

Ch5: Phase Plane Analysis

Phase Plane Concept

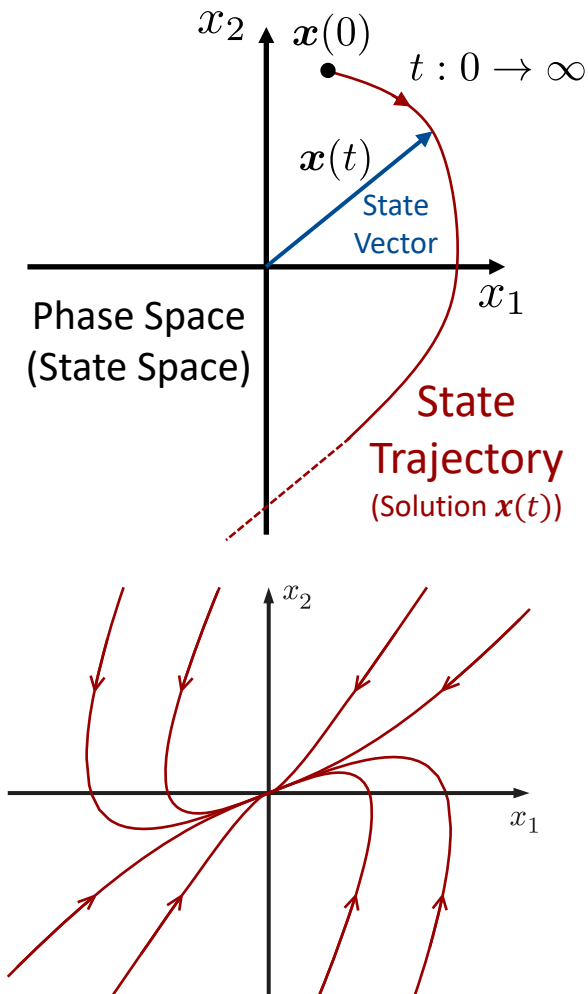
Phase Plane & Phase Portrait

- A **two-dimensional** state space plane is called the **Phase Plane**.
- Given a set of **initial conditions** $x(0)$, the solution $x(t)$ of a **second-order autonomous** system, when t varied from 0 to ∞ , can be represented geometrically as a curve (**trajectory**) in the phase plane (arrows denote the direction of motion).

$$\dot{x}(t) = f(x(t)) \implies \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

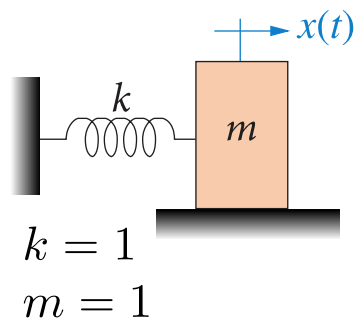
$$\text{Slope of trajectory: } \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

A **family** of phase plane trajectories corresponding to **various initial conditions** is called a **phase portrait** of a system.



Example: Phase portrait of a linear system

A mass-spring system:



x_0 : Initial position

\dot{x}_0 : Initial velocity

$$\ddot{x} + x = 0$$

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

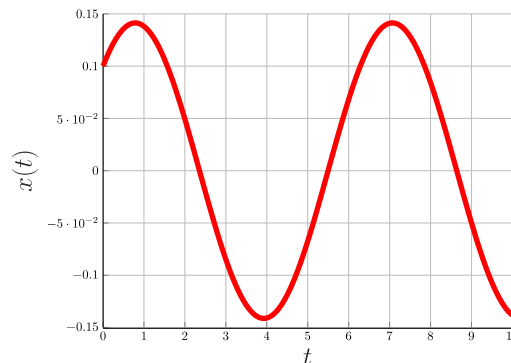
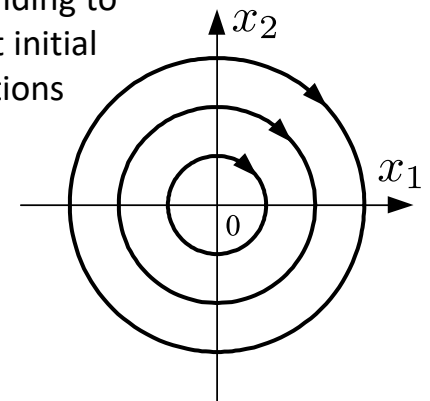
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

$$\begin{aligned} x_1 &= x_0 \cos t + \dot{x}_0 \sin t \\ x_2 &= -x_0 \sin t + \dot{x}_0 \cos t \end{aligned}$$

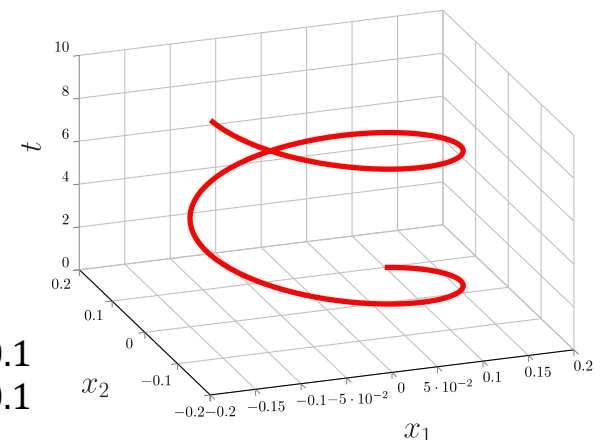
$$x_1^2 + x_2^2 = x_0^2 + \dot{x}_0^2$$

Equation of the trajectories

Corresponding to different initial conditions



$$\begin{aligned} x_0 &= 0.1 \\ \dot{x}_0 &= 0.1 \end{aligned}$$



Singular Point

An **equilibrium point** of a second-order system is called a **Singular Point**.

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) = 0 \implies \begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}$$

Slope of trajectory
at singular points:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \frac{0}{0} \quad (\text{undefined!})$$

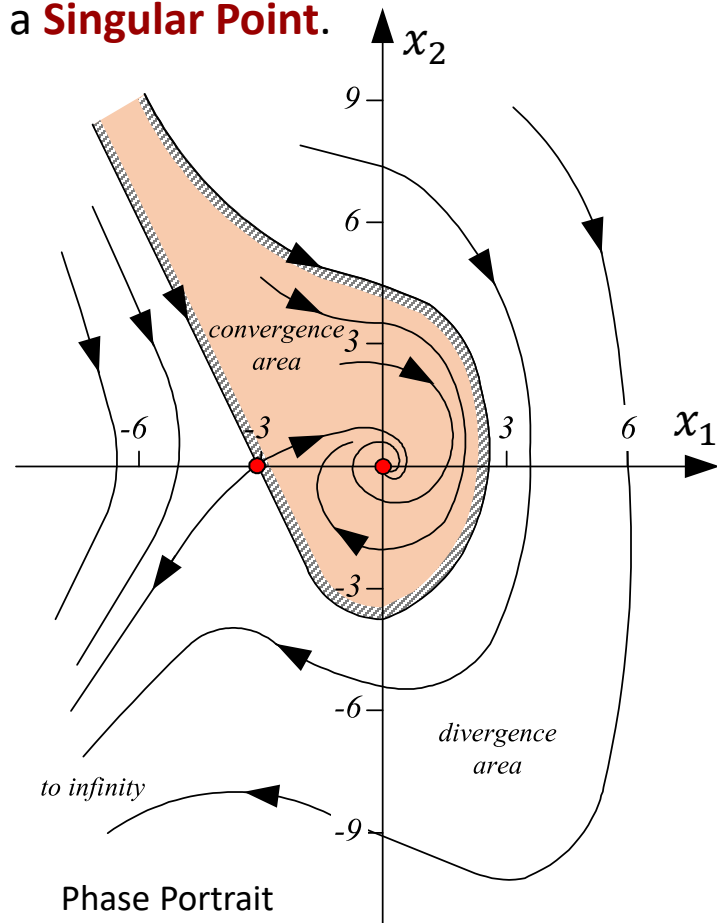
Phase portrait of a nonlinear 2nd order system:

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

The system has two singular points: $(0, 0)$, $(-3, 0)$

$$x_1 = x$$

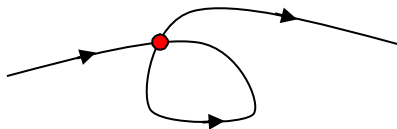
$$x_2 = \dot{x}$$



Singular Point

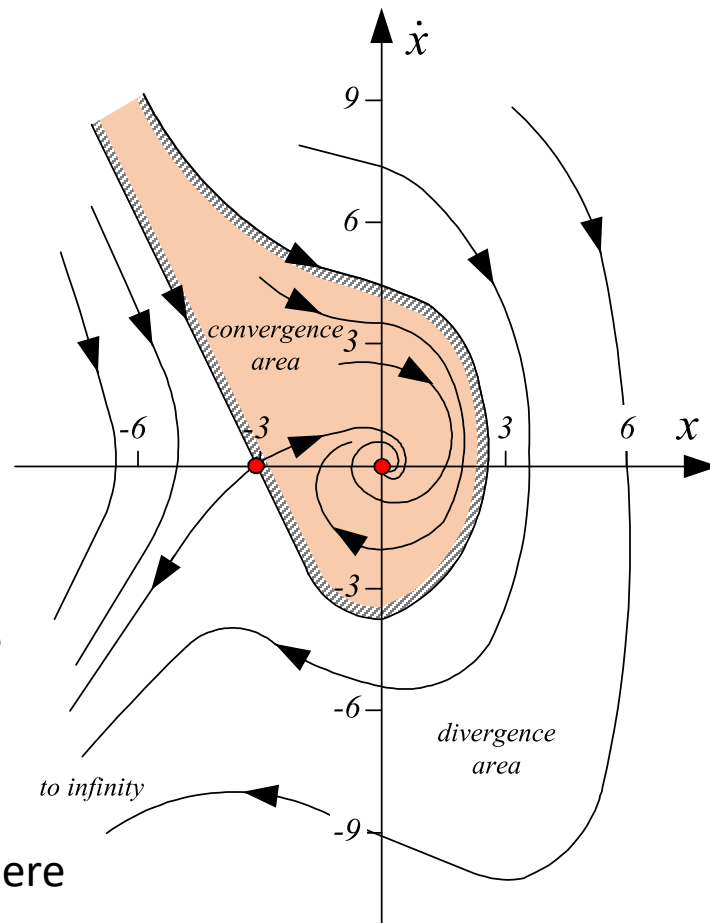
Note: The motion patterns of the system trajectories in the vicinity of the two singular points may have different natures!

Note: With the functions f_1 and f_2 assumed to be single valued, a phase trajectory **cannot intersect itself**!



Note: Singular points are very important features in the phase plane, e.g., for **linear systems**, the **stability** of the systems is uniquely characterized by the **nature of their singular points**.

Note: For **nonlinear systems**, besides singular points, there may be more complex features, such as **limit cycles**.



Phase Plane for First-order Systems

Although the phase plane method is developed primarily for second-order systems, it can also be applied to the analysis of **first-order** systems of the form

$$\dot{x} + f(x) = 0$$

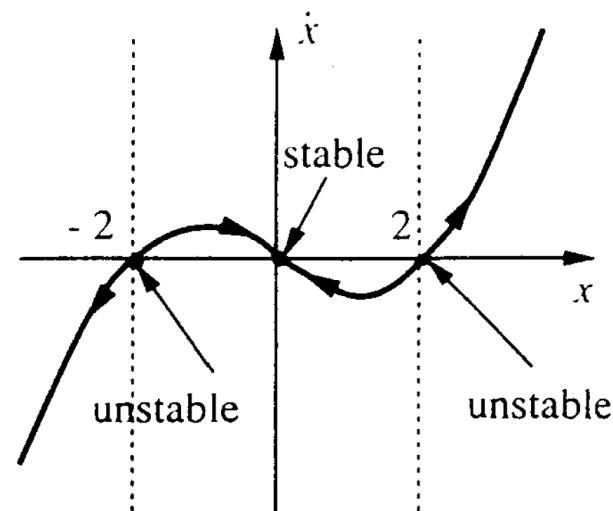
The difference now is that the phase portrait is composed of a **single trajectory**.

Example: Consider the first-order system $\dot{x} = -4x + x^3$

There are three singular points ($\dot{x} = -4x + x^3 = 0$):

$$x = 0, -2, 2$$

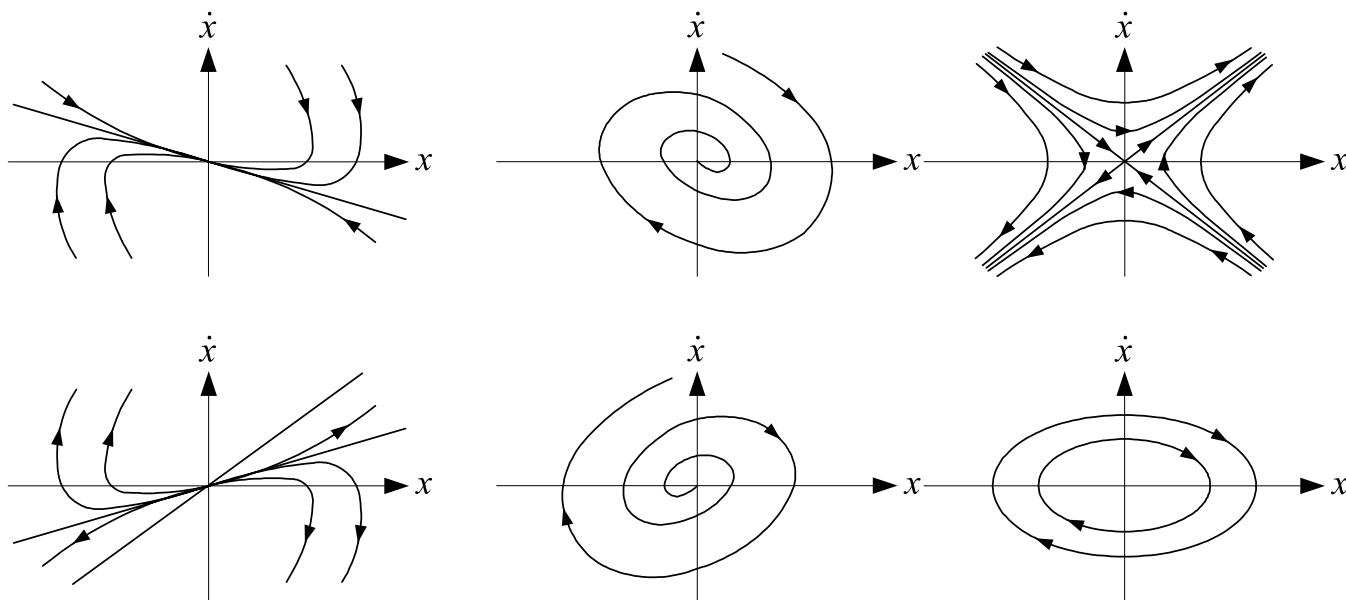
The arrows denote the direction of motion, and it is determined by the sign of \dot{x} at that point.



Phase Plane Analysis

Phase Plane Analysis

Phase plane analysis is a **graphical method** to visually examine the global behavior of **second-order** autonomous systems, i.e., **stability** and **motion patterns**.



Although the phase plane analysis is applicable only to second-order systems, it can provide **intuitive insights** about **nonlinear effects**.

Phase Plane Analysis of Linear Systems

General form of a linear second-order system:

$$\ddot{x} + a\dot{x} + bx = 0 \quad (\text{or}) \quad \begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \Rightarrow \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Solution:

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \quad \lambda_1 \neq \lambda_2$$

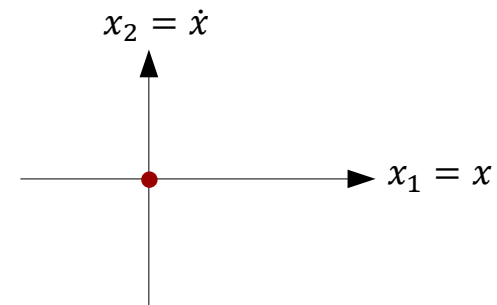
$$x(t) = k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t} \quad \lambda_1 = \lambda_2$$

$$\lambda_{1,2} = (-a \pm \sqrt{a^2 - 4b})/2$$

(solutions of the characteristic equations
[$\lambda^2 + a\lambda + b = 0$] or eigenvalues of matrix \mathbf{A}
[$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$])

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0)$$

There is only one isolated singular point at origin $\mathbf{x} = 0$, assuming $b \neq 0$ or \mathbf{A} is nonsingular ($\det(\mathbf{A}) \neq 0$). However, the trajectories in the vicinity of this singularity point can display quite different characteristics, depending on the values of a and b .



Phase Plane Analysis of Linear Systems

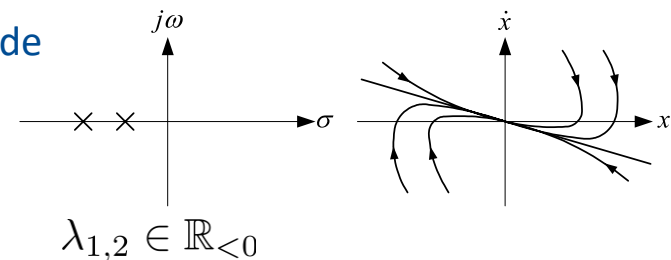
Stable/Unstable Node: Both $x(t)$ and $\dot{x}(t)$ converge to/diverge from zero **exponentially**.

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

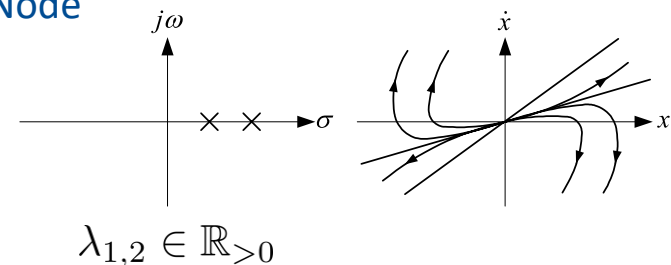
$$\lambda_{1,2} \in \mathbb{R}_{<0} \quad \text{Stable Node}$$

$$\lambda_{1,2} \in \mathbb{R}_{>0} \quad \text{Unstable Node}$$

Stable Node



Unstable Node

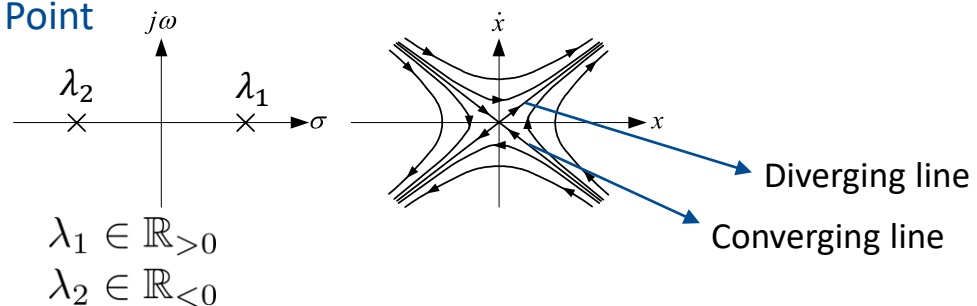


Saddle Point: Because of the unstable pole λ_1 , almost all of the system trajectories diverge to infinity.

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

$$\lambda_1 \in \mathbb{R}_{>0}, \lambda_2 \in \mathbb{R}_{<0}$$

Saddle Point



Phase Plane Analysis of Linear Systems

Stable/Unstable Focus: The trajectories encircle the origin one or more times before converging to it, unlike the situation for a stable node.

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} = K e^{\sigma t} \cos(\omega t - \phi)$$

$$(\lambda_{1,2} = \sigma \pm j\omega)$$

$\sigma \in \mathbb{R}_{<0}$ Stable Focus

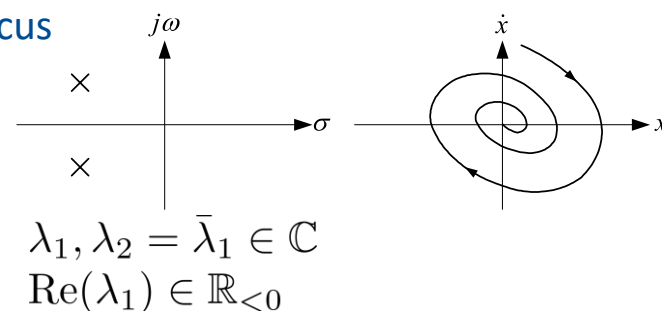
$\sigma \in \mathbb{R}_{>0}$ Unstable Focus

Center Point: All trajectories are ellipses and the singularity point is the center of these ellipses. The system trajectories neither converge to the origin nor diverge to infinity (**marginal stability**).

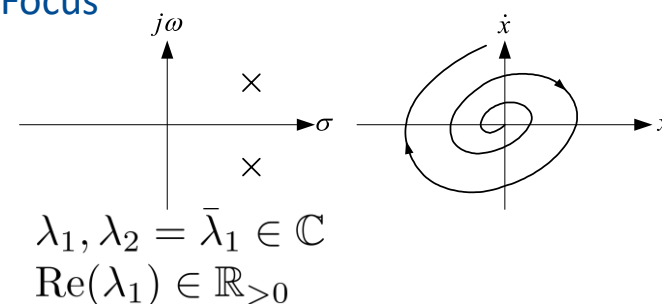
$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} = K \cos(\omega t - \phi)$$

$$(\lambda_{1,2} = \pm j\omega)$$

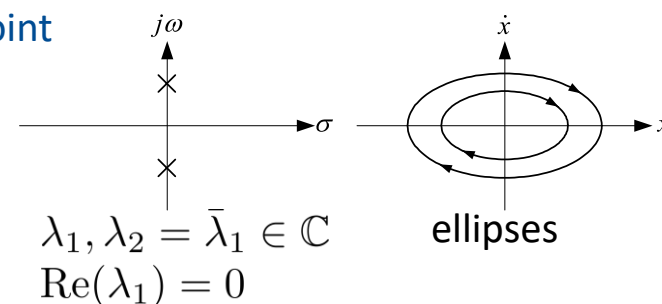
Stable Focus



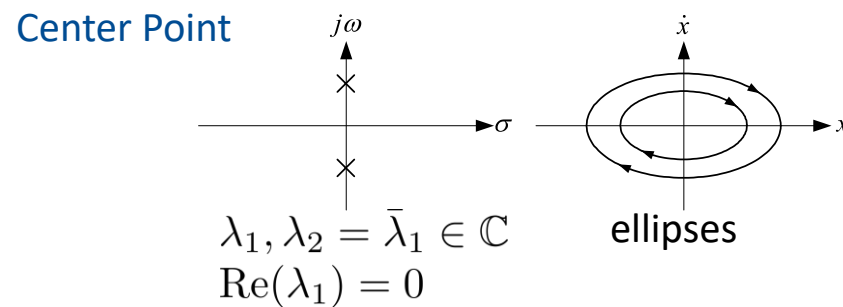
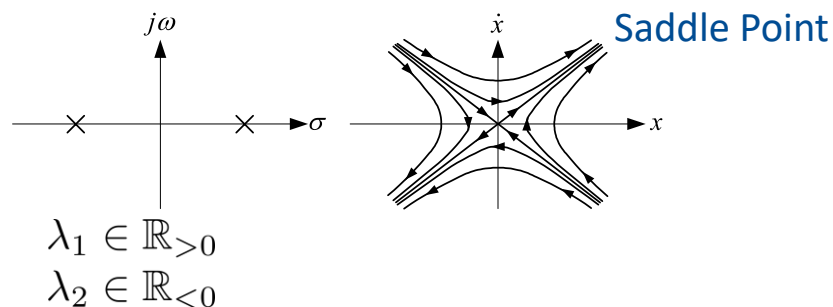
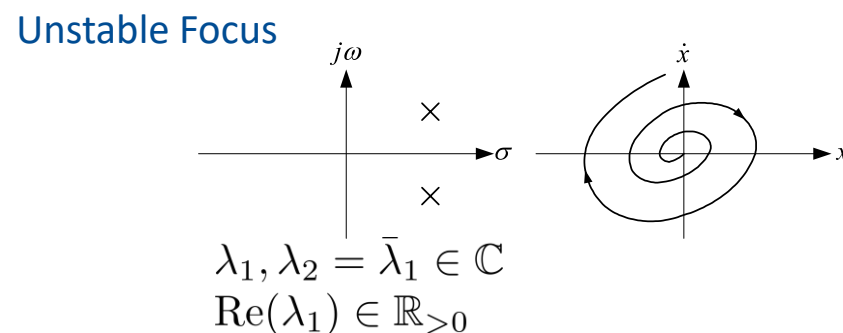
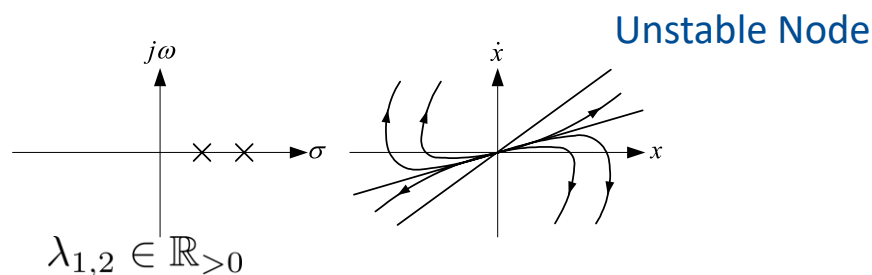
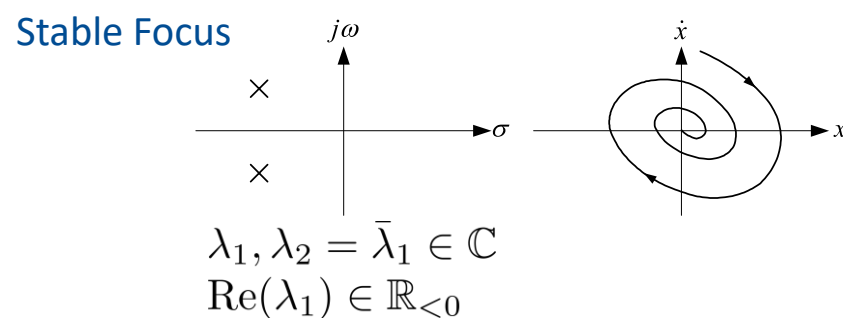
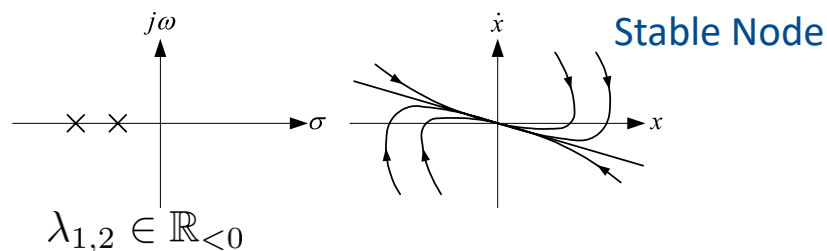
Unstable Focus



Center Point



Phase Plane Analysis of Linear Systems (review)

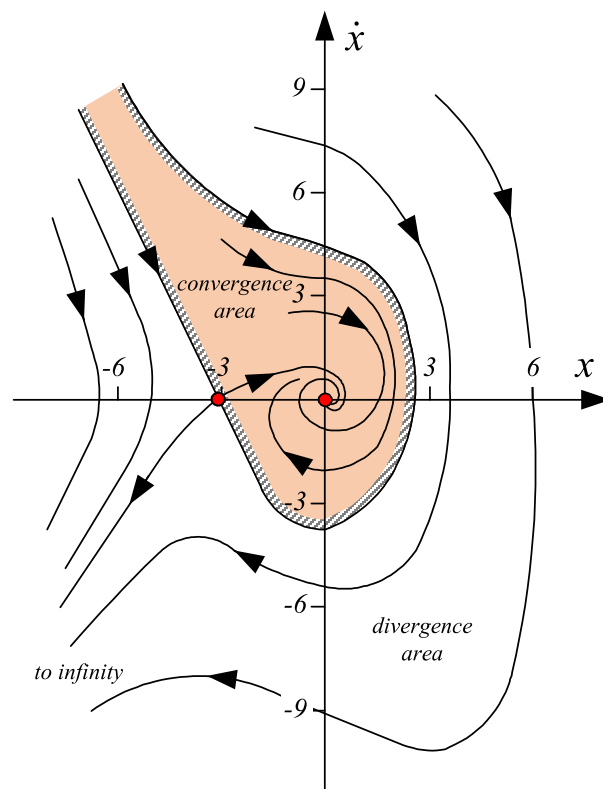


Phase Plane Analysis of Nonlinear Systems: Local Behavior

- Nonlinear systems frequently have **more than one equilibrium point**, in contrast to linear systems.
- Local behavior** of a nonlinear system can be approximated by the behavior of a linear system **in the neighborhood of** each equilibrium point.

$(0, 0)$: Stable Focus
 $(-3, 0)$: Saddle Point

This behavior can be determined via **linearization** of the nonlinear system with respect to each equilibrium point.



Linearization

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \longrightarrow \begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \xrightarrow{\text{Taylor expansion about } \mathbf{x}_e = [x_{e1}, x_{e2}]^T} \left[f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \right]$$

$$\begin{aligned} \dot{x}_1 &= f_1(x_{e1}, x_{e2}) + a_{11}(x_1 - x_{e1}) + a_{12}(x_2 - x_{e2}) + \text{H.O.T} \\ \dot{x}_2 &= f_2(x_{e1}, x_{e2}) + a_{21}(x_1 - x_{e1}) + a_{22}(x_2 - x_{e2}) + \text{H.O.T} \end{aligned}$$

(Higher Order Terms)

$\mathbf{f}(\mathbf{x}_e) = \mathbf{0}$ Change of variables: $\bar{x}_1 = (x_1 - x_{e1})$ In the vicinity of \mathbf{x}_e
 $\bar{x}_2 = (x_2 - x_{e2})$

Linearized

state equation:

$$\dot{\bar{\mathbf{x}}} = \mathbf{A}\bar{\mathbf{x}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \bar{\mathbf{x}} = \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right] \bigg|_{\mathbf{x}=\mathbf{x}_e} \bar{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \bigg|_{\mathbf{x}=\mathbf{x}_e} \bar{\mathbf{x}}$$

Jacobian of \mathbf{f}

Example

$$\ddot{\theta} + \frac{c}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta = 0 \quad x_1 = \theta, \quad x_2 = \dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ -\frac{c}{ml^2}x_2 - \frac{g}{l}\sin x_1 \end{bmatrix}$$

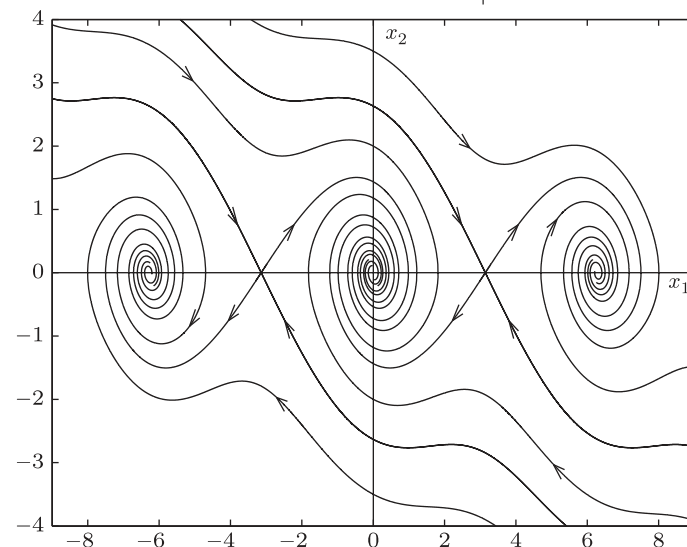
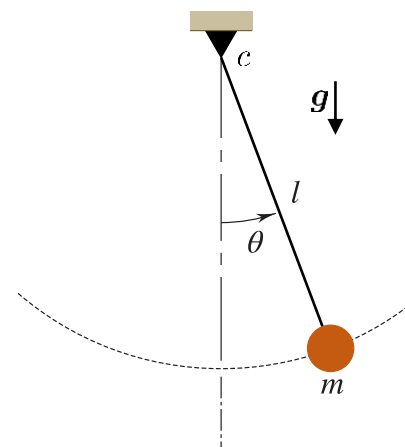
$$\begin{aligned} x_2 &= 0 \\ \sin(x_1) &= 0 \end{aligned}$$

It has an infinite number of isolated singular points: $(n\pi, 0)$
 for $n = 0, \pm 1, \pm 2, \dots$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -0.3 \end{bmatrix}$$

$$A_1|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -0.3 \end{bmatrix} \quad \text{Stable Focus (vertical down position)}$$

$$A_2|_{(\pi,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -0.3 \end{bmatrix} \quad \text{Saddle Point (vertical up position)}$$



Limit Cycle

Let's plot phase portrait of the Van der Pol equation:

$$\ddot{x} - \mu (1 - x^2) \dot{x} + x = 0, \quad \mu = 1$$

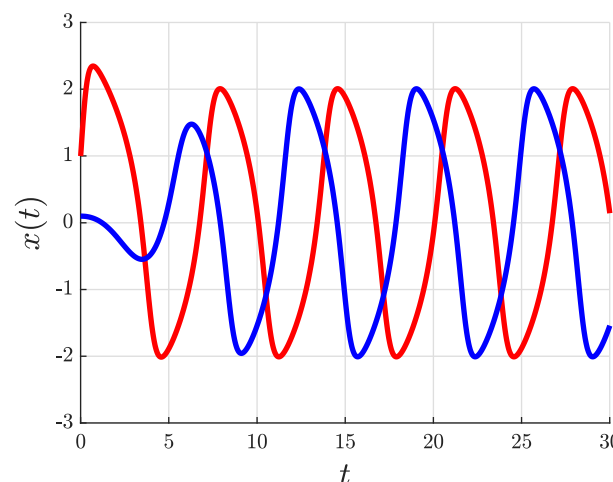
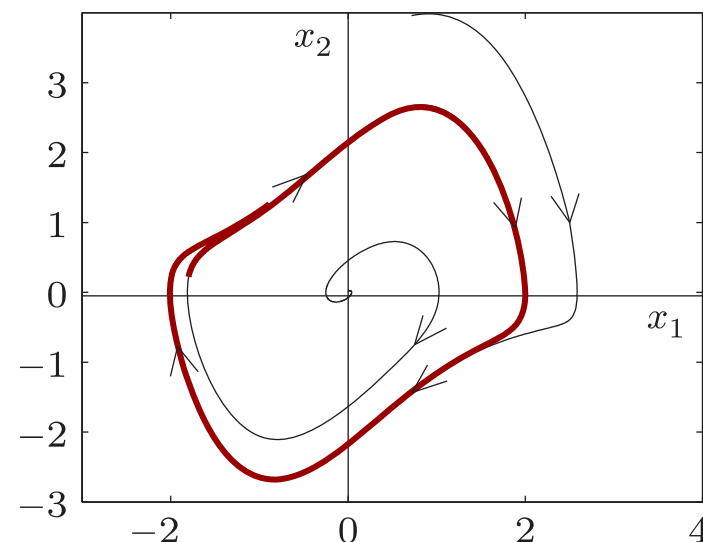
- An unstable node at the origin.
- A closed curve!



All trajectories inside & outside the curve tend to this curve. A motion started on this curve will stay on it forever, circling periodically around the origin.

This closed curve correspond to oscillations of **fixed amplitude** and **fixed period without external excitation** and **independent of initial conditions**, which is called **Limit Cycle** (LC) or **Self-Excited Oscillations**.

Limit cycles are **unique** features of nonlinear systems.

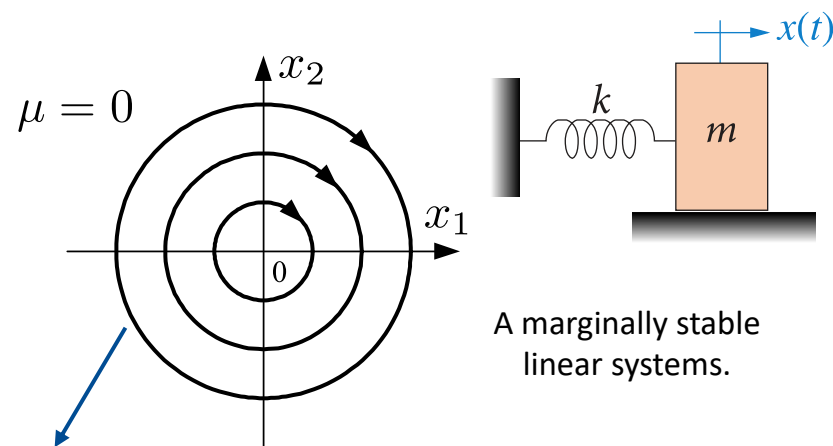
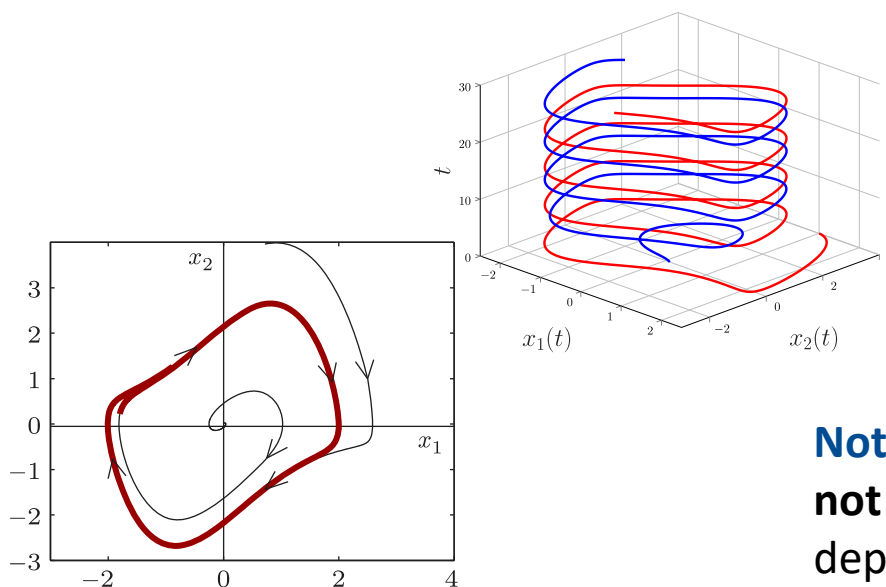


Limit Cycle

A **Limit Cycle** is defined as an isolated closed curve.

Indicates the limiting nature of the cycle (nearby trajectories converging or diverging from it)

Indicates the periodic nature of the motion.



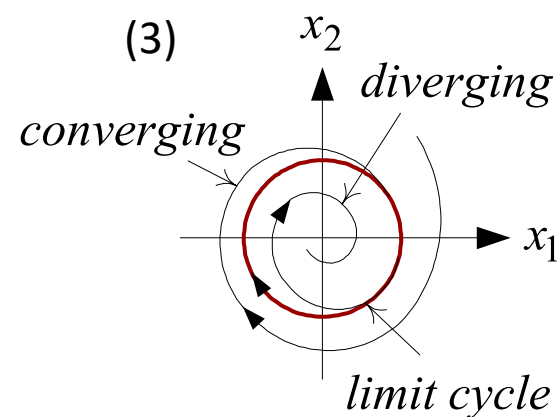
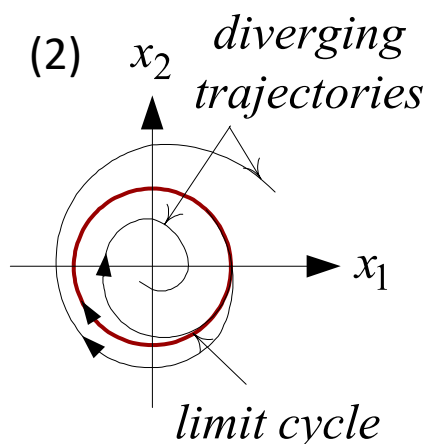
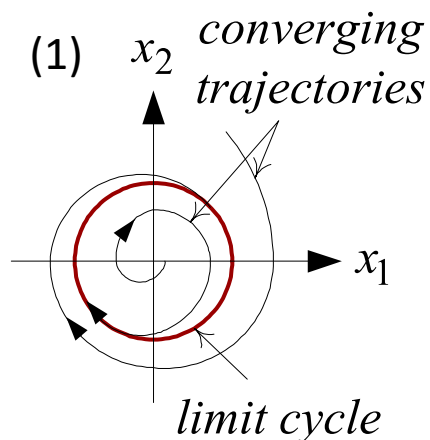
A marginally stable linear systems.

Note: These are not limit cycles, because they are **not isolated** and the amplitude of the oscillations depends on the initial conditions.

Limit Cycles

Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, there are three kinds of limit cycles:

- 1) **Stable Limit Cycles:** All trajectories in the vicinity of the LC converge to it as $t \rightarrow \infty$.
- 2) **Unstable Limit Cycles:** All trajectories in the vicinity of the LC diverge from it as $t \rightarrow \infty$.
- 3) **Semi-stable Limit Cycles:** Some of the trajectories in the vicinity of the LC converge to it, while the others diverge from it as $t \rightarrow \infty$.



Example

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{cases}$$

By introducing
polar coordinates

$$r^2 = x_1^2 + x_2^2$$

$$\tan \theta = x_2/x_1$$

$$\dot{r} = -r(r^2 - 1)$$

$$\dot{\theta} = -1$$

When the state starts on the unit circle $r = 1$, the $\dot{r} = 0$. This implies that the state will circle around the origin. When $r < 1$, then $\dot{r} > 0$. This implies that the state tends to the circle from inside. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from outside. Therefore, the **unit circle is a stable limit cycle**.

Constructing Phase Portraits

Constructing Phase Portraits

Although phase portraits are routinely computer-generated, it is still practically useful to learn how to roughly sketch phase portraits or quickly verify the plausibility of computer outputs.

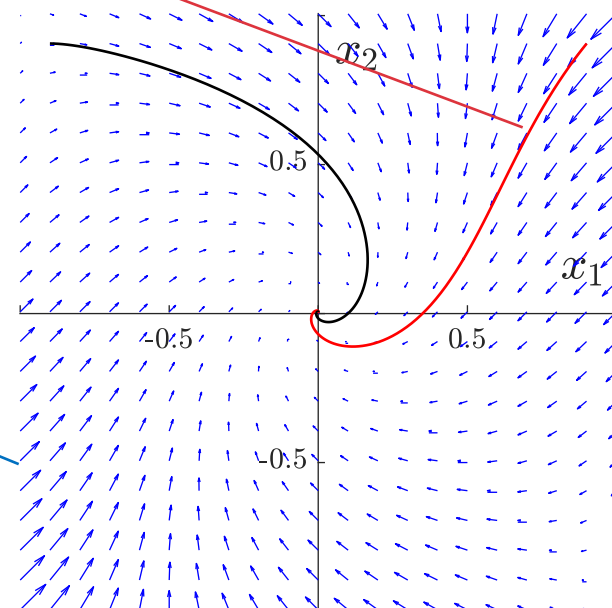
MATLAB Code

```
% Phase Trajectory
opts = odeset('RelTol',1e-6,'AbsTol',1e-6);
[t,x] = ode45(@func,[0 10],[0.9; 0.9],opts);

function dxdt = func(t,x)
dxdt = [-x(1) - 2*x(2)*x(1)^2 + x(2); -x(1) - x(2)];
end
```

```
% Phase Portrait
[x1, x2] = meshgrid(-1:0.1:1, -1:0.1:1);
x1dot = -x1 - 2 * x2 .* x1.^2 + x2;
x2dot = -x1 - x2;
quiver(x1,x2,x1dot,x2dot)
```

$$\begin{aligned}\dot{x}_1 &= -x_1 - 2x_2x_1^2 + x_2 \\ \dot{x}_2 &= -x_1 - x_2\end{aligned}$$



Two simple methods are **Analytical Method** and **Isoclines Method**.

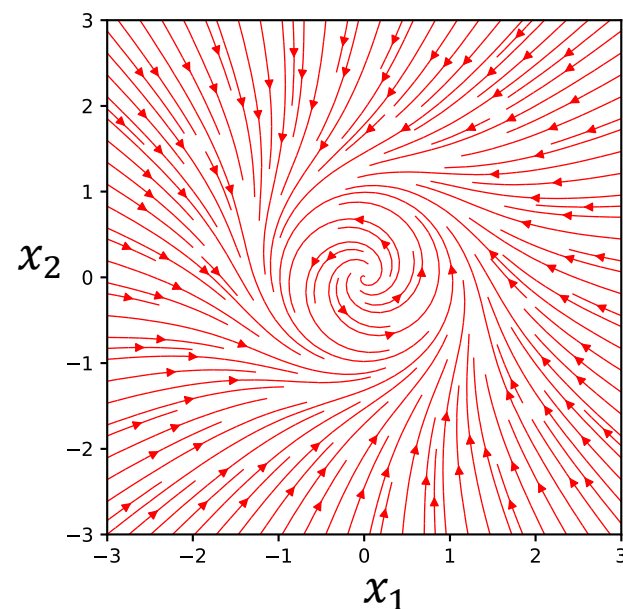
Method 1: Analytical Method

The method is based on finding a functional relation between the two phase variable x_1 and x_2 of the 2nd-order system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in the form

$$g(x_1, x_2, c) = 0$$

↓
effect of initial conditions

Plotting this relation in the phase plane for **different initial conditions** yields a phase portrait.



Note: This method is useful for some **special** nonlinear systems, particularly **piece-wise linear systems**, whose phase portraits can be constructed by piecing together the phase portraits of the related linear systems.

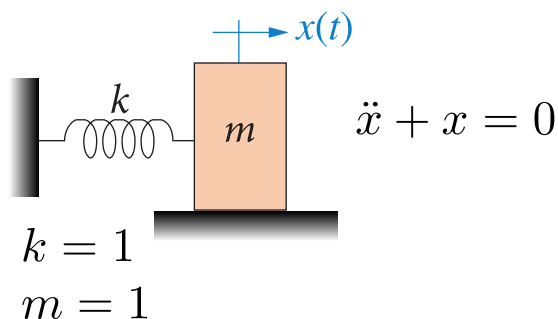
Method 1: Analytical Method

Technique 1:

$$\begin{array}{ccc}
 \dot{x}_1 = f_1(x_1, x_2) & \rightarrow & x_1 = g_1(t) \\
 \dot{x}_2 = f_2(x_1, x_2) & & x_2 = g_2(t)
 \end{array}
 \xrightarrow{\text{Eliminating time } t \text{ from these equations}}
 g(x_1, x_2, c) = 0$$

\downarrow
 effect of initial conditions

Example: A mass-spring system



x_0 : Initial length
 \dot{x}_0 : Initial velocity

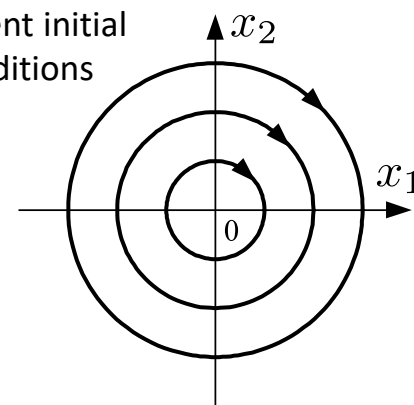
$$\begin{array}{ll}
 x_1 = x & \dot{x}_1 = x_2 \\
 x_2 = \dot{x} & \dot{x}_2 = -x_1
 \end{array}$$

$$\begin{array}{l}
 x_1 = x_0 \cos t + \dot{x}_0 \sin t \\
 x_2 = -x_0 \sin t + \dot{x}_0 \cos t
 \end{array}$$

$$x_1^2 + x_2^2 = x_0^2 + \dot{x}_0^2$$

Equation of the trajectories

Corresponding to
different initial
conditions



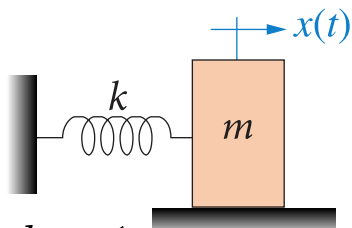
Method 1: Analytical Method

Technique 2:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \rightarrow \frac{dx_1}{dx_2} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \rightarrow g(x_1, x_2, c) = 0$$

effect of initial conditions

Example: A mass-spring system



$$k = 1$$

$$m = 1$$

x_0 : Initial length

\dot{x}_0 : Initial velocity

$$\ddot{x} + x = 0$$

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

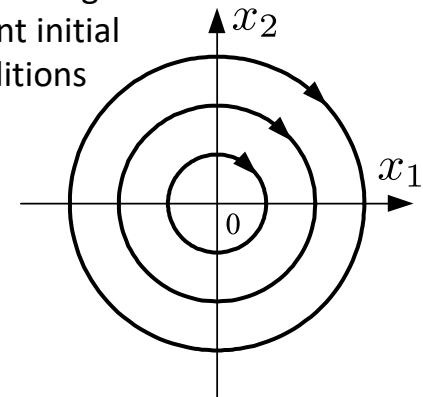
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

$$\frac{dx_1}{dx_2} = \frac{x_2}{-x_1} \rightarrow -x_1 dx_1 = x_2 dx_2$$

$$x_1^2 + x_2^2 = x_0^2 + \dot{x}_0^2$$

Equation of the trajectories

Corresponding to
different initial
conditions



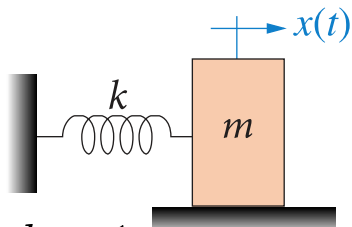
Method 2: Isoclines Method

An **isocline** is defined to be the locus of the points with a given tangent slope α .

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha \rightarrow f_2(x_1, x_2) = \alpha f_1(x_1, x_2) \quad (\text{isocline equation})$$

All points on this curve have the same tangent slope α .

Example: A mass-spring system



$$\ddot{x} + x = 0$$

$$k = 1$$

$$m = 1$$

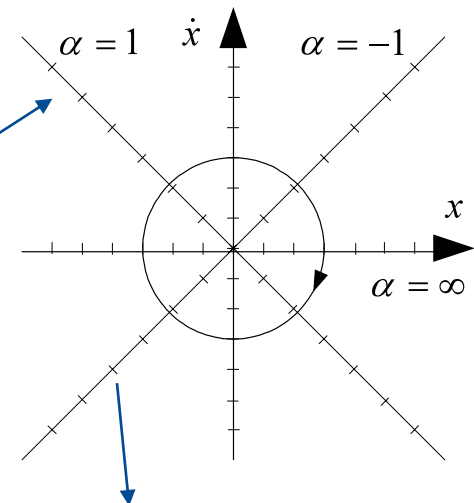
$$\begin{aligned} x_1 &= x & \rightarrow & \dot{x}_1 = x_2 \\ x_2 &= \dot{x} & \rightarrow & \dot{x}_2 = -x_1 \end{aligned}$$

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha$$

$$\alpha x_2 = -x_1$$

We assume that the tangent slopes are locally constant. Therefore, a trajectory starting from any point in the field of directions can be found by connecting a sequence of line segments.

isoclines



Short line segments with slope α to generate a field of directions (same scales should be used for the x_1, x_2 axes)

Example: Van der Pol Equation

$$\ddot{x} + 0.2(x^2 - 1)\dot{x} + x = 0$$

$$\frac{dx_2}{dx_1} = -\frac{0.2(x_1^2 - 1)x_2 + x_1}{x_2} = \alpha$$

$$0.2(x_1^2 - 1)x_2 + x_1 + \alpha x_2 = 0 \quad (\text{isocline equation})$$

All points on this curve have the same tangent slope α .

By taking α of different values, different isoclines can be obtained.

* For connecting the segments, we can first determine the type of the equilibrium points and check if there is a limit cycle.

The trajectories starting from both outside and inside converge to the limit cycle.

