Ch7: Trajectory Generation

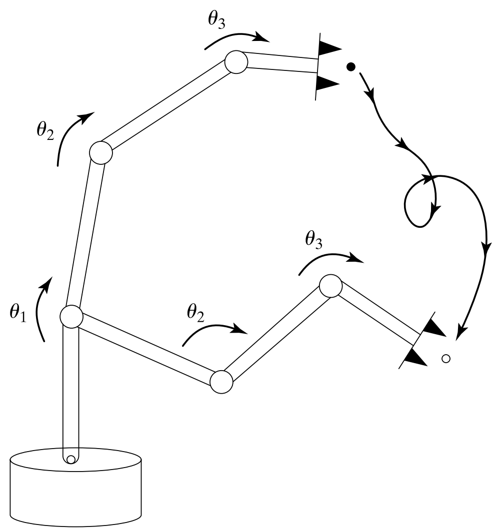
Path and Trajectory

Path and Trajectory

Path $c(s)$ is a purely geometric description of the sequence of configurations achieved by the robot: $c\colon \underbrace{[0,1]}_{} \to \mathbb{C}$

$s \in [0,1]$: scalar path parameter (0 at the start and 1 at the end of the path)

A point in the robot's C-space



- As s increases from 0 to 1, the robot moves along the path.

Time Scaling $s(t)$ specifies the times when those robot configurations are reached:

$$s\colon [0, T] \to [0,1]$$

Trajectory $c\big(s(t)\big)$ or $c(t)$ specifies the robot configuration as a function of time, i.e., the combination of a path and a time scaling.

$$c(t) \qquad \dot{c} = \frac{dc}{ds}\dot{s} \qquad \ddot{c} = \frac{dc}{ds}\ddot{s} + \frac{d^2c}{ds^2}\dot{s}^2$$

Remarks

- In the presence of **path constraints** (e.g., obstacles), **trajectory planning in the task space** may be advisable, because these constraints are typically better described in the task space.

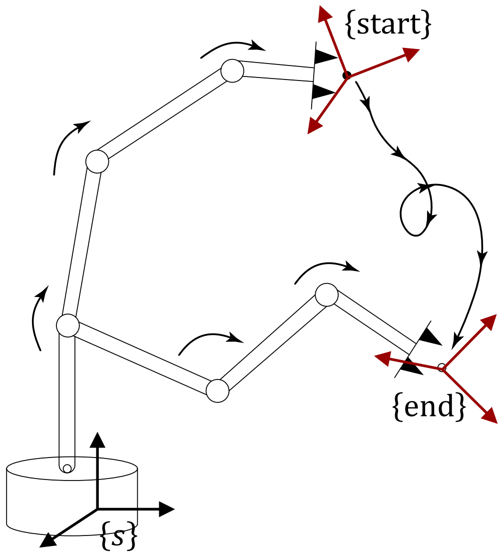
- In the presence of **joint limits**, **motion near singular configuration**, and **redundant DOFs**, trajectory planning in the task space may involve problems difficult to solve and it may be advisable to **plan trajectory in the joint space** to satisfy the constraints imposed on the trajectory.

- Since the control action on the robot is carried out in the joint space, if the trajectory planning is performed in the task space ($\boldsymbol{x}(t)$ or $\boldsymbol{T}(t)$), we have to use **inverse kinematics** to reconstruct the corresponding time sequence of joint variables $\boldsymbol{\theta}(t)$ along the path.

Point-to-Point Trajectories

Point-to-Point Motion

Point-to-Point motion is the simplest type of motion which is from rest at one configuration (start) to rest at another configuration (end).



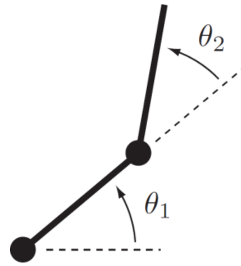
The path for point-to-point motion from a start configuration to an end configuration can be constructed in either **joint space** or **task space**.

(1) Point-to-Point Straight-Line Path in Joint Space

Straight-Line Path in Joint Space: $\boldsymbol{\theta}(s) = \boldsymbol{\theta}_{\text{start}} + s(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$

$s \in [0,1]$, $\boldsymbol{\theta} \in \mathbb{R}^n$ (n: number of joints)

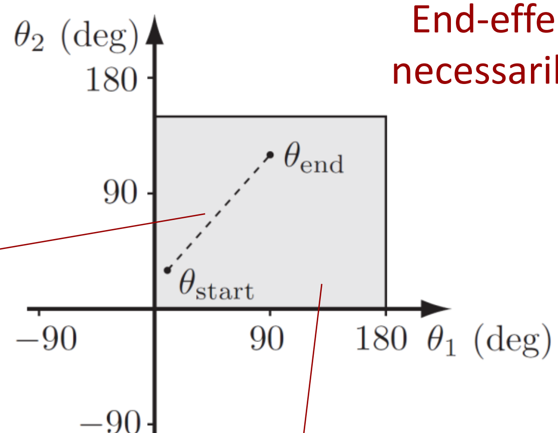
Example: A 2R robot with joint limits: $0° \le \theta_1 \le 180°, 0° \le \theta_2 \le 150°$.

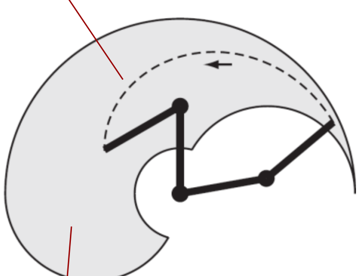


End-effector path is not necessarily a straight-line.

A straight line in **J-space** (joint limits and singularity avoidance are satisfied)

$(\boldsymbol{\theta}_{\min} \le \boldsymbol{\theta} \le \boldsymbol{\theta}_{\max})$





J-space is a convex set

Workspace is not necessarily a convex set.

(2.1) Point-to-Point Straight-Line Path in Task Space (in Cartesian Space $\mathbb{R}^3$)

- If the EE frame is represented by a minimum set of coordinates, i.e., $\boldsymbol{x} \in \mathbb{R}^m$:

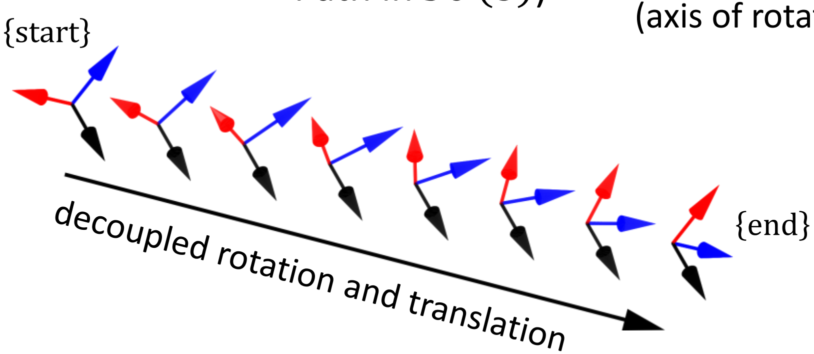
$$\boldsymbol{x}(s) = \boldsymbol{x}_{\text{start}} + s(\boldsymbol{x}_{\text{end}} - \boldsymbol{x}_{\text{start}})$$

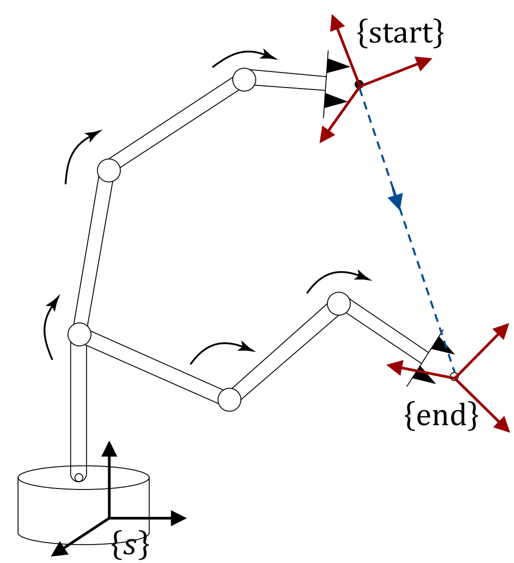
- If the EE frame is represented by position vector $\boldsymbol{p} \in \mathbb{R}^3$ and the rotation matrix $\boldsymbol{R} \in SO(3)$:

$$\begin{cases} \boldsymbol{p}(s) = \boldsymbol{p}_{\text{start}} + s(\boldsymbol{p}_{\text{end}} - \boldsymbol{p}_{\text{start}}) \\ \boldsymbol{R}(s) = \boldsymbol{R}_{\text{start}} \exp\Big(\underbrace{\log\big(\underbrace{\boldsymbol{R}_{\text{start}}^{\mathrm{T}}\boldsymbol{R}_{\text{end}}}_{\boldsymbol{R}_{\text{start,end}}}\big)}_{[\hat{\boldsymbol{\omega}}_{\text{start}}]\phi}\, s\Big) \end{cases}$$

(A Straight-Line Path in $SO(3)$)

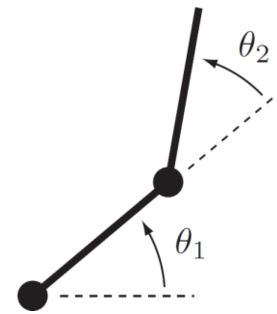
(axis of rotation is constant in {start})



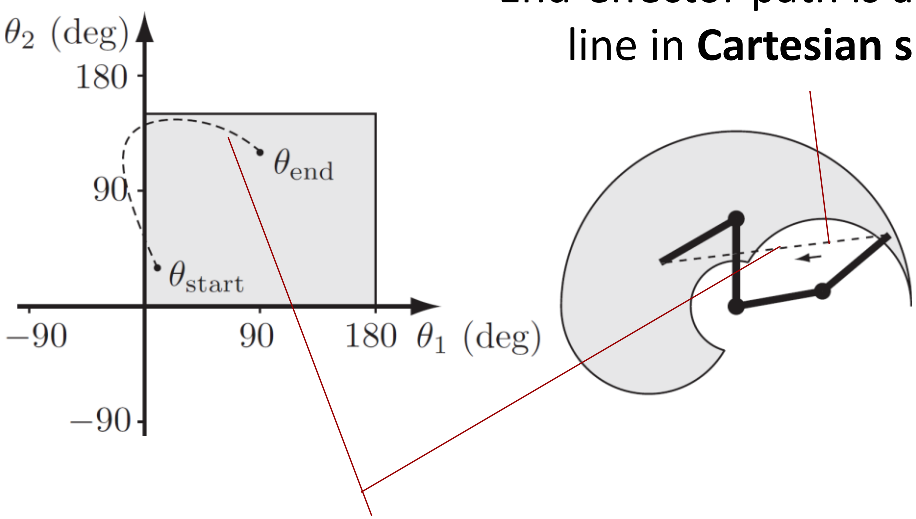


(2.1) Point-to-Point Straight-Line Path in Task Space (in Cartesian Space $\mathbb{R}^3$) (cont.)

Example: A 2R robot with joint limits: $0° \leq \theta_1 \leq 180°, 0° \leq \theta_2 \leq 150°$.



End-effector path is a straight-line in **Cartesian space**.



- The path in the J-space may violate the **joint limits**.
- The path may pass near a **kinematic singularity** (where joint velocities become large).

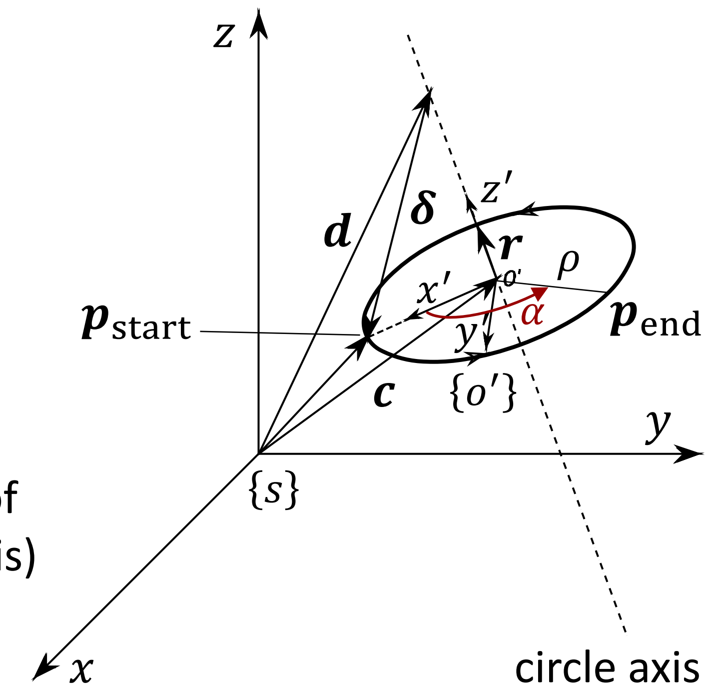
(2.1’) Point-to-Point Circular Path in Task Space (in Cartesian Space $\mathbb{R}^3$)

$\boldsymbol{r}$: unit vector of the circle axis
$\boldsymbol{d}$: a point along the circle axis
$\boldsymbol{p}_{\text{start}}$: of a point on the circle
$\boldsymbol{c}$: center of the circle
ρ: radius of the circle

$$\boldsymbol{\delta} = \boldsymbol{p}_{\text{start}} - \boldsymbol{d}$$

$$\boldsymbol{c} = \boldsymbol{d} + (\boldsymbol{\delta}^T \boldsymbol{r})\boldsymbol{r}$$

$$\rho = \|\boldsymbol{p}_{\text{start}} - \boldsymbol{c}\|$$



Path corresponding motion of position vector $\boldsymbol{p} \in \mathbb{R}^3$ of EE along the circle by an angle α (measured from x'-axis) when s goes from 0 to 1:

$$\boldsymbol{p}(s) = \boldsymbol{c} + \boldsymbol{R}_{so'} \begin{bmatrix} \rho\cos(\alpha s) \\ \rho\sin(\alpha s) \\ 0 \end{bmatrix}$$

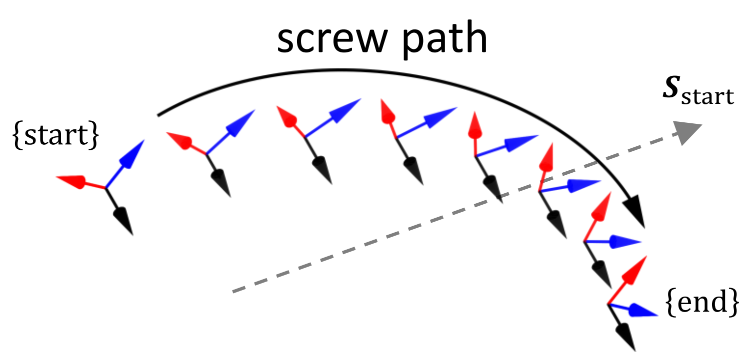
(2.2) Point-to-Point Straight-Line Path in Task Space (in SE(3))

If EE frame is represented by $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) \in SE(3)$:

$$\boldsymbol{T}(s) = \boldsymbol{T}_{\text{start}} \exp(\underbrace{\log(\underbrace{\boldsymbol{T}_{\text{start}}^{-1} \boldsymbol{T}_{\text{end}}}_{\boldsymbol{T}_{\text{start,end}}})}_{[\boldsymbol{S}_{\text{start}}]\phi}\, s)$$

A Straight-Line Path in $SE(3)$

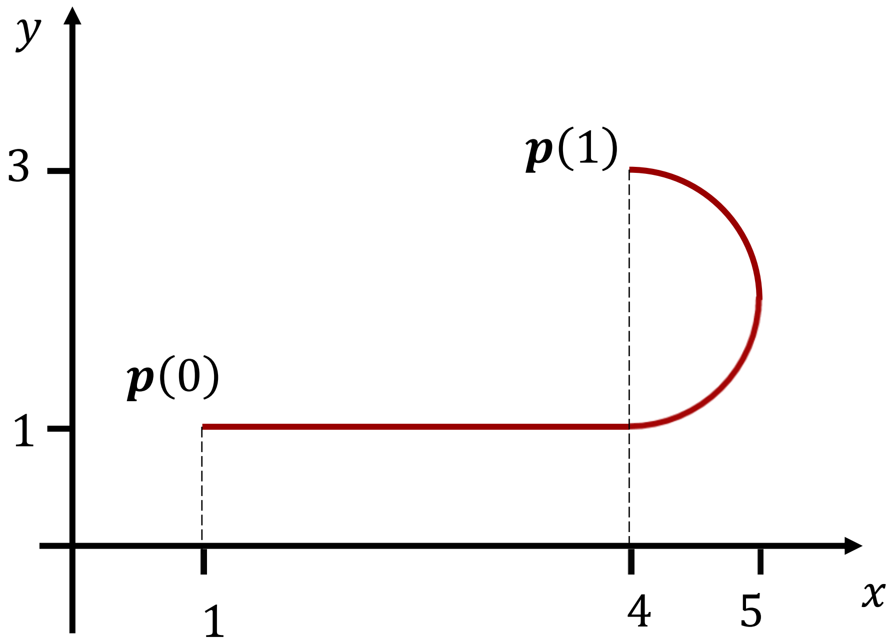
This path is equivalent to a constant screw motion of the EE frame (simultaneous rotation about and translation along a <u>fixed</u> screw axis $\boldsymbol{S}_{\text{start}}$) in Cartesian space $\mathbb{R}^3$.



Note that the origin of the EE frame does not generally follow a straight line in Cartesian space $\mathbb{R}^3$.

Example

Give an expression for the path $\boldsymbol{p}(s) = \big(x(s), y(s)\big) \in \mathbb{R}^2$, $s \in [0,1]$.



Time Scaling a Path

A time scaling $s(t)$ of a path should ensure that the motion is appropriately smooth, and it should satisfy any constraints on joint velocities, accelerations, or torques or EE velocities and accelerations.

∗ Joint velocities and accelerations for straight-line path in joint space:

$$\dot{\boldsymbol{\theta}} = \dot{s}(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}}) \qquad \ddot{\boldsymbol{\theta}} = \ddot{s}(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$$

∗ EE velocities and accelerations for straight-line path in task space parametrized by a minimum set of coordinates $\boldsymbol{x} \in \mathbb{R}^m$:

$$\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{ds}\dot{s} = \dot{s}(\boldsymbol{x}_{\text{end}} - \boldsymbol{x}_{\text{start}}) \qquad \ddot{\boldsymbol{x}} = \frac{d^2\boldsymbol{x}}{ds^2}\ddot{s} = \ddot{s}(\boldsymbol{x}_{\text{end}} - \boldsymbol{x}_{\text{start}})$$

The most common methods for time-scaling $s(t)$:

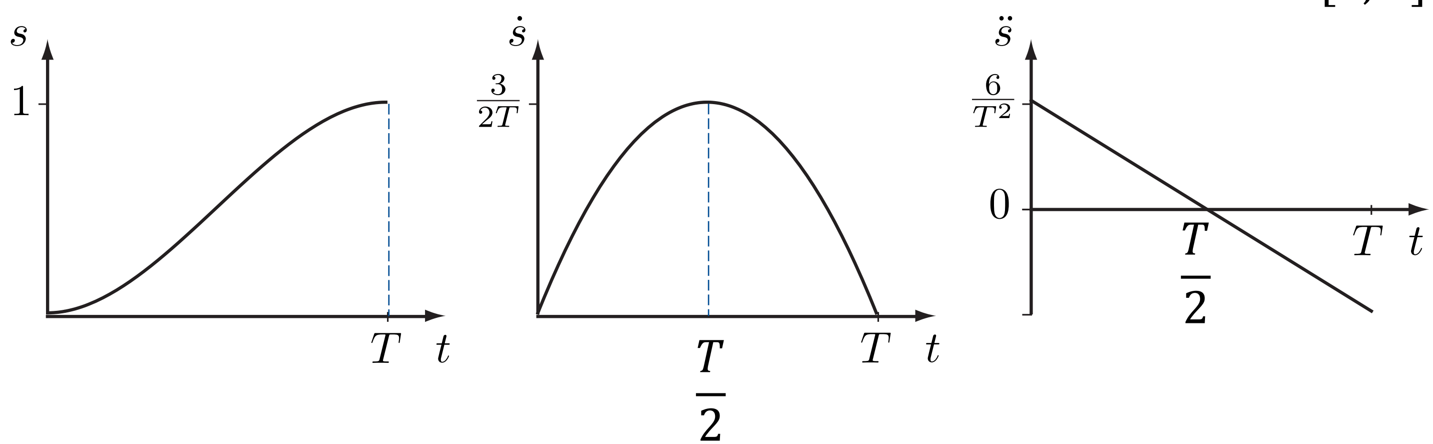
1. 3rd-Order Polynomial Position Profile
2. 5th-Order Polynomial Position Profile
3. Trapezoidal Velocity Profile
4. S-Curve Velocity Profile

1. 3rd-Order Polynomial Position Profile

Time scaling $s(t)$ using 3rd-order polynomial position profile with the motion time T:

$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \xrightarrow[\text{(constraints)}]{\substack{s(0)=0\\ \dot{s}(0)=0\\ s(T)=1\\ \dot{s}(T)=0}} s(t) = \left(\frac{3}{T^2}\right)t^2 + \left(-\frac{2}{T^3}\right)t^3$$

$$t \in [0, T]$$



1. 3rd-Order Polynomial Position Profile (cont.)

For a straight-line path in joint space (i.e., $\boldsymbol{\theta}(s) = \boldsymbol{\theta}_{\text{start}} + s(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$):

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}_{\text{start}} + \left(\frac{3t^2}{T^2} - \frac{2t^3}{T^3}\right)(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}}) \longrightarrow \begin{cases} \dot{\boldsymbol{\theta}}(t) = \left(\frac{6t}{T^2} - \frac{6t^2}{T^3}\right)(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}}) \\ \ddot{\boldsymbol{\theta}}(t) = \left(\frac{6}{T^2} - \frac{12t}{T^3}\right)(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}}) \end{cases}$$

$$\dot{\boldsymbol{\theta}}_{\max} = \dot{\boldsymbol{\theta}}\Big|_{t=T/2} = \frac{3}{2T}(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$$

(maximum joint velocities)

$$\ddot{\boldsymbol{\theta}}_{\max/\min} = \ddot{\boldsymbol{\theta}}\Big|_{t=0 \text{ or } T} = \pm\left|\frac{6}{T^2}(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})\right|$$

(maximum joint accelerations and decelerations)

Note: If there are given limits on the maximum joint velocities and accelerations (i.e., $|\dot{\boldsymbol{\theta}}| \leq \dot{\boldsymbol{\theta}}_{\text{limit}}$, $|\ddot{\boldsymbol{\theta}}| \leq \ddot{\boldsymbol{\theta}}_{\text{limit}}$), we can solve for the minimum possible motion time T that satisfies both constraints.

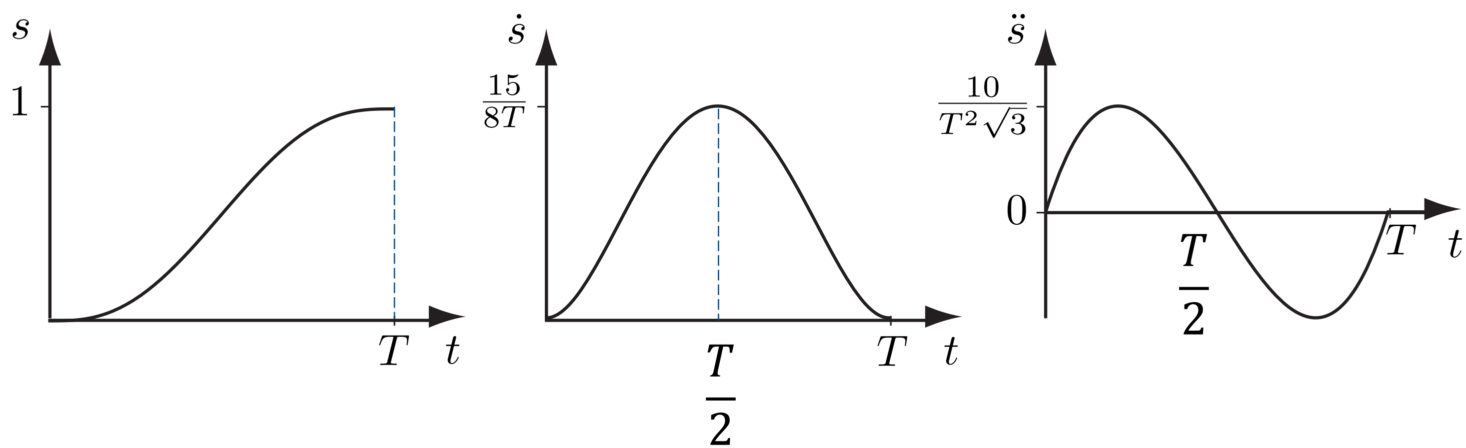
Note: For a straight-line path in task space parametrized by a minimum set of coordinates $\boldsymbol{x} \in \mathbb{R}^m$, simply replace $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$ by $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, and $\ddot{\boldsymbol{x}}$.

2. 5th-Order Polynomial Position Profile

The discontinuous jump in acceleration at both $t = 0$ and $t = T$ of the 3rd-order polynomial position profile (which implies an infinite jerk d^3s/dt^3) may cause vibration of the robot.

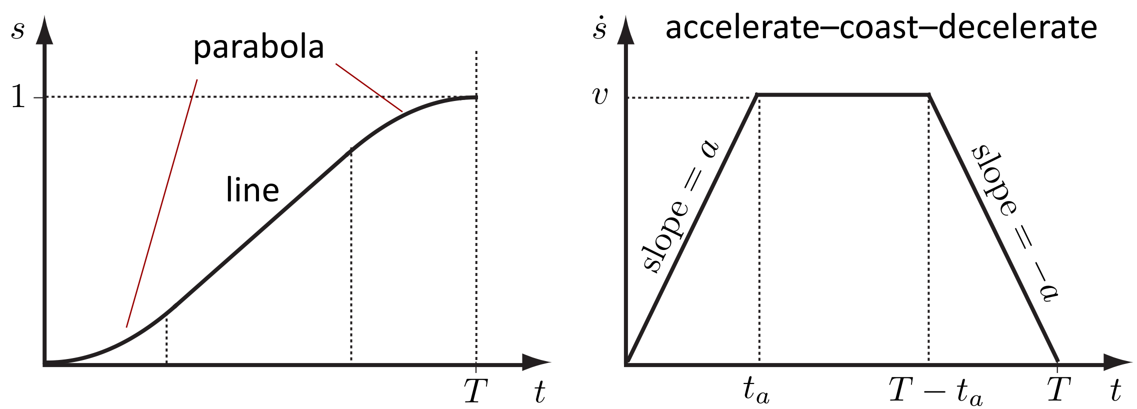
This problem can be solved by adding two constraints $\ddot{s}(0) = \ddot{s}(T) = 0$ and using a 5th-order polynomial position profile for time scaling as

$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \qquad t \in [0, T]$$



3. Trapezoidal Velocity Profile

This motion consists of a constant acceleration phase $\ddot{s} = a$ of time t_a, followed by a constant velocity phase $\dot{s} = v$ of time $T - 2t_a$, followed by a constant deceleration phase $\ddot{s} = -a$ of time t_a.



$$\begin{cases} t_a = \dfrac{v}{a} \\ \displaystyle\int_0^T \dot{s}\, dt = s(T) = 1 \end{cases}$$

Disadvantage: It is not as smooth as the 3rd-order position profile time scaling.

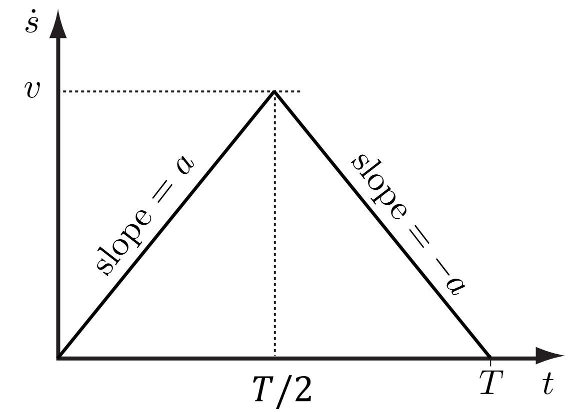
Advantage: If there are known constant limits on the joint velocities $\dot{\boldsymbol{\theta}}_{\text{limit}} \in \mathbb{R}^n$ and joint accelerations $\ddot{\boldsymbol{\theta}}_{\text{limit}} \in \mathbb{R}^n$ this motion with the largest v and a satisfying

$$|(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})v| \leq \dot{\boldsymbol{\theta}}_{\text{limit}}$$
$$|(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})a| \leq \ddot{\boldsymbol{\theta}}_{\text{limit}}$$

is the fastest straight-line motion possible (i.e., minimum T).

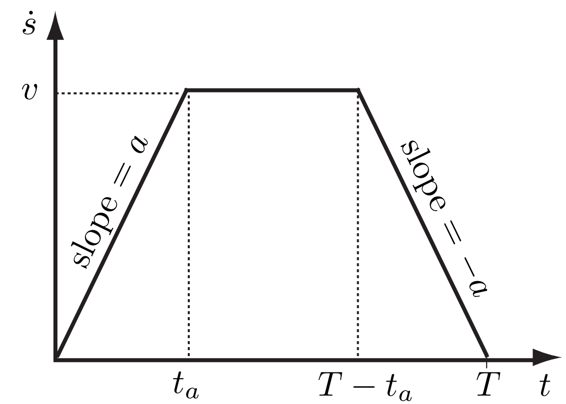
3. Trapezoidal Velocity Profile (cont.)

accelerate–decelerate "bang-bang" motion



After choosing v and a:

- If $v^2/a \geq 1$, the motion never reaches the velocity v (or reaches only at $T/2$) and the velocity profile is triangular.
- If $v^2/a < 1$, the motion reaches the velocity v and the velocity profile is trapezoidal:



$$s(t) = \begin{cases} at^2/2 & 0 \leq t \leq v/a \\ vt - \dfrac{v^2}{2a} & v/a < t \leq T - v/a \\ \dfrac{2avT - 2v^2 - a^2(t-T)^2}{2a} & T - v/a < t \leq T \end{cases}$$

$$\dot{s}(t) = \begin{cases} at & 0 \leq t \leq v/a \\ v & v/a < t \leq T - v/a \\ -a(t-T) & T - v/a < t \leq T \end{cases} \qquad \ddot{s}(t) = \begin{cases} a & 0 \leq t \leq v/a \\ 0 & \dfrac{v}{a} < t \leq T - v/a \\ -a & T - v/a < t \leq T \end{cases}$$

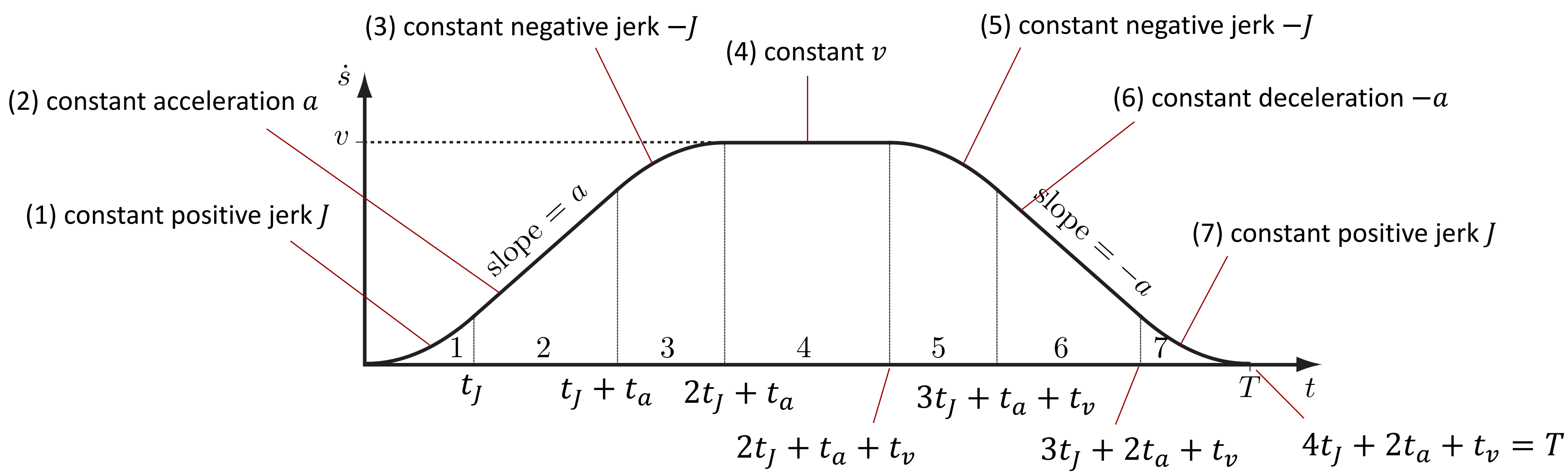
Note: Only two of v, a, and T can be chosen independently, since they must satisfy $s(T) = 1$.

* If v, a are given, is derived from $s(T) = 1$ as: $T = (a + v^2)/(va)$

4. S-Curve Velocity Profile

The discontinuous jump in acceleration at $t \in \{0, t_a, T - t_a, T\}$ of the trapezoidal velocity profile (which implies an infinite jerk d^3s/dt^3) may cause vibration of the robot.

This problem can be solved by using a smoother S-curve velocity profile for time scaling.

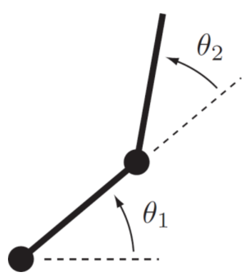


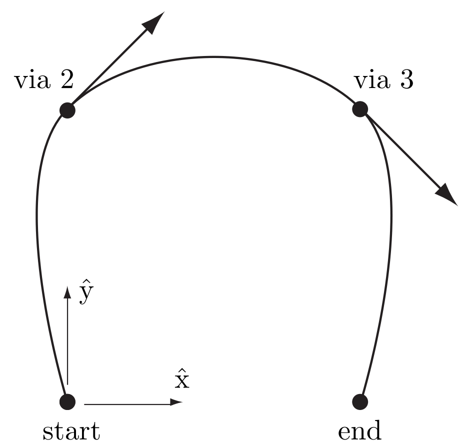
It is fully specified by 7 quantities: $t_J, t_a, t_v, T, J, a, v$. The constraints are $t_J J = a$, $J t_J^2 + a t_a = v$, $4t_J + 2t_a + t_v = T$, $s(T) = 1$. Therefore, 3 of the 7 quantities can be specified independently.

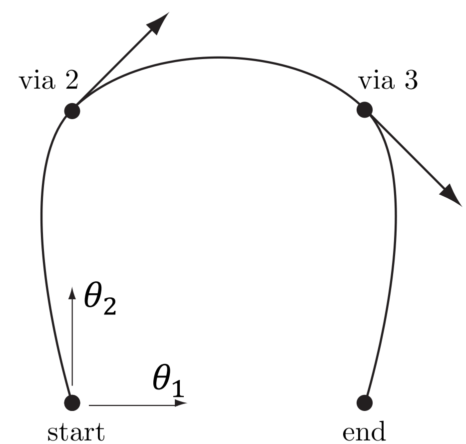
Polynomial Via Point Trajectories

Polynomial Via Point Trajectories

If the goal is that the trajectories pass through a sequence of via points at specified times without a strict specification on the shape of path between consecutive points, a simple solution is to use polynomial interpolation to find the trajectories in the joint space $\boldsymbol{\theta}(t)$ or in the task space $\boldsymbol{x}(t)$ directly without first specifying a path $\boldsymbol{c}(s)$ and then a time scaling $s(t)$.







The most common methods:

1. Using Cubic Polynomials
2. Using Linear and Quadratic Polynomials (B-Spline)

Polynomial Via Point Trajectories Using Cubic Polynomials

Let's focus on a single trajectory β:

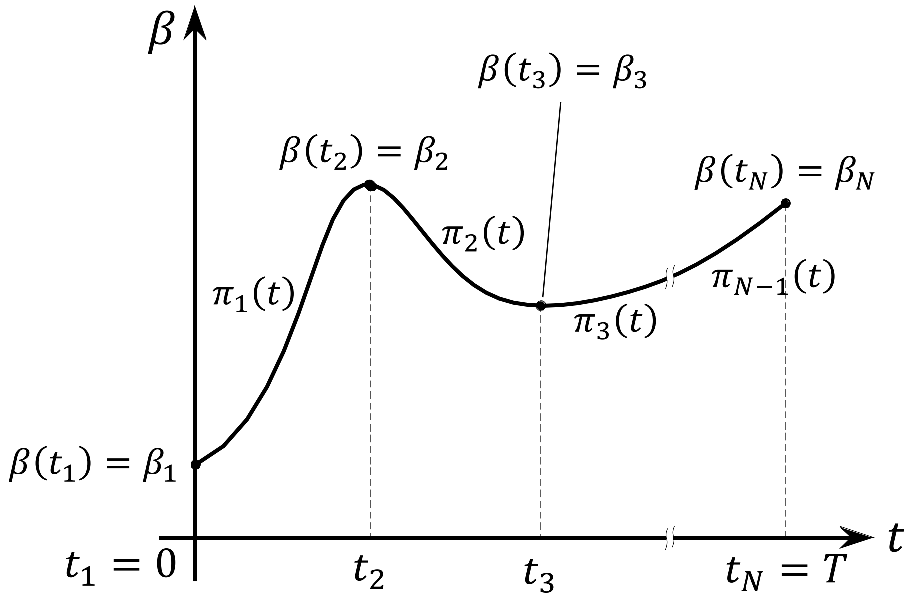
$$\beta(t) = \begin{cases} \pi_1(t) & 0 \leq t \leq t_2 \\ \vdots & \vdots \\ \pi_{N-1}(t) & t_{N-1} \leq t \leq T \end{cases}$$

# of via points: N

# of segments: $N-1$

$$\pi_k(t) = a_{k,0} + a_{k,1}t + a_{k,2}t^2 + a_{k,3}t^3$$

Position: $\beta(t_k) = \beta_k \quad k \in \{1, \dots, N\}$



(a) Cubic Polynomials with Imposed Velocities at Via Points

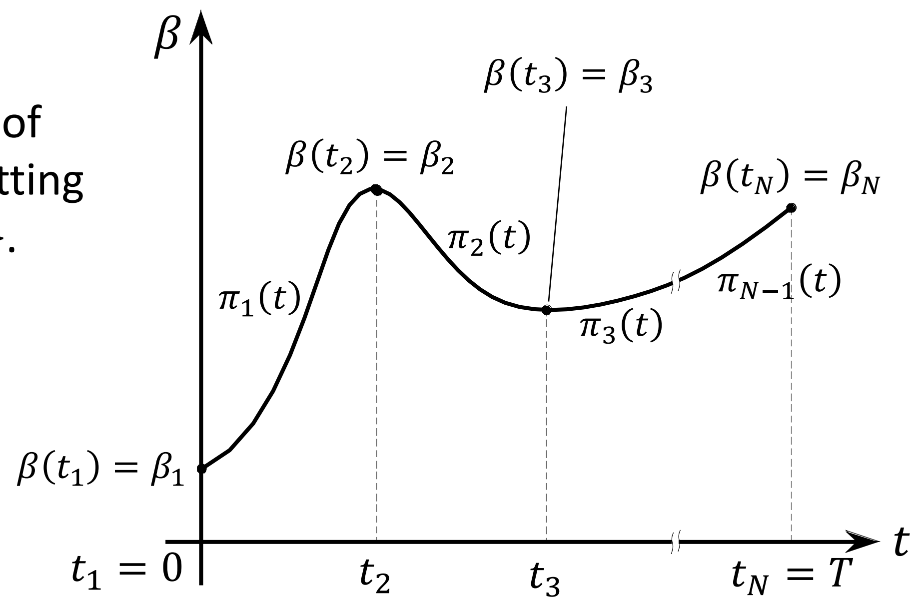
The desired velocity at each via point is given: $\dot{\beta}(t_k) = \dot{\beta}_k \qquad k \in \{1, \dots, N\}$

$$\begin{cases} \pi_k(t_k) = \beta_k \\ \pi_k(t_{k+1}) = \beta_{k+1} \\ \dot{\pi}_k(t_k) = \dot{\beta}_k \\ \dot{\pi}_k(t_{k+1}) = \dot{\beta}_{k+1} \end{cases} \qquad k \in \{1, \dots, N-1\} \quad \Rightarrow$$

$N - 1$ systems of four equations that can be solved <u>independently</u>.

- Typically, $\dot{\beta}(0) = \dot{\beta}(T) = 0$ and continuity of velocity at the path points is ensured by setting $\dot{\pi}_k(t_{k+1}) = \dot{\pi}_{k+1}(t_{k+1}), k \in \{1, \dots, N-2\}$.

Note: The approach is easily generalized to the use of 5th-order polynomials and specification of the accelerations at the via points.



(b) Cubic Polynomials with Computed Velocities at Via Points

Velocity at each point is computed by the assumption that the trajectory between two consecutive points is a linear segment. Thus, the velocities $\dot{\beta}_k$ can be computed as:

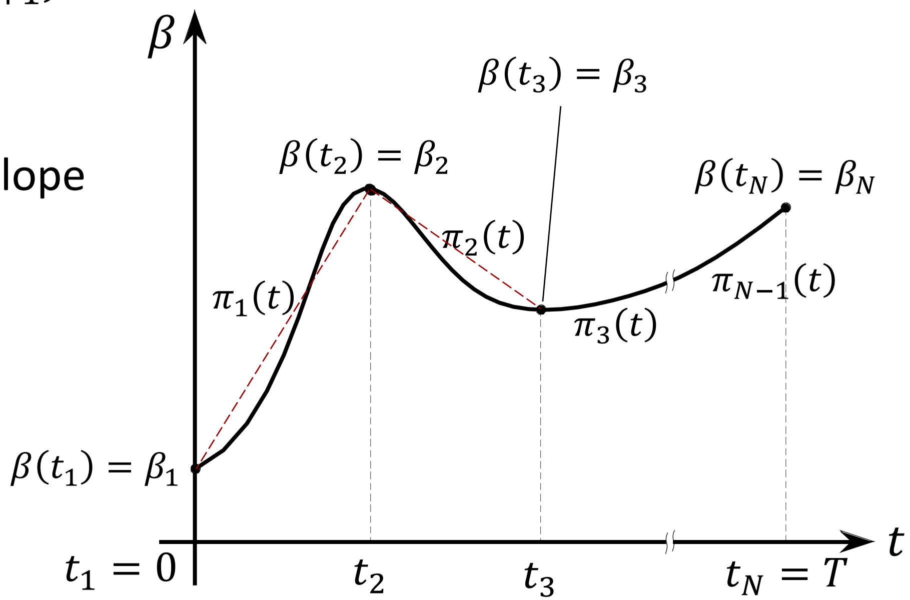
$$\dot{\beta}_1 = 0$$

$$\dot{\beta}_k = \begin{cases} 0 & \mathrm{sgn}(v_k) \neq \mathrm{sgn}(v_{k+1}) \\ \frac{1}{2}(v_k + v_{k+1}) & \mathrm{sgn}(v_k) = \mathrm{sgn}(v_{k+1}) \end{cases} \qquad k \in \{2, \dots, N-1\}$$

$$\dot{\beta}_N = 0,$$

where $v_k = (\beta_k - \beta_{k-1})/(t_k - t_{k-1})$ is the slope of the segment in the time interval $[t_{k-1}, t_k]$.

Then, we use the method in (a).



(c) Cubic Polynomials with Continuous Velocities and Accelerations at Via Points (Splines)

Both (a) and (b) do not ensure continuity of accelerations at the via points. In this method:

$$\begin{cases} \pi_k(t_k) = \beta_k \\ \pi_k(t_k) = \pi_{k-1}(t_k) \\ \dot{\pi}_k(t_k) = \dot{\pi}_{k-1}(t_k) \\ \ddot{\pi}_k(t_k) = \ddot{\pi}_{k-1}(t_k) \end{cases} \qquad k \in \{2, \dots, N-1\}$$

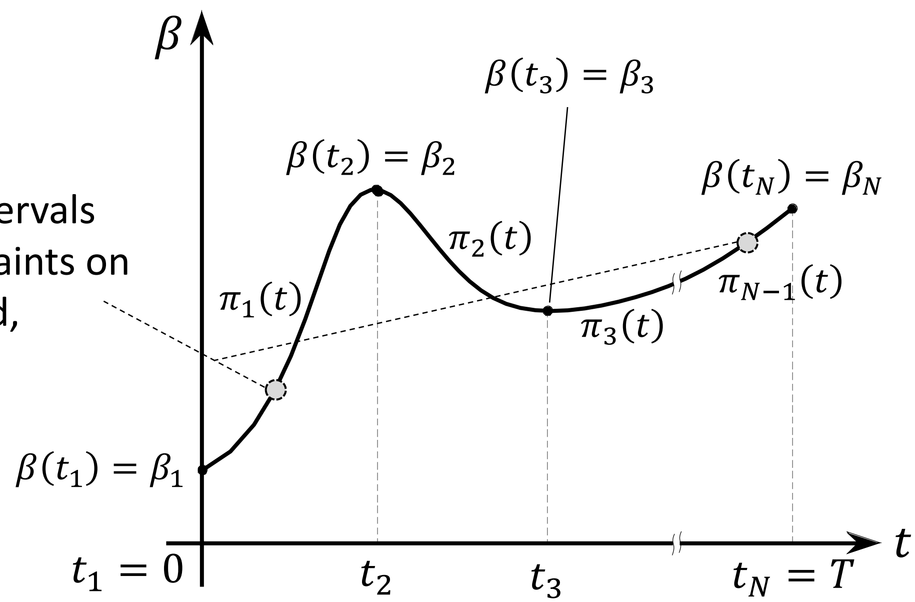
$$\pi_1(0) = \beta_1 \qquad \pi_{N-1}(T) = \beta_N$$
$$\dot{\pi}_1(0) = \dot{\beta}_1 \qquad \dot{\pi}_{N-1}(T) = \dot{\beta}_N$$
$$\ddot{\pi}_1(0) = \ddot{\beta}_1 \qquad \ddot{\pi}_{N-1}(T) = \ddot{\beta}_N$$

# of cubic polynomials: $N-1$, # of Unknowns: $4(N-1)$, # of Equations: $4(N-2)+6$!

Solutions:

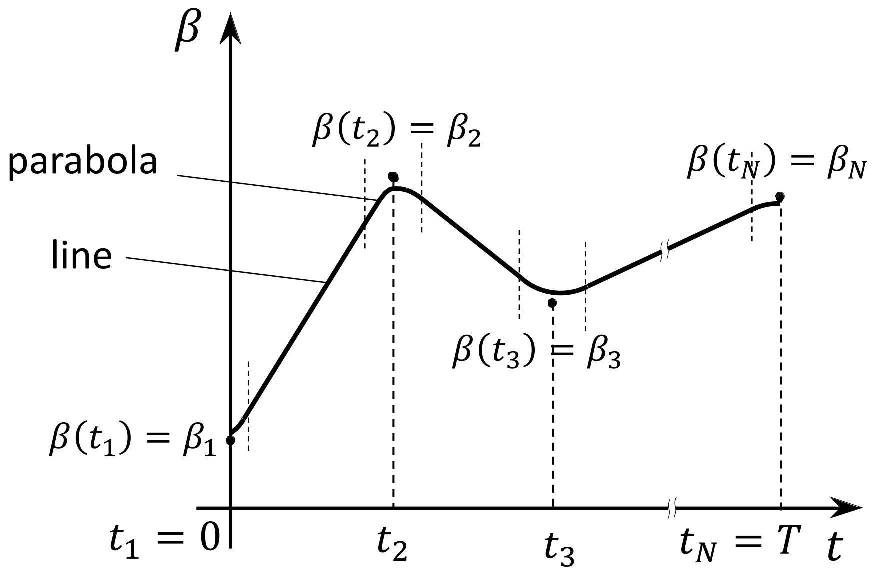
1. Eliminating $\ddot{\pi}_1(0) = \ddot{\beta}_1$ and $\ddot{\pi}_{N-1}(T) = \ddot{\beta}_N$.
2. Using 4th-order polynomials only for π_1 and π_{N-1}.
3. Introducing <u>two virtual points</u> arbitrarily in the intervals $[t_1, t_2]$ and $[t_{N-1}, t_N]$, for which continuity constraints on position, velocity and acceleration can be imposed, without specifying the actual positions:

   # of cubic polynomials: $N+1$
   # of unknowns: $4(N+1)$
   # of Equations: $4(N-2)+6+3+3$



Polynomial Via Point Trajectories Using Linear and Quadratic Polynomials (B-Spline)

The entire trajectory is composed of a sequence of <u>linear</u> and <u>quadratic polynomials</u>.



- Trajectory does not ensure continuity of accelerations at the via points.
- The path does not pass exactly through the via points.
- The path stay within convex hull of the via points (This can be important to ensure that joint limits or workspace obstacles are respected).
- This is an application of the trapezoidal velocity profile law to the via points problem.