Ch6: Time Response

Contents:

Introduction

First-Order Systems

Second-Order Systems

Underdamped Second-Order Systems

Using MATLAB and Control System Toolbox

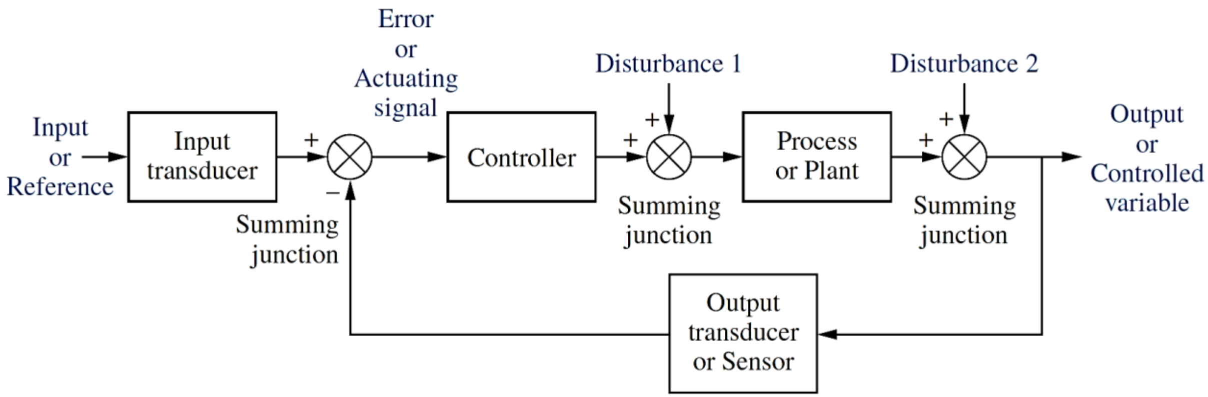
Higher-Order Systems & Systems with Zeros

Introduction

Introduction

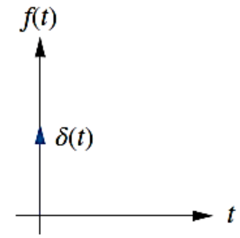
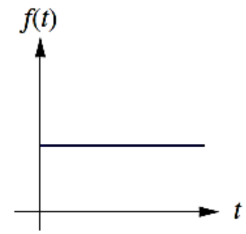
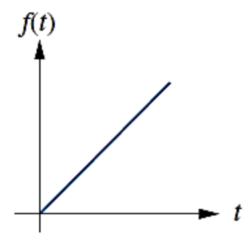
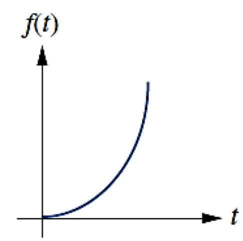
After obtaining a mathematical representation of a system, it is **analyzed** for its performance, i.e., **transient response**, **steady-state response**, and **stability**, to see if these characteristics yield the desired behavior.

- In practice, the input signal to a control system is not known ahead of time and is random in nature. Hence, in analysis and design of control systems, we must have a basis for comparing the performance of various control systems.
- This basis may be set up by specifying particular **test input signals** (that the system will be subjected to most frequently under normal operation) and comparing the responses of various systems (with zero initial condition) to these input signals.



Typical Test Input Signals

The commonly used **test input signals**: **impulse**, **step**, **ramp**, **parabola**, **sinusoidal**, **white noise**.

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)dt = 1$		Transient response Modeling
$\frac{d}{dt} \upharpoonleft\!\downharpoonright \int$				
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
$\frac{d}{dt} \upharpoonleft\!\downharpoonright \int$				
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
$\frac{d}{dt} \upharpoonleft\!\downharpoonright \int$				
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error

Order of a System

Order of a system is

- The order of the equivalent differential equation representing the system.

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

- The highest power of s in the denominator of the transfer function **after** cancellation of common factors in the numerator.

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- The number of simultaneous first-order equations required for the state-space representation of the system (i.e., the dimension of vector $\boldsymbol{x}$ or the number of states).

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

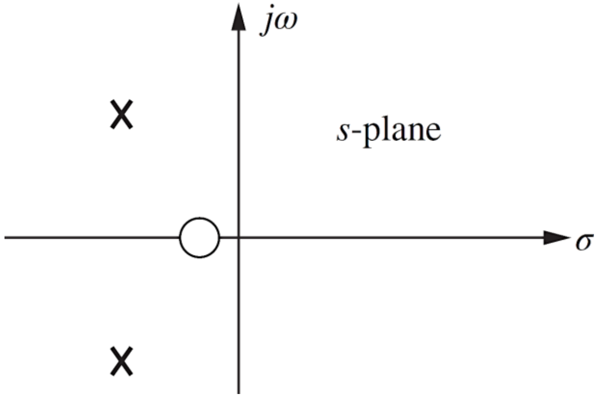
Poles & Zeros of a Transfer Function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Poles of a Transfer Function (p_i) are (1) the values of the Laplace transform variable, s, that cause the transfer function to become infinite or (2) any roots of the denominator of the transfer function that are common to roots of the numerator (i.e., before cancelation).

Zeros of a Transfer Function (z_i) are (1) the values of the Laplace transform variable, s, that cause the transfer function to become zero, or (2) any roots of the numerator of the transfer function that are common to roots of the denominator (i.e., before cancelation).

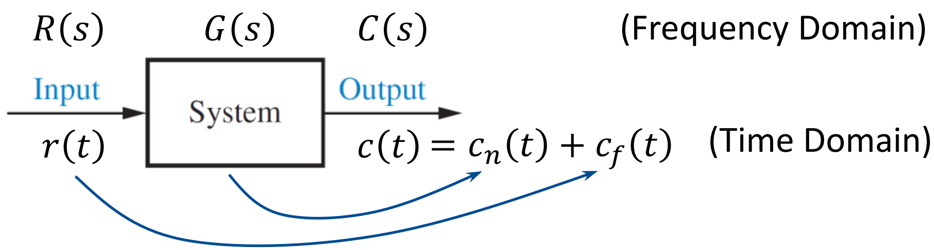
Symbol of pole on the complex s-plane: ×
Symbol of zero on the complex s-plane: o



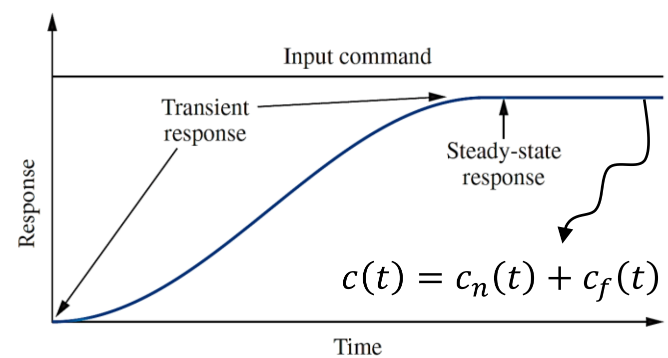
Forced & Natural Responses vs Transient & Steady-State Responses

The total time response $c(t)$ of a linear system is the sum of two responses:

1) **Natural Response** (or **homogeneous** solution) $c_n(t)$ which depends <u>only on the system</u>, not the input.
2) **Forced Response** (or **particular** solution) $c_f(t)$ which depends <u>only on the input</u>, not the system.

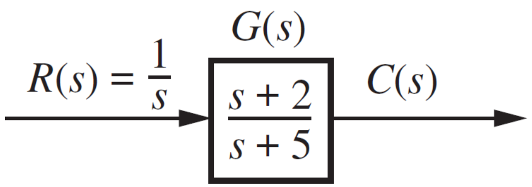


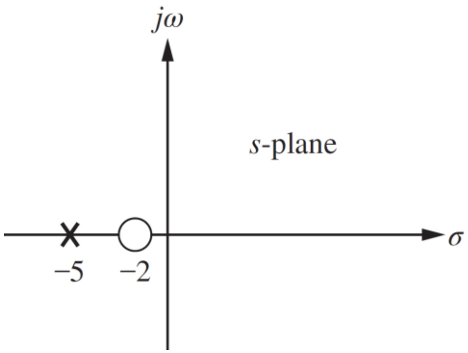
By considering the **response plot**, it is inferred that the response can consist of two parts, the **Transient** response (the way it goes from the initial state to the final state) and **Steady-State** response (the way it behaves as time approaches infinity).



❖ **Natural Response** contributes more to the **Transient Response** and **Force Response** contributes more to the **Steady-State Response**.

Example: Time Response of a System





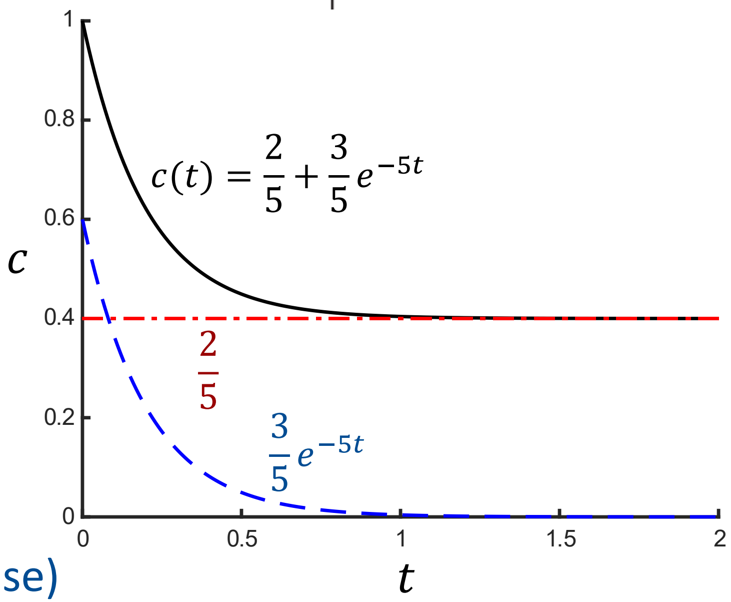
Pole: $s = -5$
Zero: $s = -2$

The unit step response of the system:

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{K_1}{s} + \frac{K_2}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

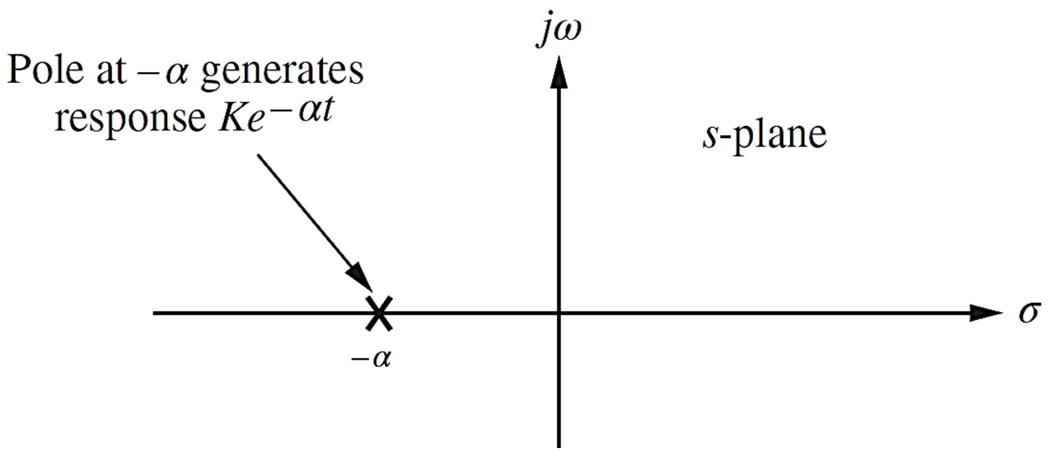
$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

(Forced response) (Natural response)



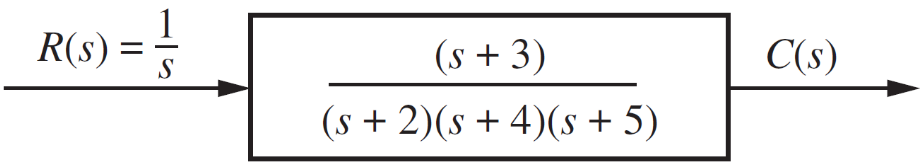
Important Conclusions

a. Poles of the transfer function determine the form of the **natural response** (exponential, sinusoid, ...),
b. Poles of the input function determine the form of the **forced response**,
c. Zeros and poles of the input and transfer function contribute to the amplitudes (or residues), i.e., K_1 and K_2, of both the forced and natural responses,
d. Non-zero poles **on the real axis** ($-\alpha$) generate **exponential** responses ($e^{-\alpha t}$). Therefore, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero



Example: Evaluating Response Using Poles

Specify the forced and natural parts of the solution.



Answer:

$$c(t) = \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}}$$

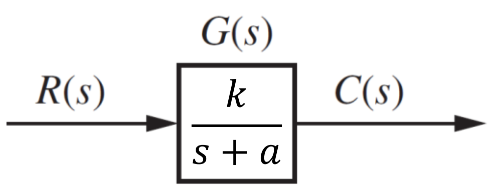
First-Order Systems

First-Order Systems

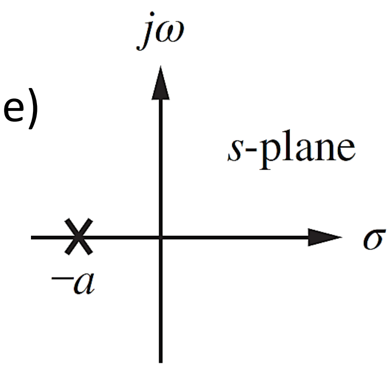
Consider a **First-Order System** (without zeros) with zero initial conditions.

[Physical Examples: RC circuit, thermal system]

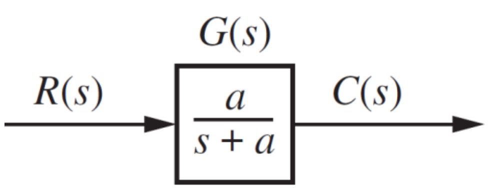
- General Form:



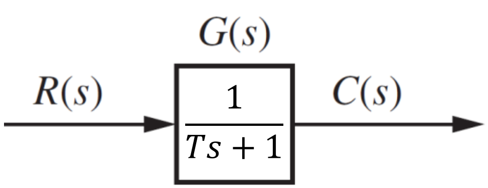
(One pole)



- Standard Form (used to normalize response):



or
$\equiv$



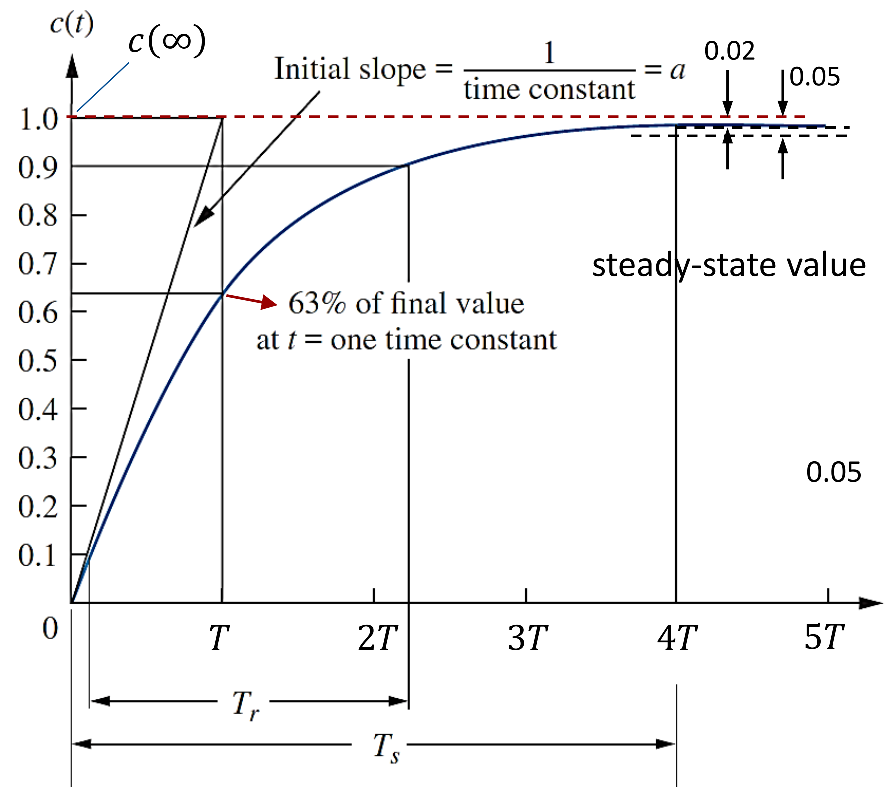
Unit-Step Response

$R(s) = 1/s$

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$
$$= 1 - e^{-t/T}$$

$c(\infty) = 1$



Transient Response Specifications:

1. Time Constant T: The time required for the step response to rise to 63% of the steady-state value (final value).

$$T = t\Big|_{c(t)=0.63c(\infty)} = \frac{1}{a}$$

$$c(t)\Big|_{t=1/a} = 1 - e^{-1} \approx 0.63$$

- The initial rate of change (slope) of response is a or $1/T$.

$$\frac{dc(t)}{dt}\Big|_{t=0} = a = \frac{1}{T}$$

Unit-Step Response

2. Rise Time T_r: The time required for the step response to go from 10% to 90% of the steady-state value.

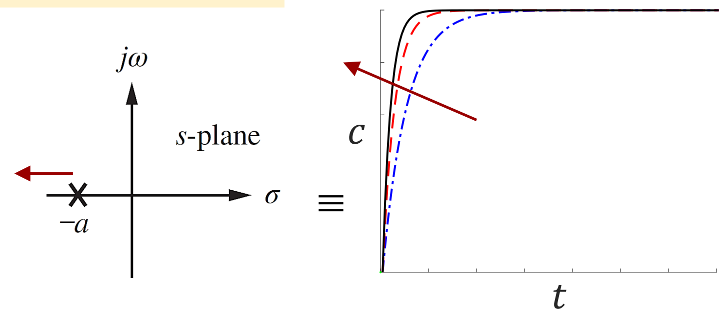
$$T_r = t\Big|_{c(t)=0.9c(\infty)} - t\Big|_{c(t)=0.1c(\infty)} = \frac{-\ln(0.1)}{a} - \frac{-\ln(0.9)}{a} \approx \frac{2.30}{a} - \frac{0.10}{a} = \frac{2.2}{a}$$

$$\Rightarrow \boxed{T_r = \frac{2.2}{a} = 2.2T}$$

3. Settling Time T_s: The time required for the step response to **reach and stay** within 2% (or 5%) of the steady-state value.

$$T_s = t\Big|_{c(t)=0.98c(\infty)} = \frac{-\ln(0.02)}{a} \approx \frac{4}{a} \quad \Rightarrow \quad \boxed{T_s = \frac{4}{a} = 4T}$$

Note: The farther to the left the pole $(-a)$ from the imaginary axis $j\omega$, the faster the transient response.



Example

For the given the transfer function. Find the time constant T, settling time T_s, and rise time T_r.

$$G(s) = \frac{100}{s+50}$$

Answer:

$$T = \frac{1}{a} = \frac{1}{50} = 0.02\text{s}$$

$$T_s = \frac{4}{a} = \frac{4}{50} = 0.08\text{s}$$

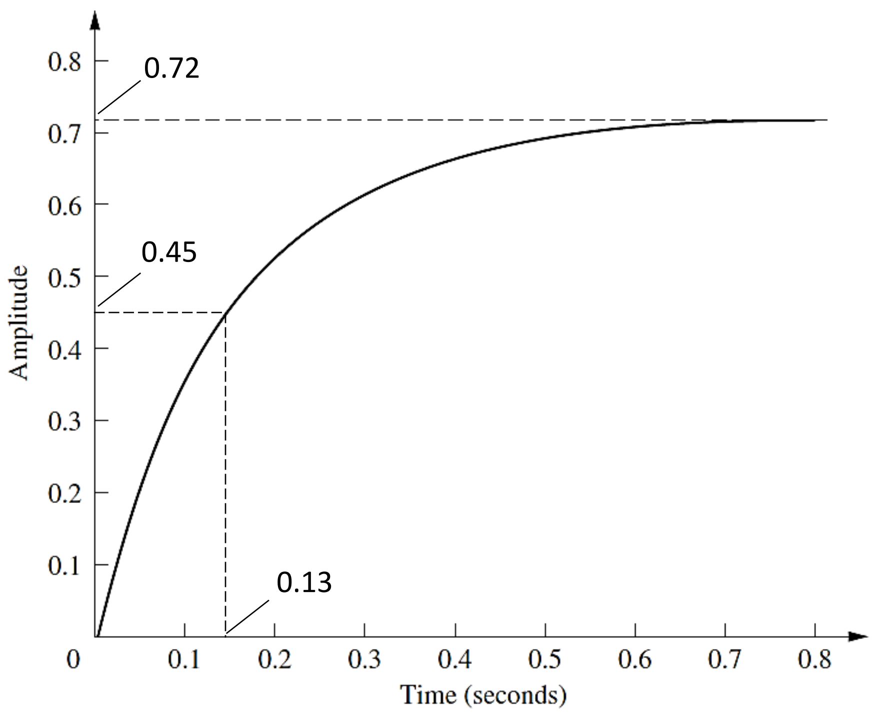
$$T_r = \frac{2.2}{a} = \frac{2.2}{50} = 0.044\text{s}$$

Note: T, T_r, and T_s are independent of the value of k in the general form of a first order system $\frac{k}{s+a}$.

Example: Determining Transfer Function via Testing

Often it is not possible or practical to obtain a system's transfer function analytically. Therefore, the system's step response can lead to a representation of the system.

The unit-step response of an unknown system is given. Determine its transfer function.



Answer:

$$G(s) = \frac{5.54}{s + 7.7}$$

Unit-Ramp Response

$$R(s) = 1/s^2$$

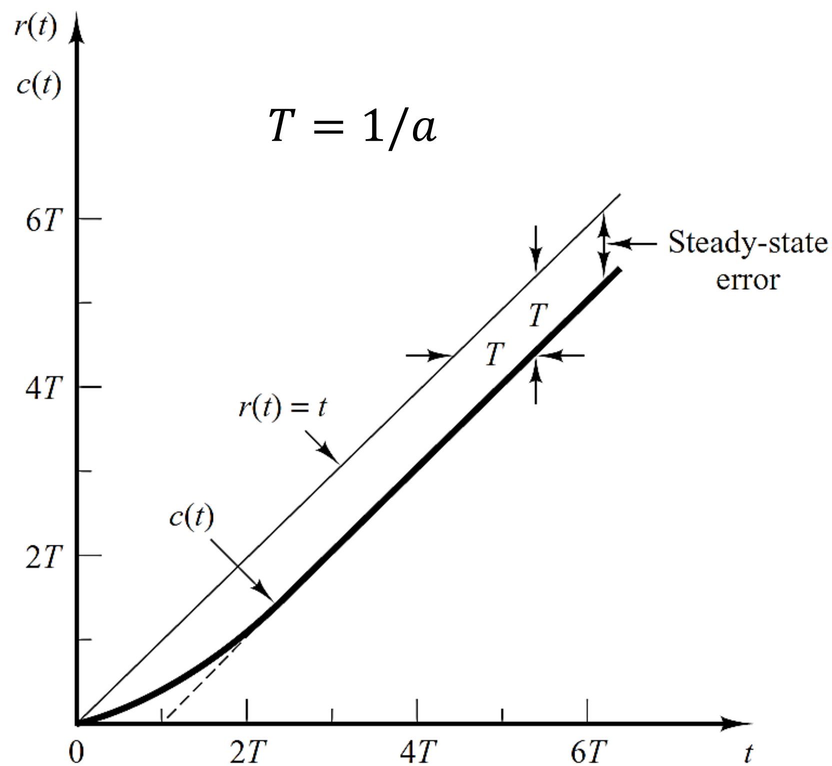
$$C(s) = R(s)G(s) = \frac{a}{s^2(s+a)}$$

$$= \frac{1}{s^2} - \frac{1/a}{s} + \frac{1/a}{s+a}$$

$$c(t) = c_f(t) + c_n(t) = t - \frac{1}{a} + \frac{1}{a}e^{-at}$$

$$e(t) = r(t) - c(t) = \frac{1}{a}(1 - e^{-at})$$

$$\lim_{t\to\infty} e(t) = \frac{1}{a} = T$$



Note: The farther to the left the pole ($-a$) from the imaginary axis $j\omega$, the smaller the steady-state error in following the ramp input.

Unit-Impulse Response

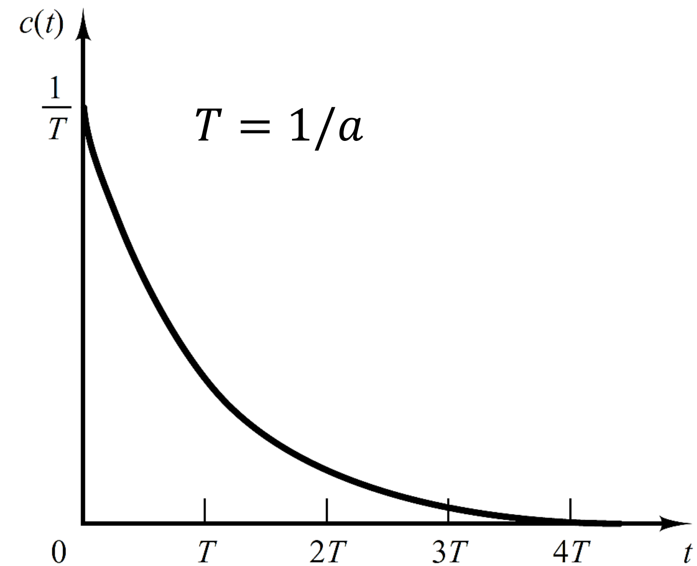
$$R(s) = 1$$

$$C(s) = R(s)G(s) = \frac{a}{s+a}$$

$$c(t) = \cancelto{0}{c_f(t)} + c_n(t) = ae^{-at}$$

Note:

$$\begin{matrix} T\dot{x} + x = \delta(t), \\ x(0^-) = 0 \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} T\dot{w} + w = 0, \\ w(0^+) = 1/T \end{matrix}$$



An Important Property of LTI Systems:

Comparing the system responses to these three inputs clearly indicates that the response to the **derivative/integral** of an input signal r can be obtained by **differentiating/integrating** the response c of the system to the original input signal r.

$$\boxed{\begin{matrix} r \to c \\ dr/dt \to dc/dt \\ \int r \to \int c \end{matrix}}$$

$$r_{\text{impulse}} \underset{d/dt}{\overset{\int}{\rightleftharpoons}} r_{\text{step}} \underset{d/dt}{\overset{\int}{\rightleftharpoons}} r_{\text{ramp}} \quad \Longrightarrow \quad c_{\text{impulse}} \underset{d/dt}{\overset{\int}{\rightleftharpoons}} c_{\text{step}} \underset{d/dt}{\overset{\int}{\rightleftharpoons}} c_{\text{ramp}}$$

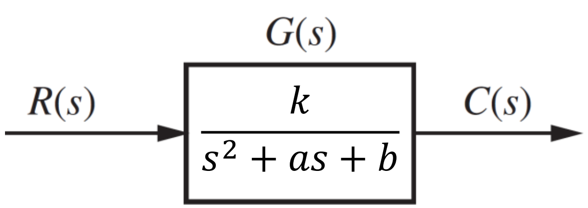
Second-Order Systems

Second-Order Systems

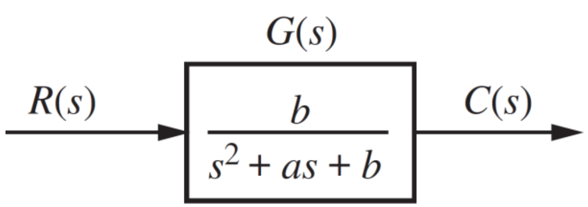
Consider a **Second-Order System** (without zeros) with zero initial conditions.

[Physical Examples: RLC circuit, Mass-Spring-Damper System, DC Motor]

- General Form:



- Standard Form (used to normalize response):

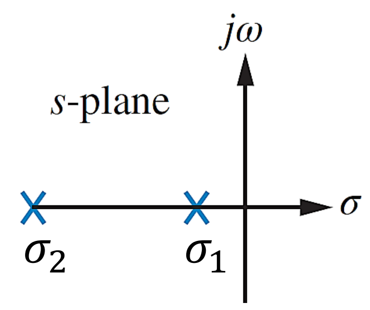


- Unlike first-order systems, changes in the parameters (a,b) of a second-order system can totally change the **form** of the response. Therefore, depending on the location of two poles of the second-order system, it can exhibit different types of responses: **Overdamped**, **Underdamped**, **Undamped**, and **Critically Damped**.

Unit-Step Response

1. Overdamped Response

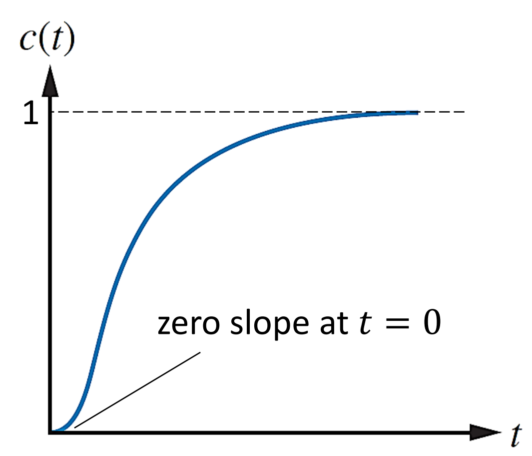
When $G(s)$ has two real poles at σ_1 and σ_2:



$$C(s) = R(s)G(s)$$

$$C(s) = \frac{1}{s} \cdot \frac{b}{s^2 + as + b} = \frac{1}{s} \cdot \frac{b}{(s + \sigma_1)(s + \sigma_2)} \qquad \begin{cases} \sigma_1 + \sigma_2 = a \\ \sigma_1 \sigma_2 = b \end{cases}$$

$$= \frac{K_0}{s} + \frac{K_1}{s + \sigma_1} + \frac{K_2}{s + \sigma_2}$$



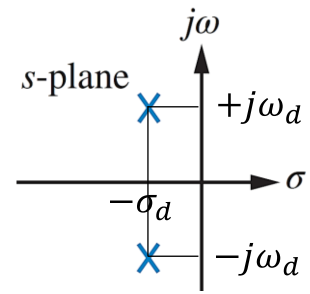
$$c(t) = c_f(t) + c_n(t) = 1 + K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

- It is called overdamped due to absorption of a large amount of energy in the system.

Unit-Step Response

2. Underdamped Response

When $G(s)$ has two complex poles at $-\sigma_d \pm j\omega_d$:



$$C(s) = R(s)G(s)$$

$$C(s) = \frac{1}{s} \cdot \frac{b}{s^2 + as + b} = \frac{K_0}{s} + \frac{K_1(s+\sigma_d) + K_2\omega_d}{(s+\sigma_d)^2 + {\omega_d}^2}$$

$$\begin{cases} \sigma_d = a/2 \\ {\omega_d}^2 = b - (a/2)^2 \end{cases}$$

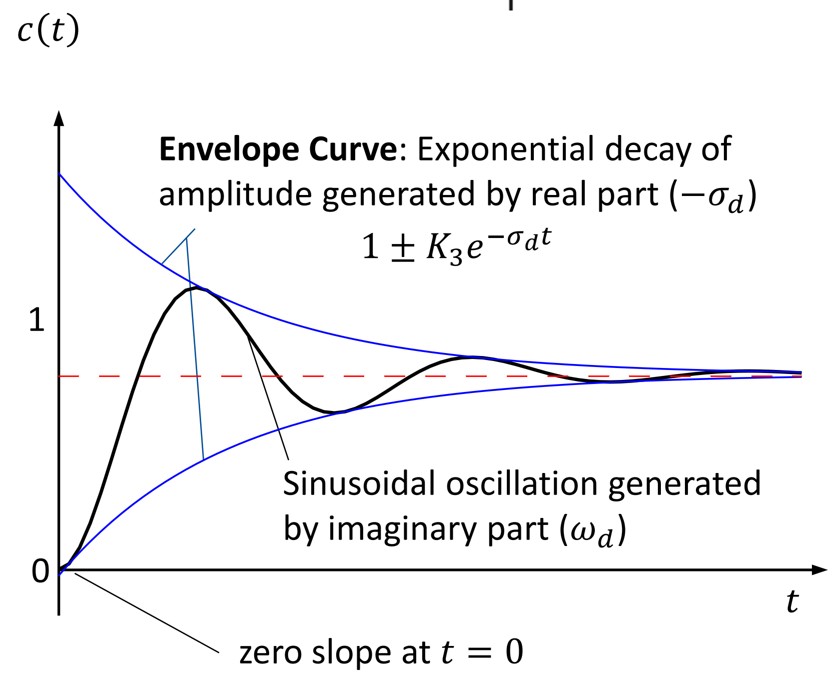
$$c(t) = c_f(t) + c_n(t)$$

$$= 1 + e^{-\sigma_d t}(K_1 \cos\omega_d t + K_2 \sin\omega_d t)$$

$$= 1 + K_3 e^{-\sigma_d t}\cos(\omega_d t - \phi)$$

$$= 1 + K_3 e^{-\sigma_d t}\sin(\omega_d t + \bar{\phi})$$

$$\left(K_3 = \sqrt{K_1^2 + K_2^2}, \phi = \tan^{-1}\frac{K_2}{K_1}, \bar{\phi} = \tan^{-1}\frac{K_1}{K_2}\right)$$

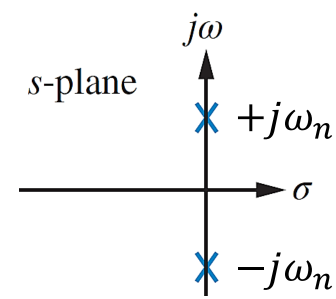


- This is an exponentially damped sinusoidal response.

Unit-Step Response

3. Undamped Response

When $G(s)$ has two imaginary poles at $\pm j\omega_n$:



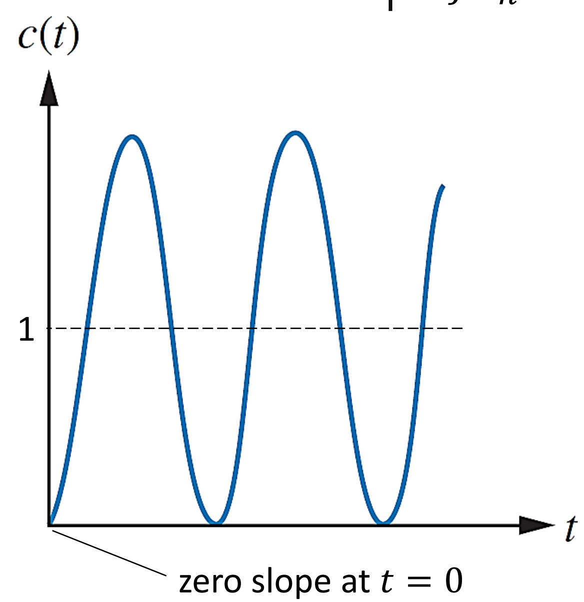
$$C(s) = R(s)G(s)$$

$$C(s) = \frac{1}{s} \cdot \frac{b}{s^2 + as + b} = \frac{K_0}{s} + \frac{K_1 s + K_2 \omega_n}{s^2 + {\omega_n}^2} \qquad \begin{cases} a = 0 \\ {\omega_n}^2 = b \end{cases}$$

$$c(t) = c_f(t) + c_n(t) = 1 + K_1 \cos \omega_n t + K_2 \sin \omega_n t$$

$$= 1 + K_3 \cos(\omega_n t - \phi) = 1 + K_3 \sin(\omega_n t + \bar{\phi})$$

$$\left(K_3 = \sqrt{K_1^2 + K_2^2}, \phi = \tan^{-1}\frac{K_1}{K_2}, \bar{\phi} = \tan^{-1}\frac{K_2}{K_1}\right)$$

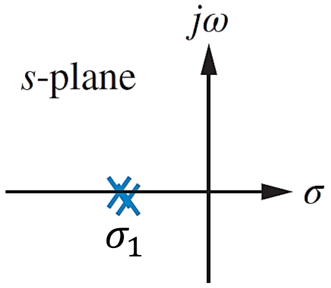


- This is a sinusoidal response whose frequency is ω_n (i.e., location of the imaginary poles).

Unit-Step Response

4. Critically-Damped Response

When $G(s)$ has two repeated real poles at σ_1:

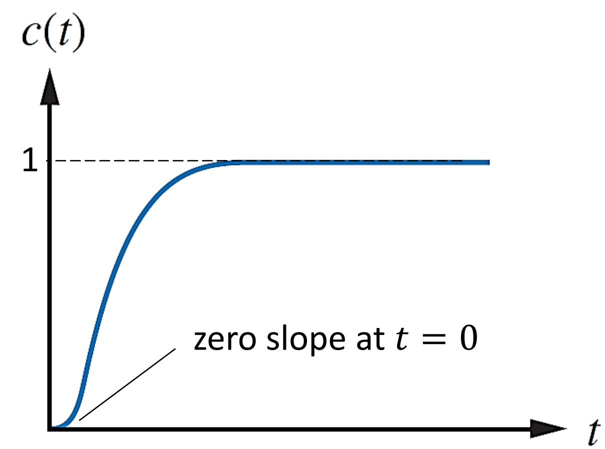


$$C(s) = R(s)G(s)$$

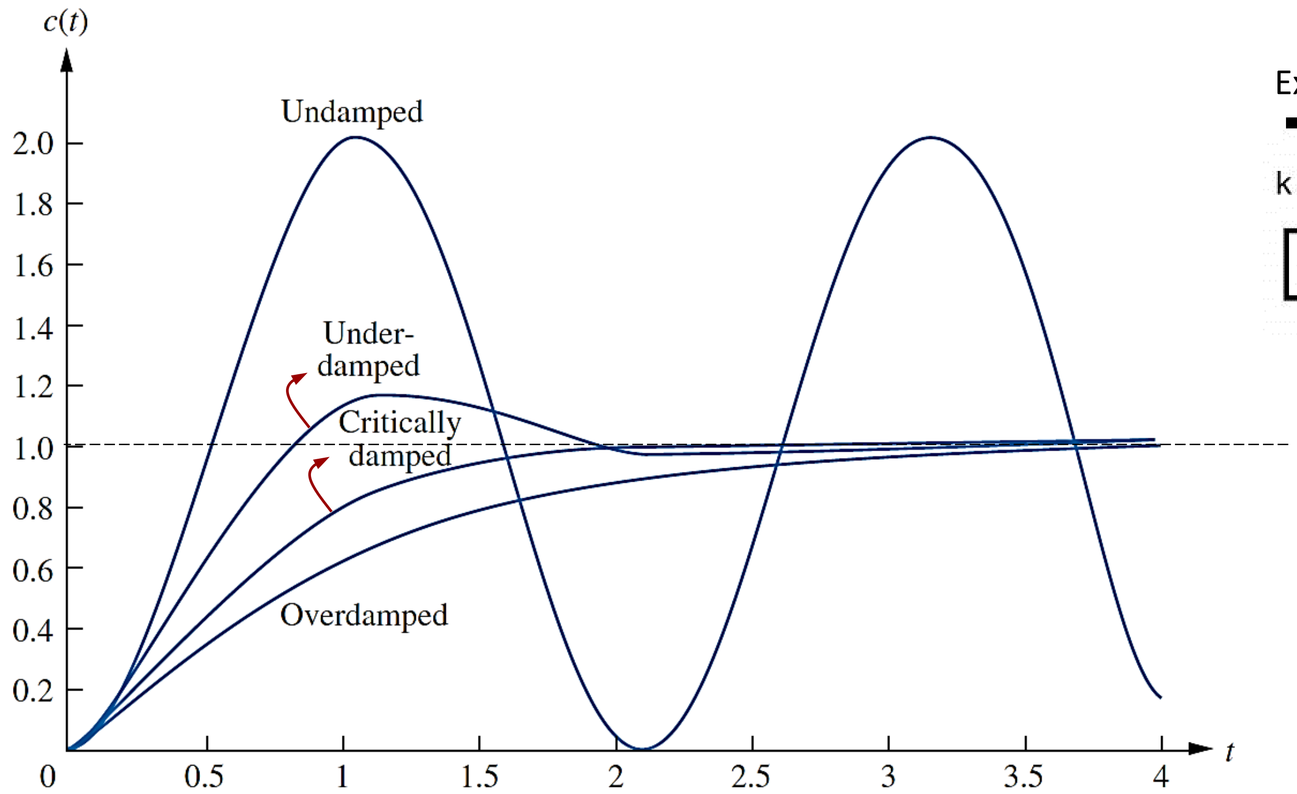
$$C(s) = \frac{1}{s}\cdot\frac{b}{s^2+as+b} = \frac{1}{s}\cdot\frac{b}{(s+\sigma_1)^2} \qquad \begin{cases} 2\sigma_1 = a \\ \sigma_1^2 = b \end{cases}$$

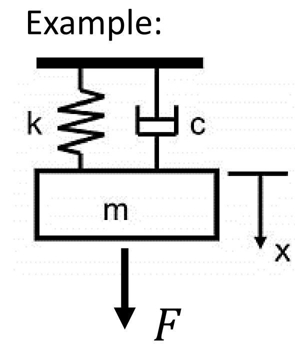
$$= \frac{K_0}{s} + \frac{K_1}{(s+\sigma_1)} + \frac{K_2}{(s+\sigma_1)^2}$$

$$c(t) = c_f(t) + c_n(t) = 1 + K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$



Unit-Step Response: Summery





❖ **Note**: The **critically damped** case is the division between the **overdamped** cases and the **underdamped** cases and is the **fastest response possible without overshoot**.

A New Definition of Second-Order Systems

Two physically meaningful specifications/parameters used to describe the characteristics of the second-order transient response:

- **Natural Frequency** (ω_n): It is the frequency of system oscillation without damping.

$$G(s) = \frac{b}{s^2 + as + b}$$ Without damping, the poles would be on the $j\omega$-axis $\rightarrow$ $a = 0$ $\rightarrow$ $\omega_n = \sqrt{b}$

- **Damping Ratio** (ζ): $$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (red/s)}} \rightarrow \zeta = \frac{\sigma_d}{\omega_n} = \frac{a/2}{\omega_n} = \frac{a}{2\sqrt{b}}$$

$$\Rightarrow \quad \begin{aligned} b &= \omega_n^2 \\ a &= 2\zeta\omega_n \end{aligned}$$

Therefore, standard form of a second-order system in terms of ζ and ω_n can be written as

$$G(s) = \frac{b}{s^2 + as + b} \longrightarrow G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

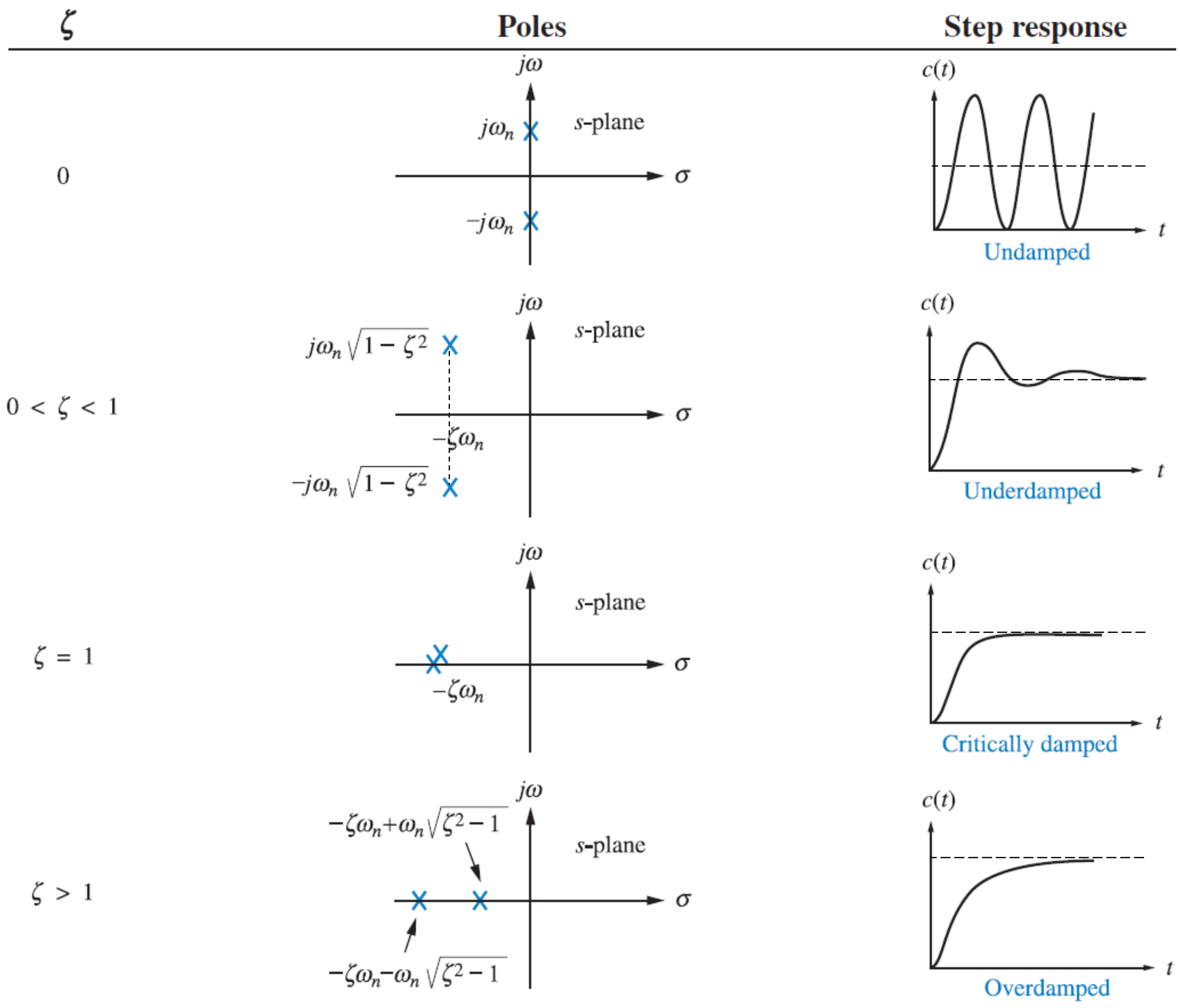
Poles: $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Step Response of Second-Order Systems Using the New Definition

The step response can be classified based on the value of ζ:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Poles: $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$



Example

For the given the transfer function, find ζ and ω_n.

$$G(s) = \frac{360}{s^2 + 4.2s + 36}$$

Answer: $\omega_n = 6, \;\; \zeta = 0.35$

Underdamped Second-Order Systems

Step Response of Underdamped Second-Order Systems

Underdamped second-order systems (i.e., when $0 < \zeta < 1$) are common in physical problems. Thus, a detailed description of these systems is necessary for both analysis and design.

$$C(s) = R(s)G(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$c(t) = 1 - e^{-\sigma_d t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) = 1 - \frac{e^{-\sigma_d t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi)$$

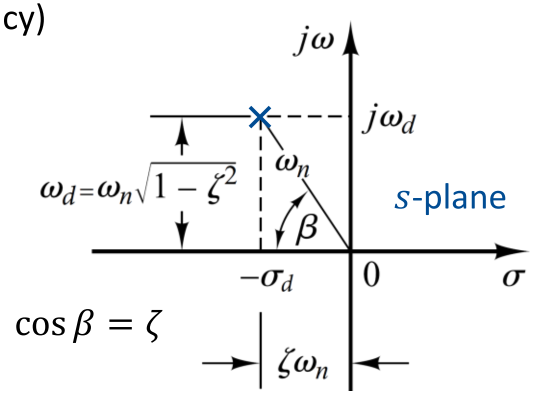
$$= 1 - \frac{e^{-\sigma_d t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \bar{\phi})$$

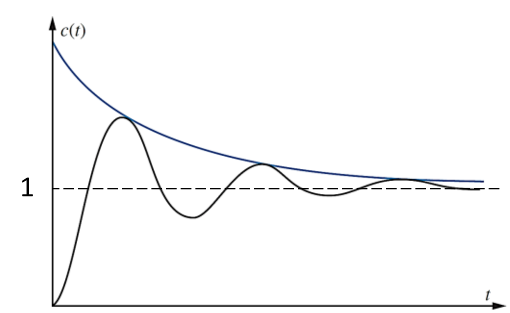
$\sigma_d = \zeta\omega_n$ (Exponential Damping Frequency or Attenuation)

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ (Damped Natural Frequency)

$$\phi = \tan^{-1}\left(\zeta/\sqrt{1-\zeta^2}\right)$$

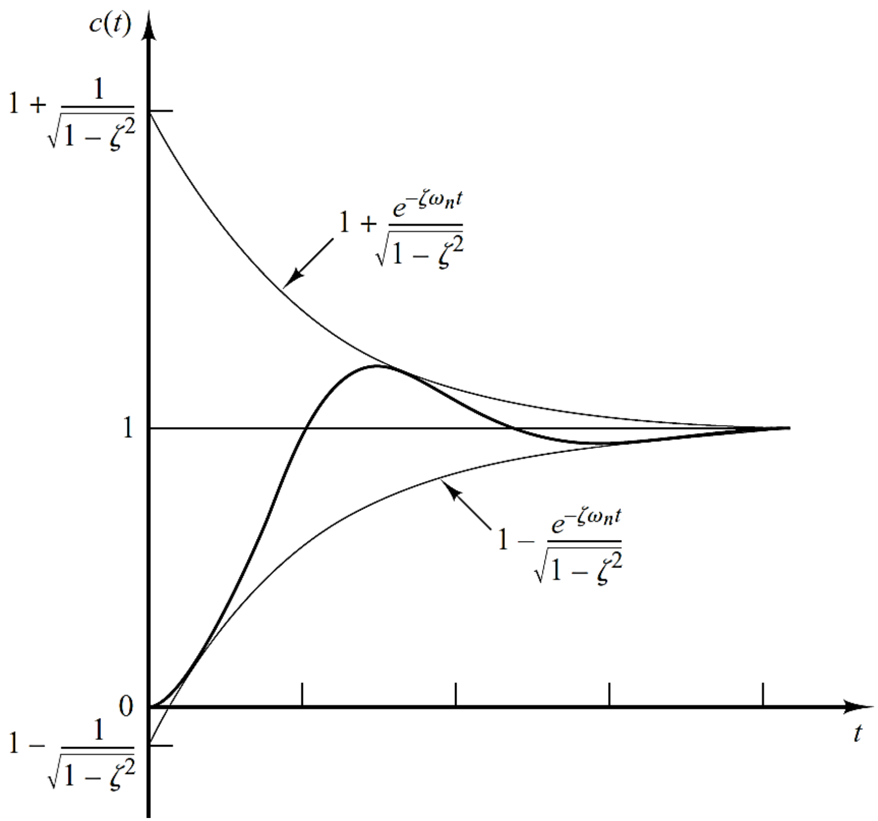
$$\bar{\phi} = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$$



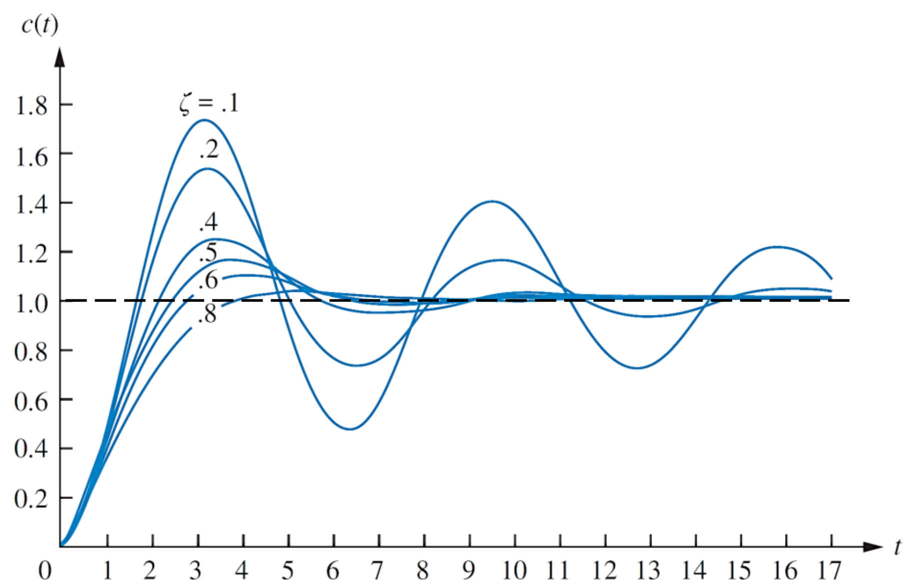


Step Response of Underdamped Second-Order Systems

Pair of envelope curves for the unit-step response:



Unit-step response for different damping ratio ζ values:



Parameters of Step Response of Underdamped Second-Order Systems

The parameters defined for the step input response of underdamped second-order systems:

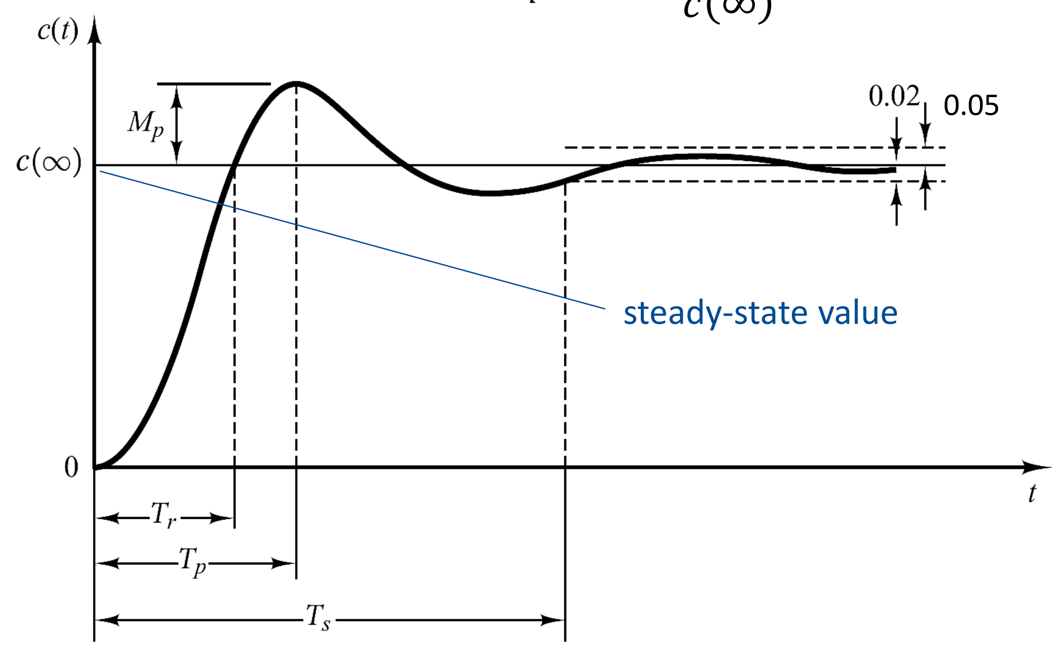
1. **Peak Time** T_p: The time required for the response to reach the **first** (or maximum) peak.

2. **Maximum Overshoot** M_p: The percentage of the steady-state value that the response overshoots the steady-state value at the peak time T_p.

$$M_p = \frac{c(T_p) - c(\infty)}{c(\infty)} \times 100$$

3. **Settling Time** T_s: The time required for the response to **reach and stay** within 2% (or 5%) of the steady-state value.

4. **Rise Time** T_r: The time required for the response to go from 0% to 100% (or 10% to 90% or 5% to 95%) of the steady-state value.



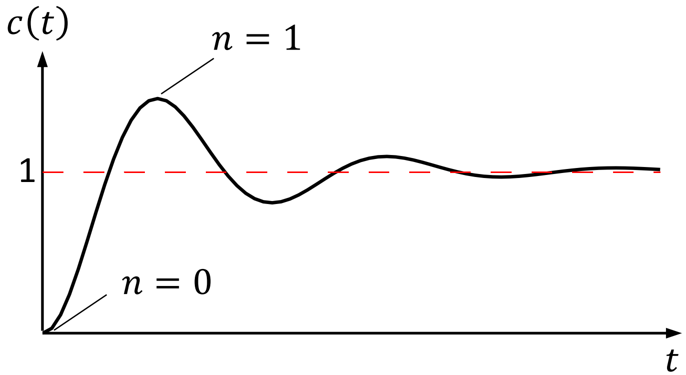
Calculation of Parameters: T_p

Peak Time T_p:

$$T_p = t\Big|_{\dot{c}(t)=0} \quad \Rightarrow \quad \dot{c}(t) = \frac{d\left(1 - e^{-\sigma_d t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right)}{dt} = 0 \quad \Rightarrow$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma_d t}\sin\omega_d t = 0 \quad \Rightarrow \quad \omega_d t = n\pi$$

Each value of n yields the time for local maxima or minima. Letting $n = 0$ yields $t = 0$. The first peak is found by letting $n = 1$:



$$\boxed{T_p = \frac{\pi}{\omega_d}}$$

Calculation of Parameters: M_p

Maximum Overshoot M_p:

$$M_p = \frac{c(T_p) - c(\infty)}{c(\infty)} \times 100$$

$$c(T_p) = c\left(\frac{\pi}{\omega_d}\right) = 1 - e^{-\sigma_d \pi/\omega_d}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) = 1 + e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}$$

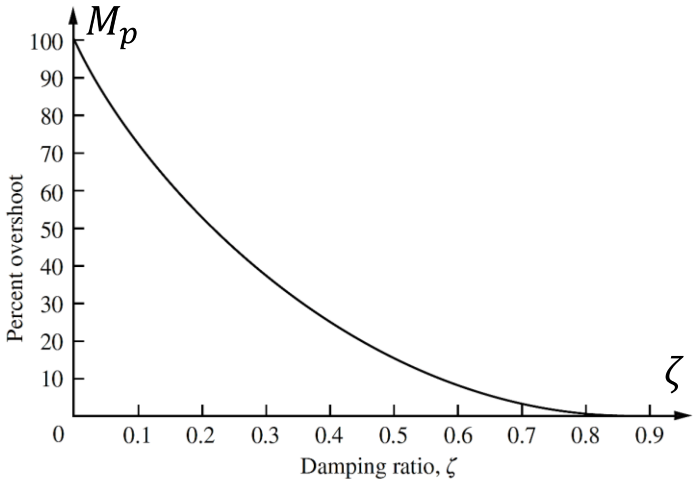
$c(\infty) = 1$ (For the unit step)

$\Rightarrow$

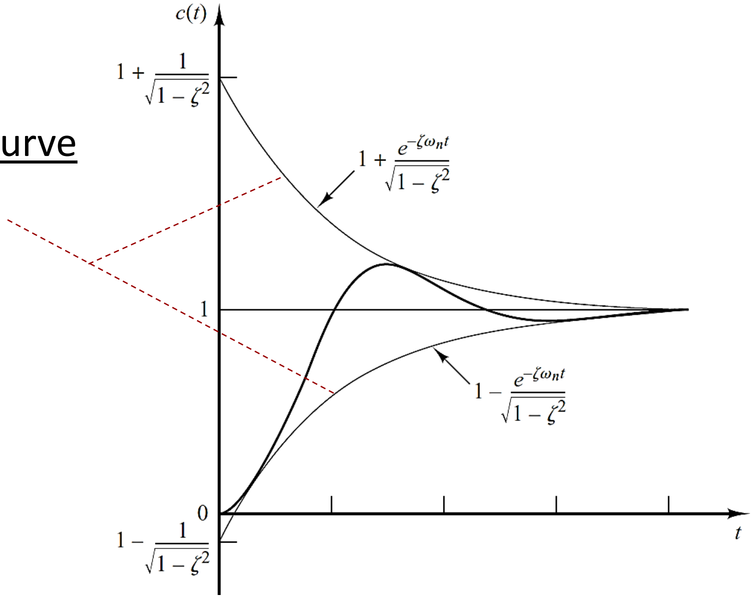
$$\boxed{M_p = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100}$$

(or)

$$\boxed{\zeta = \frac{-\ln(M_p/100)}{\sqrt{\pi^2 + \ln^2(M_p/100)}}}$$



Calculation of Parameters: T_s



Settling Time T_s:

We can find T_s by finding the time it takes for the envelope curve to reach 2% or 5% of the steady-state value.

- For 2% of the steady-state value:

$$1 - \left(1 - \frac{e^{-\sigma_d t}}{\sqrt{1-\zeta^2}}\right) = 0.02 \quad \Rightarrow \quad \frac{e^{-\sigma_d t}}{\sqrt{1-\zeta^2}} = 0.02$$

$$\Rightarrow \quad T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\sigma_d}$$

Since for $0 < \zeta < 0.9$, we have $3.91 < -\ln\left(0.02\sqrt{1-\zeta^2}\right) < 4.74$

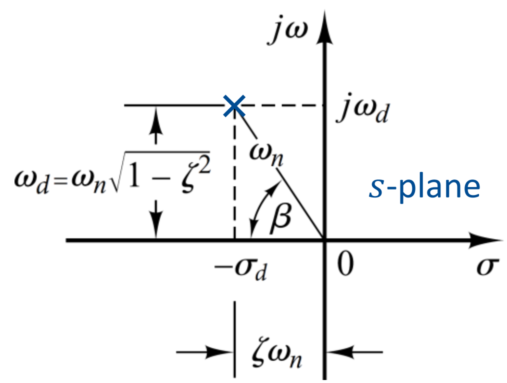
$$\Rightarrow \quad \boxed{T_s \approx \frac{4}{\sigma_d}}$$

- Similarly, for 5% of the steady-state value: $\boxed{T_s \approx \frac{3}{\sigma_d}}$

Calculation of Parameters: T_r

Rise Time T_r:

- For 10% to 90%, a precise analytical relationship cannot be found.
- For 0% to 100%:



$$\cos\beta = \zeta$$

$$T_r = t\Big|_{c(t)=c(\infty)} - t\Big|_{c(t)=0} \quad\Rightarrow\quad c(T_r) = 1 \quad\Rightarrow$$

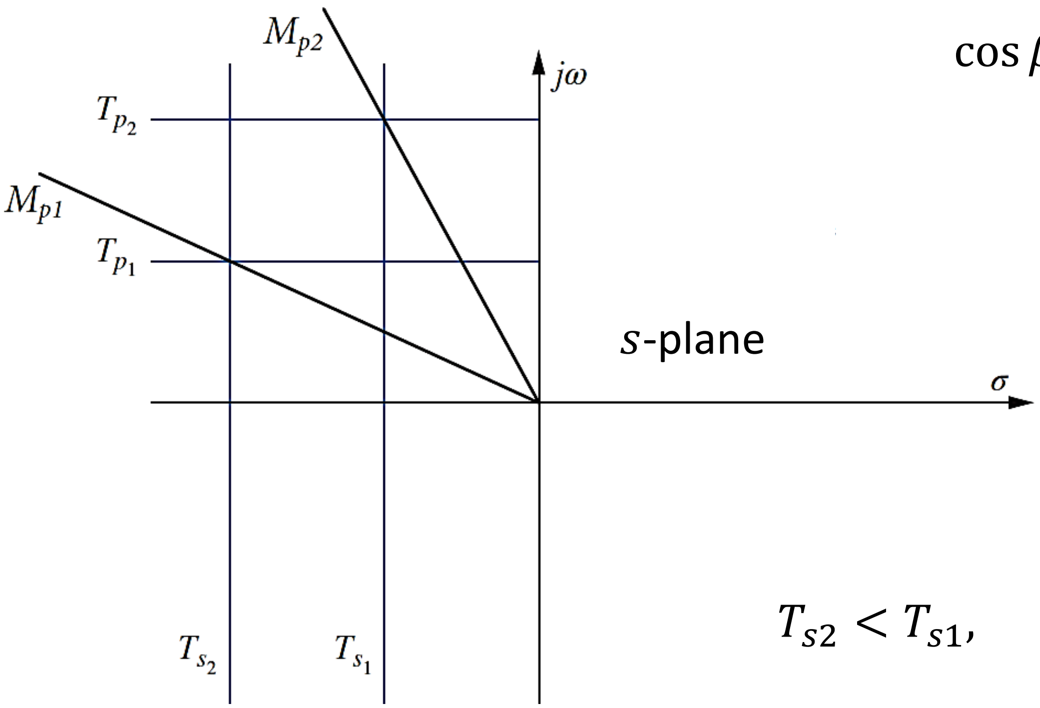
$$1 - e^{-\sigma_d T_r}\left(\cos\omega_d T_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d T_r\right) = 1 \quad\Rightarrow$$

$$\cos\omega_d T_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d T_r = 0 \quad\Rightarrow\quad \boxed{T_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right) = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma_d}\right) = \frac{\pi-\beta}{\omega_d}}$$

(from above figure)

Lines of Constant T_s, T_p, and M_p

- **Vertical** lines on the s-plane are lines of constant **Settling Time**, since $T_s = 4/\sigma_d$.
- **Horizontal** lines on the s-plane are lines of constant **Peak Time**, since $T_p = \pi/\omega_d$.
- **Radial** lines on the s-plane are lines of constant ζ and **Maximum Overshoot**, since



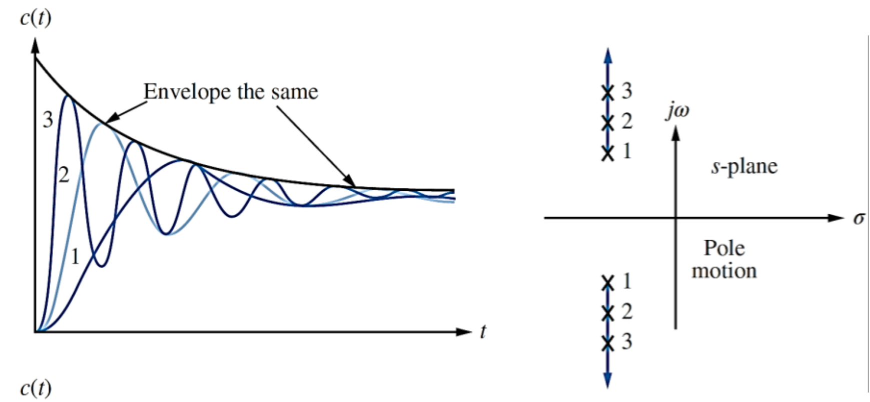
$$\cos\beta = \zeta\,, \qquad M_p = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

$$T_{s2} < T_{s1}, \qquad T_{p2} < T_{p1}, \qquad M_{p1} < M_{p2}$$

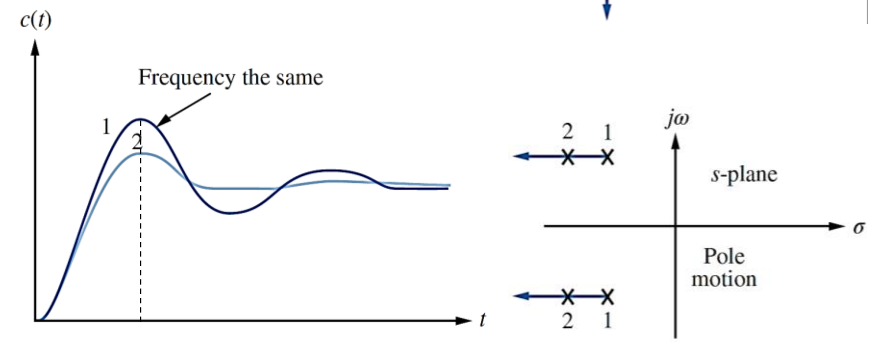
Step Responses of Underdamped 2nd-Order Systems as Poles Move

$T_r = (\pi - \beta)/\omega_d,\ T_p = \pi/\omega_d,\ T_s = 4/\sigma_d,\ \cos\beta = \zeta$

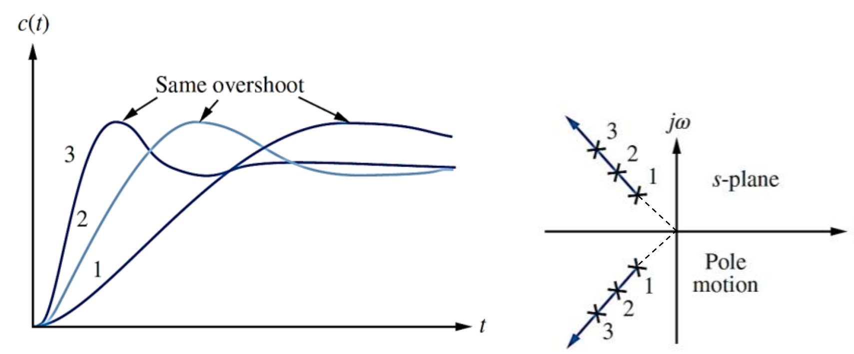
As the poles move in a vertical direction upward, ω_d and M_p increase, T_p and T_r decrease, and T_s remains constant.



As the poles move in a horizontal direction to the left, M_p and T_s decrease (the response damps out more rapidly), T_r increases, ω_d and T_p remain constant.



As the poles move diagonally away from the origin, ω_d increases, T_r, T_s, and T_p decrease, and M_p remains constant.

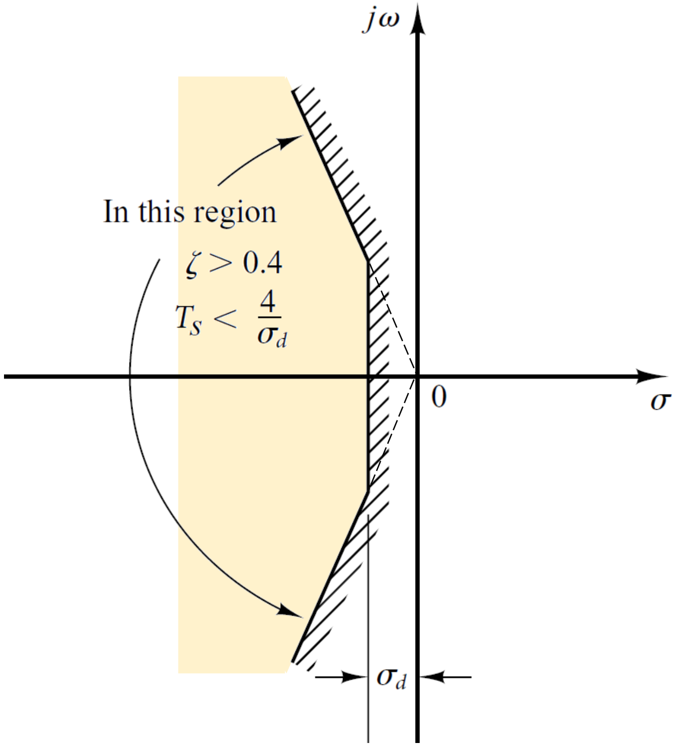


Some Comments

- Except for certain applications where oscillations cannot be tolerated, it is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus, for a **desirable transient response** of a second-order system, the damping ratio (ζ) must be **between 0.4 and 0.8**. Values of $\zeta < 0.4$ yield excessive overshoot and values of $\zeta > 0.8$ responds slowly.

- To guarantee specific, desired transient response characteristics, it is necessary that the poles of the system lie in a particular region in the complex plane.

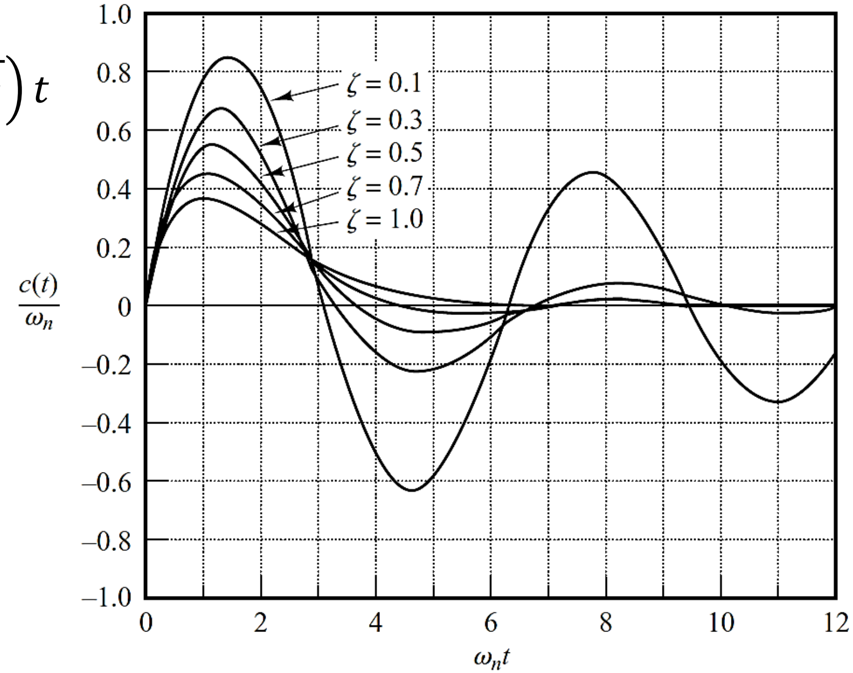
  For example, in this figure, a desired region for poles of a 2nd-order system to have fast ($T_s < 4/\sigma$), yet well-damped ($\zeta > 0.4$), transient response characteristics, is shown.



Impulse Response of Second-Order Systems

Impulse Response of second-order systems can be derived by **differentiating** the corresponding unit-step response, since the unit-impulse function is the time derivative of the unit-step function.

- $0 \leq \zeta < 1:\quad c(t) = \dfrac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\right) t$
- $\zeta = 1:\quad c(t) = \omega_n^2 t e^{-\omega_n t}$
- $\zeta > 1:\quad c(t) = \dfrac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} - \dfrac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$

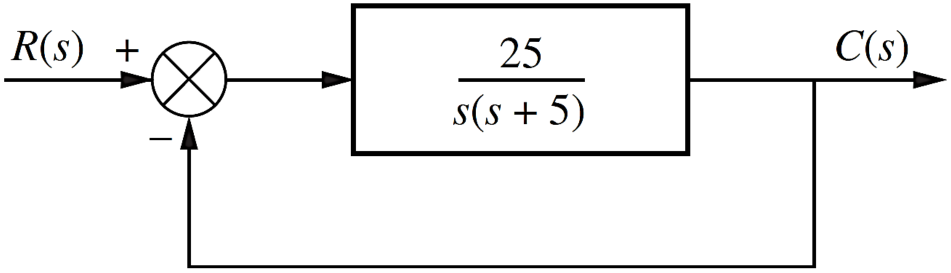


Note:

$$\ddot{x} + a\dot{x} + bx = b\delta(t),\quad x(0^-) = 0, \dot{x}(0^-) = 0 \quad \Leftrightarrow \quad \ddot{w} + a\dot{w} + bw = 0,\quad \dot{w}(0^+) = b, w(0^+) = 0$$

Example

Find the peak time, maximum overshoot, and settling time (to reach 2% of the steady-state value).

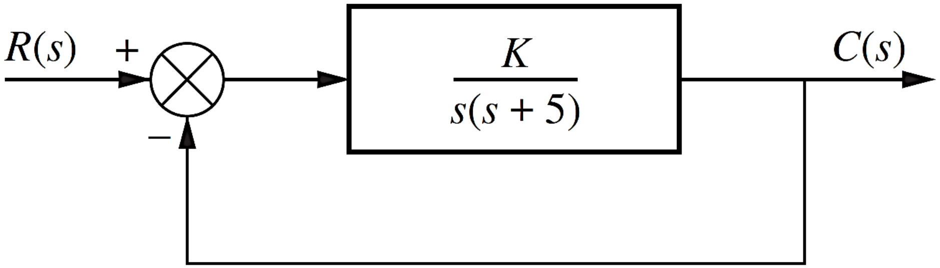


Answer:

$T_p = 0.726\ \mathrm{s}$, $M_p = 16.303$, $T_s = 1.6\ \mathrm{s}$

Example

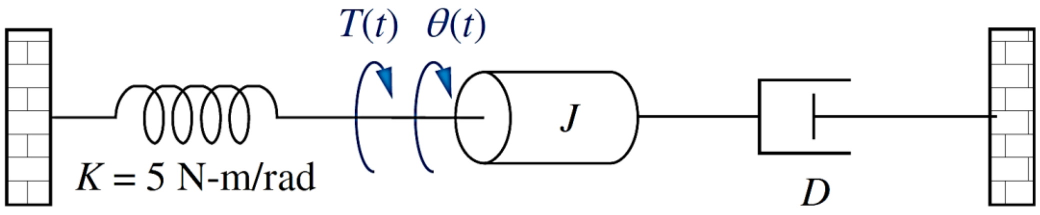
Design the value of gain K for the shown feedback control system so that the system will respond to a step input with a 10% overshoot.



Answer: $K = 17.9$

Example

Given the system shown, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque $T(t)$.



Answer:

$J = 0.26\ \mathrm{kgm}^2 \qquad D = 1.04\ \mathrm{Nms/rad}$

Using MATLAB and Control System Toolbox

Time Response of Systems Using step and impulse

For Unit-Step Response:

```
num = 4;
den = [1 1 4];
T1 = tf(num,den);

% Unit-Step Response
step(T1)

% Unit-Step Response for a given time range
figure
t_range = 0:0.1:15;
step(T1,t_range)

% or
figure
[y,t] = step(T1);
plot(t,y)
```

```
% Time response of multiple systems
T2 = tf(1,[1 2 1]);
T3 = tf(4,[1 4 4]);
T4 = tf(4,[1 1 4]);
figure
step(T2,T3,T4)
% or
figure
step(T2,'r-',T3,'k-.',T4,'b--')
```

- When t is not explicitly included in the step commands, the time vector is automatically determined.

For Unit-Impulse Response:

Method 1: Substitute the command "step" with command "impulse".

Method 2: The unit-impulse response of $G(s)$ is the same as the unit-step response of $sG(s)$.

Time Response of Systems Using lsim

For Unit-Ramp Response:
Method 1: The unit-ramp response of $G(s)$ is the same as the unit-step response of $G(s)/s$.
Method 2: Use the general command "lsim".

```
lsim(sys,r,t)
y = lsim(sys,r,t)
```

The command "lsim" is used to obtain the response to an **arbitrary input**.

```
num = 4;
den = [1 1 4];
T1 = tf(num,den);
t = 0:0.1:10;
r = t; % ramp input
y = lsim(T1,r,t);
figure
plot(t,r,'-',t,y,'-.')
```

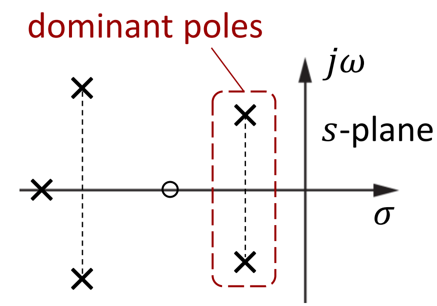
To find step-response characteristics (e.g., T_p, T_s, ...) use stepinfo.

```
stepinfo(sys)
```

Higher-Order Systems

Dominant Poles

- If a system has more than two poles or has zeros, **analytical** expressions for T_s, T_p, and M_p cannot be found and the transient response of these systems to any given input can be obtained by a **computer simulation**. However,
  - If the system has just two **complex dominant poles**, it can be approximated as a **second-order system** and all the derived formulas for T_s, T_p, and M_p of the second-order systems can be used.
  - Transient response of all other higher-order systems can be considered using **Root Locus** techniques (that will be covered in the next chapters).



- In this section, the effect of **adding a pole** to a second-order system $T(s)$ is investigated.

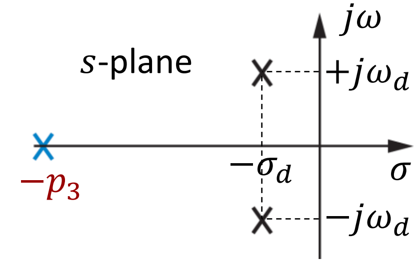
$$T'(s) = \frac{T(s)}{(s+p_3)} = \frac{k}{(s^2+as+b)(s+p_3)}$$

System Response with an Additional Pole

Consider the unit-step response of a three-pole system $T'(s)$ with two complex dominant poles $-\sigma_d \pm j\omega_d$ and a real pole p_3:

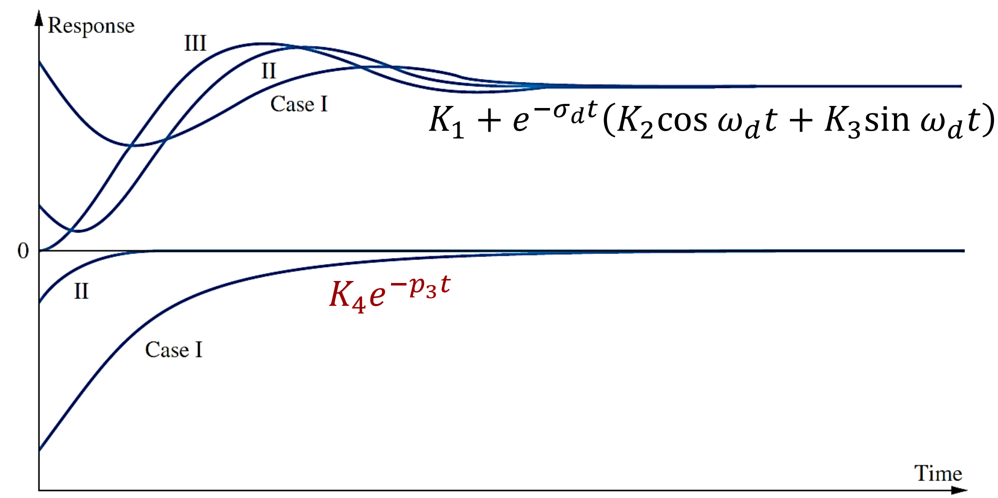
$$C(s) = \frac{1}{s} \cdot \frac{k}{(s^2 + as + b)(s + p_3)} = \frac{K_1}{s} + \frac{K_2(s + \sigma_d) + K_3\omega_d}{(s + \sigma_d) + \omega_d^2} + \frac{K_4}{s + p_3}$$

$$c(t) = K_1 + e^{-\sigma_d t}(K_2 \cos \omega_d t + K_3 \sin \omega_d t) + K_4 e^{-p_3 t}$$



Comparison of the component parts of $c(t)$ for 3 cases:

Case I: p is near dominant complex dominant poles (i.e., $p_3 > \sigma_d$).
Case II: p is far from dominant complex dominant poles (i.e., $p_3 \gg \sigma_d$).
Case III: p is at infinity (i.e., $p_3 = \infty$, and consequently, $K_4 = 0$).



System Response with an Additional Pole

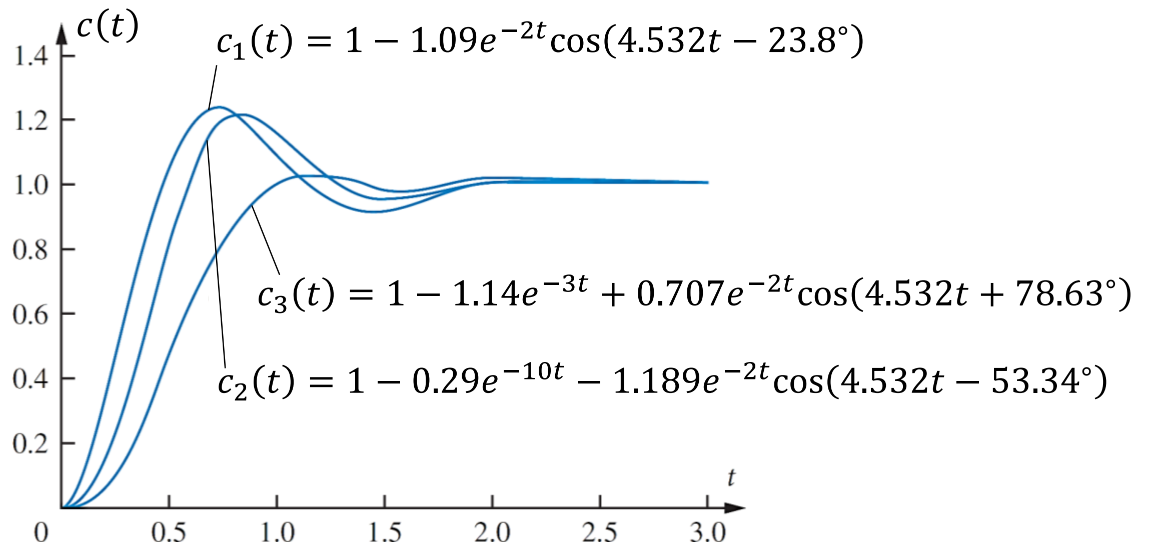
Therefore, if the term $K_4 e^{-p_3 t}$ decays to an insignificant value at the time of the first overshoot, then T_s, T_p, and M_p will be generated by the underdamped second-order step response component and the system can be represented as a **pure second-order system**.

Example:

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542}$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.542)}$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.542)}$$



Note: It can be assumed that if the real pole p_3 is **five times** farther to the left than the dominant poles (i.e., $p_3 > 5\sigma_d$), the system is represented by its dominant second-order pair of poles.

Note: The magnitude of residue of the non-dominant pole (K_4) will also decrease as the pole is moved farther into the left of $j\omega$-axis.

Example

Determine the validity of a second-order approximation for each of these two transfer functions:

a) $G(s) = \dfrac{700}{(s+15)(s^2+4s+100)}$

b) $G(s) = \dfrac{360}{(s+4)(s^2+2s+90)}$

Answer: a) Valid, b) invalid

Systems with Zeros

System Response with a Zero

Let $C(s)$ be the response of a system $T(s)$, with a constant in the numerator. If we add a zero to the transfer function, i.e., $T'(s) = (s+z)T(s)$, the **response** will consist of **two parts**:

$$C'(s) = (s+z)C(s) = sC(s) + zC(s)$$

$$c'(t) = \dot{c}(t) + zc(t)$$

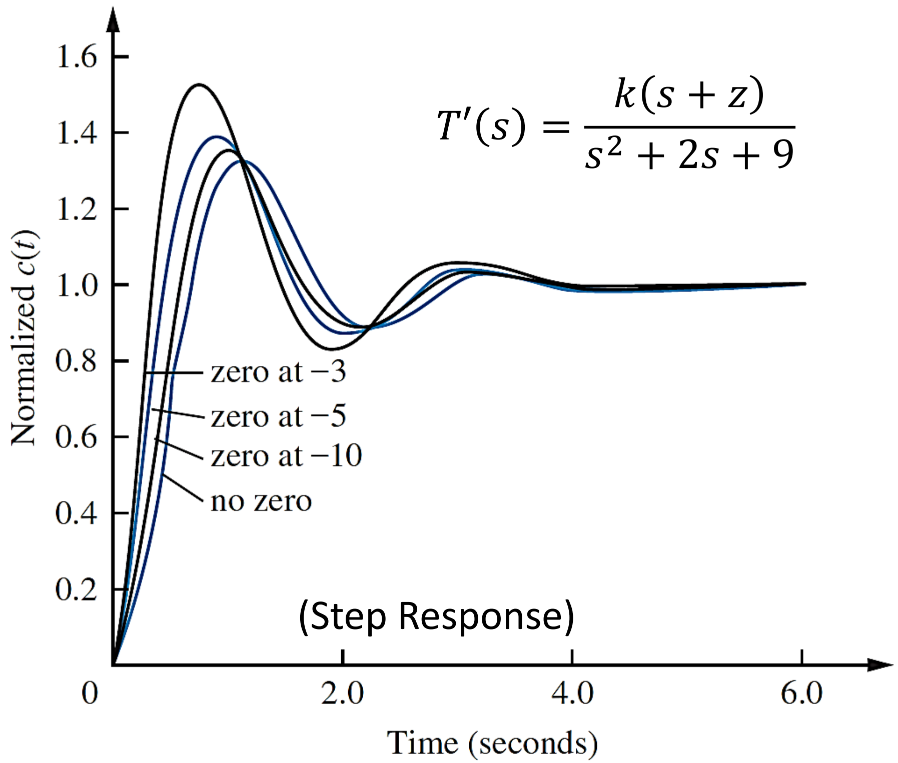
(Derivative of the original response) (A scaled version of the original response)

- In this section, the effect of **adding a zero** to a second-order system $T(s)$ is investigated.

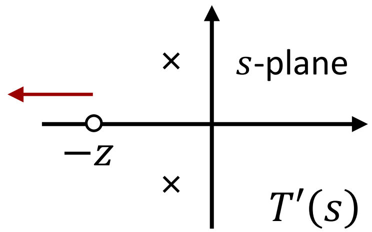
$$T'(s) = T(s)(s+z) = \frac{k(s+z)}{s^2 + as + b}$$

System Response with a Negative Zero

Let $T'(s)$ be a second-order system with a **negative real zero** z (i.e., z is placed in the **L**eft **H**alf-**P**lane or **LHP**). As the z moves away from the dominant poles to the left, the derivative term $\dot{c}(t)$ contributes less to the response $c'(t)$ and the response approaches to the form of the second-order system and the zero acts as a simple gain factor (i.e., $c'(t) \approx zc(t)$).



$$T'(s) = \frac{k(s+z)}{s^2+2s+9}$$



$z > 0$

$$T'(s) = \frac{k(s+z)}{s^2+as+b}$$

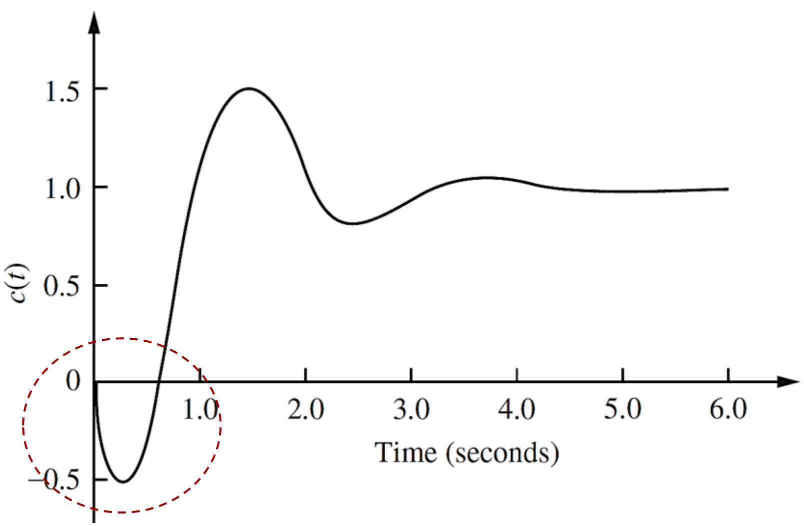
$$c'(t) = \dot{c}(t) + zc(t)$$

For step response, $c(t) > 0$, $\dot{c}(t) > 0$, $zc(t) > 0$, then $c'(t) > 0$.

- As z becomes smaller (closer to the dominant poles), the derivative term contributes more to the response and maximum overshoot increases.

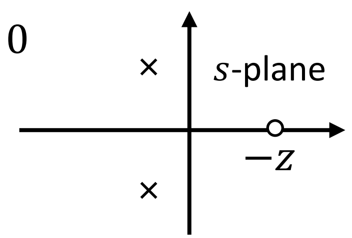
System Response with a Positive Zero
(Minimum-Phase & Nonminimum-Phase Systems)

Let $T'(s)$ be a second-order system with a **positive real zero** z (i.e., z is placed in the **R**ight **H**alf-**P**lane or **RHP**). In this case, the step response first begins to turn toward the negative direction even though the steady-state value is positive.



$z < 0$

$$T'(s) = \frac{k(s+z)}{s^2 + as + b}$$



$$c'(t) = \dot{c}(t) + zc(t)$$

For step response, $c(t) > 0$, $\dot{c}(t) > 0$, but since $z < 0$, then $zc(t) < 0$.

❖ If all the poles and zeros of a system lie in the LHP, then the system is called **Minimum Phase**. If a system has at least one pole or zero in the RHP, then the system is called **Non-Minimum-Phase** (**NMP**) phase.

Pole-Zero Cancellation

Consider a three-pole system with a zero.

$$G(s) = \frac{k(s+z)}{(s^2+as+b)(s+p_3)}$$

- If the pole term, $(s+p_3)$, and the zero term, $(s+z)$, cancel out (i.e., $z = p_3$), the system is **approximated as a second-order system**.

- If the zero at $-z$ is **very close** to the pole at $-p_3$ (i.e., $z \approx p_3$), then the amplitude of the exponential decay $e^{-p_3 t}$ in the step-input response is much smaller than the other amplitudes and the system can be **approximated as a second-order system**. <u>Hence, a pair of closely located poles and zeros will effectively cancel each other.</u>

$$C(s) = \frac{1}{s}G(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \cancelto{\approx 0}{\frac{0.033}{s+4.01}} \longrightarrow c_2(t) = 0.87 - 5.3e^{-5t} + 4.4e^{-6t} + \cancelto{\approx 0}{0.033e^{-4.01t}}$$

$$G(s) = \frac{26.25\cancel{(s+4)}}{\cancel{(s+4.01)}(s+5)(s+6)}$$