

Ch4: Rigid-Body Motions – Transformation

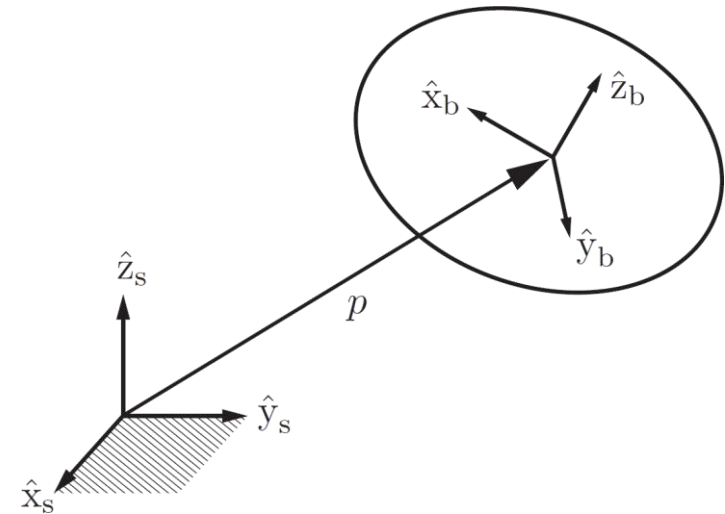
Rigid-Body Motions

Homogeneous Transformation Matrices

Rigid-body configuration can be represented by the pair (\mathbf{R}, \mathbf{p}) ($\mathbf{R} \in SO(3)$, $\mathbf{p} \in \mathbb{R}^3$). We can package (\mathbf{R}, \mathbf{p}) into a single 4×4 matrix as

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

(an implicit representation of the C-space)



Special Euclidean Group $SE(n)$

The **Special Euclidean Group** $SE(3)$, also known as the **group of rigid-body motions** or **homogeneous transformation matrices** in \mathbb{R}^3 , is the set of all 4×4 real matrices \mathbf{T} of the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \mathbf{T} \in SE(3) \\ \mathbf{R} \in SO(3) \\ \mathbf{p} \in \mathbb{R}^3 \end{array}$$

$$\mathbf{T} = (\mathbf{R}, \mathbf{p}) \in SE(3) \quad SE(3) = \{(\mathbf{R}, \mathbf{p}) \mid \mathbf{R} \in SO(3), \mathbf{p} \in \mathbb{R}^3\}$$

The **special Euclidean group** $SE(2)$ is the set of all 3×3 real matrices \mathbf{T} of the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & p_1 \\ \sin \theta & \cos \theta & p_2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \mathbf{T} \in SE(2) \\ \mathbf{R} \in SO(2) \\ \mathbf{p} \in \mathbb{R}^2 \\ \theta \in [0, 2\pi) \end{array}$$

- $SE(2)$ is a subgroup of $SE(3)$: $SE(2) \subset SE(3)$

Properties of Transformation Matrices

$SE(3)$ (or $SE(2)$) is a **matrix (Lie) group** (and the group operation \bullet is matrix multiplication).

Closure: $T_1 T_2 \in SE(3)$

Associative: $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ (but generally not commutative, $T_1 T_2 \neq T_2 T_1$)

Identity: $\exists I \in SE(3)$ such that $TI = IT = T$

Inverse: $\exists T^{-1} \in SE(3)$ such that $TT^{-1} = T^{-1}T = I$

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

Note: T preserves both distances and angles.

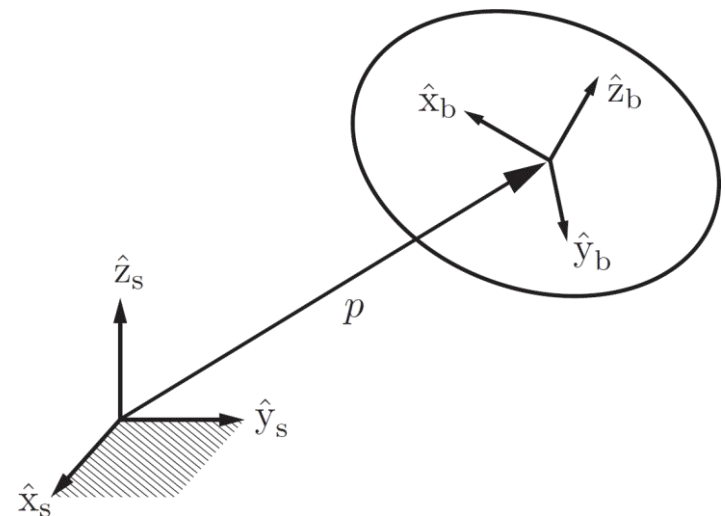
Uses of Transformation Matrices (1)

(1) Representing configuration (position and orientation) of a frame relative to another frame.

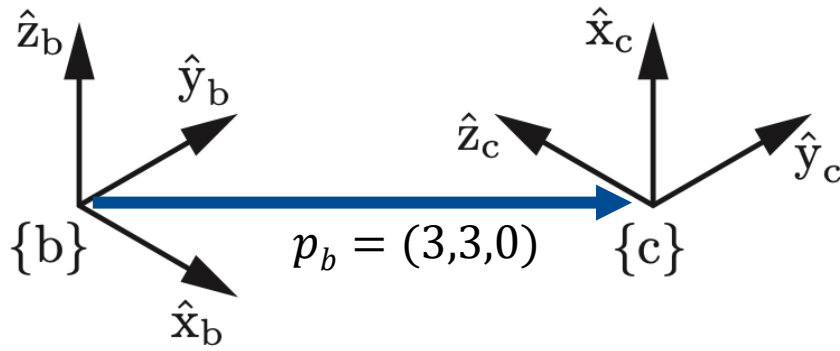
Notation: T_{sb} is the configuration of $\{b\}$ relative to $\{s\}$.

$$T_{sb} = \begin{bmatrix} R_{sb} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$T_{sb}T_{bs} = I \quad \text{or} \quad T_{bs} = T_{sb}^{-1} = \begin{bmatrix} R_{sb}^T & -R_{sb}^T \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$



Example



Uses of Transformation Matrices (2)

(2) Changing the reference frame of a vector or frame.

Subscript Cancellation Rule:

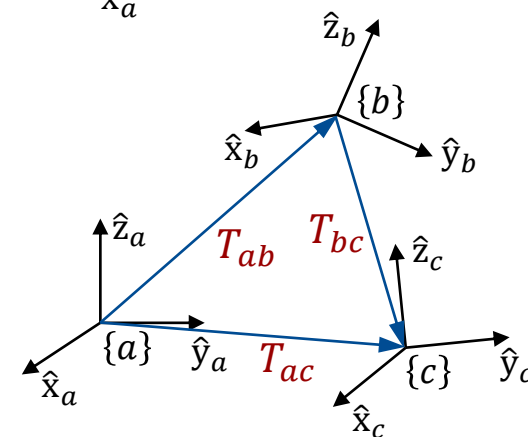
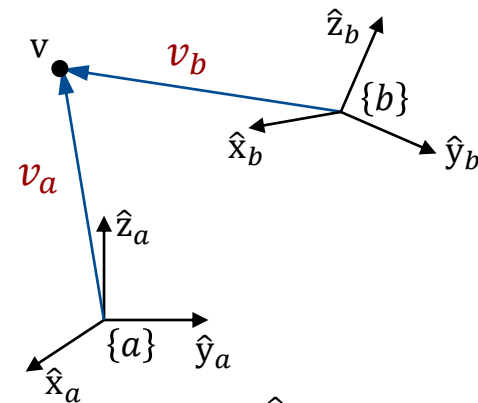
$$\mathbf{T}_{ab} \mathbf{v}_b = \mathbf{T}_{a\cancel{b}} \mathbf{v}_{\cancel{b}} = \mathbf{v}_a$$

$$\mathbf{T}_{ab} \mathbf{T}_{bc} = \mathbf{T}_{a\cancel{b}} \mathbf{T}_{\cancel{b}c} = \mathbf{T}_{ac}$$

\mathbf{T}_{ab} can be viewed as a mathematical operator that changes the reference frame from $\{b\}$ to $\{a\}$.

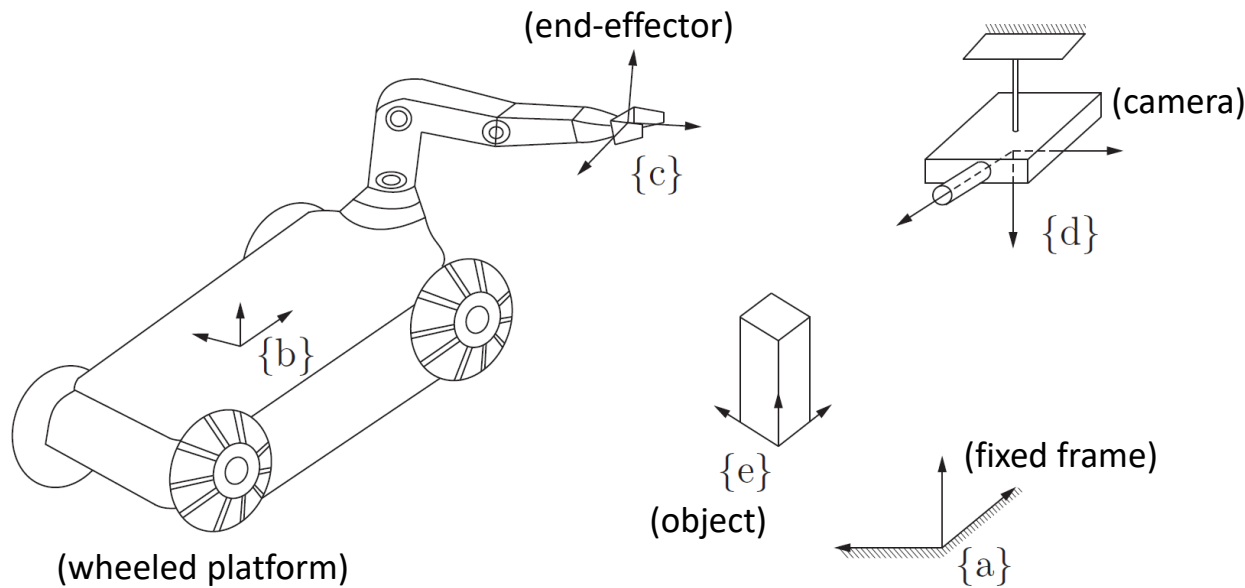
Note: To calculate $\mathbf{T}\mathbf{v}$, we append a “1” to \mathbf{v} and it is called **homogeneous coordinates** representation of \mathbf{v} .

$$\mathbf{v} = [v_1 \ v_2 \ v_3 \ 1]^T$$



Example

A robot arm mounted on a wheeled mobile platform moving in a room, and a camera fixed to the ceiling. The robot must pick up an object with body frame $\{e\}$. What is the configuration of the object relative to the robot hand, T_{ce} , given T_{db} , T_{de} , T_{bc} , and T_{ad} ?



Uses of Transformation Matrices (3)

(3) Displacing (rotating and translating) a vector or frame.

$$\mathbf{T} = (\mathbf{R}, \mathbf{p}) = (\text{Rot}(\hat{\boldsymbol{\omega}}, \theta), \mathbf{p}) = \text{Trans}(\mathbf{p})\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

$$\text{Trans}(\mathbf{p}) = \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

\mathbf{T} can be viewed as a **mathematical operator** that rotates a frame or vector about a unit axis $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$ by an amount θ + translating it by \mathbf{p} .

Uses of Transformation Matrices (3) (cont.)

- Rotation of vector \mathbf{v} about a unit axis $\hat{\omega}$ (expressed in the same frame) by an amount θ and translation of it by \mathbf{p} (expressed in the same frame) is vector \mathbf{v}' expressed in the same frame:

$$\mathbf{v}' = \mathbf{T}\mathbf{v} = \text{Trans}(\mathbf{p})\text{Rot}(\hat{\omega}, \theta)\mathbf{v}$$

← Interpretation

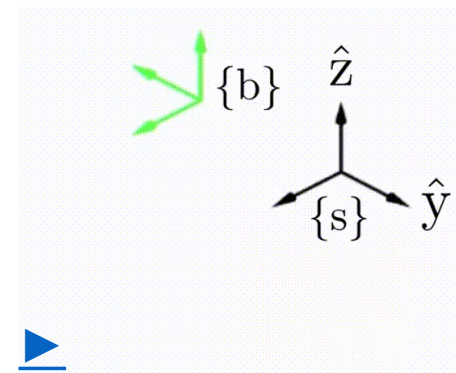
- Fixed-frame Transformation:**

2. Translating it by \mathbf{p} in $\{s\}$ to get $\{b'\}$

1. Rotating $\{b\}$ by θ about $\hat{\omega}$ in $\{s\}$ (this can move $\{b\}$ origin)

$$\mathbf{T}_{sb'} = \mathbf{T}\mathbf{T}_{sb} = \text{Trans}(\mathbf{p})\text{Rot}(\hat{\omega}, \theta)\mathbf{T}_{sb}$$

← Interpretation



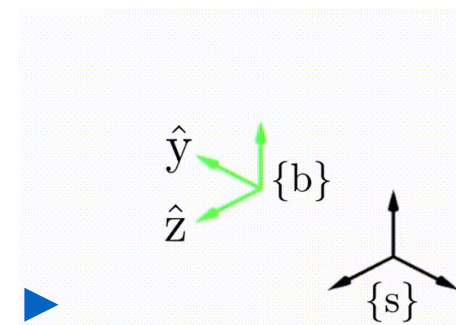
- Body-frame Transformation:**

1. Translating $\{b\}$ by \mathbf{p} in $\{b\}$

2. Rotating it by θ about $\hat{\omega}$ in the new body frame to get $\{b''\}$

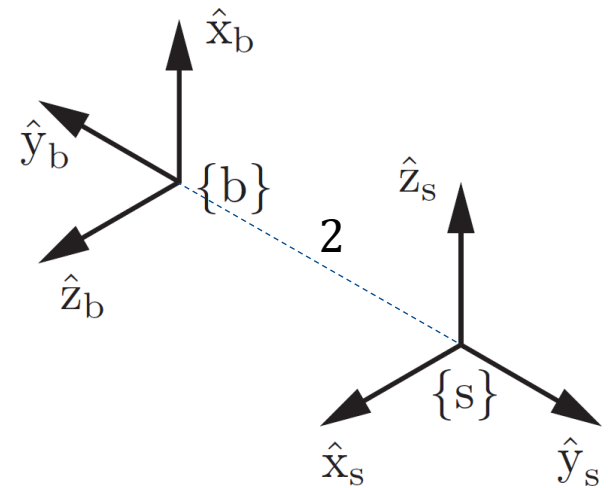
$$\mathbf{T}_{sb''} = \mathbf{T}_{sb}\mathbf{T} = \mathbf{T}_{sb}\text{Trans}(\mathbf{p})\text{Rot}(\hat{\omega}, \theta)$$

← Interpretation



Example

Fixed-frame and body-frame transformations corresponding to $\hat{\omega} = (0,0,1)$, $\theta = 90^\circ$, and $\mathbf{p} = (0,2,0)$.

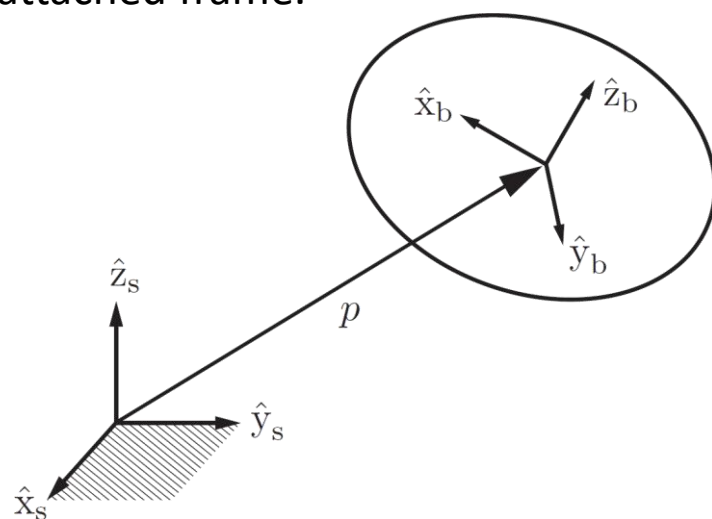


Twists

Spatial Velocity or Twist

Let's find both the linear and angular velocities of a frame fixed to a moving body. Body Frame $\{b\}$ is instantaneously coincident with this body-attached frame.

$$\mathbf{T}(t) = \begin{bmatrix} \mathbf{R}(t) & \mathbf{p}(t) \\ \mathbf{0} & 1 \end{bmatrix} : \mathbf{T}_{sb} \text{ at time } t$$



A rigid body's velocity can be represented simply as a point in \mathbb{R}^6 , defined by three angular velocities and three linear velocities, which together we call a **Spatial Velocity** or **Twist**.

Body Twist

Similar to $R^{-1}\dot{R} = [\omega_b]$, let's compute $T^{-1}\dot{T}$: $(R = R_{sb}, T = T_{sb})$

$$\begin{aligned}
 T^{-1}\dot{T} &= \begin{bmatrix} R^T & -R^T p \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ \mathbf{0} & 0 \end{bmatrix} \\
 &= \begin{bmatrix} R^T \dot{R} & R^T \dot{p} \\ \mathbf{0} & 0 \end{bmatrix} \\
 &= \begin{bmatrix} [\omega_b] & v_b \\ \mathbf{0} & 0 \end{bmatrix} \xrightarrow[\substack{v_b \in \mathbb{R}^3 \\ [\omega_b] \in so(3)}]{\text{red arrow}} \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6
 \end{aligned}$$

\mathcal{V}_b is defined as **Body Twist**
(or spatial velocity in the body frame)

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$$

The set of all 4×4 matrices of this form is called $se(3)$ and comprises the 4×4 matrix representations of the **body twists** associated with the rigid-body configurations $SE(3)$.

($se(3)$ is called the Lie algebra of the Lie group $SE(3)$)

Spatial Twist

Similar to $\dot{R}R^{-1} = [\omega_s]$, let's compute $\dot{T}T^{-1}$: $(R = R_{sb}, T = T_{sb})$

$$\begin{aligned}
 \dot{T}T^{-1} &= \begin{bmatrix} \dot{R} & \dot{p} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ \mathbf{0} & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \dot{R}R^T & \dot{p} - \dot{R}R^T p \\ \mathbf{0} & 0 \end{bmatrix} \\
 &= \begin{bmatrix} [\omega_s] & v_s \\ \mathbf{0} & 0 \end{bmatrix} \xrightarrow[\substack{v_s \in \mathbb{R}^3 \\ [\omega_s] \in so(3)}]{} \mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6
 \end{aligned}$$

\mathcal{V}_s is defined as **Spatial Twist**
(or spatial velocity in the space frame)

$$\dot{T}T^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$$

The set of all 4×4 matrices of this form is called $se(3)$ and comprises the 4×4 matrix representations of the **spatial twists** associated with the rigid-body configurations $SE(3)$.

Adjoint Map

~~$$\mathbf{v}_s = \mathbf{T}_{sb} \mathbf{v}_b$$

\downarrow
 4×4

\downarrow
 6×6~~

$$\begin{aligned} [\mathbf{v}_b] &= \mathbf{T}^{-1} \dot{\mathbf{T}} \\ [\mathbf{v}_s] &= \dot{\mathbf{T}} \mathbf{T}^{-1} \end{aligned} \quad \longrightarrow \quad [\mathbf{v}_s] = \mathbf{T} [\mathbf{v}_b] \mathbf{T}^{-1} \longrightarrow$$

$$[\mathbf{v}_s] = \begin{bmatrix} \mathbf{R}[\boldsymbol{\omega}_b] \mathbf{R}^T & -\mathbf{R}[\boldsymbol{\omega}_b] \mathbf{R}^T \mathbf{p} + \mathbf{R} \mathbf{v}_b \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \xrightarrow[\substack{[\boldsymbol{\omega}] \mathbf{p} = -[\mathbf{p}] \boldsymbol{\omega} \\ \mathbf{p}, \boldsymbol{\omega} \in \mathbb{R}^3}]{\mathbf{R}[\boldsymbol{\omega}] \mathbf{R}^T = [\mathbf{R} \boldsymbol{\omega}]} = \begin{bmatrix} [\mathbf{R} \boldsymbol{\omega}_b] & [\mathbf{p}] \mathbf{R} \boldsymbol{\omega}_b + \mathbf{R} \mathbf{v}_b \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\Rightarrow \mathbf{v}_s = \begin{bmatrix} \boldsymbol{\omega}_s \\ \mathbf{v}_s \end{bmatrix} = \begin{bmatrix} \mathbf{R} \boldsymbol{\omega}_b \\ [\mathbf{p}] \mathbf{R} \boldsymbol{\omega}_b + \mathbf{R} \mathbf{v}_b \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ [\mathbf{p}] \mathbf{R} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}_b \\ \mathbf{v}_b \end{bmatrix} = [\text{Ad}_T] \mathbf{v}_b$$

$$[\text{Ad}_T] = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ [\mathbf{p}] \mathbf{R} & \mathbf{R} \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

Adjoint Map associated with \mathbf{T}
or Adjoint Representation of \mathbf{T}

$$\mathbf{v}_s = [\text{Ad}_{T_{sb}}] \mathbf{v}_b = \text{Ad}_{T_{sb}}(\mathbf{v}_b)$$

$$\text{Similarly, } \mathbf{v}_b = [\text{Ad}_{T_{bs}}] \mathbf{v}_s = \text{Ad}_{T_{bs}}(\mathbf{v}_s)$$

Adjoint Map Properties

- Let $\mathbf{T}_1, \mathbf{T}_2 \in SE(3)$ and $\mathcal{V} = (\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}^6$. Then,

$$[\text{Ad}_{\mathbf{T}_1}][\text{Ad}_{\mathbf{T}_2}]\mathcal{V} = [\text{Ad}_{\mathbf{T}_1\mathbf{T}_2}]\mathcal{V} \quad \text{or} \quad \text{Ad}_{\mathbf{T}_1}(\text{Ad}_{\mathbf{T}_2}(\mathcal{V})) = \text{Ad}_{\mathbf{T}_1\mathbf{T}_2}(\mathcal{V})$$

- For any $\mathbf{T} \in SE(3)$, $[\text{Ad}_{\mathbf{T}}]^{-1} = [\text{Ad}_{\mathbf{T}^{-1}}]$
- For any two frames $\{c\}$ and $\{d\}$, a twist represented as \mathcal{V}_c in $\{c\}$ is related to its representation \mathcal{V}_d in $\{d\}$ by

$$\mathcal{V}_c = [\text{Ad}_{\mathbf{T}_{cd}}]\mathcal{V}_d \qquad \mathcal{V}_d = [\text{Ad}_{\mathbf{T}_{dc}}]\mathcal{V}_c$$

(changing the reference frame of a twist)

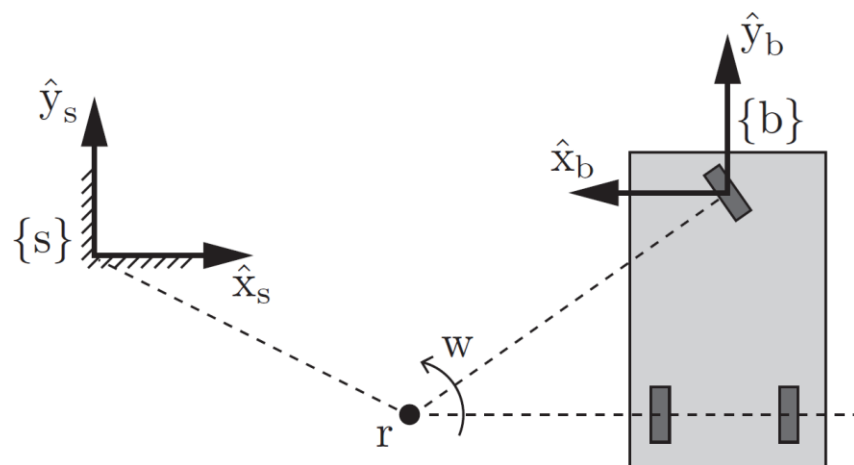
- $[\text{Ad}_{\mathbf{T}}]$ is always invertible.

Example

Consider a three-wheeled car with a single steerable front wheel, driving on a plane. The angle of the front wheel of the car causes the car's motion to be a pure angular velocity 2 rad/s about an axis out of the page at the point r in the plane. Find \mathcal{V}_s and \mathcal{V}_b when

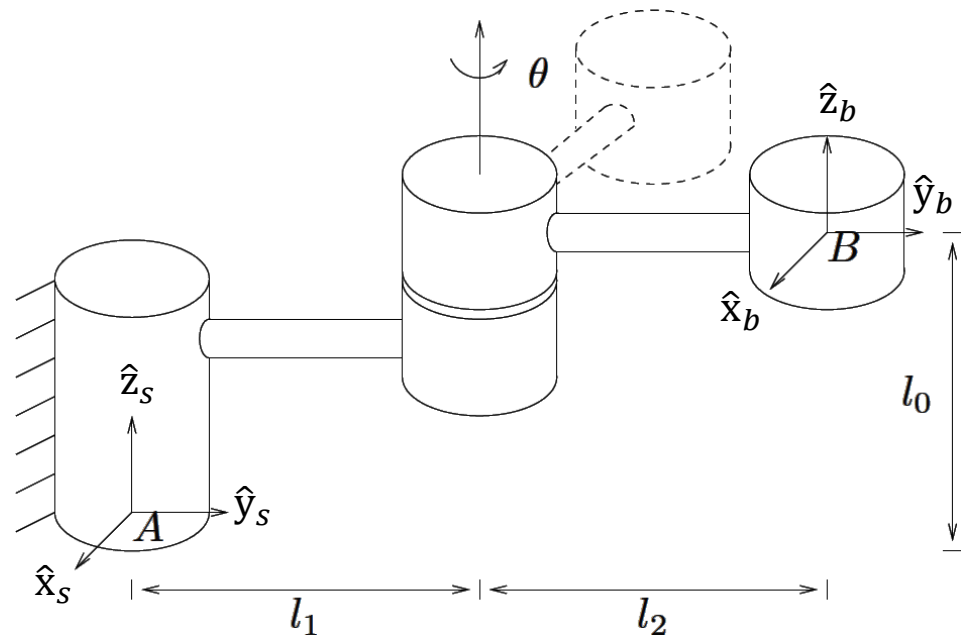
$$r_s = (2, -1, 0)$$

$$r_b = (2, -1.4, 0)$$



Example

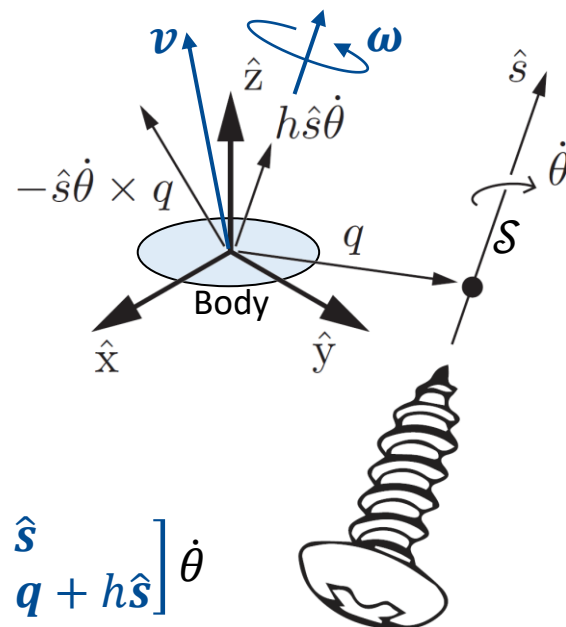
Find \mathcal{V}_s and \mathcal{V}_b for the shown one degree of freedom manipulator.



Screw Interpretation of a Twist

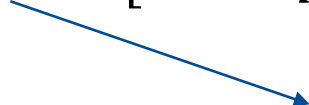
Any rigid-body velocity or twist \mathcal{V} is equivalent to the instantaneous velocity $\dot{\theta}$ about some screw axis \mathcal{S} (i.e., rotating about the axis while also translating along the axis).

A screw axis \mathcal{S} represented by a point $\mathbf{q} \in \mathbb{R}^3$ on the axis, a unit vector $\hat{\mathbf{s}} \in S^2$ in the direction of the axis, and a pitch $h \in \mathbb{R}$ (linear velocity along the axis divided by angular velocity $\dot{\theta}$ about the axis) as $\{\mathbf{q}, \hat{\mathbf{s}}, h\}$.



Thus, twist \mathcal{V} can be represented as

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \omega \times (-\mathbf{q}) + h\omega \\ -\hat{\mathbf{s}}\dot{\theta} \times \mathbf{q} + h\dot{\theta}\hat{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \mathbf{q} + h\hat{\mathbf{s}} \end{bmatrix} \dot{\theta}$$



Due to rotation about \mathcal{S}
(which is in the plane orthogonal to $\hat{\mathbf{s}}$)

Due to translation along \mathcal{S}
(which is in the direction of $\hat{\mathbf{s}}$)

Representation of Screw Axis

Now, instead of representing the screw axis \mathcal{S} as $\{\mathbf{q}, \hat{\mathbf{s}}, h\}$ (with the non-uniqueness of \mathbf{q}), we represent a “unit” screw axis as a vector as

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} \in \mathbb{R}^6 \quad \text{where} \quad \mathbf{v} = \mathbf{S}\dot{\theta} \in \mathbb{R}^6 \quad \mathbf{S}_\omega, \mathbf{S}_v \in \mathbb{R}^3$$

- Finding \mathbf{S} and $\{\mathbf{q}, \hat{\mathbf{s}}, h\}$ by having \mathbf{v} :

(a) If $\|\boldsymbol{\omega}\| \neq 0$ (\equiv rotation with/without translation along $\hat{\mathbf{s}}$):

$$\begin{aligned} \mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} &= \mathbf{v} / \|\boldsymbol{\omega}\| = \begin{bmatrix} \boldsymbol{\omega} / \|\boldsymbol{\omega}\| \\ \mathbf{v} / \|\boldsymbol{\omega}\| \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \mathbf{q} + h\hat{\mathbf{s}} \end{bmatrix} \\ &= \begin{bmatrix} \text{angular velocity when } \dot{\theta} = 1 \\ \text{linear velocity of origin when } \dot{\theta} = 1 \end{bmatrix} \end{aligned}$$

Pitch h is finite.

$$h = \mathbf{S}_\omega^T \mathbf{S}_v = \boldsymbol{\omega}^T \mathbf{v} / \|\boldsymbol{\omega}\|^2$$

$$\hat{\mathbf{s}} = \mathbf{S}_\omega = \boldsymbol{\omega} / \|\boldsymbol{\omega}\|, \quad \|\mathbf{S}_\omega\| = 1$$

$\dot{\theta} = \|\boldsymbol{\omega}\|$ is interpreted as angular velocity about $\hat{\mathbf{s}}$

To find \mathbf{q} , use $\mathbf{v} - h\boldsymbol{\omega} = -\boldsymbol{\omega} \times \mathbf{q}$
or $(\mathbf{S}_v - h\mathbf{S}_\omega = -\mathbf{S}_\omega \times \mathbf{q})$

(b) If $\|\boldsymbol{\omega}\| = 0$ (\equiv pure translation along $\hat{\mathbf{s}}$):

$$\begin{aligned} \mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} &= \mathbf{v} / \|\mathbf{v}\| = \begin{bmatrix} \mathbf{0} \\ \mathbf{v} / \|\mathbf{v}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{s}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} \\ \text{normalized linear velocity of origin} \end{bmatrix} \end{aligned}$$

Pitch h is infinite, $\|\mathbf{S}_\omega\| = 0$

$$\hat{\mathbf{s}} = \mathbf{S}_v = \mathbf{v} / \|\mathbf{v}\|, \quad \|\mathbf{S}_v\| = 1$$

$\dot{\theta} = \|\mathbf{v}\|$ is interpreted as linear velocity along $\hat{\mathbf{s}}$

Screw Axis Properties

- ❖ Since a screw axis \mathcal{S} is just a normalized twist, the 4×4 matrix representation $[\mathcal{S}]$ of $\mathcal{S} = (\mathcal{S}_\omega, \mathcal{S}_v)$ is

$$[\mathcal{S}] = \begin{bmatrix} [\mathcal{S}_\omega] & \mathcal{S}_v \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$$

$$\mathcal{V} = \mathcal{S}\dot{\theta} \in \mathbb{R}^6 \quad \Rightarrow \quad [\mathcal{V}] = [\mathcal{S}]\dot{\theta} \in se(3)$$

- ❖ For any two frames $\{c\}$ and $\{d\}$, a screw axis represented as \mathcal{S}_c in $\{c\}$ is related to its representation \mathcal{S}_d in $\{d\}$ by:

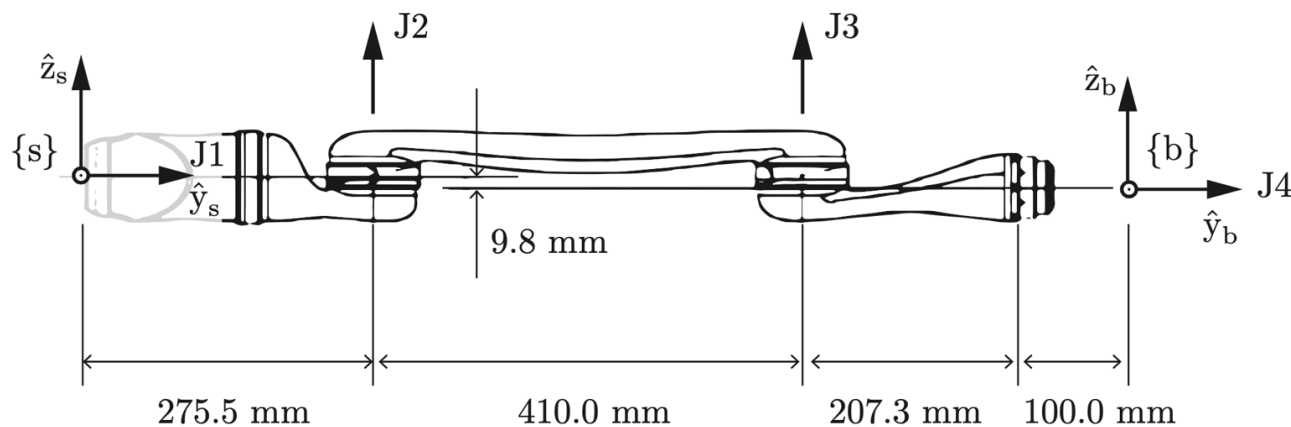
$$\mathcal{S}_c = [\text{Ad}_{T_{cd}}]\mathcal{S}_d$$

$$\mathcal{S}_d = [\text{Ad}_{T_{dc}}]\mathcal{S}_c$$

(changing the reference frame of a screw axis)

Example

Kinova lightweight 4-dof arm:



What are the screw axis \mathcal{S}_b and \mathcal{S}_s for J4 and J2?

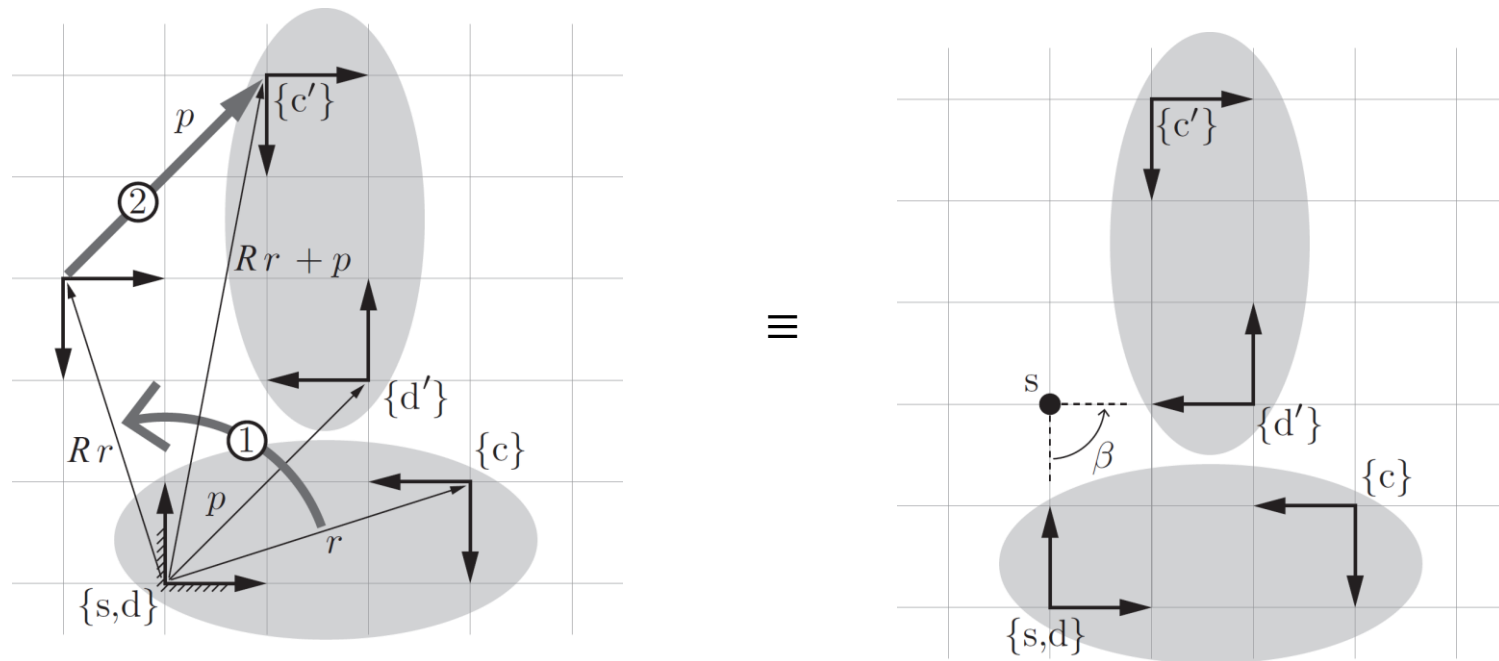
Exponential Coordinate Representation of Rigid-Body Motion

Screw Motion

Instead of viewing a displacement as a rotation followed by a translation, both rotation and translation can be performed simultaneously.

Planar example of a screw motion:

The displacement can be viewed as a rotation of $\beta = 90^\circ$ about a fixed point s .



Exponential Coordinates of Rigid-Body Motions

Chasles–Mozzi theorem states that every rigid-body displacement can be expressed as a finite rotation and translation about a fixed screw axis in space.

This theorem motivates a six-parameter representation of a configuration (or a homogeneous transformation $\mathbf{T} \in SE(3)$) called the **exponential coordinates** as $\mathbf{S}\theta \in \mathbb{R}^6$, where \mathbf{S} is the screw axis and θ is the distance that must be traveled along the screw axis to take a frame from the origin \mathbf{I} to \mathbf{T} .

Note: \mathbf{T} is equivalent to the displacement obtained by rotating a frame from \mathbf{I} about \mathbf{S}

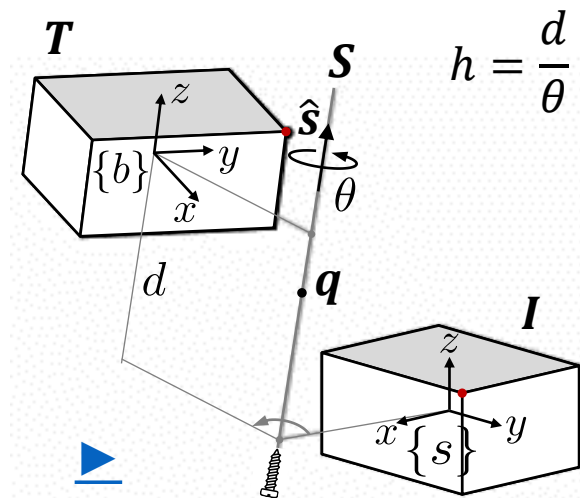
- by an angle θ , or
- at a speed $\dot{\theta} = 1$ rad/s for θ s, or
- at a speed $\dot{\theta} = \theta$ for 1s, or
- by constant twist \mathbf{V} for 1s.
($\mathbf{V}t = \mathbf{S}\theta$)

Constant Screw Motion:

A rotation θ + a translation d about/along a fixed screw axis \mathbf{S} .

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \mathbf{q} + h\hat{\mathbf{s}} \end{bmatrix} \quad (\text{for rotation with/without translation along } \hat{\mathbf{s}})$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{s}} \end{bmatrix} \quad (\text{for pure translation along } \hat{\mathbf{s}})$$



Exponential Coordinates of Rigid-Body Motions

As with rotations, we can define a matrix exponential (exp) and matrix logarithm (log). For any transformation matrix $\mathbf{T} \in SE(3)$, we can always find a screw axis $\mathbf{S} = (\mathbf{S}_\omega, \mathbf{S}_v) \in \mathbb{R}^6$ ($\|\mathbf{S}_\omega\| = 1$ or $\mathbf{S}_\omega = \mathbf{0}$, $\|\mathbf{S}_v\| = 1$) and scalar $\theta \in \mathbb{R}$ such that $\mathbf{T} = e^{[\mathbf{S}]\theta}$.

$$\begin{aligned} \text{exp:} \quad & [\mathbf{S}]\theta \in se(3) \rightarrow \mathbf{T} \in SE(3) & : & e^{[\mathbf{S}]\theta} = \mathbf{T} = (\mathbf{R}, \mathbf{p}) \\ \text{log:} \quad & \mathbf{T} \in SE(3) \rightarrow [\mathbf{S}]\theta \in se(3) & : & \log(\mathbf{T}) = [\mathbf{S}]\theta \end{aligned}$$

$\mathbf{S}\theta \in \mathbb{R}^6$: Exponential coordinates of $\mathbf{T} \in SE(3)$

$[\mathbf{S}]\theta = [\mathbf{S}\theta] \in se(3)$: Matrix logarithm of \mathbf{T} (inverse of the matrix exponential)

Note: \mathbf{T} and \mathbf{S} have the same base.

Matrix Exponential

$$\text{exp: } [\mathbf{S}]\theta \in se(3) \rightarrow \mathbf{T} \in SE(3) \quad : \quad e^{[\mathbf{S}]\theta} = \mathbf{T} = (\mathbf{R}, \mathbf{p})$$

❖ Finding $\mathbf{T} = (\mathbf{R}, \mathbf{p})$ by having $\mathbf{S} = (\mathbf{S}_\omega, \mathbf{S}_v)$ and θ :

$$e^{[\mathbf{S}]\theta} = \begin{bmatrix} e^{[\mathbf{S}_\omega]\theta} & \mathbf{G}(\theta)\mathbf{S}_v \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\mathbf{G}(\theta) = \mathbf{I}\theta + (1 - \cos \theta)[\mathbf{S}_\omega] + (\theta - \sin \theta)[\mathbf{S}_\omega]^2 \in \mathbb{R}^{3 \times 3}$$

Matrix Exponential: Remark

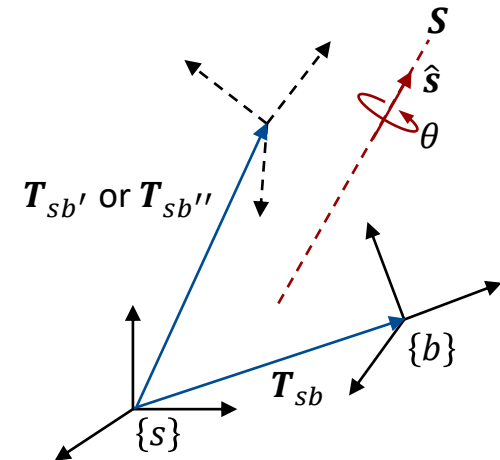
For a given \mathcal{S}_s or \mathcal{S}_b : $(\mathcal{S}_s = [\text{Ad}_{T_{sb}}]\mathcal{S}_b)$

\mathcal{S} is expressed in $\{s\}$

Fixed-frame displacement: $T_{sb'} = e^{[\mathcal{S}_s]\theta} T_{sb}$

Body-frame displacement: $T_{sb''} = T_{sb} e^{[\mathcal{S}_b]\theta}$

\mathcal{S} is expressed in $\{b\}$



Matrix Logarithm

$$\log: \quad \mathbf{T} \in SE(3) \quad \rightarrow \quad [\mathbf{S}]\theta \in se(3) \quad : \quad \log(\mathbf{T}) = [\mathbf{S}]\theta$$

❖ Finding $\mathbf{S} = (\mathbf{S}_\omega, \mathbf{S}_v)$ and $\theta \in [0, \pi]$ by having $\mathbf{T} = (\mathbf{R}, \mathbf{p})$:

(a) If $\text{tr} \mathbf{R} = 3$ (or $\mathbf{R} = \mathbf{I}$), then set $\mathbf{S}_\omega = \mathbf{0}$, $\mathbf{S}_v = \mathbf{p}/\|\mathbf{p}\|$, and $\theta = \|\mathbf{p}\|$.

(b) Otherwise, use the matrix logarithm $\log(\mathbf{R}) = [\mathbf{S}_\omega]\theta$ to determine \mathbf{S}_ω ($\hat{\boldsymbol{\omega}}$ in the $SO(3)$ algorithm) and $\theta \in [0, \pi]$. Then, \mathbf{S}_v is calculated as

$$\mathbf{S}_v = \mathbf{G}^{-1}(\theta)\mathbf{p}$$

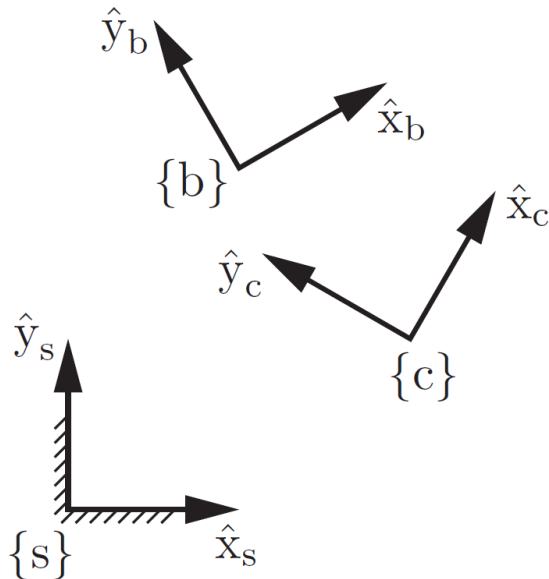
$$\mathbf{G}^{-1}(\theta) = \frac{1}{\theta}\mathbf{I} - \frac{1}{2}[\mathbf{S}_\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\mathbf{S}_\omega]^2 \in \mathbb{R}^{3 \times 3}$$

Example

The initial frame $\{b\}$ and final frame $\{c\}$ are given. Find the screw motion that displaces the frame at T_{sb} to T_{sc} .

$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Wrenches

Wrench

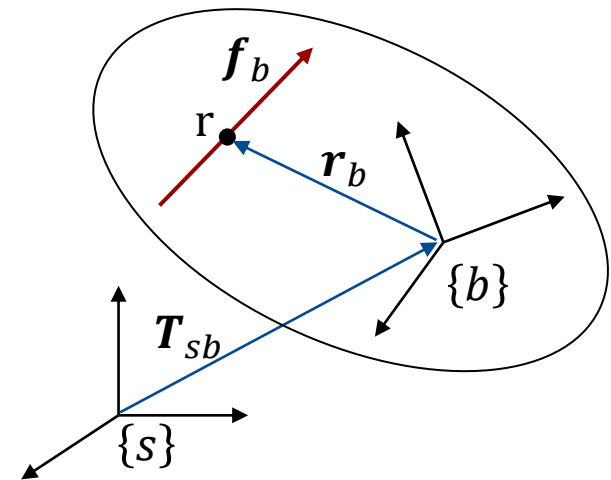
Consider a linear force \mathbf{f} acting on a rigid body at a point \mathbf{r} . Both $\mathbf{f}_b \in \mathbb{R}^3$ and $\mathbf{r}_b \in \mathbb{R}^3$ are represented in $\{b\}$. This force creates a torque or moment $\mathbf{m}_b \in \mathbb{R}^3$ in $\{b\}$ as

$$\mathbf{m}_b = \mathbf{r}_b \times \mathbf{f}_b$$

We can package the moment and force together in a single six-dimensional vector called **wrench** (or **spatial force**) in $\{b\}$ as

$$\mathcal{F}_b = \begin{bmatrix} \mathbf{m}_b \\ \mathbf{f}_b \end{bmatrix} \in \mathbb{R}^6$$

$$\mathcal{F}_s = ?$$



Wrench

The **power** is a coordinate-independent quantity, i.e., the power generated (or dissipated) by an $(\mathcal{F}, \mathcal{V})$ pair (wrench and twist) must be the same regardless of the frame in which it is represented:

$$(\mathcal{V} \cdot \mathcal{F} = \text{power})$$

$$\mathcal{V}_s^T \mathcal{F}_s = \mathcal{V}_b^T \mathcal{F}_b = \text{power}$$

$$(\mathcal{V}_b = [\text{Ad}_{T_{bs}}] \mathcal{V}_s)$$

$$\begin{aligned} \mathcal{V}_s^T \mathcal{F}_s &= ([\text{Ad}_{T_{bs}}] \mathcal{V}_s)^T \mathcal{F}_b \\ &= \mathcal{V}_s^T [\text{Ad}_{T_{bs}}]^T \mathcal{F}_b \end{aligned}$$

Since this must hold for all \mathcal{V}_s

$$\mathcal{F}_s = [\text{Ad}_{T_{bs}}]^T \mathcal{F}_b$$

spatial wrench

body wrench

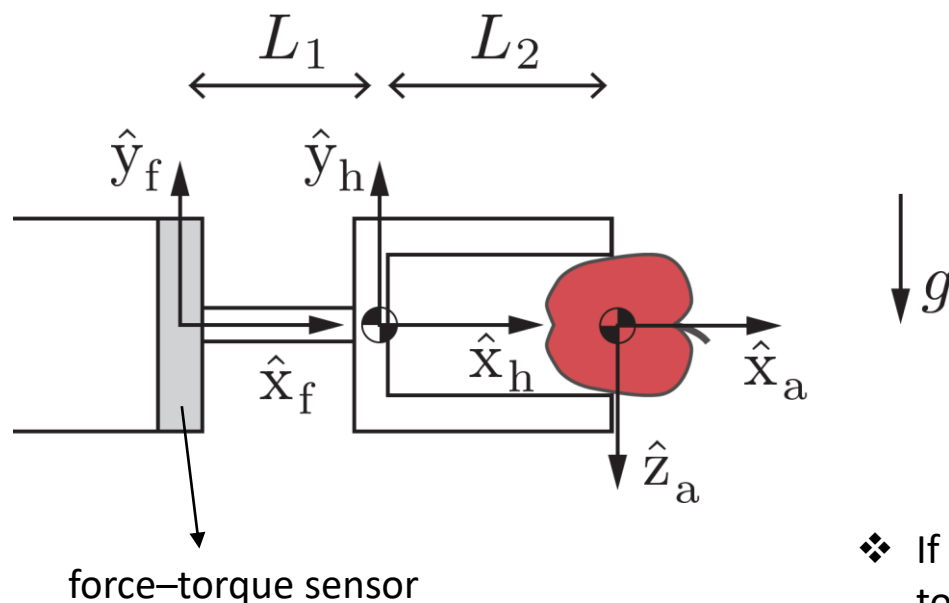
$$\mathcal{F}_a = [\text{Ad}_{T_{ba}}]^T \mathcal{F}_b$$

$$\mathcal{F}_b = [\text{Ad}_{T_{ab}}]^T \mathcal{F}_a$$

(changing the reference frame of a wrench)

Example

The robot hand shown is holding an apple with a mass of 0.1 kg in a gravitational field $g=10 \text{ m/s}^2$. The mass of the hand is 0.5 kg, $L_1=10 \text{ cm}$, and $L_2=15 \text{ cm}$. What is the force and torque measured by the six-axis force–torque sensor between the hand and the robot arm?



- ❖ If more than one wrench acts on a rigid body, the total wrench on the body is simply the vector sum of the individual wrenches, provided that the wrenches are expressed in the same frame.

Review

Rigid-Body Motions

Rotations	Rigid-Body Motions
$\mathbf{R} \in SO(3)$: 3×3 matrices $\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}, \det(\mathbf{R}) = 1$	$\mathbf{T} \in SE(3)$: 4×4 matrices $\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix},$ where $\mathbf{R} \in SO(3), \mathbf{p} \in \mathbb{R}^3$
$\mathbf{R}^{-1} = \mathbf{R}^T$	$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$
Change of coordinate frame: $\mathbf{R}_{ab} \mathbf{R}_{bc} = \mathbf{R}_{ac}, \mathbf{R}_{ab} \mathbf{p}_b = \mathbf{p}_a$	Change of coordinate frame: $\mathbf{T}_{ab} \mathbf{T}_{bc} = \mathbf{T}_{ac}, \mathbf{T}_{ab} \mathbf{p}_b = \mathbf{p}_a$

Rigid-Body Motions

Rotations	Rigid-Body Motions
<p>Rotating a frame $\{b\}$:</p> $\mathbf{R} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$ <p>$\mathbf{R}_{sb'} = \mathbf{R}\mathbf{R}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$</p> <p>$\mathbf{R}_{sb''} = \mathbf{R}_{sb}\mathbf{R}$: rotate θ about $\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}$</p>	<p>Displacing a frame $\{b\}$:</p> $\mathbf{T} = \begin{bmatrix} \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$ <p>$\mathbf{T}_{sb'} = \mathbf{T}\mathbf{T}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ (moves $\{b\}$ origin), translate \mathbf{p} in $\{s\}$</p> <p>$\mathbf{T}_{sb''} = \mathbf{T}_{sb}\mathbf{T}$: translate \mathbf{p} in $\{b\}$, rotate θ about $\hat{\boldsymbol{\omega}}$ in new body frame</p>
<p>Unit rotation axis is $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$, where $\ \hat{\boldsymbol{\omega}}\ = 1$</p>	<p>“Unit” screw axis is $\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\ \mathbf{S}_\omega\ = 1$ or (ii) $\ \mathbf{S}_\omega\ = 0, \ \mathbf{S}_v\ = 1$</p>
	<p>For a screw axis $\{\mathbf{q}, \hat{\mathbf{s}}, h\}$ with finite h,</p> $\mathbf{S} = \begin{bmatrix} \mathbf{S}_\omega \\ \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \mathbf{q} + h\hat{\mathbf{s}} \end{bmatrix}$
<p>Angular velocity is $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}$</p>	<p>Twist is $\mathcal{V} = \mathbf{S}\dot{\theta}$</p>

Rigid-Body Motions

Rotations	Rigid-Body Motions
<p>For any $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$,</p> $[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ <p>Properties: For any $\boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)$:</p> $[\boldsymbol{\omega}] = -[\boldsymbol{\omega}]^T, [\boldsymbol{\omega}]\boldsymbol{x} = -[\boldsymbol{x}]\boldsymbol{\omega},$ $[\boldsymbol{\omega}][\boldsymbol{x}] = ([\boldsymbol{x}][\boldsymbol{\omega}])^T, \boldsymbol{R}[\boldsymbol{\omega}]\boldsymbol{R}^T = [\boldsymbol{R}\boldsymbol{\omega}]$	<p>For any $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} \in \mathbb{R}^6$ or $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$,</p> $[\boldsymbol{v}] = \begin{bmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3),$ $[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$
$\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s], \quad \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b] \quad (\boldsymbol{R} := \boldsymbol{R}_{sb})$	$\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\boldsymbol{v}_s], \quad \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{v}_b] \quad (\boldsymbol{T} := \boldsymbol{T}_{sb})$
	$[\text{Ad}_{\boldsymbol{T}}] = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ [\boldsymbol{p}]\boldsymbol{R} & \boldsymbol{R} \end{bmatrix} \in \mathbb{R}^{6 \times 6}$ <p>Properties: $[\text{Ad}_{\boldsymbol{T}}]^{-1} = [\text{Ad}_{\boldsymbol{T}^{-1}}],$</p> $[\text{Ad}_{\boldsymbol{T}_1}][\text{Ad}_{\boldsymbol{T}_2}] = [\text{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]$
<p>Change of coordinate frame:</p> $\hat{\boldsymbol{\omega}}_a = \boldsymbol{R}_{ab}\hat{\boldsymbol{\omega}}_b, \quad \boldsymbol{\omega}_a = \boldsymbol{R}_{ab}\boldsymbol{\omega}_b$	<p>Change of coordinate frame:</p> $\boldsymbol{S}_a = [\text{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b, \quad \boldsymbol{v}_a = [\text{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{v}_b$

Rigid-Body Motions

Rotations	Rigid-Body Motions
$\hat{\omega}_s = R_{sb} \hat{\omega}_b$	$S_s = [\text{Ad}_{T_{sb}}] S_b, \mathcal{V}_s = [\text{Ad}_{T_{sb}}] \mathcal{V}_b, [\text{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$
Exponential coordinate for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	Exponential coordinate for $T \in SE(3)$: $S\theta \in \mathbb{R}^6$
$\exp: [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$ $R = \text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta}$ $R = I + \sin \theta [\hat{\omega}] + (1 - \cos \theta) [\hat{\omega}]^2$ (Rodrigues' formula for rotations)	$\exp: [S]\theta \in se(3) \rightarrow T \in SE(3)$ $T = e^{[S]\theta}$ $T = \begin{bmatrix} e^{[S_\omega]\theta} & G(\theta)S_v \\ 0 & 1 \end{bmatrix}$ $G(\theta) = I\theta + (1 - \cos \theta)[S_\omega] + (\theta - \sin \theta)[S_\omega]^2$
$\log: R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$ $\log(R) = [\hat{\omega}]\theta$	$\log: T \in SE(3) \rightarrow [S]\theta \in se(3)$ $\log(T) = [S]\theta$
Moment change of coordinate frame: $m_a = R_{ab} m_b$	Wrench change of coordinate frame: $\mathcal{F}_a = (m_a, f_a) = [\text{Ad}_{T_{ba}}]^T \mathcal{F}_b$