MEC549: Robot Dynamics and Control

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Ch1: Summary of Linear Algebra & Robot Kinematics

Linear Algebra

Basic Notation

$\forall$

$\exists$

$\in$

$\implies$

$\iff$

$:=$

$\dot{x}$

$f: \mathcal{D} \to \mathcal{R}$

$\mathbb{R}$

$\mathbb{R}_+, \mathbb{R}_{++}$

$|x|$

Vector

$\boldsymbol{x} \in \mathbb{R}^n$: (an n-dimensional real vector in the column format)

$\mathbb{R}^n$: n-dimensional real space (Euclidian Space)

$\boldsymbol{x}^T$:

Vector Norm

General Definition: Given $\boldsymbol{x} \in \mathbb{R}^n$, vector norm $\|\boldsymbol{x}\| \in \mathbb{R}_+$ is defined such that

- $\|\boldsymbol{x}\| > 0$ when $\boldsymbol{x} \neq \mathbf{0}$ and $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \mathbf{0}$.
- $\|k\boldsymbol{x}\| = |k|\|\boldsymbol{x}\|, \ \forall k \in \mathbb{R}$.
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|, \ \forall \boldsymbol{y} \in \mathbb{R}^n$.

❖ The p-norm (or ℓ_p-norm) of $\boldsymbol{x}$ for $p \in \mathbb{R}, p \geq 1$ is defined as $\|\boldsymbol{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

e.g. $\|\boldsymbol{x}\|_2 = \|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$ (Euclidean Norm)

Special case: $\|\boldsymbol{x}\|_\infty := \max_i |x_i|$

Schwartz Inequality: $|\boldsymbol{x}^T\boldsymbol{y}| \leq \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

Unit Vector: $\|\widehat{\boldsymbol{x}}\|_2 = \widehat{\boldsymbol{x}}^T\widehat{\boldsymbol{x}} = 1, \qquad \widehat{\boldsymbol{x}} = \boldsymbol{x}/\|\boldsymbol{x}\|_2$

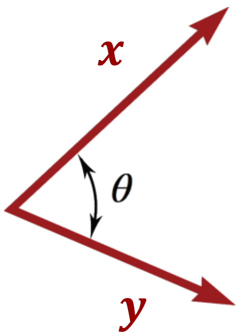
Dot Product or Scalar Product or Inner Product

Dot Product or Scalar Product or Inner Product of two vectors $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^n$ is a scalar defined as

(Algebraic Definition) $$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i = \boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{x}$$

(Geometric Definition) $$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \cos\theta$$

$$(0 \le \theta \le 180°)$$



Orthogonal Vectors:

Matrix

$\boldsymbol{A} \in \mathbb{R}^{m \times n}$ (an m by n dimensional real matrix)

$\boldsymbol{A}^T$

Matrix-vector multiplication $\boldsymbol{Ax}$ as linear combination of columns of $\boldsymbol{A}$:

Particular Matrices

Square Matrix:

- Upper Triangular
- Lower Triangular
- Diagonal
    - Identity Matrix
- Null Matrix

Symmetric Matrix:

Skew-symmetric Matrix:

Partitioned Matrix: A matrix whose elements are matrices (blocks) of proper dimensions.

Matrix Operations

Trace of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$: $\operatorname{tr}(\boldsymbol{A})$

Sum of matrices: $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$

Symmetric and skew-symmetric part of a square matrix $\boldsymbol{A}$:

Product of matrices: $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$

Determinant of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$: $\det(\boldsymbol{A})$

Singular and Nonsingular Matrices:

Matrix Operations

Rank of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$: $\mathrm{rank}(\boldsymbol{A})$

Inverse of $\boldsymbol{A}$: $\boldsymbol{A}^{-1}$

Orthogonal Matrix:

Linearly Independent Vectors $\boldsymbol{x}_i \in \mathbb{R}^m, i = 1, \dots, n$

Derivative of $\boldsymbol{A}(t) \in \mathbb{R}^{m \times n}$: $\dfrac{d}{dt}\boldsymbol{A}(t) = \dot{\boldsymbol{A}}(t)$

Derivative of $\boldsymbol{A}^{-1}(t) \in \mathbb{R}^{n \times n}$:

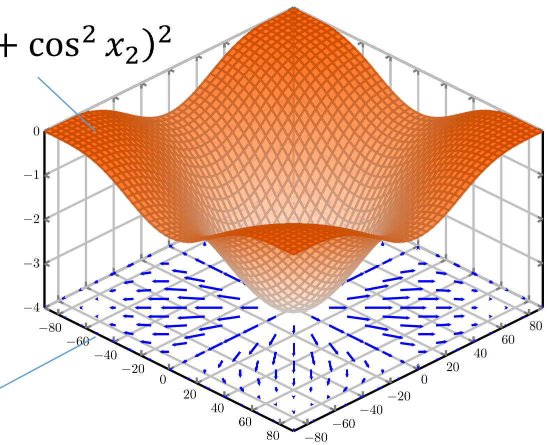
Gradient

For a **scalar function** $f: \mathbb{R}^n \to \mathbb{R}$ which is differentiable with respect to the elements x_i of $\boldsymbol{x} \in \mathbb{R}^n$, its **gradient** with respect to $\boldsymbol{x}$ is an n-dimensional column vector $\boldsymbol{\nabla}_{\boldsymbol{x}} f \in \mathbb{R}^n$ as:

(nabla symbol and pronounced "del")

$$\boldsymbol{\nabla}_{\boldsymbol{x}} f(\boldsymbol{x}) = \left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^T = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}(\boldsymbol{x}) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\boldsymbol{x}) \end{bmatrix}$$

$f(x_1, x_2) = -(\cos^2 x_1 + \cos^2 x_2)^2$



The gradient depicted as a projected vector field.

If $\boldsymbol{x}(t)$ is a differentiable function with respect to t:

$$\dot{f}(\boldsymbol{x}) = \frac{d}{dt} f\big(\boldsymbol{x}(t)\big) = \frac{\partial f}{\partial \boldsymbol{x}} \frac{d\boldsymbol{x}}{dt} = \frac{\partial f}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = \boldsymbol{\nabla}_{\boldsymbol{x}}^T f(\boldsymbol{x}) \dot{\boldsymbol{x}} \qquad \text{(Chain Rule)}$$

Jacobian

For a **vector function** $\boldsymbol{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose elements f_i are differentiable with respect to the elements x_i of $\boldsymbol{x} \in \mathbb{R}^n$, its **Jacobian** with respect to $\boldsymbol{x}$ is matrix $\boldsymbol{J_f} \in \mathbb{R}^{m \times n}$ as:

$$\boldsymbol{J_f}(\boldsymbol{x}) = \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \frac{\partial f_2(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \\ \frac{\partial f_m(\boldsymbol{x})}{\partial \boldsymbol{x}} \end{bmatrix}$$

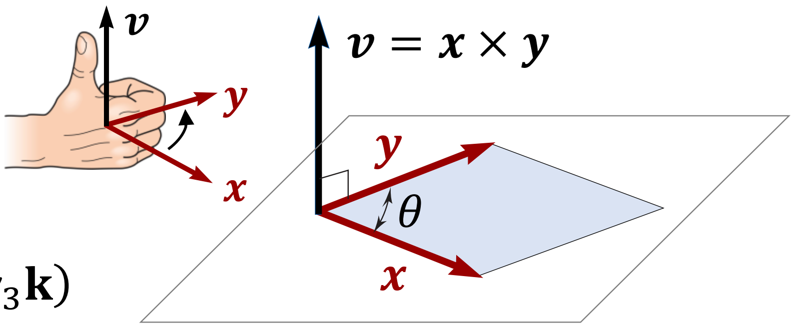
If $\boldsymbol{x}(t)$ is a differentiable function with respect to t:

$$\dot{\boldsymbol{f}}(\boldsymbol{x}) = \frac{d}{dt}\boldsymbol{f}\big(\boldsymbol{x}(t)\big) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \boldsymbol{J_f}(\boldsymbol{x})\dot{\boldsymbol{x}} \qquad \text{(Chain Rule)}$$

Cross Product or Vector Product

Cross product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ (in the Euclidean space) is defined as a <u>vector</u> $\boldsymbol{v} = \boldsymbol{x} \times \boldsymbol{y} \in \mathbb{R}^3$ that is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$ ($\boldsymbol{v} \perp \boldsymbol{x}$, $\boldsymbol{v} \perp \boldsymbol{y}$), with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

$$\|\boldsymbol{v}\|_2 = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \sin\theta \qquad (0 \le \theta \le 180°)$$



$$\boldsymbol{v} = \boldsymbol{x} \times \boldsymbol{y} = (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) \times (y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})$$

Coordinate notation

$$\boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3)$$

$$\boldsymbol{v} = \boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Matrix notation

Cross Product as a Matrix-Vector Multiplication

Cross product $\boldsymbol{x} \times \boldsymbol{y}$ ($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$) can be thought of as a multiplication of a vector by a 3×3 skew-symmetric matrix as

$$\boldsymbol{x} \times \boldsymbol{y} = \underbrace{\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}}_{[\boldsymbol{x}]} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [\boldsymbol{x}]\boldsymbol{y} = -[\boldsymbol{y}]\boldsymbol{x}$$

The matrix $[\boldsymbol{x}]$ is a <u>3x3 skew-symmetric matrix representation</u> of $\boldsymbol{x}$. $[\boldsymbol{x}] = -[\boldsymbol{x}]^T$

Eigenvalues and Eigenvectors

If the vector resulting from the linear transformation $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ on a vector $\boldsymbol{u}$ has the same direction of $\boldsymbol{u}$ (with $\boldsymbol{u} \neq \boldsymbol{0}$), then $\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}$.

For each square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ there exist n **eigenvalues** (in general, complex numbers) denoted by $\lambda_i(\boldsymbol{A})$, $i = 1, \dots, n$ that satisfy

(characteristic equation) $\det(\boldsymbol{A} - \lambda_i(\boldsymbol{A})\boldsymbol{I}) = 0$ $\qquad \boldsymbol{I} = \mathrm{diag}(1) \in \mathbb{R}^{n\times n}$

- If $\boldsymbol{A} = \boldsymbol{A}^T$, then $\lambda_i(\boldsymbol{A}) \in \mathbb{R}$, $i = 1, \dots, n$.

Eigenvectors $\boldsymbol{u}_i$ associated with the eigenvalues λ_i satisfy $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{u}_i = \boldsymbol{0} \qquad i = 1, \dots, n$

- If the eigenvectors $\boldsymbol{u}_i$ of $\boldsymbol{A}$ are linearly independent, matrix $\boldsymbol{U}$ formed by the column vectors $\boldsymbol{u}_i$ is invertible and $\boldsymbol{\Lambda} = \boldsymbol{U}^{-1}\boldsymbol{A}\boldsymbol{U}$ where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. If $\boldsymbol{A}$ is symmetric, $\boldsymbol{U}$ is orthogonal ($\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$) and $\boldsymbol{\Lambda} = \boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U}$.

  $\Rightarrow$ Eigendecomposition: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$ and if $\boldsymbol{A}$ is symmetric $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$.

- $\det(\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i$
- $\lambda(\boldsymbol{A}^T) = \lambda(\boldsymbol{A})$
- $\lambda(\boldsymbol{A}^{-1}) = 1/\lambda(\boldsymbol{A})$

Matrix Norm

General Definition: Given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, vector norm $||\boldsymbol{A}|| \in \mathbb{R}_+$ is defined such that

- $||\boldsymbol{A}|| > 0$ when $\boldsymbol{A} \neq \mathbf{0}$ and $||\boldsymbol{A}|| = 0$ iff $\boldsymbol{A} = \mathbf{0}$.
- $||k\boldsymbol{A}|| = |k|||\boldsymbol{A}||, \ \forall k \in \mathbb{R}$.
- $||\boldsymbol{A} + \boldsymbol{B}|| \leq ||\boldsymbol{A}|| + ||\boldsymbol{B}||, \ \forall \boldsymbol{B} \in \mathbb{R}^{n\times n}$.
- $||\boldsymbol{AB}|| \leq ||\boldsymbol{A}||||\boldsymbol{B}||, \ \forall \boldsymbol{B} \in \mathbb{R}^{n\times n}$.

The p-norm of $\boldsymbol{A}$ (induced by vector p-norms) for $0 \leq p \leq \infty$ is defined as

$$||\boldsymbol{A}||_p = \sup_{||\boldsymbol{x}||_p=1} ||\boldsymbol{Ax}||_p = \sup_{\boldsymbol{x}\neq 0} \frac{||\boldsymbol{Ax}||_p}{||\boldsymbol{x}||_p} \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

Matrix Norm (cont.)

In the special cases of $p = 1,2,\infty$, these norms can be computed/estimated by:

- $\|\boldsymbol{A}\|_1 = \max\limits_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$ (the max. absolute column sum of $\boldsymbol{A}$)
- $\|\boldsymbol{A}\|_2 = \sqrt{\lambda_{\max}(\boldsymbol{A}^T\boldsymbol{A})} = \sigma_{\max}(\boldsymbol{A})$ (Spectral Norm) $\xrightarrow{\text{If } \boldsymbol{A} = \boldsymbol{A}^T} \begin{cases} \|\boldsymbol{A}\|_2 = \max\limits_i |\lambda_i(\boldsymbol{A})| \\ \|\boldsymbol{A}^{-1}\|_2 = 1/\min\limits_i |\lambda_i(\boldsymbol{A})| \end{cases}$

  (the square root of the maximum eigenvalue of $\boldsymbol{A}^T\boldsymbol{A}$, or the largest singular value of $\boldsymbol{A}$)

- $\|\boldsymbol{A}\|_\infty = \max\limits_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$ (the max. absolute row sum of $\boldsymbol{A}$)
- **Frobenius Norm**: $\|\boldsymbol{A}\|_F = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = \sqrt{\operatorname{tr}(\boldsymbol{A}^T\boldsymbol{A})}$

$\|\boldsymbol{A}\boldsymbol{x}\|_2 \le \|\boldsymbol{A}\|_2 \|\boldsymbol{x}\|_2$, $\quad \|\boldsymbol{A}\|_2 \le \|\boldsymbol{A}\|_F$

Quadratic Form

A **Quadratic Form** is a polynomial with terms all of degree two:

$$Q(x) = ax^2$$
$$Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$$
$$Q(x_1, x_2, x_3) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_2x_3 + ex_3^2 + fx_1x_3$$

The quadratic form associated with a $n \times n$ matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is the function $Q: \mathbb{R}^n \to \mathbb{R}$ such that $Q(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ for all $\boldsymbol{x}$.

- The quadratic function associated with a skew-symmetric matrix $\boldsymbol{A}_{ss}$ is always **zero**.

$$\boldsymbol{A}_{ss} \text{ is skew-symmetric} \qquad \Leftrightarrow \qquad \boldsymbol{x}^T\boldsymbol{A}_{ss}\boldsymbol{x} = 0 \quad (\forall \boldsymbol{x})$$

- Each quadratic function $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is always equal to a quadratic function with the symmetric part of matrix.

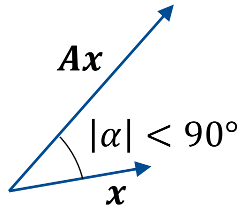
$$Q(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T(\boldsymbol{A}_s + \boldsymbol{A}_{ss})\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{A}_s\boldsymbol{x}$$

- If $\boldsymbol{A} = \boldsymbol{A}^T$: $\quad \boldsymbol{\nabla}_x Q(\boldsymbol{x}) = \left(\dfrac{\partial Q(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T = 2\boldsymbol{A}\boldsymbol{x}\,, \qquad \dot{Q}(x) = \dfrac{d}{dt}Q(\boldsymbol{x}(t)) = 2\boldsymbol{x}^T\boldsymbol{A}\dot{\boldsymbol{x}} + \boldsymbol{x}^T\dot{\boldsymbol{A}}\boldsymbol{x}$

Definite and Semi-Definite Matrices

A square not necessarily symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is

- **Positive Definite** (PD or $\boldsymbol{A} > 0$) if $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} > 0$ for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$.
- **Positive Semi-Definite** (PSD or $\boldsymbol{A} \geq 0$) if $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
- **Negative Definite** (ND or $\boldsymbol{A} < 0$) if $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} < 0$ for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$.
- **Negative Semi-Definite** (NSD or $\boldsymbol{A} \leq 0$) if $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \leq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
- **Indefinite** if $\boldsymbol{A}$ neither positive semi-definite nor negative semi-definite.



Geometric Interpretation of the Positive Definiteness of $\boldsymbol{A}$.

- A square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is negative definite if $-\boldsymbol{A}$ is positive definite and it is negative semidefinite if $-\boldsymbol{A}$ is positive semidefinite.

- A **necessary** condition for $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ to be PD [or PSD] is that its diagonal elements be strictly positive [or nonnegative].

- Since $\boldsymbol{x}^T \boldsymbol{A}_{ss} \boldsymbol{x} = 0$, the test for the definiteness of $\boldsymbol{A}$ may be done by considering only its symmetric part.

Definite and Semi-Definite Matrices (cont.)

$\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is symmetric and PD [or PSD]. $\Leftrightarrow$ (Sylvester's theorem) Principal minors (i.e., a_{11}, $a_{11}a_{22} - a_{21}a_{12}$, ..., det $\boldsymbol{A}$) all are strictly positive [or nonnegative]. $\Leftrightarrow$ All its eigenvalues are strictly positive [or nonnegative].

$\downarrow$

- Any symmetric PD matrix $\boldsymbol{A} = \boldsymbol{A}^T > 0$ is always full-rank and nonsingular.

- Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric PD matrix and $\lambda_{\min}$, $\lambda_{\max}$ be the minimum and maximum eigenvalues of $\boldsymbol{A}$. For any $\boldsymbol{x} \in \mathbb{R}^n$,

$(\boldsymbol{x}^T\boldsymbol{x} = \|x\|_2^2)$

$$\lambda_{\min}(\boldsymbol{A})\boldsymbol{x}^T\boldsymbol{x} \le \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le \lambda_{\max}(\boldsymbol{A})\boldsymbol{x}^T\boldsymbol{x} \qquad \text{(Rayleigh–Ritz Theorem)}$$

- Semi-definiteness implies that $\mathrm{rank}(\boldsymbol{A}) = r < n$, and thus r eigenvalues of $\boldsymbol{A}$ are positive/negative and $n - r$ are 0.

- A matrix inequality of the form $\boldsymbol{A}_1 > \boldsymbol{A}_2$, where $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{n\times n}$ means that $\boldsymbol{A}_1 - \boldsymbol{A}_2 > 0$, i.e., $\boldsymbol{A}_1 - \boldsymbol{A}_2$ is PD. Similar notations apply to the concepts of PSD, ND, NSD.

Rigid-Body Motions

Rigid-Body Motions

Rotations	Rigid-Body Motions
$\boldsymbol{R} \in SO(3)$: 3×3 matrices $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \det(\boldsymbol{R}) = 1$	$\boldsymbol{T} \in SE(3)$: 4×4 matrices $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix},$ where $\boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3$
$\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$	$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$
Change of coordinate frame: $\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{ac}, \ \boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$	Change of coordinate frame: $\boldsymbol{T}_{ab}\boldsymbol{T}_{bc} = \boldsymbol{T}_{ac}, \ \boldsymbol{T}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$

Rigid-Body Motions

Rotations	Rigid-Body Motions
Rotating a frame $\{\mathrm{b}\}$: $$\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$ $\boldsymbol{R}_{sb'} = \boldsymbol{R}\boldsymbol{R}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ $\boldsymbol{R}_{sb''} = \boldsymbol{R}_{sb}\boldsymbol{R}$: rotate θ about $\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}$	Displacing a frame $\{\mathrm{b}\}$: $$\boldsymbol{T} = \begin{bmatrix} \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$ $\boldsymbol{T}_{sb'} = \boldsymbol{T}\boldsymbol{T}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ (moves $\{\mathrm{b}\}$ origin), translate $\boldsymbol{p}$ in $\{\mathrm{s}\}$ $\boldsymbol{T}_{sb''} = \boldsymbol{T}_{sb}\boldsymbol{T}$: translate $\boldsymbol{p}$ in $\{\mathrm{b}\}$, rotate θ about $\hat{\boldsymbol{\omega}}$ in new body frame
Unit rotation axis is $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$, where $\lVert\hat{\boldsymbol{\omega}}\rVert = 1$	"Unit" screw axis is $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\lVert\boldsymbol{S}_\omega\rVert = 1$ or (ii) $\lVert\boldsymbol{S}_\omega\rVert = 0$, $\lVert\boldsymbol{S}_v\rVert = 1$
	For a screw axis $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$ with finite h, $$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}} \\ -\hat{\boldsymbol{s}} \times \boldsymbol{q} + h\hat{\boldsymbol{s}} \end{bmatrix}$$
Angular velocity is $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}$	Twist is $\boldsymbol{\mathcal{V}} = \boldsymbol{S}\dot{\theta}$

Rigid-Body Motions

Rotations	Rigid-Body Motions
For any $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, $$[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$ **Properties**: For any $\boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)$: $$[\boldsymbol{\omega}] = -[\boldsymbol{\omega}]^{\mathrm{T}}, [\boldsymbol{\omega}]\boldsymbol{x} = -[\boldsymbol{x}]\boldsymbol{\omega},$$ $$[\boldsymbol{\omega}][\boldsymbol{x}] = ([\boldsymbol{x}][\boldsymbol{\omega}])^T, \boldsymbol{R}[\boldsymbol{\omega}]\boldsymbol{R}^{\mathrm{T}} = [\boldsymbol{R}\boldsymbol{\omega}]$$	For any $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} \in \mathbb{R}^6$ or $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$, $$[\boldsymbol{\mathcal{V}}] = \begin{bmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix} \in se(3),$$ $$[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$$
$\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s],\ \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b] \quad (\boldsymbol{R} \coloneqq \boldsymbol{R}_{sb})$	$\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\boldsymbol{\mathcal{V}}_s],\ \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{\mathcal{V}}_b] \quad (\boldsymbol{T} \coloneqq \boldsymbol{T}_{sb})$
	$$[\mathrm{Ad}_{\boldsymbol{T}}] = \begin{bmatrix} \boldsymbol{R} & \mathbf{0} \\ [\boldsymbol{p}]\boldsymbol{R} & \boldsymbol{R} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$ **Properties**: $[\mathrm{Ad}_{\boldsymbol{T}}]^{-1} = [\mathrm{Ad}_{\boldsymbol{T}^{-1}}]$, $[\mathrm{Ad}_{\boldsymbol{T}_1}][\mathrm{Ad}_{\boldsymbol{T}_2}] = [\mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]$
Change of coordinate frame: $$\hat{\boldsymbol{\omega}}_a = \boldsymbol{R}_{ab}\hat{\boldsymbol{\omega}}_b, \quad \boldsymbol{\omega}_a = \boldsymbol{R}_{ab}\boldsymbol{\omega}_b$$	Change of coordinate frame: $$\boldsymbol{S}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b, \quad \boldsymbol{\mathcal{V}}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{\mathcal{V}}_b$$

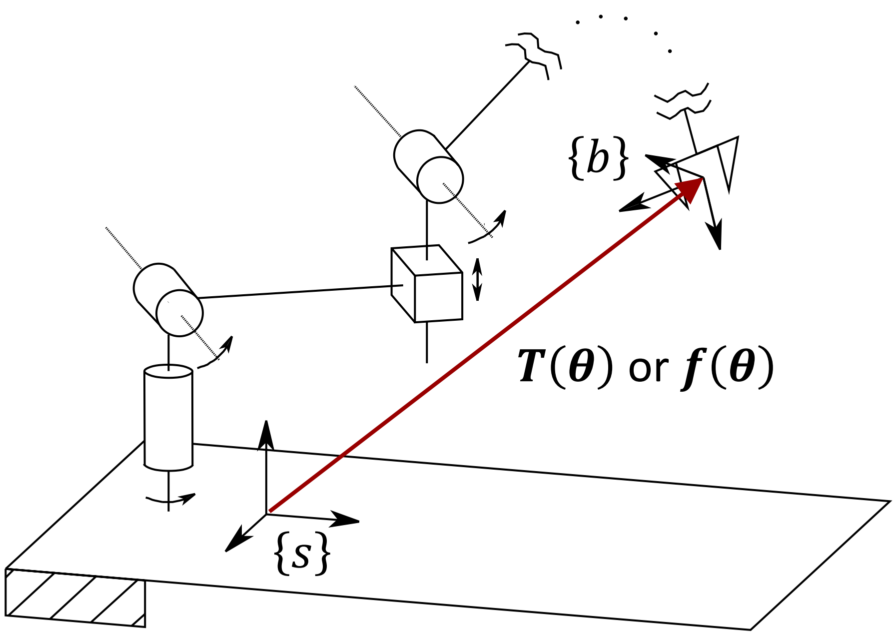
Rigid-Body Motions

Rotations	Rigid-Body Motions
$\hat{\boldsymbol{\omega}}_s = \boldsymbol{R}_{sb}\hat{\boldsymbol{\omega}}_b$	$\boldsymbol{S}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{S}_b$, $\boldsymbol{\mathcal{V}}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{\mathcal{V}}_b$, $[\mathrm{Ad}_{\boldsymbol{T}}] = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ [\boldsymbol{p}]\boldsymbol{R} & \boldsymbol{R} \end{bmatrix}$
Exponential coordinate for $\boldsymbol{R} \in SO(3)$: $\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$	Exponential coordinate for $\boldsymbol{T} \in SE(3)$: $\boldsymbol{S}\theta \in \mathbb{R}^6$
$\exp: [\hat{\boldsymbol{\omega}}]\theta \in so(3) \to \boldsymbol{R} \in SO(3)$ $\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = e^{[\hat{\boldsymbol{\omega}}]\theta}$ $\boldsymbol{R} = \boldsymbol{I} + \sin\theta[\hat{\boldsymbol{\omega}}] + (1-\cos\theta)[\hat{\boldsymbol{\omega}}]^2$ (Rodrigues' formula for rotations)	$\exp: [\boldsymbol{S}]\theta \in se(3) \to \boldsymbol{T} \in SE(3)$ $\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$ $\boldsymbol{T} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix}$ $\boldsymbol{G}(\theta) = \boldsymbol{I}\theta + (1-\cos\theta)[\boldsymbol{S}_\omega] + (\theta - \sin\theta)[\boldsymbol{S}_\omega]^2$
$\log: \boldsymbol{R} \in SO(3) \to [\hat{\boldsymbol{\omega}}]\theta \in so(3)$ $\log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta$	$\log: \boldsymbol{T} \in SE(3) \to [\boldsymbol{S}]\theta \in se(3)$ $\log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$
Moment change of coordinate frame: $\boldsymbol{m}_a = \boldsymbol{R}_{ab}\boldsymbol{m}_b$	Wrench change of coordinate frame: $\boldsymbol{\mathcal{F}}_a = (\boldsymbol{m}_a, \boldsymbol{f}_a) = [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{\mathrm{T}}\boldsymbol{\mathcal{F}}_b$

Forward/Velocity/Inverse Kinematics

Forward Kinematics

The forward kinematics of a robot refers to the calculation of the position and orientation (**pose**) of its end-effector frame from its joint positions $\boldsymbol{\theta}$.



- "Geometric" forward kinematics:

Given $\boldsymbol{\theta} \in \mathbb{R}^n$, Find $\boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$

$$\boldsymbol{T}: \mathbb{R}^n \rightarrow SE(3)$$

(Using PoE or D-H Method)

$$\boldsymbol{T}(\boldsymbol{\theta}) = e^{[\boldsymbol{S}_1]\theta_1} \dots e^{[\boldsymbol{S}_{n-1}]\theta_{n-1}} e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M}$$

where $\boldsymbol{M} = \boldsymbol{T}_{sb}(\boldsymbol{0}) \in SE(3)$ and $\boldsymbol{S}_1, \dots, \boldsymbol{S}_n$ are screw axes expressed in $\{s\}$ when $\boldsymbol{\theta} = \boldsymbol{0}$.

- "Minimum-Coordinate" forward kinematics:

Given $\boldsymbol{\theta} \in \mathbb{R}^n$, Find $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$

$$(m \leq n) \qquad \boldsymbol{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Velocity Kinematics

- $\mathcal{V}_s = \begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\mathcal{V}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Geometric Jacobian

- $\begin{bmatrix} \dot{\boldsymbol{\phi}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \boldsymbol{\phi} = (\alpha, \beta, \gamma)$
- $\begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,q}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \boldsymbol{q} = (q_0, q_1, q_2, q_3)$
- $\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \boldsymbol{r} = \hat{\boldsymbol{\omega}}\theta$

Analytic Jacobian

- $\boldsymbol{J}_s(\boldsymbol{\theta}) = [\boldsymbol{J}_{s1} \quad \boldsymbol{J}_{s2}(\boldsymbol{\theta}) \quad \cdots \quad \boldsymbol{J}_{sn}(\boldsymbol{\theta})]$, $\boldsymbol{J}_{si}(\boldsymbol{\theta}) = \left[\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}\right] \boldsymbol{\mathcal{S}}_i \qquad i = 2, \ldots, n,\ \boldsymbol{J}_{s1} = \boldsymbol{\mathcal{S}}_1$

- $\boldsymbol{J}_b(\boldsymbol{\theta}) = [\mathrm{Ad}_{T_{bs}}]\boldsymbol{J}_s(\boldsymbol{\theta})$

- Statics: $\boldsymbol{\tau} = \boldsymbol{J}_b^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_b$, $\boldsymbol{\tau} = \boldsymbol{J}_s^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_s$

- In singular configuration $\boldsymbol{\theta}^*$, $\boldsymbol{J}(\boldsymbol{\theta}^*) \in \mathbb{R}^{r \times n}$ is rank-deficient, i.e., $\mathrm{rank}\big(\boldsymbol{J}(\boldsymbol{\theta}^*)\big) < r$.

Inverse Kinematics

The inverse kinematics of a robot refers to the calculation of the joint coordinates $\boldsymbol{\theta}$ from the position and orientation (**pose**) of its end-effector frame.

- “Geometric” inverse kinematics:

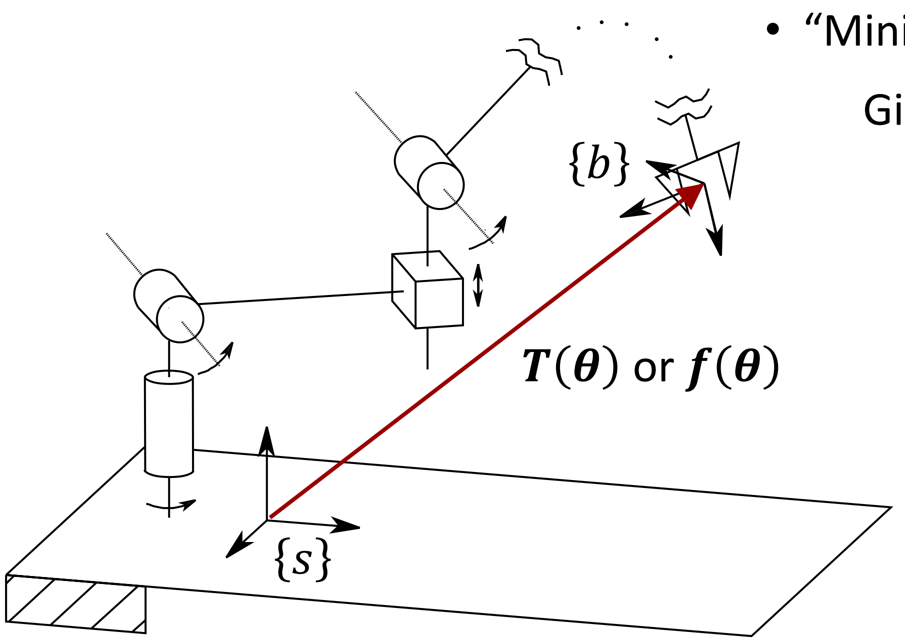
  Given $\boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$, Find $\boldsymbol{\theta} \in \mathbb{R}^n$

- “Minimum-Coordinate” inverse kinematics:

  Given $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$, Find $\boldsymbol{\theta} \in \mathbb{R}^n$

  - **Analytic Methods**: Finding closed-form solutions using algebraic or geometric intuition intuitions.
  - **Iterative Numerical Methods**: For instance, using Newton–Raphson method:

$$\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \boldsymbol{J}^{\dagger}(\boldsymbol{\theta}^i)\boldsymbol{e} = \boldsymbol{\theta}^i + \boldsymbol{J}^{\dagger}(\boldsymbol{\theta}^i)\left(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^i)\right)$$

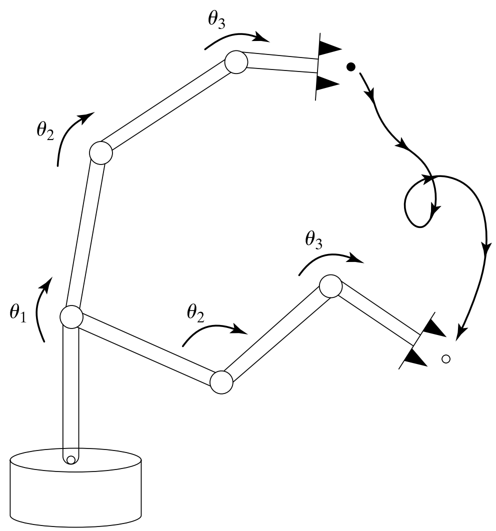


Trajectory Generation

Trajectory Generation: Path & Time Scaling

Trajectory $\mathcal{C}(s(t))$ or $\mathcal{C}(t)$ specifies the robot configuration as a function of time, i.e., the combination of a **path** $\mathcal{C}(s)$ and a **time scaling** $s(t)$.

$$\mathcal{C}: [0,1] \rightarrow \mathbb{C} \qquad s: [0, t_f] \rightarrow [0,1]$$



- Straight-Line Path in Joint Space: $\boldsymbol{\theta}(s) = \boldsymbol{\theta}_{\text{start}} + s(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$

- Straight-Line Path in Task Space:

(1) $\boldsymbol{x}(s) = \boldsymbol{x}_{\text{start}} + s(\boldsymbol{x}_{\text{end}} - \boldsymbol{x}_{\text{start}}) \in \mathbb{R}^m$

(2) $\boldsymbol{p}(s) = \boldsymbol{p}_{\text{start}} + s(\boldsymbol{p}_{\text{end}} - \boldsymbol{p}_{\text{start}}) \in \mathbb{R}^3$

$\boldsymbol{R}(s) = \boldsymbol{R}_{\text{start}} \exp\left(\log\left(\boldsymbol{R}_{\text{start}}^{\mathrm{T}} \boldsymbol{R}_{\text{end}}\right) s\right) \in SO(3)$

(3) $\boldsymbol{T}(s) = \boldsymbol{T}_{\text{start}} \exp(\log(\boldsymbol{T}_{\text{start}}^{-1} \boldsymbol{T}_{\text{end}}) s) \in SE(3)$

Examples of Time Scaling:

- 3rd-Order, 5th-Order Polynomial Position Profile $\begin{cases} s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \end{cases}$
- Trapezoidal/S-Curve Velocity Profile
- Polynomial Via Point Trajectories