Ch10: Method of Virtual Work

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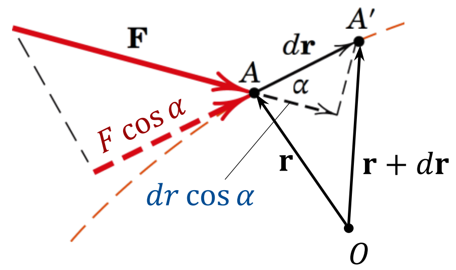
Work of a Force during an Infinitesimal Translation

Consider a particle that moves from a point A to a neighboring point A' while a force $\mathbf{F}$ is acting on it. The work dU done by the force $\mathbf{F}$ on the body during the infinitesimal displacement $d\mathbf{r}$ is defined as



$$\boxed{dU = \mathbf{F}\cdot d\mathbf{r}} = \underbrace{(F\cos\alpha)dr = (dr\cos\alpha)F}$$

Scalar (dot) Product

This can be considered as either the component of the force in the direction of the displacement times the displacement, or the component of the displacement in the direction of the force times the force.

Work is a scalar quantity that can either be positive or negative.

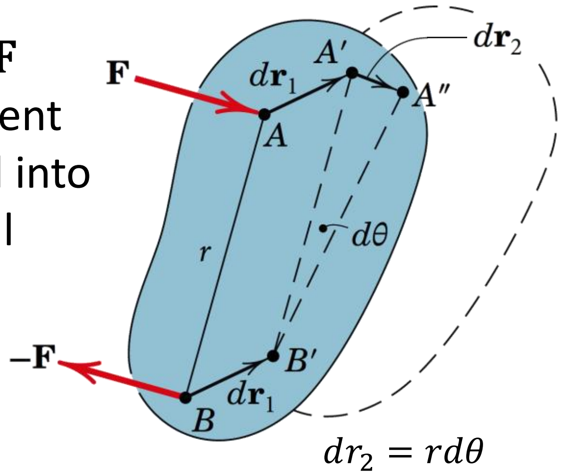
- If $\mathbf{F}$ has the same direction as $d\mathbf{r}$ ($\alpha = 0$), the work dU is positive.
- If $\mathbf{F}$ has a direction opposite to that of $d\mathbf{r}$ ($\alpha = \pi$), the work dU is negative.
- If $\mathbf{F}$ is perpendicular to $d\mathbf{r}$ ($\alpha = \pi/2$), the work dU is zero.

To distinguish between units of moment and work

❖ Unit of Work in the SI: joule J $\overbrace{(1\ \text{N·m} = 1\ \text{J})}$
❖ Unit of Work in the USCS: ft·lb

Work of a Couple Moment during an Infinitesimal Rotation

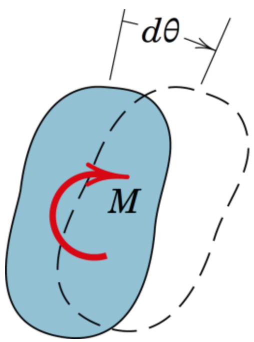
Consider a rigid body which is acted upon by the couple forces $\mathbf{F}$ and $-\mathbf{F}$ that produce a couple moment $\mathbf{M}$. Any small displacement of the rigid body bringing A and B into A'' and B' can be divided into two parts: **translation** $d\mathbf{r}_1$, and **rotation** through an infinitesimal angle $d\theta$ about B'.



$$dU = dU_{\mathbf{F}} + dU_{-\mathbf{F}} = \mathbf{F}\cdot d\mathbf{r}_1 + \mathbf{F}\cdot d\mathbf{r}_2 - \mathbf{F}\cdot d\mathbf{r}_1$$

$$= \mathbf{F}\cdot d\mathbf{r}_2 = Frd\theta = Md\theta$$

Thus, work of a couple moment is $\boxed{dU = Md\theta}$



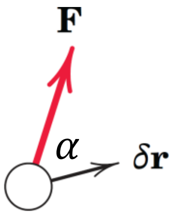
Note: $d\theta$ should be expressed in radians.

Note: If M and $d\theta$ have the same sense, the work is **positive**; however, if they have the opposite sense, the work will be **negative**.

Virtual Work

Consider an imaginary or **virtual displacement** $\delta\mathbf{r}$ of a particle in static equilibrium. The work δU of any force $\mathbf{F}$ on the particle along this virtual displacement is known as the **virtual work**:

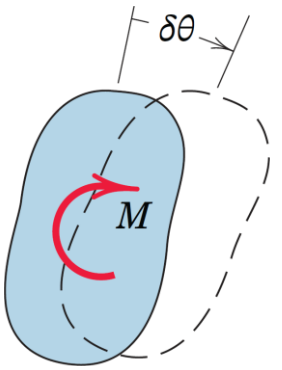
$$\delta U = \mathbf{F} \cdot \delta\mathbf{r} = F\cos\alpha\,\delta r$$



(The term **virtual** is used to indicate that the displacement does not really exist but only is assumed to exist.)

Similarly, the virtual work done by a couple M during a virtual angular displacement $\delta\theta$ is:

$$\delta U = M\delta\theta$$



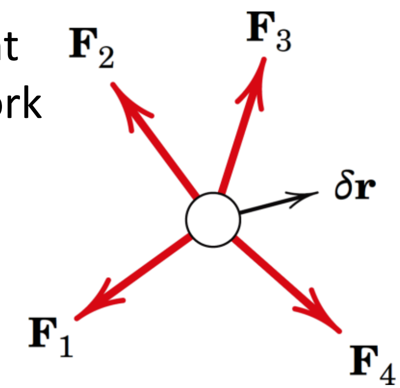
Note: dr (or $d\theta$) refers to an actual infinitesimal movement and can be integrated, whereas δr (or $\delta\theta$) refers to an infinitesimal virtual or assumed movement and cannot be integrated. Mathematically, both quantities are **first-order differential** quantities.

Principle of Virtual Work

Principle of Virtual Work

(Equilibrium of a Particle)

Consider the particle in equilibrium. For an assumed virtual displacement $\delta\mathbf{r}$ of the particle away from its **equilibrium** position, the total virtual work done on the particle can be written as



$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \mathbf{F}_3 \cdot \delta\mathbf{r} + \cdots = \Sigma\mathbf{F} \cdot \delta\mathbf{r} = 0$$

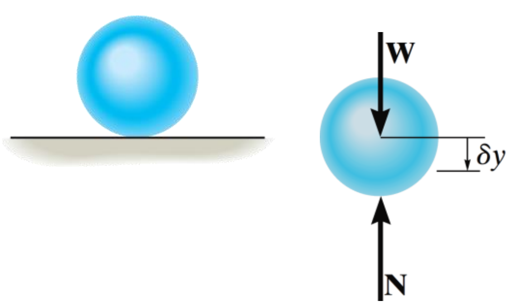
(since $\Sigma\mathbf{F} = \mathbf{0}$)

$$\Downarrow$$

★ If a particle is in equilibrium, the total virtual work of the forces acting on the particle for **any** virtual displacement of the particle is **zero**.

$$\boxed{\delta U = 0}$$

Example: If we "imagine" the ball is displaced downwards by a virtual amount δy:



$$\delta U = W\delta y - N\delta y$$
$$= (W - N)\delta y = 0$$
$$\Rightarrow W = N$$

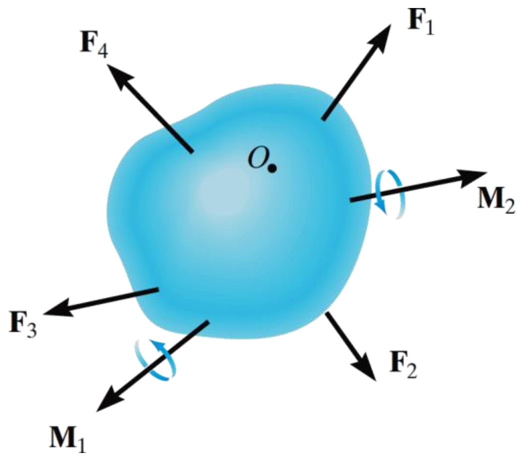
We get the same results using equilibrium equation $\Sigma F_y = 0$.

Principle of Virtual Work

(Equilibrium of a Rigid Body)

★ If a rigid body is in equilibrium, the total virtual work of the **external forces/moments** acting on the rigid body for **any** virtual displacement of the body is **zero**.

$$\delta U = 0$$

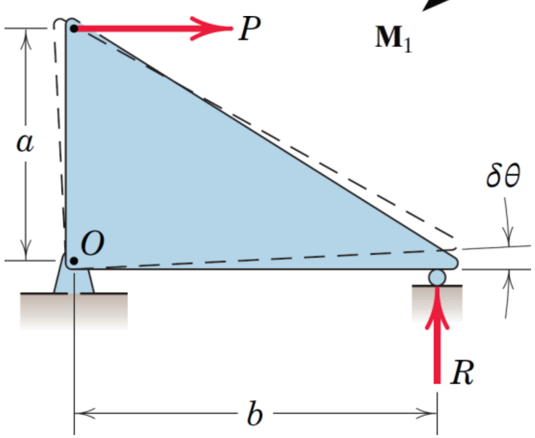


Example: If we "imagine" the hinged plate of negligible weight is rotated a virtual amount $\delta\theta$:

$$\delta U = -Pa\delta\theta + Rb\delta\theta = 0$$

$$\Rightarrow \quad Rb = Pa$$

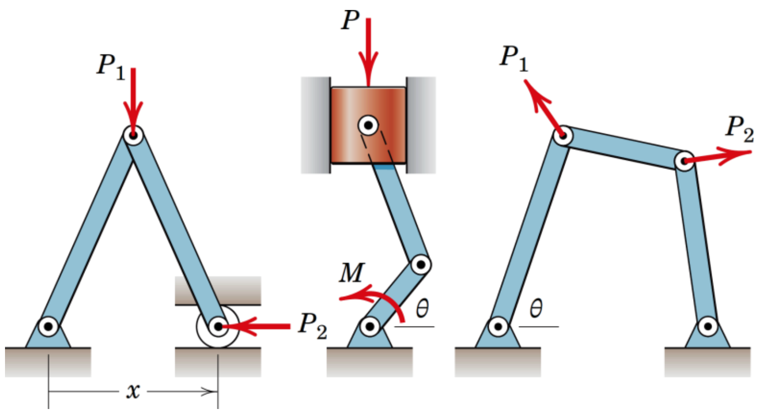
We get the same results using equilibrium equation $\Sigma M_O = 0$.



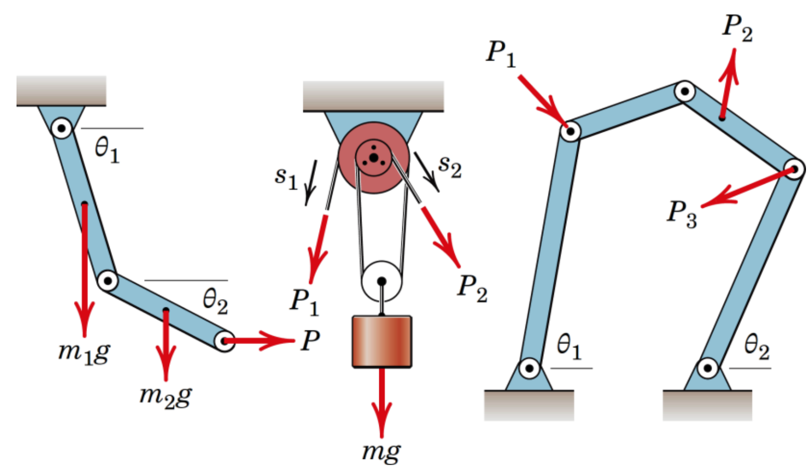
Degree of Freedom

The number of **degrees of freedom** (**DOF**) of a mechanical system is the number of **independent coordinates** needed to specify completely the configuration of the system.

Examples of one-degree-of-freedom systems:



Examples of two-degree-of-freedom systems:



Principle of Virtual Work

(Equilibrium of an Ideal System of Connected Rigid Bodies)

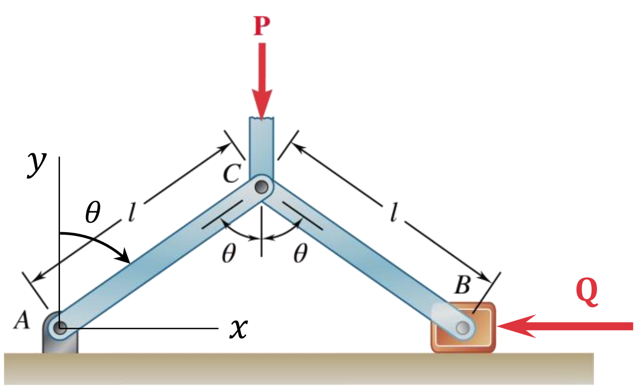
The method of virtual work is **particularly effective** for solving equilibrium problems that involve an **ideal system** of **several connected rigid bodies**.

The friction in the system is negligible.

★ In these systems, the total virtual work of the external forces/moments applied to the system for **any** virtual displacement consistent with the constraints of the system is **zero**.

$$\boxed{\delta U = 0}$$

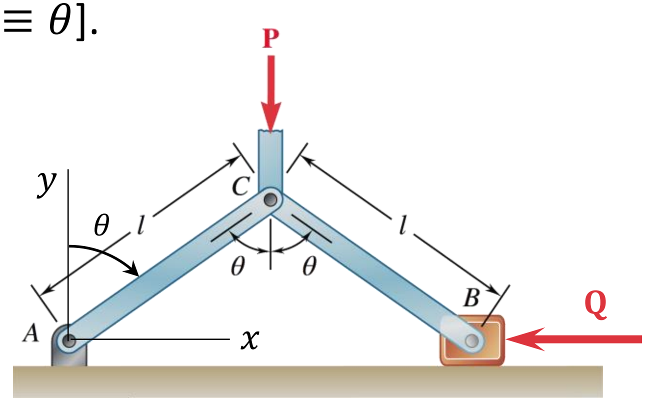
Example: Find a relation between P and Q when the mechanism is in equilibrium.

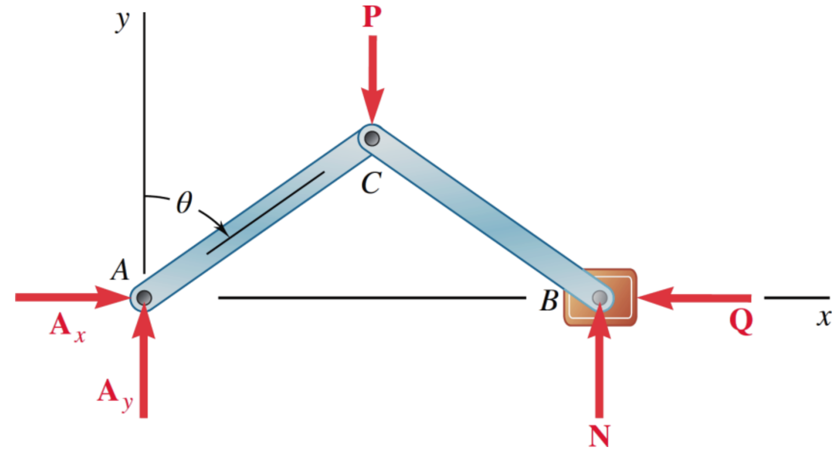


Principle of Virtual Work

(Equilibrium of an Ideal System of Connected Rigid Bodies) (cont.)

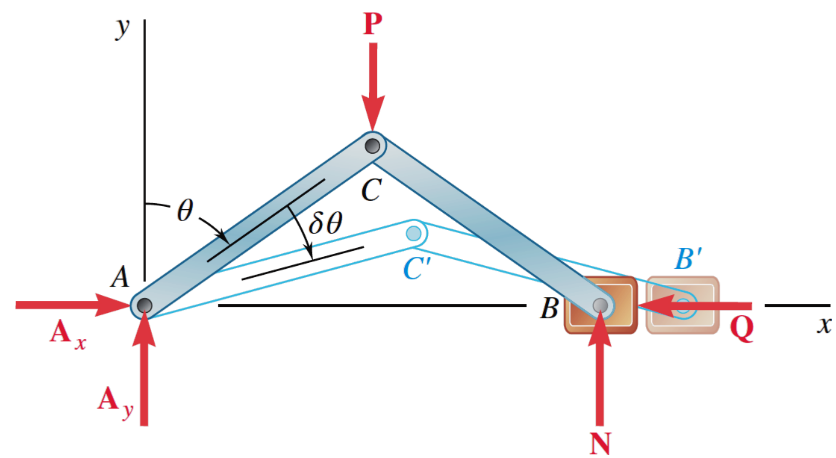
(1) Draw the free-body diagram of the system and define a coordinate q for virtual displacement [here: $q \equiv \theta$].





(2) Draw the system when it undergoes a **positive virtual displacement** δq [here: $\delta q \equiv \delta\theta$].

Note: The virtual displacements must be **consistent** with the **constraints** imposed by the supports and connections.

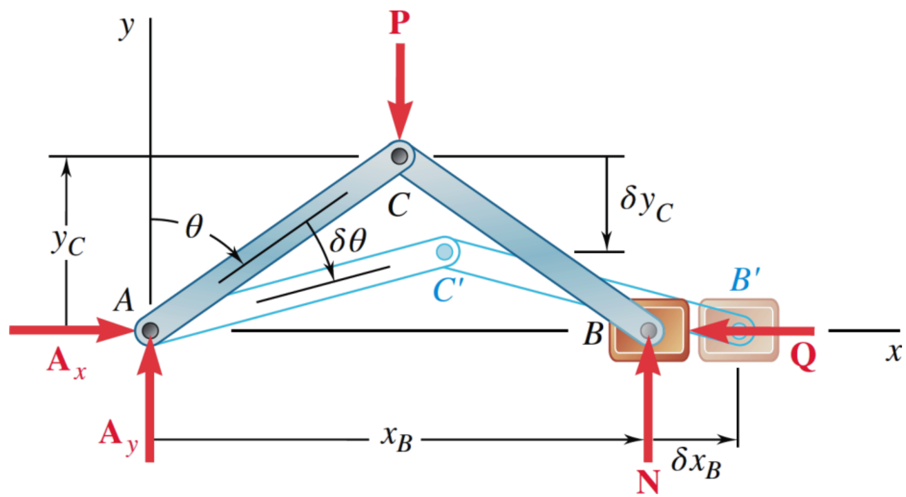


Principle of Virtual Work

(Equilibrium of an Ideal System of Connected Rigid Bodies) (cont.)

(3) Indicate position coordinates s of the point of application of all the **forces which do work**. s should be parallel to the line of action of these forces, measured from a fixed point [here: $s \equiv x_B, y_C$].

(4) Indicate the virtual displacement δs of all the **forces/moments which do work** [here: δx_B, δy_C].

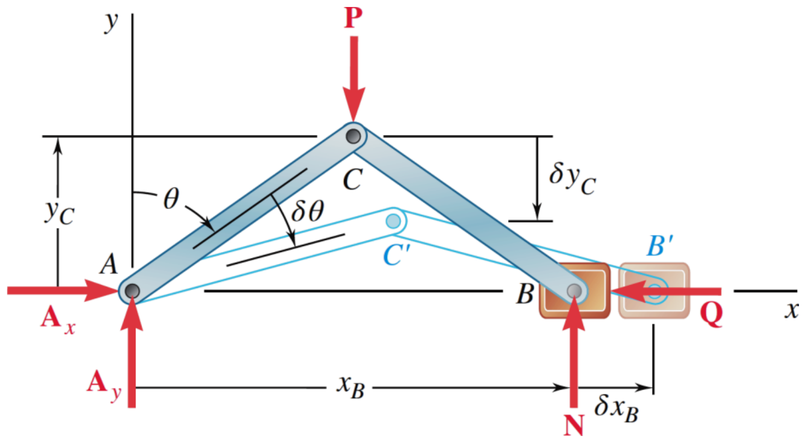


Note: The total work of the internal forces at the connections (like C) is zero; Moreover, $\delta U_{\text{Reactions}} = \delta U_{\mathbf{A}_x} = \delta U_{\mathbf{A}_y} = \delta U_{\mathbf{N}} = 0$. Thus, only $\mathbf{P}$ and $\mathbf{Q}$ do work in this virtual displacement.

Principle of Virtual Work

(Equilibrium of an Ideal System of Connected Rigid Bodies) (cont.)

(5) Relate each of the position coordinates s to the coordinate q; then, differentiate these expressions in order to express each virtual displacement δs in terms of δq.



$$x_B = 2l\sin\theta \longrightarrow \delta x_B = 2l\cos\theta\,\delta\theta$$

$$y_C = l\cos\theta \longrightarrow \delta y_C = -l\sin\theta\,\delta\theta$$

(6) Find the total virtual work of the system in equilibrium and set $\delta U = 0$ to solve for the unknown force, couple moment, or equilibrium position q.

Note: If a force or couple moment is in the same direction as the positive virtual displacement δs or δq, the work is **positive**. Otherwise, it is **negative**.

$$\delta U_P = P|\delta y_C| = Pl\sin\theta\,\delta\theta$$

$$\delta U_Q = -Q|\delta y_B| = -2Ql\cos\theta\,\delta\theta$$

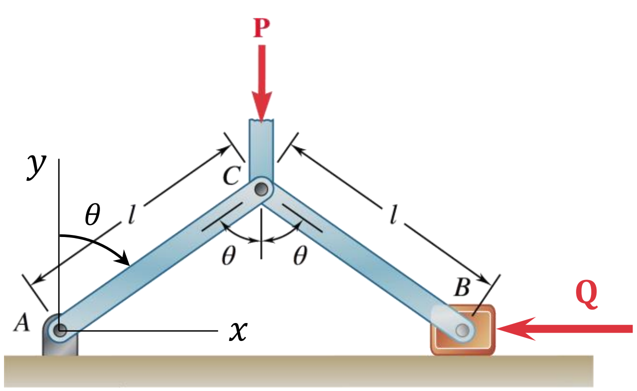
$$\delta U = \delta U_P + \delta U_Q = Pl\sin\theta\,\delta\theta - 2Ql\cos\theta\,\delta\theta = 0 \longrightarrow Q = \frac{1}{2}P\tan\theta$$

Remarks

The advantages of the method of virtual work over the conventional equilibrium equations:

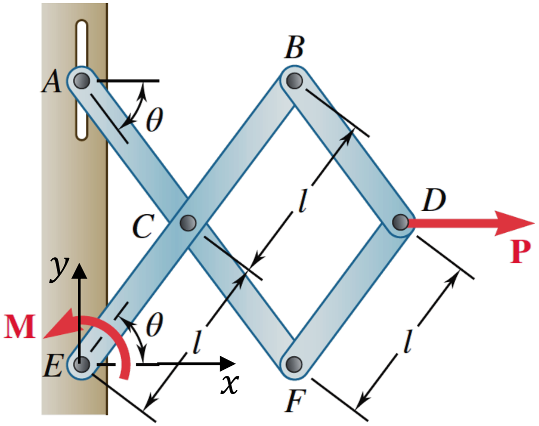
- The relations between the external forces/moments can be determined **directly** without reference to the reactions at supports.
- It is **not necessary** to **dismember** the systems in order to establish the relations between the forces/moments acting on each member.

- Try to solve the same problem using the methods of Chapter 6 and compare the solution and results.



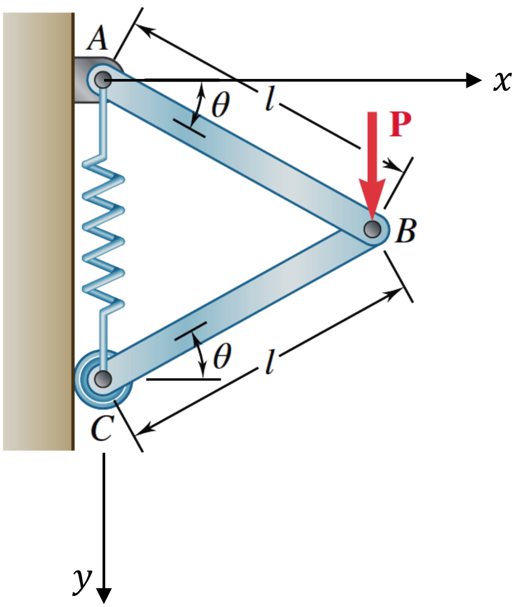
Sample Problem 10.1

Using the method of virtual work, determine the magnitude of the couple $\mathbf{M}$ required to maintain the equilibrium of the mechanism shown.



Sample Problem 10.2

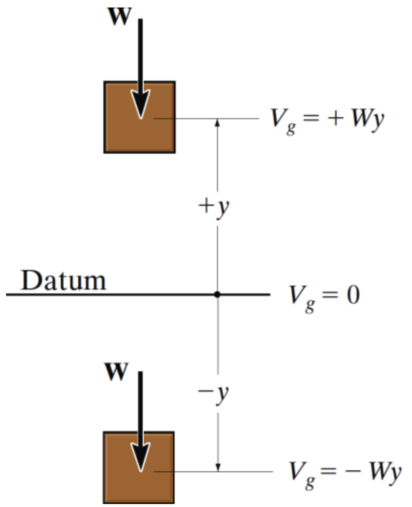
Using the method of virtual work, determine a relation between θ and P that corresponds to the equilibrium position of the mechanism. The unstretched length of the spring is h, and the spring constant is k. Neglect the weight of the mechanism.



Potential-Energy Method for Equilibrium

Potential Energy

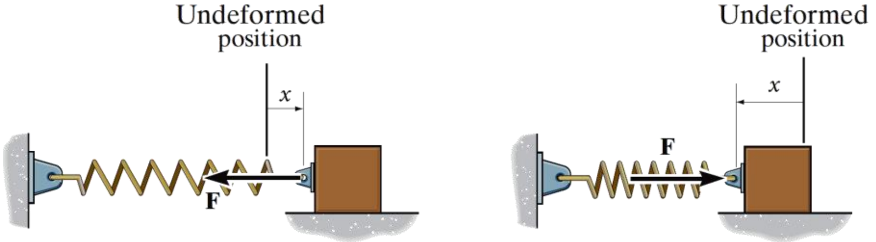
- **Gravitational Potential Energy** is defined as $V_g = Wy$
  - y is the distance of the body center of gravity from an **arbitrarily** datum (level).



Note: V_g can be positive, negative, or zero depending on the chosen datum.

- **Elastic Potential Energy** (for a linear spring $F = kx$) is defined as $V_e = \dfrac{1}{2}kx^2$
  - x is the deformation of the spring measured from its unstretched position.
  - k is the spring constant (Unit: N/m or lb/ft or lb/in.)

Note: V_e is always positive.



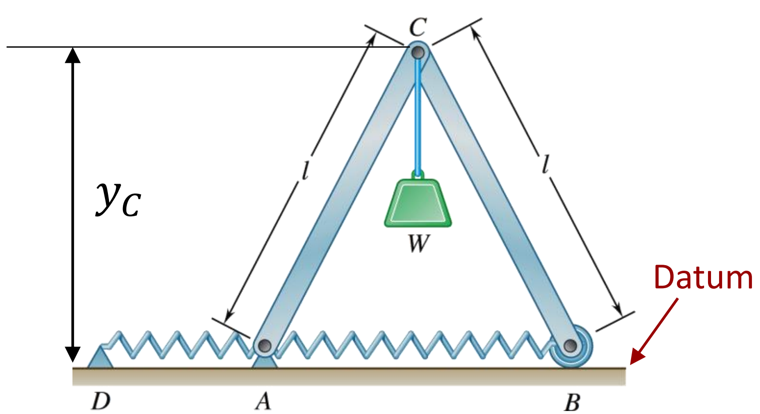
Note: Unit of the potential energy is the same as unit of work, i.e., J or ft·lb or in·lb.

Potential Energy (cont.)

Note: If a body is subjected to both gravitational and elastic forces, the total potential energy V of the body is

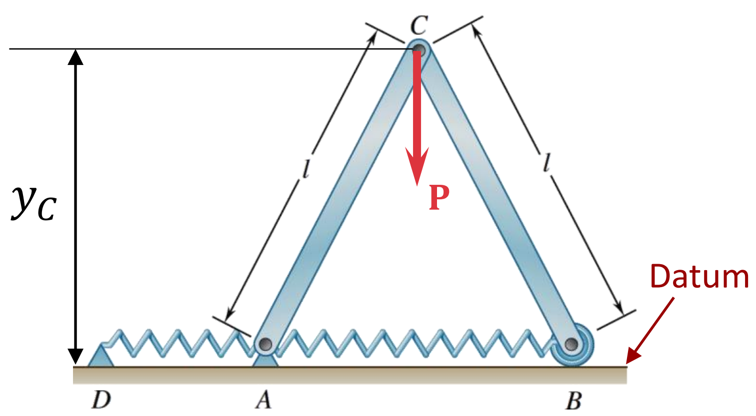
$$V = V_g + V_e$$

Note: We can use the potential energy V_g with <u>any vertical force</u> $\mathbf{P}$ of constant magnitude directed downward, i.e., $V_P = Py$ (where y is measured is from an arbitrarily datum):



$$V_g = Wy$$

Gravitational Potential Energy



$$V_P = Py$$

Potential Energy of Force $\mathbf{P}$

Potential Energy and Equilibrium
(Potential-Energy Method)

Consider an ideal (frictionless) one-degree-of-freedom system in equilibrium. If q is an independent coordinate specifying completely the configuration of the system, then the <u>total potential energy</u> V is a function of q, i.e., $V = V(q)$.

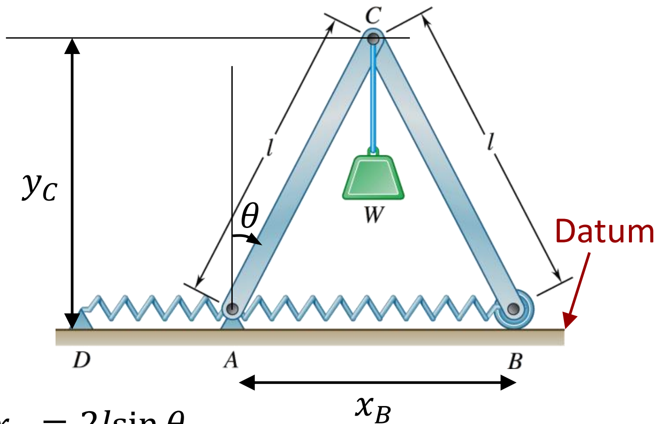
For this system, the principle of virtual work ($\delta U = 0$) in terms of potential energy can be expressed as

$$\boxed{\frac{dV(q)}{dq} = 0}$$

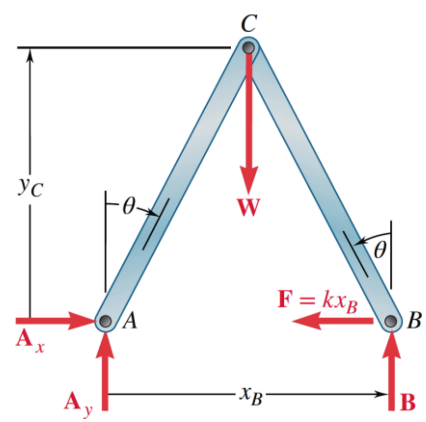
- This equation can be used to determine the **equilibrium positions** and the **stability** of a system.

Potential-Energy Method: Example

Consider a mechanism carrying a load $\mathbf{W}$. A spring BD connects B to a fixed point D and the natural length of the spring is equal to AD (the friction forces and the weights of the members are negligible).



$$x_B = 2l\sin\theta$$
$$y_C = l\cos\theta$$



By choosing an arbitrary datum and also θ as an independent coordinate, the total potential energy of the system is

$$V = V_g + V_e = Wy_C + \frac{1}{2}kx_B^2 = Wl\cos\theta + 2kl^2\sin^2\theta$$

The positions of equilibrium of the system is obtained by

$$\frac{dV(\theta)}{d\theta} = 0 \;\rightarrow -Wl\sin\theta + 4kl^2\sin\theta\cos\theta$$
$$\rightarrow l\sin\theta(4kl\cos\theta - W) = 0$$

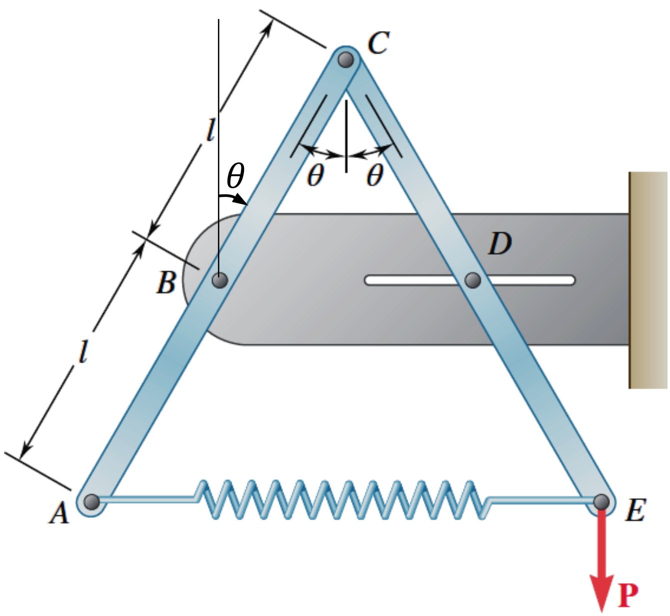
There are two positions of equilibrium:

$$\theta = 0, \qquad \theta = \cos^{-1}(W/4kl) \qquad (W \leq 4kl)$$

Which one is stable?

Problem 10.29/10.59

Two rods AC and CE are connected by a pin at C and by a spring AE. The constant of the spring is k, and the spring is unstretched when $\theta = 30°$. For the loading shown, derive an equation in P, θ, l, and k that must be satisfied when the system is in equilibrium, **(a)** using the Method of Virtual Work, **(b)** using the Potential-Energy Method.



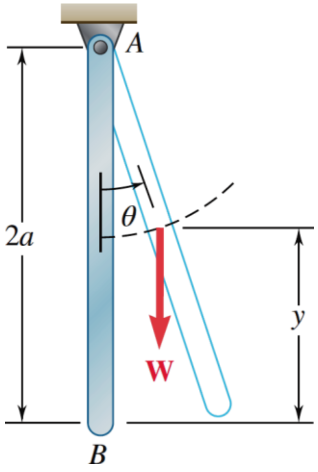
Stability of Equilibrium

Stability of Equilibrium

Consider three uniform rods with a length of $2a$ and weight W in **equilibrium**. Each rod is slightly displaced (disturbed) from its position of equilibrium.

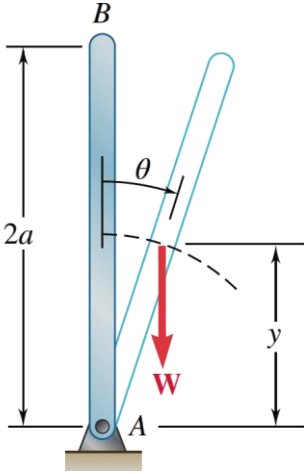
Stable Equilibrium:

- System has a tendency to **return** to its original position.
- The potential energy V of the system is at its **minimum**.



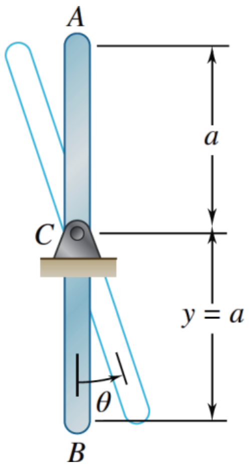
Unstable Equilibrium:

- System has a tendency to **keep moving away** from its original position.
- The potential energy V of the system is at its **maximum**.



Neutral Equilibrium:

- System has a tendency to **remains in its new position**.
- The potential energy V of the system is **constant**.



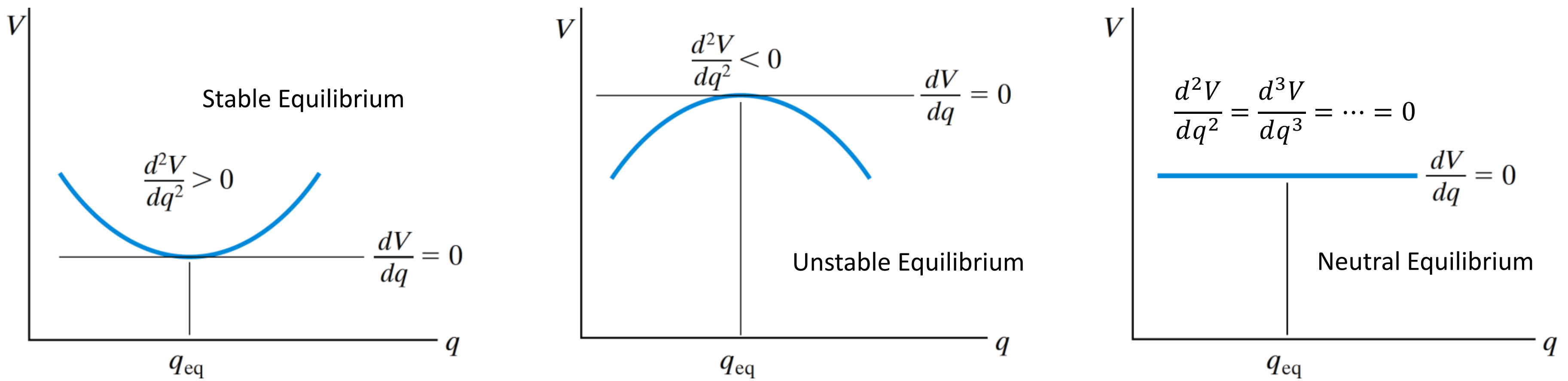
Stability of Equilibrium (cont.)

The conditions for the stability of a system with one degree of freedom <u>at the equilibrium position</u> q_{eq}:

If $\dfrac{dV}{dq} = 0, \quad \dfrac{d^2V}{dq^2} > 0 \Rightarrow$ **Stable Equilibrium**

If $\dfrac{dV}{dq} = 0, \quad \dfrac{d^2V}{dq^2} < 0 \Rightarrow$ **Unstable Equilibrium**

If $\dfrac{dV}{dq} = 0, \quad \dfrac{d^2V}{dq^2} = \dfrac{d^3V}{dq^3} = \cdots = 0 \quad \Rightarrow$ **Neutral Equilibrium**



Sample Problem 10.4

A 10-kg block is attached to the rim of a 300-mm-radius disk as shown. Knowing that spring BC is unstretched when $\theta = 0$, using the Potential-Energy Method, determine the position or positions of equilibrium, and state in each case whether the equilibrium is stable, unstable, or neutral.