

# Ch3: Modeling of Dynamic Systems – Part 2

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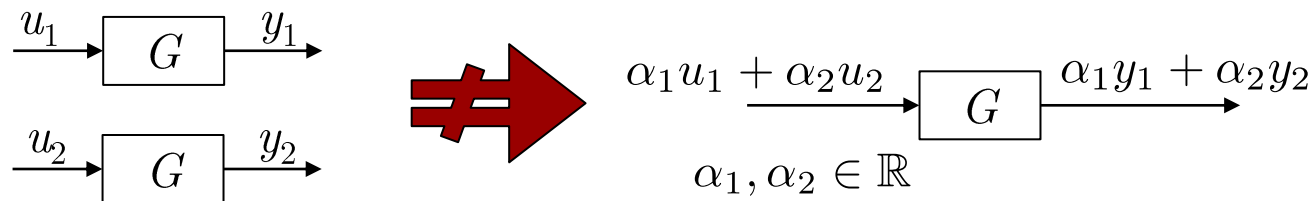
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# Nonlinear Systems

# Nonlinear Systems

A system is nonlinear if the **principle of superposition** does **not** apply.

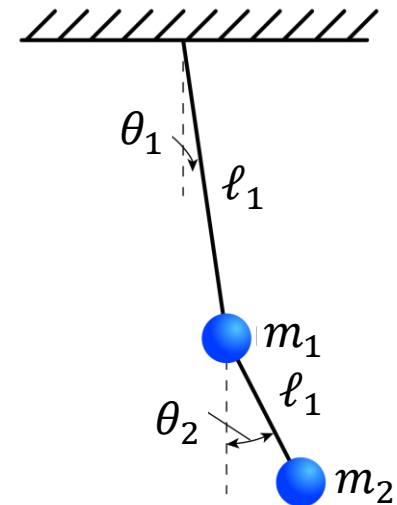


For example, in the **dynamic equations of robots** usually the nonlinear terms  $\sin$ ,  $\cos$ , and squares of velocities appears.

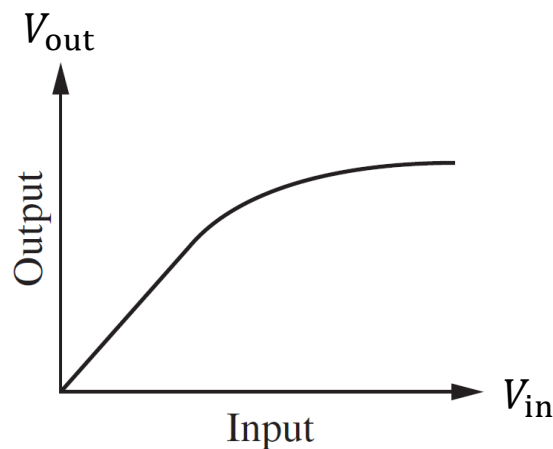
Double-Pendulum:

$$(m_1 + m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin\theta_1 = 0$$

$$m_2\ell_2\ddot{\theta}_2 + m_2\ell_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - m_2\ell_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + m_2g\sin\theta_2 = 0$$

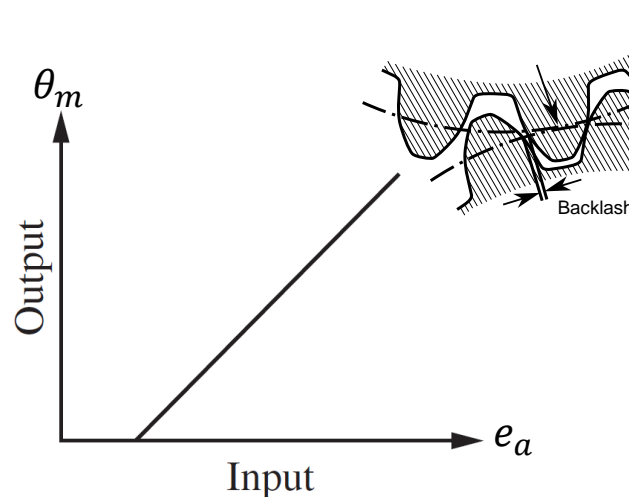


# Examples of Physical Nonlinearities



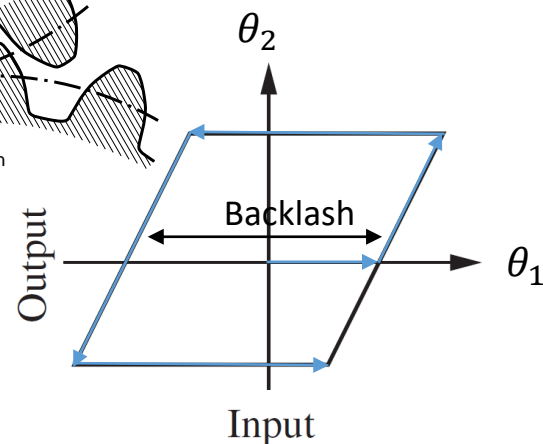
## Amplifier Saturation

An electronic amplifier is linear over a specific range but exhibits the nonlinearity called saturation at high input voltages.



## Motor Dead Zone

A motor that does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called dead zone.



## Backlash in Gears

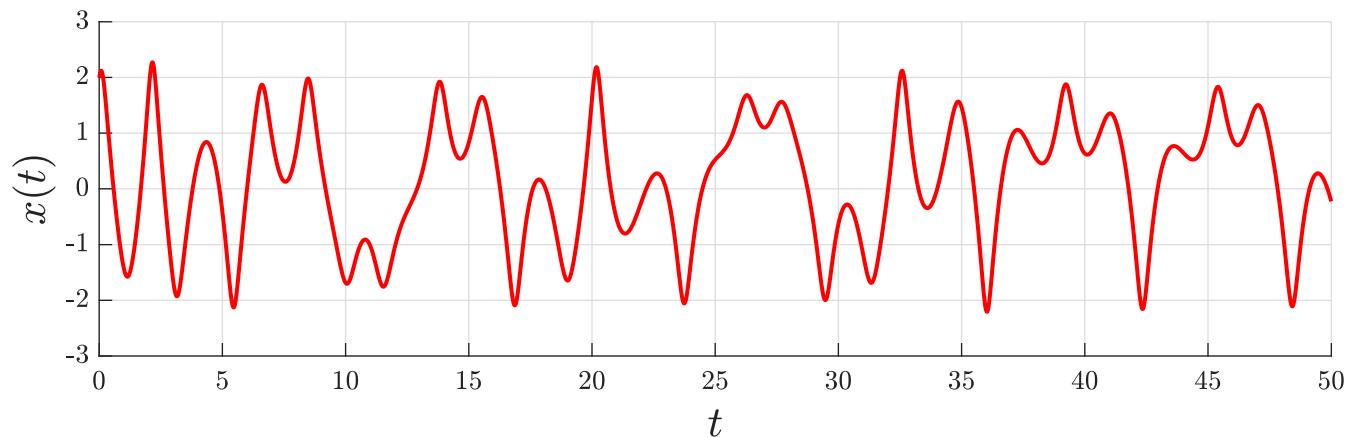
Gears that do not fit tightly exhibit a nonlinearity called backlash which the input moves over a small range without the output responding.

- Nonlinearities can be classified in terms of their mathematical properties, as **continuous** and **discontinuous**. Because discontinuous nonlinearities cannot be locally approximated by linear functions, they are also called **hard nonlinearities** (e.g., backlash, hysteresis, or stiction).

# Nonlinear System Behavior: Chaos

- In the steady state, **sinusoidal inputs** to a stable LTI system generate a sinusoidal outputs of the same frequency (but different in amplitude and phase angle from the input). By contrast, the output of a nonlinear system may display sinusoidal, periodic, or chaotic behaviors.

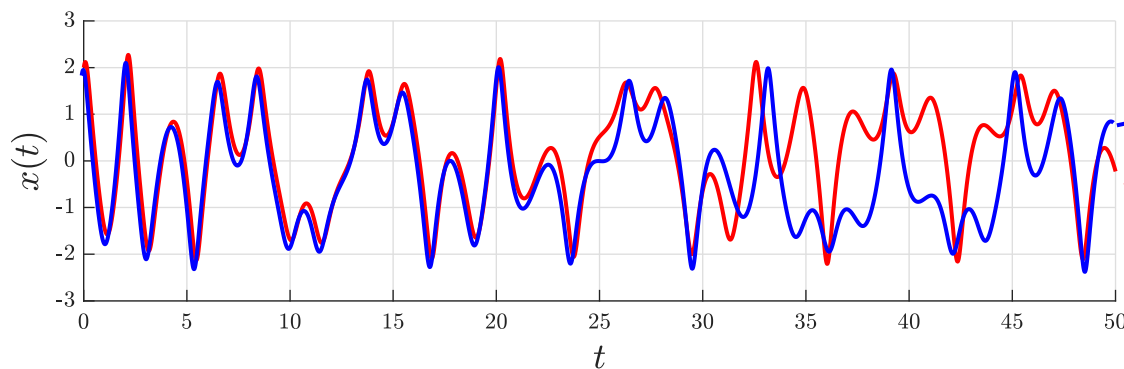
$$\ddot{x} + 0.1\dot{x} + x^5 = u(t), \quad u(t) = 6 \sin t \quad \begin{aligned} x(0) &= 2, \\ \dot{x}(0) &= 3 \end{aligned}$$



# Nonlinear System Behavior: Chaos

- For stable linear systems, small differences in initial conditions can only cause small differences in output. However, output of strongly nonlinear systems is extremely sensitive to **initial conditions**.

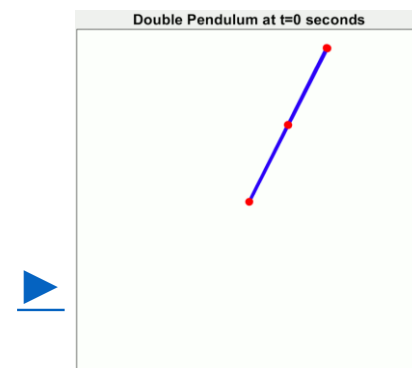
$$\ddot{x} + 0.1\dot{x} + x^5 = u(t), \quad u(t) = 6 \sin t$$



$$x(0) = 2.01, \\ \dot{x}(0) = 3.01$$

$$x(0) = 2, \\ \dot{x}(0) = 3$$

- Starting the pendulum from a slightly different initial condition would result in a vastly different trajectory.

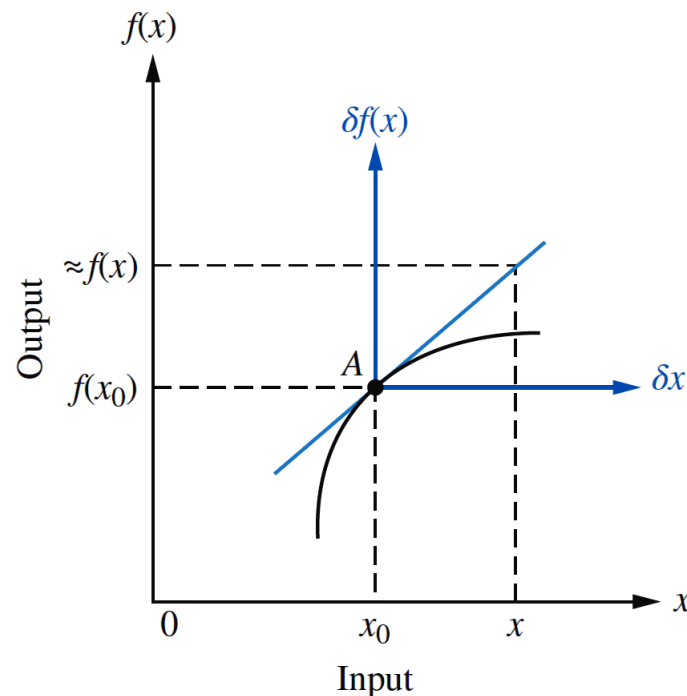


# Linearization of Nonlinear Systems

# Linearization of Nonlinear Systems

In control engineering, a normal operation of the system may be around an **equilibrium point** or a **limited operating range**. Therefore, it is possible to approximate the nonlinear system by an equivalent linear system within the limited operating range.

- Linear approximations simplify the analysis and design of a system.

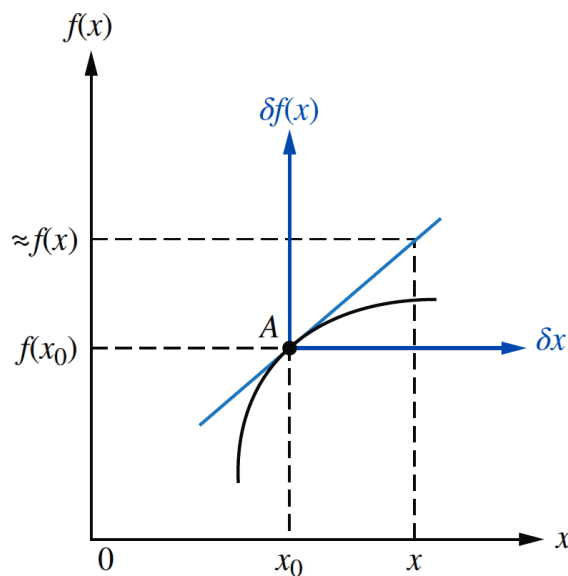




# Linear Approximation of Nonlinear Mathematical Models

The linearization procedure is based on (1) the expansion of nonlinear function  $f(x)$  into a **Taylor Series** about the operating point  $A (x_0, y_0 = f(x_0))$  and (2) the retention of only the linear term.

**Note:** Since the variables deviate only slightly from the operating condition  $(x - x_0)$ , higher-order terms of the Taylor series expansion can be neglected.



$$y = f(x)$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \dots$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

(A straight-line relationship)

Expressing this straight line in frame  $\delta x - \delta f(x)$ :

$$\begin{cases} \delta x = x - x_0 \\ \delta f(x) = f(x) - f(x_0) \end{cases} \Rightarrow \delta f(x) = f'(x_0)\delta x$$

# Example

Linearize  $f(x) = 5 \cos x$  about  $x = \pi/2$ .

## Method 1:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) \approx 5 \cos\left(\frac{\pi}{2}\right) - 5 \sin\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$$

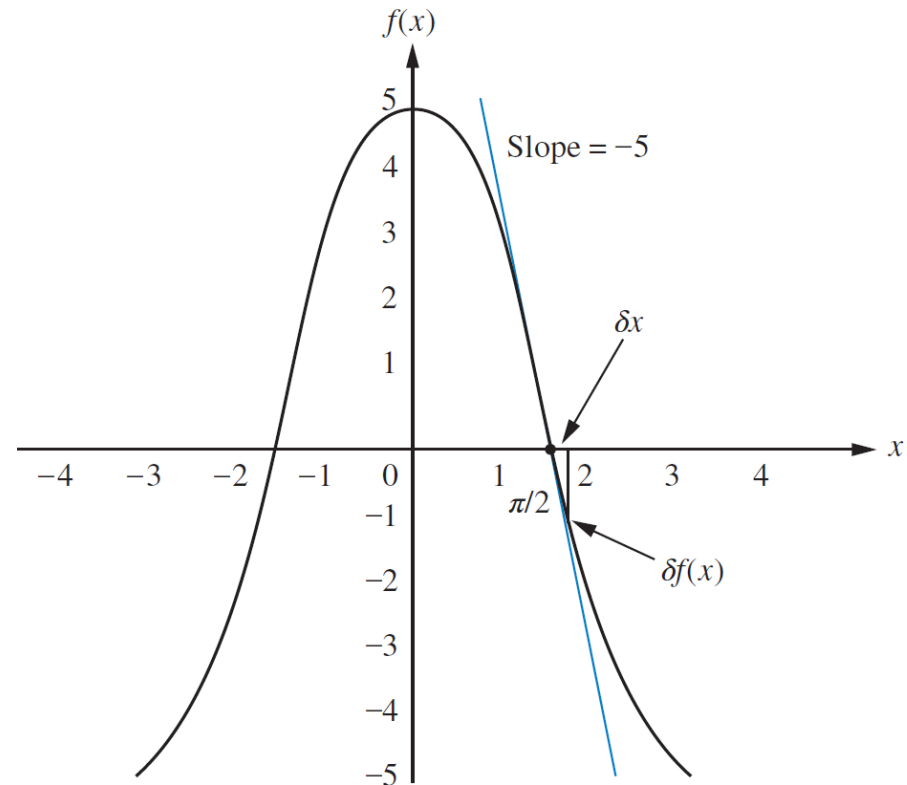
$$f(x) \approx -5\left(x - \frac{\pi}{2}\right)$$

## Method 2:

For small deviation about  $\frac{\pi}{2}$ :  $x = \frac{\pi}{2} + \delta x$

$$f(x) = 5 \cos\left(\delta x + \frac{\pi}{2}\right) = -5 \sin \delta x$$

$$f(x) \approx -5\delta x$$



# Example

Linearize  $\ddot{x} + 2\dot{x} + \cos x = 0$  for small deviations about  $x = \pi/4$ .

# State-Space Representation

# Some Definitions

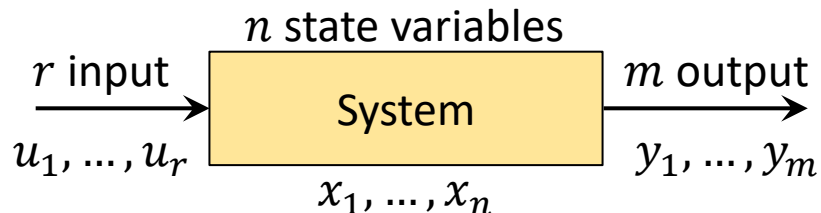
**Linear Combination:** A linear combination of  $n$  variables,  $x_i$ , is given by

$$S = k_1x_1 + k_2x_2 + \cdots + k_nx_n \quad , \quad k_i = \text{constant}, \quad i = 1, \dots, n$$

**Linear Independence:** A set of variables is said to be linearly independent if none of the variables can be written as a linear combination of the others.

**System Variable:** Any variable that responds to an input or initial conditions in a system.

**State Variables:** The **smallest set of linearly independent** system variables  $(x_1, \dots, x_n)$  such that knowledge of these variables at  $t = t_0$ , together with knowledge of the input  $(\mathbf{u}(t))$  for  $t \geq t_0$ , completely determines the behavior of the system for any time  $t \geq t_0$ .



# State-Space Representation

**State-space Representation** is a mathematical model of a physical system as a set of input  $\mathbf{u}(t) \in \mathbb{R}^r$ , output  $\mathbf{y}(t) \in \mathbb{R}^m$ , and state variables  $\mathbf{x}(t) \in \mathbb{R}^n$  related by  $n$  simultaneous first-order differential equations.

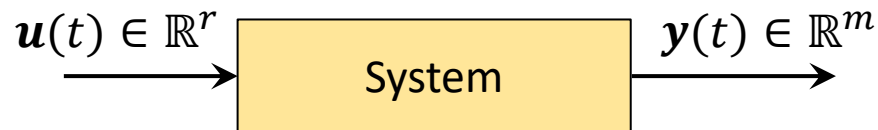
$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

State Equation

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Output Equation

$\mathbf{f}$  and  $\mathbf{g}$  are vector functions.

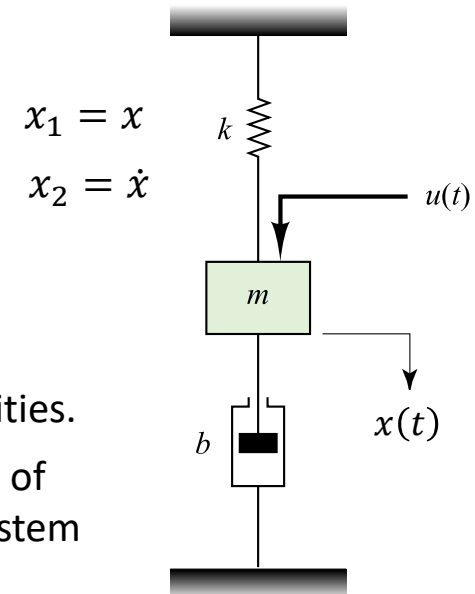


The number of states ( $n$ ) is the **order** of the system.

State Vector:  $\mathbf{x}(t) \in \mathbb{R}^n$

**Note:** State variables need not be physically measurable or observable quantities.

**Note:** The choice of state variables of a system is not unique, but the number of states is unique. For all invertible  $\mathbf{T} \in \mathbb{R}^{n \times n}$ ,  $\bar{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$  can be also the system state variables.



# State-Space Representation

General Form:

MIMO, Nonlinear, Time  
Variant (General Form )

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$$

MIMO, Linear,  
Time Variant

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

MIMO, Linear, Time  
Invariant

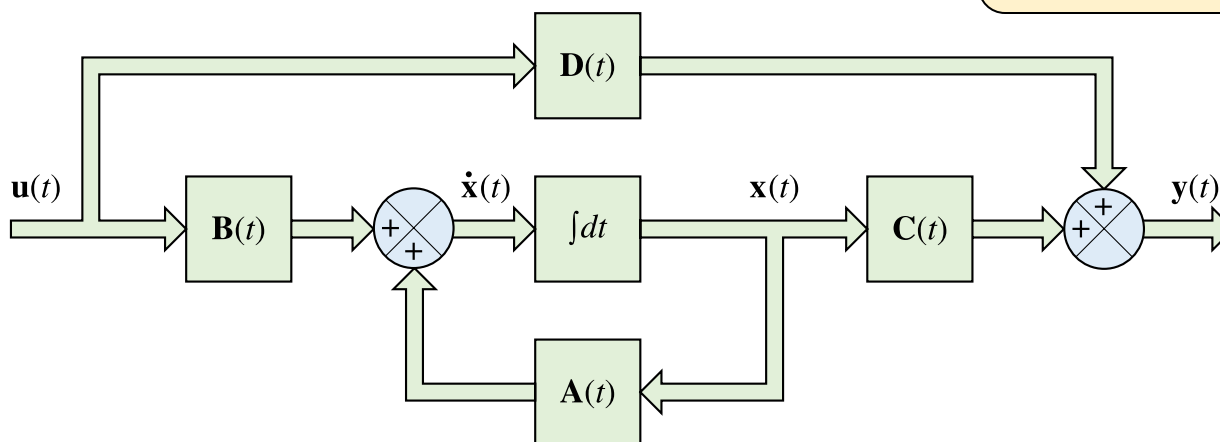
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$\mathbf{A}$ : State matrix,  
 $\mathbf{C}$ : Output matrix,

$\mathbf{B}$ : Input matrix  
 $\mathbf{D}$ : Feedforward matrix

SISO, Linear, Time Invariant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$



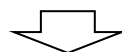
# State-Space Representation of LTI Systems

Consider a general,  $n$ th-order, linear differential equation with constant coefficients:

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_0 x = b_0 u$$

- An  $n$ th-order differential equation can be converted to  $n$  simultaneous **first-order differential equations**.
- There are many ways to do this conversion and obtain state-space representations of systems, such as **phase-variable** form, **controllable** canonical form, **observable** canonical form, **diagonal** canonical form, and **Jordan** canonical form.

A convenient way to choose state variables is to choose  $x(t)$  and its  $(n - 1)$  derivatives as the state variables, which are called **phase variables**.





# State-Space Representation of LTI Systems

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_0 x = b_0 u$$

$$\begin{aligned} x_1 &= x \\ x_2 &= \frac{dx}{dt} \\ &\vdots \\ x_n &= \frac{d^{n-1} x}{dt^{n-1}} \end{aligned}$$

Differentiating

$$\begin{aligned} \dot{x}_1 &= \frac{dx}{dt} \\ \dot{x}_2 &= \frac{d^2 x}{dt^2} \\ &\vdots \\ \dot{x}_n &= \frac{d^n x}{dt^n} \end{aligned}$$

Substituting

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_{n-1}x_n - \dots - a_0x_1 + b_0u \end{aligned}$$

Vector-Matrix Form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad \dots \quad 0]$$

(Output can be the first state)

# Example

The external force  $u(t)$  is the input to the system, and the displacement  $x(t)$  of the mass, measured from the equilibrium position in the absence of the external force, is the output. Find the state equations.

## Solution:

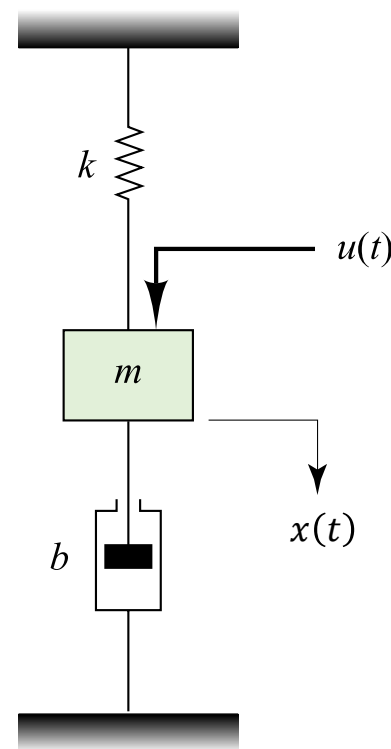
$$m\ddot{x} + b\dot{x} + kx = u$$

Let's define:  $\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}(-kx - b\dot{x}) + \frac{1}{m}u \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \\ y &= x = x_1 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$



# Example

Find the state equations. What is the output equation if the output is  $z_1(t)$ ?

**Solution:**

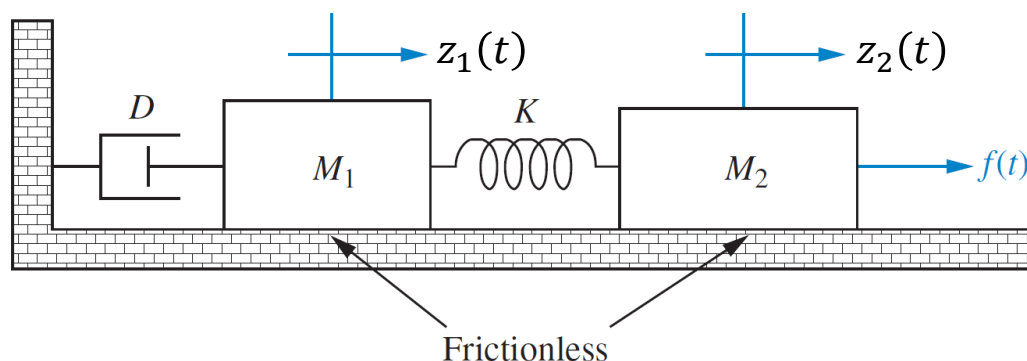
$$M_1 \ddot{z}_1 + D \dot{z}_1 + K z_1 - K z_2 = 0$$

$$M_2 \ddot{z}_2 - K z_1 + K z_2 = f(t)$$

$$\begin{array}{l} x_1 = z_1 \\ x_2 = \dot{z}_1 \\ x_3 = z_2 \\ x_4 = \dot{z}_2 \end{array} \quad \Rightarrow \quad \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{K}{M_1} x_1 - \frac{D}{M_1} x_2 + \frac{K}{M_1} x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = +\frac{K}{M_2} x_1 - \frac{K}{M_2} x_3 + \frac{1}{M_2} f(t) \end{array}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

$$y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



# Converting from SS to a TF

Deriving the transfer function from the state-space equations:

$$\begin{array}{l} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{array} \xrightarrow[\text{assuming zero initial conditions}]{\text{Laplace transform}} \begin{array}{l} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{array} \rightarrow \begin{array}{l} \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \end{array}$$

( $\mathbf{I}$  is the identity matrix)

$\underbrace{[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]}$   
Transfer Function Matrix

Transfer Function for a SISO system which  $\mathbf{U}(s) = U(s)$  and  $\mathbf{Y}(s) = Y(s)$ :

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

# Example

Obtain the transfer function  $Y(s)/U(s)$  from the state-space equations of the system shown in the previous example.

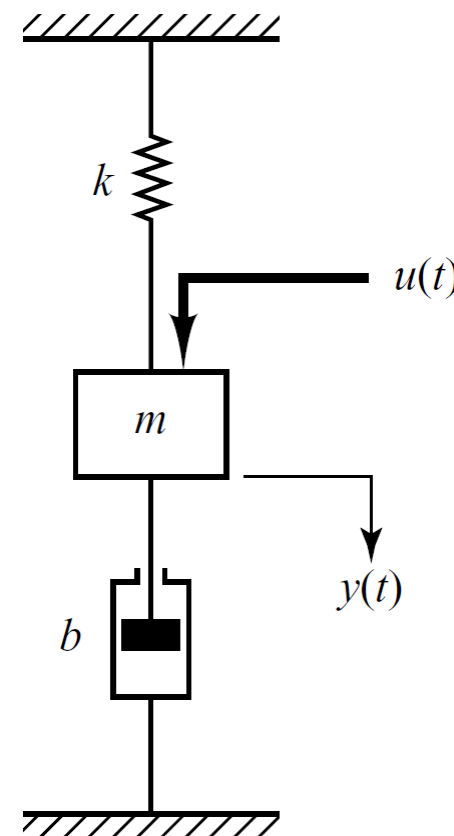
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Solution:**

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$G(s) = \frac{1}{ms^2 + bs + k}$$



# Converting a TF to SS

To convert a transfer function into state-space equations in phase-variable form, first convert the transfer function to a **differential equation** by cross-multiplying and taking the inverse Laplace transform, assuming zero initial conditions.

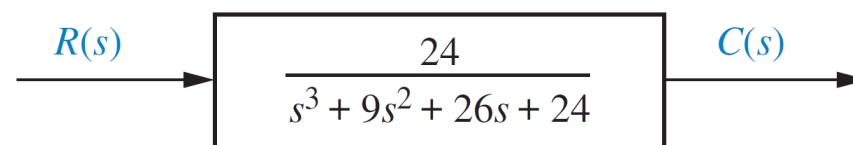
$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} \quad \longrightarrow \quad \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = b_0 u$$

Then, convert this  $n$ th-order differential equation to  $n$  simultaneous first-order differential equations.

# Example

Find the state-space representation in phase-variable form.

**Solution:**



$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

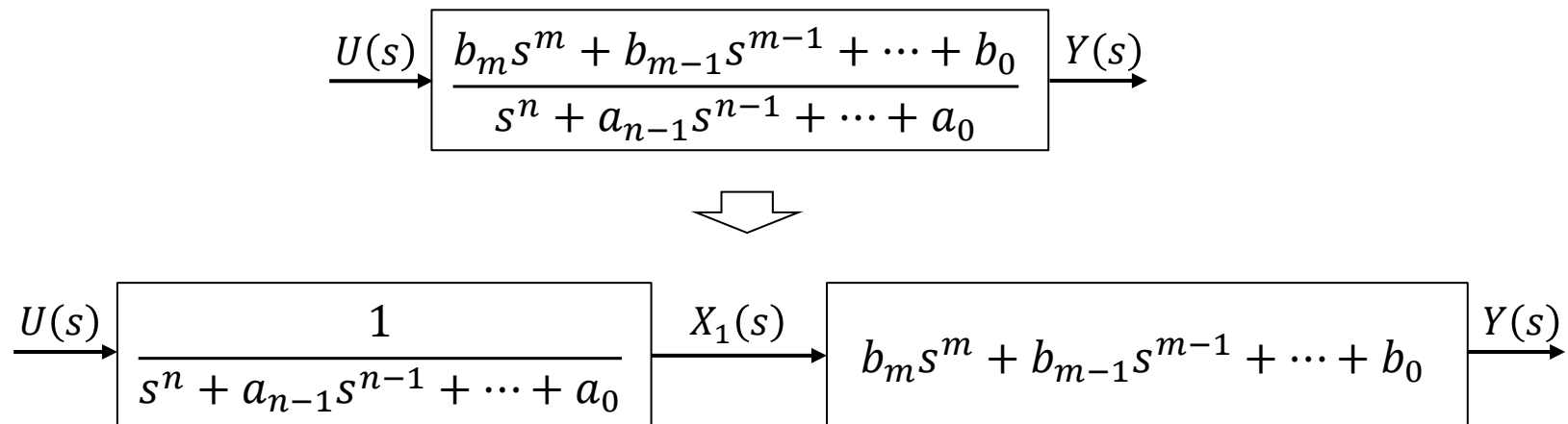
$$\ddot{c} + 9\dot{c} + 26c = 24r$$

$$\begin{array}{lcl} x_1 = c & \longrightarrow & \dot{x}_1 = x_2 \\ x_2 = \dot{c} & & \dot{x}_2 = x_3 \\ x_3 = \ddot{c} & & \dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \\ & & y = c = x_1 \end{array}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r \quad y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Converting a TF to SS

If a transfer function has a polynomial in  $s$  in the numerator, separate the transfer function into two cascaded transfer functions; the first is the denominator and the second is just the numerator.

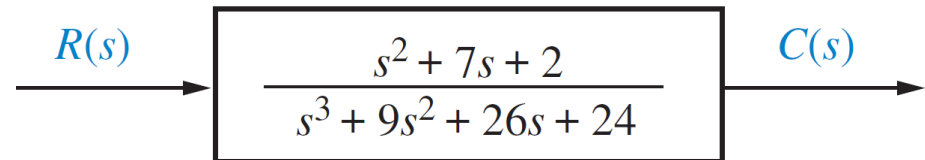


- The first transfer function with just the denominator is converted to the phase-variable representation in state space.
- The second transfer function with just the numerator yields the output equation.

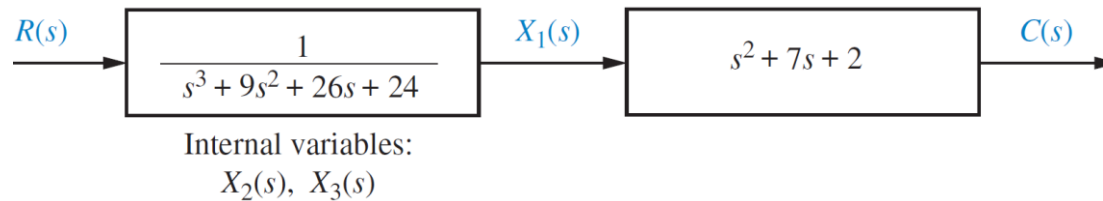


# Example

Find the state-space representation of the transfer function.



**Solution:**



From previous example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$C(s) = (s^2 + 7s + 2)X_1(s)$$

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1 \xrightarrow{\begin{matrix} x_1 = x_1 \\ \dot{x}_1 = x_2 \\ \ddot{x}_1 = x_3 \end{matrix}} y = c(t) = x_3 + x_2 + 2x_1 \longrightarrow y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Using MATLAB and Control System Toolbox

# Transfer-Function Representation Using tf, zpk

## %% Transfer-Function Representation, Method 1

```
F1 = tf([3], [1 2 5 0]);  
F2 = tf([1], [1 0]);  
F3 = tf([1 0], [1]);  
F4 = tf(16, poly([0 -1 -1]));
```

### Method 1

$$F_1(s) = \frac{3}{s(s^2 + 2s + 5)}$$

## %% Transfer-Function Representation, Method 2

```
s = tf('s');  
F1 = 3/(s^3 + 2*s^2 + 5*s);  
F2 = 1/s;  
F3 = s;  
F4 = 16/(s*(s+1)^2);
```

### Method 2

$$F_2(s) = \frac{1}{s}$$

$$F_3(s) = s$$

$$F_4(s) = \frac{16}{s(s+1)^2}$$

## %% Transfer-Function Representation, Method 3

```
F1 = zpk([], [0 -1+2i -1-2i], [3]);  
F2 = zpk([], [0], [1]);  
F3 = zpk([0], [], [1]);  
F4 = zpk([], [0 -1 -1], [16]);
```

### Method 3

# State-Space Representation Using ss and Conversions Using tf2ss, ss2tf

## %% State-Space Representation

```
A = [-4 -1.5; 4 0];  
B = [2 0]';  
C = [1.5 0.625];  
D = 0;  
T_ss = ss(A,B,C,D);
```

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y &= [1.5 \quad 0.625] \mathbf{x}\end{aligned}$$

## % converting SS to TF

```
T_tf = tf(T_ss);
```

$$T(s) = \frac{3s + 5}{s^2 + 4s + 6}$$

## %% TF to SS, SS to TF

```
num = [1 7 2];  
den = [1 9 26 24];  
[A, B, C, D] = tf2ss(num, den);  
T1 = ss(A,B,C,D);  
T1 = tf(T1);
```

$$T_1(s) = T_2(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

## % For SISO systems

```
[num,den] = ss2tf(A,B,C,D);  
T2 = tf(num,den);
```