

MEC529: Introduction to Robotics (Theory and Applications)

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Ch1: Motivational Example

Linear Algebra

Vector

A **coordinate free vector** is a geometric quantity with a length and a direction.

Given a reference frame, \mathbf{v} can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then, the vector \mathbf{v} can be represented by its coordinates \boldsymbol{v} in the reference frame.

- \mathbf{v} refers to a physical quantity in the underlying space.
- \boldsymbol{v} is a representation of \mathbf{v} that depends on the choice of coordinate frame.

Vector

$\boldsymbol{x} \in \mathbb{R}^n$: (an n -dimensional vector) \mathbb{R}^n : n -dimensional real space (Euclidian Space)

$\boldsymbol{x}, \boldsymbol{x}^T$:

Norm of a vector \boldsymbol{x} :

Point and Its Coordinate

- **Point:** p denotes a point in the physical space
- A point p can be represented by as a vector from frame origin to p
- \mathbf{p} denotes the coordinate of a point p
- The coordinate \mathbf{p} depends on the choice of reference frame

Matrix

$\mathbf{A} \in \mathbb{R}^{m \times n}:$

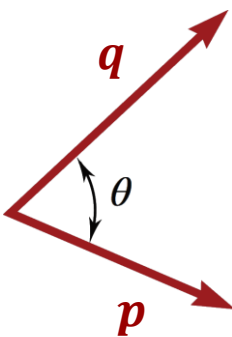
Symmetric matrix:

Skew-symmetric matrix:

Matrix vector multiplication as linear combination of columns:

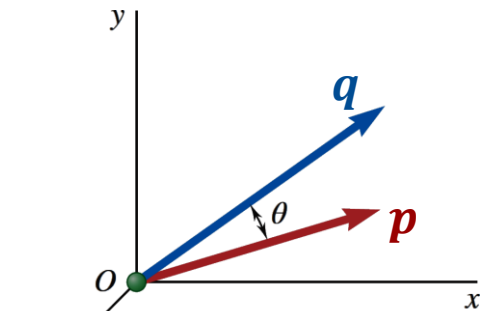
Dot Product (or Scalar Product or Inner Product)

Dot Product of two vectors $\mathbf{p} \in \mathbb{R}^n, \mathbf{q} \in \mathbb{R}^n$ is defined as the scalar $\mathbf{p} \cdot \mathbf{q}$.

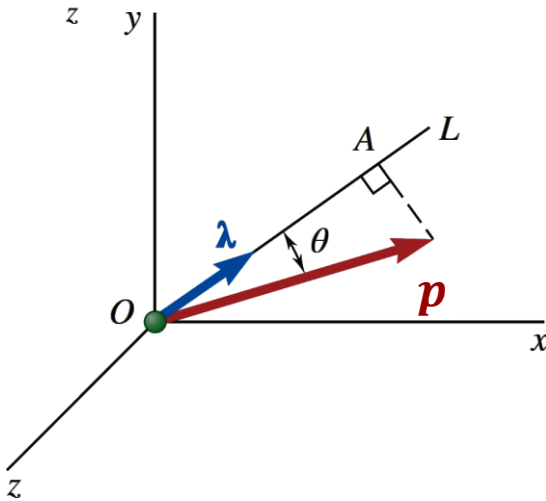


Applications of Dot Product

(1) Finding angle formed between two given vectors $\mathbf{p} \in \mathbb{R}^n, \mathbf{q} \in \mathbb{R}^n$ (or intersecting lines):



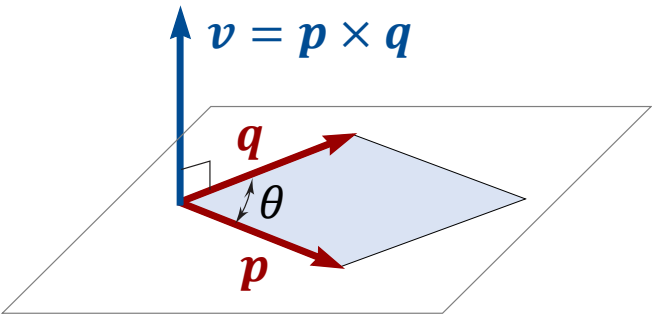
(2) Finding projection of a vector $\mathbf{p} \in \mathbb{R}^n$ on a given axis or directed line:



$\boldsymbol{\lambda}$: unit vector of line L

Cross Product (or Vector Product)

Cross product of $\mathbf{p} \in \mathbb{R}^3$, $\mathbf{q} \in \mathbb{R}^3$ is defined as the vector $\mathbf{v} = \mathbf{p} \times \mathbf{q}$ where $\mathbf{v} \in \mathbb{R}^3$.



Cross Product (or Vector Product)

Coordinate notation: $\boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{q} = (p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}) \times (q_x \mathbf{i} + q_y \mathbf{j} + q_z \mathbf{k})$

Matrix notation: $\boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}$

Cross Product as a Matrix-Vector Multiplication

Cross product $\mathbf{p} \times \mathbf{q}$ ($\mathbf{p} \in \mathbb{R}^3$, $\mathbf{q} \in \mathbb{R}^3$) can be thought of as a multiplication of vector \mathbf{q} by a 3x3 skew-symmetric matrix $[\mathbf{p}]$.

$$\mathbf{p} \times \mathbf{q} = \underbrace{\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}}_{[\mathbf{p}]} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = [\mathbf{p}]\mathbf{q}$$

$$\mathbf{p} = (p_x, p_y, p_z), \mathbf{q} = (q_x, q_y, q_z)$$

The matrix $[\mathbf{p}]$ is a 3x3 skew-symmetric matrix representation of \mathbf{p} .

$$[\mathbf{p}] = -[\mathbf{p}]^T$$

2R Planar Manipulator

RR (or 2R) Planar Manipulator

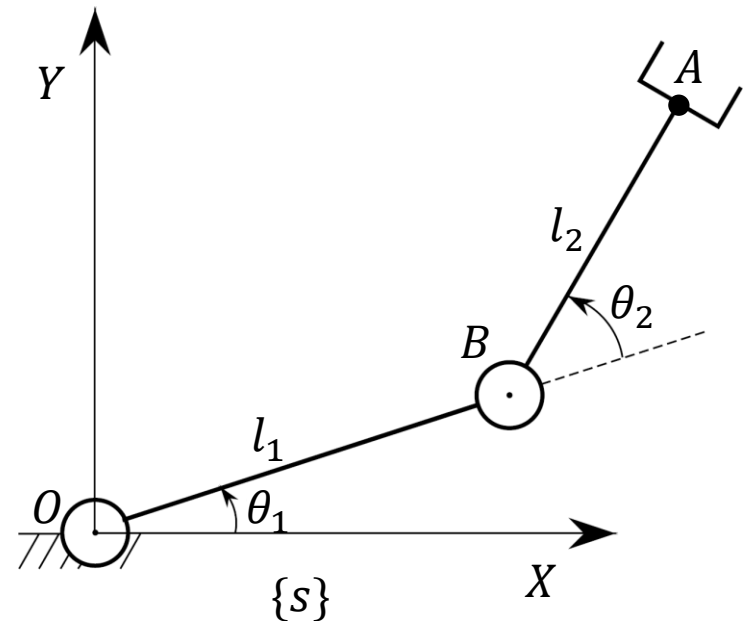
(θ_1, θ_2) : Joint angles

(x, y) : Position of end-effector (point A)

$\{s\}$: Base frame of manipulator

l_1 : Length of link 1

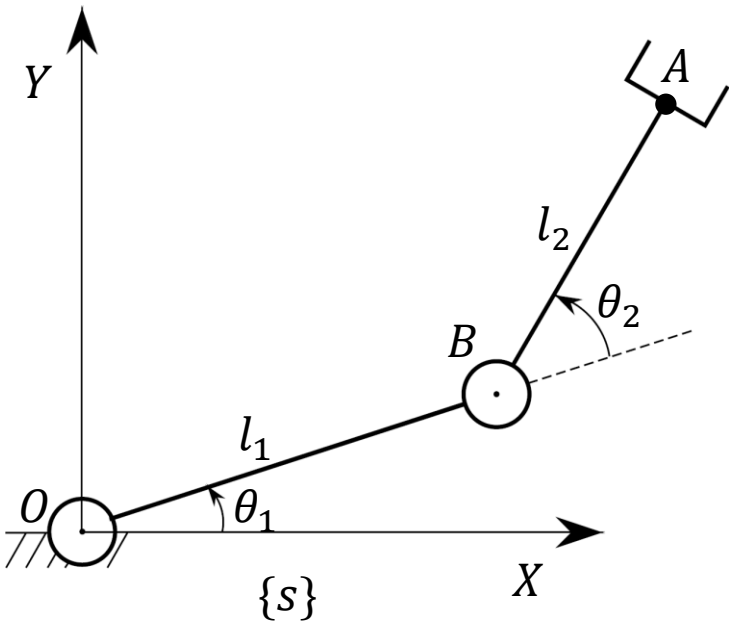
l_2 : Length of link 2



Position Kinematics

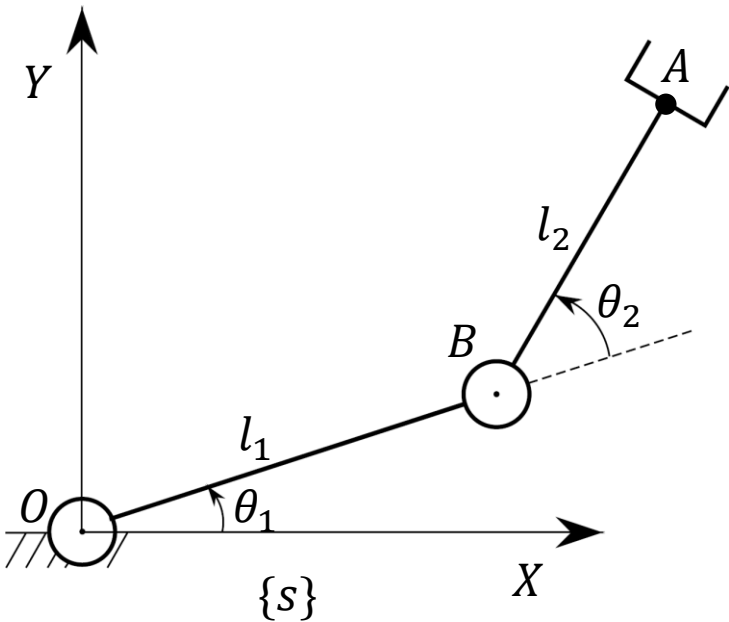
Relation between Joint Angles and End-Effector Position

What is the relationship between the joint angles, (θ_1, θ_2) , and the position of the end effector point A , (x, y) , in the base frame $\{s\}$?



Forward (Direct) Position Kinematics

Given the joint angles, (θ_1, θ_2) , of the 2R robot, find the position, (x, y) of the end-effector point A , in the base frame $\{s\}$.



Forward (Direct) Position Kinematics

$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$: Vector of joint angles

$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix}$: Position vector of end-effector point

$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \equiv f_1(\theta_1, \theta_2)$$

$$y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \equiv f_2(\theta_1, \theta_2)$$

$$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$

More abstractly, the forward kinematics map is

$$\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{\theta})$$

where \boldsymbol{f} is a vector function.

$$\boldsymbol{f}(\boldsymbol{\theta}) = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$

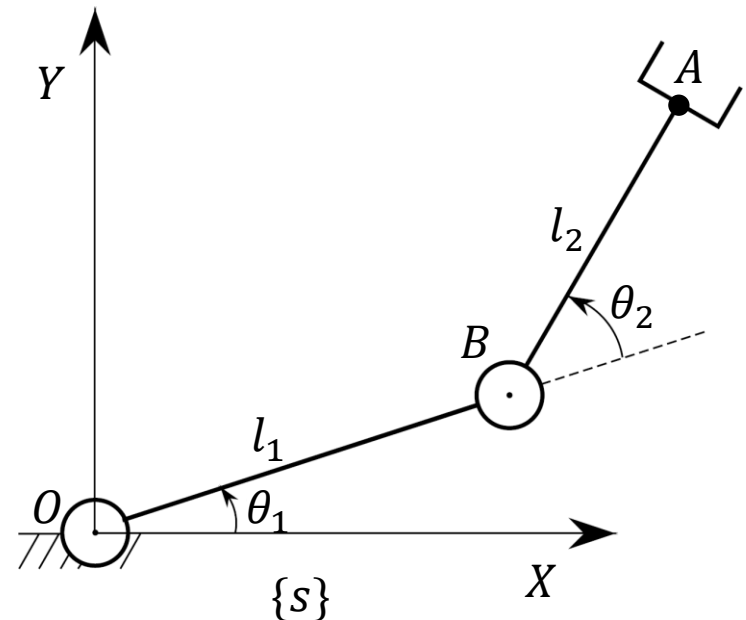
Inverse Position Kinematics

Given the position, (x, y) , of the end effector point A , find the joint angles, (θ_1, θ_2) so that the position (x, y) is reached.

In other words, from the equations

$$\begin{aligned}x &= l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)\end{aligned}$$

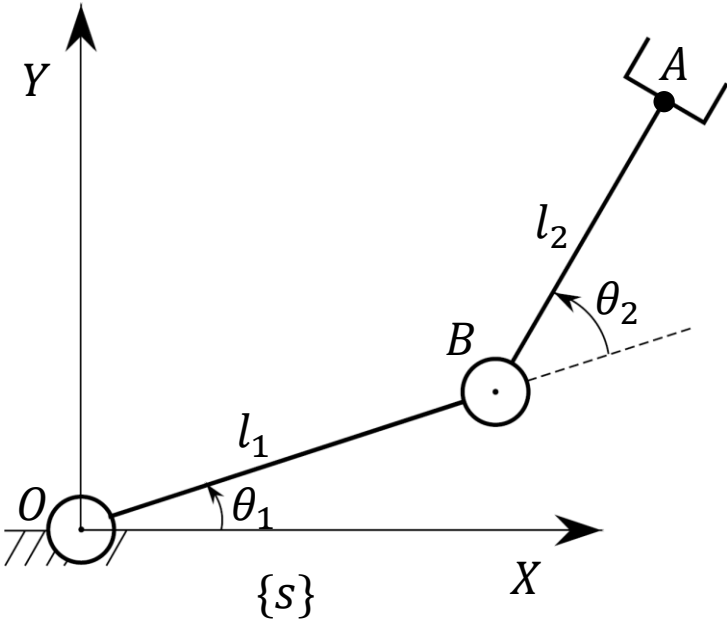
Find θ_1 and θ_2 as a function of x and y .



Numerical Example (Exercise)

Forward and Inverse Position Kinematics:

$$\begin{aligned}
 x &= l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\
 y &= l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)
 \end{aligned}$$



$$\begin{aligned}
 \theta_2 &= \text{atan2} \left(\pm \sqrt{1 - u^2}, u \right) \\
 \theta_1 &= \text{atan2}(y, x) - \text{atan2}(l_2 \sin \theta_2, l_1 + l_2 \cos \theta_2)
 \end{aligned}$$

$$u = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

Velocity Kinematics

Relation between Joint Angle Rates and End-Effector Velocity

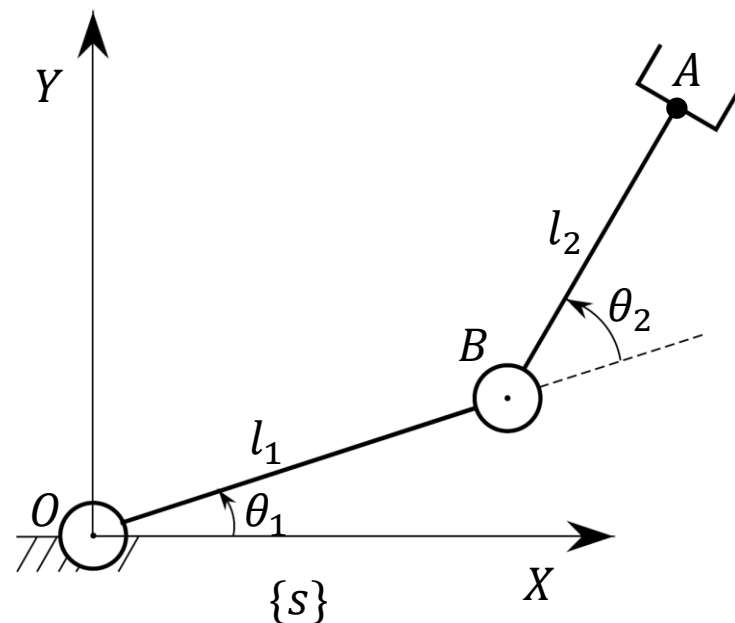
What is the relationship between the joint angle rates of motion $(\dot{\theta}_1, \dot{\theta}_2)$, and the velocity of the end effector point (v_x, v_y) ?

$\dot{\theta}_1 = \frac{d\theta_1}{dt}$: Rate of change of angle of joint 1.

$\dot{\theta}_2 = \frac{d\theta_2}{dt}$: Rate of change of angle of joint 2.

$v_x = \frac{dx}{dt} = \dot{x}$: x -component of velocity of point A.

$v_y = \frac{dy}{dt} = \dot{y}$: y -component of velocity of point A.



Relation between Joint Angle Rates and End-Effector Velocity

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The manipulator (analytic) Jacobian is:

$$J(\theta_1, \theta_2) = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$: Vector of joint angle rates.

$v = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$: Velocity of end-effector point.

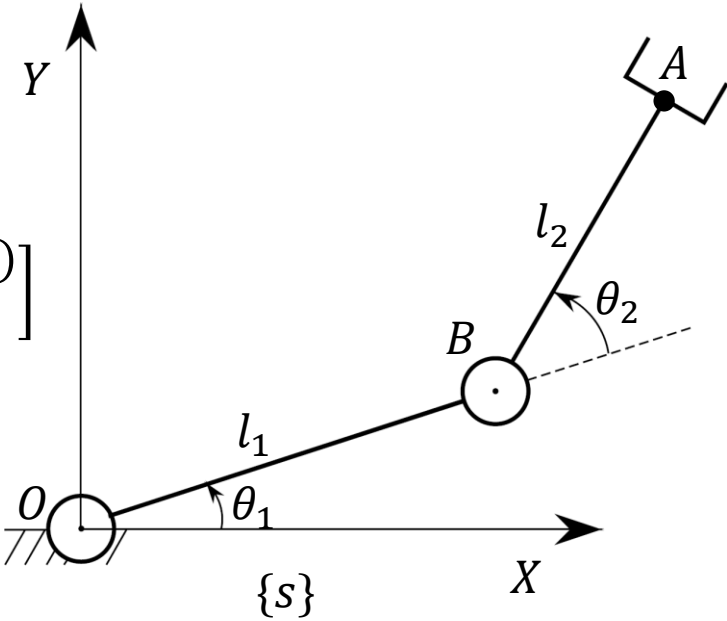
The velocity kinematics equations in vector-matrix form is: $v = J(\theta)\dot{\theta}$

Forward (Direct) Velocity Kinematics

Given the configuration of the robot, θ , and the joint angle rates, $\dot{\theta}$, compute the velocity, v of the end effector.

$$v = J(\theta)\dot{\theta}$$

$$J(\theta) = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

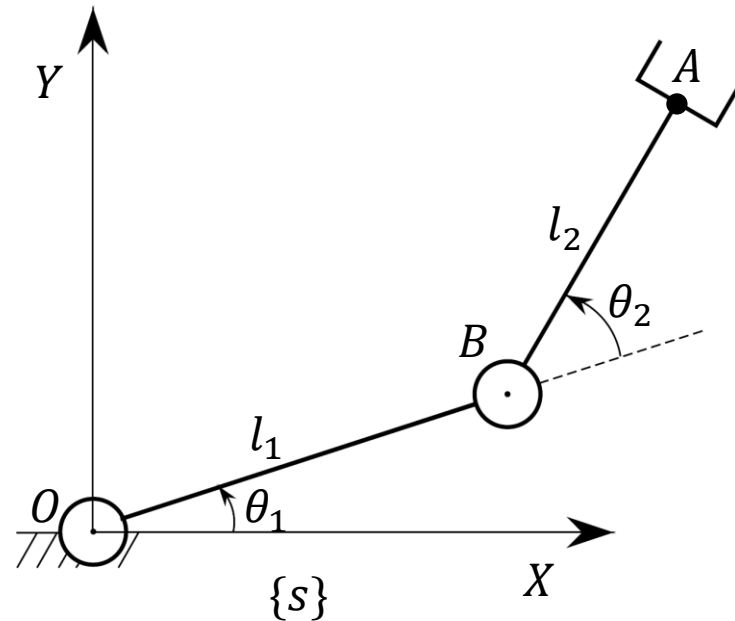


Inverse Velocity Kinematics

Given the configuration of the robot, θ , and the velocity, v , of the end effector, compute the joint angle rates, $\dot{\theta}$.

$$\dot{\theta} = J^{-1}(\theta)v$$

assuming $J^{-1}(\theta)$ exists or the Jacobian matrix is invertible at the configuration θ .

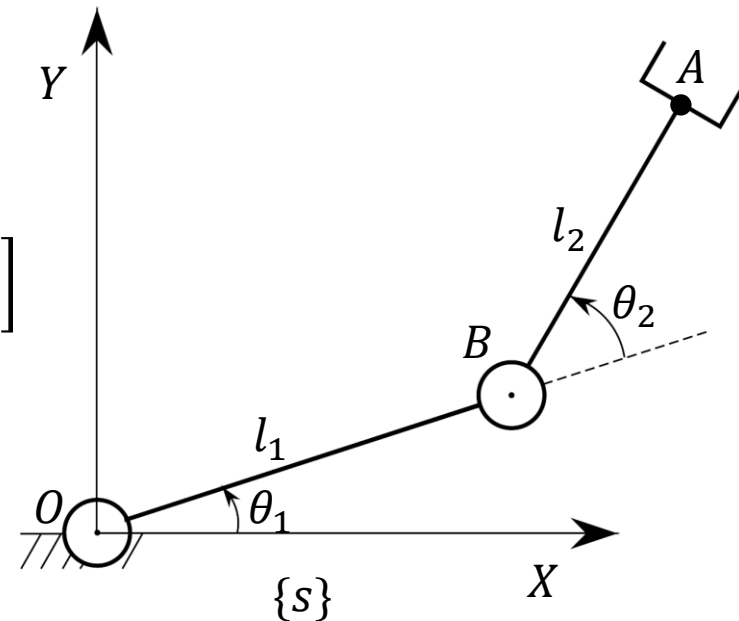


Kinematic Singularities

The configuration $\boldsymbol{\theta}$ at which the Jacobian, $\boldsymbol{J}(\boldsymbol{\theta})$ of a manipulator loses rank is called a **kinematic singularity** or **singular configuration** of the manipulator.

For a 2R manipulator, the Jacobian, $\boldsymbol{J}(\boldsymbol{\theta})$ losing rank implies $\det(\boldsymbol{J}(\boldsymbol{\theta})) = 0$.

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$



Physical Implications of Kinematic Singularities

Why should we care about singular (or almost singular) configurations?

Jacobian in Multi-Variable Calculus

Let $\boldsymbol{x} \in \mathbb{R}^n$ (be an n -dimensional vector) vary with time t . What is $\frac{d\boldsymbol{x}}{dt}$?

$$\boldsymbol{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a **scalar function** (a function that maps a vector of dimension n to a scalar.
 What is $\frac{df}{dt}$?

Jacobian in Multi-Variable Calculus

Let $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a **vector function** (a function that maps a vector of dimension n to a vector of dimension m).

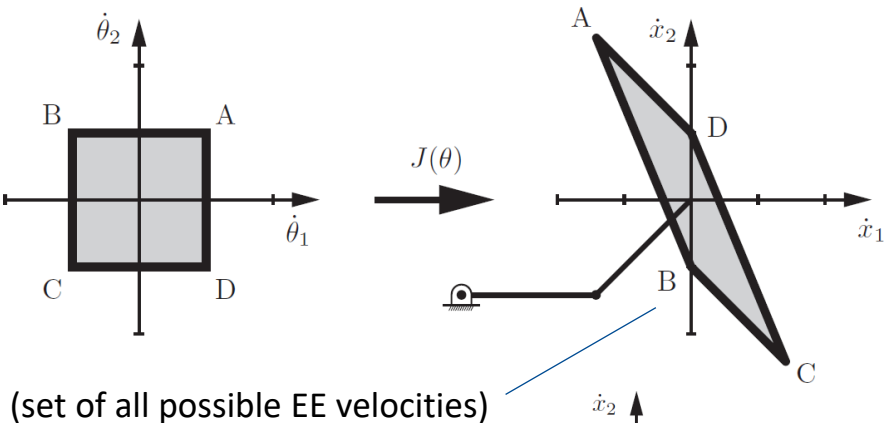
What is $\frac{d\mathbf{f}}{dt}$?

$$\frac{d\mathbf{f}}{dt} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}}_{\text{Jacobian } \frac{\partial \mathbf{f}}{\partial \mathbf{x}}} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

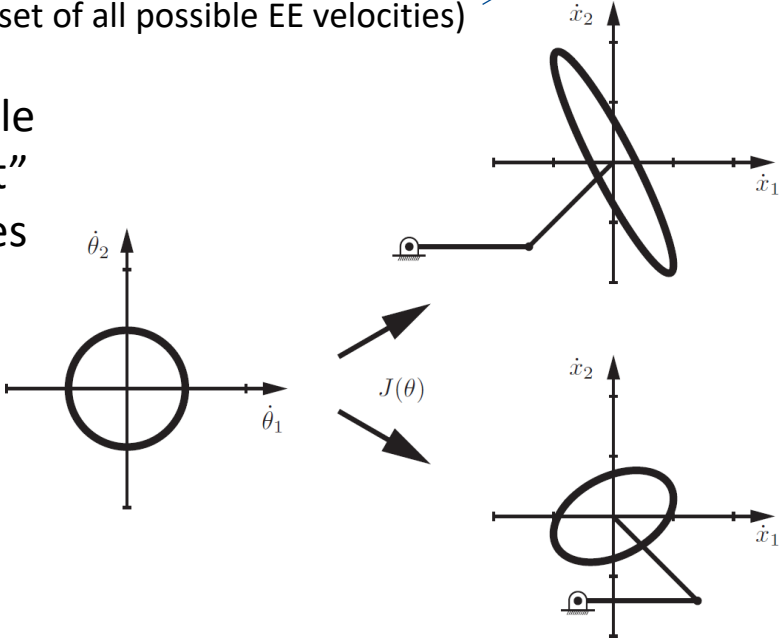
* You can also obtain the manipulator Jacobian using this formula.

Velocity Manipulability Ellipsoid

The Jacobian can be used to map bounds on the rotational speed of the joints (which is a polygon) to bounds on \mathbf{v} .



The Jacobian can be also used to map a unit circle of joint velocities in the $\theta_1 - \theta_2$ -plane (“iso-effort” contour) to an ellipse in the space of EE velocities (this ellipse is called the **velocity manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the tip move in arbitrary directions.

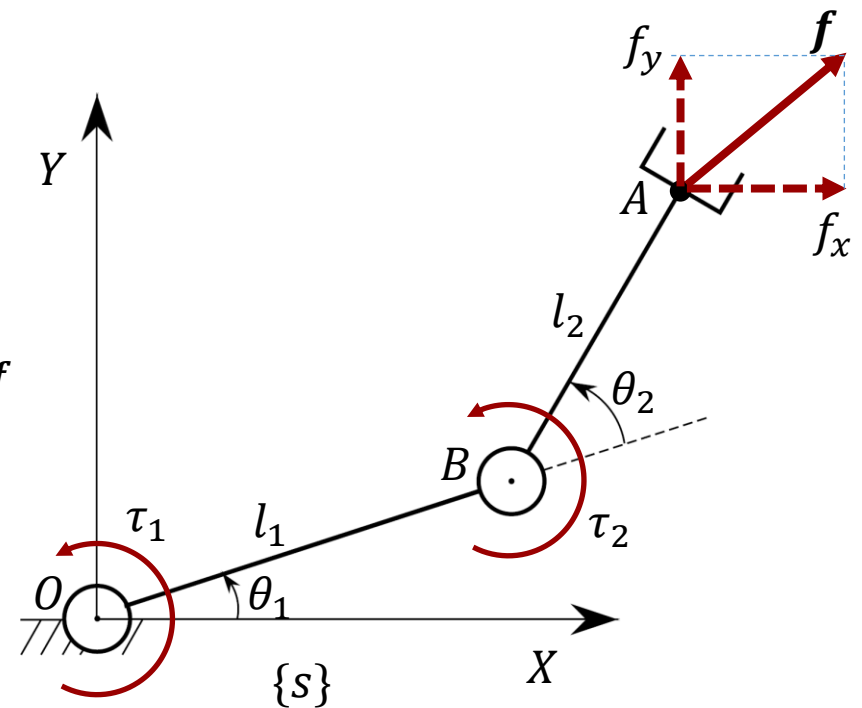
Statics

Statics

What is the relationship between the applied force \mathbf{f} and the joint torques $\boldsymbol{\tau}$ such that the manipulator is at equilibrium at a given configuration $\boldsymbol{\theta}$?

$\mathbf{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$: Force acting at end-effector point A

$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$: Vector of joint torques required to resist \mathbf{f}



(Assume that $\mathbf{g} = \mathbf{0}$)

Statics

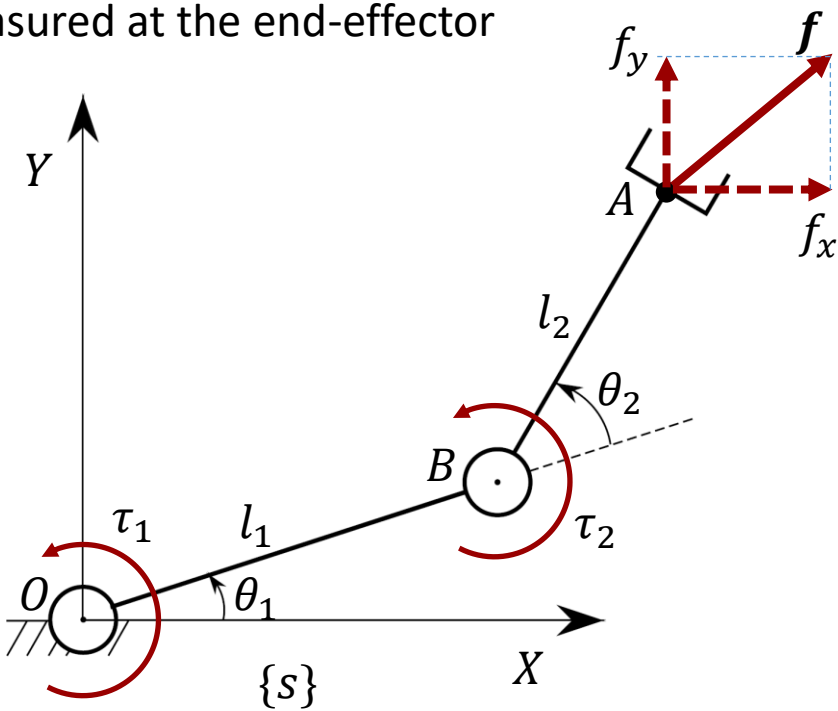
$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ -l_2 \sin(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^T \boldsymbol{f}$$

Statics

A more general method to derive a relation between \mathbf{f} and $\boldsymbol{\tau}$.

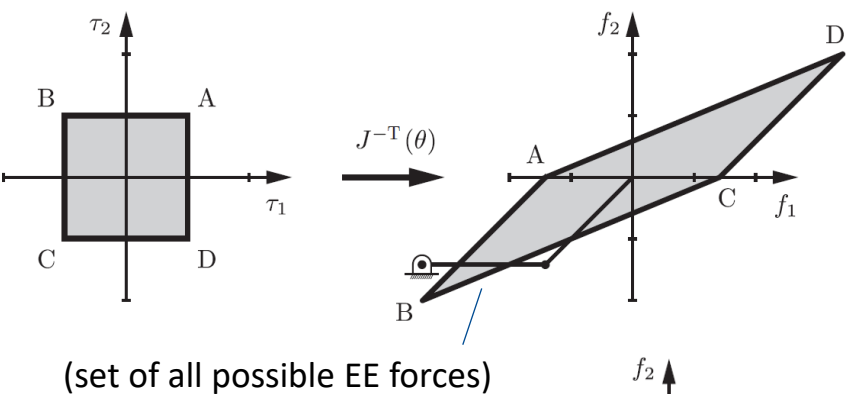
Principle of conservation of power:
power generated at the joints = power measured at the end-effector



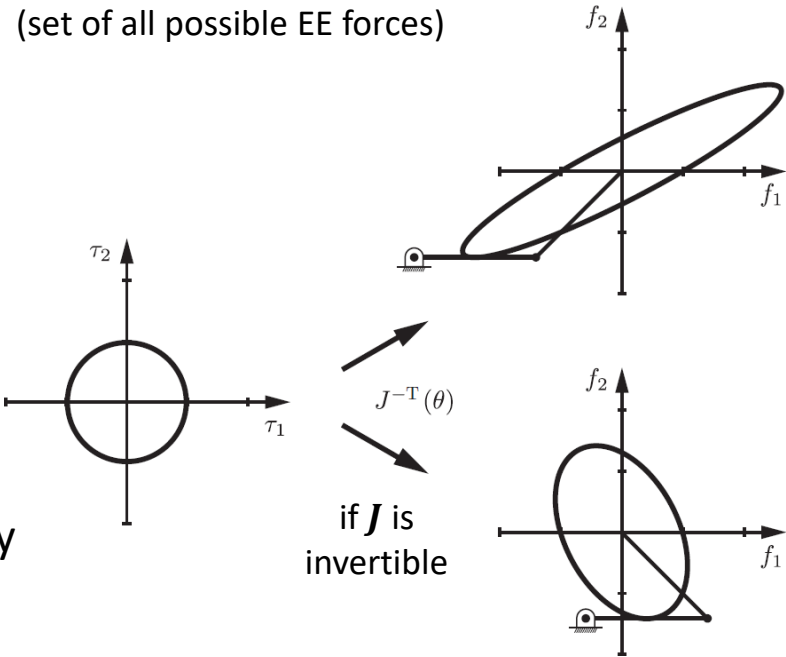
(Assume that $\mathbf{g} = \mathbf{0}$)

Force Manipulability Ellipsoid

Since $\mathbf{f} = (\mathbf{J}(\boldsymbol{\theta})^T)^{-1}\boldsymbol{\tau} = \mathbf{J}(\boldsymbol{\theta})^{-T}\boldsymbol{\tau}$, Jacobian transpose inverse can be used to map bounds on the joint torques (which is a polygon) to bounds on end-effector force \mathbf{f} .



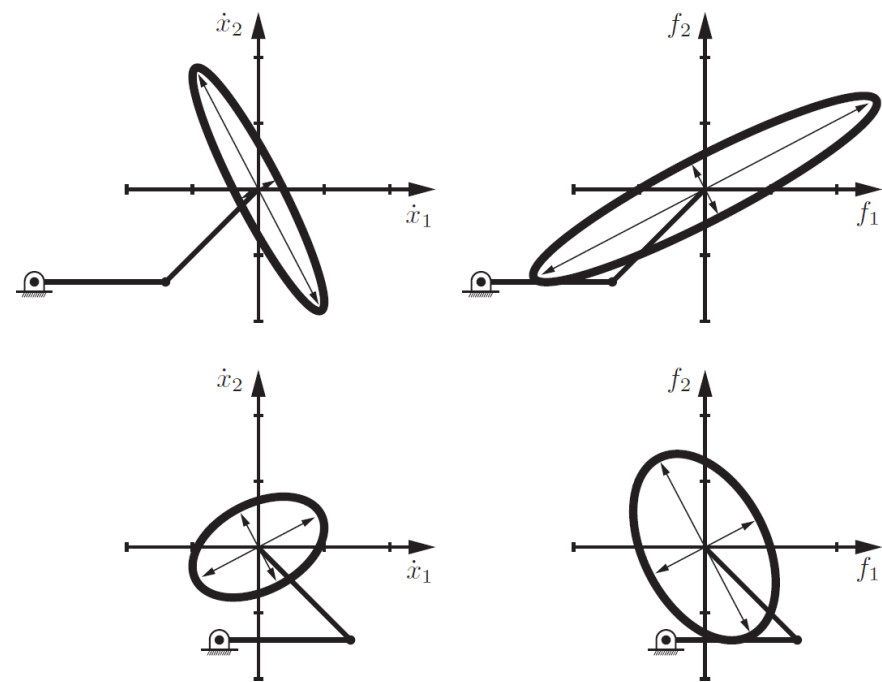
The Jacobian transpose inverse can be also used to map a unit circle of joint torques in the τ_1 - τ_2 -plane (“iso-effort” contour) to an ellipse in the space of EE forces (this ellipse is called the **force manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the EE generate forces in arbitrary directions.

Kineto-Statics Duality

If it is easy to generate a tip velocity in a given direction then it is difficult to generate a force in that same direction, and vice versa.



$$v = J(\theta)\dot{\theta}$$

$$\tau = J(\theta)^T f$$

At a singularity, EE motion capability becomes zero in one or more directions, and it can resist infinite force in one or more directions.