Ch2: Statics of Particles

Contents:

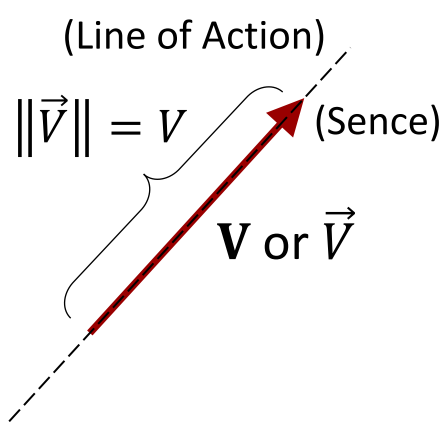
Vector Operations

Scalars and Vectors

A **Scalar** is any <u>positive or negative physical quantity</u> that can be completely specified by its **magnitude** (e.g., time, length, area, volume, speed, mass, density, pressure, temperature, energy, work, or power).

A **Vector** is any <u>physical quantity</u> that requires both a **<u>magnitude</u>** and a **<u>direction</u>** for its complete description (e.g., force, displacement, velocity, acceleration, or momentum).

$$\|\mathbf{V}\| = \|\vec{V}\| = V$$



A vector is shown graphically by an **arrow**.

Magnitude: Length of the arrow

Direction: Defined by **<u>Line of Action</u>** and **<u>Sense</u>**

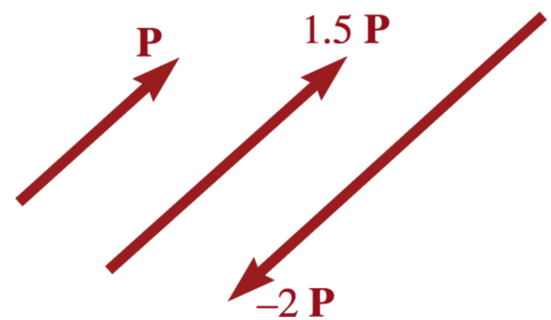
Line of Action: Indicated by An infinite straight line

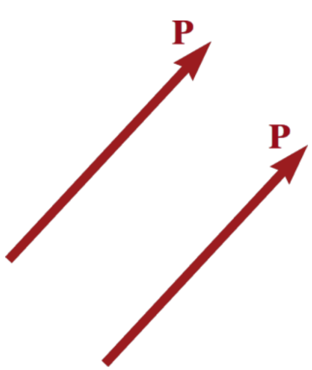
Sense: An arrowhead

It is characterized by an angle in 2D (or two angles in 3D), formed with some fixed axis.

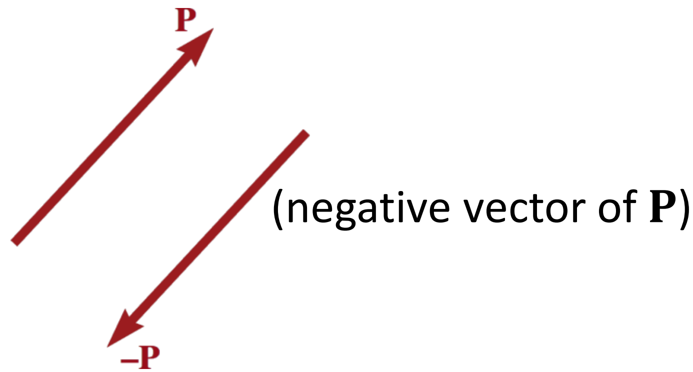
Multiplication of a Vector by a Scalar

The product $k\mathbf{P}$ of a scalar k and a vector $\mathbf{P}$ is defined as a vector having the same direction as $\mathbf{P}$ (if k is positive) or a direction opposite to that of $\mathbf{P}$ (if k is negative) and a magnitude equal to the product of P and the absolute value of k, i.e., $|k|P$.





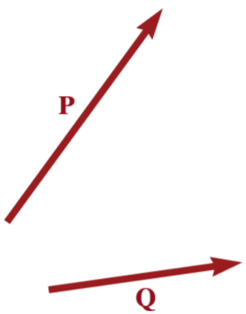
Equal vectors

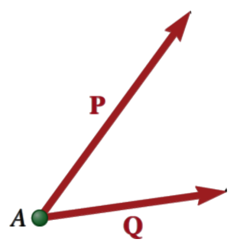


Equal and Opposite vectors

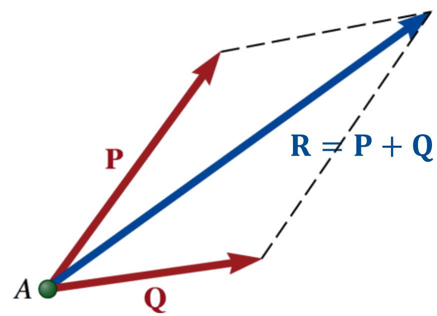
Addition of Two Vectors

By definition, vectors add according to the **Parallelogram Law**.





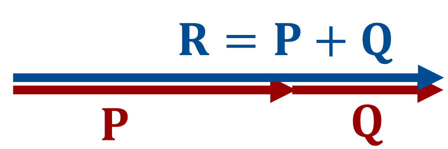
Make them **Concurrent** by join the tails at arbitrary point A.



Construct a **Parallelogram**. The diagonal that passes through A represents the sum of the vectors.

This single equivalent vector $\mathbf{R}$ is called the **Resultant** of the original vectors $\mathbf{P}$ and $\mathbf{Q}$.

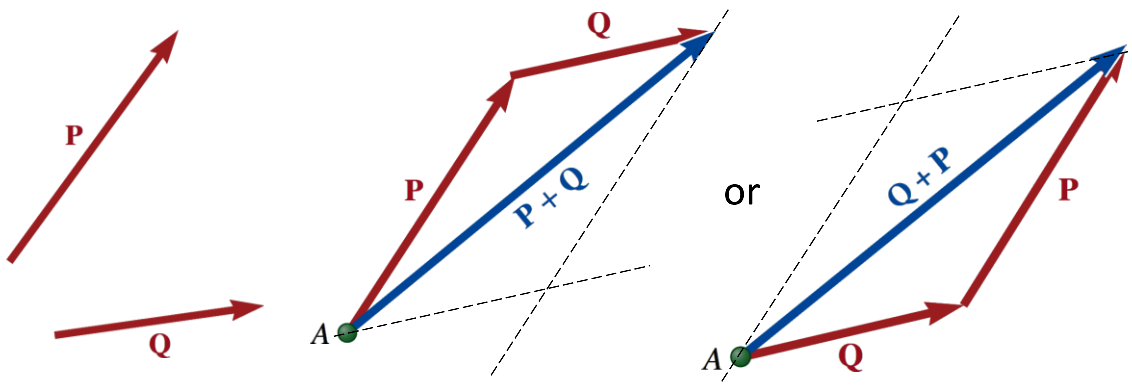
Note: The magnitude of the vector $\mathbf{R} = \mathbf{P} + \mathbf{Q}$ is not, in general, equal to $R = P + Q$. However, there is a <u>special case</u> that if the two vectors $\mathbf{P}$ and $\mathbf{Q}$ are **Collinear** (i.e., both have the same line of action):



$$R = P + Q$$

Addition of Two Vectors (Alternative Method)

Triangle Rule: An alternative method for determining the sum of two vectors.



(half of the parallelogram)

Arrange $\mathbf{P}$ and $\mathbf{Q}$ in tip-to-tail fashion and then connect the tail of $\mathbf{P}$ (or $\mathbf{Q}$) with the tip of $\mathbf{Q}$ (or $\mathbf{P}$).

$\Rightarrow$ Vector addition is **Commutative**:

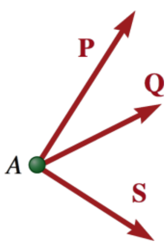
$$\boxed{\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}}$$

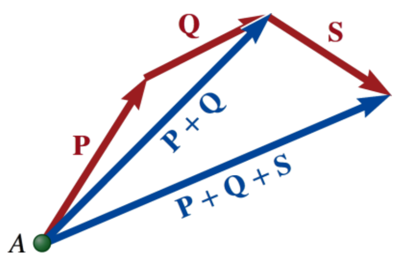
Addition of More Than Two Vectors

The triangle rule (or parallelogram law) is applied **repeatedly** to successive pairs of vectors until all of the given vectors are replaced by a single vector.

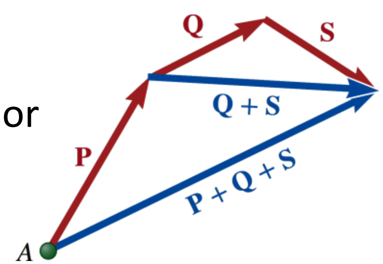
$$\mathbf{P}+\mathbf{Q}+\mathbf{S}=(\mathbf{P}+\mathbf{Q})+\mathbf{S}$$

(Coplanar Concurrent Vectors)



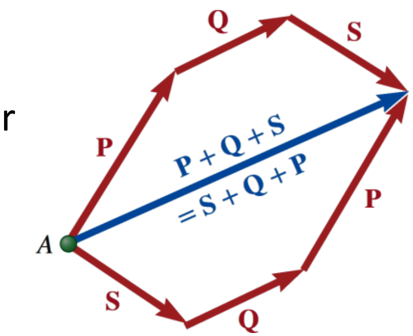


or



(repeated application of the triangle rule)

or



The order in which we add the vectors is immaterial.

$$\boxed{\mathbf{P}+\mathbf{Q}+\mathbf{S}=(\mathbf{P}+\mathbf{Q})+\mathbf{S}=\mathbf{P}+(\mathbf{Q}+\mathbf{S})}$$

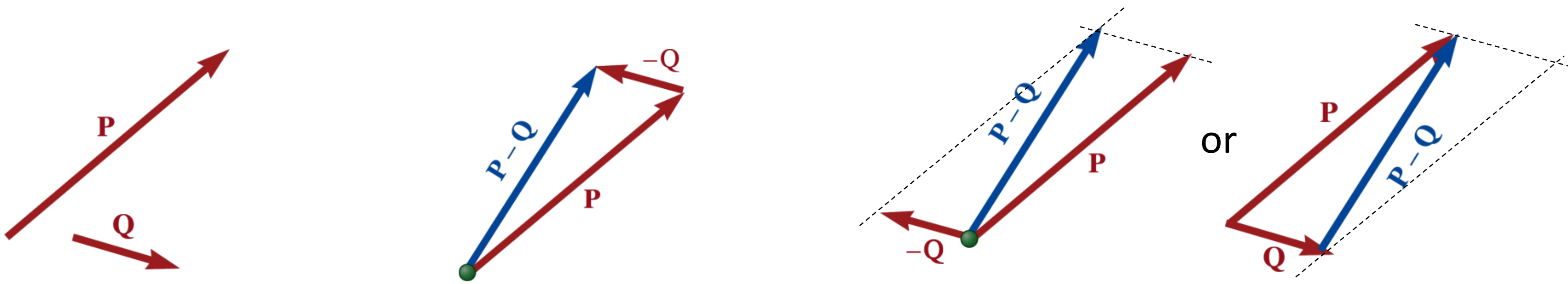
Vector addition is **Associative**.

Polygon Rule: By arranging the given vectors in tip-to-tail fashion and connecting the tail of the first vector with the tip of the last one.

Subtraction of Vectors

Subtraction is defined as **a special case of addition**. Therefore, the rules of vector addition also apply to vector subtraction:

$$\mathbf{P} - \mathbf{Q} = \mathbf{P} + (-\mathbf{Q})$$

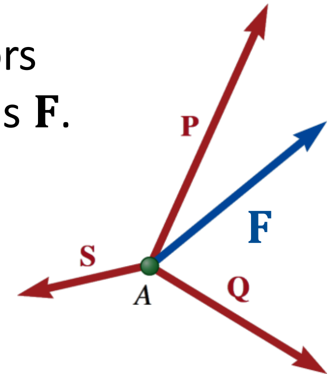


Triangle Rule

Parallelogram Law

Resolution of a Vector into Components

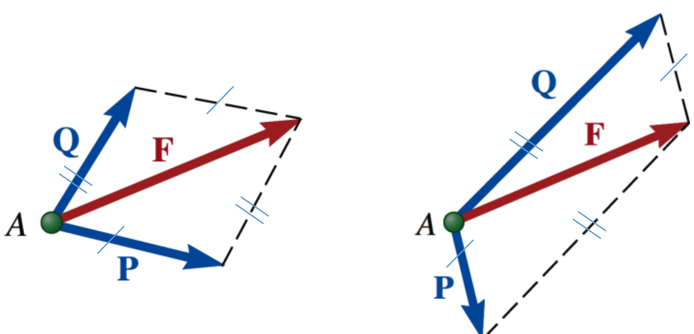
Vector **F** can be resolved into an infinite number of possible sets of vectors (called **Components** of **F**), such that the resultant of all the components is **F**.



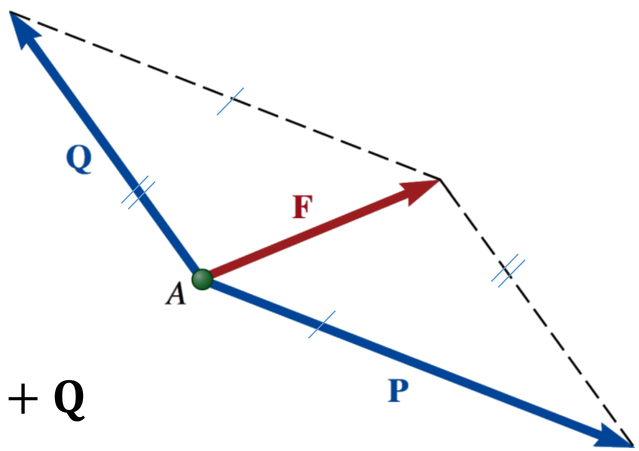
Sets of **two components** are the most common sets in mechanics (and they form a parallelogram).

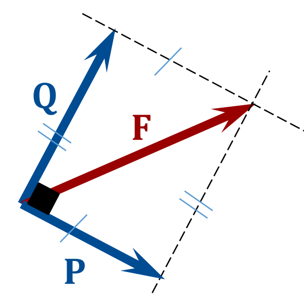
For Example,
Four (among infinite) possible two-component sets for a given vector **F**:

$$\mathbf{F} = \mathbf{P} + \mathbf{Q}$$



$$\mathbf{F} = \mathbf{P} + \mathbf{Q}$$



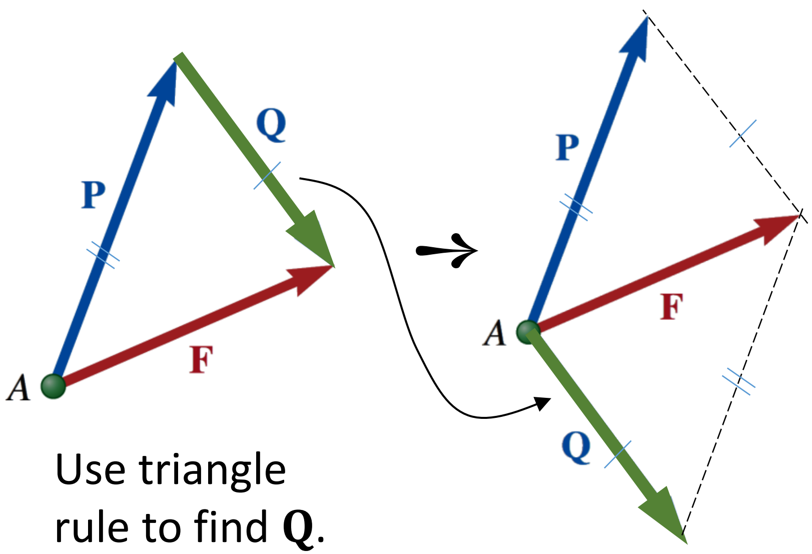


Rectangular Components

Resolution of a Vector into Components (cont.)

We can resolve a vector into **two <u>unique</u> components** by having some information about the components. It is done graphically by drawing the appropriate **parallelogram** or **triangle** that satisfies the given conditions.

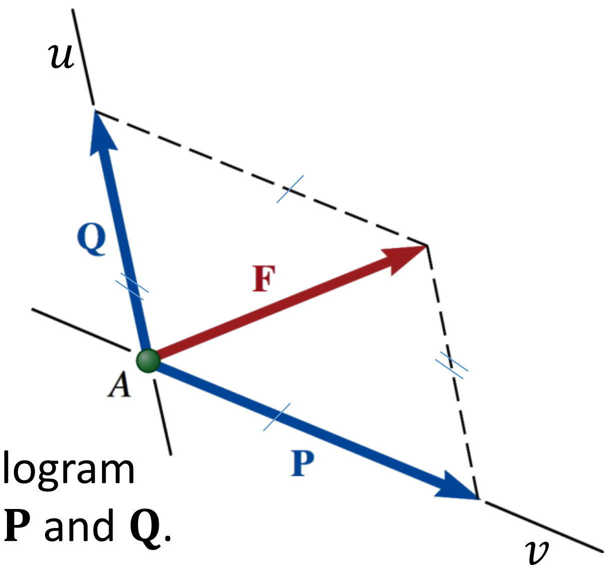
<u>Ex. 1</u>: One of the two components (say $\mathbf{P}$) of vector $\mathbf{F}$ is known.



Use triangle rule to find $\mathbf{Q}$.

$$\mathbf{F} = \mathbf{P} + \mathbf{Q}$$

<u>Ex. 2</u>: Lines of action of the components (say u, v) of vector $\mathbf{F}$ are known.



Use parallelogram law to find $\mathbf{P}$ and $\mathbf{Q}$.

$$\mathbf{F} = \mathbf{P} + \mathbf{Q}$$

Adding Forces in a Plane

Force on a Particle

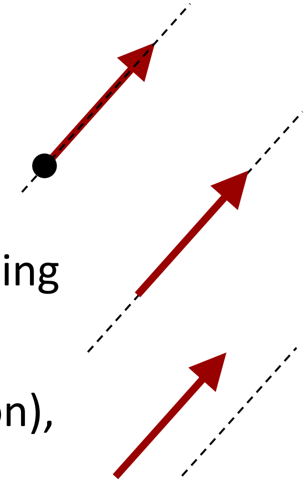
A **force** represents the **action** of one body on another. It can be exerted by actual contact, (like a push or a pull) or at a distance (like gravitational or magnetic forces).

Experimental evidence has shown that **a force is a vector quantity** since it is characterized by its **magnitude**, its **direction**, and its **point of application**, and it adds according to the parallelogram law.

Newton (N) [SI], Pound (lb) [USCS]

Three types of vectors:

- **Fixed Vector** (cannot be moved), e.g., forces acting on a particle.
- **Sliding Vector** (can be moved along its line of action), e.g., forces acting on a rigid body.
- **Free Vector** (can be moved freely in space, parallel to its line of action), e.g., couple.



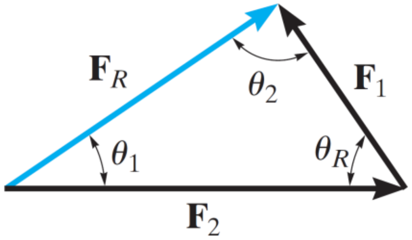
In this chapter, we assume all forces acting on a given body ("particle") act at the same point, i.e., forces are **fixed vectors** and **concurrent**.

Addition of Concurrent Forces

Two methods to solve the problems concerning the **resultant of forces**:

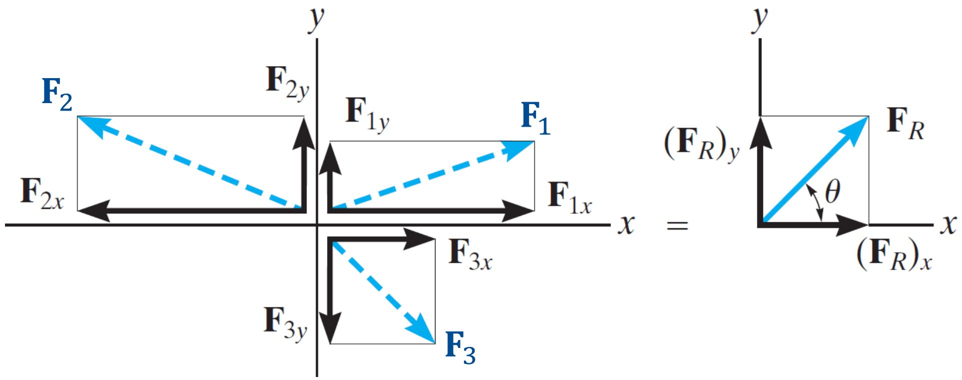
(1) Trigonometric Method:

- It is more convenient when only two forces are involved.
- In this method, we use Triangle Rule (Parallelogram Law) + Sine/Cosine laws.



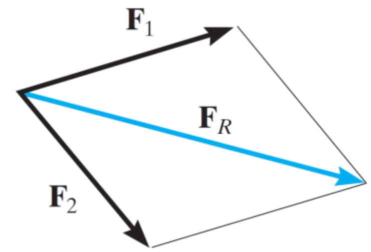
(2) Analytic Method:

- It is more convenient when more than two forces are involved.
- In this method, we use rectangular components of the forces.
- It is a general solution and the most common approach.

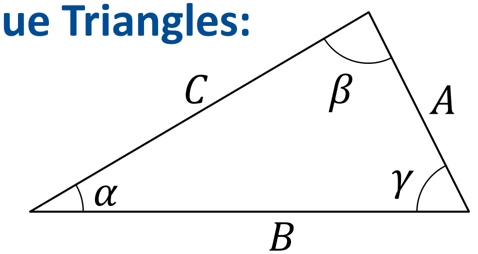


(1) Trigonometric Method

When <u>only two forces</u> are involved, the Triangle Rule (or Parallelogram Law) and Sine/Cosine laws can be used.



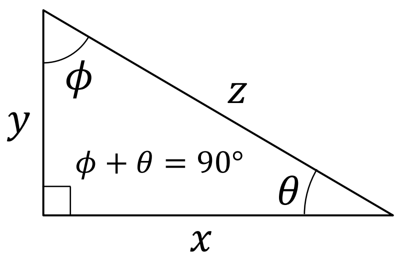
(I) Oblique Triangles:



Law of Sines: $\dfrac{A}{\sin\alpha} = \dfrac{B}{\sin\beta} = \dfrac{C}{\sin\gamma}$

Law of Cosines: $C^2 = A^2 + B^2 - 2AB\cos\gamma$

(II) Right Triangles:

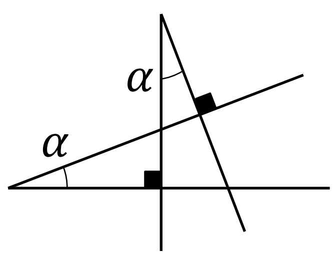


$$\sin(\theta) = \frac{y}{z}, \cos(\theta) = \frac{x}{z}$$

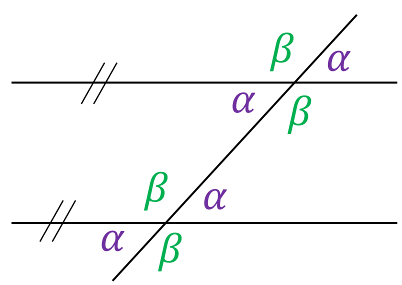
$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$x^2 + y^2 = z^2$$

(III) Angles with Perpendicular Sides:

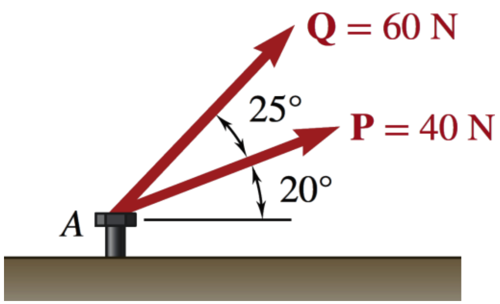


(IV) Parallel lines cut by a transversal:



Sample Problem 2.1

Two forces $\mathbf{P}$ and $\mathbf{Q}$ act on a bolt A. Determine their resultant (magnitude and direction). Use trigonometric method.



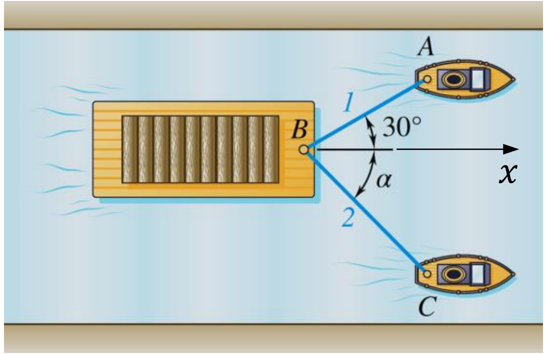
Sample Problem 2.2

Two tugboats are pulling a barge. If the resultant of the forces exerted by the tugboats is a 5000-lb force directed along the axis of the barge (x-axis), determine

(a) the tension in each of the ropes, given that $\alpha = 45°$,

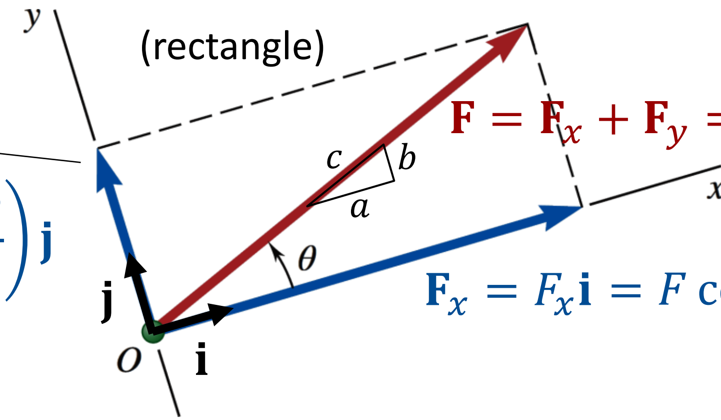
(b) the value of α for which the tension in rope 2 is a minimum.

Use trigonometric method.



Rectangular Components of a Force in a Plane

When a force is resolved into two components along two perpendicular axes (e.g., x and y), the components are called **rectangular components**.



$$\mathbf{F}_y = F_y\mathbf{j} = F\sin\theta\,\mathbf{j} = F\left(\frac{b}{c}\right)\mathbf{j}$$

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y = F_x\mathbf{i} + F_y\mathbf{j}$$

$$\mathbf{F}_x = F_x\mathbf{i} = F\cos\theta\,\mathbf{i} = F\left(\frac{a}{c}\right)\mathbf{i}$$

$\mathbf{F}_x$, $\mathbf{F}_y$: Vector Components of $\mathbf{F}$

$\mathbf{i}$, $\mathbf{j}$: Unit Vectors along the $+x$ and $+y$ axes, $\|\mathbf{i}\| = \|\mathbf{j}\| = 1$

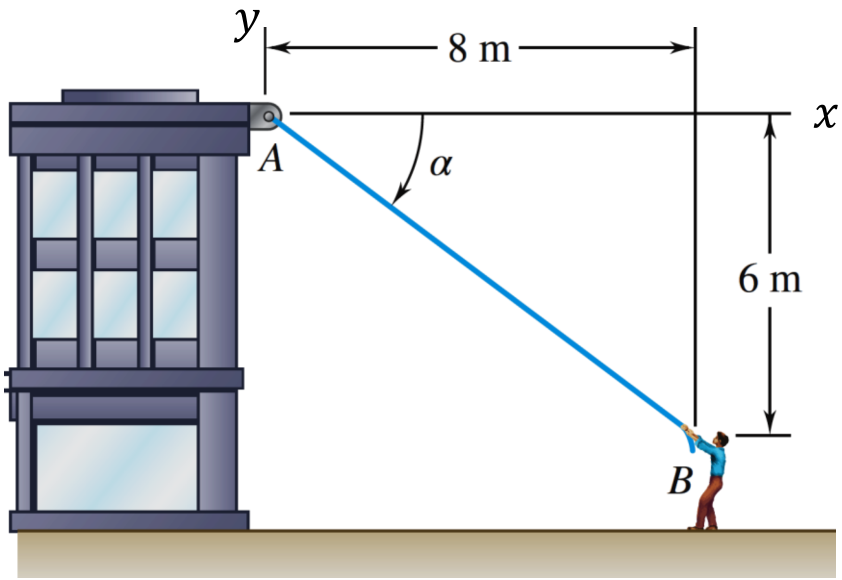
F_x, F_y: Scalar Components of $\mathbf{F}$ (can be positive or negative, depending upon the sense of $\mathbf{F}_x$ and of $\mathbf{F}_y$)

- When $\mathbf{F}_x$ and $\mathbf{F}_y$ are given, direction and magnitude of $\mathbf{F}$: $\theta = \tan^{-1}\dfrac{|F_y|}{|F_x|}$, $F = \sqrt{F_x^2 + F_y^2}$

(inverse tangent)

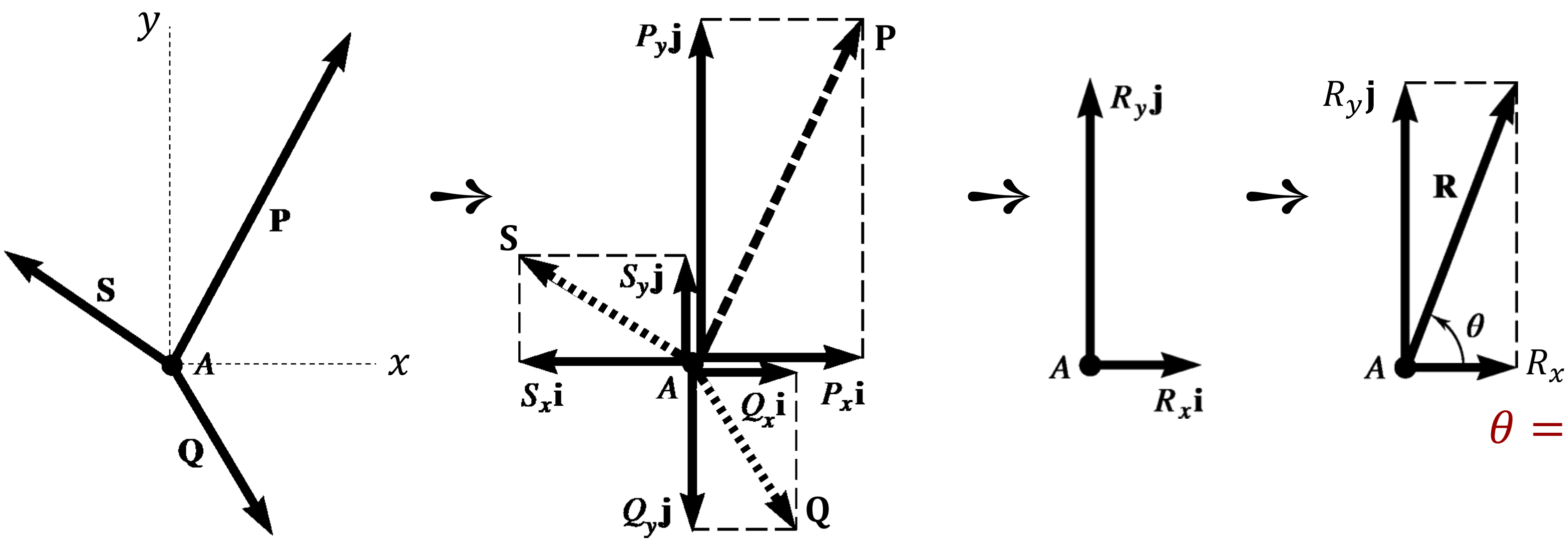
Concept Application 2.2

A man pulls with a force of 300 N on a rope attached to the top of a building. What are the horizontal and vertical components of the force exerted by the rope at point A?



(2) Analytic Method

Consider three forces $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{S}$ acting on a particle A:



$$\theta = \tan^{-1}\frac{|R_y|}{|R_x|},$$

$$R = \sqrt{R_x^2 + R_y^2}$$

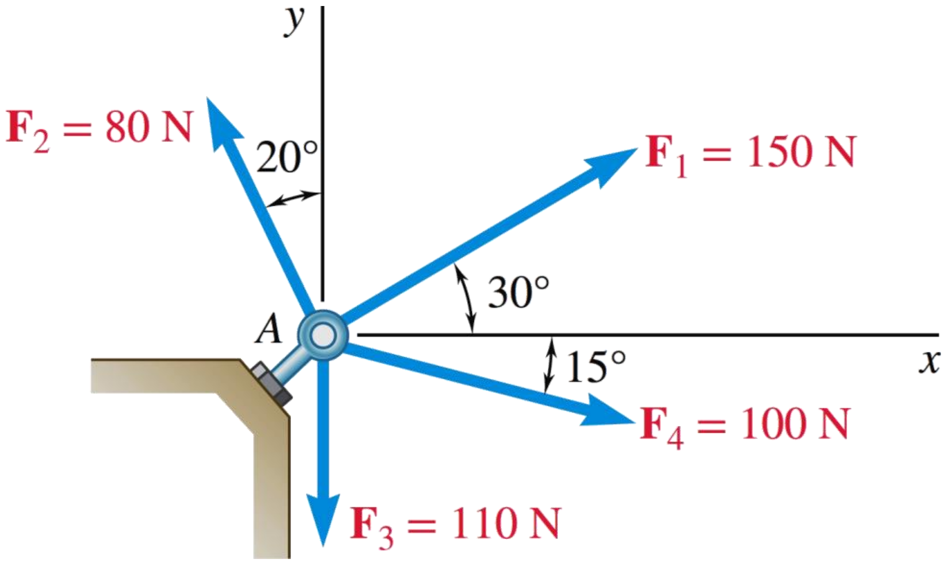
$$\mathbf{R} = \Sigma\mathbf{F} = \mathbf{P} + \mathbf{Q} + \mathbf{S} \ \Rightarrow\ R_x\mathbf{i} + R_y\mathbf{j} = P_x\mathbf{i} + P_y\mathbf{j} + Q_x\mathbf{i} + Q_y\mathbf{j} + S_x\mathbf{i} + S_y\mathbf{j}$$

$$= \underbrace{(P_x + Q_x + S_x)}_{R_x=\Sigma F_x}\mathbf{i} + \underbrace{(P_y + Q_y + S_y)}_{R_y=\Sigma F_y}\mathbf{j}$$

The resultant $\mathbf{R}$ is obtained by adding algebraically the x and y scalar components of the given forces. When three or more forces are involved, this general method is used.

Sample Problem 2.3

Four forces act on bolt A as shown. Determine the resultant of the forces on the bolt.



Equilibrium in a Plane

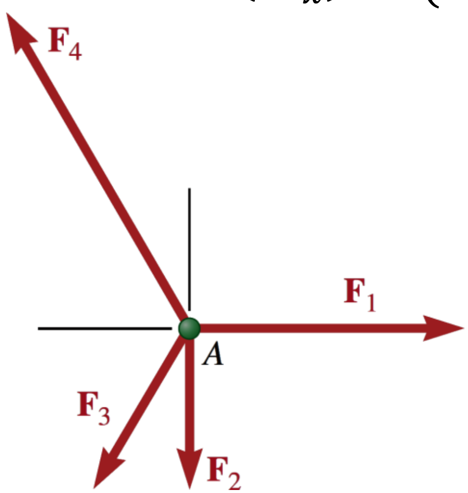
Equilibrium of a Particle

When the resultant of all the forces acting on a particle is **zero**, the particle is in **Equilibrium**.

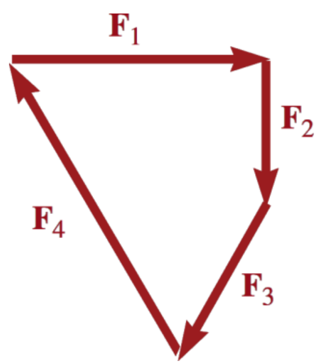
$$\mathbf{R} = \Sigma\mathbf{F} = \mathbf{0}$$ (equation of equilibrium)

(for coplanar forces)

$$(\Sigma F_x)\mathbf{i} + (\Sigma F_y)\mathbf{j} = \mathbf{0} \quad \rightarrow \quad \Sigma F_x = 0, \quad \Sigma F_y = 0$$



$\rightarrow$

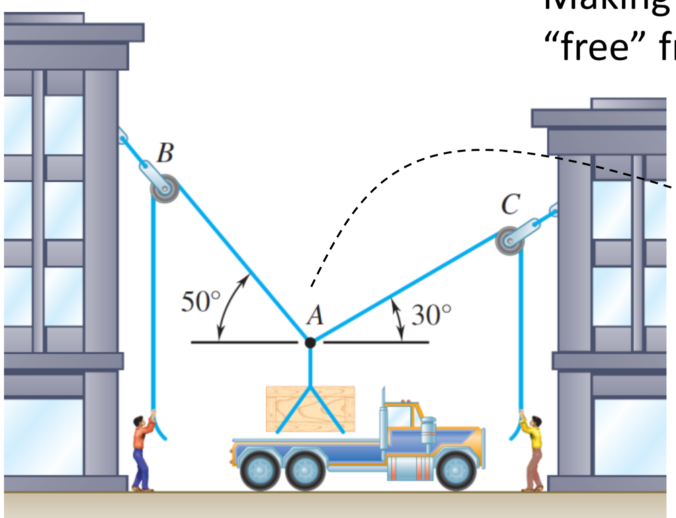


Graphical expression of the equilibrium of A is a **Closed Polygon**.

From Newton's First Law of Motion, we can conclude that a particle in equilibrium is either at rest (static equilibrium) or moving in a straight line with constant speed.

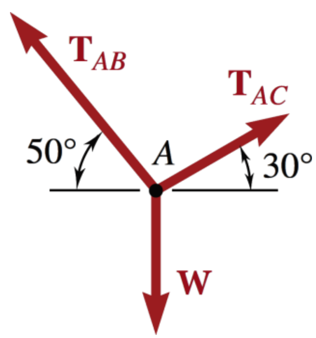
Free-Body Diagram

A drawing that shows the abject with all the <u>**forces**</u> that act on it is called a **Free-Body Diagram** (**FBD**). Drawing an accurate FBD is a <u>must</u> in the solution of problems in mechanics.

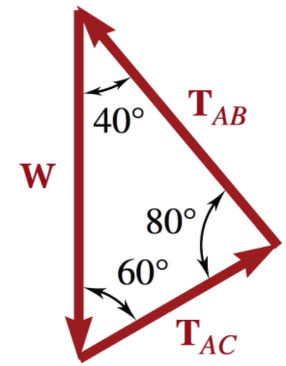


Making an object isolated and "free" from its surroundings.

FBD:



$\Rightarrow$



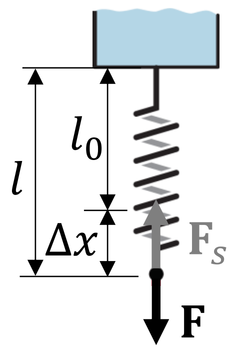
If point A is in equilibrium, the forces must form a closed triangle.

In FBD, you should indicate the magnitudes and directions (angles or dimensions) of known and unknown forces.

Free-Body Diagram

Three common types of supports encountered in particle equilibrium problems:

Linearly Elastic Springs:



l_0: Undeformed Length

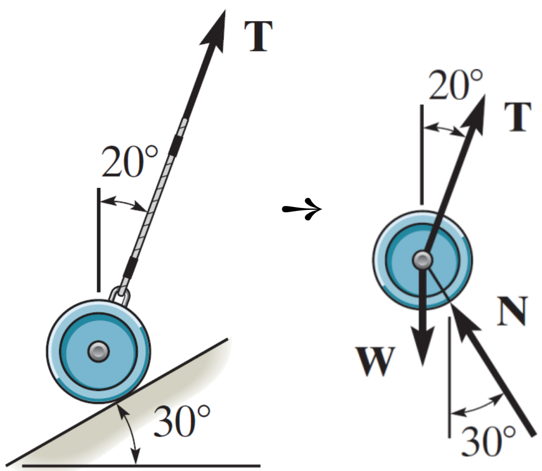
$\Delta x = l - l_0$: Elongation/Compression

k: Spring Constant or Stiffness

$$F_s = k\Delta x$$

Smooth Contact:

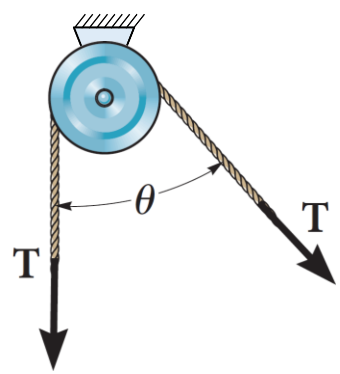
When an object rests on a smooth surface, the surface will exert a force on the object that is normal to the surface at the point of contact.



(forces are concurrent at the center)

Cables and Pulleys:

If the cable is unstretchable and its weight is negligible, and pulley is frictionless, the cable is subjected to a constant tension T throughout its length.

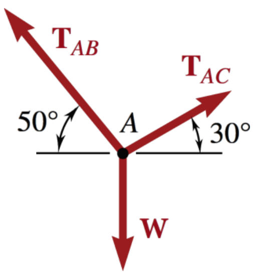


Equilibrium of a Particle

Methods to solve the problems concerning the equilibrium of a particle:

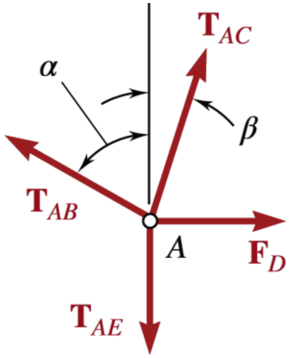
(1) Trigonometric Method:

- It is more convenient when a particle is in equilibrium under only three forces.
- In this method, we use Triangle Rule (or Parallelogram Law) + Sine/Cosine laws.



(2) Analytic Method:

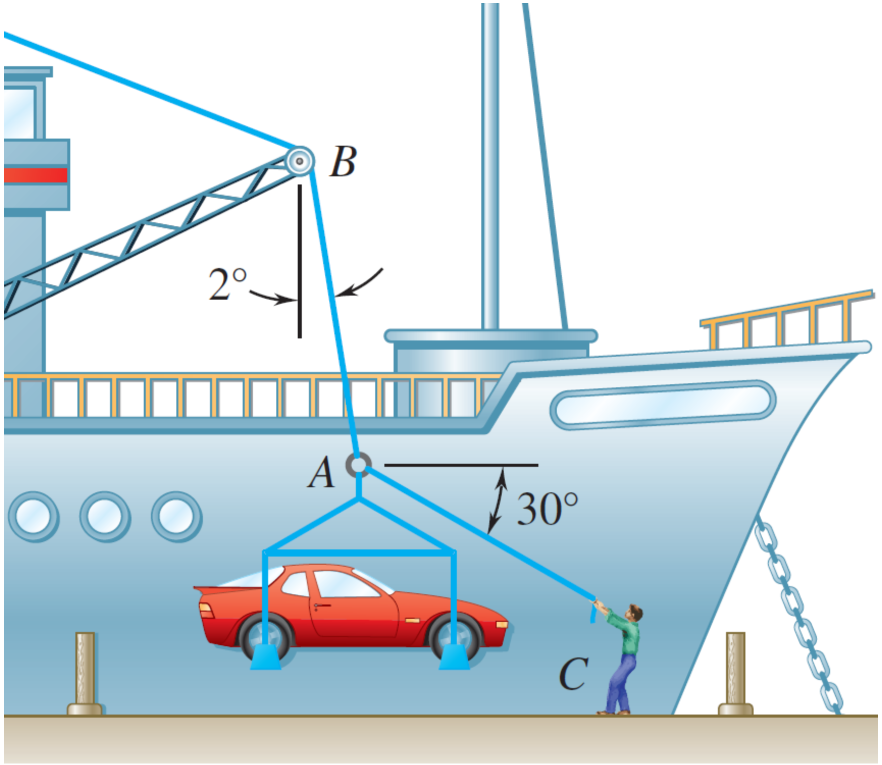
- It is more convenient when a particle is in equilibrium under more than three forces.
- In this method, we use rectangular components of the forces.
- It is a general solution and the most common approach.



∗ Regardless of the method used to solve a planar equilibrium problem, we can determine at most two unknowns.

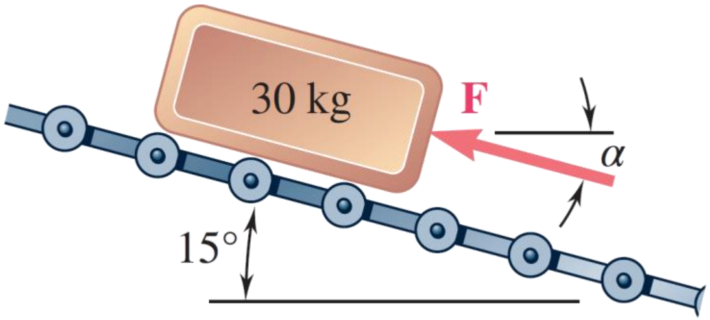
Sample Problem 2.4

In a ship-unloading operation, a 3500-lb automobile is supported by a cable. What are the tensions in the rope AC and cable AB? Use trigonometric method.



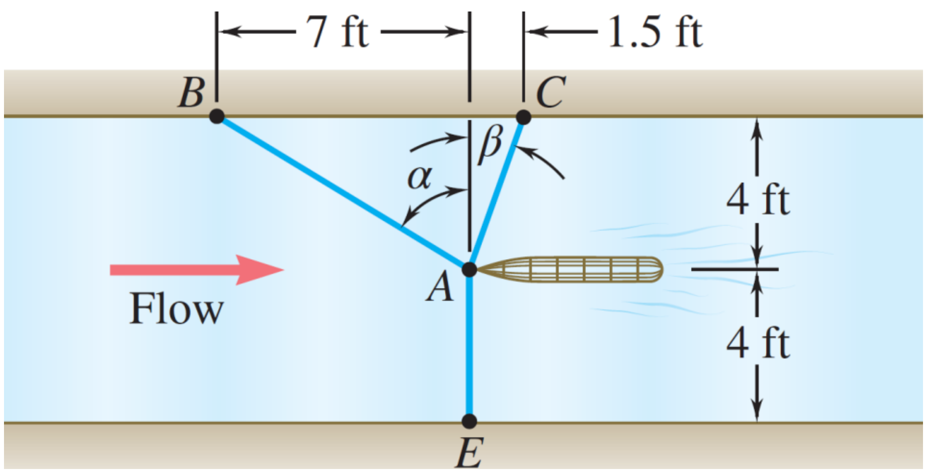
Sample Problem 2.5

Determine the magnitude and direction of the smallest force $\mathbf{F}$ that maintains the 30-kg package shown in equilibrium. Note that the force exerted by the rollers on the package is perpendicular to the incline. Use trigonometric method. $g = 9.81\ \text{m/s}^2$



Sample Problem 2.6

For a new sailboat, a designer wants to determine the drag force that may be expected at a given speed. To do so, she places a model of the proposed hull in a test channel and uses three cables to keep its bow on the centerline of the channel. Dynamometer readings indicate that for a given speed, the tension is 40 lb in cable AB and 60 lb in cable AE. Determine the drag force exerted on the hull and the tension in cable AC.

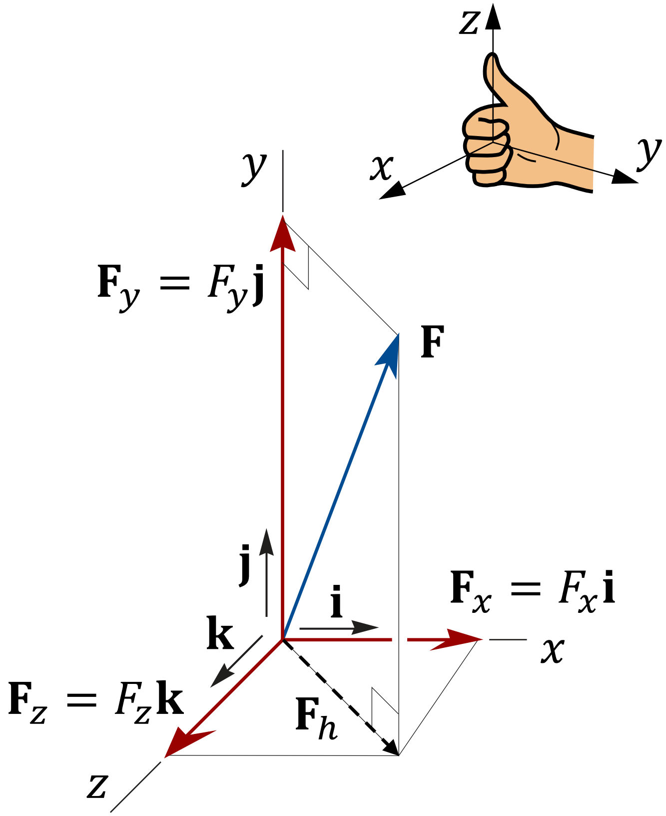


Adding Forces in Space

Rectangular Components of a Force in Space

We use a Right-Handed Coordinate System.

$$\left.\begin{aligned}\mathbf{F} &= \mathbf{F}_y + \mathbf{F}_h \\ \mathbf{F}_h &= \mathbf{F}_x + \mathbf{F}_z\end{aligned}\right\} \boxed{\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}}$$



$\mathbf{i}, \mathbf{j}, \mathbf{k}$: Unit Vectors along the $+x$, $+y$, and $+z$ axes.
F_x, F_y, F_z: Scalar Components of $\mathbf{F}$ (can be positive or negative)
$\mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z$: Vector Components of $\mathbf{F}$.

Magnitude of $\mathbf{F}$: $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$

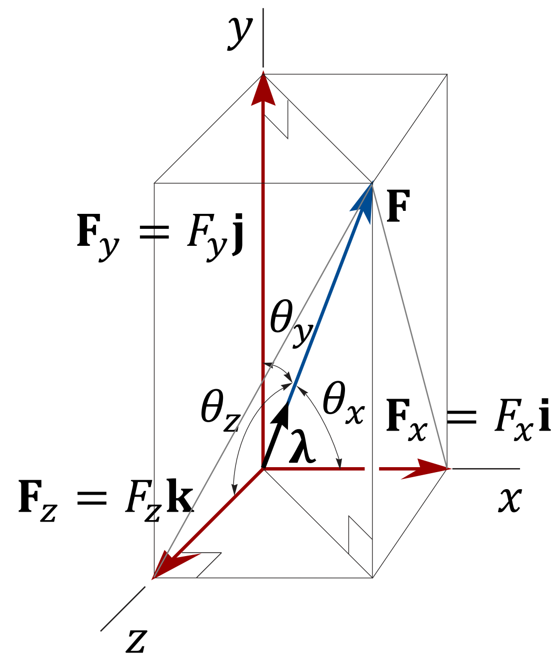
Direction of $\mathbf{F}$: It can be determined by (**1**) three angles $\theta_x, \theta_y, \theta_z$ called as **Coordinate Direction Angles**, or (**2**) two points on the line of action of the force.

(1) Coordinate Direction Angles

$\theta_x, \theta_y, \theta_z$ are the angles of the force $\mathbf{F}$ with the $+x$, $+y$, $+z$ axes ($0 \le \theta_x, \theta_y, \theta_z \le 180°$).

$F_x = F\cos\theta_x$, $F_y = F\cos\theta_y$, $F_z = F\cos\theta_z$

$\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ are called Direction Cosines of $\mathbf{F}$.



$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$$= F\big(\underbrace{\cos\theta_x\,\mathbf{i} + \cos\theta_y\,\mathbf{j} + \cos\theta_z\,\mathbf{k}}_{\boldsymbol{\lambda} = \lambda_x\mathbf{i} + \lambda_y\mathbf{j} + \lambda_z\mathbf{k}}\big) = F\boldsymbol{\lambda}$$

$\|\mathbf{F}\| = F\|\boldsymbol{\lambda}\| \rightarrow \|\boldsymbol{\lambda}\| = 1 \rightarrow$ $\boldsymbol{\lambda}$: Unit Vector along the line of action of $\mathbf{F}$

- **Note**: Since $\|\boldsymbol{\lambda}\| = 1$, $\theta_x, \theta_y, \theta_z$ are not independent:

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

(Relationship among direction cosines)

← (If only two of the coordinate angles are known, the third angle can be found.)

(2) Force Directed along a Line

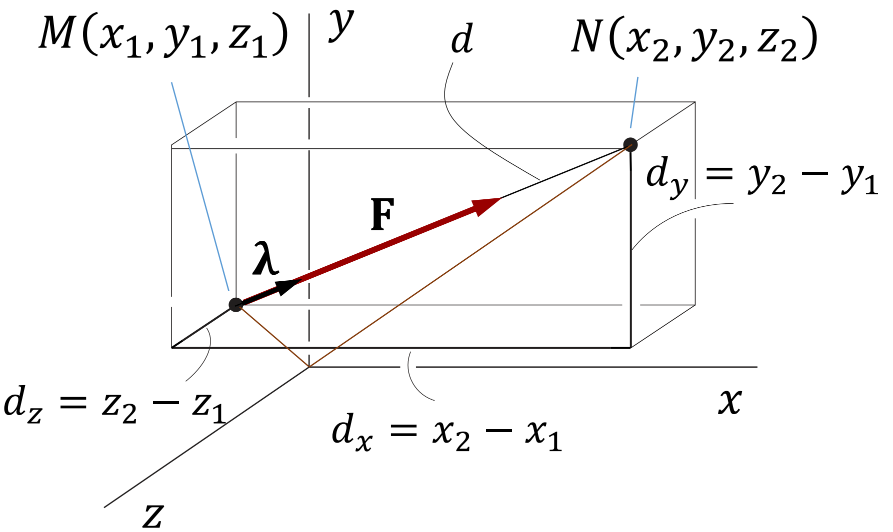
The line of action of $\mathbf{F}$ is determined by the two points M and N.

$$\boldsymbol{\lambda} = \frac{\overrightarrow{MN}}{\|\overrightarrow{MN}\|} = \frac{\vec{N} - \vec{M}}{\|\vec{N} - \vec{M}\|}$$

$$= \frac{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$= \frac{d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}}{d}$$

$$= \lambda_x\mathbf{i} + \lambda_y\mathbf{j} + \lambda_z\mathbf{k} \qquad \Rightarrow \qquad \mathbf{F} = F\boldsymbol{\lambda}$$



d: distance from M to N

Addition of Concurrent Forces

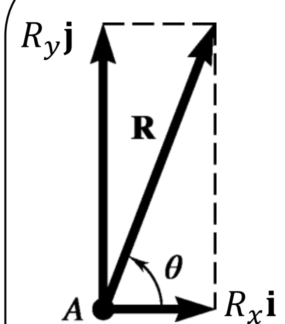
The resultant **R** of two or more forces in space can be determine using **Analytic Method**. **Trigonometric Method** is generally not practical in the case of forces in space.

$$\mathbf{R} = \Sigma\mathbf{F} = (\Sigma F_x)\mathbf{i} + \left(\Sigma F_y\right)\mathbf{j} + (\Sigma F_z)\mathbf{k} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$$

Magnitude: $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$

Direction:
$$\begin{cases} \theta_x = \cos^{-1}\dfrac{R_x}{R} \\ \theta_y = \cos^{-1}\dfrac{R_y}{R} \\ \theta_z = \cos^{-1}\dfrac{R_z}{R} \end{cases}$$

Recall the resultant **R** of forces in plane:



$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$

$$R = \sqrt{R_x^2 + R_y^2}$$

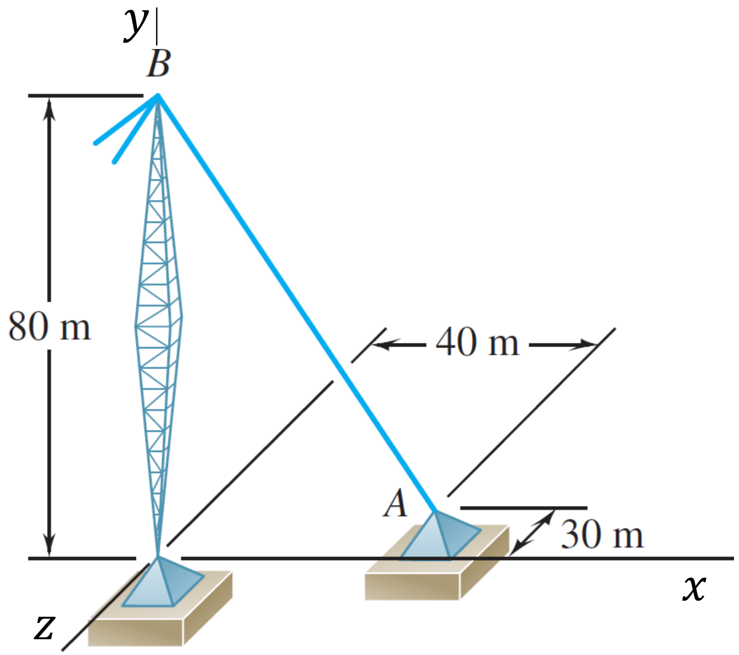
$$\theta = \tan^{-1}\frac{|R_y|}{|R_x|}$$

Sample Problem 2.7

A tower guy wire is anchored by means of a bolt at A. The tension in the wire is 2500 N. Determine

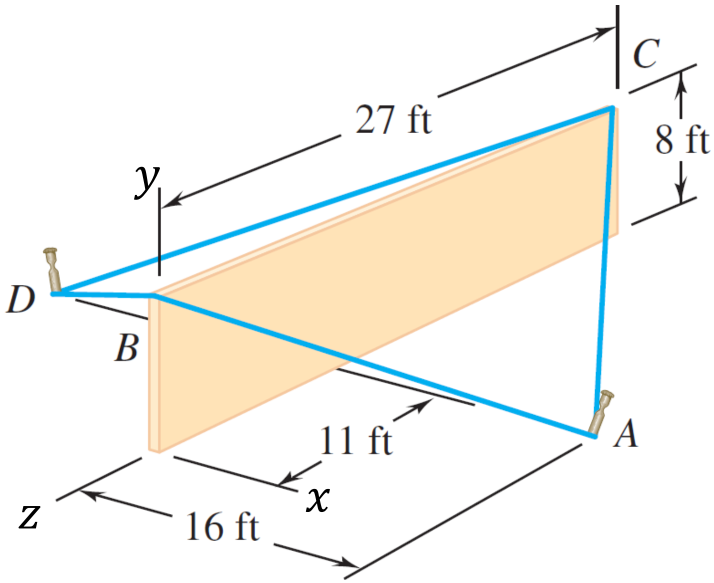
(a) the components F_x, F_y, and F_z of the force acting on the bolt at A,

(b) the angles θ_x, θ_y, and θ_z defining the direction of the force.



Sample Problem 2.8

A wall section of precast concrete is temporarily held in place by the cables shown. If the tension is 840 lb in cable AB and 1200 lb in cable AC, determine the magnitude and direction of the resultant of the forces exerted by cables AB and AC on stake A.



Equilibrium in Space

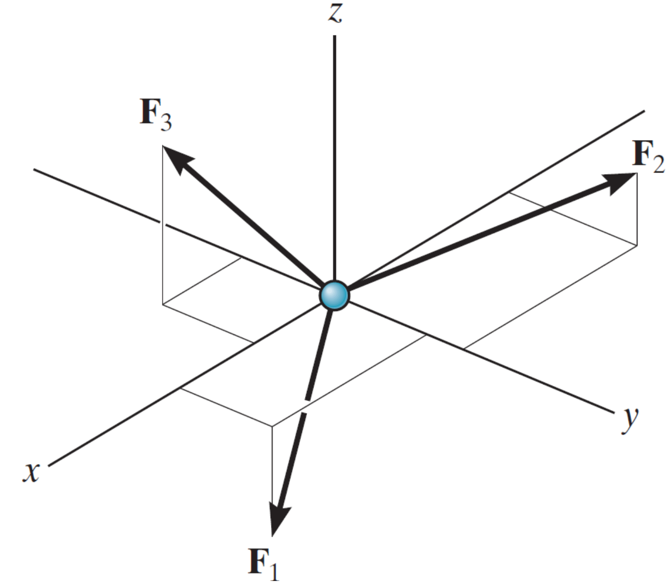
Equilibrium of a Particle

When the resultant of all the forces acting on a particle is **zero**, the particle is in **Equilibrium**.

$$\mathbf{R} = \Sigma\mathbf{F} = \mathbf{0}$$ (equation of equilibrium)

$$(\Sigma F_x)\mathbf{i} + (\Sigma F_y)\mathbf{j} + (\Sigma F_z)\mathbf{k} = \mathbf{0} \quad \rightarrow \quad \Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma F_z = 0$$

The first step in solving three-dimensional equilibrium problems is to draw a **free-body diagram** showing the particle in equilibrium and all of the forces acting on it.



Note that using these equations, we can determine at most three unknowns.

Sample Problem 2.9

A 200-kg cylinder is hung by means of two cables AB and AC that are attached to the top of a vertical wall. A horizontal force $\mathbf{P}$ perpendicular to the wall holds the cylinder in the position shown. Determine the magnitude of $\mathbf{P}$ and the tension in each cable ($g = 9.81$ m/s²).