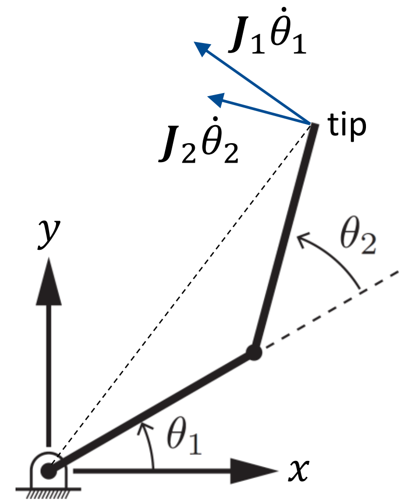
Ch6: Velocity Kinematics and Statics

Manipulator Jacobian

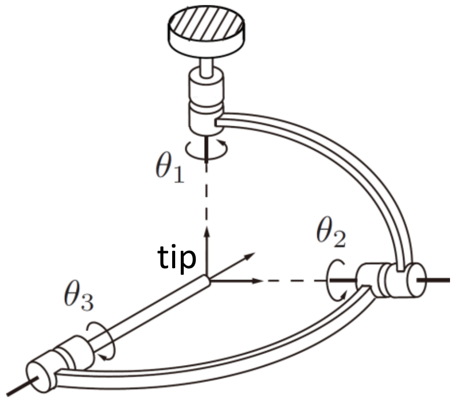
Manipulator Jacobian

In a 2R planar robot, we saw that $\boldsymbol{v}_{tip}$ is the linear velocity of the end-effector frame

$$\boldsymbol{v}_{tip} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \boldsymbol{J}(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = [\boldsymbol{J}_1 \quad \boldsymbol{J}_2] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \boldsymbol{J}_1 \dot{\theta}_1 + \boldsymbol{J}_2 \dot{\theta}_2$$



In a pure orienting devices such as a wrist, $\boldsymbol{v}_{tip}$ is the angular velocity of the end-effector frame.



- Thus, $\boldsymbol{v}_{tip}$ determine the specific form of the Jacobian.

Space and Body Manipulator Jacobians

Let's assume that the configuration of the end-effector is expressed as $\boldsymbol{T} \in SE(3)$ and its velocity is expressed as a twist $\boldsymbol{\mathcal{V}} \in \mathbb{R}^6$ in the fixed base frame $\{s\}$ or the end-effector body frame $\{b\}$.

❖ The Jacobian is derived based on the following general idea:
Given the configuration $\boldsymbol{\theta} \in \mathbb{R}^n$ of the robot, $\boldsymbol{J}_i(\boldsymbol{\theta}) \in \mathbb{R}^6$, which is column i of $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$, is the twist $\boldsymbol{\mathcal{V}}$ when the robot is in an arbitrary configuration $\boldsymbol{\theta}$ (not in home configuration $\boldsymbol{\theta} = \mathbf{0}$), $\dot{\theta}_i = 1$, and all other joint velocities are zero.

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = [\boldsymbol{J}_1 \quad \boldsymbol{J}_2 \quad \cdots \quad \boldsymbol{J}_n]\dot{\boldsymbol{\theta}}$$

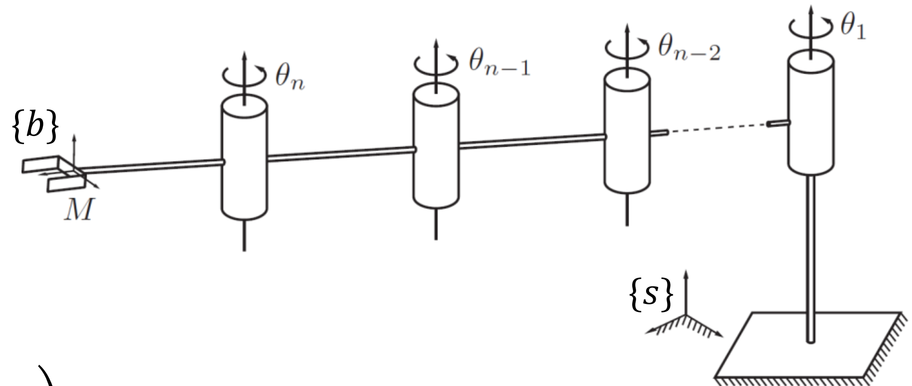
$\dot{\boldsymbol{\theta}}$: joint velocities

- If each column $\boldsymbol{J}_i(\boldsymbol{\theta})$ is expressed in the fixed space frame $\{s\}$: $\Rightarrow$ Space Jacobian $\quad \boldsymbol{\mathcal{V}}_s = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- If each column $\boldsymbol{J}_i(\boldsymbol{\theta})$ is expressed in the end-effector frame $\{b\}$: $\Rightarrow$ Body Jacobian $\quad \boldsymbol{\mathcal{V}}_b = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Space Jacobian

Consider an n-link open chain:

$$\boldsymbol{T}(\boldsymbol{\theta}) = e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M}$$ forward kinematics



$$[\boldsymbol{\mathcal{V}}_s] = \dot{\boldsymbol{T}}\boldsymbol{T}^{-1} \qquad (\boldsymbol{T} = \boldsymbol{T}_{sb})$$

$$\dot{\boldsymbol{T}} = \left(\frac{d}{dt} e^{[\boldsymbol{S}_1]\theta_1}\right) \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + e^{[\boldsymbol{S}_1]\theta_1} \left(\frac{d}{dt} e^{[\boldsymbol{S}_2]\theta_2}\right) \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + \cdots$$

$$= [\boldsymbol{S}_1]\dot{\theta}_1 e^{[\boldsymbol{S}_1]\theta_1} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + e^{[\boldsymbol{S}_1]\theta_1} [\boldsymbol{S}_2]\dot{\theta}_2 e^{[\boldsymbol{S}_2]\theta_2} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + \cdots$$

$$\boldsymbol{T}^{-1} = \boldsymbol{M}^{-1} e^{-[\boldsymbol{S}_n]\theta_n} \dots e^{-[\boldsymbol{S}_1]\theta_1}$$

$$[\boldsymbol{\mathcal{V}}_s] = [\boldsymbol{S}_1]\dot{\theta}_1 + e^{[\boldsymbol{S}_1]\theta_1}[\boldsymbol{S}_2]e^{-[\boldsymbol{S}_1]\theta_1}\dot{\theta}_2 + e^{[\boldsymbol{S}_1]\theta_1}e^{[\boldsymbol{S}_2]\theta_2}[\boldsymbol{S}_3]e^{-[\boldsymbol{S}_2]\theta_2}e^{-[\boldsymbol{S}_1]\theta_1}\dot{\theta}_3 + \cdots$$

$$\boldsymbol{A}[\boldsymbol{S}_i]\boldsymbol{A}^{-1} = [[\mathrm{Ad}_{\boldsymbol{A}}]\boldsymbol{S}_i]$$

$$\boldsymbol{\mathcal{V}}_s = \underbrace{\boldsymbol{S}_1}_{\boldsymbol{J}_{s1}} \dot{\theta}_1 + \underbrace{\left[\mathrm{Ad}_{e^{[\boldsymbol{S}_1]\theta_1}}\right]\boldsymbol{S}_2}_{\boldsymbol{J}_{s2}} \dot{\theta}_2 + \underbrace{\left[\mathrm{Ad}_{e^{[\boldsymbol{S}_1]\theta_1}e^{[\boldsymbol{S}_2]\theta_2}}\right]\boldsymbol{S}_3}_{\boldsymbol{J}_{s3}} \dot{\theta}_3 + \cdots$$

$$\boldsymbol{\mathcal{V}}_s = \boldsymbol{J}_{s1}\dot{\theta}_1 + \boldsymbol{J}_{s2}(\boldsymbol{\theta})\dot{\theta}_2 + \cdots + \boldsymbol{J}_{sn}(\boldsymbol{\theta})\dot{\theta}_n$$

Space Jacobian (cont.)

$$\mathcal{V}_s = [J_{s_1} \quad J_{s_2}(\boldsymbol{\theta}) \quad \cdots \quad J_{s_n}(\boldsymbol{\theta})] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = J_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

The **space Jacobian** $J_s(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$. The ith column of $J_s(\boldsymbol{\theta})$ is

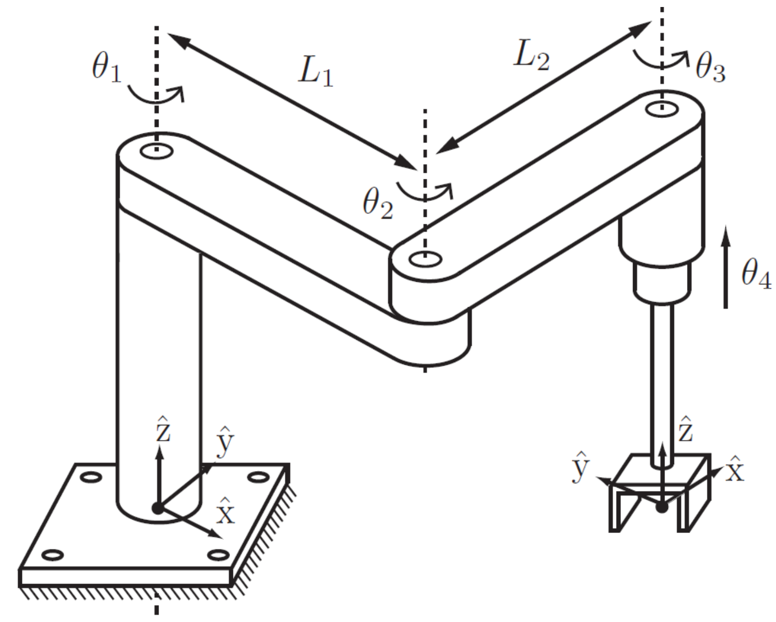
$$J_{si}(\boldsymbol{\theta}) = \left(\boldsymbol{\omega}_{si}(\boldsymbol{\theta}), \boldsymbol{v}_{si}(\boldsymbol{\theta})\right) = \left[\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}\right] \mathcal{S}_i \qquad \begin{array}{c} J_{s1} = \mathcal{S}_1 \\ i = 2, \dots, n \end{array}$$

Screw axis describing the ith joint axis (expressed in the fixed space frame $\{s\}$) after the joints $1, \dots, i-1$ move from their zero position to the current values $\theta_1, \dots, \theta_{i-1}$.

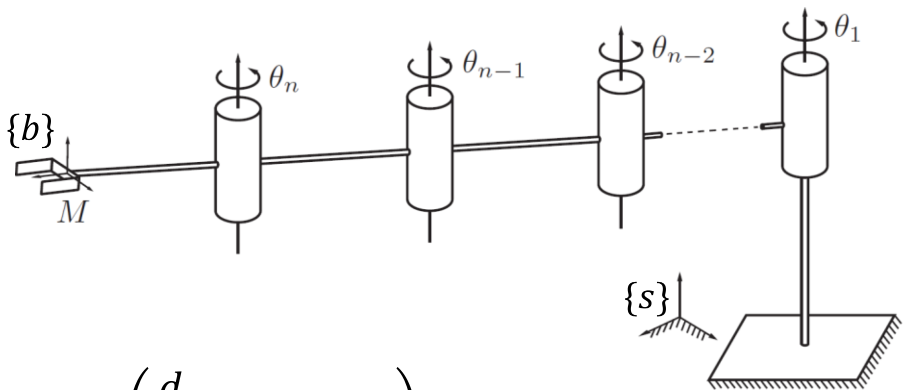
Screw axis describing the ith joint axis (expressed in the fixed space frame $\{s\}$) when the robot in its home or zero position $\boldsymbol{\theta} = \mathbf{0}$.

- The space Jacobian is independent of the choice of the end-effector frame $\{b\}$.

Example: Space Jacobian of a Spatial RRRP Robot



Body Jacobian



Consider an n-link open chain:

$$\boldsymbol{T}(\boldsymbol{\theta}) = \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}e^{[\boldsymbol{\mathcal{B}}_2]\theta_2}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}$$

(forward kinematics)

$$[\boldsymbol{\mathcal{V}}_b] = \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} \qquad (\boldsymbol{T} = \boldsymbol{T}_{sb})$$

$$\dot{\boldsymbol{T}} = \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}\left(\frac{d}{dt}e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\right) + \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots\left(\frac{d}{dt}e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}\right)e^{[\boldsymbol{\mathcal{B}}_n]\theta_n} + \cdots$$

$$= \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}[\boldsymbol{\mathcal{B}}_n]\dot{\theta}_n + \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}[\boldsymbol{\mathcal{B}}_{n-1}]e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_{n-1} + \cdots$$
$$+\boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}[\boldsymbol{\mathcal{B}}_1]e^{[\mathcal{B}_2]\theta_2}\dots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1$$

$$\boldsymbol{T}^{-1} = e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\dots e^{-[\boldsymbol{\mathcal{B}}_1]\theta_1}\boldsymbol{M}^{-1}$$

$$[\boldsymbol{\mathcal{V}}_b] = [\boldsymbol{\mathcal{B}}_n]\dot{\theta}_n + e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}[\boldsymbol{\mathcal{B}}_{n-1}]e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_{n-1} + \cdots + e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\dots e^{-[\boldsymbol{\mathcal{B}}_2]\theta_2}[\boldsymbol{\mathcal{B}}_1]e^{[\boldsymbol{\mathcal{B}}_2]\theta_2}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_1$$

$$\boldsymbol{A}^{-1}[\boldsymbol{\mathcal{B}}_i]\boldsymbol{A} = [[\mathrm{Ad}_{\boldsymbol{A}^{-1}}]\boldsymbol{\mathcal{B}}_i]$$

$$\boldsymbol{\mathcal{V}}_b = \underbrace{\boldsymbol{\mathcal{B}}_n}_{\boldsymbol{J}_{bn}}\dot{\theta}_n + \underbrace{\left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}}\right]\boldsymbol{\mathcal{B}}_{n-1}}_{\boldsymbol{J}_{b,n-1}}\dot{\theta}_{n-1} + \cdots + \underbrace{\left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\cdots e^{-[\boldsymbol{\mathcal{B}}_2]\theta_2}}\right]\boldsymbol{\mathcal{B}}_1}_{\boldsymbol{J}_{b1}}\dot{\theta}_1$$

$$\boldsymbol{\mathcal{V}}_b = \boldsymbol{J}_{b1}(\boldsymbol{\theta})\dot{\theta}_1 + \cdots + \boldsymbol{J}_{bn-1}(\boldsymbol{\theta})\dot{\theta}_{n-1} + \boldsymbol{J}_{bn}\dot{\theta}_n$$

Body Jacobian (cont.)

$$\boldsymbol{\mathcal{V}}_b = [\boldsymbol{J}_{b_1}(\boldsymbol{\theta}) \quad \cdots \quad \boldsymbol{J}_{b_{n-1}}(\boldsymbol{\theta}) \quad \boldsymbol{J}_{b_n}] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

The **body Jacobian** $\boldsymbol{J}_b(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ to the end-effector (or body) twist $\boldsymbol{\mathcal{V}}_b$. The ith column of $\boldsymbol{J}_b(\boldsymbol{\theta})$ is

$$\boldsymbol{J}_{bi}(\boldsymbol{\theta}) = \left(\boldsymbol{\omega}_{bi}(\boldsymbol{\theta}), \boldsymbol{v}_{bi}(\boldsymbol{\theta})\right) = \left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\cdots e^{-[\boldsymbol{\mathcal{B}}_{i+1}]\theta_{i+1}}}\right]\boldsymbol{\mathcal{B}}_i$$

$$\boldsymbol{J}_{bn} = \boldsymbol{\mathcal{B}}_n$$
$$i = n-1, \dots, 1$$

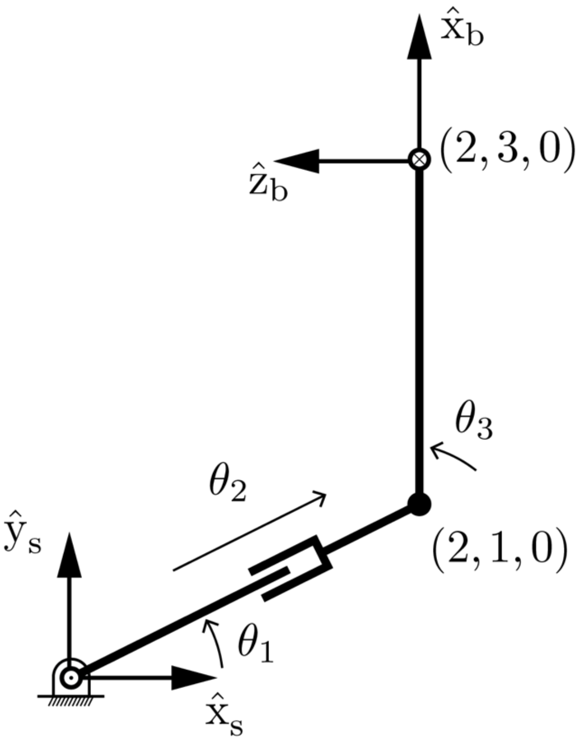
Screw axis describing the ith joint axis (expressed in the end-effector frame $\{b\}$) after the joints $i+1, \dots, n$ move from their zero position to the current values $\theta_n, \dots, \theta_{i+1}$.

Screw axis describing the ith joint axis (expressed in the end-effector frame $\{b\}$) when the robot in its home or zero position $\boldsymbol{\theta} = \boldsymbol{0}$.

- The body Jacobian is independent of the choice of the space frame $\{s\}$.

Example

Find the space and body Jacobians in the given configuration.



Relationship between Space and Body Jacobian

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} \qquad \mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$$

$$\mathcal{V}_b = J_b(\theta)\dot{\theta} \qquad \mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s = \mathrm{Ad}_{T_{bs}}(\mathcal{V}_s)$$

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} \quad \Rightarrow \quad [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = J_s(\theta)\dot{\theta} \quad \Rightarrow \quad [\mathrm{Ad}_{T_{bs}}][\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta}$$

$$\Rightarrow \quad \mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta} \quad \Rightarrow \quad J_b(\theta)\dot{\theta} = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta} \quad \xRightarrow{\forall \dot{\theta} \neq 0}$$

$$\boxed{J_b(\theta) = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)}$$

Similarly, $\boxed{J_s(\theta) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta)}$

Note: The space and body Jacobians, and the space and body twists, are similarly related by the adjoint map because each column of the space or body Jacobian corresponds to a twist.

Alternative Notions of Jacobian

There exist alternative notions of the Jacobian that are based on the representation of the end-effector configuration using a minimum set of coordinates $\boldsymbol{x}$ corresponding to a specific robot task space (which is a subspace of $SE(3)$), the different representations of rotations (e.g., Euler angles $\boldsymbol{\phi}$, unit quaternions $\boldsymbol{q}$, or exponential coordinates $\boldsymbol{r}$), or the different definitions of the end-effector velocities.

- $\boldsymbol{\mathcal{V}}_s = \begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Geometric Jacobian

- $\begin{bmatrix} \dot{\boldsymbol{\phi}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\quad \boldsymbol{\phi} = (\alpha, \beta, \gamma)$
- $\begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,q}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\quad \boldsymbol{q} = (q_0, q_1, q_2, q_3)$
- $\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\quad \boldsymbol{r} = \hat{\boldsymbol{\omega}}\theta$

If the end-effector velocity is represented by the time derivative of the coordinates, the Jacobian is called the **Analytic Jacobian** $\boldsymbol{J}_a$.

Example

Find the relationship between the space Jacobian $\boldsymbol{J}_s$ and geometric Jacobian $\boldsymbol{J}_g$ as defined as follows.

$$\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

Example

Find the relationship between the body Jacobian $\boldsymbol{J}_b$ in the body frame and an analytic Jacobian $\boldsymbol{J}_a$ that uses exponential coordinates $\boldsymbol{r} = \hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3(\|\hat{\boldsymbol{\omega}}\| = 1, \theta \in [0, \pi])$ to represent the orientation of the end-effector frame $\{b\}$ in the fixed frame $\{s\}$ and three coordinates $\boldsymbol{p} \in \mathbb{R}^3$for the position of the origin of the end-effector frame $\{b\}$ in the fixed frame $\{s\}$.

$$\boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

- Note that $\boldsymbol{\omega}_b = \boldsymbol{A}(\boldsymbol{r})\dot{\boldsymbol{r}}$ where $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{I} - \frac{1-\cos\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^2}[\boldsymbol{r}] + \frac{\|\boldsymbol{r}\| - \sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^3}[\boldsymbol{r}]^2$

and we assume that the matrix $\boldsymbol{A}(\boldsymbol{r})$ is invertible.

Velocity Kinematics

Velocity Kinematics and Kinematic Redundancy

In general, depending on the dimension of task space (i.e., $r \leq 6$), the differential kinematics equation can be represented as

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

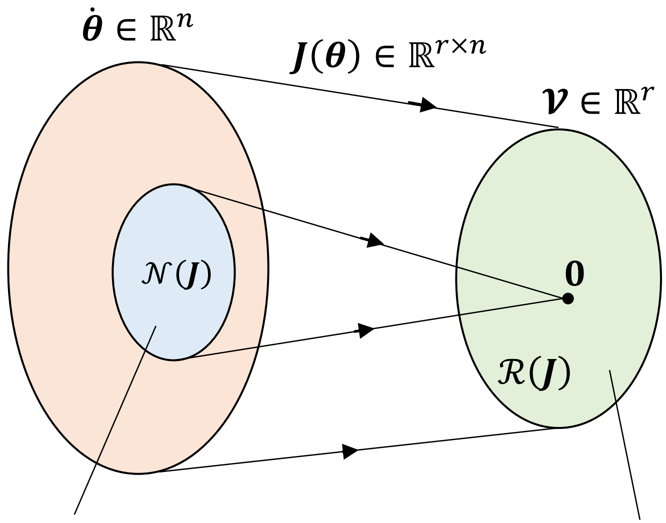
where now $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$ is end-effector velocity for the specific task, $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$, and $\boldsymbol{J} \in \mathbb{R}^{r \times n}$ is the corresponding Jacobian matrix that can be extracted from a $6 \times n$ geometric or analytic Jacobian.

Assume that the robot is not at a singular configuration,

- If $n \geq r$, then any arbitrary twists $\boldsymbol{\mathcal{V}}$ can be achieved (the robot have enough joints).
- If $n > r$, then an arbitrary twist $\boldsymbol{\mathcal{V}}$ places r constraints on the joint rates, and the remaining $n - r$ freedoms (redundancy) correspond to **internal motions** of the robot that are not evident in the motion of the end-effector.
- If $n < r$, then arbitrary twists $\boldsymbol{\mathcal{V}}$ cannot be achieved (the robot does not have enough joints).

Velocity Kinematics and Kinematic Redundancy

If $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ $(n \geq r)$ is full rank:



$\dot{\boldsymbol{\theta}}$ that does not produce any $\boldsymbol{\mathcal{V}}$ due to $n-r$ redundancy
$\dim \mathcal{N}(\boldsymbol{J}) = n-r$

$\boldsymbol{\mathcal{V}}$ that can be generated by $\dot{\boldsymbol{\theta}}$
$\dim \mathcal{R}(\boldsymbol{J}) = r$

$$\mathrm{Null}(\boldsymbol{J}) = \mathcal{N}(\boldsymbol{J}) = \{\dot{\boldsymbol{\theta}} \,|\, \boldsymbol{J}\dot{\boldsymbol{\theta}} = \mathbf{0}\}$$

$$\dim \mathcal{R}(\boldsymbol{J}) + \dim \mathcal{N}(\boldsymbol{J}) = n$$

Preliminary: Solving $\boldsymbol{Ax} = \boldsymbol{b}$

Consider $\boldsymbol{Ax} = \boldsymbol{b}$ ($\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{x} \in \mathbb{R}^{n}$, $\boldsymbol{b} \in \mathbb{R}^{m}$).

- ❖ If $\boldsymbol{A}$ is invertible, i.e., square and full rank, $\mathrm{rank}(\boldsymbol{A}) = n = m$: $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

- ❖ If $\boldsymbol{A}$ is not invertible, i.e., $\boldsymbol{A}$ is not square ($n \neq m$) or rank deficient ($\mathrm{rank}(\boldsymbol{A}) < \min(m, n)$), $\boldsymbol{Ax} = \boldsymbol{b}$ can still be solved (or approximately solved) for $\boldsymbol{x}$ with the Moore–Penrose pseudoinverse $\boldsymbol{A}^{\dagger}$: $\boldsymbol{x}^{*} = \boldsymbol{A}^{\dagger}\boldsymbol{b}$.

  - If $n > m$ (Fat): There can be an infinite number of solutions $\boldsymbol{x}$ to $\boldsymbol{Ax} = \boldsymbol{b}$. Among all solutions, $\boldsymbol{x}^{*}$ minimizes the Euclidean norm of $\boldsymbol{x}$ (i.e., $\|\boldsymbol{x}^{*}\| \leq \|\boldsymbol{x}\|$). If $\boldsymbol{A}$ is full rank, i.e., $\mathrm{rank}(\boldsymbol{A}) = \min(m, n)$:

$$\boldsymbol{A}^{\dagger} = \boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\right)^{-1} \qquad \text{(right inverse, } \boldsymbol{A}\boldsymbol{A}^{\dagger} = \boldsymbol{I}\text{)}$$

  - If $n < m$ (Tall): There is a unique $\boldsymbol{x}$ or there is no $\boldsymbol{x}$ that exactly satisfies $\boldsymbol{Ax} = \boldsymbol{b}$, then $\boldsymbol{x}^{*}$ minimizes the Euclidean norm of the error, (i.e., $\|\boldsymbol{Ax}^{*} - \boldsymbol{b}\| \leq \|\boldsymbol{Ax} - \boldsymbol{b}\|$). If $\boldsymbol{A}$ is full rank, i.e., $\mathrm{rank}(\boldsymbol{A}) = \min(m, n)$:

$$\boldsymbol{A}^{\dagger} = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}} \qquad \text{(left inverse, } \boldsymbol{A}^{\dagger}\boldsymbol{A} = \boldsymbol{I}\text{)}$$

Inverse Velocity Kinematics

Given a desired twist $\boldsymbol{\mathcal{V}}$, what joint velocities $\dot{\boldsymbol{\theta}}$ are needed?

- If $\boldsymbol{J}$ is square ($n = r$) and full rank $\mathrm{rank}(\boldsymbol{J}) = r$, (i.e., not at a singular configuration), then $\boldsymbol{J}$ is invertible and

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}(\boldsymbol{\theta})\boldsymbol{\mathcal{V}}$$

- If $\boldsymbol{J}$ is not square and $n > r$ (redundant robot), then infinite solutions exist, and we can formulate the problem as a constrained linear optimization problem.

  The solution that locally minimizes the norm of joint velocities is $\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{\dagger}\boldsymbol{\mathcal{V}}$

  In case $\boldsymbol{J}$ is full rank: $\boldsymbol{J}^{\dagger} = \boldsymbol{J}^{T}\left(\boldsymbol{J}\boldsymbol{J}^{T}\right)^{-1}$

  $\boldsymbol{J}^{\dagger}$ is the right Moore–Penrose Pseudoinverse of $\boldsymbol{J}$.

Statics of Open Chains

Statics of Open Chains

Principle of conservation of power:

power generated at the joints = (power measured at the end-effector) + (power to move the robot)

At static equilibrium, no power is being used to move the robot, thus:

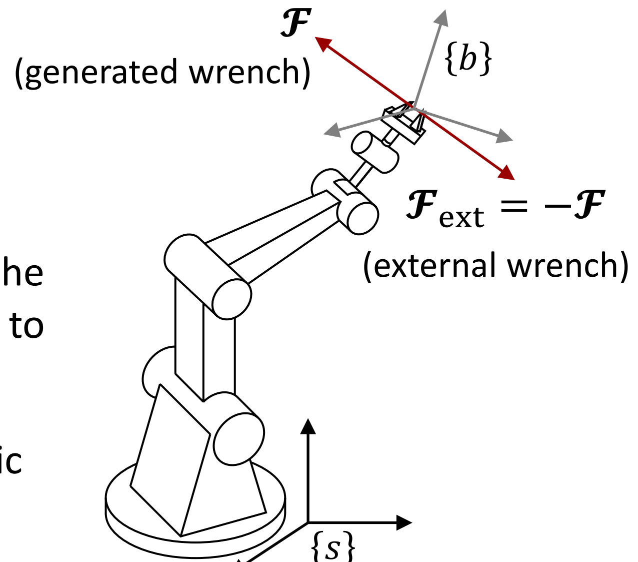
$$\boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{F}}_b^{\mathrm{T}}\boldsymbol{\mathcal{V}}_b \qquad \dot{\boldsymbol{\theta}} \to \mathbf{0}$$

$\boldsymbol{\tau}$: vector of joint torques

$$\boldsymbol{\mathcal{V}}_b = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\boldsymbol{\tau} = \boldsymbol{J}_b^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_b$$

Similarly, $\boldsymbol{\tau} = \boldsymbol{J}_s^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_s$



Note: If an external wrench $\boldsymbol{\mathcal{F}}_{\text{ext}}$ is applied to the end-effector when the robot is at equilibrium, $\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}}$ calculates the joint torques $\boldsymbol{\tau}$ needed to generate the opposing wrench $\boldsymbol{\mathcal{F}}$, keeping the robot at equilibrium.

Note: If the robot has to support itself against gravity to maintain static equilibrium, these torques must be added to the torques that offset gravity.

Statics and Kinematic Redundancy

In general, depending on the dimension of task space (i.e., $r \leq 6$),, the static equation can be represented as

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$$

where now $\boldsymbol{\mathcal{F}} \in \mathbb{R}^r$ is end-effector forces/moments for the specific task, $\boldsymbol{\tau} \in \mathbb{R}^n$, and $\boldsymbol{J}^T \in \mathbb{R}^{n\times r}$ is the corresponding Jacobian matrix that can be extracted from a $6 \times n$ geometric or analytic Jacobian.

Assume that the robot is not at a singular configuration,

- If $n = r$ and the embedding the end-effector in concrete will immobilize the robot.
- If $n > r$, then the robot is redundant, and even if the end-effector is embedded in concrete, the joint torques may cause internal motions of the links. The static equilibrium assumption is no longer satisfied, and we need to include dynamics to know what will happen to the robot.

Singularity Analysis

Kinematic Singularity

The configurations at which the robot's end-effector loses the ability to move instantaneously in one or more directions is called a **Kinematic Singularity**. In these directions, the robot can resist arbitrary wrenches.

❖ In singular configuration $\boldsymbol{\theta}^*$, $\boldsymbol{J}(\boldsymbol{\theta}^*) \in \mathbb{R}^{r\times n}$ is rank-deficient, i.e., $\operatorname{rank}\big(\boldsymbol{J}(\boldsymbol{\theta}^*)\big) < r$.

$\downarrow$ To check rank-deficiency

- If $r = 6$ and $n \geq 6$, $\boldsymbol{J} \in \mathbb{R}^{6\times n}$ can be $\boldsymbol{J}_s, \boldsymbol{J}_b, \boldsymbol{J}_g$, or $\boldsymbol{J}_a$.
- For other cases, $\boldsymbol{J} \in \mathbb{R}^{r\times n}$ can be only $\boldsymbol{J}_g$ or $\boldsymbol{J}_a$.

❖ The kinematic singularities are independent of the choice of fixed frame $\{s\}$ and end-effector frame $\{b\}$.

❖ In the neighborhood of a singularity, small velocities $\boldsymbol{\mathcal{V}}$ in the task space may cause large velocities $\dot{\boldsymbol{\theta}}$ in the joint space.

❖ Since $[\mathrm{Ad}_{\boldsymbol{T}}]$ is always invertible and $\boldsymbol{J}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{J}_b$, $\boldsymbol{J}_b$ and $\boldsymbol{J}_s$ always have the same rank.

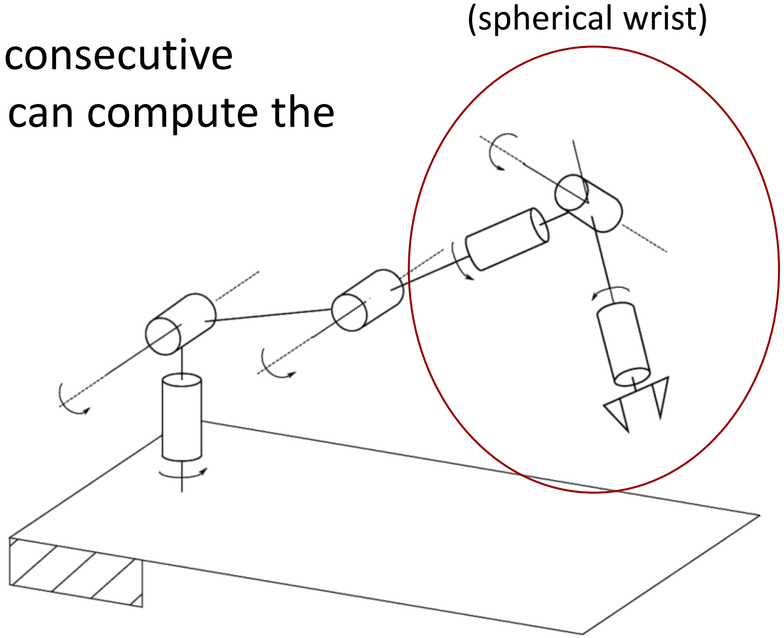
Kinematic Singularity

Singularities can be classified into:

- **Boundary Singularities**: They occur when the manipulator is either outstretched or retracted (it is easy to avoid).
- **Internal Singularities**: They occur anywhere inside the reachable workspace (it is hard to avoid).

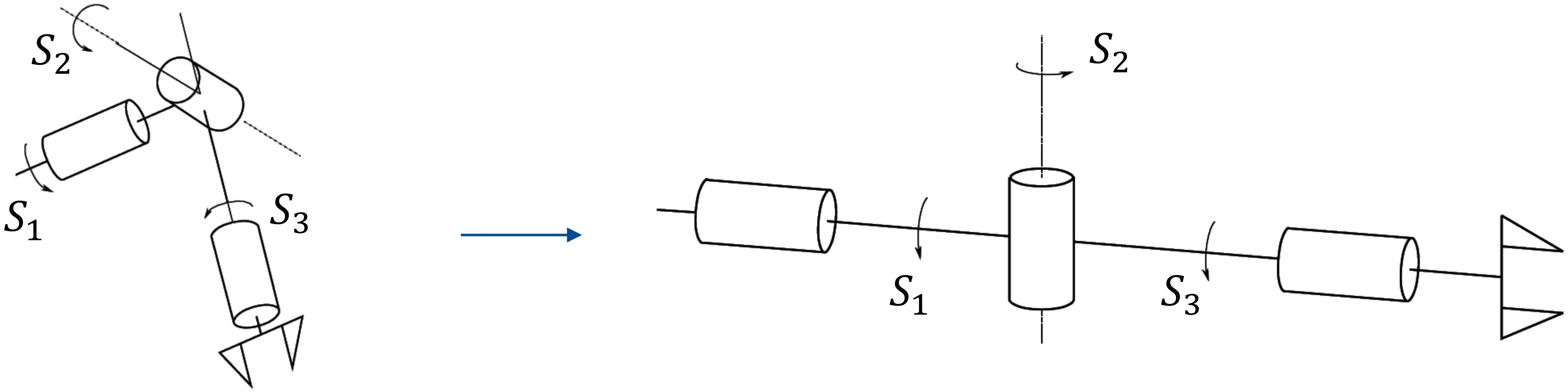
For manipulators having a spherical wrist (i.e., three consecutive revolute joint axes intersect at a common point), we can compute the singular configurations in two steps:

- Computation of **wrist singularities** resulting from the motion of the spherical wrist.
- Computation of **arm singularities** resulting from the motion of the first 3 or more links.



An Example of Wrist Singularity

The singularity occurs when S_1 and S_3 are aligned.

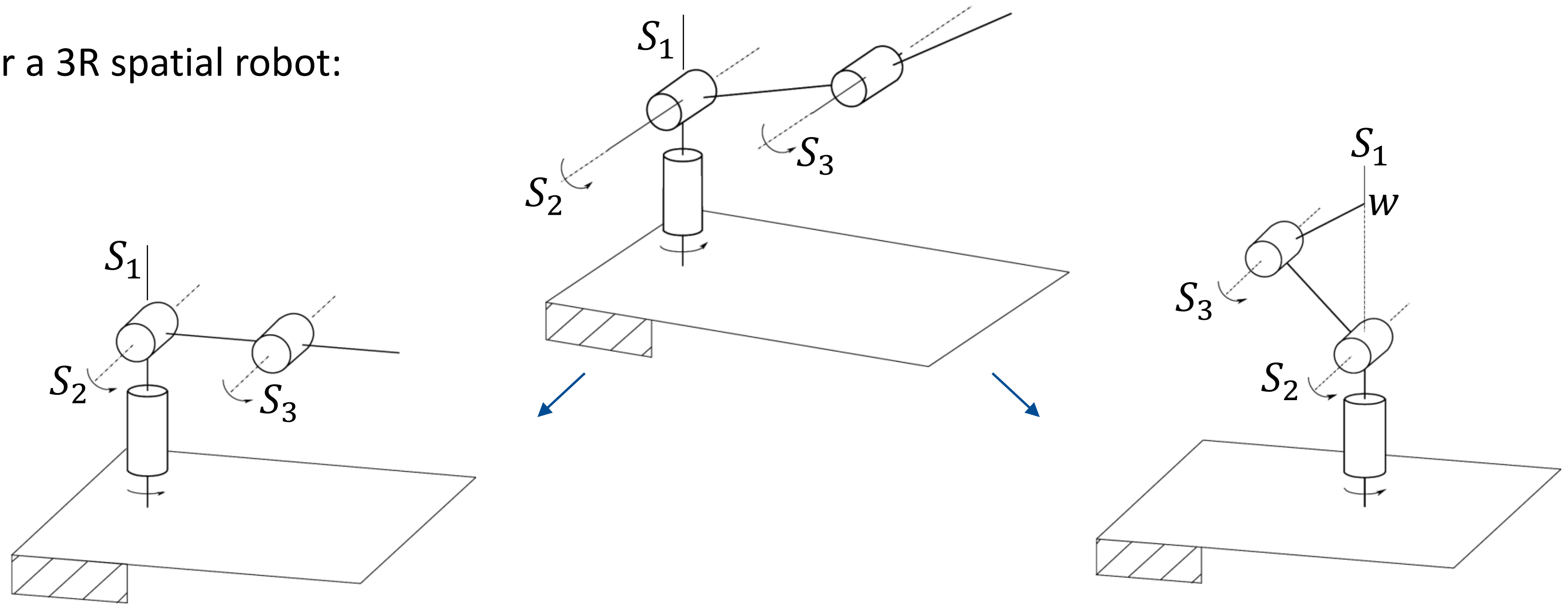


- In this configuration, the wrist cannot rotate about the axis orthogonal to S_1 and S_2.
- Rotations of equal magnitude about opposite directions on S_1 and S_3 do not produce any end-effector rotation.

Note: Wrist singularity is naturally described in the joint space and can be encountered anywhere inside the manipulator reachable workspace.

Examples of Arm Singularities

For a 3R spatial robot:



Elbow Singularity: when the elbow is outstretched or retracted.

Shoulder Singularity: when the wrist point (w) lies on axis S_1 (the whole axis S_1 describes a continuum of singular configurations).

Note: Arm singularity is well identified in the task space, and thus, they can be suitably avoided in the end-effector trajectory planning stage.

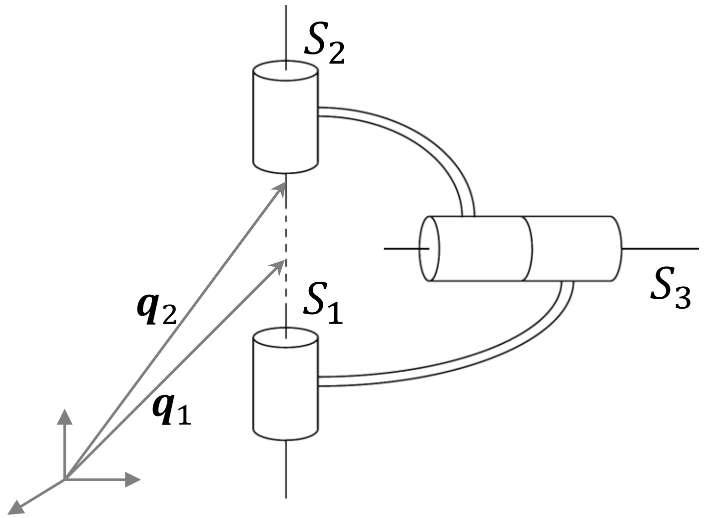
Examples of Common Singular Configurations ($n \geq 3$)

Case I: Two Collinear Revolute Joint Axes

$$J_{s1}(\theta) = \begin{bmatrix} \omega_{s1} \\ -\omega_{s1} \times q_1 \end{bmatrix} \qquad J_{s2}(\theta) = \begin{bmatrix} \omega_{s2} \\ -\omega_{s2} \times q_2 \end{bmatrix}$$

$$\left.\begin{aligned} \boldsymbol{\omega}_{s1} &= \boldsymbol{\omega}_{s2} \\ \boldsymbol{\omega}_{s1} \times \boldsymbol{q}_1 &= \boldsymbol{\omega}_{s1} \times \boldsymbol{q}_1 \end{aligned}\right\} \boldsymbol{J}_{s1} = \boldsymbol{J}_{s2}$$

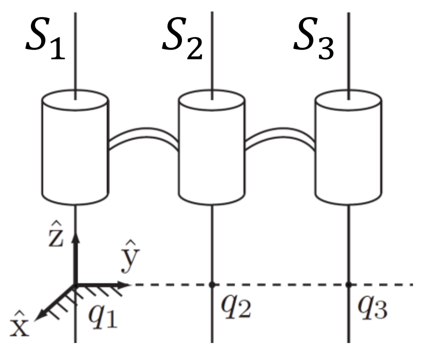
The set $\{\boldsymbol{J}_{s1}, \boldsymbol{J}_{s2}, \dots\}$ cannot be linearly independent.



Case II: Three Coplanar and Parallel Revolute Joint Axes

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \underbrace{\begin{matrix} \omega_{s1} \\ -\omega_{s1} \times q_2 \end{matrix}}_{\boldsymbol{u}} & \underbrace{\begin{matrix} \omega_{s1} \\ -\omega_{s1} \times q_3 \end{matrix}}_{\alpha \boldsymbol{u}} & \cdots \\ 0 & & & \cdots \end{bmatrix}$$

The set $\{\boldsymbol{J}_{s1}, \boldsymbol{J}_{s2}, \boldsymbol{J}_{s3}, \dots\}$ cannot be linearly independent.

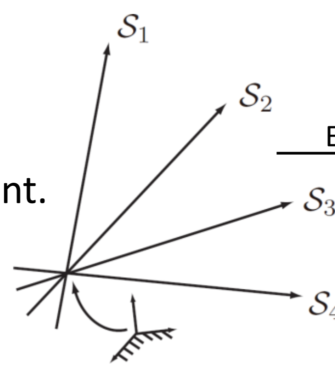


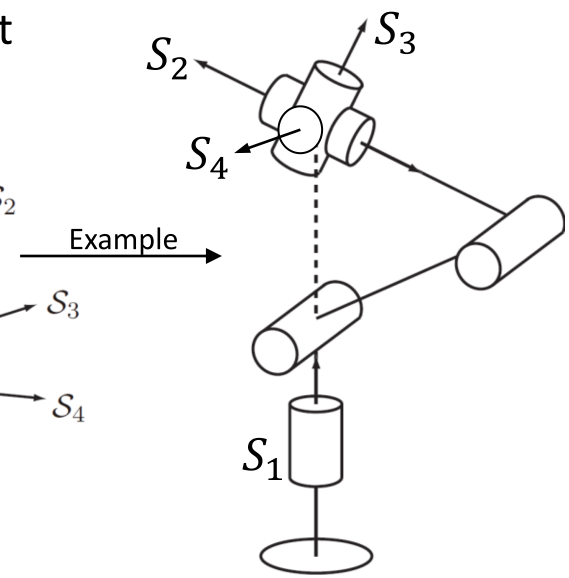
Examples of Common Singular Configurations ($n \geq 4$)

Case III: Four Revolute Joint Axes Intersecting at a Common Point

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s2} & \omega_{s3} & \omega_{s4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

The set $\{\boldsymbol{J}_{s1}, \boldsymbol{J}_{s2}, \boldsymbol{J}_{s3}, \boldsymbol{J}_{s4}, \ldots\}$ cannot be linearly independent.





Case IV: Four Coplanar Revolute Joints

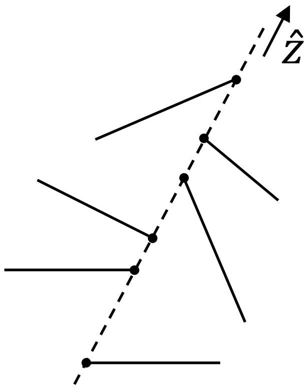
$$\omega_{si} = \begin{bmatrix} \omega_{six} \\ \omega_{siy} \\ 0 \end{bmatrix} \qquad q_i = \begin{bmatrix} q_{ix} \\ q_{iy} \\ 0 \end{bmatrix} \qquad -\omega_{si} \times q_i = \begin{bmatrix} 0 \\ 0 \\ \omega_{siy} q_{ix} - \omega_{six} q_{iy} \end{bmatrix}$$

$$\begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_{s1y} q_{1x} - \omega_{s1x} q_{1y} & \omega_{s2y} q_{2x} - \omega_{s2x} q_{2y} & \omega_{s3y} q_{3x} - \omega_{s3x} q_{3y} & \omega_{s4y} q_{4x} - \omega_{s4x} q_{4y} \end{bmatrix}$$

The set $\{\boldsymbol{J}_{s1}, \boldsymbol{J}_{s2}, \boldsymbol{J}_{s3}, \boldsymbol{J}_{s4}, \ldots\}$ cannot be linearly independent.

Examples of Common Singular Configurations ($n \geq 6$)

Case V: Six Revolute Joints Intersecting a Common Line



$$-\omega_{si} \times q_i = (\omega_{siy} q_{iz}, -\omega_{six} q_{iz}, 0)$$

$$\begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} & \omega_{s5x} & \omega_{s6x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} & \omega_{s5y} & \omega_{s6y} \\ \omega_{s1z} & \omega_{s2z} & \omega_{s3z} & \omega_{s4z} & \omega_{s5z} & \omega_{s6z} \\ \omega_{s1y} q_{1z} & \omega_{s2y} q_{2z} & \omega_{s3y} q_{3z} & \omega_{s4y} q_{4z} & \omega_{s5y} q_{5z} & \omega_{s6y} q_{6z} \\ -\omega_{s1x} q_{1z} & -\omega_{s2x} q_{2z} & -\omega_{s3x} q_{3z} & -\omega_{s4x} q_{4z} & -\omega_{s5x} q_{5z} & -\omega_{s6x} q_{6z} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The set $\{\boldsymbol{J}_{s1}, \boldsymbol{J}_{s2}, \boldsymbol{J}_{s3}, \boldsymbol{J}_{s4}, \boldsymbol{J}_{s5}, \boldsymbol{J}_{s6}, \dots\}$ cannot be linearly independent.

Manipulability

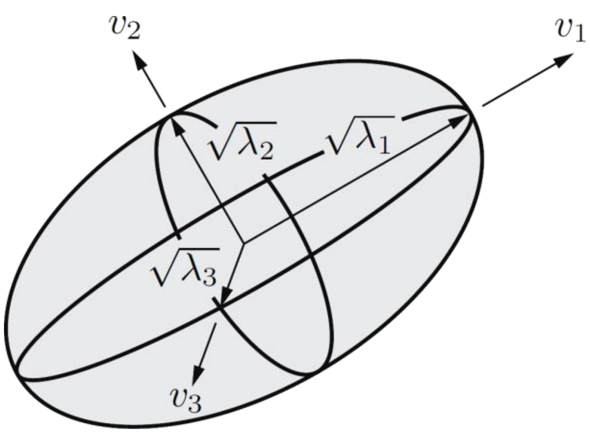
Preliminary: Ellipsoid Representation

For any symmetric positive-definite $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, the set of vectors $\boldsymbol{x} \in \mathbb{R}^m$ satisfying $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = 1$ defines an ellipsoid (function of $\boldsymbol{x}$) in the m-dimensional space.

Assume that $\boldsymbol{v}_i \in \mathbb{R}^m$ are Eigenvectors and $\lambda_i \in \mathbb{R}$ are Eigenvalues of $\boldsymbol{A}^{-1}$ $(i = 1, \dots, m)$.

Therefore, for the ellipsoid,

- Directions of the principal axes are $\boldsymbol{v}_i$,
- Lengths of the principal semi-axes are $\sqrt{\lambda_i}$,
- Volume is proportional to $\sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{\det(\boldsymbol{A}^{-1})}$.



Velocity Manipulability Ellipsoid

The **velocity manipulability ellipsoid** corresponds to the end-effector velocities $\boldsymbol{\mathcal{V}}$ for joint rates $\dot{\boldsymbol{\theta}}$ satisfying $\|\dot{\boldsymbol{\theta}}\| = \dot{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = 1$.

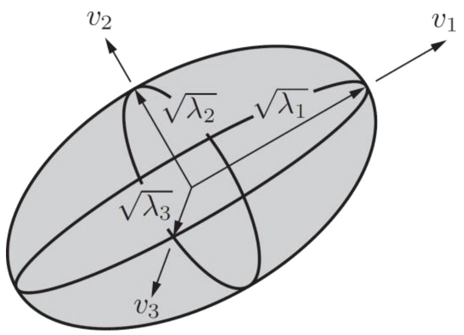
At a nonsingular configuration:

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \boldsymbol{\mathcal{V}} \in \mathbb{R}^r, \dot{\boldsymbol{\theta}} \in \mathbb{R}^n, \boldsymbol{J} \in \mathbb{R}^{r\times n}$$

$$\boldsymbol{J}^+ = \boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1} \qquad n \geq r$$

$$\begin{aligned} 1 &= \dot{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\theta}} \\ &= (\boldsymbol{J}^+\boldsymbol{\mathcal{V}})^{\mathrm{T}}(\boldsymbol{J}^+\boldsymbol{\mathcal{V}}) \\ &= \boldsymbol{\mathcal{V}}^{\mathrm{T}}\boldsymbol{J}^{+\mathrm{T}}\boldsymbol{J}^+\boldsymbol{\mathcal{V}} \\ &= \boldsymbol{\mathcal{V}}^{\mathrm{T}}\left(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\right)^{-1}\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{\mathcal{V}} \end{aligned}$$

(An ellipsoid function of $\boldsymbol{\mathcal{V}}$)



$\boldsymbol{A} = \boldsymbol{J}\boldsymbol{J}^{\mathrm{T}} \in \mathbb{R}^{r\times r}$ is square, symmetric, and positive definite, as is $\boldsymbol{A}^{-1}$.

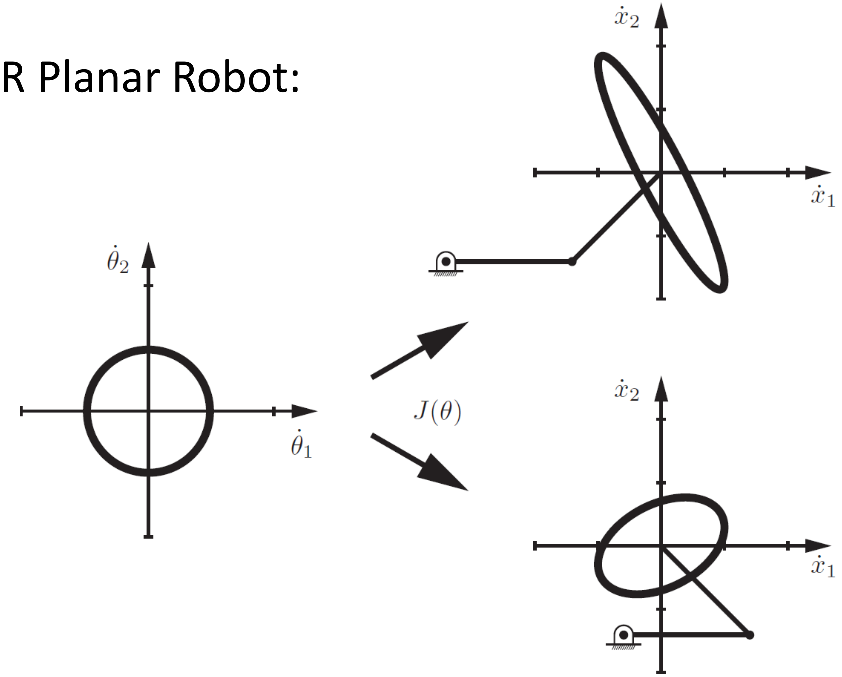
Assume that $\boldsymbol{v}_i \in \mathbb{R}^r$ are eigenvectors and $\lambda_i \in \mathbb{R}$ are eigenvalues of $\boldsymbol{A} = \boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}$ $(i = 1, \ldots, r)$.

- Directions of the principal axes: $\boldsymbol{v}_i$
- Lengths of the principal semi-axes: $\sqrt{\lambda_i}$
- Volume is proportional to $\sqrt{\lambda_1\lambda_2\cdots\lambda_r} = \sqrt{\det(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}})} \xrightarrow{n=r} = |\det(\boldsymbol{J})|$

Velocity Manipulability Ellipsoid (cont.)

- It is used to visualize and characterize how close a nonsingular configuration of a robot is to being singular.
- Along the direction of the **major axis** of the ellipsoid, the end-effector can move at **large velocity**, while along the direction of the **minor axis small end-effector velocities** are obtained.

For a 2R Planar Robot:



Velocity Manipulability Measures

Manipulability measures:

$$A = JJ^{\mathrm{T}}$$

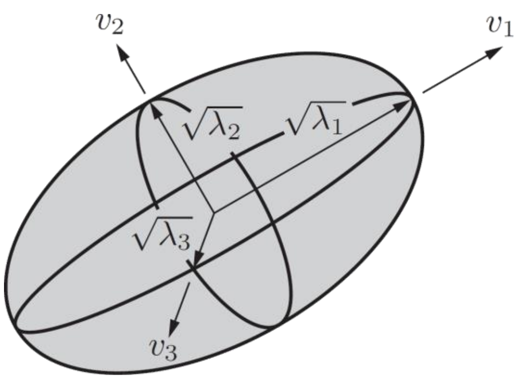
(1) The ratio of the largest to smallest principal semi-axes:

$$\mu_1(A) = \frac{\sqrt{\lambda_{\max}(A)}}{\sqrt{\lambda_{\min}(A)}} = \sqrt{\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}} \geq 1$$

(2) The ratio of the largest to smallest eigenvalues:

$$\mu_2(A) = \mu_1(A)^2 = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \geq 1$$

(condition number of A)



(3) The volume of the ellipsoid (proportional to $\sqrt{\lambda_1 \lambda_2 \ldots}$):

$$\mu_3(A) = \sqrt{\lambda_1 \lambda_2 \cdots} = \sqrt{\det(A)}$$

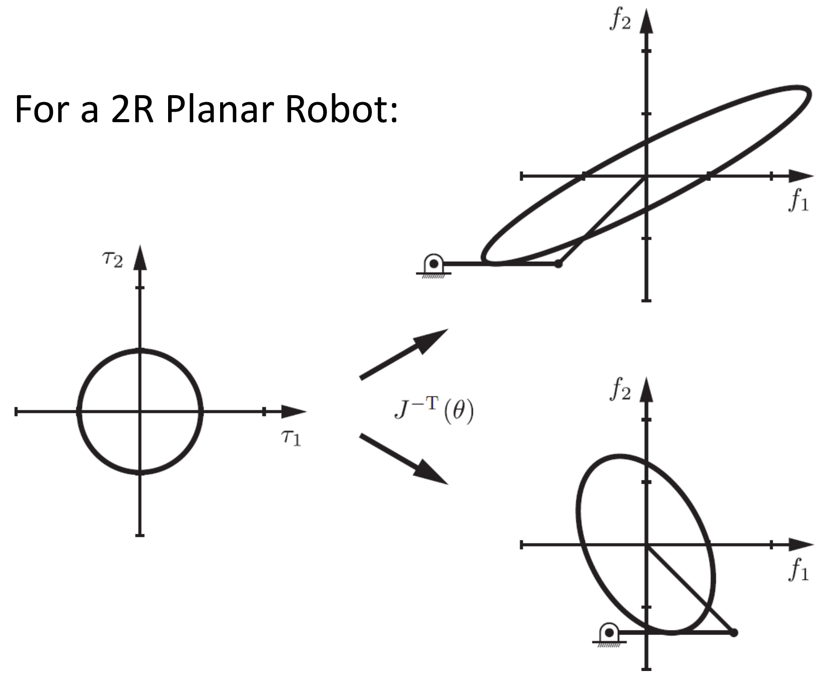
Force Manipulability Ellipsoid

The **force manipulability ellipsoid** corresponds to forces $\boldsymbol{\mathcal{F}}$ generated at the end-effector by joint rates $\boldsymbol{\tau}$ satisfying $\|\boldsymbol{\tau}\| = \boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\tau} = 1$.

$$\boldsymbol{\tau} = \boldsymbol{J}^{T}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$$

$$1 = \boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\tau} = \boldsymbol{\mathcal{F}}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\mathcal{F}}$$

For a 2R Planar Robot:



- The principal axes of the force manipulability ellipsoid coincide with the principal axes of the velocity manipulability ellipsoid.
- The lengths of the respective principal semi-axes are in inverse proportion $(1/\sqrt{\lambda_i})$.

Visualizing Manipulability Ellipsoids

If it is desired to geometrically visualize manipulability in a space of dimension greater than 3, it is worth separating the components of linear velocity (or force) from those of angular velocity (or moment), also avoiding problems due to nonhomogeneous dimensions of the relevant quantities (e.g., m/s vs rad/s).

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{J}_\omega(\boldsymbol{\theta}) \\ \boldsymbol{J}_v(\boldsymbol{\theta}) \end{bmatrix} \in \mathbb{R}^{6\times n}$$

$\boldsymbol{J}_\omega(\boldsymbol{\theta}) \in \mathbb{R}^{3\times n} \rightarrow$ angular velocity/moment ellipsoids

$\boldsymbol{J}_v(\boldsymbol{\theta}) \in \mathbb{R}^{3\times n} \rightarrow$ linear velocity/force ellipsoids

- When calculating the linear-velocity manipulability ellipsoid, it generally makes more sense to use the body Jacobian $\boldsymbol{J}_b$ or geometric Jacobian $\boldsymbol{J}_g$ instead of the space Jacobian $\boldsymbol{J}_s$, since we are usually interested in the linear velocity of a point at the origin of the end-effector frame rather than that of a point at the origin of the fixed-space frame.