot (\delta t)_{\ddot{\theta}}$
    end
$\boldsymbol{\theta}[i + 1] = \boldsymbol{\theta}$
$\dot{\boldsymbol{\theta}}[i + 1] = \dot{\boldsymbol{\theta}}$
end
```

$(\delta t)_{\ddot{\theta}} = (\delta t)_\tau / n_{\text{res}}$: timestep for motion simulation and computing $\ddot{\boldsymbol{\theta}}$.

n_{res}: Integration resolution.

First-order Euler Integration

$(\delta t)_\tau, (\delta t)_{\ddot{\theta}} \in \mathbb{R}^+$

Properties of Dynamic Model

Properties of Robot Dynamic Equations

Fundamental properties of the dynamic model of n-DOF open-chain robots are of particular importance in the study of control systems for robot manipulators.

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}\big(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\big) + \boldsymbol{g}(\boldsymbol{\theta}) \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}\big(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\big)\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta})\end{aligned}$$

$\boldsymbol{q} \in \mathbb{R}^n$: Joint Variables

$\boldsymbol{\tau} \in \mathbb{R}^n$: Joint Torques/Forces

$\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$: Mass Matrix

$\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$: Gravitational Terms

$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{n\times n}$: Coriolis Matrix

$\boldsymbol{c}\big(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\big) \in \mathbb{R}^n$: Coriolis and Centripetal Terms (velocity-product term or quadratic velocity term)

Properties of Inertia Matrix $\boldsymbol{M}(\boldsymbol{\theta})$

- The total kinetic energy $\mathcal{K} \in \mathbb{R}_+$ of an open-chain robot: $\mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

- $\boldsymbol{M}(\boldsymbol{\theta})$ **depends** only on $\boldsymbol{\theta}$.
- $\boldsymbol{M}(\boldsymbol{\theta})$ is always **symmetric** and **positive-definite**.
- $\boldsymbol{M}^{-1}(\boldsymbol{\theta})$ always exist.
- $\boldsymbol{M}(\boldsymbol{\theta})$ is bounded above and below: $\mu_1\boldsymbol{I} \le \boldsymbol{M}(\boldsymbol{\theta}) \le \mu_2\boldsymbol{I}$ $\quad \forall\boldsymbol{\theta} \in \mathbb{R}^n, \mu_1, \mu_2 \in \mathbb{R}_{++}$, $\boldsymbol{I} = \mathrm{diag}(1) \in \mathbb{R}^n$

$$\frac{1}{\mu_1}\boldsymbol{I} \ge \boldsymbol{M}^{-1}(\boldsymbol{\theta}) \ge \frac{1}{\mu_2}\boldsymbol{I}$$

- If the arm is revolute, μ_1, μ_2 are constants, and if the arm has prismatic joints, μ_1, μ_2 may depend on $\boldsymbol{\theta}$.

- This property can also be expressed as $m_1 \le \|\boldsymbol{M}(\boldsymbol{\theta})\| \le m_2 \qquad \forall\boldsymbol{\theta} \in \mathbb{R}^n$

$\|\cdot\|$ is any matrix norm, $m_1, m_2 \in \mathbb{R}_{++}$

Properties of Coriolis & Centripetal Terms

- $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is quadratic in $\dot{\boldsymbol{\theta}}$.
- $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})|_{\dot{\boldsymbol{\theta}}=\mathbf{0}} = \mathbf{0}$.
- $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ can be bounded above by a quadratic function of $\dot{\boldsymbol{\theta}}$: $\|\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\| \leq c_b\|\dot{\boldsymbol{\theta}}\|^2$

$\|\cdot\|$ is any vector norm, $c_b \in \mathbb{R}_+$, $\forall \boldsymbol{\theta},\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$

- If the arm is revolute, c_b is constant, and if the arm has prismatic joints, c_b may depend on $\boldsymbol{\theta}$.

- If $\|\cdot\|$ is 2-norm: $c_b = n^2\left(\max_{k,i,j,\boldsymbol{\theta}}\left|\Gamma_{k_{ij}}(\boldsymbol{\theta})\right|\right)$

- Matrix $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ may be not unique, but the vector $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$ is indeed unique.

- In general, $\dot{\boldsymbol{\theta}}^T(\dot{\boldsymbol{M}} - 2\boldsymbol{C})\dot{\boldsymbol{\theta}} = \mathbf{0}$.

- We can always find the standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ that $\boldsymbol{S}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{M}} - 2\boldsymbol{C} \in \mathbb{R}^{n\times n}$ is **skew symmetric**, i.e., $\boldsymbol{x}^T(\dot{\boldsymbol{M}} - 2\boldsymbol{C})\boldsymbol{x} = \mathbf{0}$, $\forall \boldsymbol{x} \in \mathbb{R}^n$. (Passivity Property)

- For a standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$, $\dot{\boldsymbol{M}} = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})^T$.

Properties of Coriolis & Centripetal Terms

- We can find the standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ as $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = 1/2\left(\dot{\boldsymbol{M}} + \boldsymbol{U}^T - \boldsymbol{U}\right)$

$$\boldsymbol{U}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \left(\boldsymbol{I}_n \otimes \dot{\boldsymbol{\theta}}^T\right)\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{\theta}} \in \mathbb{R}^{n\times n}, \boldsymbol{I}_n = \mathrm{diag}(1) \in \mathbb{R}^n$$

  - We can find 2 other choices of $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ as $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{M}} - 1/2\boldsymbol{U}$

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{U}^T - 1/2\boldsymbol{U}$$

Let define **Kronecker Product** of two metrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{p\times q}$ as

$$\boldsymbol{A} \otimes \boldsymbol{B} = \left[a_{ij}\boldsymbol{B}\right] \in \mathbb{R}^{mp\times nq}$$

For instance, for $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$: $\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & a_{13}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & a_{23}\boldsymbol{B} \\ a_{31}\boldsymbol{B} & a_{32}\boldsymbol{B} & a_{33}\boldsymbol{B} \end{bmatrix}$

Also, for $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{R}^{m\times n}$, $\boldsymbol{x} \in \mathbb{R}^p$, let define the matrix derivative as $\dfrac{\partial \boldsymbol{A}}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{\partial \boldsymbol{A}}{\partial x_1} \\ \vdots \\ \dfrac{\partial \boldsymbol{A}}{\partial x_p} \end{bmatrix} \in \mathbb{R}^{mp\times n}$

Properties of Gravitational Terms $g(q)$

- Let $\mathcal{P} \in \mathbb{R}_+$ be the total potential energy of an open-chain robot. Then, $\boldsymbol{g}(\boldsymbol{\theta}) = \dfrac{\partial \mathcal{P}}{\partial \boldsymbol{\theta}}$

- $\boldsymbol{g}(\boldsymbol{\theta})$ **depends** only on $\boldsymbol{\theta}$.
- $\boldsymbol{g}(\boldsymbol{\theta})$ is bounded above: $\|\boldsymbol{g}(\boldsymbol{\theta})\| \leq g_b \qquad \forall \boldsymbol{\theta} \in \mathbb{R}^n$

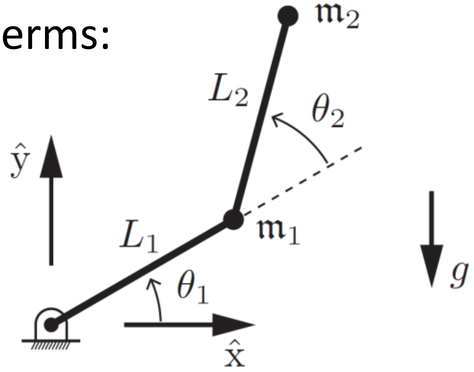
  $\|\cdot\|$ is any vector norm, $g_b \in \mathbb{R}_+$

  - If the arm is revolute, g_b is constant, and if the arm has prismatic joints, g_b may depend on $\boldsymbol{\theta}$.

- $\displaystyle\int_0^{t_f} \boldsymbol{g}(\boldsymbol{\theta}(t))^T \dot{\boldsymbol{\theta}}(t)dt = \mathcal{P}\left(\boldsymbol{\theta}(t_f)\right) - \mathcal{P}(\boldsymbol{\theta}(0))$

Example

Dynamic equations of a planar 2R open-chain in absence of friction terms:



$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta})$$

$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix}$$

$$\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} \qquad \boldsymbol{g}(\boldsymbol{\theta}) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2) L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2 \cos(\theta_1 + \theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Find the bounds on the $\boldsymbol{M}(\boldsymbol{\theta})$, $\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, $\boldsymbol{g}(\boldsymbol{\theta})$. Suppose that the joint angles θ_1 and θ_2 are limited by $\pm\pi/2$.

Note: The selection of a suitable norm is not always straightforward. This choice often depends simply on which norm makes it possible to evaluate the bounds. Usually, 1-norm is an easy choice.

Example (cont.)

- The induced 1-norm for $\boldsymbol{M}(\boldsymbol{\theta})$:

$$\|\boldsymbol{M}(\boldsymbol{\theta})\|_1 = \left|\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right| + \left|\mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\right|$$

$$m_1 \leq \|\boldsymbol{M}(\boldsymbol{\theta})\|_1 \leq m_2$$

$$m_2 = (\mathfrak{m}_1 + \mathfrak{m}_2)L_1^2 + 2\mathfrak{m}_2L_2^2 + 3\mathfrak{m}_2L_1L_2$$
$$m_1 = (\mathfrak{m}_1 + \mathfrak{m}_2)L_1^2 + 2\mathfrak{m}_2L_2^2$$

- The 1-norm of $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$: $\left\|\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\right\|_1 = \left|\mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\right| + \left|\mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2\right|$

$$\leq \mathfrak{m}_2L_1L_2\left(|\dot{\theta}_1| + |\dot{\theta}_2|\right)^2 \equiv c_b\left\|\dot{\boldsymbol{\theta}}\right\|_1^2$$

$$\left\|\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\right\| \leq c_b\left\|\dot{\boldsymbol{\theta}}\right\|^2$$

$$c_b = \mathfrak{m}_2L_1L_2$$

- The 1-norm of $\boldsymbol{g}(\boldsymbol{\theta})$: $\|\boldsymbol{g}(\boldsymbol{\theta})\|_1 = |(\mathfrak{m}_1 + \mathfrak{m}_2)L_1g\cos\theta_1 + \mathfrak{m}_2gL_2\cos(\theta_1 + \theta_2)|$

$$+|\mathfrak{m}_2gL_2\cos(\theta_1 + \theta_2)|$$
$$\leq (\mathfrak{m}_1 + \mathfrak{m}_2)gL_1 + 2\mathfrak{m}_2gL_2 \equiv g_b$$

$$\|\boldsymbol{g}(\boldsymbol{\theta})\| \leq g_b$$

Example (cont.)

- We can find the standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ where $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$ as:

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = 1/2\left(\dot{\boldsymbol{M}} + \boldsymbol{U}^T - \boldsymbol{U}\right) = \begin{bmatrix} -\dot{\theta}_2\mathfrak{m}_2L_1L_2\sin\theta_2 & -\left(\dot{\theta}_1+\dot{\theta}_2\right)\mathfrak{m}_2L_1L_2\sin\theta_2 \\ \dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2 & 0 \end{bmatrix}$$

where $\boldsymbol{U}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \left(\boldsymbol{I}_n \otimes \dot{\boldsymbol{\theta}}^T\right)\dfrac{\partial \boldsymbol{M}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} 0 & 0 \\ -\left(2\dot{\theta}_1+\dot{\theta}_2\right)\mathfrak{m}_2L_1L_2\sin\theta_2 & -\dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2 \end{bmatrix}.$

- Two other choices of $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ are

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{M}} - 1/2\boldsymbol{U} = \begin{bmatrix} -2\dot{\theta}_2\mathfrak{m}_2L_1L_2\sin\theta_2 & -\dot{\theta}_2\mathfrak{m}_2L_1L_2\sin\theta_2 \\ \left(\dot{\theta}_1 - 1/2\dot{\theta}_2\right)\mathfrak{m}_2L_1L_2\sin\theta_2 & 1/2\dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2 \end{bmatrix}$$

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{U}^T - 1/2\boldsymbol{U} = \begin{bmatrix} 0 & -\left(2\dot{\theta}_1+\dot{\theta}_2\right)\mathfrak{m}_2L_1L_2\sin\theta_2 \\ \left(\dot{\theta}_1 + 1/2\dot{\theta}_2\right)\mathfrak{m}_2L_1L_2\sin\theta_2 & 1/2\dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2 \end{bmatrix}$$

Example (cont.)

Matrix of Christoffel symbols of the first kind $\boldsymbol{\Gamma}(\boldsymbol{\theta})$:

$$\boldsymbol{\Gamma}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\Gamma}_1(\boldsymbol{\theta}) \\ \boldsymbol{\Gamma}_2(\boldsymbol{\theta}) \end{bmatrix} \qquad \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{\theta}}^{\mathrm{T}} \boldsymbol{\Gamma}(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}} = \begin{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \overbrace{\begin{bmatrix} 0 & -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 \\ -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 & -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 \end{bmatrix}}^{\boldsymbol{\Gamma}_1(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \underbrace{\begin{bmatrix} \mathfrak{m}_2 L_1 L_2 \sin\theta_2 & 0 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\Gamma}_2(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \end{bmatrix}$$

Using $\boldsymbol{\Gamma}_1(\boldsymbol{\theta})$ and $\boldsymbol{\Gamma}_2(\boldsymbol{\theta})$, we can find c_b in $\left\|\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\right\| \leq c_b \left\|\dot{\boldsymbol{\theta}}\right\|^{\mathbf{2}}$ when $\|\cdot\|$ is 2-norm by

$$c_b = n^2 \left(\max_{k,i,j,\boldsymbol{\theta}} \left| \Gamma_{k_{ij}}(\boldsymbol{\theta}) \right| \right) \qquad \begin{aligned} &\max_{\theta} \left|\Gamma_{1_{11}}(\boldsymbol{\theta})\right| = 0\,, && \max_{\theta} \left|\Gamma_{2_{11}}(\boldsymbol{\theta})\right| = \mathfrak{m}_2 L_1 L_2 \\ &\max_{\theta} \left|\Gamma_{1_{12}}(\boldsymbol{\theta})\right| = \mathfrak{m}_2 L_1 L_2\,, && \max_{\theta} \left|\Gamma_{2_{12}}(\boldsymbol{\theta})\right| = 0 \\ &\max_{\theta} \left|\Gamma_{1_{21}}(\boldsymbol{\theta})\right| = \mathfrak{m}_2 L_1 L_2\,, && \max_{\theta} \left|\Gamma_{2_{21}}(\boldsymbol{\theta})\right| = 0 \\ &\max_{\theta} \left|\Gamma_{1_{22}}(\boldsymbol{\theta})\right| = \mathfrak{m}_2 L_1 L_2\,, && \max_{\theta} \left|\Gamma_{2_{22}}(\boldsymbol{\theta})\right| = 0. \end{aligned}$$

$$\Rightarrow c_b = 4\mathfrak{m}_2 L_1 L_2$$

Linearity in Dynamic Parameters

An important property of the dynamic model of an open-chain manipulator is the linearity with respect to a suitable set of parameters $\boldsymbol{\pi} \in \mathbb{R}^p$, including dynamic parameters (mass m_i, first moment of inertia $m_i l_{C_x,i}, m_i l_{C_y,i}, m_i l_{C_z,i}$, the six components of inertia matrix $\boldsymbol{I}_{b,i}$) and friction parameters ($F_{v,i}$, $F_{s,i}$) as

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s\,\mathbf{sgn}(\dot{\boldsymbol{\theta}}) = \boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})\boldsymbol{\pi}$$

$\boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \in \mathbb{R}^{n\times p}$ is called **regressor**.

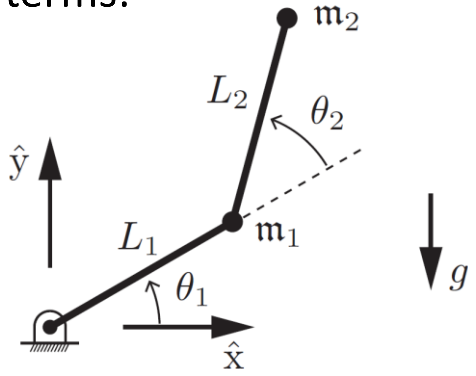
- This property is useful in **Adaptive Control**, where some or all the parameters maybe unknown.
- Note that $p \leq 12n$, since not all the dynamic/friction parameters appear in dynamic equations or are unknown.

Example

Dynamic equations of a planar 2R open chain in presence of friction terms:

$$\begin{aligned}\tau_1 = &\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right)\ddot{\theta}_1 \\ &+\mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_2 - \mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ &+(\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) + F_{v,1}\dot{\theta}_1 + F_{s,1}\,\mathrm{sgn}\,\dot{\theta}_1\,,\end{aligned}$$

$$\begin{aligned}\tau_2 = &\,\mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_1 + \mathfrak{m}_2L_2^2\ddot{\theta}_2 + \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2 \\ &+\mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) + F_{v,2}\dot{\theta}_2 + F_{s,2}\,\mathrm{sgn}\,\dot{\theta}_2\,.\end{aligned}$$



If the set of unknow parameters $\boldsymbol{\pi}$ is defined as $\boldsymbol{\pi} = \left[\mathfrak{m}_1, \mathfrak{m}_2, F_{s,1}, F_{v,1}, F_{s,2}, F_{v,2}\right]^T$,

Find $\boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$ where $\boldsymbol{\tau} = \boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})\boldsymbol{\pi}$.

Example

We can find $\boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$ as

$$\boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & 0 & 0 \\ 0 & Y_{22} & 0 & 0 & Y_{25} & Y_{26} \end{bmatrix}$$

$$\begin{aligned}
Y_{11} &= L_1^2\ddot{\theta}_1 + gL_1\cos\theta_1 \\
Y_{12} &= [L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2]\ddot{\theta}_1 + [L_2^2 + L_1L_2\cos\theta_2]\ddot{\theta}_2 \\
&\quad -L_1L_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 + gL_1\cos\theta_1 + gL_2\cos(\theta_1 + \theta_2) \\
Y_{13} &= \operatorname{sgn}(\dot{\theta}_1) \\
Y_{14} &= \dot{\theta}_1 \\
Y_{22} &= [L_2^2 + L_1L_2\cos\theta_2]\ddot{\theta}_1 + L_2^2\ddot{\theta}_2 + L_1L_2\dot{\theta}_1^2\sin\theta_2 + gL_2\cos(\theta_1 + \theta_2) \\
Y_{25} &= \operatorname{sgn}(\dot{\theta}_2) \\
Y_{26} &= \dot{\theta}_2
\end{aligned}$$

Dynamics in Task Space

Dynamics in Task Space

- For Geometric Representation of end-effector frame, i.e., $\boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$:

Joint-space Dynamics:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}\big(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\big) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

$$\begin{cases} \boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} & \text{(twist of the end-effector)} \\ \dot{\boldsymbol{\mathcal{V}}} = \dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{J}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} \end{cases}$$

Assumption:
$\boldsymbol{J}(\boldsymbol{\theta})$ is invertible

$$\longrightarrow \begin{cases} \dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}} \\ \ddot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{\mathcal{V}}} - \boldsymbol{J}^{-1}\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}} \end{cases}$$

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\big(\boldsymbol{J}^{-1}\dot{\boldsymbol{\mathcal{V}}} - \boldsymbol{J}^{-1}\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}\big) + \boldsymbol{c}(\boldsymbol{\theta},\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

Pre-multiply both sides by $\boldsymbol{J}^{-\mathrm{T}}$

$$\xrightarrow[\boldsymbol{\mathcal{F}} = \boldsymbol{J}^{-\mathrm{T}}\boldsymbol{\tau}]{}$$

$$\boldsymbol{\mathcal{F}} = \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{c}_C(\boldsymbol{\theta},\boldsymbol{\mathcal{V}}) + \boldsymbol{g}_C(\boldsymbol{\theta}) + \boldsymbol{\mathcal{F}}_{\mathrm{tip}} \quad \text{(Task-space Dynamics)}$$

$$\boldsymbol{M}_C(\boldsymbol{\theta}) = \boldsymbol{J}^{-\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{J}^{-1},$$

$$\boldsymbol{c}_C(\boldsymbol{\theta},\boldsymbol{\mathcal{V}}) = \boldsymbol{J}^{-\mathrm{T}}\boldsymbol{c}(\boldsymbol{\theta},\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}) - \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}, \qquad \boldsymbol{g}_C(\boldsymbol{\theta}) = \boldsymbol{J}^{-\mathrm{T}}\boldsymbol{g}(\boldsymbol{\theta})$$

Dynamics in Task Space

By considering $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$:

$$\begin{aligned}\boldsymbol{c}_C(\boldsymbol{\theta},\boldsymbol{\mathcal{V}}) &= \boldsymbol{J}^{-\mathrm{T}}\boldsymbol{c}(\boldsymbol{\theta},\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}) - \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}} \\ &= \boldsymbol{J}^{-\mathrm{T}}\boldsymbol{C}(\boldsymbol{\theta},\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}})\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}} - \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}} \\ &= \left(\boldsymbol{J}^{-\mathrm{T}}\boldsymbol{C}(\boldsymbol{\theta},\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}) - \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{J}}\right)\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}} \\ &= \boldsymbol{C}_C(\boldsymbol{\theta},\boldsymbol{\mathcal{V}})\boldsymbol{\mathcal{V}}\end{aligned}$$

Note: In general, we cannot replace the dependence on $\boldsymbol{\theta}$ by a dependence on the end-effector configuration $\boldsymbol{T}$ because there may be multiple solutions to the inverse kinematics, and the dynamics depends on the specific joint configuration $\boldsymbol{\theta}$.

$$\boldsymbol{J}(\boldsymbol{\theta}) = [\boldsymbol{J}_1(\boldsymbol{\theta}),\dots,\boldsymbol{J}_n(\boldsymbol{\theta})]$$

Note: For finding $\dot{\boldsymbol{J}}(\boldsymbol{\theta})$, let $\boldsymbol{J}_i(\boldsymbol{\theta})$ denote the ith column of $\boldsymbol{J}(\boldsymbol{\theta}) = [\boldsymbol{J}_1(\boldsymbol{\theta}),\dots,\boldsymbol{J}_n(\boldsymbol{\theta})]$, thus:

$$\dot{\boldsymbol{J}}(\boldsymbol{\theta}) = \frac{d}{dt}\boldsymbol{J}(\boldsymbol{\theta}) = \left[\frac{d}{dt}\boldsymbol{J}_1(\boldsymbol{\theta}),\dots,\frac{d}{dt}\boldsymbol{J}_n(\boldsymbol{\theta})\right] \quad \text{where} \quad \frac{d}{dt}\boldsymbol{J}_i(\boldsymbol{\theta}) = \sum_{j=1}^{n}\frac{\partial \boldsymbol{J}_i}{\partial \theta_j}\dot{\theta}_j$$

- If $\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{J}_s(\boldsymbol{\theta})$:

$$\frac{\partial \boldsymbol{J}_i}{\partial \theta_j} = \begin{cases}\left[\mathrm{ad}_{\boldsymbol{J}_j}\right]\boldsymbol{J}_i & i > j \\ \boldsymbol{0} & i \le j\end{cases}$$

- If $\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{J}_b(\boldsymbol{\theta})$:

$$\frac{\partial \boldsymbol{J}_i}{\partial \theta_j} = \begin{cases}\left[\mathrm{ad}_{\boldsymbol{J}_i}\right]\boldsymbol{J}_j & i < j \\ \boldsymbol{0} & i \ge j\end{cases}$$

Dynamics in Task Space

- For Minimum-Coordinate Representation of end-effector frame, i.e., $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$:

In a similar way, by using $\boldsymbol{J}_a(\boldsymbol{\theta})$ where $\dot{\boldsymbol{x}} = \boldsymbol{J}_a(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and $\boldsymbol{\tau} = \boldsymbol{J}_a^T(\boldsymbol{\theta})\boldsymbol{F}$, dynamic equation in T-space can be written as

$$\boldsymbol{F} = \boldsymbol{M}_C(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{c}_C(\boldsymbol{\theta},\dot{\boldsymbol{x}}) + \boldsymbol{g}_C(\boldsymbol{\theta}) + \boldsymbol{F}_{\text{tip}} = \boldsymbol{M}_C(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{C}_C(\boldsymbol{\theta},\dot{\boldsymbol{x}})\dot{\boldsymbol{x}} + \boldsymbol{g}_C(\boldsymbol{\theta}) + \boldsymbol{F}_{\text{tip}}$$

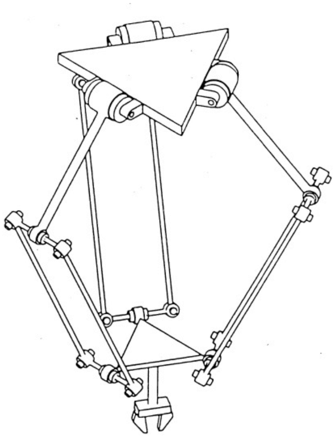
Note: All the properties of the J-space dynamic model carry over to the T-space dynamic model as long as $\boldsymbol{J}$ (or $\boldsymbol{J}_a$) is nonsingular. For instance,

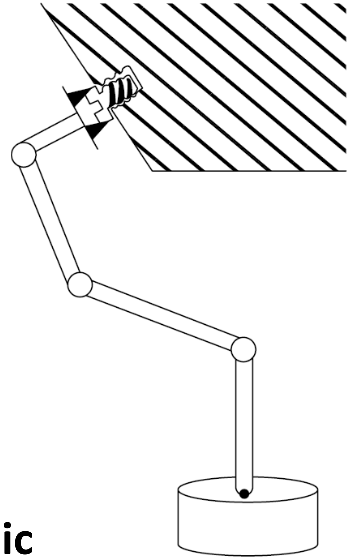
- $\boldsymbol{M}_C$ is symmetric and positive definite.
- $\boldsymbol{S}_c = \dot{\boldsymbol{M}}_C - 2\boldsymbol{C}_C$ is skew-symmetric (if $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ is in standard form).
- Property of linearity in the parameters: $\boldsymbol{F} = \boldsymbol{M}_C(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{c}_C(\boldsymbol{\theta},\dot{\boldsymbol{x}}) + \boldsymbol{g}_C(\boldsymbol{\theta}) = \overbrace{\boldsymbol{J}^{-\mathrm{T}}\boldsymbol{Y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}})}^{\boldsymbol{Y}_C}\boldsymbol{\pi}$

Constrained Dynamics

Constrained Dynamics

Sometimes robots are subject to a set of constraints on their motion.





Consider the case where the n-joint robot is subject to a set of k **holonomic constraints** or **nonholonomic Pfaffian velocity constraints** of the form:

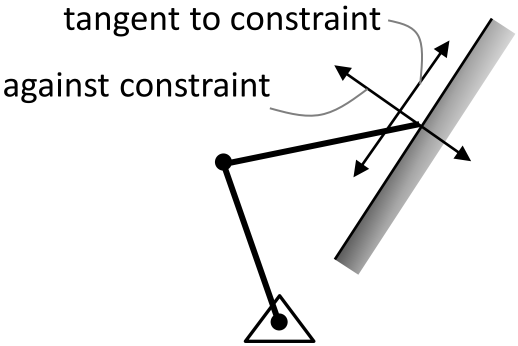
$$\boldsymbol{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0}, \qquad \boldsymbol{A}(\boldsymbol{\theta}) \in \mathbb{R}^{k\times n}$$

Assumption: These k **equality constraints** are **workless**, meaning that the forces that enforce these constraints do no work on the robot.

Constrained Dynamics

Space of joint torques/forces $\boldsymbol{\tau}$ is divided into (I) an $(n-k)$-dimensional subspace that affects the motion of the robot, but not the constraint force (tangent to constraint) and (II) a k-dimensional subspace that affects the constraint force, but not the motion (against the constraints).

$$\boldsymbol{\tau} = \underbrace{\boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}_{\text{For moving the robot tangent to constraint}} + \underbrace{\boldsymbol{\tau}_{\text{con}}}_{\text{For acting against the constraints}}$$



Workless Constraints Assumption: → $\boldsymbol{\tau}_{\text{con}}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = 0$, $\mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0}$ ⇨ $\boldsymbol{\tau}_{\text{con}}$ is a linear combination of the columns of $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})$ ⇨ $\boldsymbol{\tau}_{\text{con}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda}$

$\boldsymbol{\lambda} \in \mathbb{R}^k$

($\boldsymbol{\lambda}$: Lagrange multipliers)

Constrained Equations of Motion

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda} \quad (1)$$

$$\mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0} \xrightarrow{\text{(or)}} \dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \mathbf{0} \quad (2)$$

⇨ $n+k$ equations, $n+k$ variables (for ID: $\boldsymbol{\lambda},\boldsymbol{\tau}$, for FD: $\boldsymbol{\lambda},\ddot{\boldsymbol{\theta}}$)

Constrained Dynamics

Thus, the robot has $n-k$ velocity freedoms and k force freedoms.

For finding $\boldsymbol{\lambda}$: (1) $\rightarrow$ $\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{h} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}\right)$ (3)

(2), (3) $\rightarrow$ $\boldsymbol{\lambda} = \left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{A}\boldsymbol{M}^{-1}(\boldsymbol{\tau}-\boldsymbol{h}) + \dot{\boldsymbol{A}}\dot{\boldsymbol{\theta}}\right)$ (4)

By eliminating $\boldsymbol{\lambda}$ in (1) using (4), $n+k$ constrained equations of motion can be reduce to the dynamics projected to the $(n-k)$-dimensional space tangent to the constraints, i.e., (5):

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda} \\ \mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0} \overset{\text{(or)}}{\rightarrow} \dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \mathbf{0} \end{cases} \Rightarrow$$

$$\boldsymbol{P}\boldsymbol{\tau} = \boldsymbol{P}\left(\boldsymbol{M}\ddot{\boldsymbol{\theta}} + \boldsymbol{h}\right) \quad (5)$$

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{A}\boldsymbol{M}^{-1} \in \mathbb{R}^{n\times n}$$

$\boldsymbol{I}$: $n\times n$ identity matrix $\quad \operatorname{rank}(\boldsymbol{P}) = n-k$

Note: If the constraint acts at the end-effector of an open-chain robot, then

$$\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda}$$

$\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$: wrench the end-effector applies to the constraint

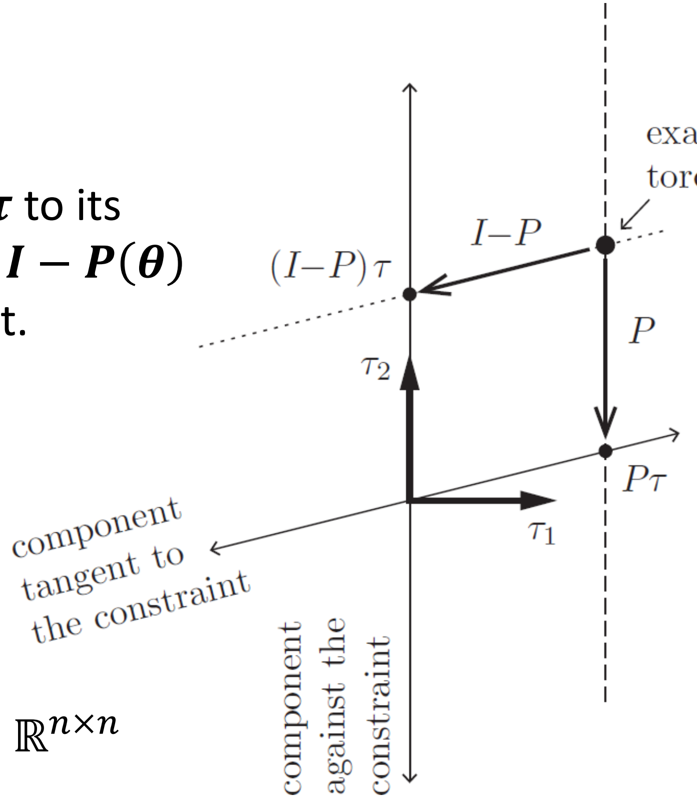
If $\boldsymbol{J}\ (\boldsymbol{\theta})$ is invertible: $\boldsymbol{\mathcal{F}}_{\mathrm{tip}} = \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda}$

Constrained Dynamics

Therefore,

$$\boldsymbol{\tau} = \underbrace{\boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}_{\text{For moving the robot tangent to constraint}} + \underbrace{\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda}}_{\text{For acting against the constraints } (\boldsymbol{\tau}_{\text{con}})} \qquad \equiv \qquad \boldsymbol{\tau} = \boldsymbol{P}(\theta)\boldsymbol{\tau} + \big(\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{\theta})\big)\boldsymbol{\tau}$$

Matrix $\boldsymbol{P}(\theta)$ is a **projection matrix** that projects $\boldsymbol{\tau}$ to its component tangent to the constraint, the matrix $\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{\theta})$ projects $\boldsymbol{\tau}$ to its component against the constraint.


example joint torques τ
$I-P$
$(I-P)\tau$
P
τ_2
$P\tau$
τ_1
component tangent to the constraint
component against the constraint

Note: (5) can be also written as

$$\boldsymbol{P}_{\ddot{\theta}}\ddot{\boldsymbol{\theta}} = \boldsymbol{P}_{\ddot{\theta}}\boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{h})$$

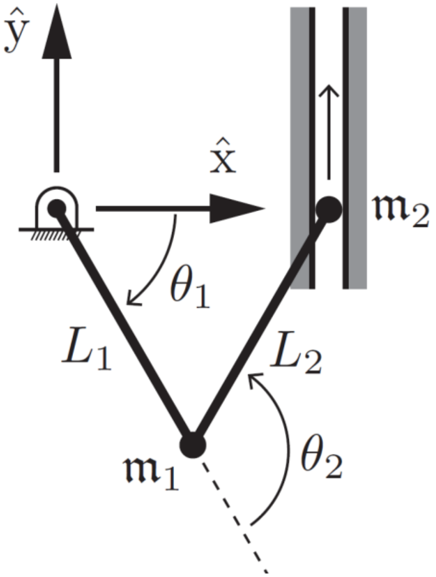
$$\boldsymbol{P}_{\ddot{\theta}} = \boldsymbol{M}^{-1}\boldsymbol{P}\boldsymbol{M} = \boldsymbol{I} - \boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{A} \in \mathbb{R}^{n\times n}$$

$$\operatorname{rank}(\boldsymbol{P}_{\ddot{\theta}}) = n - k$$

Example

Consider a 2R robot whose tip is constrained to move in a frictionless linear channel at $x = 1$. The lengths of each link are $L_1 = L_2 = 1$, and the point masses at the ends of each link are $\mathfrak{m}_1 = \mathfrak{m}_2 = 1$. For simplicity, assume that $g = 0$. Consider the case where $(\theta_1, \theta_2) = (-\pi/3, 2\pi/3)$ as shown, and the tip is moving with the velocity $(\dot{x}, \dot{y}) = (0,1)$ m/s at the current configuration.

- Solve the constrained forward dynamics for $\ddot{\boldsymbol{\theta}} = (\ddot{\theta}_1, \ddot{\theta}_2)$ and λ when $\boldsymbol{\tau} = (\tau_1, \tau_2)$.
- Find the task-space constraint force $\boldsymbol{f}_{\text{tip}} = \left(f_x, f_y\right)$.
- Solve the constrained inverse dynamics for $\boldsymbol{\tau}$ given a $\ddot{\boldsymbol{\theta}}$ satisfying the constraint (i.e., $\dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \mathbf{0}$) and λ satisfying a desired force $\boldsymbol{f}_{\text{tip}} = \left(f_x, f_y\right) = (f, 0)$ against the channel.
- Find the projection $\boldsymbol{P}$.



$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix}$$

$$\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} \qquad \boldsymbol{g}(\boldsymbol{\theta}) = \mathbf{0}$$

Solution

There are $n = 2$ joint coordinates and $k = 1$ constraint.

If the tip of the robot is at (x, y), the robot's forward kinematics can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{c}_1 + \mathrm{c}_{12} \\ \mathrm{s}_1 + \mathrm{s}_{12} \end{bmatrix} \longrightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -\mathrm{s}_1 - \mathrm{s}_{12} & -\mathrm{s}_{12} \\ \mathrm{c}_1 + \mathrm{c}_{12} & \mathrm{c}_{12} \end{bmatrix}}_{\boldsymbol{J}(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \underbrace{\begin{bmatrix} -\dot{\theta}_1\mathrm{c}_1 - (\dot{\theta}_1 + \dot{\theta}_2)\mathrm{c}_{12} & -(\dot{\theta}_1 + \dot{\theta}_2)\mathrm{c}_{12} \\ -\dot{\theta}_1\mathrm{s}_1 - (\dot{\theta}_1 + \dot{\theta}_2)\mathrm{s}_{12} & -(\dot{\theta}_1 + \dot{\theta}_2)\mathrm{s}_{12} \end{bmatrix}}_{\dot{\boldsymbol{J}}(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The constraint is $\mathrm{c}_1 + \mathrm{c}_{12} = 1$. It is holonomic and can be written as

$$\underbrace{\begin{bmatrix} -\mathrm{s}_1 - \mathrm{s}_{12} & -\mathrm{s}_{12} \end{bmatrix}}_{\boldsymbol{A}(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{\boldsymbol{A}}(\boldsymbol{\theta}) = \begin{bmatrix} -\dot{\theta}_1\mathrm{c}_1 - (\dot{\theta}_1 + \dot{\theta}_2)\mathrm{c}_{12} & -(\dot{\theta}_1 + \dot{\theta}_2)\mathrm{c}_{12} \end{bmatrix}$$

Solution

$(\theta_1, \theta_2) = (-\pi/3, 2\pi/3)$ and $(\dot{x}, \dot{y}) = (0,1)$ m/s imply that $(\dot{\theta}_1, \dot{\theta}_2) = (1,0)$ rad/s. Thus,

$$\boldsymbol{A}(\boldsymbol{\theta}) = [0 \quad -0.866] \qquad \dot{\boldsymbol{A}}(\boldsymbol{\theta}) = [-1 \quad -0.5]$$

$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \qquad \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} 0 \\ 0.866 \end{bmatrix}$$

Constrained forward dynamics:

$$\lambda = \left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{A}\boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{h}) + \dot{\boldsymbol{A}}\dot{\boldsymbol{\theta}}\right) \longrightarrow \lambda = 0.289\tau_1 - 1.155\tau_2 - 0.167$$

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{h} - \boldsymbol{A}^{\mathrm{T}}\lambda\right) \longrightarrow \begin{aligned} \ddot{\theta}_1 &= 0.5\tau_1 + 0.289 \\ \ddot{\theta}_2 &= -1.155 \end{aligned}$$

Solution

Task-space constraint forces: $\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{f}_{\mathrm{tip}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda$

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & -0.866 \\ 1 & 0.5 \end{bmatrix}$$

Since $\boldsymbol{J}\ (\boldsymbol{\theta})$ is invertible: $\boldsymbol{f}_{\mathrm{tip}} = \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda$

$$\boldsymbol{f}_{\mathrm{tip}} = \begin{bmatrix} 0.577 & -1.155 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -0.866 \end{bmatrix}\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda = \begin{bmatrix} 0.289\tau_1 - 1.155\tau_2 - 0.167 \\ 0 \end{bmatrix}$$

This agrees with our understanding that the robot can only apply forces against the constraint in the f_x-direction.

Note: If $\boldsymbol{\tau} = \mathbf{0}$, the task-space constraint force is $\boldsymbol{f}_{\mathrm{tip}} = (-0.167, 0)$, meaning that the robot's tip pushes to the left on the constraint while the constraint pushes back equally to the right to enforce the constraint. In the absence of the constraint, the acceleration of the tip of the robot would have a component to the left.

Solution

Constrained inverse dynamics:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda$$

$$\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{f}_{\mathrm{tip}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda \longrightarrow \begin{bmatrix}0 & -0.866\\ 1 & 0.5\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}f\\ 0\end{bmatrix} = \begin{bmatrix}0\\ -0.866\end{bmatrix}\lambda \longrightarrow f = \lambda$$

$$\dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \boldsymbol{0} \longrightarrow \ddot{\theta}_2 = -1.155,\ \ \forall\ddot{\theta}_1 \longrightarrow \ddot{\boldsymbol{\theta}} = \left(\ddot{\theta}_1, -1.155\right)$$

$$\Rightarrow \quad \begin{aligned}\tau_1 &= 2\ddot{\theta}_1 - 0.578\\ \tau_2 &= 0.5\ddot{\theta}_1 - 0.866f - 0.289\end{aligned}$$

Projection $\boldsymbol{P}$:

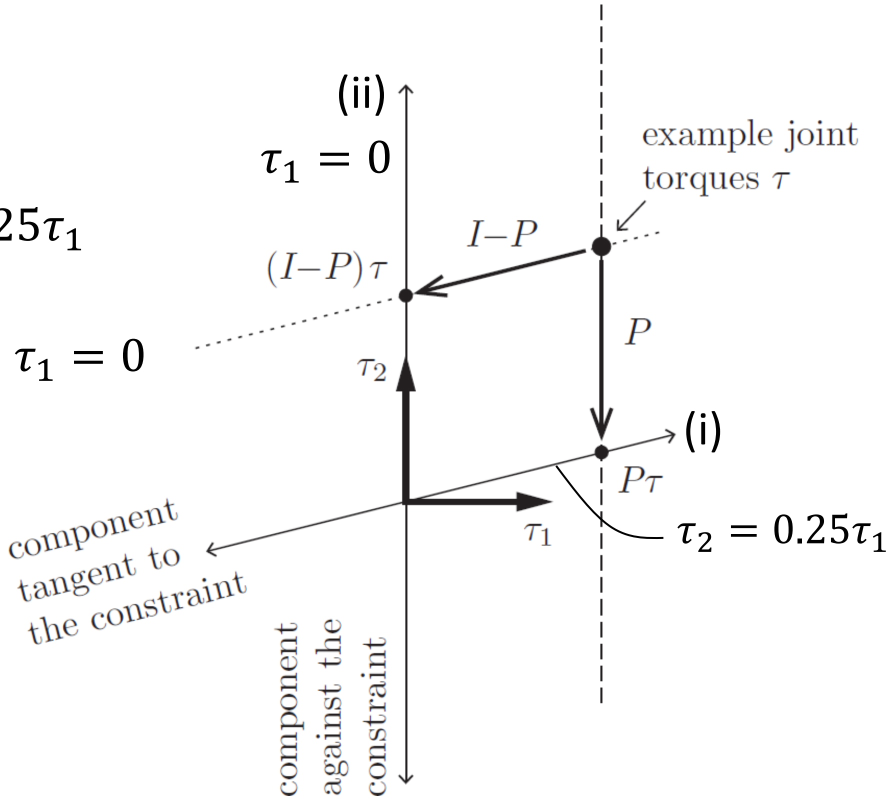
$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{A}\boldsymbol{M}^{-1} \in \mathbb{R}^{n\times n} \longrightarrow \boldsymbol{P} = \begin{bmatrix}1 & 0\\ 0.25 & 0\end{bmatrix}, \qquad \boldsymbol{I} - \boldsymbol{P} = \begin{bmatrix}0 & 0\\ -0.25 & 1\end{bmatrix}$$

Solution

$$\boldsymbol{\tau} = \boldsymbol{P}(\theta)\boldsymbol{\tau} + \left(\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{\theta})\right)\boldsymbol{\tau}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0.25 & 0\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}\tau_1\\ 0.25\tau_1\end{bmatrix} \rightarrow \tau_2 = 0.25\tau_1$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ -0.25 & 1\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}0\\ -0.25\tau_1 + \tau_2\end{bmatrix} \rightarrow \tau_1 = 0$$


(ii)
$\tau_1 = 0$
example joint torques τ
$I{-}P$
$(I{-}P)\tau$
P
τ_2
(i)
$P\tau$
τ_1
$\tau_2 = 0.25\tau_1$
component tangent to the constraint
component against the constraint

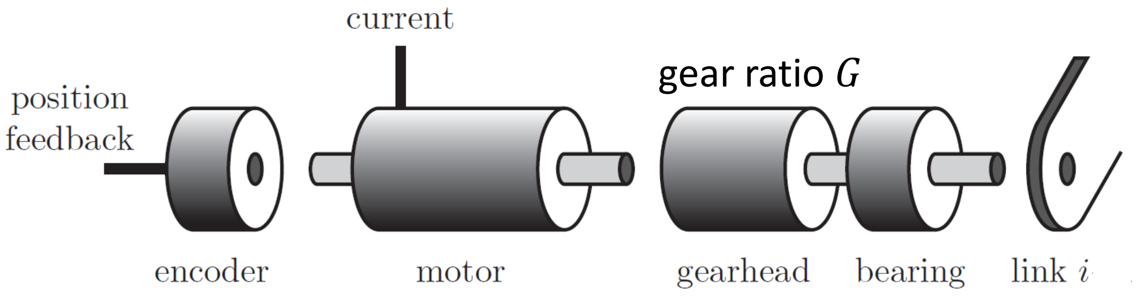
At the current state, (i) joint torques lying in the one-dimensional subspace $\tau_2 = 0.25\tau_1$ do not affect the constraint force, and (ii) joint torques lying in the one-dimensional subspace $\tau_1 = 0$ do not affect the motion of the robot.

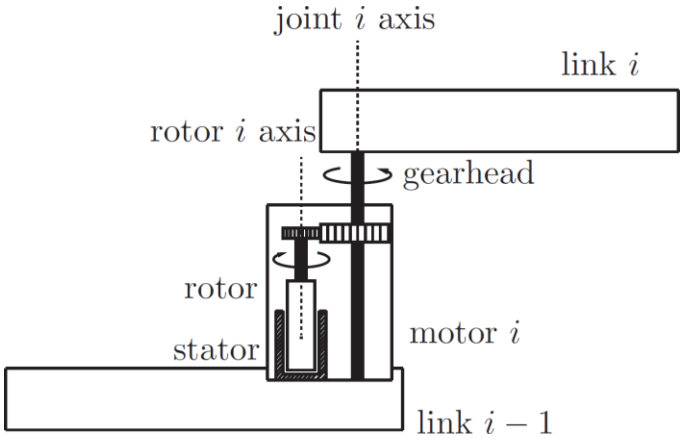
Miscellaneous Topics

Actuation and Gearing

In practice, there are many **types of actuators** (e.g., electric, hydraulic, and pneumatic) and **mechanical power transformers** (e.g., gearheads), and the actuators can be located at the joints themselves or remotely, with **mechanical power transmitted** by cables or timing belts.

Each combination of these has its own characteristics that can play a significant role in the "extended dynamics" mapping the actual control inputs to the motion of the robot.


current
position feedback
gear ratio G
encoder
motor
gearhead
bearing
link i

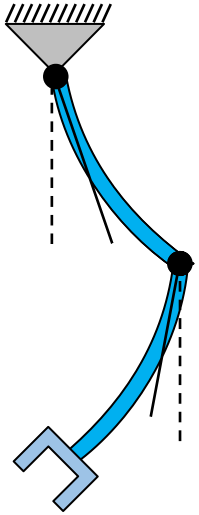

joint i axis
link i
rotor i axis
gearhead
rotor
stator
motor i
link $i-1$

- stator attached to link $i-1$
- rotor attached to link i through gearhead

Joint and Link Flexibility

In practice, a robot’s joints and links are likely to exhibit some flexibility and vibrations.

Flexible joints and links introduce extra states to the dynamics of the robot, significantly complicating the dynamics and control.



Dynamic Parameter Identification

Using the dynamic equations of the manipulators for simulation and control purposes needs a good knowledge of dynamic parameters (e.g., m_i,$m_i l_{C_x,i}$, $m_i l_{C_y,i}$, $m_i l_{C_z,i}$, $\boldsymbol{I}_{b,i}$, $F_{v,i}$, $F_{s,i}$).

The objective of **Dynamic Parametric Identification** is to obtain the numerical values of different dynamic parameters.

Different methods for parameter identification:

- **Using CAD models** (→ inaccurate due to simplification, unable to find friction parameters)
- **Dismantling components** of the manipulator and perform measurements to find the dynamic parameters (→ not easy to implement)
- **Using identification techniques** which exploit the **property of linearity** of the dynamic model of open-chain manipulators. (→ accurate)

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s \,\mathbf{sgn}\left(\dot{\boldsymbol{\theta}}\right) = \boldsymbol{Y}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}\right)\boldsymbol{\pi}$$

$$\boldsymbol{\pi} \in \mathbb{R}^p \qquad \boldsymbol{Y}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n\times p}$$

Dynamic Parameter Identification

In these techniques, we impose a suitable motion trajectory, and compute the parameter vector $\boldsymbol{\pi}$ from the measurements of joint torques $\boldsymbol{\tau}$ and evaluation of the matrix $\boldsymbol{Y}$ at some time instants $t_1, \dots, t_N$ along the trajectory (typically, $Nn \gg p$).

$$\bar{\boldsymbol{\tau}} = \begin{bmatrix} \boldsymbol{\tau}(t_1) \\ \vdots \\ \boldsymbol{\tau}(t_N) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}(t_1) \\ \vdots \\ \boldsymbol{Y}(t_N) \end{bmatrix} \boldsymbol{\pi} = \bar{\boldsymbol{Y}} \boldsymbol{\pi}$$

$\boldsymbol{\pi} \in \mathbb{R}^p$

$\bar{\boldsymbol{\tau}} \in \mathbb{R}^{Nn}$

$\bar{\boldsymbol{Y}} \in \mathbb{R}^{Nn \times p}$

By a least-squares technique:

$$\boldsymbol{\pi} = \underbrace{(\bar{\boldsymbol{Y}}^T \bar{\boldsymbol{Y}})^{-1} \bar{\boldsymbol{Y}}^T}_{\text{left pseudo-inverse of } \bar{\boldsymbol{Y}}} \bar{\boldsymbol{\tau}}$$

Remarks:

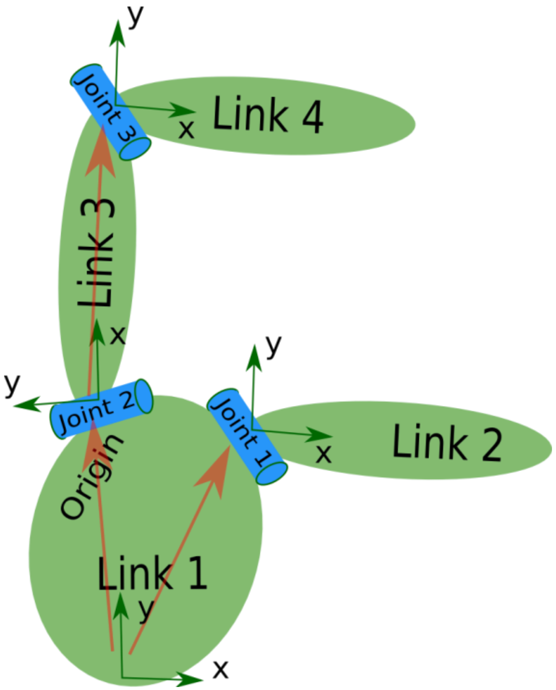
- Assumption: Kinematic parameters in $\boldsymbol{Y}$ are known with good accuracy, e.g., after kinematic calibration.
- $\ddot{\boldsymbol{q}}$ needs to be calculated using measurements of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$.
- It is possible to identify only the dynamic parameters of the manipulator that contribute to the dynamic model.
- Some parameters can be identified in linear combinations whenever they do not appear isolated in the equations.
- Trajectories should not excite any unmodelled dynamic effects such as joint elasticity or link flexibility.
- The technique can be extended to the parameter identification of an unknown payload at the end-effector.

Universal Robot Description Format

The Universal Robot Description Format (URDF) is an XML (eXtensible Markup Language) file format used by the Robot Operating System (ROS) to describe the kinematics (in defining joints), inertial properties, and link geometry of robots (in defining links) of open-chain robots.

```
<joint name="joint1" type="continuous">
        <parent link="link1"/>
        <child link="link2"/>
        <origin xyz=“0.5 0.3 0" rpy="0 0 0" />
        <axis xyz="-0.9 0.15 0" />
</joint>

<link name="link1">
        <inertial>
                <mass value=“1"/>
                <origin rpy="0.1 0 0" xyz="0 0 0"/>
                <inertia ixx="0.004" ixy="0" ixz="0"
                        iyy="0.004" iyz="0" izz="0.007"/>
        </inertial>
</link>
```


y
Joint 3
x
Link 4
Link 3
x
y
y
Joint 2
Joint 1
x
Link 2
Origin
Link 1
y
x

Link

URDF: Defining Joints

Joints connect two links: a parent link and a child link. The reference frame of each (child) link $\{L_i\}$ is located (at the bottom of the link) on the joint's axis.

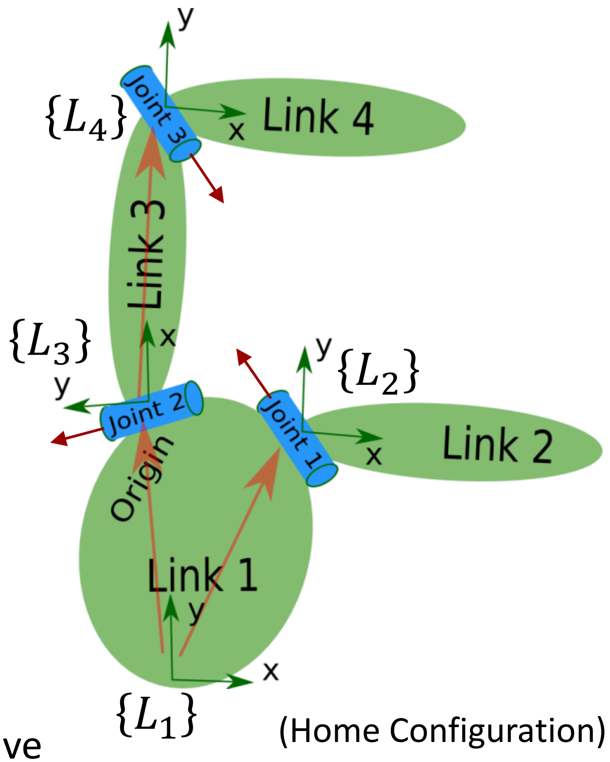
```
<joint name="joint3" type="continuous">
        <parent link="link3"/>
        <child link="link4"/>
        <origin xyz=“0.5 0 0" rpy="0 0 -1.57" />
        <axis xyz="0.707 -0.707 0" />
</joint>
```

: $\{L_4\}$ w.r.t. $\{L_3\}$

: in $\{L_4\}$

“origin” frame defines the pose of the child link frame relative to the parent link frame when the joint variable is zero.

“axis” defines the joint’s axis, a unit vector expressed in the child link’s frame, in the direction of positive rotation for a revolute joint or positive translation for a prismatic joint.



(Home Configuration)

URDF: Defining Links

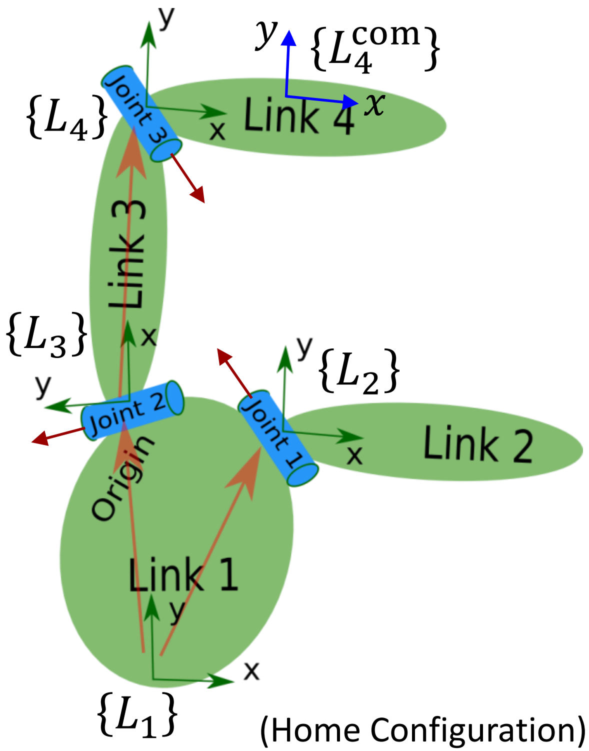
```
<link name=" link4">
        <inertial>
                <mass value=“1"/>
                <origin xyz=“0.1 0 0" rpy="0 0 0"/>
                <inertia ixx="0.004" ixy="0" ixz="0"
                        iyy="0.004" iyz="0" izz="0.007"/>
        </inertial>
        <visual>
                <geometry>
                        <mesh filename=“.../link1.stl" />
                </geometry>
                <material name="DarkGrey">
                        <color rgba="0.3 0.3 0.3 1.0"/>
                </material>
        </visual>
</link>
```

$: \{L_4^{\text{com}}\}$ w.r.t. $\{L_4\}$

$:$ in $\{L_4^{\text{com}}\}$

“inertia” six elements of inertia matrix relative to the link’s center of mass.

“origin” frame defines the position and orientation of a frame at the link’s center of mass relative to the link’s frame at its joint.


y
y
$\{L_4^{\text{com}}\}$
$\{L_4\}$
Joint 3
x
Link 4
x
Link 3
$\{L_3\}$
x
y
Joint 2
y
$\{L_2\}$
Joint 1
x
Link 2
Origin
Link 1
y
x
$\{L_1\}$

(Home Configuration)