Ch10: Design via Root Locus

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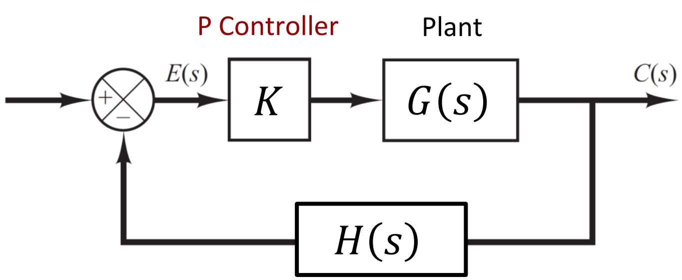
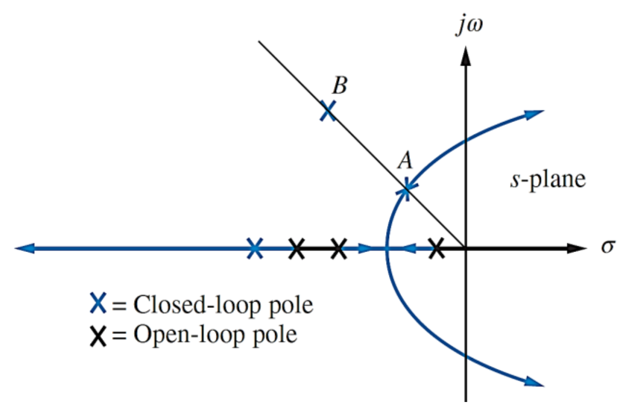
Introduction

Root Locus and P Controllers

The root locus can be sketched quickly to get a general idea of the changes in stability and transient response generated by changes in gain K. However, we are limited to those transient responses that exist **along** the root locus (say point A) by changing K.



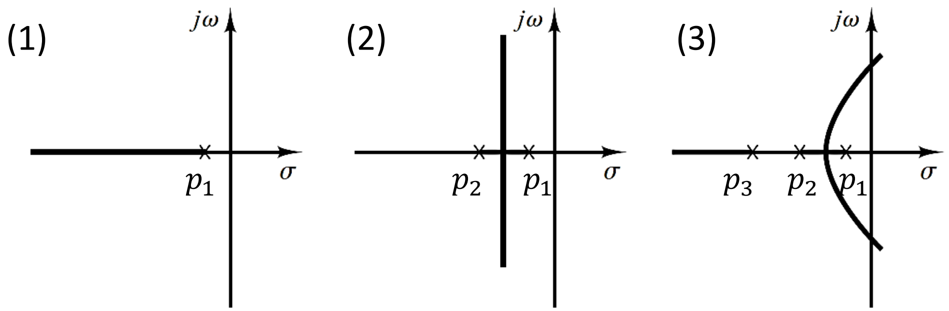


Limitations:

- A desired transient response that is not on the root locus (say point B) can not be achieved.
- The transient response and steady-state error specifications cannot be usually met at the **same time** by adjusting only one gain K.

Effects of the Addition of Poles and Zeros

The addition of a **pole** to the open-loop transfer function $G(s)H(s)$ has the effect of pulling the root locus to the **right**, tending to lower the system's relative stability and to slow down the settling of the response.

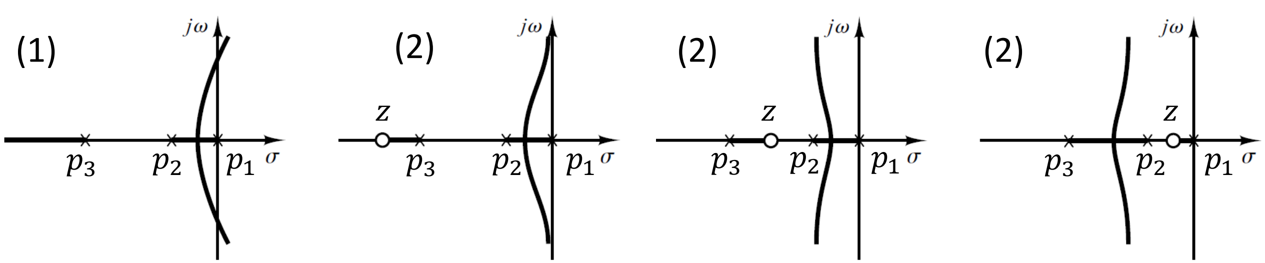


(1) $G(s)H(s) = \dfrac{1}{s+p_1}$

(2) $G(s)H(s) = \dfrac{1}{(s+p_1)(s+p_2)}$

(3) $G(s)H(s) = \dfrac{1}{(s+p_1)(s+p_2)(s+p_3)}$

The addition of a **zero** to the open-loop transfer function $G(s)H(s)$ has the effect of pulling the root locus to the **left**, tending to make the system more stable and to speed up the settling of the response.

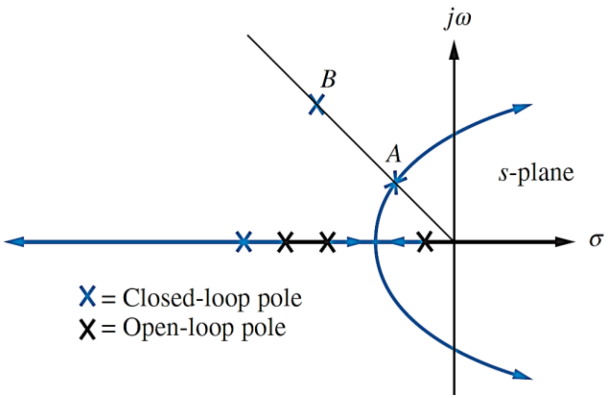


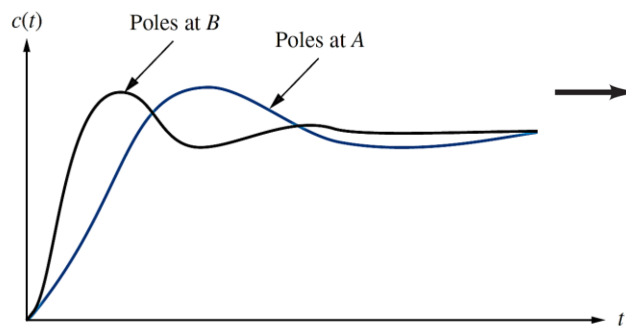
(1)
$$G(s)H(s) = \frac{1}{(s+p_1)(s+p_2)(s+p_3)}$$

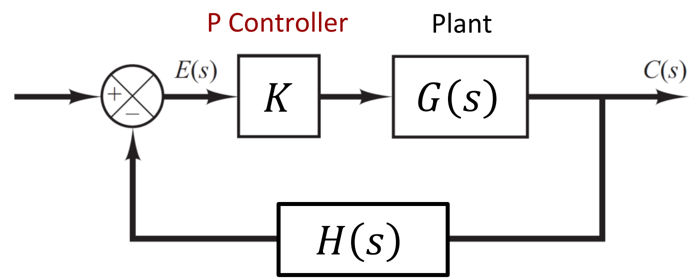
(2)
$$G(s)H(s) = \frac{(s+z)}{(s+p_1)(s+p_2)(s+p_3)}$$

Improving Transient Response & Steady-State Error

Assume that our desired transient response (M_p and T_s) is represented by point B which is not on the root locus, and we only can obtain the specified M_p at point A after a gain adjustment.



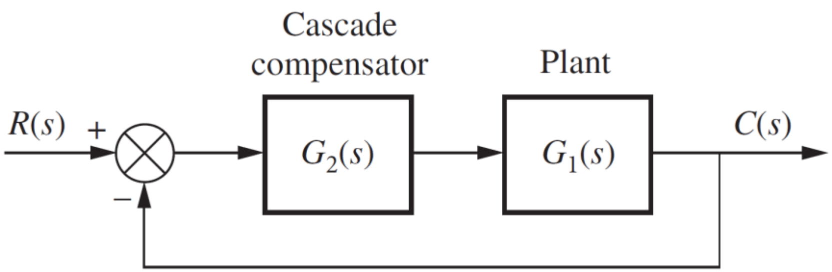


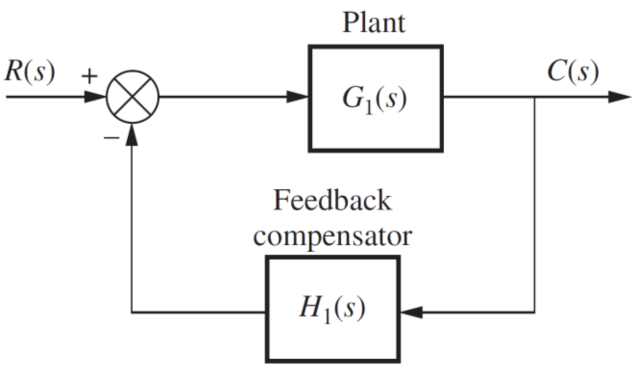


- One simple way to achieve these desired transient responses is to **compensate** the system (plant) with **additional poles and zeros**, so that the compensated system has a root locus that goes through the desired pole location for some value of gain (**reshaping the root locus**). Compensators can be also used to improve the steady-state error characteristics **independently**, without compromising between transient response and steady-state error.

Compensator Configurations

Two common configurations of compensation are **Cascade** (Series) compensation and **Feedback** (Parallel) compensation.





Both methods change the open-loop **poles and zeros**, thereby creating a new root locus that goes through the desired closed-loop pole location and/or improves the steady-state error. In general, series compensation may be simpler than parallel compensation.

Note: Adding open-loop poles (and zeros) increases the system order. Thus, we should evaluate the transient response of this higher-order system through **simulation** after the design is complete to be sure the requirements have been met.

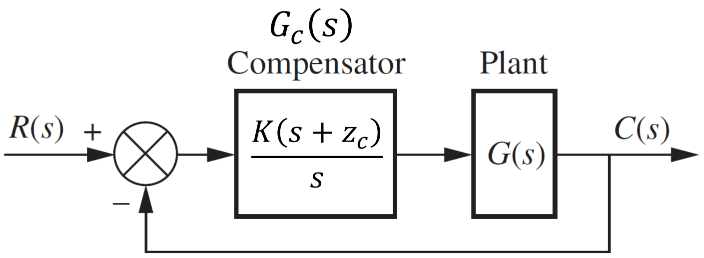
Improving Steady-State Error

Improving Steady-State Error

Two technique to improve the steady-state error of a feedback control system using cascade compensation, **without** appreciably affecting the transient response, are:

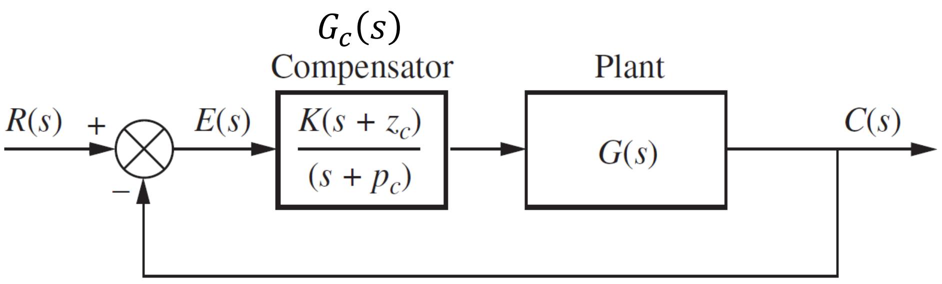
1. Ideal Integral Compensation
(Proportional-plus-Integral (PI) Controller)

$$G_c(s) = \frac{K(s+z_c)}{s} = K_P + \frac{K_I}{s}$$



2. Lag Compensation

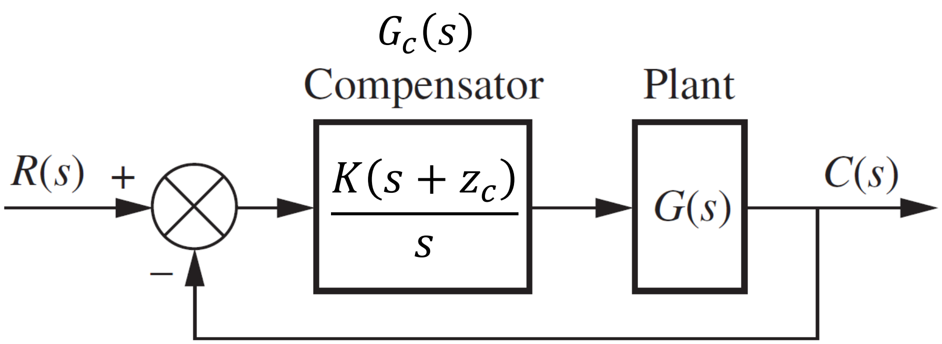
$$G_c(s) = \frac{K(s+z_c)}{(s+p_c)}$$

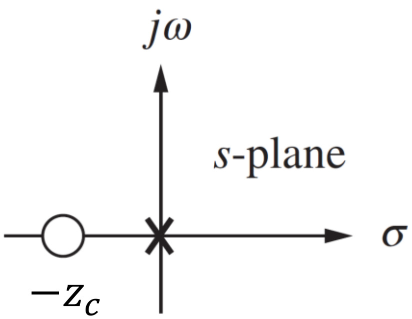


In this section, it is assumed that the closed-loop system has a pair of dominant closed-loop poles (i.e., the system can be approximated by a second-order system).

Ideal Integral Compensation

This technique uses a pure integrator to place an open-loop, forward-path pole at the origin, which results in increasing the system type by one and reducing the steady-state error to zero (For example, a Type 0 system responding to a step input with a finite error responds with zero error if the system type is increased by one).

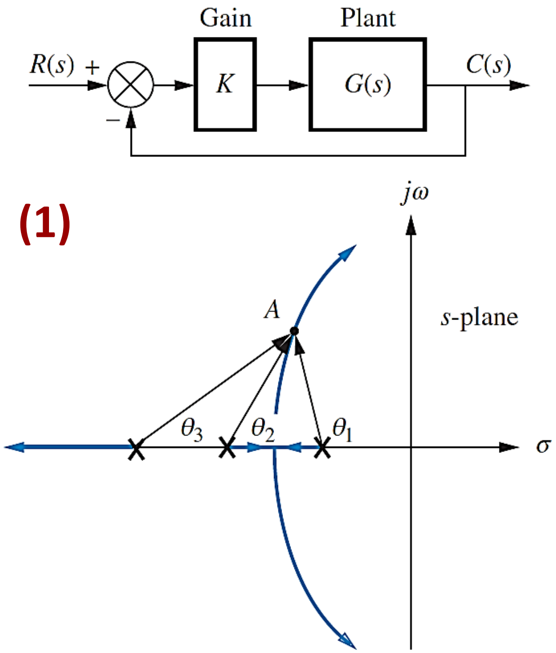




$$G_c(s) = \frac{K(s+z_c)}{s}$$

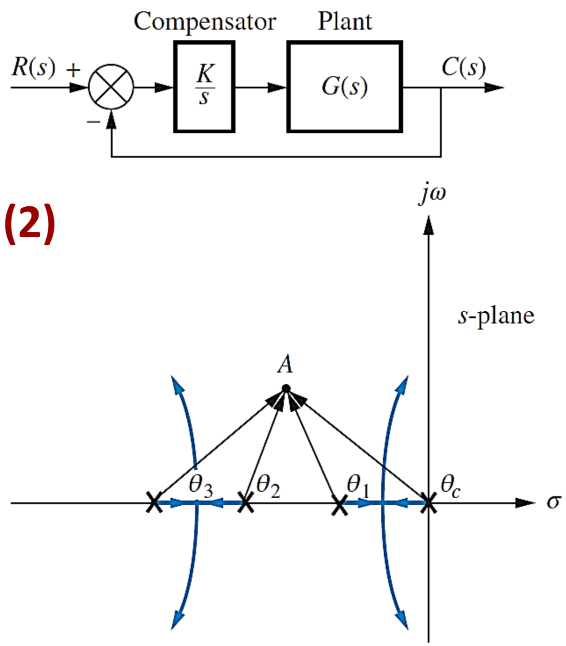
Ideal Integral Compensation

(1)



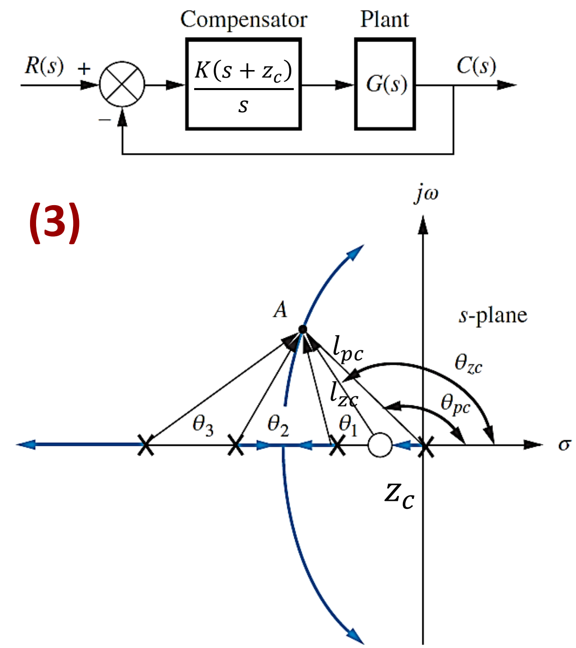
$$-\theta_1-\theta_2-\theta_3=(2k+1)180°$$

(2)



$$-\theta_1-\theta_2-\theta_3-\theta_c\neq(2k+1)180°$$

(3)



$$-\theta_1-\theta_2-\theta_3-\theta_{pc}+\theta_{zc}\cong(2k+1)180°$$

A desirable transient response generated by the closed-loop poles at point A with a specific K.

Adding a pole at the origin to increase the system type but point A is no longer on the root locus.

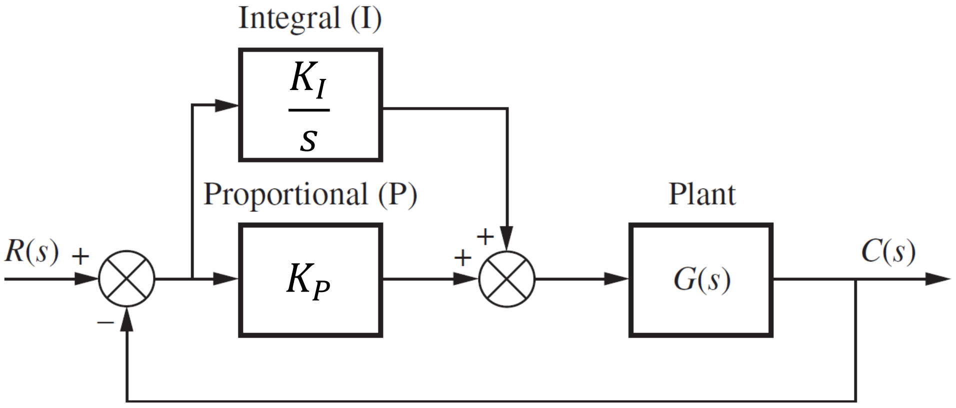
By adding a zero z_c **close** to the pole at the origin, $\theta_{z_c}-\theta_{p_c}\approx 0$, and point A is again placed on the root locus. Thus, the effect on the transient response is not significant. Since $l_{zc}\cong l_{pc}$, gain K is also about the same as before compensation.

PI Controller

Hence, an ideal integral compensator will reduce the steady-state error to **zero**, without **appreciably** affecting the transient response.

This compensator is implemented with a **Proportional-plus-Integral (PI) Controller**, where the error and the integral of the error are fed forward to the plant.

$$G_c(s) = \frac{K(s+z_c)}{s} = K_P + \frac{K_I}{s}$$

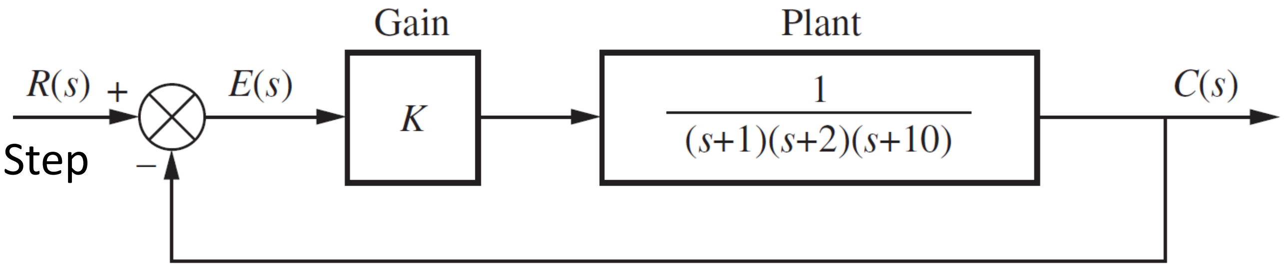


❖ This controller/compensator **must** be implemented with **active** networks, such as active amplifiers (a single zero cannot be produced by passive networks).

Example

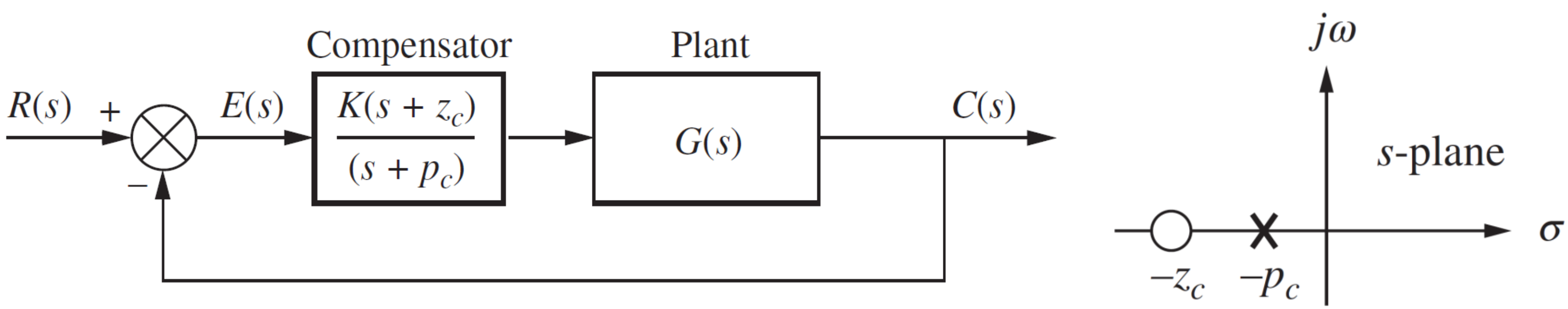
Given the following system, operating with a damping ratio of 0.174, design a PI controller (instead of the current P controller) to reduce the steady-state error to zero for a unit-step input, without appreciably affecting the transient response. We know that the line of the constant damping ratio 0.174 intersect the root locus at the point $-0.694 \pm j3.926$ with $K = 164.6$.

- Place the zero of the controller at the point -0.1.
- Roughly sketch the system response for compensated and uncompensated systems.



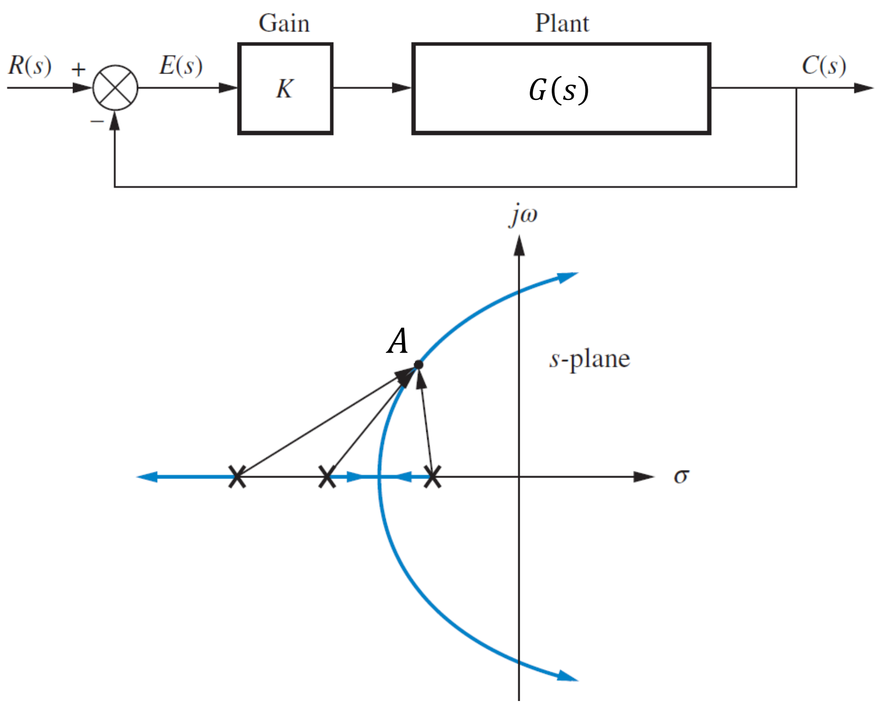
Lag Compensation

This technique places an open-loop, forward-path pole near the origin, and although it does not increase the system type to drive the steady-state error to zero, it does yield a measurable reduction in steady-state error.

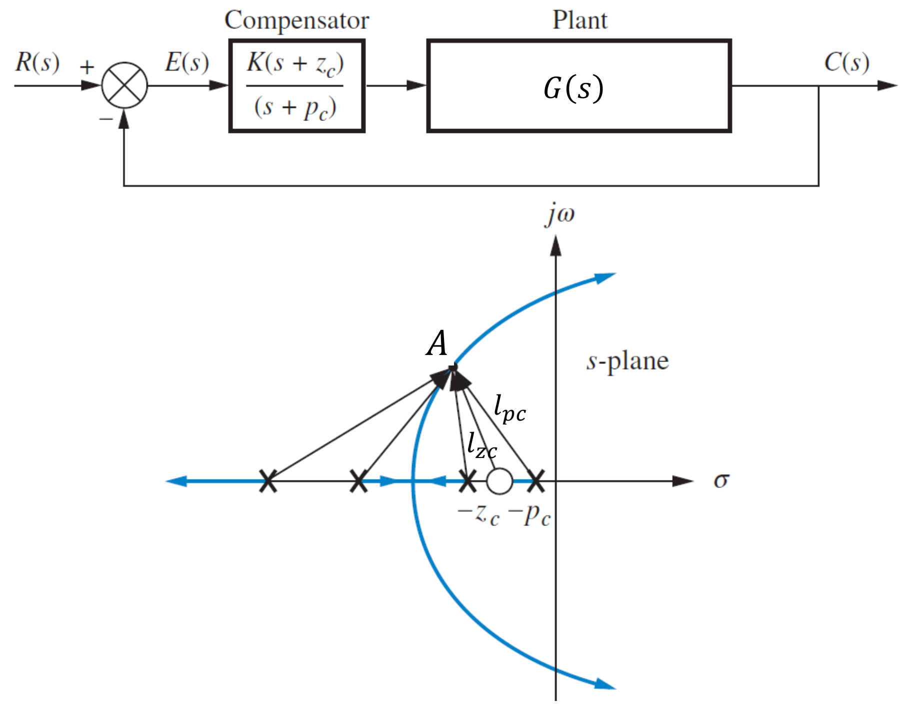


The name of this compensator comes from its frequency response characteristics.

Lag Compensation



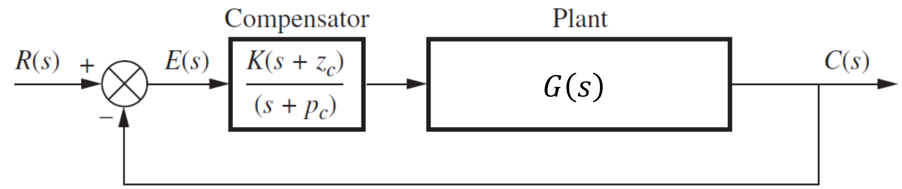
A desirable transient response generated by the closed-loop poles at point A with a specific K.



By adding a pole and a zero **close together**, then $\theta_{z_c} - \theta_{p_c} \approx 0$ and point A is still on the root locus. Thus, the effect on the transient response is not significant. Since $l_{zc} \cong l_{pc}$, gain K is about the same as before compensation.

Lag Compensation

Considering static error constant K_p:



$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$$

$$(K_p)_{\text{Conpensated}} = \left(\frac{z_1 z_2 \ldots}{p_1 p_2 \ldots}\right) K \frac{z_c}{p_c} = (K_p)_{\text{Unconpensated}} \frac{z_c}{p_c}$$

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p}$$

Improvement in the compensated system's K_p over the uncompensated system's K_p is z_c/p_c. The only way that z_c/p_c can be large is to place the compensator's pole-zero pair **close to the origin**. (For example, when $p_c = -0.001$ and $z_c = -0.01$, the pole and zero are very close and $z_c/p_c = 10$)

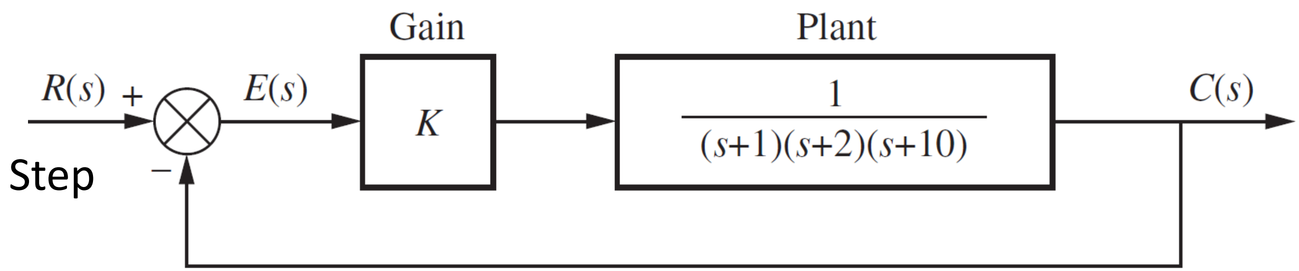
- Hence, a lag compensator will improve the static error constant by a factor equal to z_c/p_c without **appreciably** affecting the transient response.

- The compensator **can be** implemented with a less expensive **passive** network that does not require additional power sources.

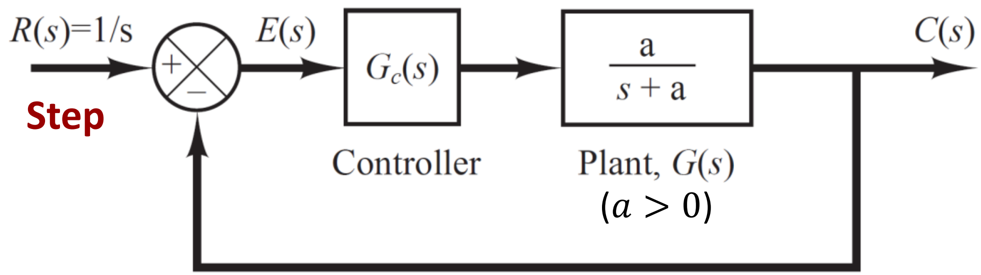
Example

Given the following system, operating with a damping ratio of 0.174, design a Lag compensator (instead of the current P controller) to improve the steady-state error by a factor of 10 for a unit-step input, without appreciably affecting transient response. We know that the line of the constant damping ratio 0.174 intersect the root locus at the point $-0.694 \pm j3.926$ with $K = 164.6$.

- Place the pole of the controller at the point -0.01.
- Roughly sketch the system response for compensated and uncompensated systems.



Example: P & I Control of a Type 0 System for a Step Input



- **Proportional (P) Controller**: $G_c(s) = K_P \quad (K_P > 0)$

$$T(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{aK_P}{s+a+aK_P} \quad \Rightarrow \quad e_{\text{step}}(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+K_P G(s)} = \frac{1}{1+K_P} \neq 0$$

(Closed-loop is stable)

❖ The steady-state error can be reduced by increasing the value of the gain K_P.

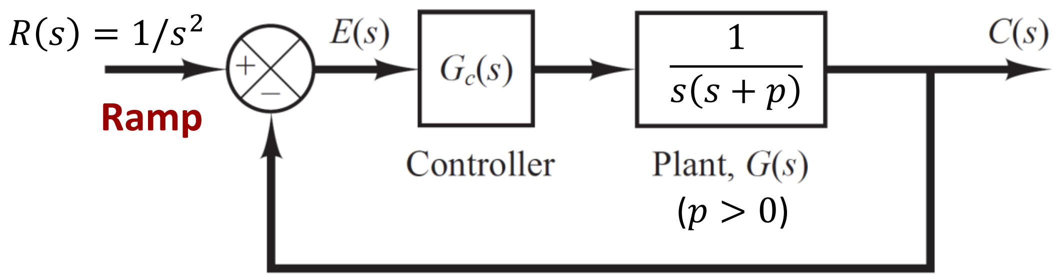
- **Integral (I) Controller**: $G_c(s) = \frac{K_I}{s} \quad (K_I > 0)$

$$T(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{aK_I}{s^2+as+aK_I} \quad \Rightarrow \quad e_{\text{step}}(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+\frac{K_I}{s}G(s)} = 0$$

(Closed-loop is stable)

❖ Integral control of the system eliminates the steady-state error in the response to the step input.
❖ The integral gain K_I can be selected purely to provide an acceptable transient response.

Example: P & I Control of a Type 1 System for a Ramp Input



- **Proportional (P) Controller**: $G_c(s) = K_P \qquad (K_P > 0)$

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K_P}{s^2 + ps + K_P} \quad \Rightarrow \quad e_{\text{ramp}}(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + K_P G(s)} = \frac{p}{K_P} \neq 0$$

(Closed-loop is stable)

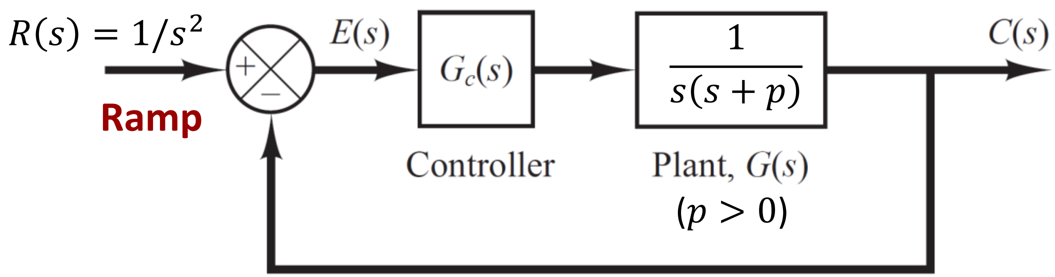
❖ The steady-state error can be reduced by increasing the value of the gain K_P. Increasing this value, however, will cause the system response to be more oscillatory.

- **Integral (I) Controller**: $G_c(s) = \frac{K_I}{s}$

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K_I}{s^3 + ps^2 + K_I}$$

(Closed-loop is **unstable**)

Example: PI Control of a Type 1 System for a Ramp Input



- **Proportional-Plus-Integral (PI) Controller:** $G_c(s) = K_P + \frac{K_I}{s}$ $\quad (K_P, K_I > 0)$

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K_P s + K_I}{s^3 + ps^2 + K_P s + K_I} \Rightarrow e_{\mathrm{ramp}}(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + \left(K_P + \frac{K_I}{s}\right) G(s)} = 0$$

If closed-loop is **stable**,

- The proportional control action tends to stabilize the system, while the integral control action tends to eliminate or reduce steady-state error in response to various inputs.
- Typically, it will cause instability if K_I is raised sufficiently high.

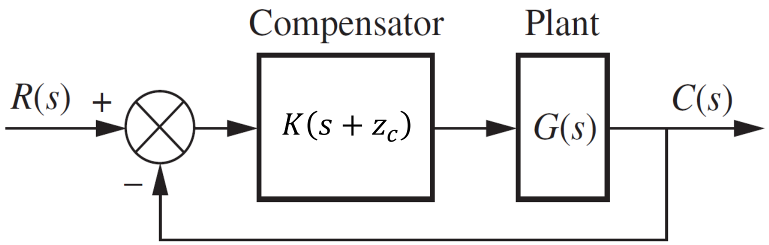
Improving Transient Response

Improving Transient Response

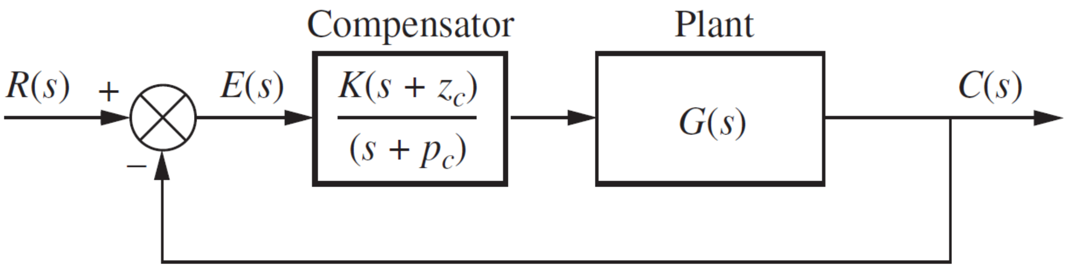
Two technique to improve the transient response of a feedback control system using cascade compensation, **without** appreciably affecting (increasing) the steady-state error, are:

1. **Ideal Derivative Compensation**
(Proportional-plus-Derivative (PD) Controller)

$$G_c(s) = K(s + z_c) = K_P + K_D s$$



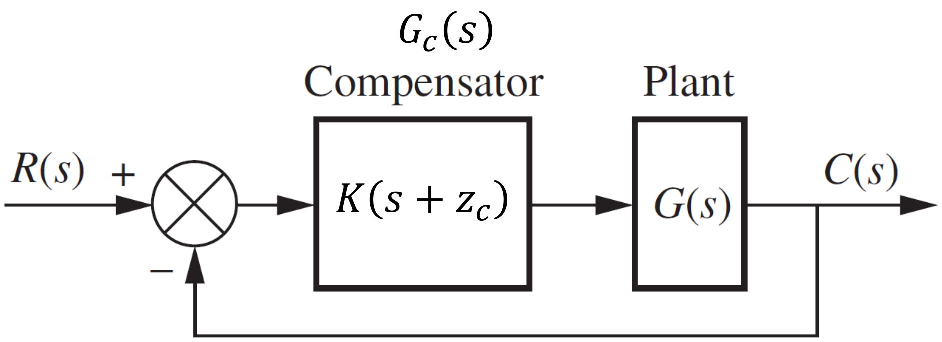
2. **Lead Compensation**

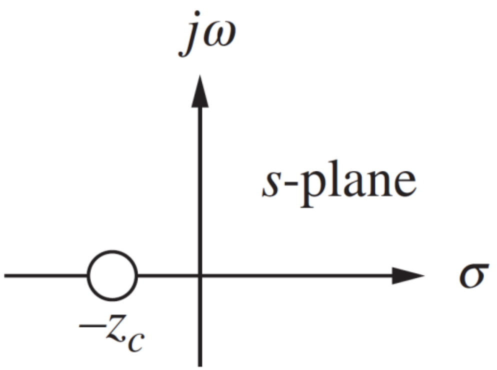


In this section, it is assumed that the closed-loop system has a pair of dominant closed-loop poles (i.e., the system can be approximated by a second-order system).

Ideal Derivative Compensation

This technique adds a zero to the open-loop, forward-path. Thus, it reshape the root locus without changing the system type

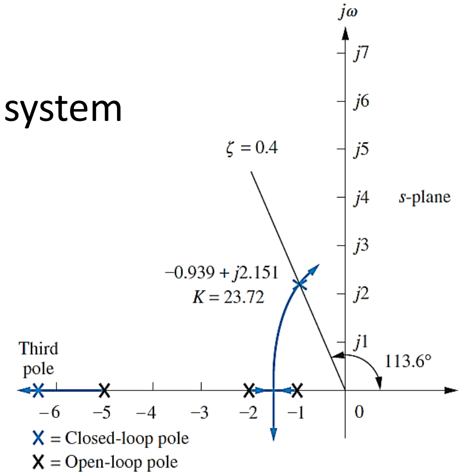




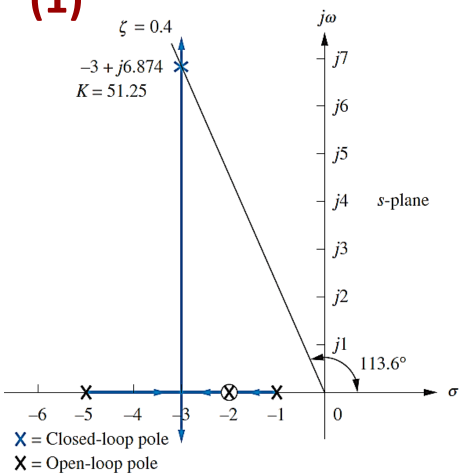
$$G_c(s) = K(s + z_c)$$

Ideal Derivative Compensation

Let's consider the effect of adding compensating zeros at -2, -3, and -4 to the system $\frac{K}{(s+1)(s+2)(s+5)}$, operating with a damping ratio of 0.4.



(1)



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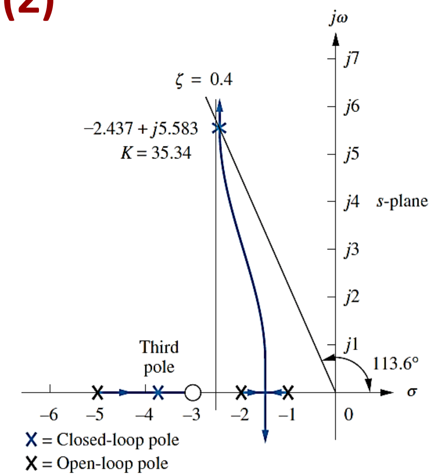

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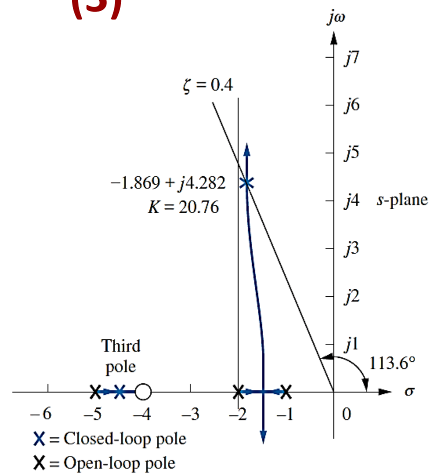
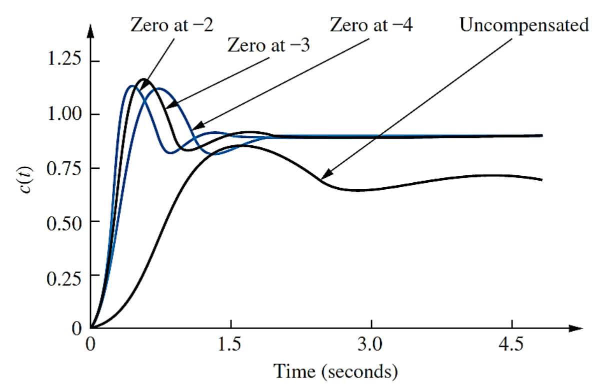


For each compensated case, the dominant, second-order poles are farther out along the 0.4 damping ratio line than the uncompensated system. Thus, in this system, adding a zero can reduce the settling time (and improve the transient response for step input).



Ideal Derivative Compensation

- Adding a single zero properly to the forward path can speed up the response and **sometimes** improve the steady-state error over the uncompensated system. Moreover, the addition of the single zero tends to reduce the number of branches of the root locus that cross into the right half-plane (i.e., it can also improve the relative stability of the system).

- **Design Method**:

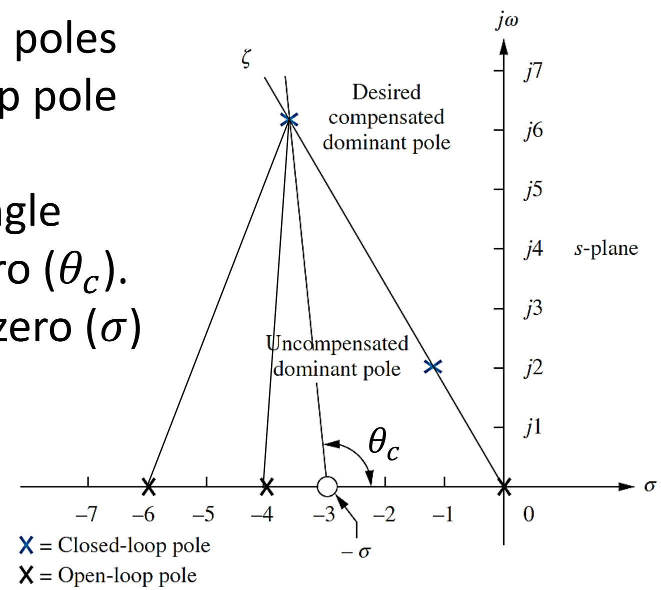
To find the proper zero (σ),

I. First, calculate the sum of the angles from the open-loop poles and zeros to a design desired point that is the closed-loop pole and yields the desired transient response.

II. Then, the difference between 180° and the calculated angle must be the angular contribution of the compensator zero (θ_c).

III. Then, Trigonometry is used to locate the position of the zero (σ) to yield the required difference in angle.

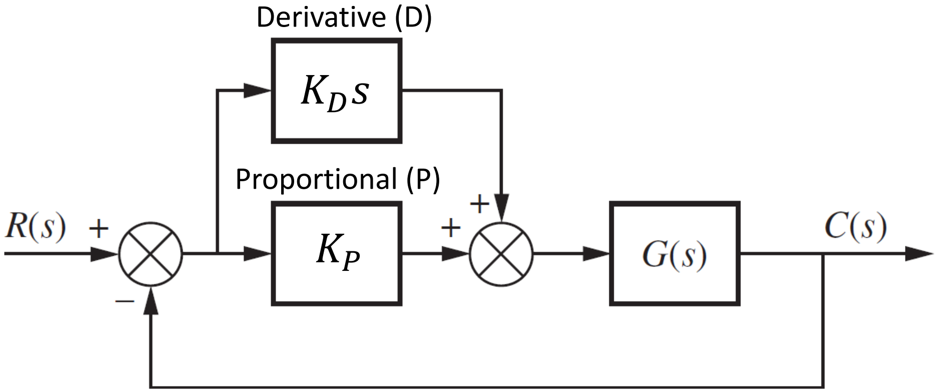
IV. $K = \dfrac{\prod \text{pole lengths}}{\prod \text{zero lengths}}$



PD Controller

This compensator is implemented with a **Proportional-plus-Derivative (PD) Controller**, where the error and the derivative of the error are fed forward to the plant.

$$G_c(s) = K(s + z_c) = K_P + K_D s$$

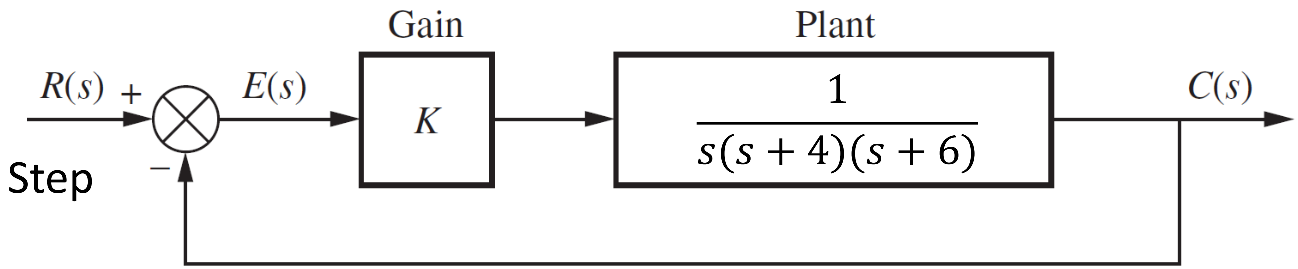


Drawback of a D Controller: Differentiation is a noisy process; although the level of the noise is low, the frequency of the noise is high compared to the signal. Thus, differentiation of high frequencies can lead to large unwanted signals or saturation of amplifiers and other components.

❖ This controller/compensator **must** be implemented with **active** networks, such as active amplifiers (a single zero cannot be produced by passive networks).

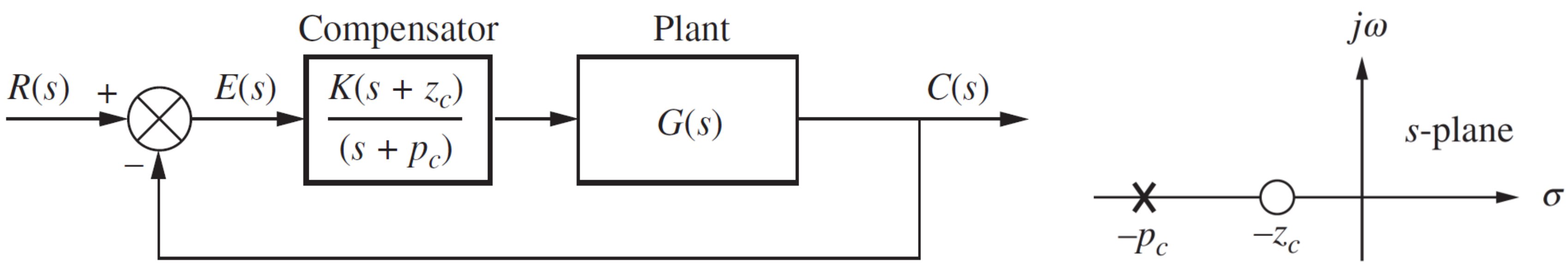
Example

Given the following system, design a PD controller (instead of the current P controller) to yield a 16% overshoot, with a threefold reduction in settling time for a unit step input. We know that the line of the constant 16% overshoot intersect the root locus at the point $-1.205 \pm j2.064$ with $K = 43.35$.



Lead Compensation

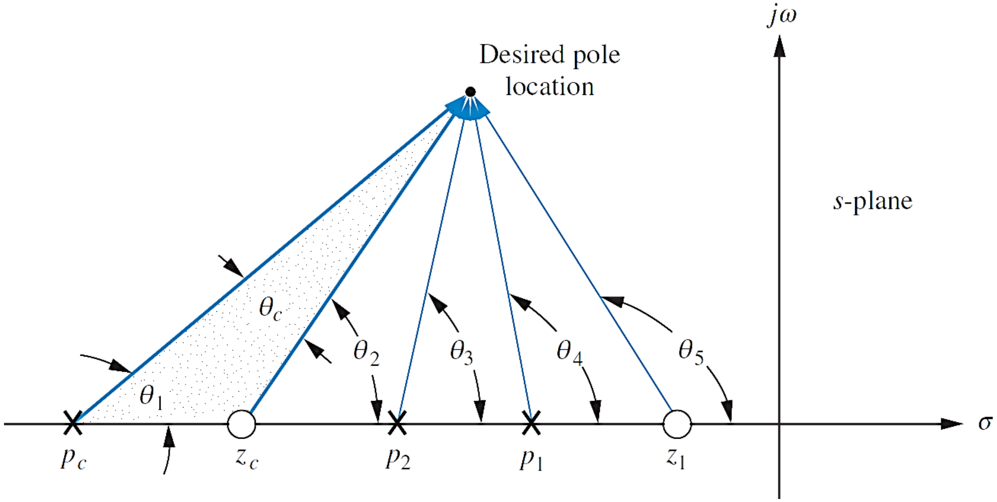
This technique approximates differentiation by adding a zero and a more distant pole to the forward-path transfer function.



The name of this compensator comes from its frequency response characteristics.

Lead Compensation

If the pole p_c is farther from the imaginary axis than the zero z_c, the angular contribution of the compensator $(\theta_2 - \theta_1)$ is still positive and thus, it approximates an equivalent single zero (just as for a single PD controller zero), with the angular contribution of $\theta_c = \theta_2 - \theta_1$.



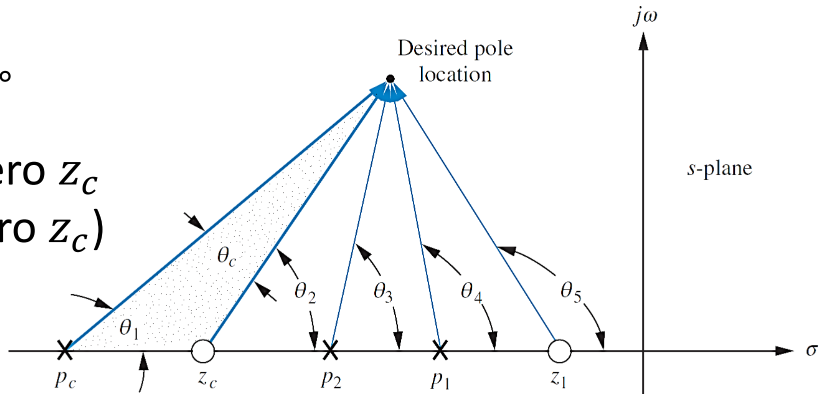
Lead Compensation

Design Method: If a desired pole location on the s-plane is selected, the sum of the angles from the uncompensated system's poles and zeros to the design point can be found. The difference between 180° and the sum of the angles must be the angular contribution required for the compensator (i.e., $\theta_c = \theta_2 - \theta_1$).

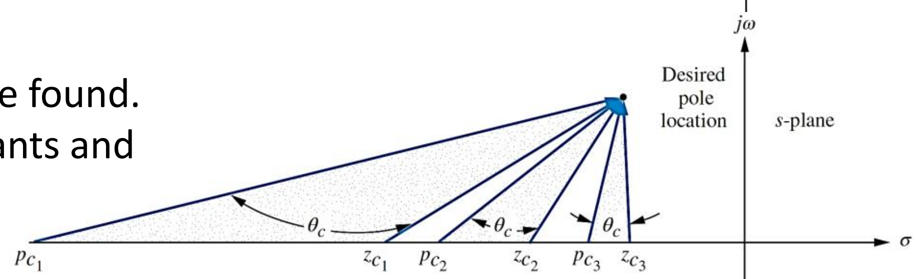
$$(\theta_2 - \theta_1) - \theta_3 - \theta_4 + \theta_5 = (2k + 1)180°$$

Thus, we can arbitrarily select a lead compensator zero z_c (or pole p_c) and find its corresponding pole p_c (or zero z_c) by using trigonometry.

$$K = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}}$$



Note: An infinite number of lead compensators can be found. The differences are in the values of static error constants and the gain K.

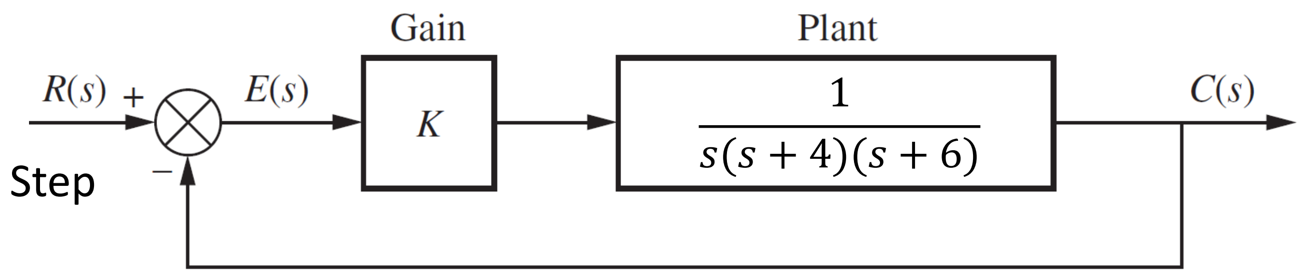


❖ This compensator **can be** implemented with a less expensive **passive** network that does not require additional power sources.

Example

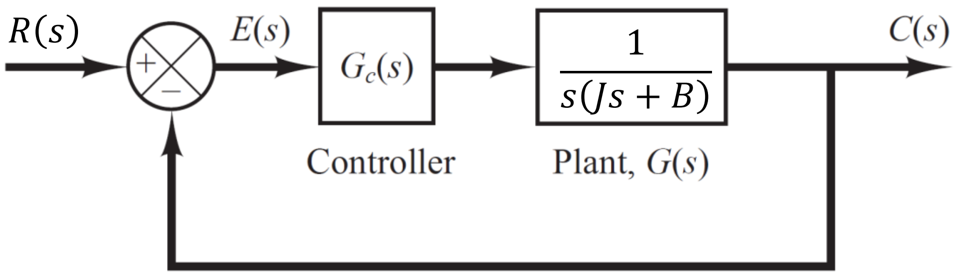
Given the following system, design a lead compensator (instead of the current P controller) that will reduce the settling time by a factor of 2 while maintaining 30% overshoot for a unit step input. We know that the line of the constant 30% overshoot intersect the root locus at the point $-1.007 \pm j2.627$ with $K = 63.21$.

- Place the zero of the controller at the point -5.



Example: PD Control of a Type 1 System for a Step & Ramp Input

Control of an inertia load with damper:



- **Proportional-Plus-Derivative (PD) Controller:** $G_c(s) = K_P + K_D s \qquad (K_P, K_D > 0)$

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K_P + K_D s}{Js^2 + (B + K_D)s + K_P}$$

(Closed-loop is stable) $\Rightarrow$

$$e_{\text{step}}(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + (K_P + K_D s)G(s)} = 0$$

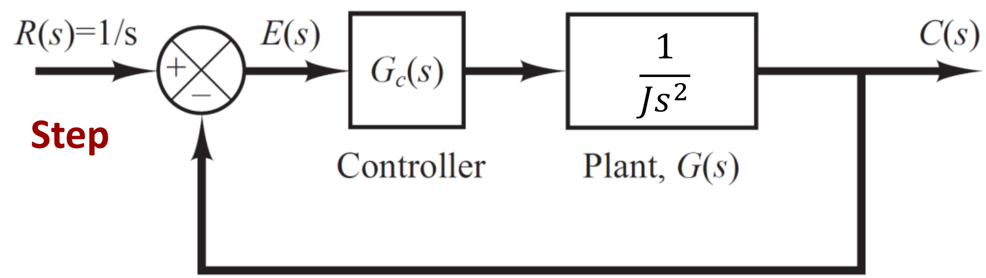
$$e_{\text{ramp}}(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + (K_P + K_D s)G(s)} = \frac{B}{K_P}$$

$$Js^2 + (B + K_D)s + K_P \equiv s^2 + 2\zeta\omega_n s + \omega_n^2$$

- A compromise between acceptable transient-response behavior and acceptable steady-state behavior can be made to select the gains.

Example: P & PD Control of a Type 2 System for a Step Input

Control of an inertia load:



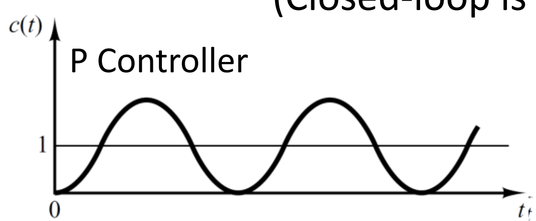
- **Proportional (P) Controller**: $G_c(s) = K_P \quad (K_P > 0) \quad \Rightarrow \quad T(s) = \dfrac{G_c(s)G(s)}{1+G_c(s)G(s)} = \dfrac{K_P}{Js^2 + K_P}$

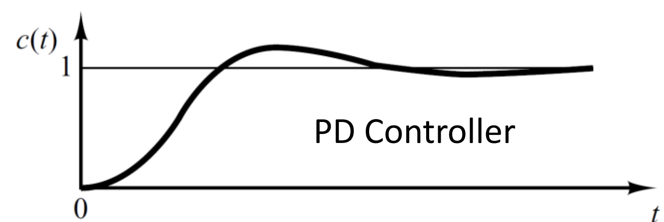
(Closed-loop is **marginally stable**)

- **Proportional-Plus-Derivative (PD) Controller**: $G_c(s) = K_P + K_D s \quad (K_P, K_D > 0)$

$$T(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{K_P + K_D s}{Js^2 + K_D s + K_P} \quad \Rightarrow \quad e_{\text{step}}(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + (K_P + K_D s)G(s)} = 0$$

(Closed-loop is stable)





(D controller introduces a damping effect)

Improving Steady-State Error and Transient Response

Improving Steady-State Error and Transient Response

The design techniques can be combined to obtain improvement in steady-state error and transient response **independently**.

Two approaches to design:

(1) First improving the transient response, then improving the steady-state error of the compensated system. (PD + PI) or (Lead + Lag) (Recommended Approach)

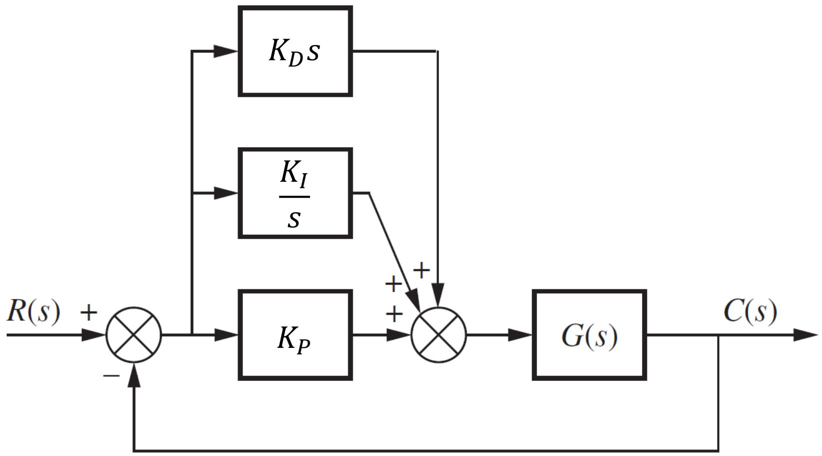
(2) First improving the steady-state error, then improving the transient response of the compensated system. (PI + PD) or (Lag + Lead)

Controller/Compensator Design

- If first an active PD controller and then an active PI controller are designed, the resulting compensator is called a **proportional-plus-integral-plus-derivative (PID) controller**.

$$G_c(s) = K\frac{(s+z_{c_I})(s+z_{c_D})}{s} = K_P + \frac{K_I}{s} + K_D s$$

(A PID controller has two zeros plus a pole at the origin.)



- If first a passive lead compensator and then a passive lag compensator are designed, the resulting compensator is called a **lag-lead compensator**.

$$G_c(s) = K\frac{(s+z_{\mathrm{lag}})(s+z_{\mathrm{lead}})}{(s+z_{\mathrm{lag}})(s+z_{\mathrm{lead}})}$$

❖ PID controllers (PD + PI) are implemented with **active** networks, such as active amplifiers, and Lead-Lag compensators (Lead + Lag) are implemented with **passive** networks.