

# Ch9: Distributed Forces: Moments of Inertia

## Contents:

Moments of Inertia of Areas (or Second Moment of Areas)

Parallel-Axis Theorem and Composite Areas

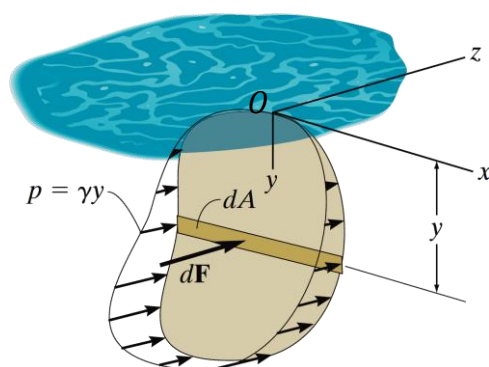
Mass Moments of Inertia

# Moments of Inertia of Areas (or Second Moment of Areas)

# Motivation

In these examples of distributed forces, the force element  $dF$  is proportional to the element of area  $dA$  on which  $dF$  acts and, at the same time, vary linearly with the distance from  $dA$  to a given axis.

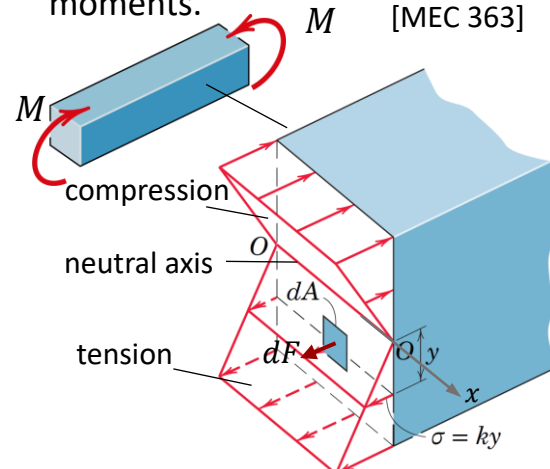
(1) A vertical surface submerged in a fluid. [MEC 364]



$$dF = p dA = \rho g y dA \rightarrow F = \rho g \int y dA$$

$$dM_x = y dF = \rho g y^2 dA \rightarrow M_x = \rho g \int y^2 dA$$

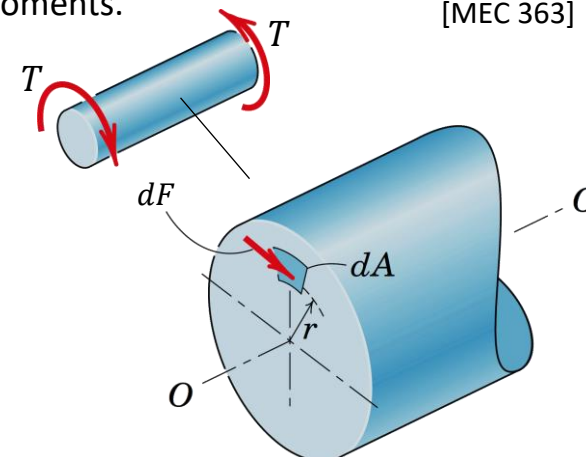
(2) A beam subjected to bending moments. [MEC 363]



$$dF = \sigma dA = k y dA$$

$$dM_x = y dF = k y^2 dA$$

(3) A circular shaft subjected to torsional moments. [MEC 363]



$$dF = \tau dA = k r dA$$

$$dM_O = r dF = k r^2 dA$$

$\int (\text{distance}) d(\text{area})$ : First Moment of an Area

(Purely mathematical properties of the area)

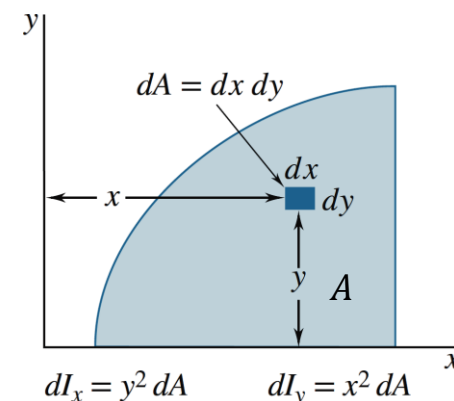
$\int (\text{distance})^2 d(\text{area})$ : Second Moment of an Area  
(or Moment of Inertia of an Area)

# Moment of Inertia of an Area (or Second Moment of an Area)

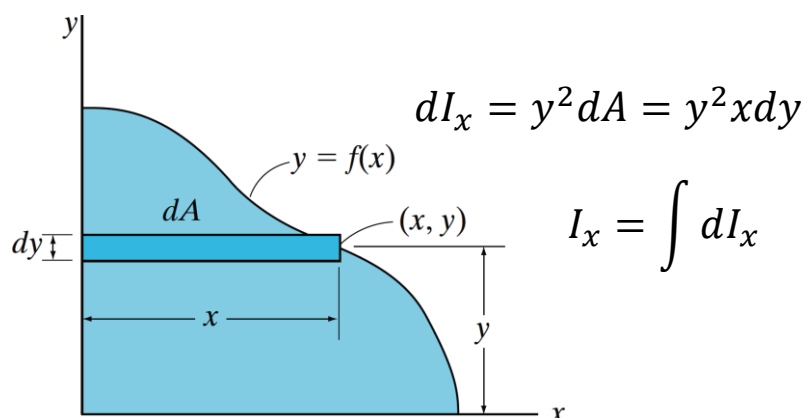
By definition, the **moments of inertia** of a differential area  $dA$  about the  $x$  and  $y$  axes are  $dI_x = y^2 dA$  and  $dI_y = x^2 dA$ , respectively. For the entire area  $A$ , the moments of inertia are determined by integration:

$$I_x = \int dI_x = \int y^2 dA$$

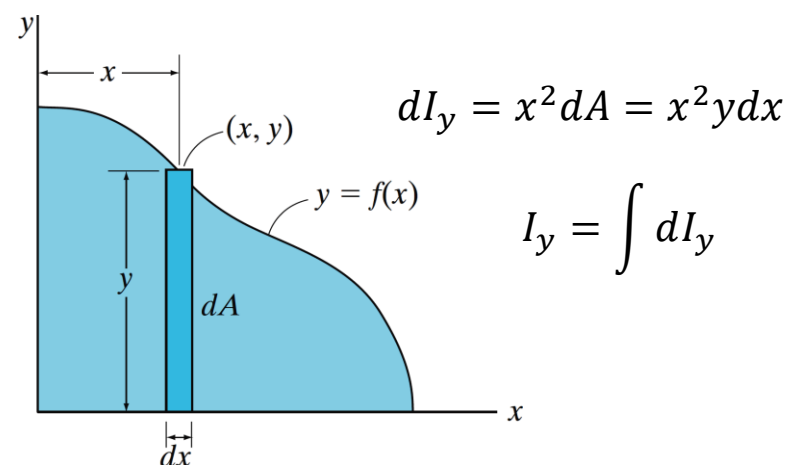
$$I_y = \int dI_y = \int x^2 dA$$



To compute  $I_x$  (or  $I_y$ ) using a single integration, choose a thin strip parallel to the  $x$  axis (or  $y$  axis), so that all points of the strip are at the same distance from the axis.



**1**  
and

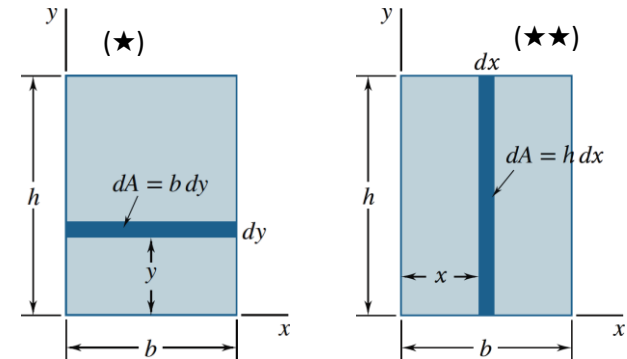


# Moment of Inertia of an Area (or Second Moment of an Area) (cont.)

**Example:** Moment of Inertia of a Rectangular Area.

$$(\star) \quad dI_x = y^2 dA = y^2 b dy \longrightarrow I_x = \int_0^h b y^2 dy = \frac{1}{3} b h^3$$

$$(\star\star) \quad dI_y = x^2 dA = x^2 h dx \longrightarrow I_y = \int_0^b x^2 h dx = \frac{1}{3} h b^3$$



**Note:** By using  $I_x$ ,  $I_y$  computed for a rectangular area, we can now use only one rectangular element  $dA$  (vertical or horizontal) to compute both moments of inertia  $I_x$  and  $I_y$  for a given area (when one side of the element  $dA$  is on the  $x$  or  $y$  axis):

②

Diagram ② shows an area under the curve  $y = f(x)$  in the first quadrant. A horizontal differential element  $dA$  of width  $x$  and height  $dy$  is shown at a distance  $y$  from the  $x$ -axis. The coordinates of the right corner of the element are  $(x, y)$ .

$$dI_x = y^2 dA = y^2 x dy$$

$$dI_y = \frac{1}{3} x^3 dy$$

$$I_x = \int dI_x$$

$$I_y = \int dI_y$$

or

③

Diagram ③ shows the same area under the curve  $y = f(x)$ . A vertical differential element  $dA$  of height  $y$  and width  $dx$  is shown at a distance  $x$  from the  $y$ -axis. The coordinates of the top corner of the element are  $(x, y)$ .

$$dI_y = x^2 dA = x^2 y dx$$

$$dI_x = \frac{1}{3} y^3 dx$$

$$I_x = \int dI_x$$

$$I_y = \int dI_y$$

# Polar Moment of Inertia of an Area

By formulating the moments of inertia of  $dA$  about the “**pole**”  $O$  or  $z$  axis, we can define the **polar moment of inertia** as

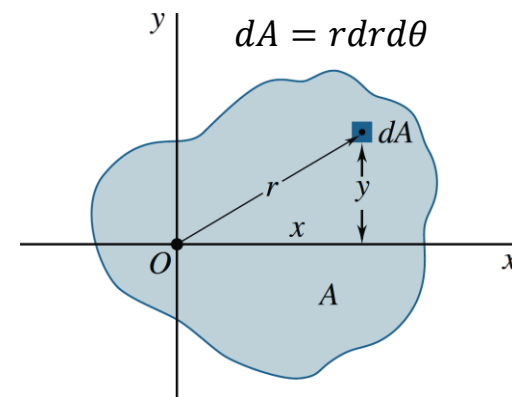
$$J_O = \int r^2 dA$$

$r$ : the perpendicular distance from  $O$  ( $z$  axis) to the element  $dA$

$$r^2 = x^2 + y^2$$

$$J_O = \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA$$

$$J_O = I_y + I_x$$

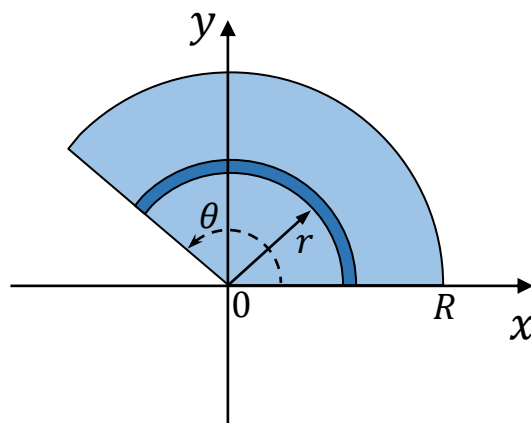


**Note:**  $I_x$ ,  $I_y$ , and  $J_O$  are always positive.

**Note:** The unit of  $I_x$ ,  $I_y$ , and  $J_O$  involve length to the power of 4, e.g.,  $\text{m}^4$ ,  $\text{mm}^4$ , or  $\text{ft}^4$ ,  $\text{in}^4$ .

# Finding Polar Moment of Inertia of an Area

- If the given area has circular symmetry, it is possible to express  $dA$  as a function of  $r$  and to compute  $J_O$  with a **single integration**.



$$dJ_O = r^2 dA = r^2 (r \theta dr)$$

$$J_O = \int dJ_O = \theta \int_0^R r^3 dr$$

- If the area lacks circular symmetry, it is usually easier first to calculate  $I_x$  and  $I_y$  and then to determine  $J_O$  from  $J_O = I_y + I_x$ .

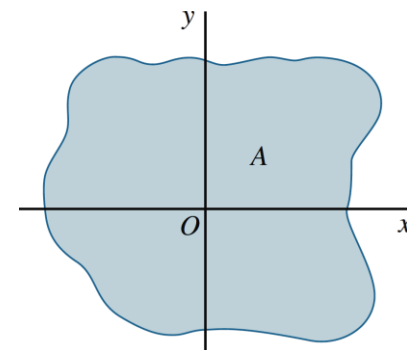
# Radius of Gyration of an Area

Consider an area  $A$  that has a moments of inertia  $I_x$ ,  $I_y$ , and  $J_O$ . The **radius of gyration**  $k_x$ ,  $k_y$ ,  $k_O$  of the area about  $x$  axis,  $y$  axis, and  $O$  ( $z$  axis) are defined as:

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

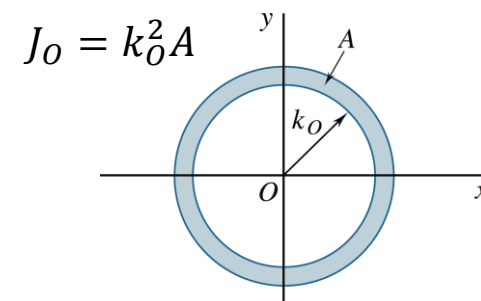
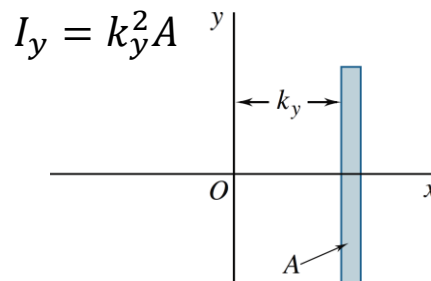
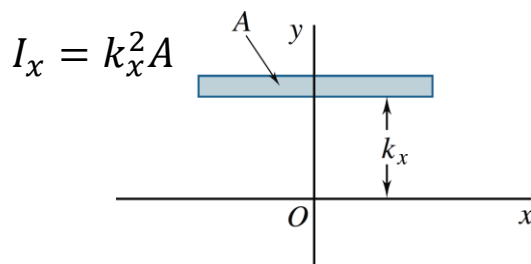
$$k_O = \sqrt{\frac{J_O}{A}}$$



**Note:** From  $J_O = I_x + I_y$ , we have:  $k_O^2 = k_x^2 + k_y^2$

**Note:** Radius of gyration has units of **length**, e.g., m, mm, or ft, in..

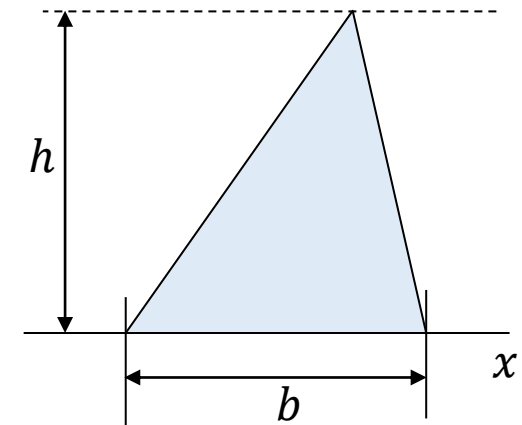
**Mathematical Interpretation of Radius of Gyration:** Imagine that the area  $A$  is concentrated into a thin strip parallel to the  $x$  axis. To have the same moment of inertia with respect to the  $x$  axis, the strip should be placed at a distance  $k_x$  from the  $x$  axis.  $k_y$  and  $k_O$  are defined in a similar way.





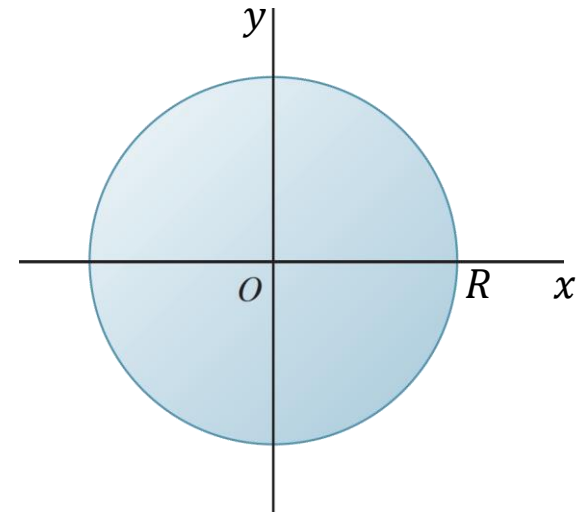
# Sample Problem 9.1

Determine the moment of inertia of a triangle with respect to its base.



# Sample Problem 9.2

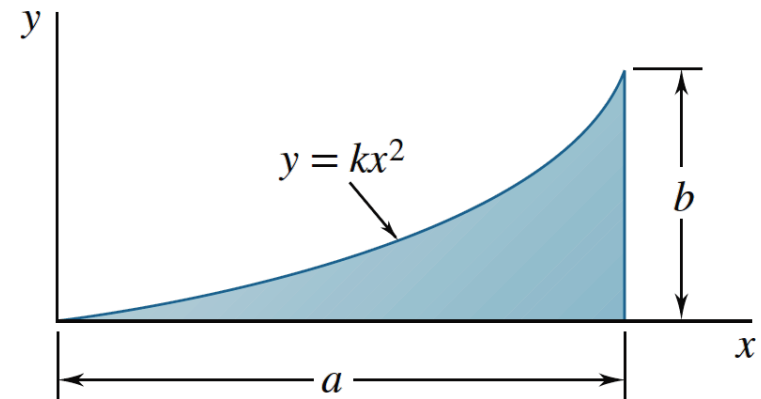
(a) Determine the centroidal polar moment of inertia of a circular area by direct integration. (b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.



# Sample Problem 9.3

(a) Determine the moment of inertia of the shaded region shown with respect to each of the coordinate axes. (b) Using the results of part a, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.

$$A = \frac{ab}{3}$$



# Parallel-Axis Theorem and Composite Areas

# Parallel-Axis Theorem for an Area

Let  $C$  be the centroid of the area  $A$ ,  $x'$  and  $y'$  axes be centroidal axes,  $\bar{I}_{x'}$  and  $\bar{I}_{y'}$  be the moment of inertia of the area about the centroidal axes, and  $x$  and  $y$  axes be two arbitrary axes parallel to centroidal axes  $x'$  and  $y'$  at distances  $d_y$  and  $d_x$ , respectively.

$$I_x = \int (y' + d_y)^2 dA = \int y'^2 dA + 2d_y \int y' dA + d_y^2 \int dA$$

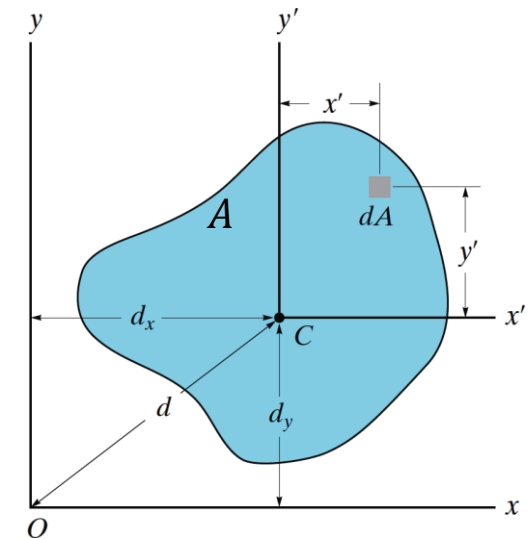
The centroid  $C$  is located on the axis  $x'$

$$I_x = \bar{I}_{x'} + Ad_y^2$$

Similarly,  $I_y = \bar{I}_{y'} + Ad_x^2$

$$\begin{cases} \bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} \\ d^2 = d_x^2 + d_y^2 \end{cases}$$

$$J_O = I_x + I_y = \bar{J}_C + Ad^2$$



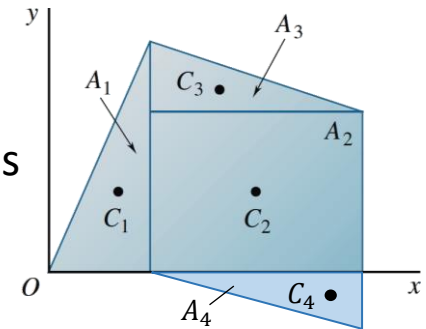
$$\begin{cases} k_x^2 = \bar{k}_{x'}^2 + d_y^2 \\ k_y^2 = \bar{k}_{y'}^2 + d_x^2 \\ k_O^2 = \bar{k}_C^2 + d^2 \end{cases}$$

★ The moment of inertia for an area about an arbitrary axis is equal to its moment of inertia about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes.

# Moments of Inertia of Composite Areas

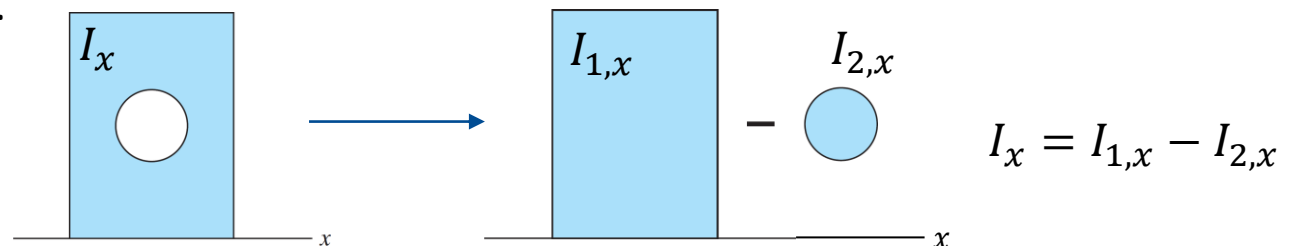
Consider a composite area  $A$  can be broken down into a sum of simple areas  $A_1, A_2, A_3, \dots$ . The moment of inertia of  $A$  about a given axis can be obtained by algebraically adding the moments of inertia of the areas  $A_1, A_2, A_3, \dots$  with respect to the same axis.

$$I_x = I_{1,x} + I_{2,x} + I_{3,x} + I_{4,x} + \dots$$



**Note:** It is usually required to first determine the **perpendicular distance** from the **centroid** of each component to the reference axis, then, use the **parallel-axis theorem** to determine the moment of inertia of the components about the same reference axis.

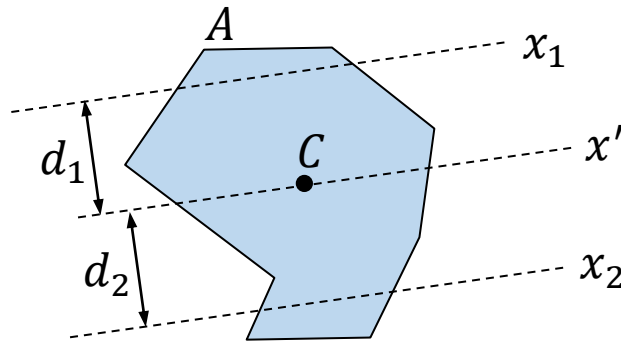
**Note:** If a composite part has an empty region (hole), its moment of inertia is found by subtracting the moment of inertia of this region from the moment of inertia of the entire part including the region.



# Remarks

**Note:** The **radius of gyration of a composite area** is not equal to the sum of the radii of gyration of the component areas. In order to determine the radius of gyration of a composite area, you must first compute the moment of inertia  $I$  of the composite area  $A$ , and then  $k = \sqrt{I/A}$ .

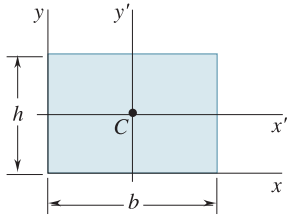
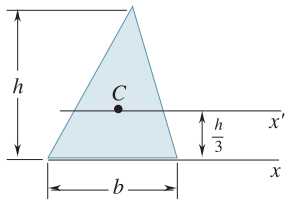
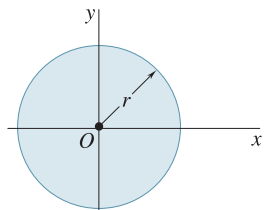
**Note:** To compute the moment of inertia of an area with respect to a **noncentroidal axis** ( $I_{x_2}$ ) when the moment of inertia of the area is **known with respect to another parallel noncentroidal axis** ( $I_{x_1}$ ), it is necessary to **first** compute the moment of inertia of the area with respect to the **centroidal axis parallel to the two given axes** ( $\bar{I}_{x'}$ ) using the parallel-axis theorem, and then use the parallel-axis theorem again to find  $I_{x_2}$ .

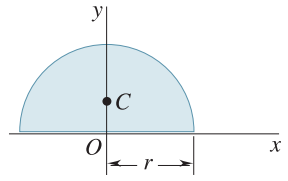
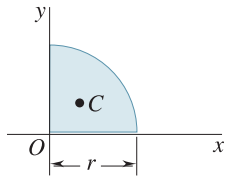
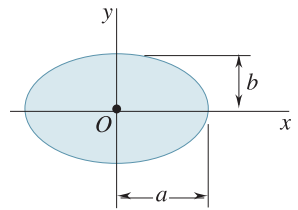


$$I_{x_1} = \bar{I}_{x'} + Ad_1^2 \quad \longrightarrow \quad \bar{I}_{x'} \quad \checkmark$$

$$I_{x_2} = \bar{I}_{x'} + Ad_2^2 \quad \longrightarrow \quad I_{x_2} \quad \checkmark$$

# Moments of Inertia of Common Shapes

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$

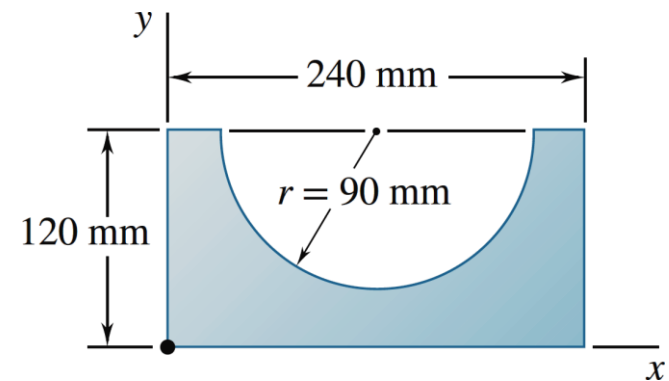
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

★ The moments of inertia of a semicircle/semiellipse and a quarter circle/ellipse can be determined by dividing the moment of inertia of a circle/ellipse by 2 and 4, respectively. Note that the moments of inertia obtained in this manner are **about the axes of symmetry** of the circle/ellipse. To obtain the centroidal moments of inertia of these shapes, use the **parallel-axis theorem**.



# Sample Problem 9.5

Determine the moment of inertia of the shaded area with respect to the  $x$  axis.



# Mass Moments of Inertia

# Mass Moments of Inertia

The mass moment of inertia of a body is a measure of the body's resistance to angular acceleration (rotational motion) [MEC 262]. The **mass moment of inertia** of a body about an axis  $AA'$  is defined as

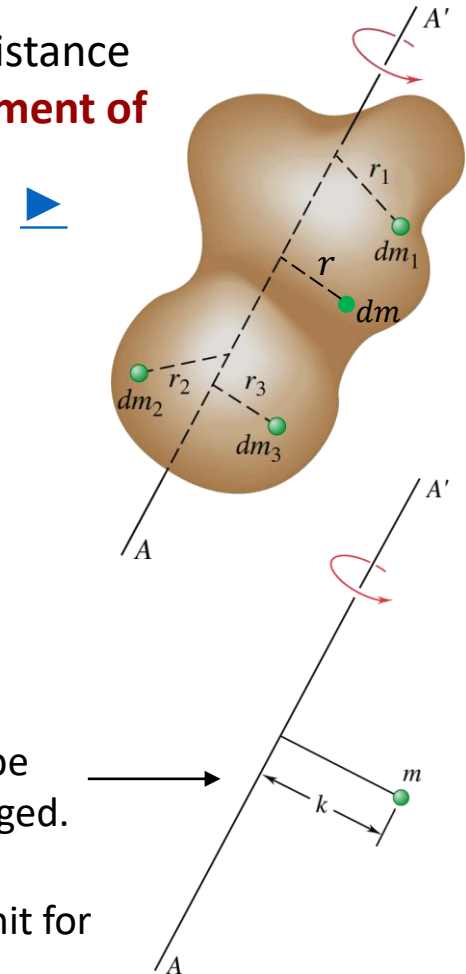
$$I = \int r^2 dm$$

**Radius of gyration** of a mass about axis  $AA'$  is defined as

$$I = k^2 m \quad \rightarrow \quad k = \sqrt{\frac{I}{m}}$$

**Note:**  $k$  represents the distance at which the entire mass of the body should be concentrated if its moment of inertia with respect to  $AA'$  is to remain unchanged.

**Note:** Unit for the mass moment of inertia  $I$  is  $\text{kg}\cdot\text{m}^2$ ,  $\text{slug}\cdot\text{ft}^2$ , or  $\text{lb}\cdot\text{ft}\cdot\text{s}^2$  and unit for the radius of gyration  $k$  is m or ft.



# Parallel-Axis Theorem for a Mass

Let  $G$  be the center of gravity/mass of the mass  $m$ ,  $G_{x'y'z'}$  be a coordinate system whose origin is at  $G$ ,  $O_{xyz}$  be a coordinate system of parallel axes whose origin is at the arbitrary point  $O$ .

$$I_x = \int r_x^2 dm = \int (y^2 + z^2) dm = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2] dm$$

$$= \int (y'^2 + z'^2) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + (\bar{y}^2 + \bar{z}^2) \int dm$$

$\int y' dm = 0$   $\int z' dm = 0$   
 $G$  is located on the axes  $y'$  and  $z'$

$$I_x = \bar{I}_x + m(\bar{y}^2 + \bar{z}^2)$$

Similarly,

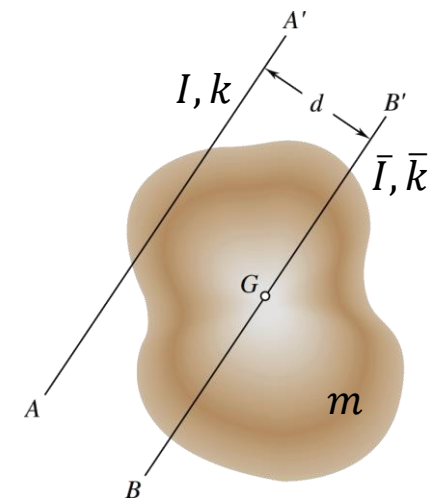
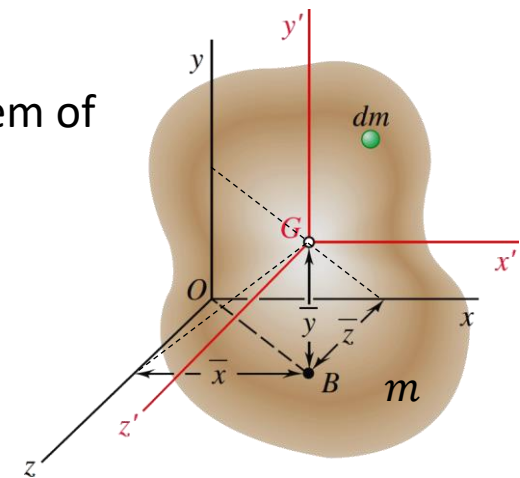
$$I_y = \bar{I}_y + m(\bar{x}^2 + \bar{z}^2)$$

$$I_z = \bar{I}_z + m(\bar{x}^2 + \bar{y}^2)$$

A general relation

$$I = \bar{I} + md^2$$

For radius of gyration:  $k^2 = \bar{k}^2 + d^2$



# Mass Moments of Inertia of Thin Plates

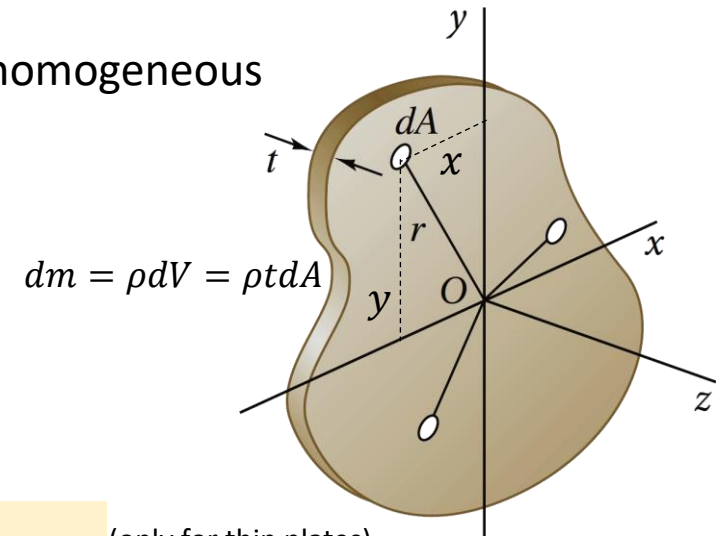
Consider a thin plate of uniform thickness  $t$ , made of a homogeneous material of density  $\rho$ .

$$I_{x,\text{mass}} = \int y^2 dm = \rho t \int y^2 dA \rightarrow I_{x,\text{mass}} = \rho t I_{x,\text{area}}$$

$$I_{y,\text{mass}} = \int x^2 dm = \rho t \int x^2 dA \rightarrow I_{y,\text{mass}} = \rho t I_{y,\text{area}}$$

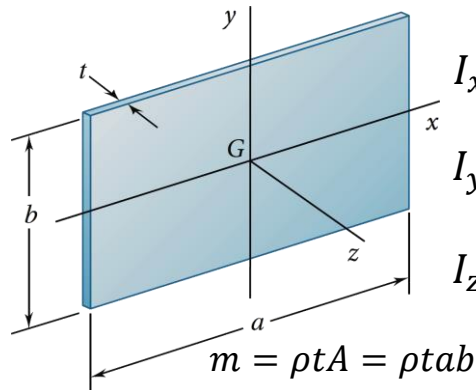
$$I_{z,\text{mass}} = \int r^2 dm = \rho t \int r^2 dA \rightarrow I_{z,\text{mass}} = \rho t J_{O,\text{area}}$$

Since  $J_{O,\text{area}} = I_{x,\text{area}} + I_{y,\text{area}} \rightarrow I_{z,\text{mass}} = I_{x,\text{mass}} + I_{y,\text{mass}}$  (only for thin plates)



**For Example:**

**A thin rectangular plate:**



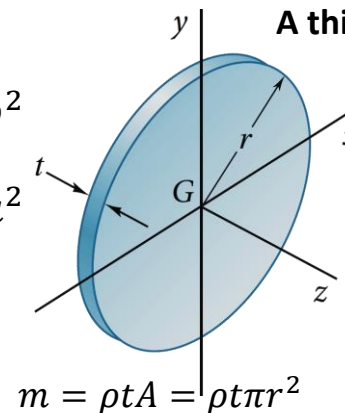
$$I_{x,\text{mass}} = \rho t \left( \frac{1}{12} ab^3 \right) = \frac{1}{12} mb^2$$

$$I_{y,\text{mass}} = \rho t \left( \frac{1}{12} ba^3 \right) = \frac{1}{12} ma^2$$

$$I_{z,\text{mass}} = \frac{1}{12} m(a^2 + b^2)$$

$$m = \rho t A = \rho tab$$

**A thin circular plate:**



$$I_{x,\text{mass}} = \rho t \left( \frac{1}{4} \pi r^4 \right) = \frac{1}{4} mr^2$$

$$I_{y,\text{mass}} = \frac{1}{4} mr^2$$

$$I_{z,\text{mass}} = \frac{1}{2} mr^2$$

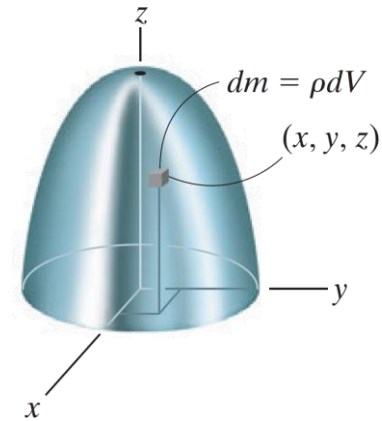
$$m = \rho t A = \rho t \pi r^2$$

# Determining Mass Moment of Inertia

In general, if the body is made of a homogeneous material with a density  $\rho$ ,

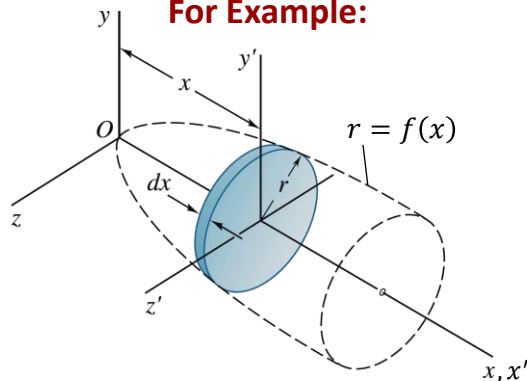
$$I = \int r^2 dm = \rho \int r^2 dV = \rho \int r^2 dx dy dz$$

Thus, it is generally necessary to perform a **triple**, or at least a **double**, **integration** which is purely a function of geometry.



However, it is usually possible to determine the body's mass moment of inertia with a **single integration**. In this cases, we first divide the body into a series of thin, parallel slabs. Then, compute the moment of inertia of the slab with respect to the given axis (use the parallel-axis theorem if necessary) and finally, **integrate** the resulting expression.

**For Example:**



$$dm = \rho dV = \rho \pi r^2 dx$$

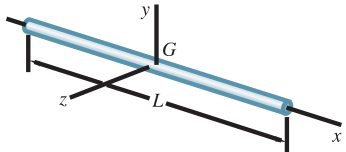
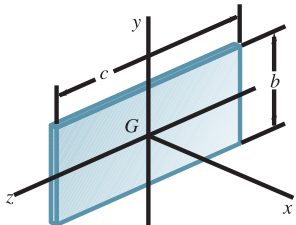
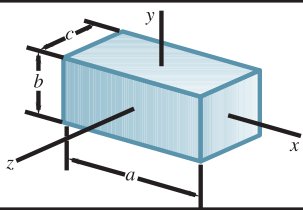
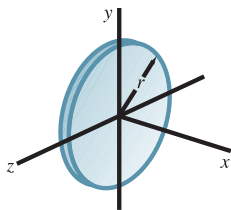
$$dI_x = dI_{x'} = \frac{1}{2} r^2 dm \quad (dI_{x'}, dI_{y'}, dI_{z'} \text{ from previous slide.})$$

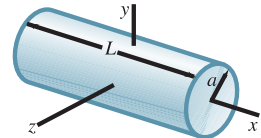
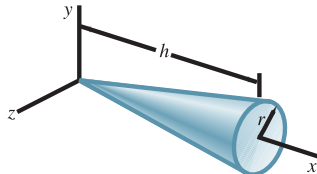
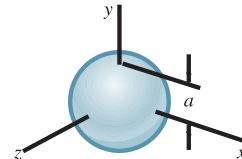
$$dI_y = dI_{y'} + x^2 dm = \left( \frac{1}{4} r^2 + x^2 \right) dm$$

$$dI_z = dI_{z'} + x^2 dm = \left( \frac{1}{4} r^2 + x^2 \right) dm$$

(You can use a similar method for **rectangular** and **triangular** slabs.)

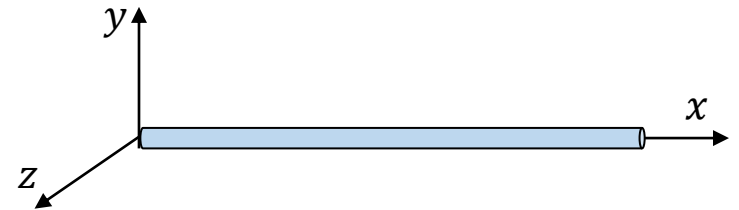
# Mass Moments of Inertia of Common Shapes

Slender rod		$I_y = I_z = \frac{1}{12} mL^2$
Thin rectangular plate		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} mc^2$ $I_z = \frac{1}{12} mb^2$
Rectangular prism		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} m(c^2 + a^2)$ $I_z = \frac{1}{12} m(a^2 + b^2)$
Thin disk		$I_x = \frac{1}{2} mr^2$ $I_y = I_z = \frac{1}{4} mr^2$

Circular cylinder		$I_x = \frac{1}{2} ma^2$ $I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$
Circular cone		$I_x = \frac{3}{10} ma^2$ $I_y = I_z = \frac{3}{5} m\left(\frac{1}{4}a^2 + h^2\right)$
Sphere		$I_x = I_y = I_z = \frac{2}{5} ma^2$

# Sample Problem 9.9

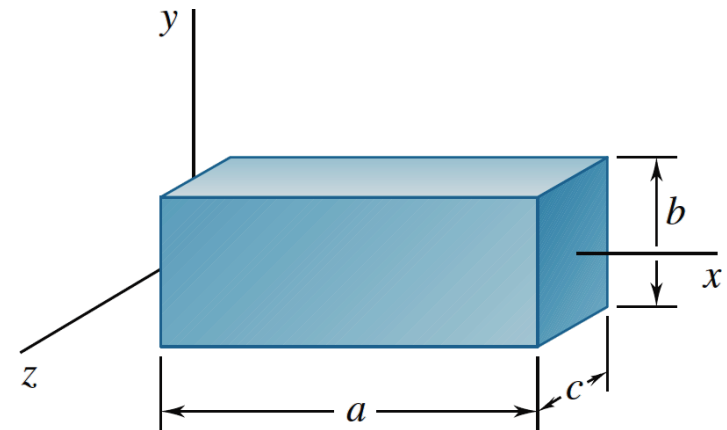
Determine the moment of inertia of a slender rod of length  $L$  and mass  $m$  with respect to an axis that is perpendicular to the rod and passes through one end.





# Sample Problem 9.10

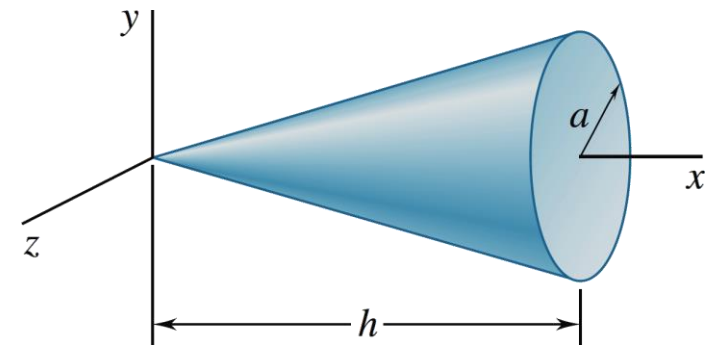
For the homogeneous rectangular prism shown, determine the moment of inertia with respect to the  $z$ -axis.



# Sample Problem 9.11

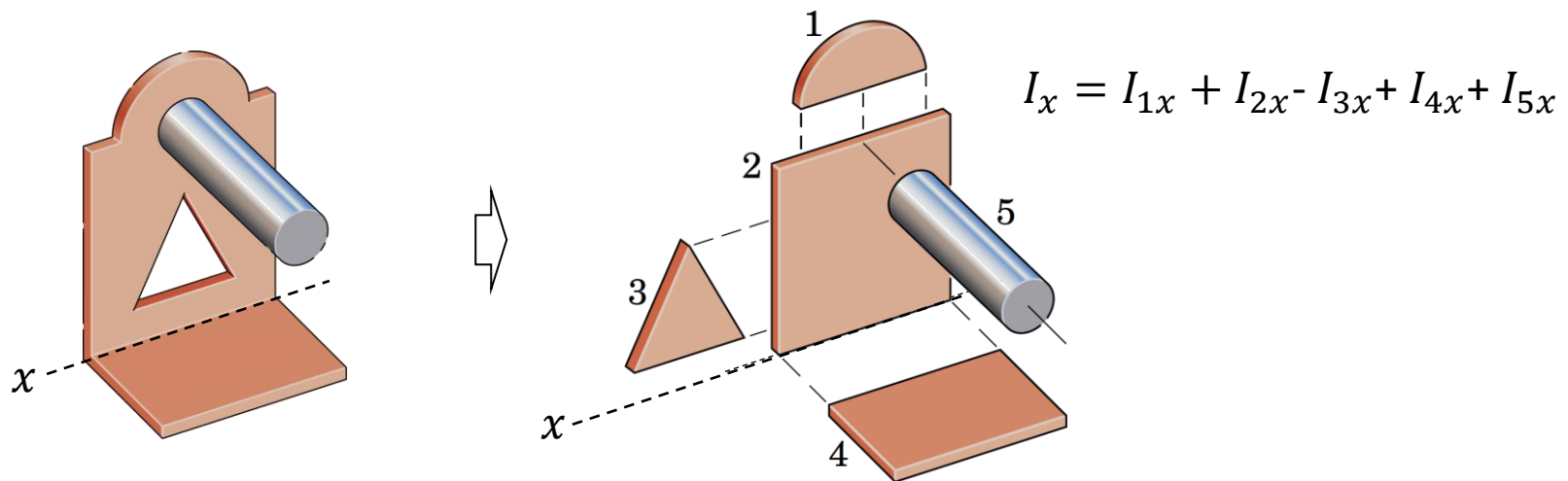
Determine the moment of inertia of a right circular cone with respect to (a) its longitudinal axis, (b) an axis through the apex of the cone and perpendicular to its longitudinal axis, (c) an axis through the centroid of the cone and perpendicular to its longitudinal axis.

$$\bar{x} = \frac{3}{4}h$$



# Moments of Inertia of Composite Bodies

In many instances, we can **divide** a body into the **common shapes** given in the tables. We can obtain the moment of inertia of the body with respect to a given axis by first computing the moments of inertia of its component parts about the given axis (you may need to use the parallel-axis theorem) and then algebraically adding them together.



**Note:** If a composite part has an empty region (hole), its moment of inertia is found by subtracting the moment of inertia of this region from the moment of inertia of the entire part including the region.

**Note:** As was the case for areas, the radius of gyration of a composite body cannot be obtained by adding the radii of gyration of its component parts. You must first compute the moment of inertia of the composite area, and then  $k = \sqrt{I/m}$ .

# Sample Problem 9.12

A steel forging consists of a 6×2×2-in. rectangular prism and two cylinders with a diameter of 2 in. and length of 3 in. as shown. Determine the moments of inertia of the forging with respect to the coordinate axes. The specific weight of steel is 490 lb/ft<sup>3</sup>.

