

# Ch8: Steady-State Errors

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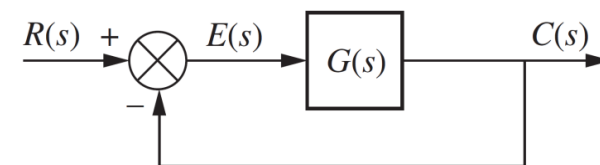
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# Definition

# Steady-State Error Definition

**Steady-State Error** is the difference between the input  $R(s)$  and the output  $C(s)$  as  $t \rightarrow \infty$  (i.e.,  $E(s) = R(s) - C(s)$ ). Note that this concept is applicable only to **stable** systems in which the **input and output have the same units**.

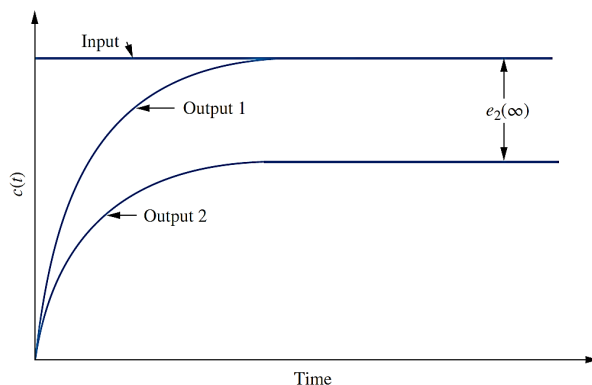
- **Test inputs**  $R(s)$  used for steady-state error analysis and design are **Step**, **Ramp**, and **Parabola**.



**Examples:**

**Step Input:**

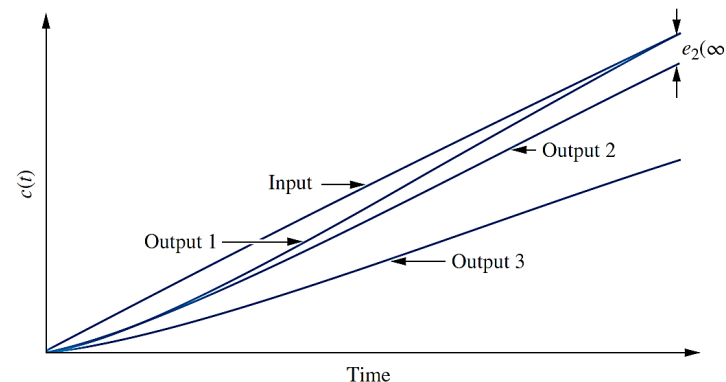
Output 1 has zero steady-state error,  
Output 2 has a finite steady-state error,  $e_2(\infty)$ .



**Note:** Errors are measured **vertically** between the input and output as  $t \rightarrow \infty$ .

**Ramp Input:**

Output 1 has zero steady-state error,  
Output 2 has a finite steady-state error,  $e_2(\infty)$ ,  
Output 3 has an infinite steady-state error.



# Sources of Steady-State Error

Steady-state errors in control systems arise from two sources:

- (1) Imperfections and nonlinearities in the system components, such as backlash in gears, static friction, or motor dead zone (when a motor will not move unless the input voltage exceeds a threshold).
- (2) The configuration (type) of open-loop transfer function of the system  $G(s)$  itself and the type of applied input  $R(s)$ . This kind of error will be investigated in this chapter.

Any physical control system inherently suffers steady-state error in response to **certain types of inputs**. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. The only way we may be able to eliminate this error is to **modify the system structure**.

# Steady-State Error for Unity Feedback Systems

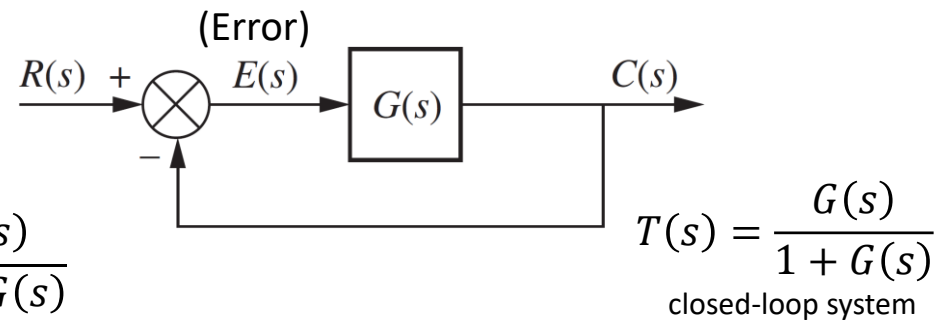
# Steady-State Error in Terms of $G(s)$

Consider the feedback control system which has **unity negative feedback** (i.e.,  $H(s) = 1$ ).

$$\left. \begin{aligned} E(s) &= R(s) - C(s) \\ C(s) &= E(s)G(s) \end{aligned} \right\} E(s) = \frac{R(s)}{1 + G(s)}$$

(or)

$$E(s) = R(s) - C(s) = R(s) - R(s)T(s) = \frac{R(s)}{1 + G(s)}$$



**Note:** You should always first make sure the closed-loop system  $T(s)$  is **stable**, using a method like the Routh-Hurwitz criterion.

Since we are interested in the final value of the error,  $e(\infty)$ , the **final value theorem**\* is used:

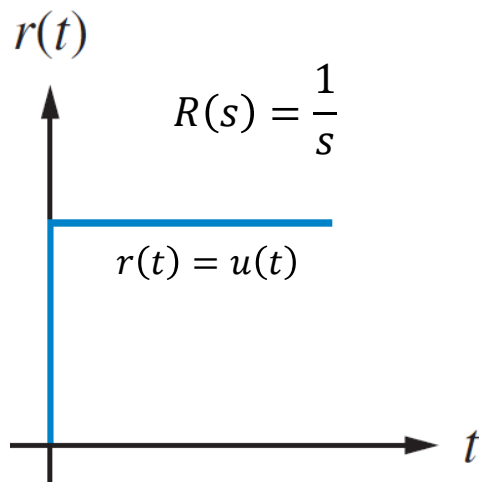
$$e(\infty) = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \quad \Rightarrow \quad e(\infty) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

\* **Note:** The final value theorem is valid only if  $E(s)$  has poles only in the left half-plane and, at most, one pole at the origin. For infinite steady-state errors, the final value theorem is also valid if  $E(s)$  has more than one pole at the origin.

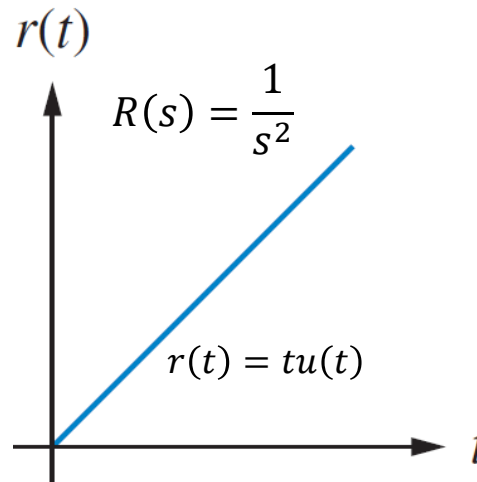
# Steady-State Error for Three Test Inputs

Now, the relationships between the open-loop system,  $G(s)$ , and the nature of the steady-state error,  $e(\infty)$ , is studied for the three test inputs.

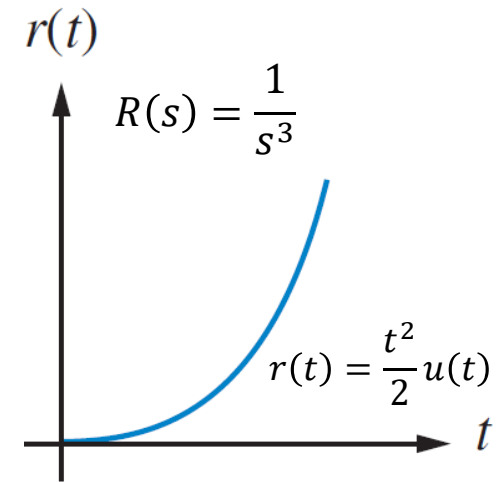
1. Step



2. Ramp



3. Parabola



# 1. Step Input

$$R(s) = \frac{1}{s} \rightarrow e_{\text{step}}(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{1}{1 + K_p}$$

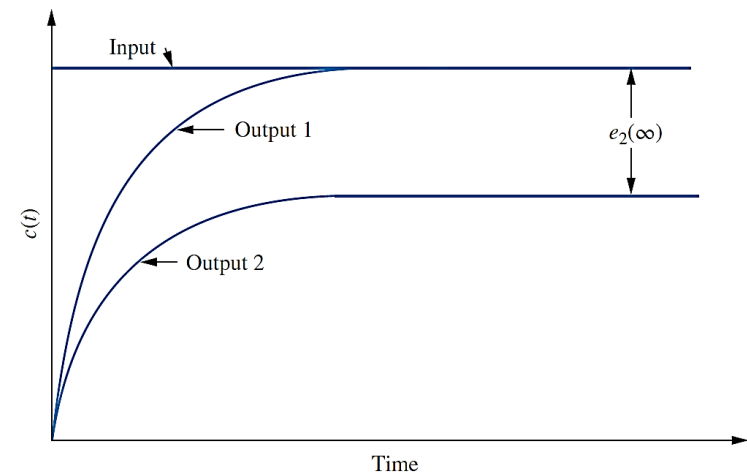
$K_p = \lim_{s \rightarrow 0} G(s)$  is called **Static Position Error Constant**.

$G(0)$  is called DC gain of the forward transfer function.

- To have zero steady-state error (Output 1), we need  $K_p = \lim_{s \rightarrow 0} G(s) = \infty$ . Hence,  $G(s)$  must have the following form:

$$G(s) = \frac{(s + z_1)(s + z_2) \cdots}{s^N (s + p_1)(s + p_2) \cdots} \quad N \geq 1$$

i.e., at least one pole must be at the origin.



- If  $N = 0$ , then  $K_p = z_1 z_2 \cdots / p_1 p_2 \cdots$  and there is a constant steady-state error (Output 2).



## 2. Ramp Input

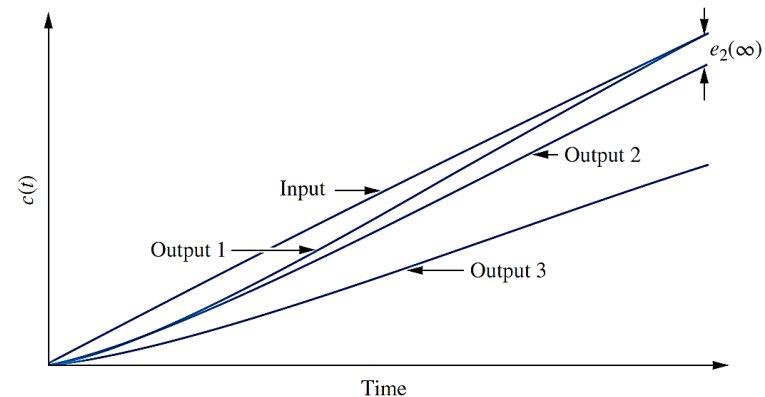
$$R(s) = \frac{1}{s^2} \rightarrow e_{\text{ramp}}(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s^2)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{K_v}$$

$K_v = \lim_{s \rightarrow 0} sG(s)$  is called **Static Velocity Error Constant**.

- To have zero steady-state error (Output 1), we need  $K_v = \lim_{s \rightarrow 0} sG(s) = \infty$ . Hence,  $G(s)$  must have the following form:

$$G(s) = \frac{(s + z_1)(s + z_2) \cdots}{s^N (s + p_1)(s + p_2) \cdots} \quad N \geq 2$$

i.e., at least two pole must be at the origin.



- If  $N = 1$ , then  $K_v = z_1 z_2 \dots / p_1 p_2 \dots$  and there is a constant steady-state error (Output 2).
- If  $N = 0$ , then  $K_v = 0$  and the steady-state error would be infinite (Output 3).

### 3. Parabolic Input

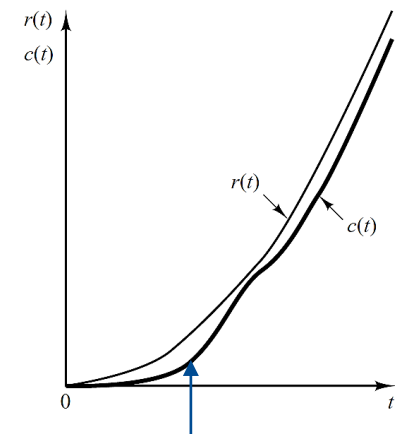
$$R(s) = \frac{1}{s^3} \rightarrow e_{\text{parabol}}(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s^3)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)} = \frac{1}{K_a}$$

$K_a = \lim_{s \rightarrow 0} s^2 G(s)$  is called **Static Acceleration Error Constant**.

- To have zero steady-state error, we need  $K_a = \lim_{s \rightarrow 0} s^2 G(s) = \infty$ . Hence,  $G(s)$  must have the following form:

$$G(s) = \frac{(s + z_1)(s + z_2) \cdots}{s^N (s + p_1)(s + p_2) \cdots} \quad N \geq 3$$

i.e., at least three pole must be at the origin.

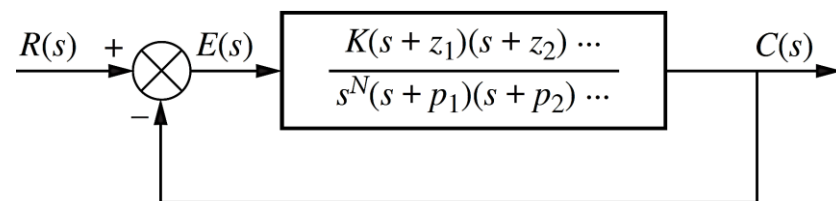


- If  $N = 2$ , then  $K_a = z_1 z_2 \dots / p_1 p_2 \dots$  and there is a constant steady-state error.
- If  $N = 1$  or  $N = 0$ , then  $K_a = 0$  and the steady-state error would be infinite.

# System Type and Steady-State Error

Since static error constants (position, velocity, and acceleration) and consequently steady-state errors depend upon the form of  $G(s)$ , **System Type** is defined to be the value of  $N$  in the denominator of  $G(s)$  in the unity feedback system.

A system  $G(s)$  is called **Type 0**, **Type 1**, **Type 2**, ..., if  $N = 0$ ,  $N = 1$ ,  $N = 2$ , ..., respectively.

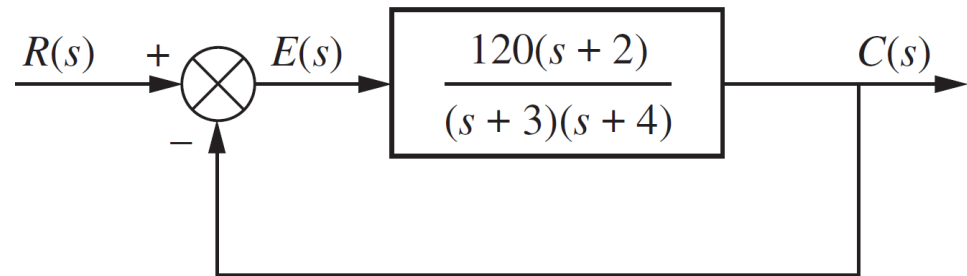


- This classification is different from that of the **order of a system**.
- As the type number is increased, **accuracy** is improved; however, **stability** is aggravated.

Input	Type 0		Type 1		Type 2	
	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$K_p = \text{Constant}$	$\frac{1}{1 + K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$K_v = 0$	$\infty$	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$K_a = 0$	$\infty$	$K_a = 0$	$\infty$	$K_a = \text{Constant}$	$\frac{1}{K_a}$

# Example

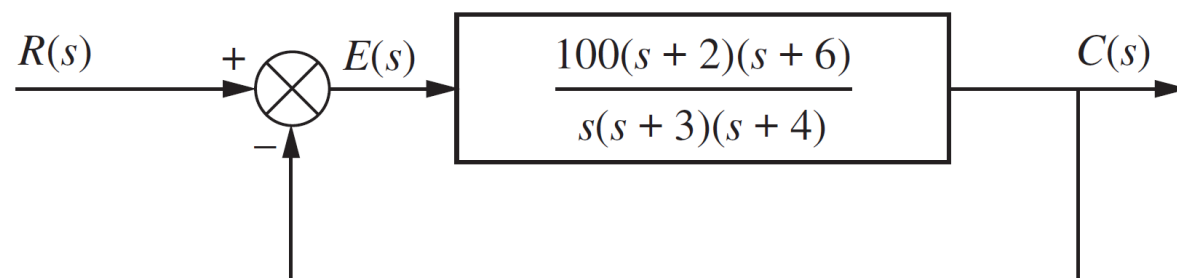
Find the steady-state errors for inputs of  $5u(t)$ ,  $5t$ , and  $5t^2$  to the system.



**Answer:**  $e_{\text{step}}(\infty) = \frac{5}{21}$ ,  $e_{\text{ramp}}(\infty) = \infty$ ,  $e_{\text{parabola}}(\infty) = \infty$

# Example

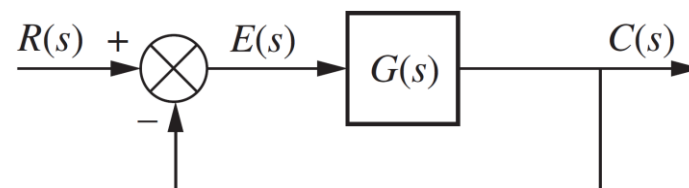
Find the steady-state errors for inputs of  $5u(t)$ ,  $5t$ , and  $5t^2$  to the system.



**Answer:**  $e_{\text{step}}(\infty) = 0, e_{\text{ramp}}(\infty) = \frac{1}{20}, e_{\text{parabola}}(\infty) = \infty$

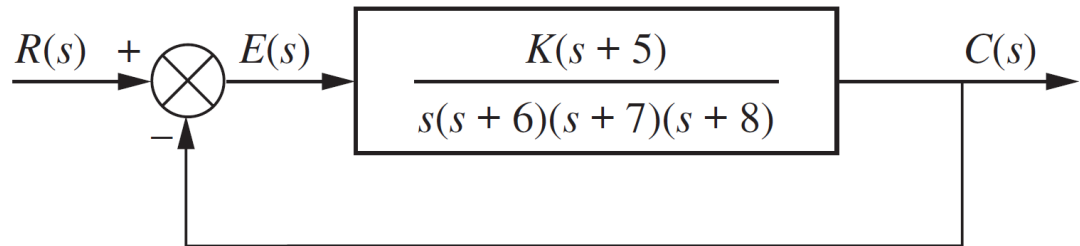
# Steady-State Error Specifications

- Similar to the parameters represent performance specifications for the **transient response** (e.g., damping ratio  $\zeta$ , settling time  $T_s$ , peak time  $T_p$ , and maximum overshoot  $M_p$ ), Static Error Constants (e.g., position constant  $K_p$ , velocity constant  $K_v$ , and acceleration constant  $K_a$ ) represent **steady-state error** performance specifications.
- The finite steady-state errors decreases as the static error constants increases.
- A wealth of information is contained within the specification of a static error constant. For example, if a control system has the specification  $K_v = 1000$ , we can draw several conclusions:
  - The system is **stable**.
  - The system is of **Type 1**.
  - A **ramp input** is the test signal.
  - The steady-state error is  $1/K_v$ .



# Example

Find the value of  $K$  so that there is 10% error in the steady state for ramp input.



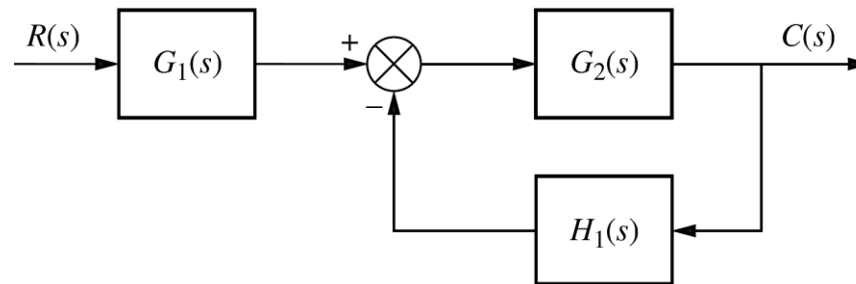
**Answer:**  $K = 672$

# Steady-State Error for Nonunity Feedback Systems



# Nonunity Feedback Systems

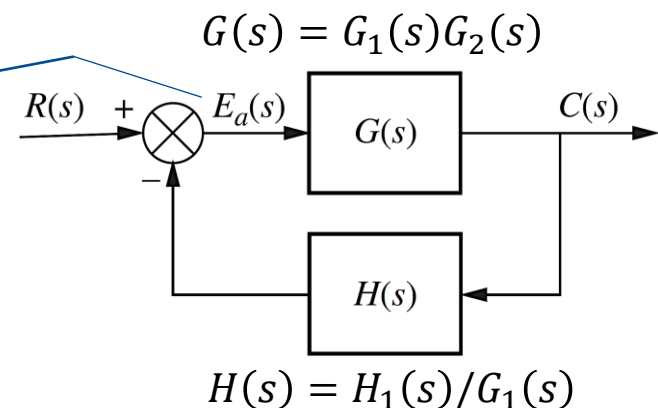
Control systems often do not have unity feedback. A general feedback system including the input transducer  $G_1(s)$ , controller and plant  $G_2(s)$ , and feedback  $H_1(s)$ , is shown.



Pushing the input transducer  $G_1(s)$  to the right past the summing junction yields another form of a general nonunity feedback system.

$E_a(s)$  is called **Actuating Signal** (not Error!):

$$\left. \begin{aligned} E_a(s) &= R(s) - C(s)H(s) \\ E_a(s)G(s) &= C(s) \end{aligned} \right\} E_a(s) = \frac{R(s)}{1 + G(s)H(s)}$$

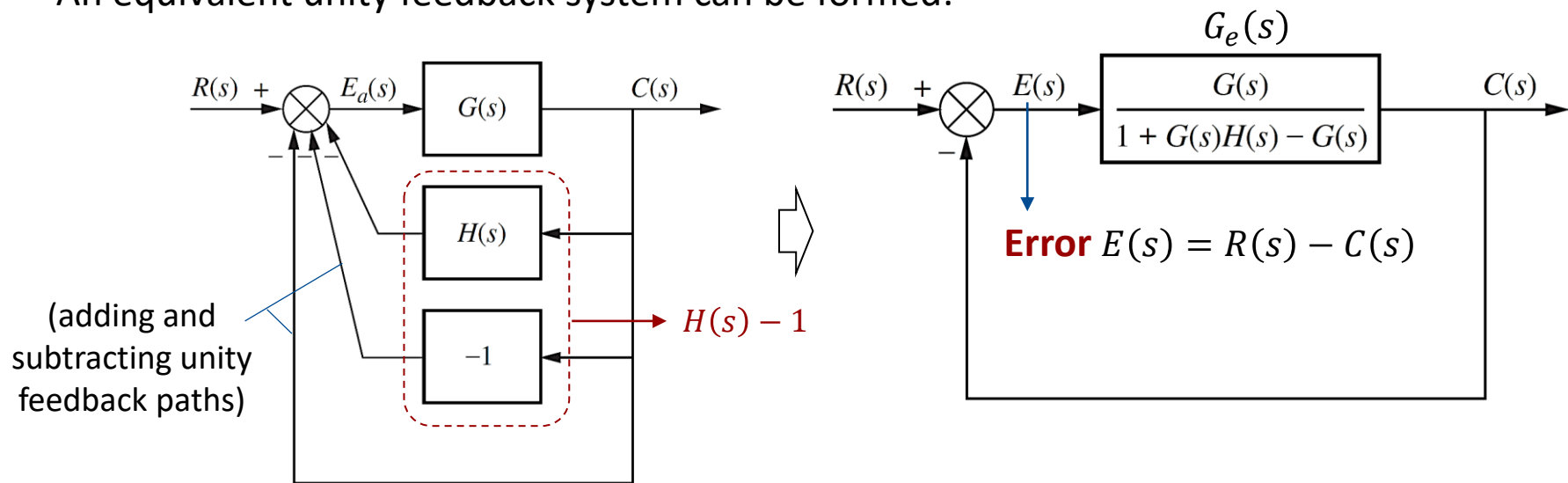


# Nonunity Feedback Systems

For Nonunity Feedback Systems we can find two steady state specifications:

1. **Steady-State Actuating Signal**  $e_a(\infty)$ : 
$$e_a(\infty) = \lim_{s \rightarrow 0} sE_a(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)H(s)}$$
2. **Steady-State Error**  $e(\infty)$  (only if input and output units are the same):

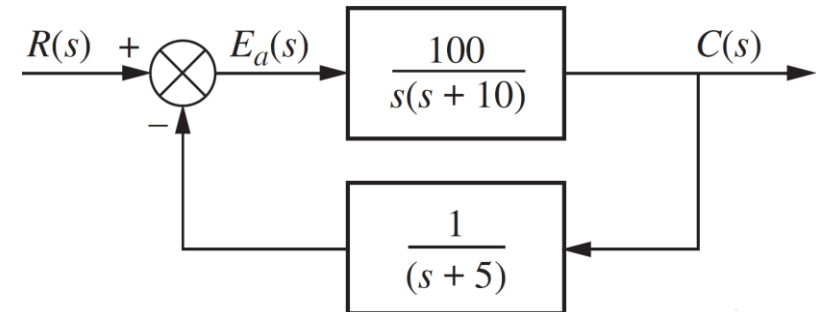
An equivalent unity feedback system can be formed:



After converting a nonunity feedback system to an equivalent unity feedback system, we can use previous methods for  $G_e(s)$  for finding the Steady-State Error  $e(\infty)$ .

# Example

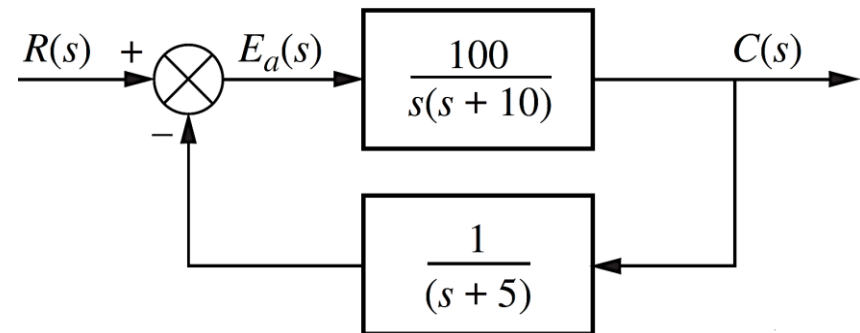
For the following system, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same.



**Answer:** Type 0,  $K_p = -5/4$ ,  $e(\infty) = -4$

# Example

Find the steady-state actuating signal for the system shown for a unit step input. Repeat for a unit ramp input.



**Answer:**  $e_{a,\text{step}}(\infty) = 0$ ,  $e_{a,\text{ramp}}(\infty) = \frac{1}{2}$

# Sensitivity

# Sensitivity

The degree to which changes in system parameters affect system transfer functions, and hence performance, is called **Sensitivity**. Ideally, parameter changes due to heat or other causes should not appreciably affect a system's performance. A system with zero sensitivity (i.e., changes in the system parameters have no effect on the transfer function) is ideal. The greater the sensitivity, the less desirable the effect of a parameter change.

- Sensitivity of function  $F$  to changes in parameter  $P$ :

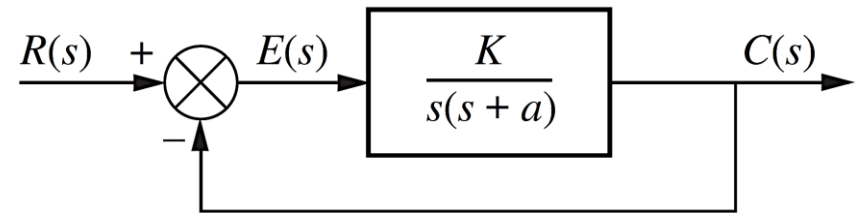
$$S_P^F = S_{F:P} = \lim_{\Delta P \rightarrow 0} \frac{\text{Fractional change in the function (F)}}{\text{Fractional change in the parameter (P)}} = \lim_{\Delta P \rightarrow 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \rightarrow 0} \frac{P}{F} \frac{\Delta F}{\Delta P}$$

$$S_P^F = S_{F:P} = \frac{P}{F} \frac{\delta F}{\delta P}$$

- In general, feedback reduces the sensitivity to parameter changes.

# Example

Given the following system, (a) calculate the sensitivity of the closed-loop transfer function to changes in the parameter  $a$ . (b) How would you reduce the sensitivity?



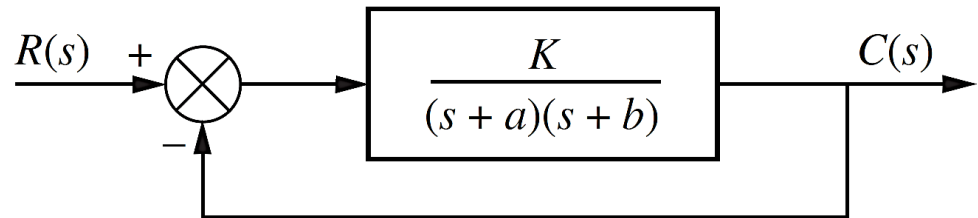
**Answer:**

$$(a) S_{T:a} = \frac{-as}{s^2 + as + K}$$

(b) By increasing  $K$ .

# Example

Find the sensitivity of the steady-state error to changes in parameter  $K$  and parameter  $a$  for the following system with a step input.



**Answer:**  $S_{e:K} = \frac{-K}{ab + K}$  ,  $S_{e:a} = \frac{K}{ab + K}$



# Using MATLAB and Control System Toolbox

# Finding Static Error Constants Using dcgain

```
s = tf('s');  
G = 500*(s+2)*(s+4)*(s+5)*(s+6)*(s+7)/...  
(s^2*(s+8)*(s+10)*(s+12));
```

% Check Stability

```
T = feedback(G,1);  
poles = pole(T)
```

% Step Input

```
Kp = dcgain(G);  
ess_step = 1/(1+Kp)
```

% Ramp Input

```
Kv = dcgain(s*G);  
ess_ramp = 1/Kv
```

% Parabolic Input

```
Ka = dcgain(s^2*G);  
ess_parabolic = 1/Ka
```

