MEC529: Introduction to Robotics (Theory and Applications)

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Ch1: Motivational Example

Linear Algebra

Vector

A **coordinate free vector** is a geometric quantity with a length and a direction.

Given a reference frame, v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then, the vector v can be represented by its coordinates $\boldsymbol{v}$ in the reference frame.

- v refers to a physical quantity in the underlying space.
- $\boldsymbol{v}$ is a representation of v that depends on the choice of coordinate frame.

Vector

$\boldsymbol{x} \in \mathbb{R}^n$: (an n-dimensional vector)

$\mathbb{R}^n$: n-dimensional real space (Euclidian Space)

$\boldsymbol{x}, \boldsymbol{x}^T$:

Norm of a vector $\boldsymbol{x}$:

Point and Its Coordinate

- **Point**: p denotes a point in the physical space
- A point p can be represented by as a vector from frame origin to p
- $\boldsymbol{p}$ denotes the coordinate of a point p
- The coordinate $\boldsymbol{p}$ depends on the choice of reference frame

Matrix

$\mathbf{A} \in \mathbb{R}^{m \times n}$:

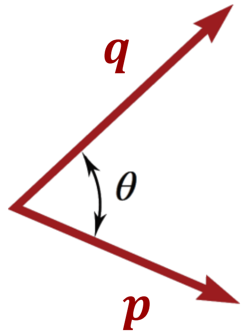
Symmetric matrix:

Skew-symmetric matrix:

Matrix vector multiplication as linear combination of columns:

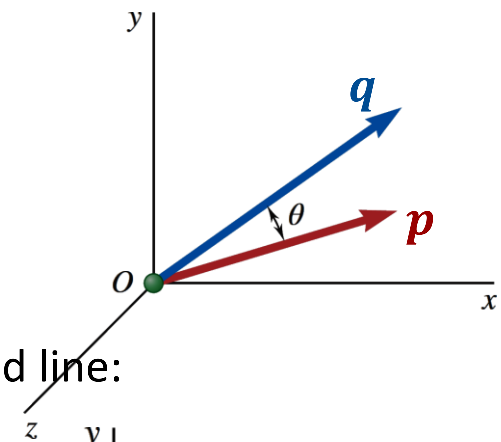
Dot Product (or Scalar Product or Inner Product)

Dot Product of two vectors $\boldsymbol{p} \in \mathbb{R}^n$, $\boldsymbol{q} \in \mathbb{R}^n$ is defined as the scalar $\boldsymbol{p} \cdot \boldsymbol{q}$.

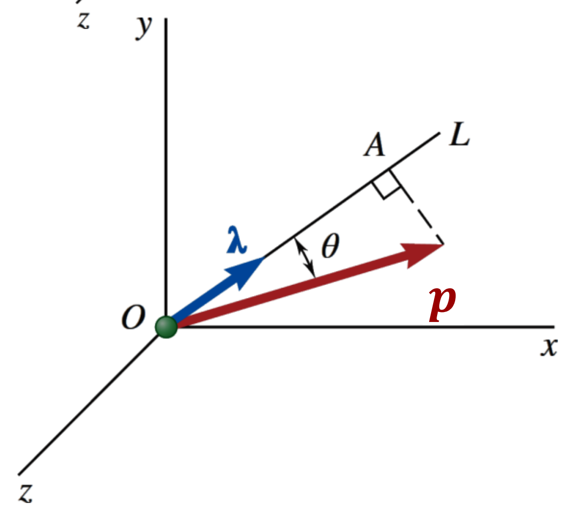


Applications of Dot Product

(1) Finding angle formed between two given vectors $\boldsymbol{p} \in \mathbb{R}^n$, $\boldsymbol{q} \in \mathbb{R}^n$ (or intersecting lines):



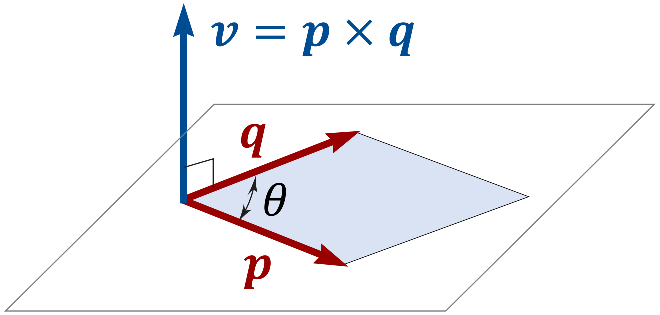
(2) Finding projection of a vector $\boldsymbol{p} \in \mathbb{R}^n$ on a given axis or directed line:



$\boldsymbol{\lambda}$: unit vector of line L

Cross Product (or Vector Product)

Cross product of $\boldsymbol{p} \in \mathbb{R}^3$, $\boldsymbol{q} \in \mathbb{R}^3$ is defined as the vector $\boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{q}$ where $\boldsymbol{v} \in \mathbb{R}^3$.



Cross Product (or Vector Product)

Coordinate notation: $\boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{q} = (p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}) \times (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k})$

Matrix notation: $\boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}$

Cross Product as a Matrix-Vector Multiplication

Cross product $\boldsymbol{p} \times \boldsymbol{q}$ ($\boldsymbol{p} \in \mathbb{R}^3$, $\boldsymbol{q} \in \mathbb{R}^3$) can be thought of as a multiplication of vector $\boldsymbol{q}$ by a 3x3 skew-symmetric matrix $[\boldsymbol{p}]$.

$$\boldsymbol{p} \times \boldsymbol{q} = \underbrace{\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}}_{[\boldsymbol{p}]} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = [\boldsymbol{p}]\boldsymbol{q}$$

$\boldsymbol{p} = (p_x, p_y, p_z), \boldsymbol{q} = (q_x, q_y, q_z)$

The matrix $[\boldsymbol{p}]$ is a <u>3x3 skew-symmetric matrix representation</u> of $\boldsymbol{p}$.

$$[\boldsymbol{p}] = -[\boldsymbol{p}]^T$$

2R Planar Manipulator

RR (or 2R) Planar Manipulator

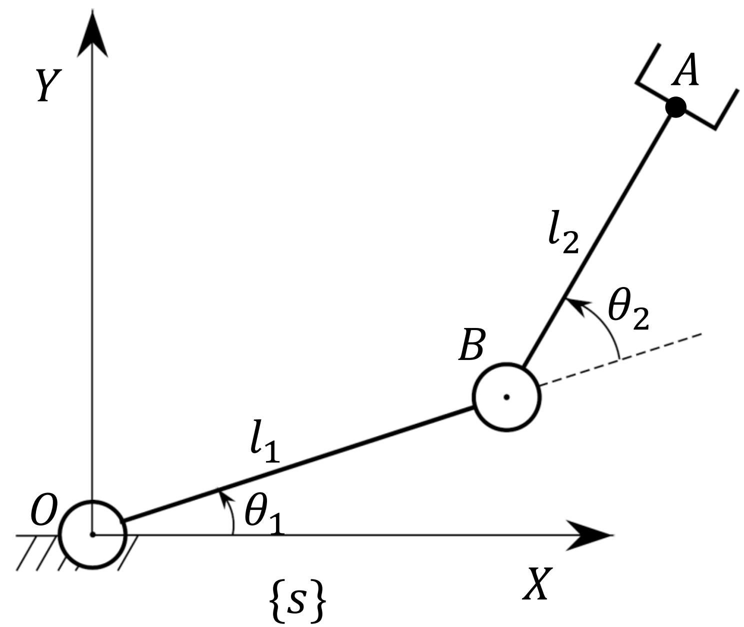
(θ_1, θ_2): Joint angles

(x, y): Position of end-effector (point A)

$\{s\}$: Base frame of manipulator

l_1: Length of link 1

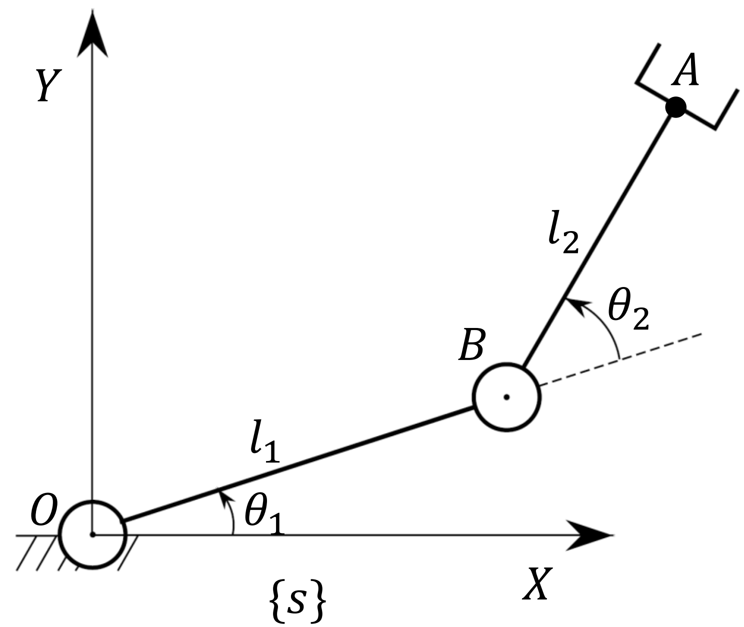
l_2: Length of link 2



Position Kinematics

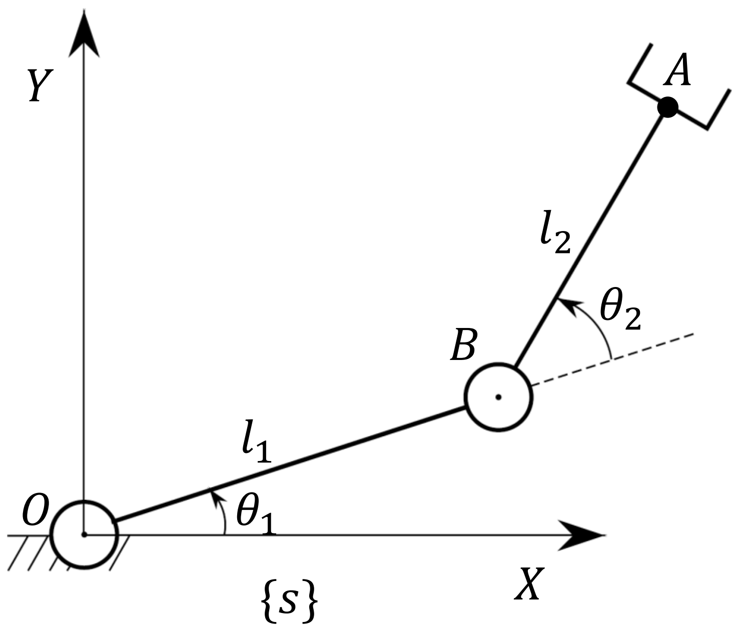
Relation between Joint Angles and End-Effector Position

What is the relationship between the joint angles, (θ_1, θ_2), and the position of the end effector point A, (x, y), in the base frame $\{s\}$?



Forward (Direct) Position Kinematics

Given the joint angles, (θ_1, θ_2), of the 2R robot, find the position, (x, y) of the end-effector point A, in the base frame $\{s\}$.



Forward (Direct) Position Kinematics

$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$: Vector of joint angles

$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix}$: Position vector of end-effector point

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \equiv f_1(\theta_1, \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \equiv f_2(\theta_1, \theta_2)$$

$$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$

More abstractly, the forward kinematics map is

$$\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{\theta})$$

$$\boldsymbol{f}(\boldsymbol{\theta}) = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$

where $\boldsymbol{f}$ is a vector function.

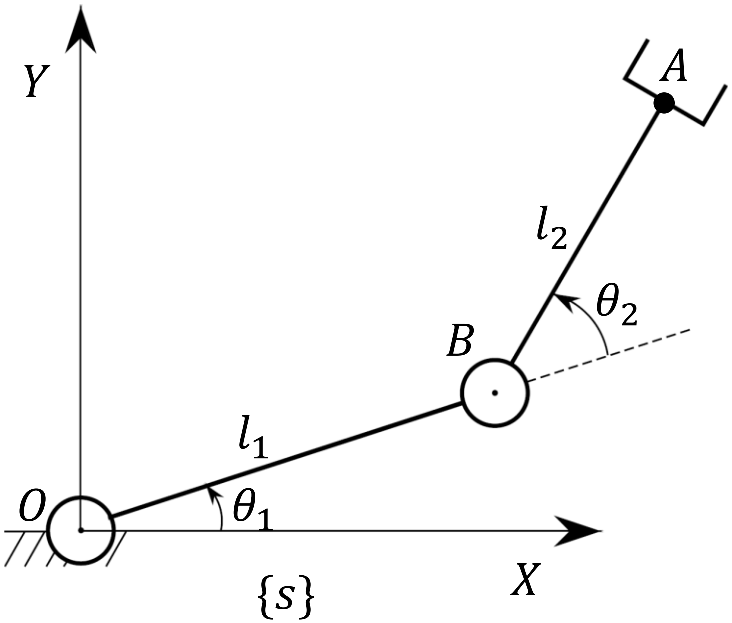
Inverse Position Kinematics

Given the position, (x, y), of the end effector point A, find the joint angles, (θ_1, θ_2) so that the position (x, y) is reached.

In other words, from the equations

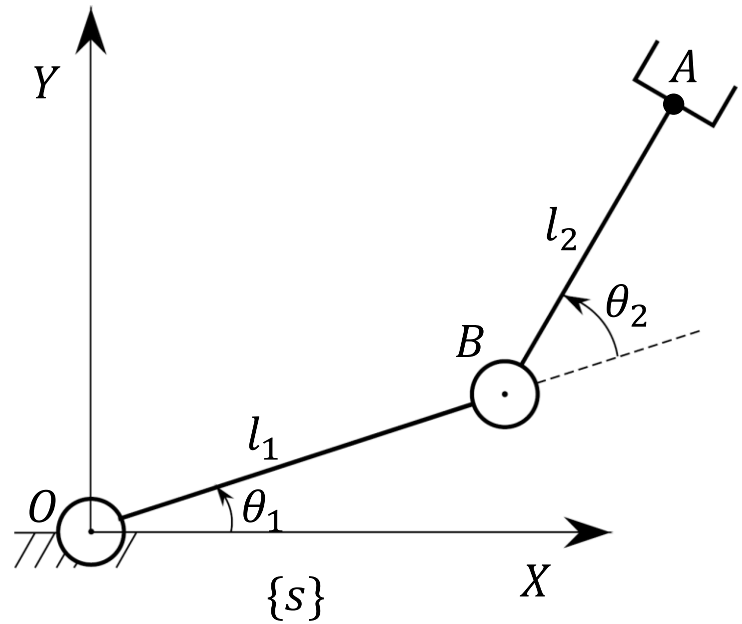
$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Find θ_1 and θ_2 as a function of x and y.



Numerical Example (Exercise)

Forward and Inverse Position Kinematics:



$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$\theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-u^2}, u\right)$$
$$\theta_1 = \operatorname{atan2}(y, x) - \operatorname{atan2}(l_2 \sin\theta_2, l_1 + l_2 \cos\theta_2)$$

$$u = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

Velocity Kinematics

Relation between Joint Angle Rates and End-Effector Velocity

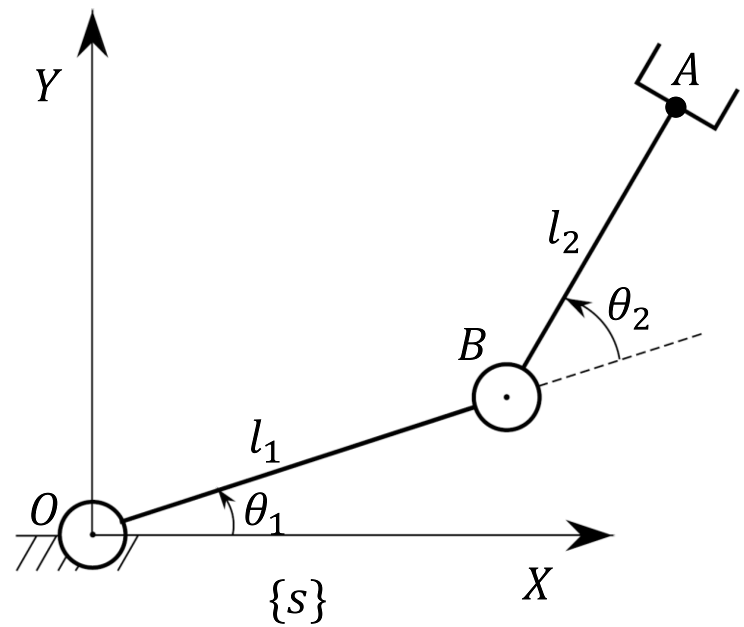
What is the relationship between the joint angle rates of motion $(\dot{\theta}_1, \dot{\theta}_2)$, and the velocity of the end effector point (v_x, v_y)?

$\dot{\theta}_1 = \dfrac{d\theta_1}{dt}$: Rate of change of angle of joint 1.

$\dot{\theta}_2 = \dfrac{d\theta_2}{dt}$: Rate of change of angle of joint 2.

$v_x = \dfrac{dx}{dt} = \dot{x}$: x-component of velocity of point A.

$v_y = \dfrac{dy}{dt} = \dot{y}$: y-component of velocity of point A.



Relation between Joint Angle Rates and End-Effector Velocity

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The manipulator (analytic) Jacobian is:

$$\boldsymbol{J}(\theta_1, \theta_2) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$: Vector of joint angle rates.

$\boldsymbol{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$: Velocity of end-effector point.

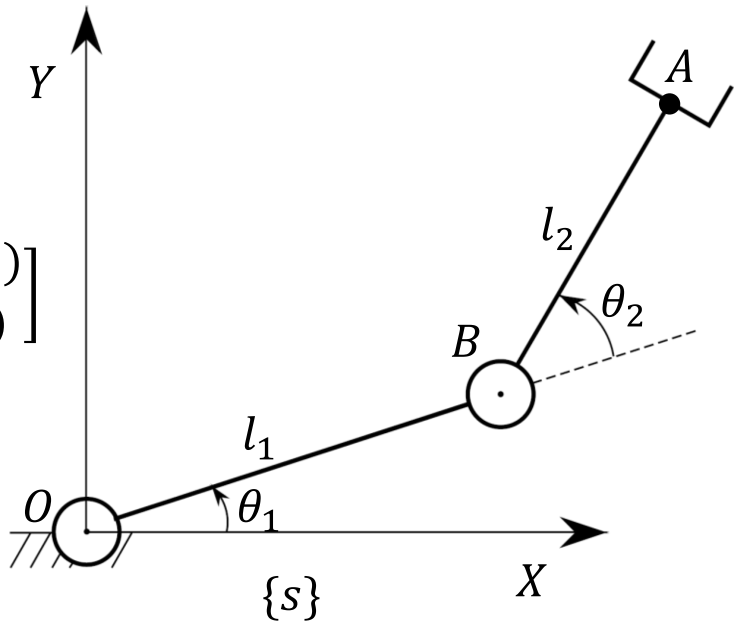
The velocity kinematics equations in vector-matrix form is: $\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Forward (Direct) Velocity Kinematics

Given the configuration of the robot, $\boldsymbol{\theta}$, and the joint angle rates, $\dot{\boldsymbol{\theta}}$, compute the velocity, $\boldsymbol{v}$ of the end effector.

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

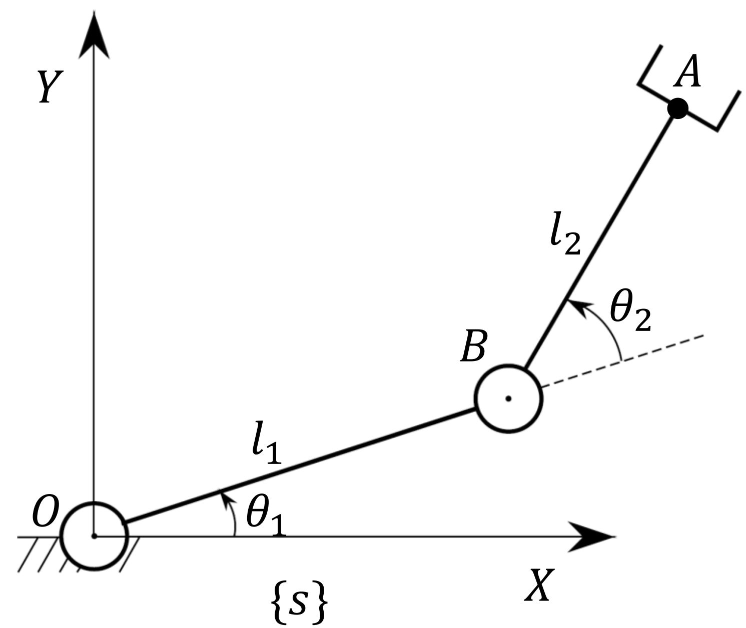


Inverse Velocity Kinematics

Given the configuration of the robot, $\boldsymbol{\theta}$, and the velocity, $\boldsymbol{v}$, of the end effector, compute the joint angle rates, $\dot{\boldsymbol{\theta}}$.

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}(\boldsymbol{\theta})\boldsymbol{v}$$

assuming $\boldsymbol{J}^{-1}(\boldsymbol{\theta})$ exists or the Jacobian matrix is invertible at the configuration $\boldsymbol{\theta}$.

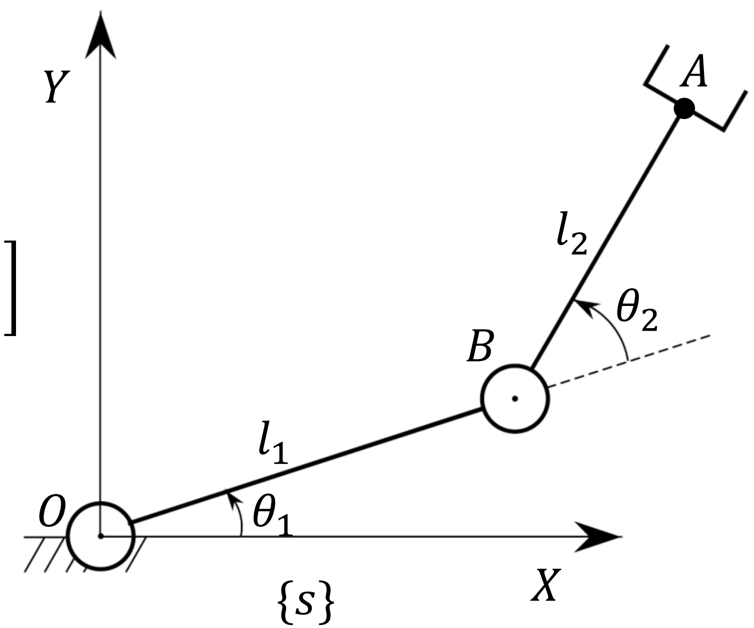


Kinematic Singularities

The configuration $\boldsymbol{\theta}$ at which the Jacobian, $\boldsymbol{J}(\boldsymbol{\theta})$ of a manipulator loses rank is called a **kinematic singularity** or **singular configuration** of the manipulator.

For a 2R manipulator, the Jacobian, $\boldsymbol{J}(\boldsymbol{\theta})$ losing rank implies $\det\big(\boldsymbol{J}(\boldsymbol{\theta})\big) = 0$.

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$



Physical Implications of Kinematic Singularities

Why should we care about singular (or almost singular) configurations?

Jacobian in Multi-Variable Calculus

Let $\boldsymbol{x} \in \mathbb{R}^n$ (be an n–dimensional vector) vary with time t. What is $\dfrac{d\boldsymbol{x}}{dt}$?

$$\boldsymbol{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a **scalar function** (a function that maps a vector of dimension n to a scalar. What is $\dfrac{df}{dt}$?

Jacobian in Multi-Variable Calculus

Let $\boldsymbol{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a **vector function** (a function that maps a vector of dimension n to a vector of dimension m).

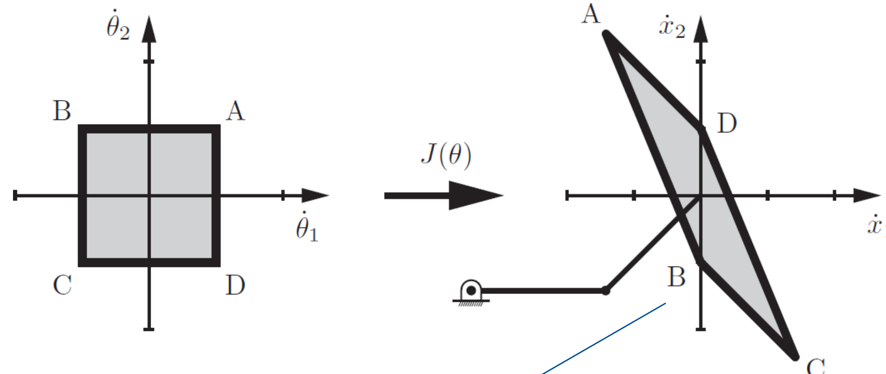
What is $\dfrac{d\boldsymbol{f}}{dt}$?

$$\frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}}_{\textbf{Jacobian} \quad \frac{\partial \mathbf{f}}{\partial \boldsymbol{x}}} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

* You can also obtain the manipulator Jacobian using this formula.

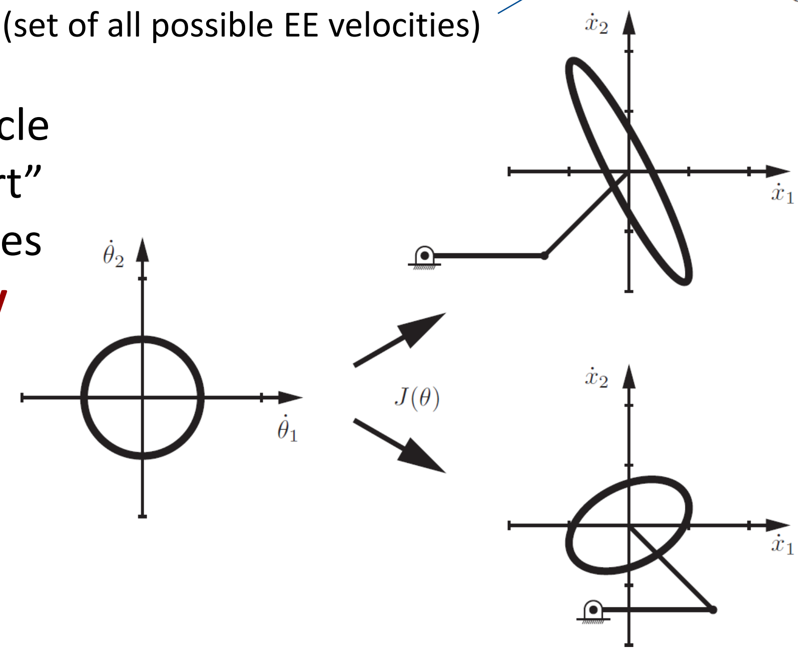
Velocity Manipulability Ellipsoid

The Jacobian can be used to map bounds on the rotational speed of the joints (which is a polygon) to bounds on $\mathbf{v}$.



(set of all possible EE velocities)

The Jacobian can be also used to map a unit circle of joint velocities in the $\theta_1 - \theta_2$-plane ("iso-effort" contour) to an ellipse in the space of EE velocities (this ellipse is called the **velocity manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the tip move in arbitrary directions.

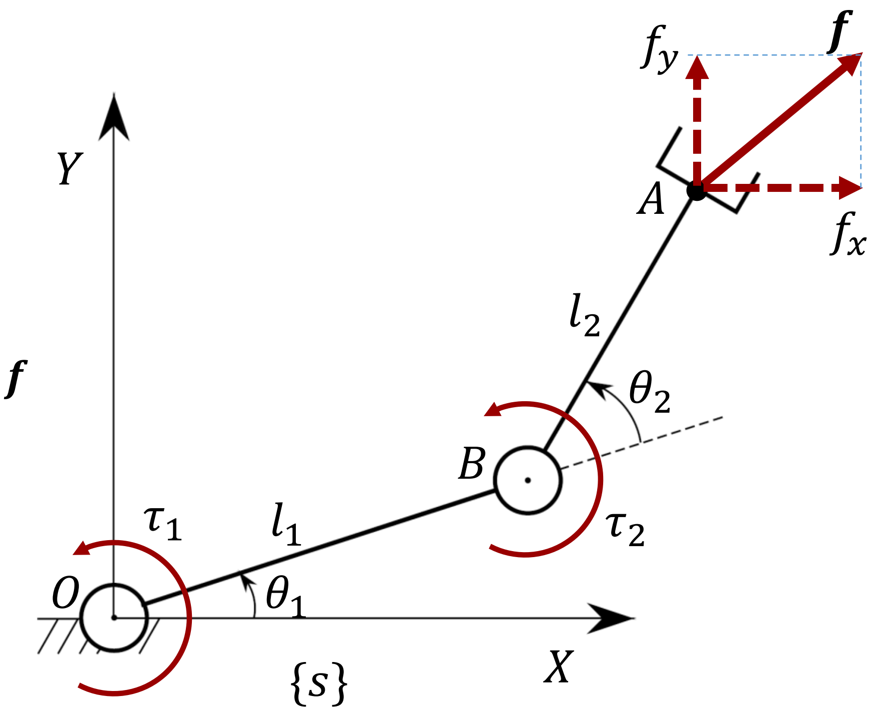
Statics

Statics

What is the relationship between the applied force $\boldsymbol{f}$ and the joint torques $\boldsymbol{\tau}$ such that the manipulator is at equilibrium at a given configuration $\boldsymbol{\theta}$?

$\boldsymbol{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$: Force acting at end-effector point A

$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$: Vector of joint torques required to resist $\boldsymbol{f}$



(Assume that $\boldsymbol{g} = \mathbf{0}$)

Statics

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}-l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2)\\ -l_2\sin(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2)\end{bmatrix}\begin{bmatrix}f_x\\ f_y\end{bmatrix}$$

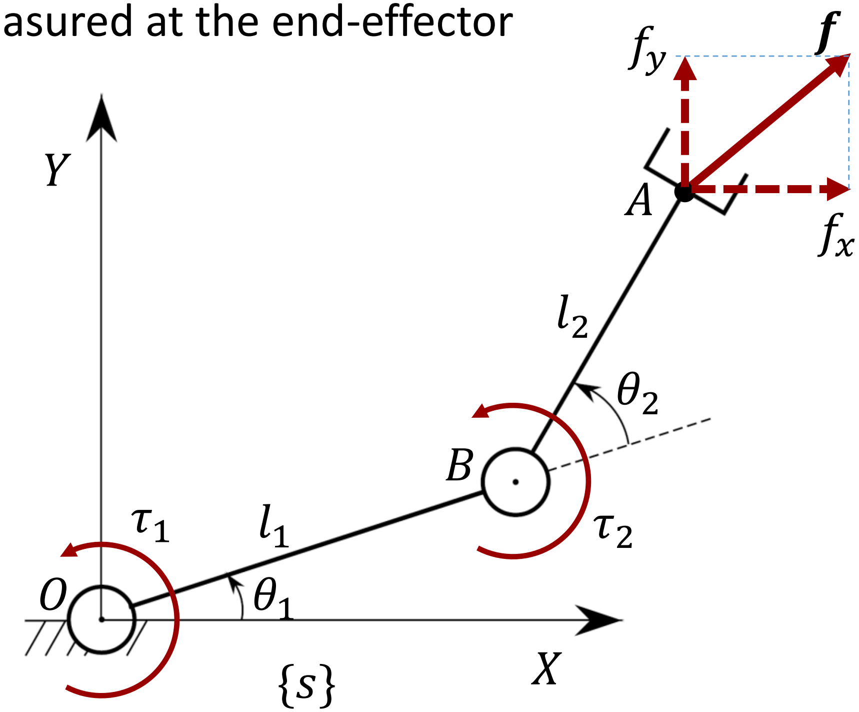
$$\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^T\boldsymbol{f}$$

Statics

A more general method to derive a relation between $\boldsymbol{f}$ and $\boldsymbol{\tau}$.

Principle of conservation of power:

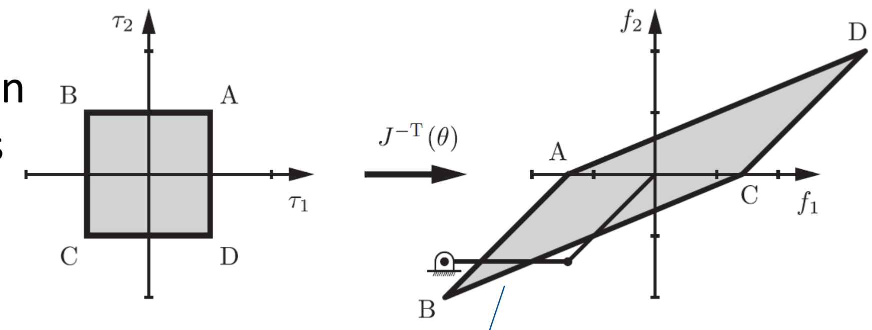
power generated at the joints = power measured at the end-effector



(Assume that $\boldsymbol{g} = \mathbf{0}$)

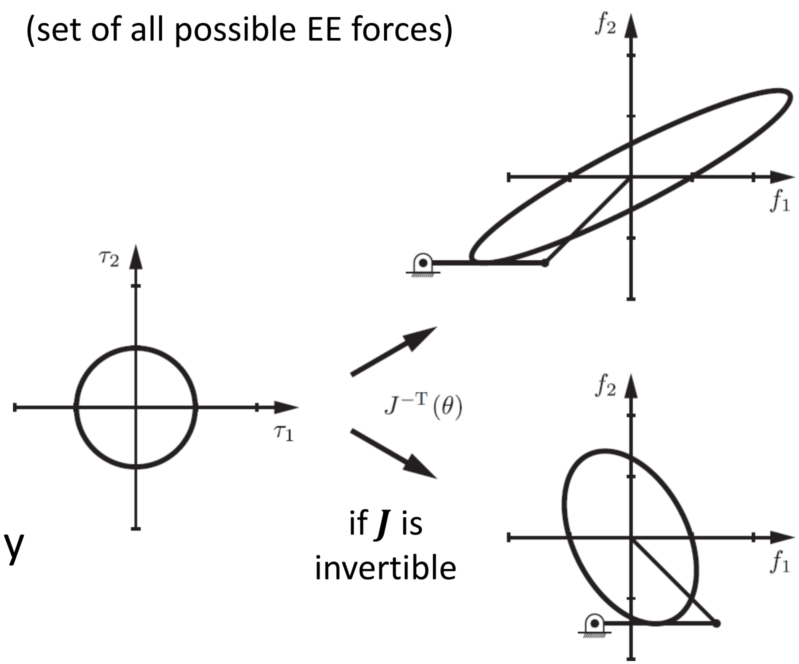
Force Manipulability Ellipsoid

Since $\boldsymbol{f} = (\boldsymbol{J}(\boldsymbol{\theta})^T)^{-1}\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^{-T}\boldsymbol{\tau}$, Jacobian transpose inverse can be used to map bounds on the joint torques (which is a polygon) to bounds on end-effector force $\boldsymbol{f}$.



(set of all possible EE forces)

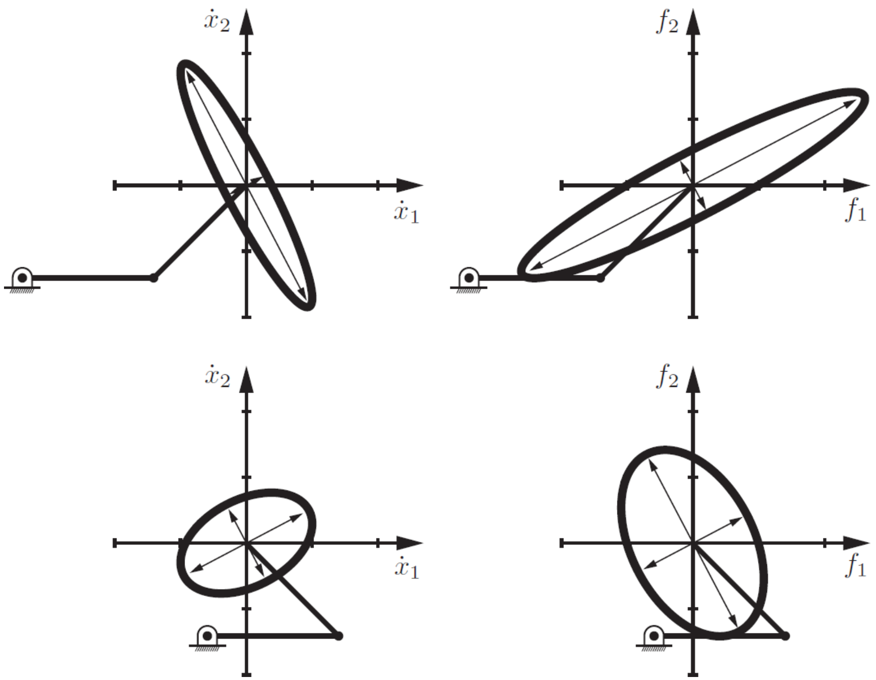
The Jacobian transpose inverse can be also used to map a unit circle of joint torques in the $\tau_1 - \tau_2$-plane ("iso-effort" contour) to an ellipse in the space of EE forces (this ellipse is called the **force manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the EE generate forces in arbitrary directions.

Kineto-Statics Duality

If it is easy to generate a tip velocity in a given direction then it is difficult to generate a force in that same direction, and vice versa.



$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^T \boldsymbol{f}$$

At a singularity, EE motion capability becomes zero in one or more directions, and it can resist infinite force in one or more directions.