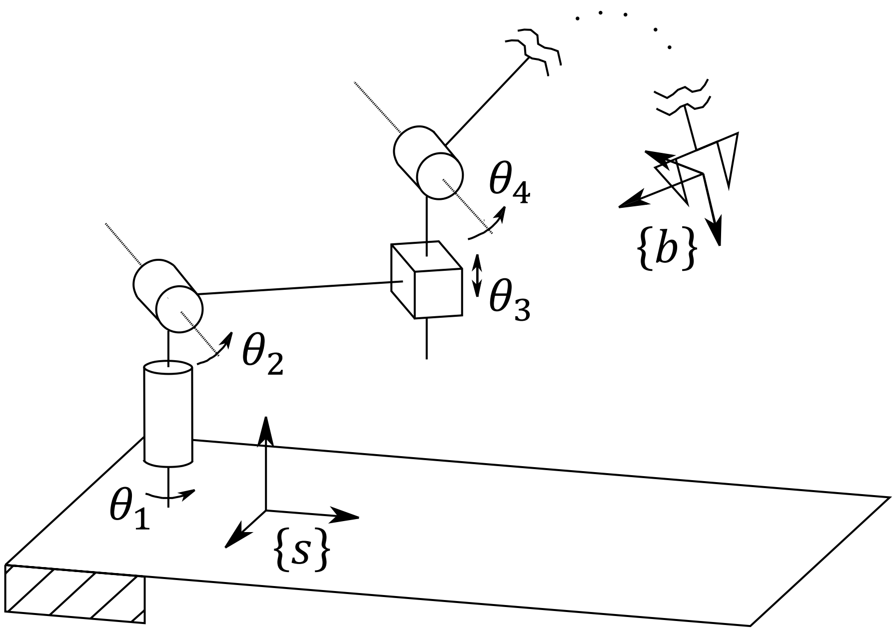
Ch5: Forward Kinematics

Forward Kinematics

Assumptions: Open-Chain Robot Manipulators

Robot manipulators are articulated mechanical systems composed of links connected by joints. In this course, we consider only n-DOF open-chain (serial) robot manipulators with revolute and/or prismatic joints.



- The generalized joint coordinate (joint position) denoted by θ_i corresponds to the angular displacement of a revolute joint or the linear displacement of a prismatic joint.
  Thus, vector of joint positions: $\boldsymbol{\theta} \in \mathbb{R}^n$

- Each joint is independently controlled through an actuator.
- The joint positions are measured by sensors placed at the actuators, that are usually located at the joints.

Forward Kinematics

The forward kinematics of a robot refers to the calculation of the position and orientation (**pose**) of its end-effector frame from its joint positions $\boldsymbol{\theta}$.

- "Geometric" forward kinematics:

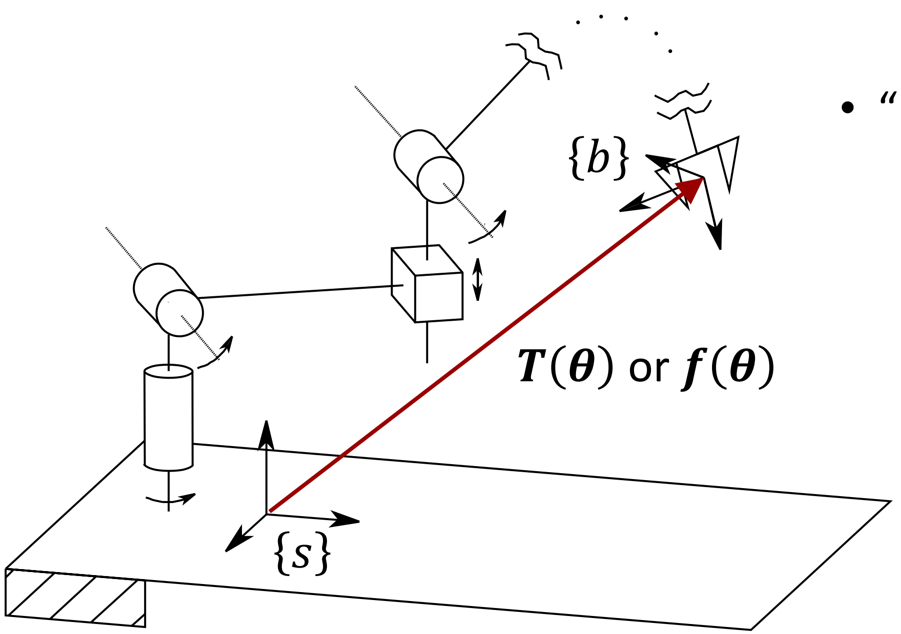
$$\text{Given } \boldsymbol{\theta} \in \mathbb{R}^n, \text{ Find } \boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$$

$$\boldsymbol{T}: \mathbb{R}^n \rightarrow SE(3)$$

- "Minimum-Coordinate" forward kinematics:

$$\text{Given } \boldsymbol{\theta} \in \mathbb{R}^n, \text{ Find } \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$$

$$(m \leq n) \qquad \boldsymbol{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$$



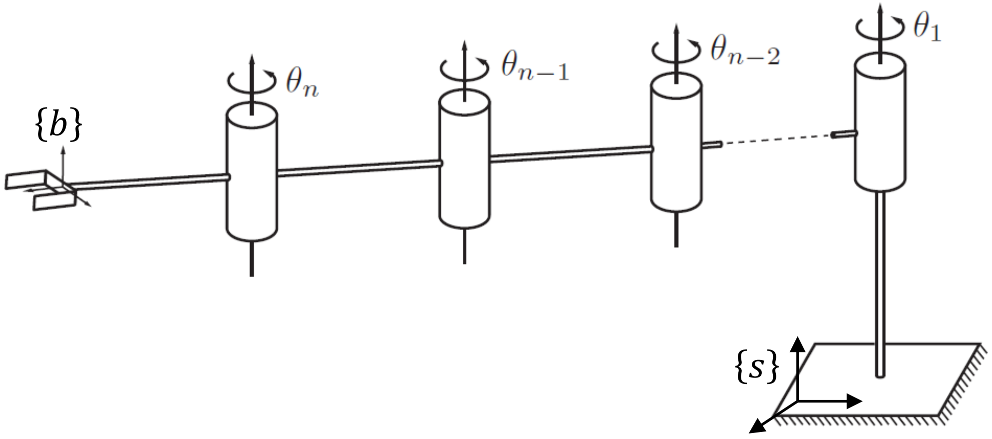
Methods of forward kinematics:
- Product of Exponentials (PoE)
- Denavit-Hartenberg (DH)

Product of Exponentials (PoE) Formulation in the Base Frame

Forward Kinematics Using Product of Exponentials (PoE) Formulation in Base Frame

Calculating the forward kinematics of an open chain using the **space form** of the PoE formula:

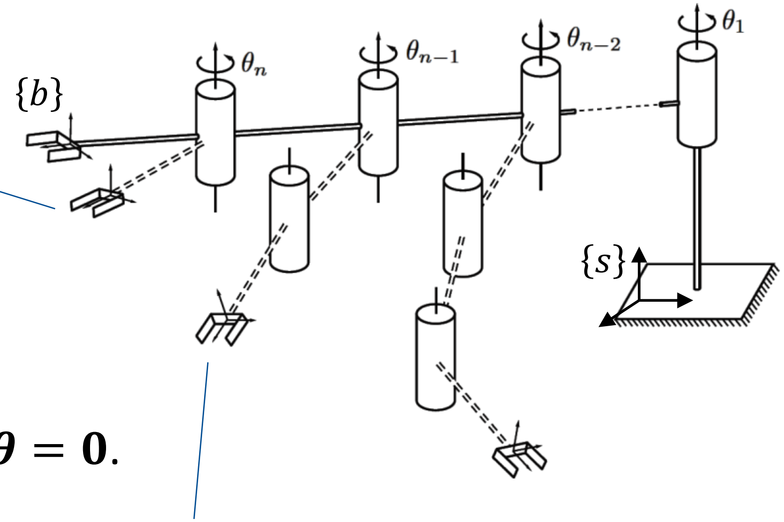
- Assign a fixed base frame $\{s\}$.
- Assign an end-effector frame $\{b\}$.
- Let $\boldsymbol{M} = \boldsymbol{T}_{sb}(\mathbf{0}) \in SE(3)$ be the configuration of $\{b\}$ relative to $\{s\}$ when the robot is in its <u>home</u> or <u>zero</u> configuration ($\boldsymbol{\theta} = \mathbf{0}$).
- Specify the direction of positive displacement (rotation for revolute joints, translation for prismatic joints) for each joint.



Forward Kinematics Using Product of Exponentials (PoE) Formulation in Base Frame (cont.)

- Suppose that joint n is displaced by θ_n (for $\theta_{1,\dots,n-1} = 0$). Then, the new configuration of $\{b\}$ is

$$\boldsymbol{T} = e^{[\boldsymbol{S}_n]\theta_n}\boldsymbol{M} \in SE(3)$$



$$\boldsymbol{S}_n = \begin{bmatrix} \boldsymbol{S}_{\omega,n} \\ \boldsymbol{S}_{v,n} \end{bmatrix} \in \mathbb{R}^6 \qquad [\boldsymbol{S}_n] = \begin{bmatrix} [\boldsymbol{S}_{\omega,n}] & \boldsymbol{S}_{v,n} \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$$

$\boldsymbol{S}_n$: Screw axis of joint n as expressed in $\{s\}$ when $\boldsymbol{\theta} = \boldsymbol{0}$.

- Suppose that joint $n-1$ is displaced by θ_{n-1} (for $\theta_{1,\dots,n-2} = 0$ and any fixed, but arbitrary, θ_n). Then, the new configuration of $\{b\}$ is

$$\boldsymbol{T} = e^{[\boldsymbol{S}_{n-1}]\theta_{n-1}}\left(e^{[\boldsymbol{S}_n]\theta_n}\boldsymbol{M}\right)$$

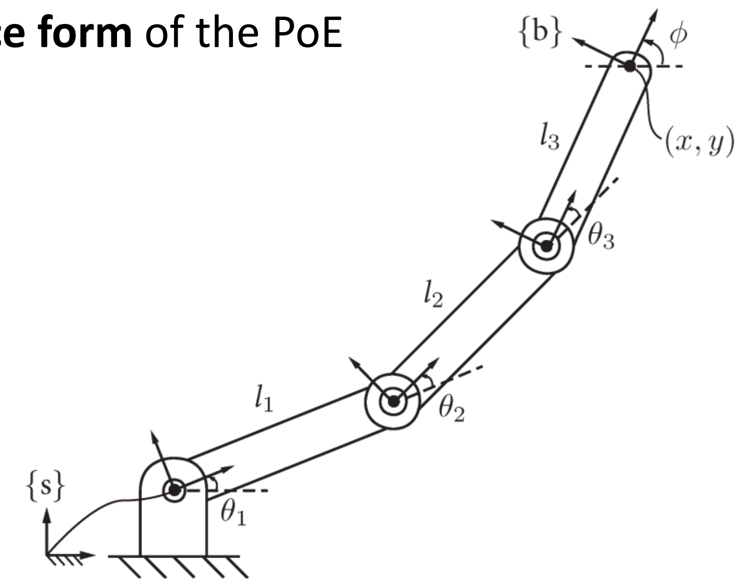
Continuing this for all the joints:

$$\boxed{\boldsymbol{T}(\boldsymbol{\theta}) = e^{[\boldsymbol{S}_1]\theta_1} \dots e^{[\boldsymbol{S}_{n-1}]\theta_{n-1}} e^{[\boldsymbol{S}_n]\theta_n}\boldsymbol{M}}$$

The screw axes $\boldsymbol{S}_1, \dots, \boldsymbol{S}_n$ expressed in $\{s\}$, corresponding to the joint motions when the robot is at its home configuration ($\boldsymbol{\theta} = \boldsymbol{0}$).

Example: 3R Planar Robot

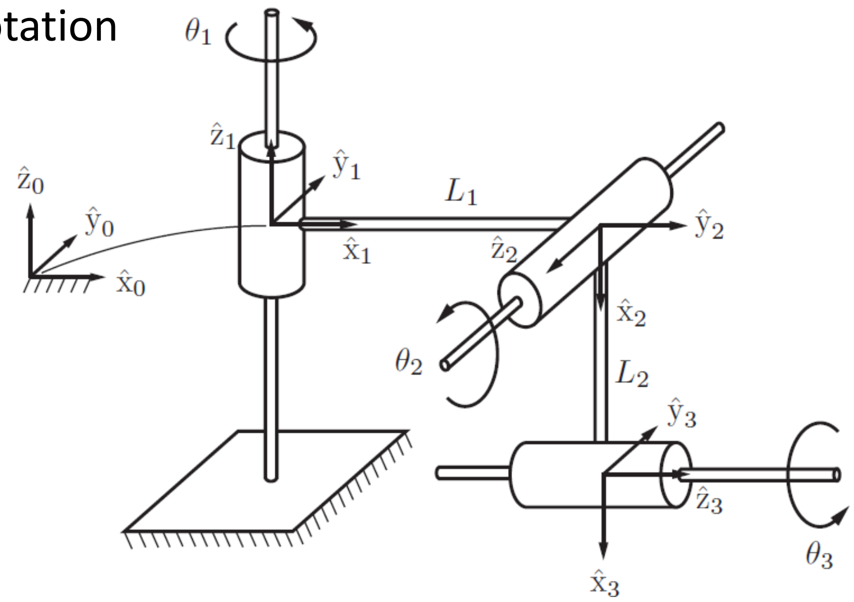
Find the Geometric forward kinematics using the **space form** of the PoE formulation.



Example: 3R Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

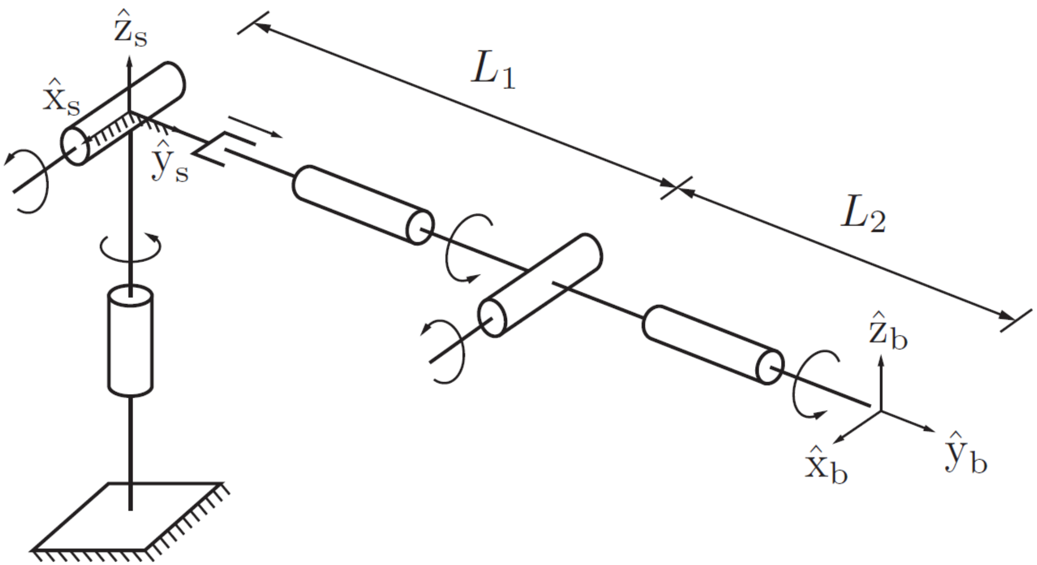
Find the Geometric forward kinematics using the **space form** of the PoE formulation.



Example: RRPRRR Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

Find the Geometric forward kinematics using the **space form** of the PoE formulation.

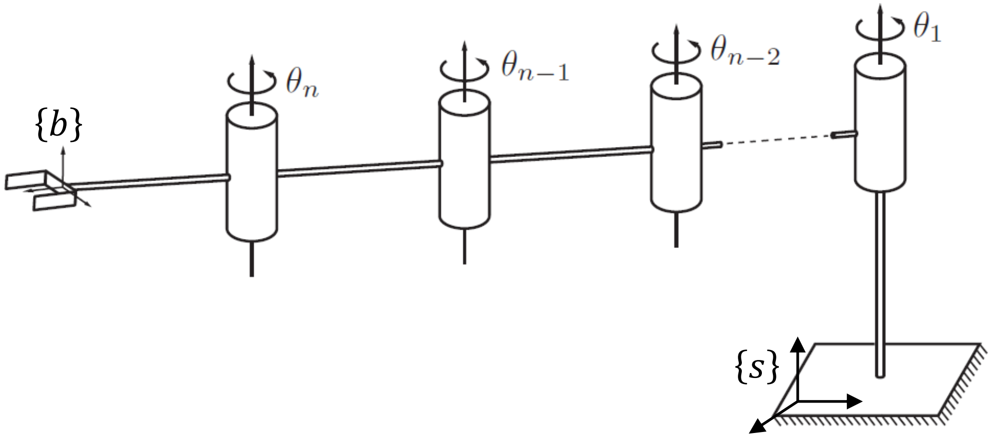


Product of Exponentials (PoE) Formulation in the End-Effector Frame

Forward Kinematics Using Product of Exponentials (PoE) Formulation in EE Frame

An alternative method to calculate the forward kinematics of an open chain is using the **body form** of the PoE formula.

- Assign a fixed base frame $\{s\}$.
- Assign an end-effector frame $\{b\}$.
- Let $\boldsymbol{M} \in SE(3)$ be the configuration of $\{b\}$ relative to $\{s\}$ when the robot is in its <u>home</u> or <u>zero</u> configuration.
- Specify the direction of positive displacement (rotation for revolute joints, translation for prismatic joints) for each joint.

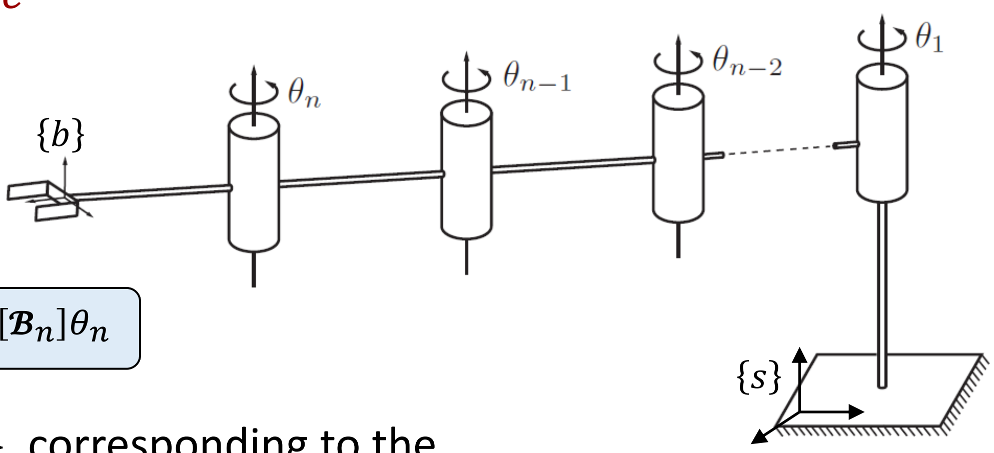


Forward Kinematics Using Product of Exponentials (PoE) Formulation in EE Frame (cont.)

We know that $e^{\boldsymbol{M}^{-1}\boldsymbol{P}\boldsymbol{M}} = \boldsymbol{M}^{-1}e^{\boldsymbol{P}}\boldsymbol{M}$, thus, $\boldsymbol{M}e^{\boldsymbol{M}^{-1}\boldsymbol{P}\boldsymbol{M}} = e^{\boldsymbol{P}}\boldsymbol{M}$.

$$
\begin{aligned}
\boldsymbol{T}(\boldsymbol{\theta}) &= e^{[\boldsymbol{\mathcal{S}}_1]\theta_1} \dots e^{[\boldsymbol{\mathcal{S}}_{n-1}]\theta_{n-1}} e^{[\boldsymbol{\mathcal{S}}_n]\theta_n} \boldsymbol{M} \\
&= e^{[\boldsymbol{\mathcal{S}}_1]\theta_1} \dots e^{[\boldsymbol{\mathcal{S}}_{n-1}]\theta_{n-1}} \boldsymbol{M} e^{\boldsymbol{M}^{-1}[\boldsymbol{\mathcal{S}}_n]\boldsymbol{M}\theta_n} \\
&= e^{[\boldsymbol{\mathcal{S}}_1]\theta_1} \dots \boldsymbol{M} e^{\boldsymbol{M}^{-1}[\boldsymbol{\mathcal{S}}_{n-1}]\boldsymbol{M}\theta_{n-1}} e^{\boldsymbol{M}^{-1}[\boldsymbol{\mathcal{S}}_n]\boldsymbol{M}\theta_n} \\
&= \boldsymbol{M} e^{\boldsymbol{M}^{-1}[\boldsymbol{\mathcal{S}}_1]\boldsymbol{M}\theta_1} \dots e^{\boldsymbol{M}^{-1}[\boldsymbol{\mathcal{S}}_{n-1}]\boldsymbol{M}\theta_{n-1}} e^{\boldsymbol{M}^{-1}[\boldsymbol{\mathcal{S}}_n]\boldsymbol{M}\theta_n} \\
&= \boldsymbol{M} e^{[\boldsymbol{\mathcal{B}}_1]\theta_1} \dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}} e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}
\end{aligned}
$$

$$
\begin{cases}
[\boldsymbol{\mathcal{B}}_i] = \boldsymbol{M}^{-1}[\boldsymbol{\mathcal{S}}_i]\boldsymbol{M} \\
\boldsymbol{\mathcal{B}}_i = [\mathrm{Ad}_{\boldsymbol{M}^{-1}}]\boldsymbol{\mathcal{S}}_i
\end{cases}
$$



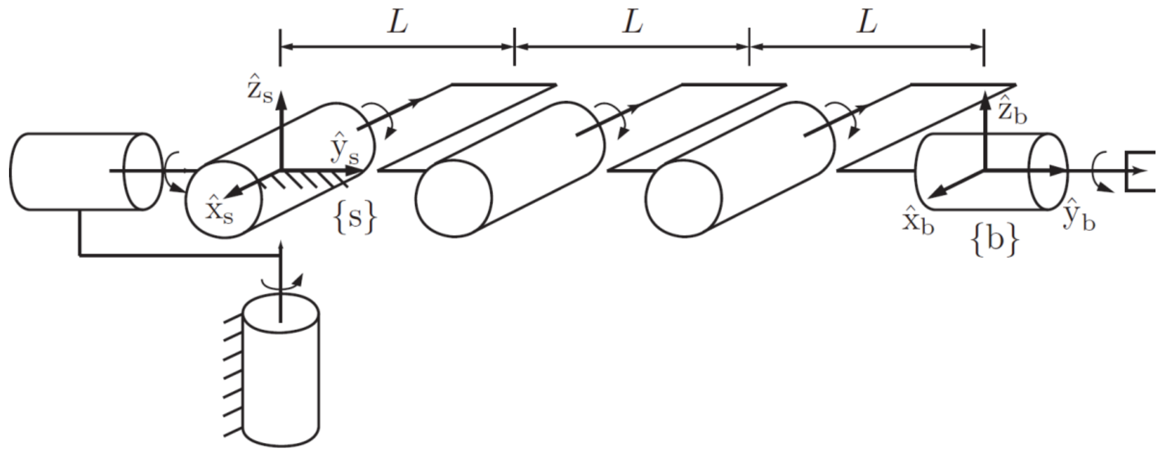
$$
\boxed{\boldsymbol{T}(\boldsymbol{\theta}) = \boldsymbol{M} e^{[\boldsymbol{\mathcal{B}}_1]\theta_1} \dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}} e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}}
$$

The screw axes $\boldsymbol{\mathcal{B}}_1, \dots, \boldsymbol{\mathcal{B}}_n$ expressed in $\{b\}$, corresponding to the joint motions when the robot is at its home configuration ($\boldsymbol{\theta} = \boldsymbol{0}$).

Example: 6R Spatial Robot

The zero position and the direction of positive rotation for each joint axis are as shown in the figure.

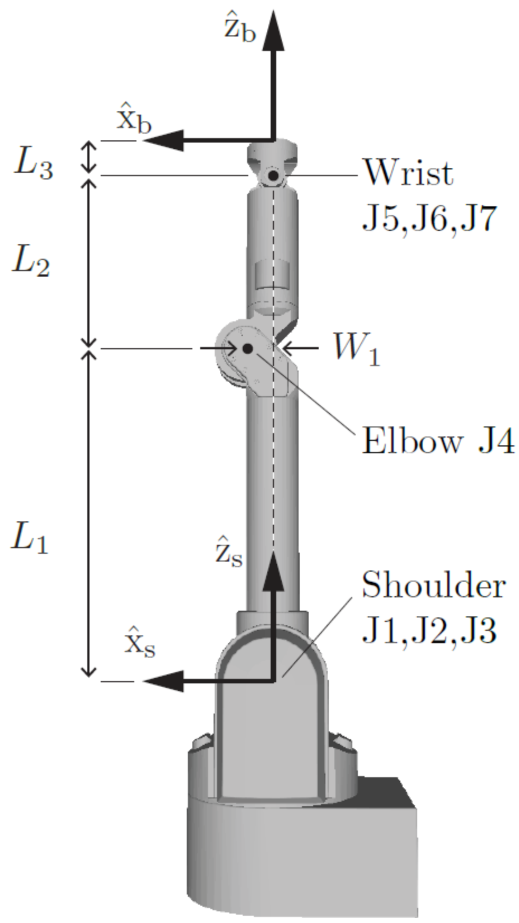
Find the Geometric forward kinematics using the **body form** of the PoE formulation.



Example: 7R Spatial Robot

At the zero configuration, axes 1, 3, 5, and 7 are along $\hat{z}_s$ and axes 2, 4, and 6 are aligned with $\hat{y}_s$. Positive rotations are given by the right-hand rule. Axes 1, 2, and 3 intersect at the origin of $\{s\}$ and axes 5, 6, and 7 intersect at the wrist. The zero configuration is singular.

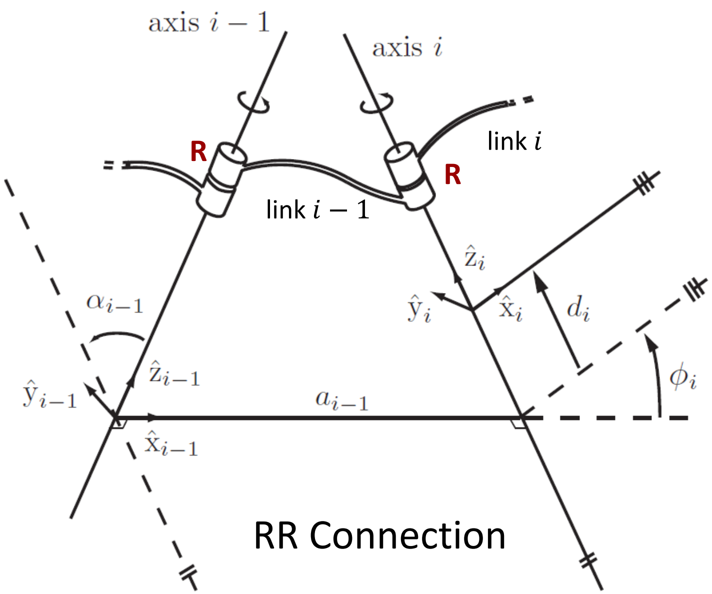
Find the Geometric forward kinematics using the **body form** of the PoE formulation.



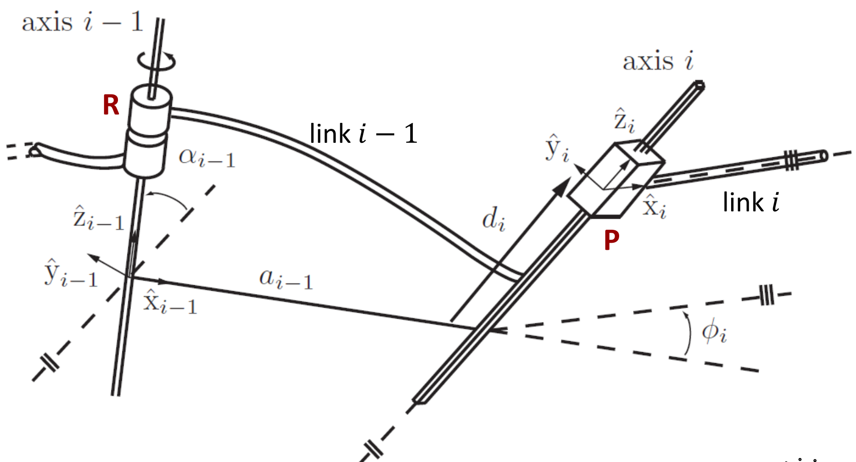
Denavit–Hartenberg Parameters

Assigning Link Frames

The basic idea of Denavit–Hartenberg (DH) method is to attach reference frames to each link of the open chain to derive the forward kinematics from the relative displacements between adjacent link frames.

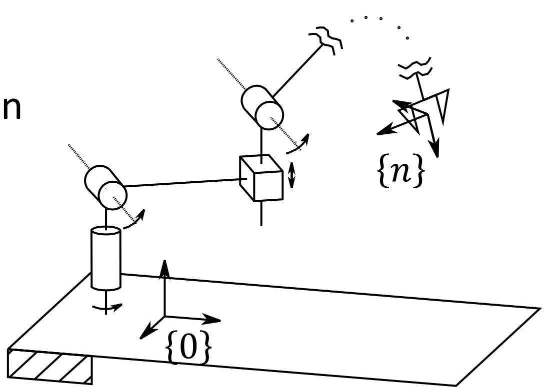


RR Connection



RP Connection

Consider an n-link open chain manipulator connected by 1 DOF joints. Attach a frame $\{0\}$ to the base, frames $\{1\}$ to $\{n\}$ to the links 1 to n (end-effector), based on the following rules:



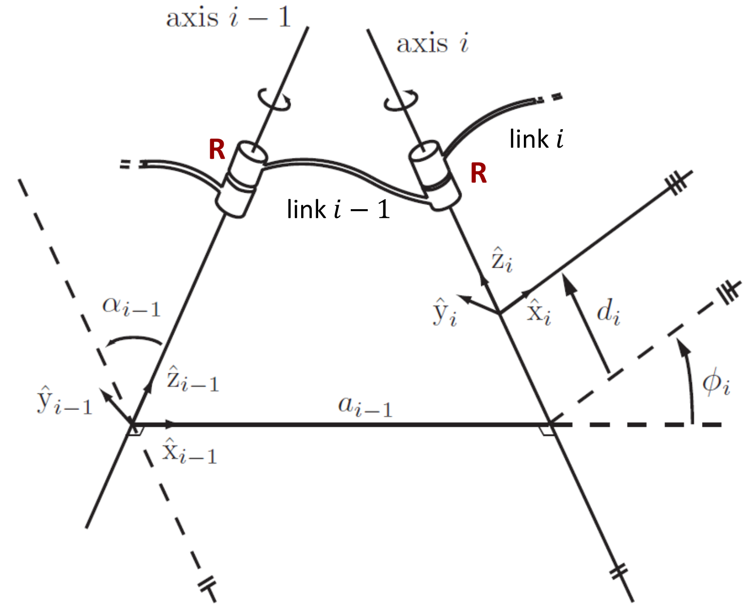
Assigning Link Frames

❶ $\hat{z}_{i-1}$ coincides with joint axis $i-1$ and $\hat{z}_i$ coincides with joint axis i, along the positive direction of rotation (by the right-hand rule) or translation.

❷ Connect the joint axes $\hat{z}_{i-1}$ and $\hat{z}_i$ by a mutually perpendicular line (if this line is not unique or fails to exist, refer to the *Special Cases*). The origin of $\{i-1\}$ is then located at the point where this line intersects joint axis $i-1$.

❸ $\hat{x}$-axis is chosen to be in the direction of the mutually perpendicular line pointing from the $(i-1)$-axis to the i-axis.

❹ $\hat{y}$-axis is determined from $\hat{x} \times \hat{y} = \hat{z}$.



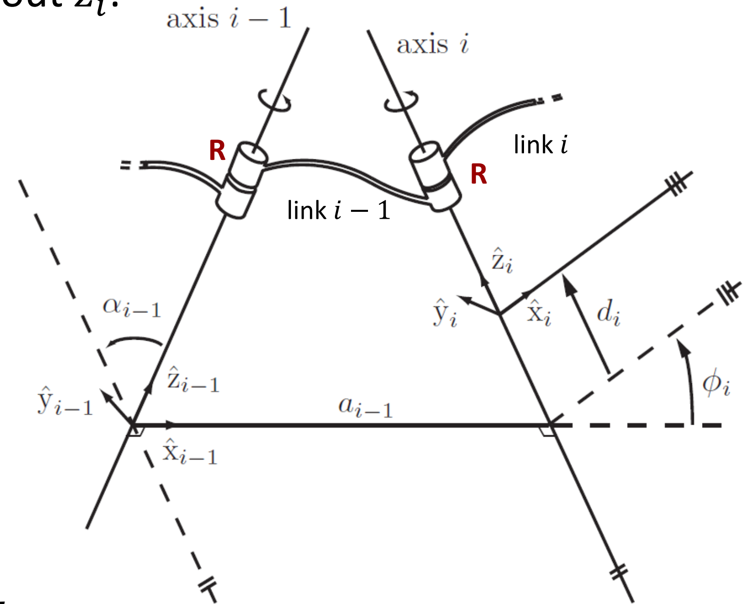
Note: $\{0\}$ is chosen to coincide with $\{1\}$ in its zero position. $\{n\}$ is attached to a point on the end-effector that makes the description of the task intuitive and/or make as many of the DH parameters as possible zero.

Denavit–Hartenberg (DH) Parameters

Four DH parameters that exactly specify $\boldsymbol{T}_{i-1,i}$:

- a_{i-1} (link length): The length of the mutually perpendicular line.
- α_{i-1} (link twist): The angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, measured about $\hat{x}_{i-1}$.
- d_i (link offset): The distance from the intersection of $\hat{x}_{i-1}$ and $\hat{z}_i$ to the origin of $\{i\}$ along $\hat{z}_i$.
- ϕ_i (joint angle): The angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, measured about $\hat{z}_i$.

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1				
2				
3				
$\vdots$				



Note: For a revolute joint ϕ_i, and for a prismatic joint d_i acts as the joint variable, and the other 3 parameters are all constant.

Manipulator Forward Kinematics

Transporting from $\{i-1\}$ to $\{i\}$:

i. A rotation of $\{i-1\}$ about its $\hat{x}$-axis by α_{i-1}.
ii. A translation of the frame along its $\hat{x}$-axis by a_{i-1}.
iii. A translation of the frame along its $\hat{z}$-axis by d_i.
iv. A rotation of the frame about its $\hat{z}$-axis by ϕ_i.

$$\boldsymbol{T}_{i-1,i} = \mathrm{Rot}(\hat{x},\alpha_{i-1})\mathrm{Trans}(\hat{x},a_{i-1})\mathrm{Trans}(\hat{z},d_i)\mathrm{Rot}(\hat{z},\phi_i) \in SE(3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & 0 \\ \sin\phi_i & \cos\phi_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i\cos\alpha_{i-1} & \cos\phi_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\ \sin\phi_i\sin\alpha_{i-1} & \cos\phi_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the forward kinematics can be expressed as

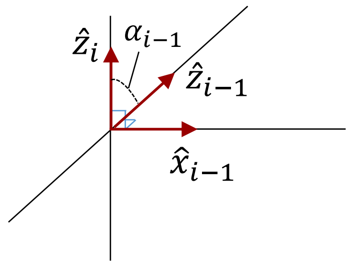
$$\boldsymbol{T}_{0n}(\theta_1,\dots,\theta_n) = \boldsymbol{T}_{01}(\theta_1)\boldsymbol{T}_{12}(\theta_2)\cdots\boldsymbol{T}_{n-1,n}(\theta_n) \in SE(3)$$

Assigning Link Frames: Special Cases

(when a mutually perpendicular line is (1) undefined or (2) not unique)

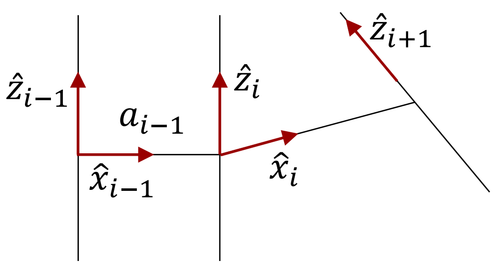
(1) When Adjacent Joint Axes $i-1$ and i Intersect:

In this case, $a_{i-1} = 0$, and we choose $\hat{x}_{i-1}$ to be perpendicular to the plane spanned by $\hat{z}_{i-1}$ and $\hat{z}_i$ (at intersection). There are two acceptable possibilities: one leads to a positive value of α_{i-1} while the other leads to a negative value.



(2) When Adjacent Joint Axes $i-1$ and i Are Parallel:

In this case there exist many possibilities for a mutually perpendicular line, all of which are valid. Choose the line that is the most physically intuitive and that results in as many zero DH parameters as possible.



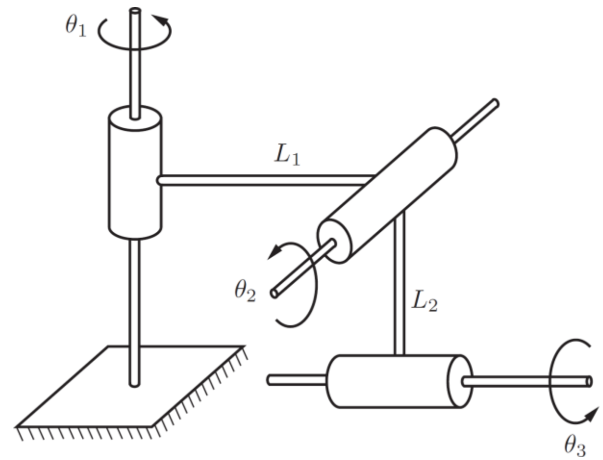
Some Remarks on DH Parameters

- In general, a minimum of six independent parameters are required to describe the relative displacement between two frames $\boldsymbol{T}_{i-1,i}$ in space (3 for the orientation and 3 for the position).
- In DH parameter representation, a minimum of four parameters are required for each transformation $\boldsymbol{T}_{i-1,i}$ (i.e., for an n-DOF open-chain robot, $4n$ DH parameters are sufficient to completely describe the forward kinematics).
- The reduction in the number of parameters is due to carefully assigning link frames based on some rules. If the link reference frames are assigned in arbitrary fashion, then more parameters are required.
- Under the DH convention, only rotations and translations along the $\hat{x}$- and $\hat{z}$-axes are allowed.

Examples

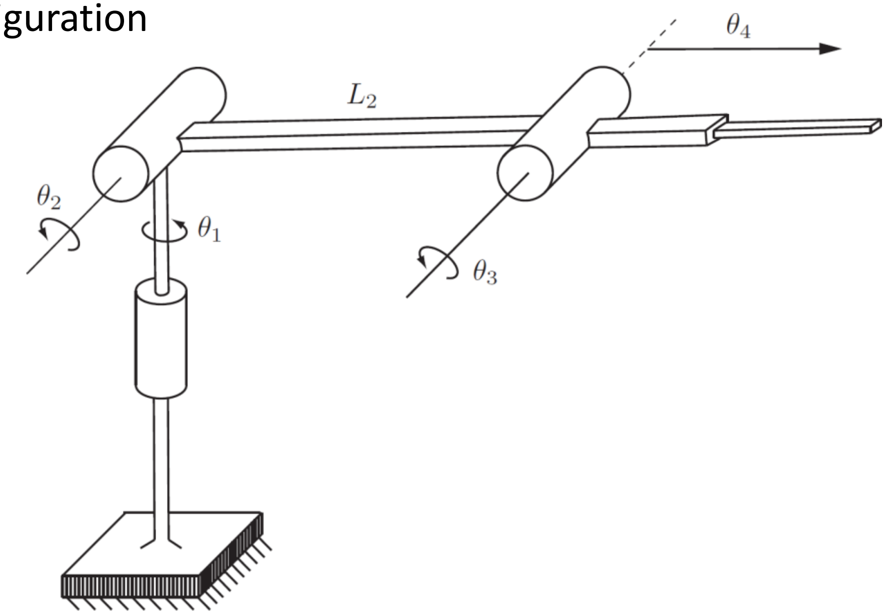
Example 1: A 3R spatial open chain in its zero configuration.

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1				
2				
3				



Example 2: A spatial RRRP open chain in its zero configuration

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1				
2				
3				
4				



Redundancy

Intrinsic and Kinematic Redundancy in Open-Chain Robot Manipulators

For an open-chain manipulator, we can define:

$\dim(\text{C–Space}) = \dim(\text{J–Space}) = n$ (Configuration Space or Joint Space)

$\dim(\text{O–Space}) = m$ (Operational Space) [$m = 3$ for planar & $m = 6$ for spatial]

$\dim(\text{T–Space}) = r$ (Task Space)

- A manipulator is **intrinsically redundant** when the dimension of the joint space is greater than the dimension of the operational space (i.e., $n > m$). Ex.: 4R Planar Robot ($n = 4, m = 3$)
- A manipulator is **kinematically redundant** when the dimension of the joint space is greater than the dimension of the task space (i.e, $n > r$) and there exist $n - r$ redundant DOFs.

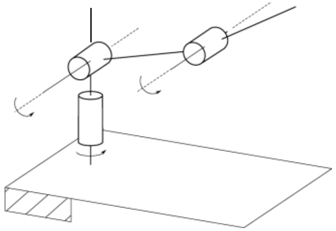
Note: A manipulator can be redundant with respect to a task and nonredundant with respect to another.

Ex: 3R Planar Robot
$$\begin{cases} n = m = 3, r = 2\ (x, y) & \text{redundant} \\ n = m = r = 3\ (x, y, \phi) & \text{nonredundant} \end{cases}$$

Ex: 3R Spatial Robot

$n = m = r = 3\ (x, y, z)$

nonredundant



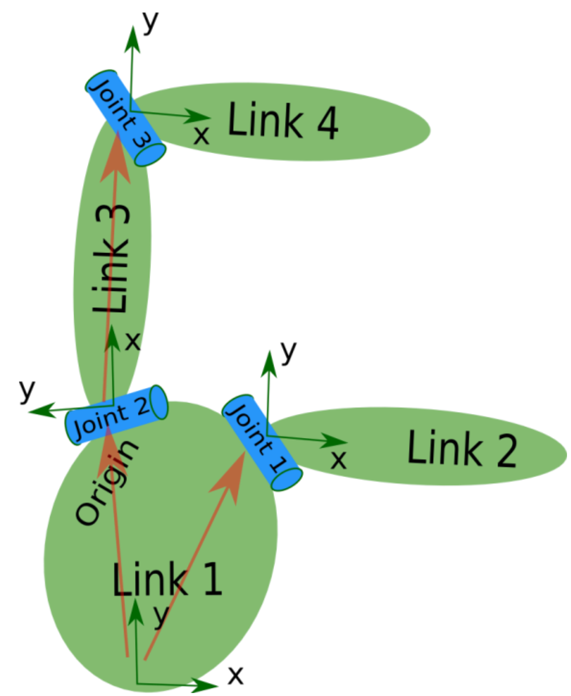
- Redundancy can provide the manipulator with dexterity and versatility in its motion. Thus, it is possible to avoid obstacles in the workspace or to optimize some objective function such as minimizing the motor power needed to hold the end-effector at that configuration.

Universal Robot Description Format (URDF)

Universal Robot Description Format

The Universal Robot Description Format (URDF) is an XML (eXtensible Markup Language) file format used by the Robot Operating System (ROS) to describe the kinematics (in defining joints), inertial properties, and link geometry of robots (in defining links) of open-chain robots in home/zero configuration.

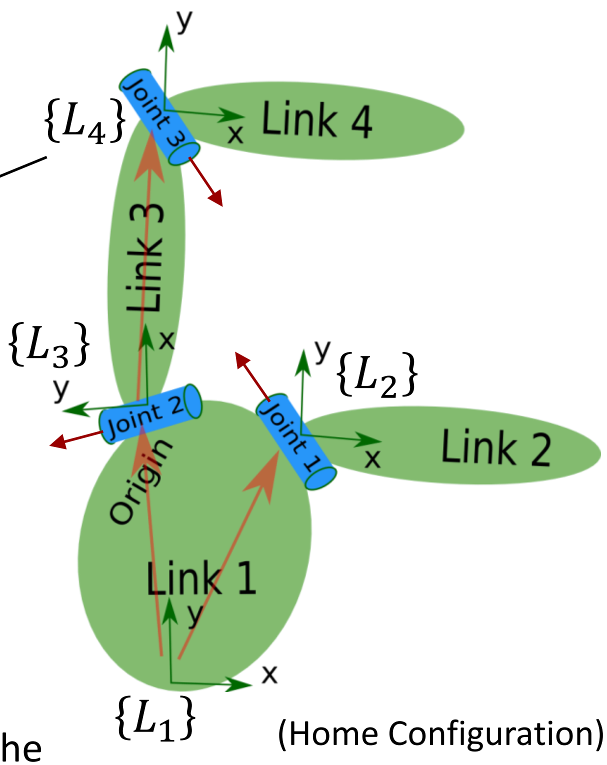
```
<joint name="joint1" type="continuous">
        <parent link="link1"/>
        <child link="link2"/>
        <origin xyz=“0.5 0.3 0" rpy="0 0 0" />
        <axis xyz="-0.9 0.15 0" />
</joint>

<link name="link1">
        <inertial>
                <mass value=“1"/>
                <origin rpy="0.1 0 0" xyz="0 0 0"/>
                <inertia ixx="0.004" ixy="0" ixz="0"
                        iyy="0.004" iyz="0" izz="0.007"/>
        </inertial>
</link>
```



URL

URDF: Defining Joints

Joints connect two links: a parent link and a child link. The reference frame of each (child) link $\{L_i\}$ is located (at the bottom of the link) on the joint's axis.




```
<joint name="joint3" type="continuous">
        <parent link="link3"/>
        <child link="link4"/>
        <origin xyz="0.5 0 0" rpy="0 0 -1.57" />    : {L4} w.r.t. {L3}
        <axis xyz="0.707 -0.707 0" />    : in {L4}
</joint>
```

"origin" defines the pose of the child link frame relative to the parent link frame when the joint variable is zero.

"axis" defines the joint's axis, a unit vector expressed in the child link's frame, in the direction of positive rotation for a revolute joint or positive translation for a prismatic joint.

URDF: Defining Links

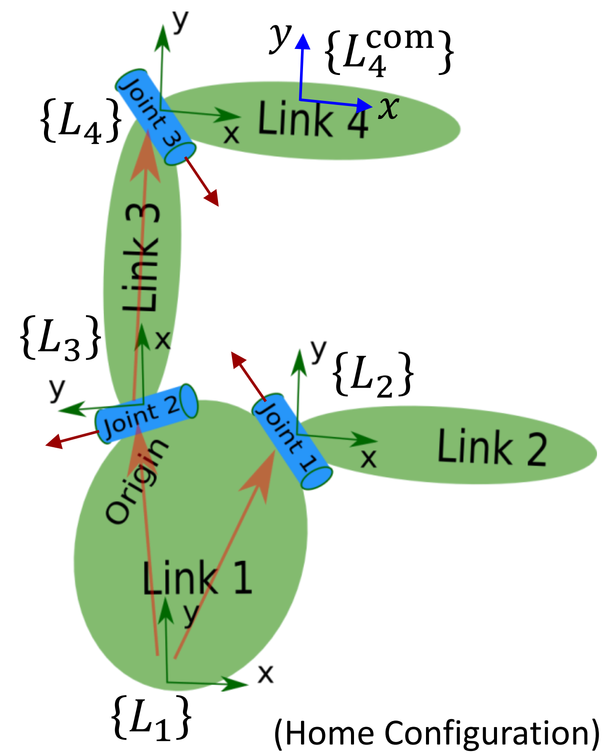
```
<link name=" link4">
    <inertial>
        <mass value="1"/>
        <origin xyz="0.1 0 0" rpy="0 0 0"/>
        <inertia ixx="0.004" ixy="0" ixz="0"
            iyy="0.004" iyz="0" izz="0.007"/>
    </inertial>
    <visual>
        <geometry>
            <mesh filename="…/link1.stl" />
        </geometry>
        <material name="DarkGrey">
            <color rgba="0.3 0.3 0.3 1.0"/>
        </material>
    </visual>
</link>
```

$\boxed{: \{L_4^{\text{com}}\} \text{ w.r.t. } \{L_4\}}$ (origin)

$\boxed{: \text{ in } \{L_4^{\text{com}}\}}$ (inertia)

"inertia" defines six elements of inertia matrix relative to the link's center of mass.

"origin" defines the position and orientation of a frame at the link's center of mass relative to the link's frame at its joint.



(Home Configuration)

URDF

```xml
<robot name="test_robot">
  <link name="link1" />
  <link name="link2" />
  <link name="link3" />
  <link name="link4" />

  <joint name="joint1" type="continuous">
    <parent link="link1"/>
    <child link="link2"/>
    <origin xyz="5 3 0" rpy="0 0 0" />
    <axis xyz="-0.9 0.15 0" />
  </joint>

  <joint name="joint2" type="continuous">
    <parent link="link1"/>
    <child link="link3"/>
    <origin xyz="-2 5 0" rpy="0 0 1.57" />
    <axis xyz="-0.707 0.707 0" />
  </joint>
```

```xml
  <joint name="joint3" type="continuous">
    <parent link="link3"/>
    <child link="link4"/>
    <origin xyz="5 0 0" rpy="0 0 -1.57" />
    <axis xyz="0.707 -0.707 0" />
  </joint>
</robot>
```