Ch4: Equilibrium of Rigid Bodies

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Equilibrium and Free-Body Diagrams

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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Equilibrium Conditions

If the **resultant force and couple moment** (at any arbitrary point O on or off the body) are both equal to **zero**, then the body is said to be in **Equilibrium**.

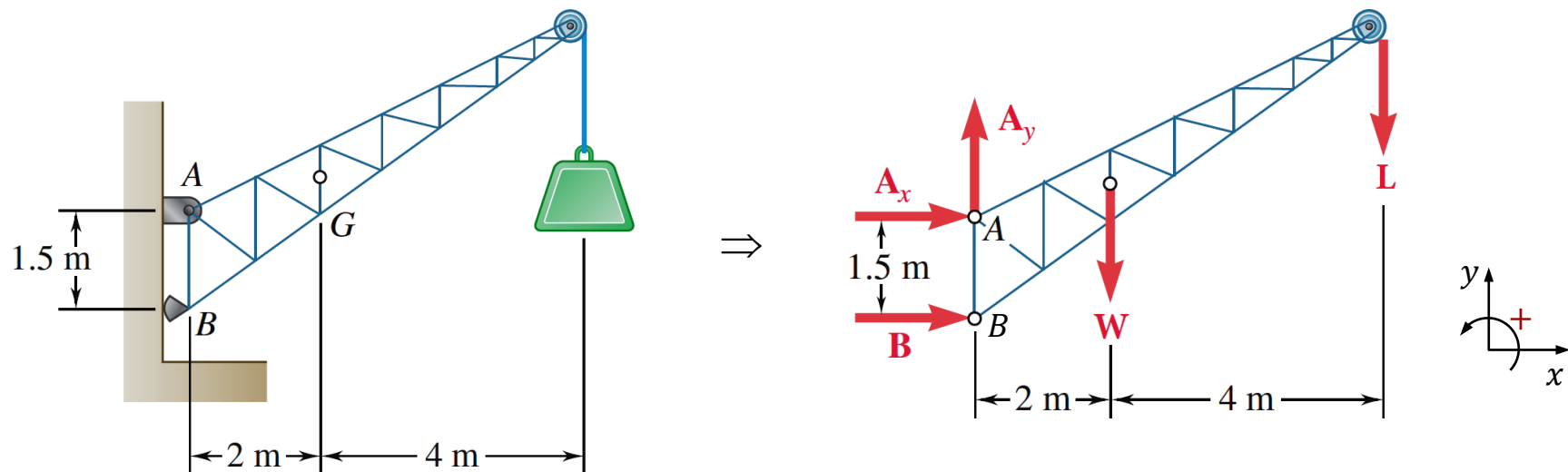$$\begin{cases} \Sigma\mathbf{F} = \mathbf{0} \\ \Sigma\mathbf{M}_O = \mathbf{0} \end{cases}$$

(External forces and couple moments)
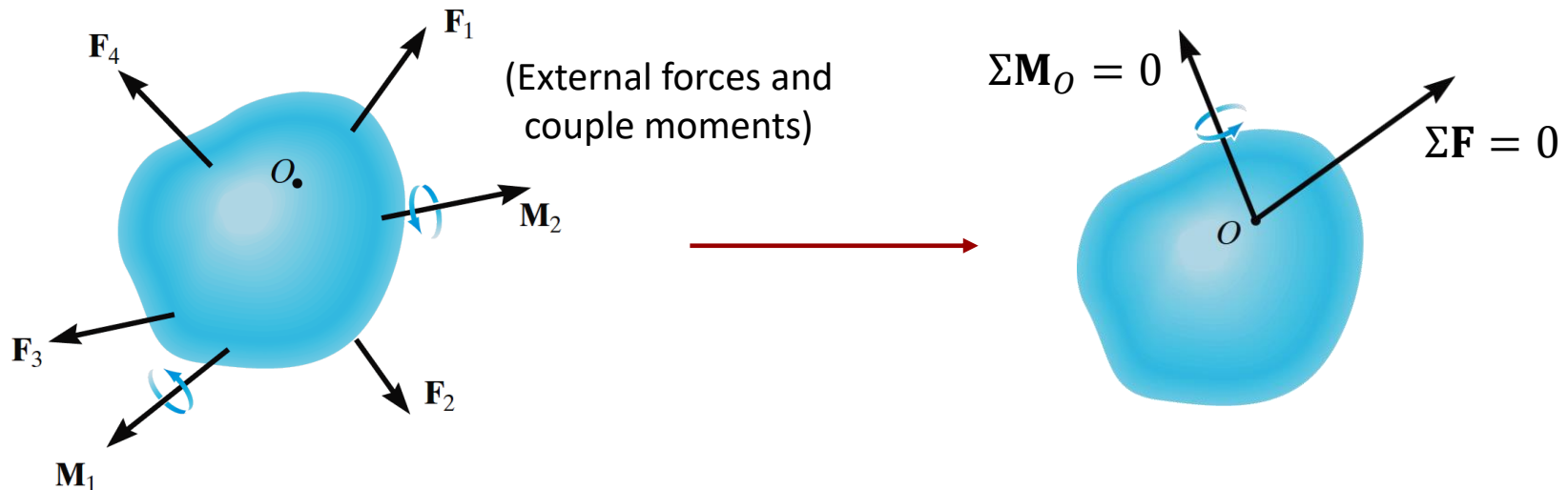
We can use these equations to determine unknown forces **applied** to the rigid body or unknown **reactions** exerted on it by its supports.
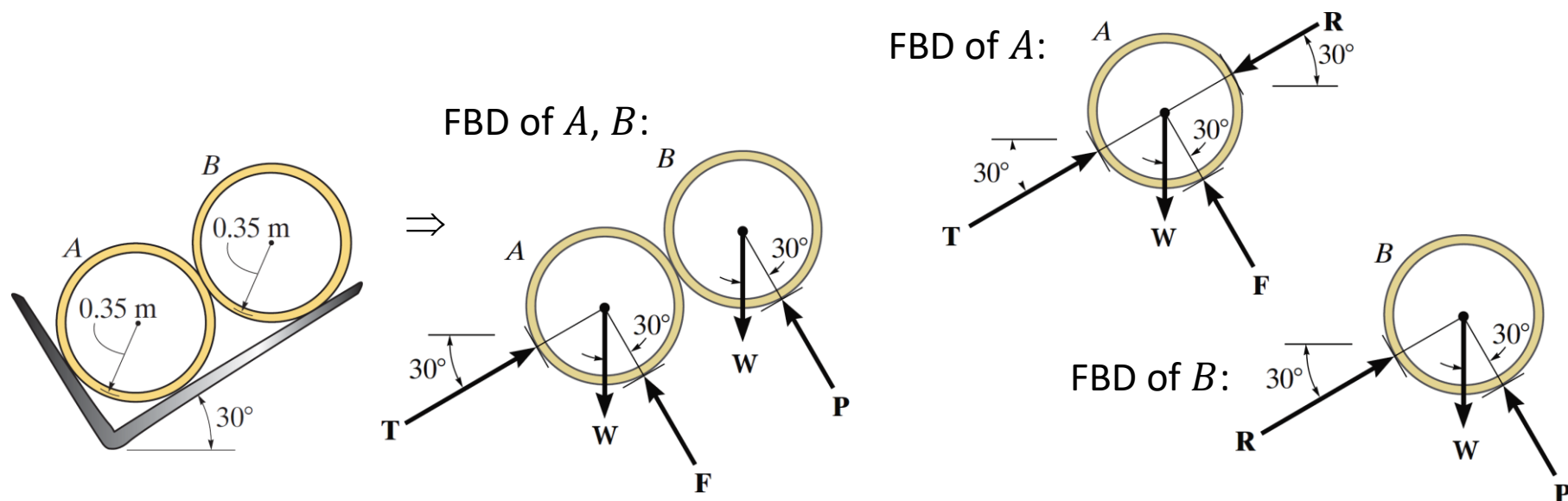
Free-Body Diagrams

In order to write the equations of equilibrium for a rigid body, we must first draw a **free-body diagram** (**FBD**) of the rigid body under consideration.

This diagram is an **isolated** sketch of the body from its surroundings. The free-body diagram should include **all the external forces and couple moments** that the surroundings exert on the body (including the reactions exerted on it by its supports, and the weight of the body) and **dimensions** (for computing moments of forces).

Equilibrium & Free-Body Diagrams | Equilibrium in 2D | Two Special Cases | Equilibrium in 3D | Statical Determinacy & Constraints

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Free-Body Diagrams

▪ If the body under consideration is made of several parts, <u>do not</u> include the **internal forces** the various parts exert on each other among the external forces, since they occur in equal but opposite collinear pairs (Newton's third law) and therefore cancel out.

▪ Include the weight **W** of the body, acting at the **center of gravity** G of the body, among the external forces. When the body is **uniform**, the center of gravity will be located at the body's **geometric center** or **centroid**.

FBD of A, B:

FBD of A:

FBD of B:

Equilibrium & Free-Body Diagrams | **Equilibrium in 2D** | Two Special Cases | Equilibrium in 3D | Statical Determinacy & Constraints

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Equilibrium in Two Dimensions

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints
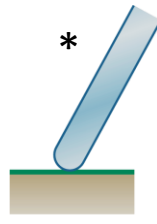○○○   ●○○○▽▽▽▽▽   ○○▽   ○○○○○○▽▽▽   ○○

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Support Reactions

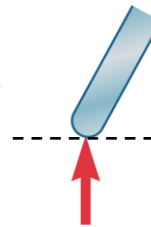- Reactions with **one unknown** (the magnitude is known, and line of action is unknown).
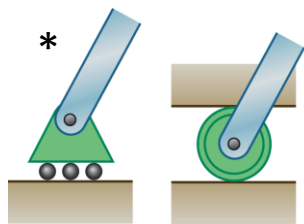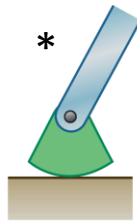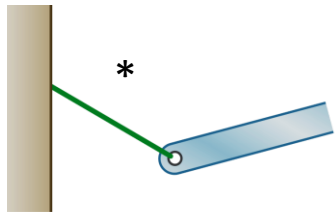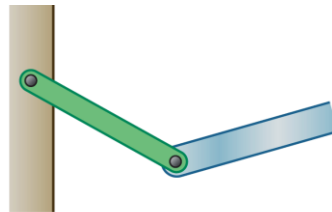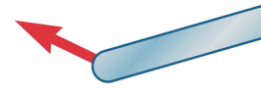

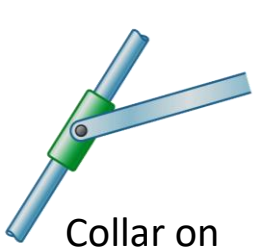Rollers   Rocker   Frictionless Surface   ⟹   Force is perpendicular to surface.
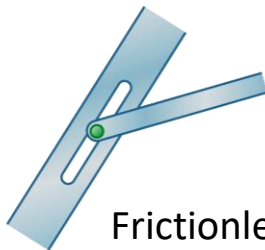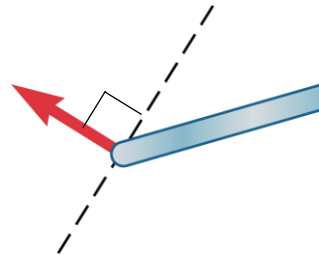
Weightless Cable   Weightless Link   ⟹   Force is along cable or link.
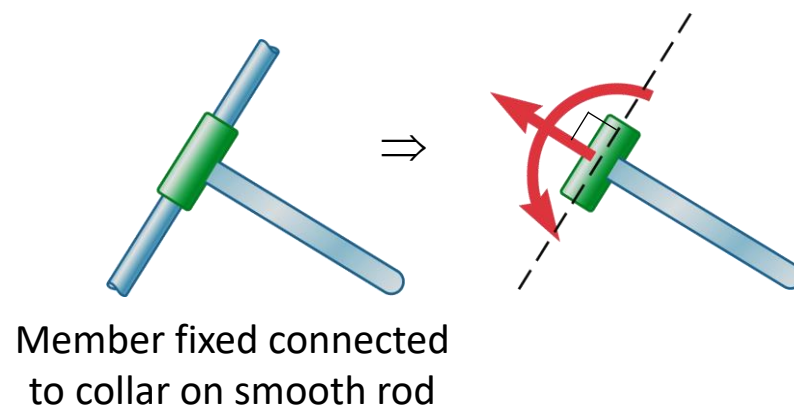
Collar on frictionless rod   Frictionless pin in slot   ⟹   Force is perpendicular to rod or slot.

(* the sense is also known)

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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Support Reactions

- Reactions with **two unknowns**.

Frictionless pin or hinge

Rough surface

or

a

Member fixed connected to collar on smooth rod

- Reactions with **three unknowns**.

or

a

Fixed support
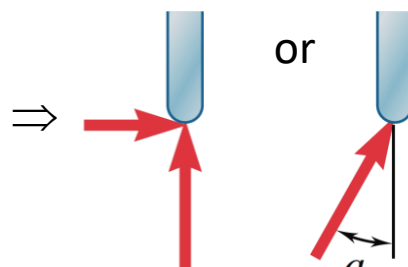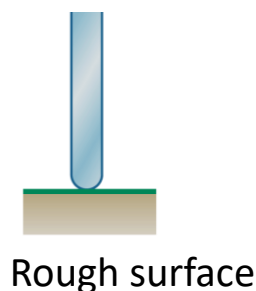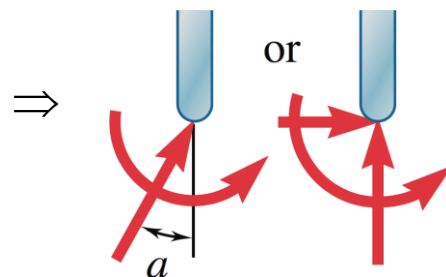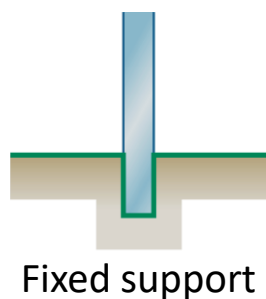
A support prevents the translation/rotation of a body in a given direction by exerting a force/couple moment on the body in the opposite direction.

Note: When the sense of an unknown force or couple is unknown, arbitrarily assume the sense; the <u>sign of the answer</u> will indicate whether the assumption is correct or not.

A positive [negative] answer means the assumption is correct [incorrect].

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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Equations of Equilibrium

If the force system acting on a rigid body lies in a single plane (say $x - y$) and any couple moments acting on the body are directed perpendicular to this plane, the force and couple system is **two-dimensional**. Equations of equilibrium:

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M_O = 0$$

(O is any point in the plane of the body)

▪ Two alternative sets of three independent equilibrium equations:

$$\Sigma F_x = 0, \quad \Sigma M_A = 0, \quad \Sigma M_B = 0$$
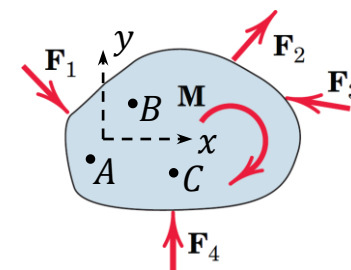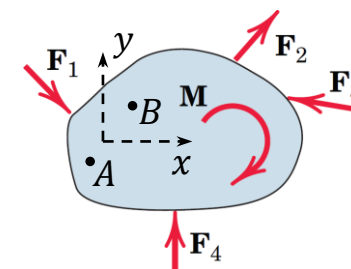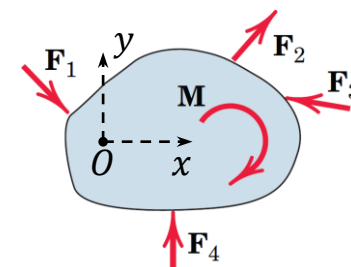
or y

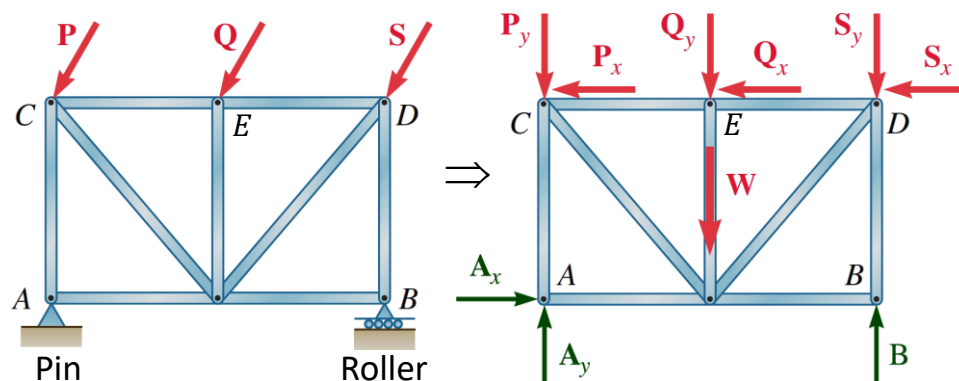Only when the line AB is not perpendicular to the x axis.

or y

$$\Sigma M_A = 0, \quad \Sigma M_B = 0, \quad \Sigma M_C = 0$$

Only when the points A, B, and C not lie along a straight line.

These three equations can be solved for **up to three unknowns**.

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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Equations of Equilibrium

For given **P**, **Q**, **S**, and **W**:

$\Sigma M_A = 0 \;\rightarrow\; B$ ✓ (It doesn't contain A_x, A_y)

$\Sigma F_x = 0 \;\rightarrow\; A_x$ ✓ (It doesn't contain A_y, B)

$\Sigma F_y = 0 \;\rightarrow\; A_y$ ✓

✦ It is desirable to choose equations of equilibrium containing **only one unknown**, because this eliminates the necessity of solving simultaneous equations. To do so,

i. **Sum moments** about the <u>point of intersection</u> of the lines of action of two unknown forces (for example, point E is not a proper choice).

ii. **Sum force components** in <u>a direction perpendicular</u> to two unknown parallel forces.

✦ Any additional equation (e.g., $\Sigma M_B = 0$) <u>is not independent</u> (does not contain any new information) and cannot be used to determine a fourth unknown (however, they can be useful for <u>checking the solution</u> obtained from the original three equations of equilibrium!).

Equilibrium & Free-Body Diagrams | **Equilibrium in 2D** | Two Special Cases | Equilibrium in 3D | Statical Determinacy & Constraints

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Sample Problem 4.1

A fixed crane has a mass of 1000 kg and is used to lift a 2400-kg crate. It is held in place by a pin at A and a rocker at B. The center of gravity of the crane is located at G. Determine the components of the reactions at A and B.

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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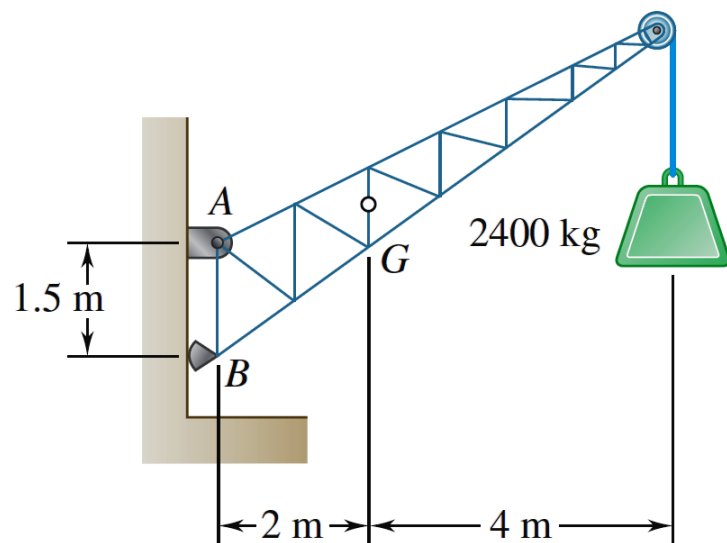
Sample Problem 4.2

Three loads are applied to a beam as shown. The beam is supported by a roller at A and by a pin at B. Neglecting the weight of the beam, determine the reactions at A and B when $P = 15$ kips.

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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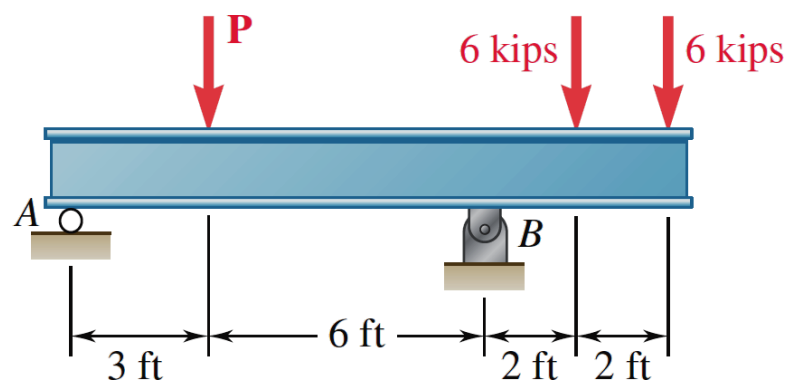
Sample Problem 4.3

A loading car is at rest on a track forming an angle of 25° with the vertical. The gross weight of the car and its load is 5500 lb, and it acts at a point 30 in. from the track, halfway between the two axles. The car is held by a cable attached 24 in. from the track. Determine the tension in the cable and the reaction at each pair of wheels.

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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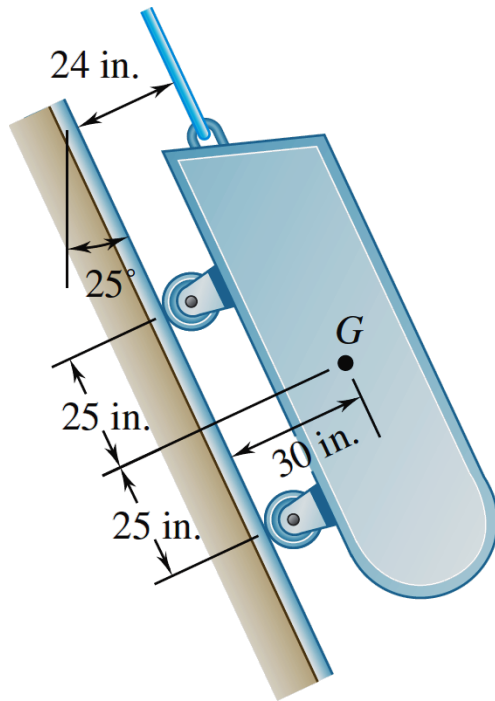
Sample Problem 4.4

The frame shown supports part of the roof of a small building. Knowing that the tension in the cable is 150 kN, determine the reaction at the fixed end E.

Equilibrium & Free-Body Diagrams   **Equilibrium in 2D**   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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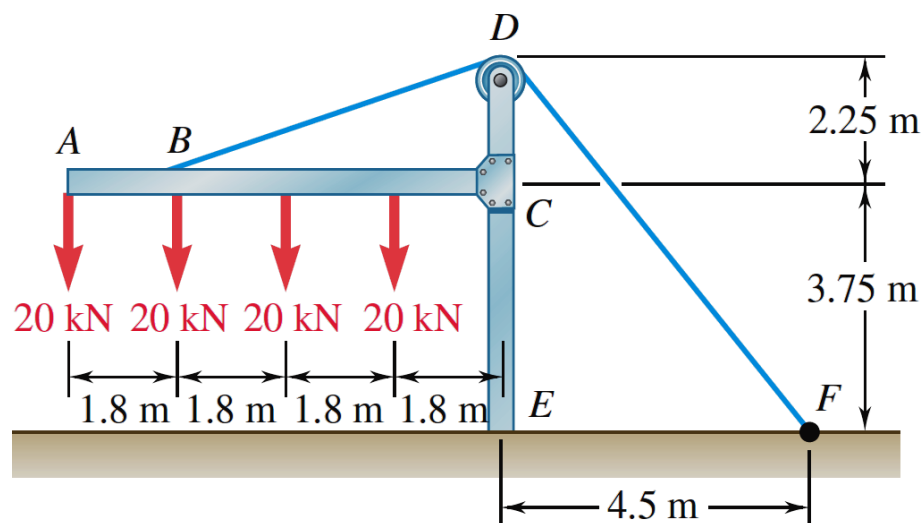
Sample Problem 4.5

A 400-lb weight is attached at A to the lever shown. The constant of the spring BC is $k = 250$ lb/in., and the spring is unstretched when $\theta = 0$. Determine the position of equilibrium.

$l = 8$ in.

A

θ

B

C

$k = 250$ lb/in.

O

$r = 3$ in.

$W = 400$ lb

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   **Two Special Cases**   Equilibrium in 3D   Statical Determinacy & Constraints

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Two Special Cases

Equilibrium & Free-Body Diagrams ○○○ | Equilibrium in 2D ○○○○▽▽▽▽▽ | **Two Special Cases** ●○▽ | Equilibrium in 3D ○○○○○○▽▽▽ | Statical Determinacy & Constraints ○○

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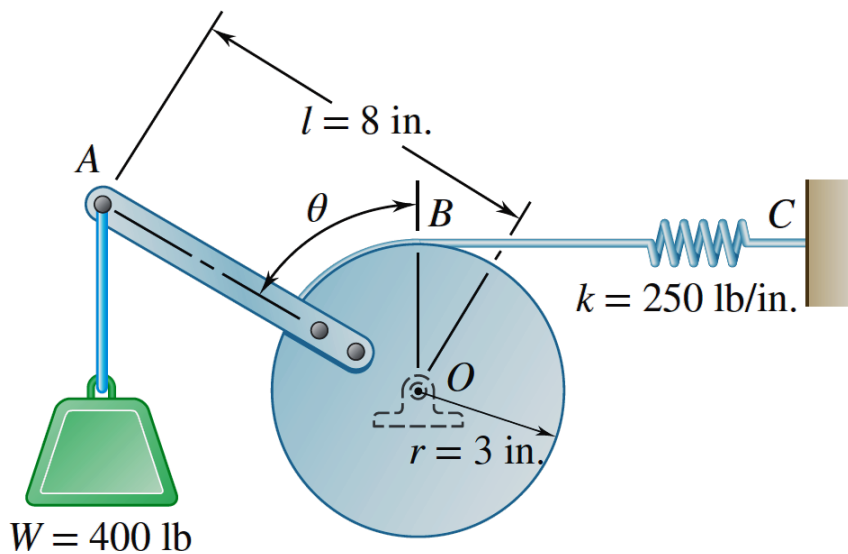
Equilibrium of a Two-Force Body

A **two-force body** is defined as a rigid body subjected to **forces acting at only two points**.
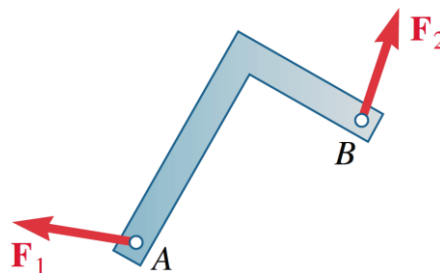
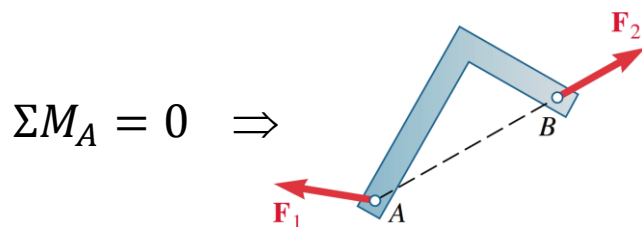If a **two-force body is in equilibrium**, the two forces must have the **same magnitude**, **opposite sense**, and the **same line of action** (directed along the line joining the two points where these forces act).

Proof: If a two-force body is in equilibrium:

$\Sigma M_A = 0 \implies$

and

$\Sigma M_B = 0 \implies$

(the line of action of $\mathbf{F}_2$ must pass through A)   (the line of action of $\mathbf{F}_1$ must pass through B)

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   **Two Special Cases**   Equilibrium in 3D   Statical Determinacy & Constraints

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Equilibrium of a Three-Force Body

A **three-force body** is defined as a rigid body subjected to **forces acting at only three points**.

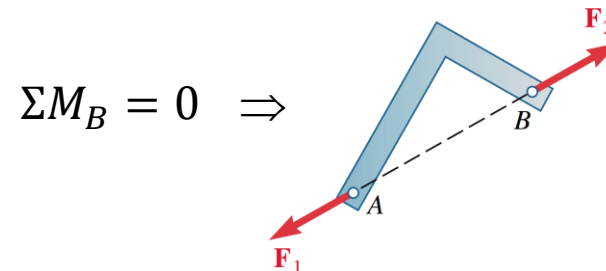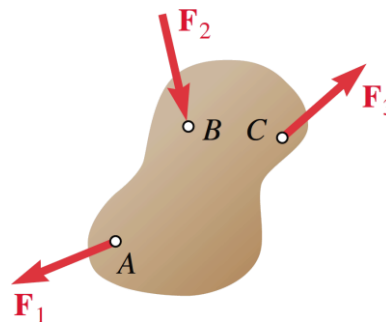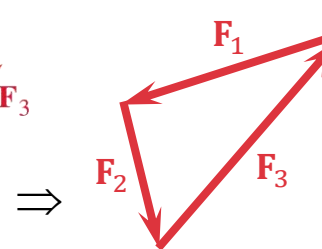If a **three-force body is in equilibrium**, the lines of action of the three forces must be either **parallel** or **concurrent** (in this case, we can use the **triangle rule**).

Proof: If a three-force body is in equilibrium:

Case 1: If the lines of action of $\mathbf{F}_2$ and $\mathbf{F}_3$ intersect at point D, then the line of action of $\mathbf{F}_1$ must also pass through point D so that $\Sigma M_D = 0$.
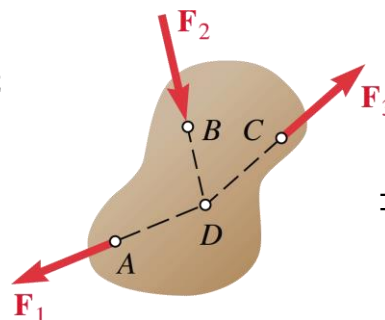
In this case, we can use the **triangle rule**.

Case 2: Lines of action are parallel.

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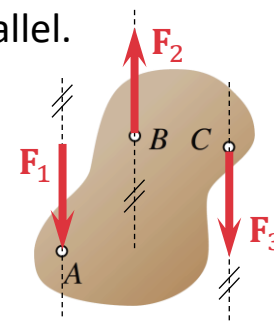
Problem 4.65

Determine the reactions at B and C when $a = 30$ mm.

Equilibrium in Three Dimensions

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   **Equilibrium in 3D**   Statical Determinacy & Constraints

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Support Reactions

Reactions with **one unknown**:

(perpendicular to surface)

$\mathbf{F}$

(along cable)

$\mathbf{F}$

Ball   Frictionless surface   Flangeless wheel/roller   Cable

Reactions with **two unknowns**:

$\mathbf{F}_y$ (perpendicular to surface)

$\mathbf{F}_z$ (parallel to axis of wheel)

Wheel support with lateral constraint (flanged wheel)

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   **Equilibrium in 3D**   Statical Determinacy & Constraints

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Support Reactions

Reactions with **three unknowns**:

Rough surface   Ball and socket

$\Rightarrow$

$\mathbf{F}_y$ $\mathbf{F}_x$ $\mathbf{F}_z$

(mutually perpendicular at point of contact)

Reactions with **six unknowns**:

Fixed support

$\Rightarrow$

$\mathbf{M}_y$ $\mathbf{F}_y$ $\mathbf{M}_x$ $\mathbf{M}_z$ $\mathbf{F}_z$ $\mathbf{F}_x$

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   **Equilibrium in 3D**   Statical Determinacy & Constraints

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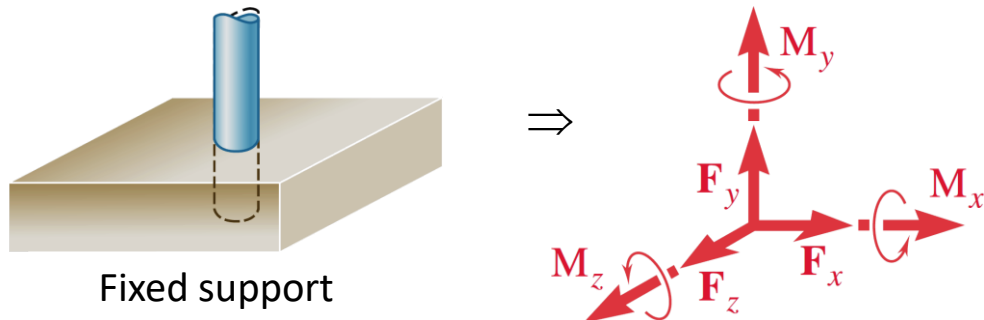
Support Reactions

Reactions with **four unknowns**:

Collar

Bearing supporting radial load only

✦ They are primarily designed to prevent only translation. Thus, only force components should be included **unless** otherwise stated.

Reactions with **five unknowns**:

Collar

Pin and bracket
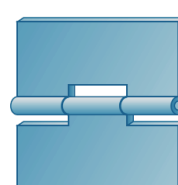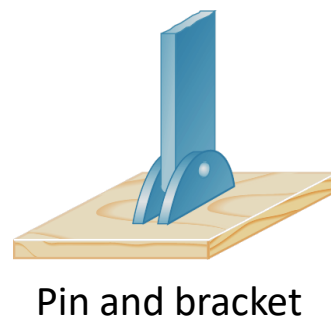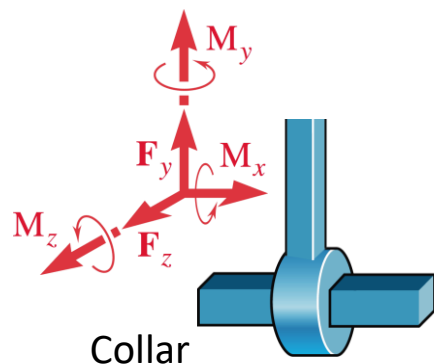
Hinge and bearing supporting axial thrust and radial load

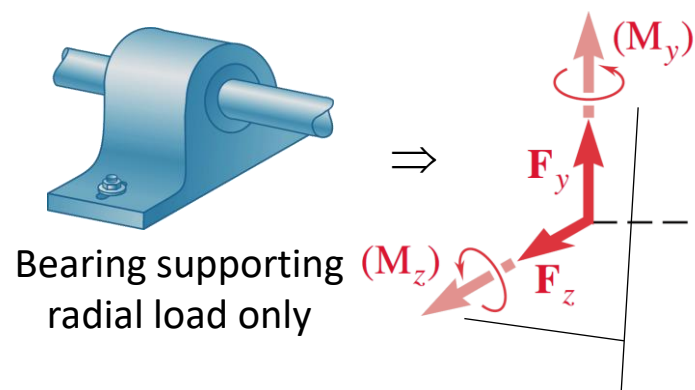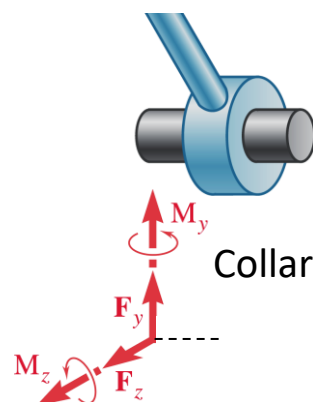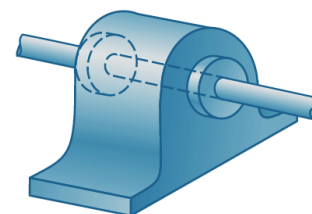Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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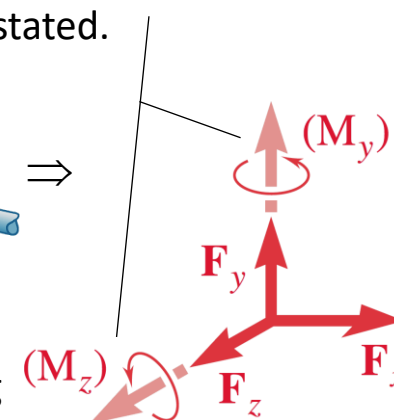
Equations of Equilibrium

The most general situation of rigid-body equilibrium occurs in three dimensions.

$$\begin{cases} \Sigma\mathbf{F} = \mathbf{0} \\ \Sigma\mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) + \Sigma\mathbf{M}_i = \mathbf{0} \end{cases}$$

(or)

$$\begin{cases} \Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0 \\ \Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0 \end{cases}$$

(O is any point located either on or off the body)

$$\Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

$$\Sigma\mathbf{M}_O = \Sigma M_x\mathbf{i} + \Sigma M_y\mathbf{j} + \Sigma M_z\mathbf{k}$$

✦ Any arbitrary **axis** aa' can be also chosen for summing moments:

$$\Sigma\mathbf{M}_{aa'} = \mathbf{0}$$

These equations can be solved for **up to six unknowns**.

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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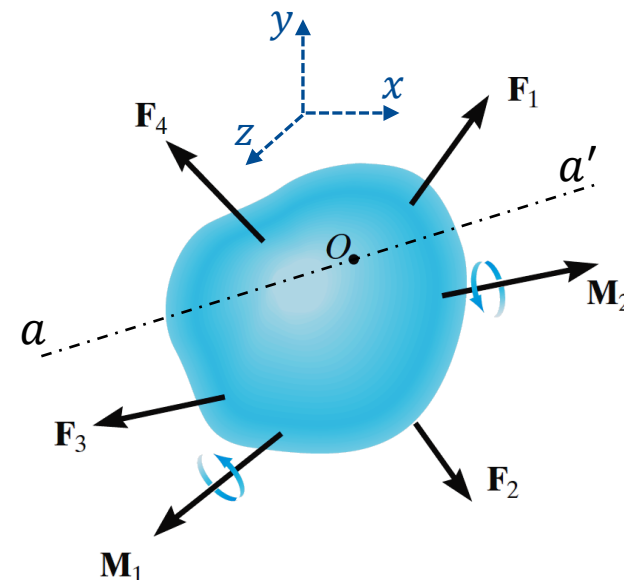
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Equations of Equilibrium

✦ It is desirable to choose equations of equilibrium containing as few unknowns as possible. Two helpful strategies:

(1) Sum moments **about a support** to eliminate three unknown reaction components.
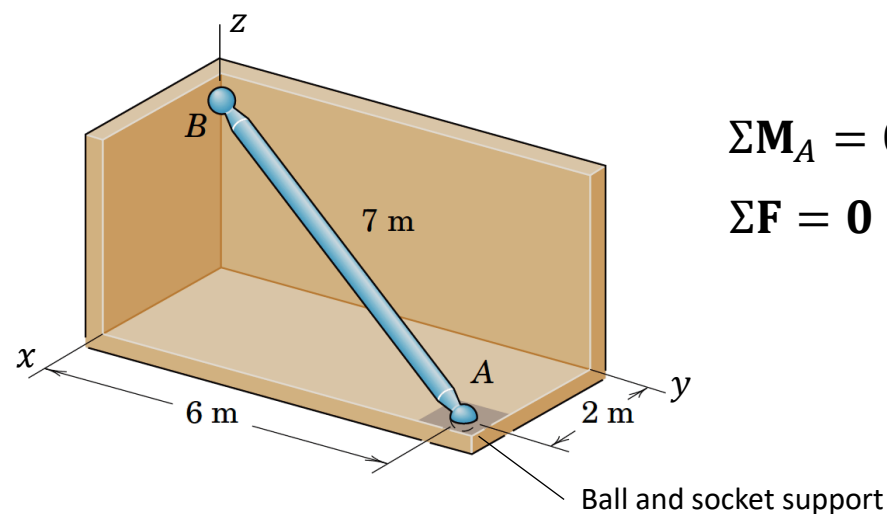
$$\Sigma \mathbf{M}_A = \mathbf{0} \;\rightarrow\; B_x, B_y \;\checkmark$$

$$\Sigma \mathbf{F} = \mathbf{0} \;\rightarrow\; A_x, A_y, A_z \;\checkmark$$

Ball and socket support
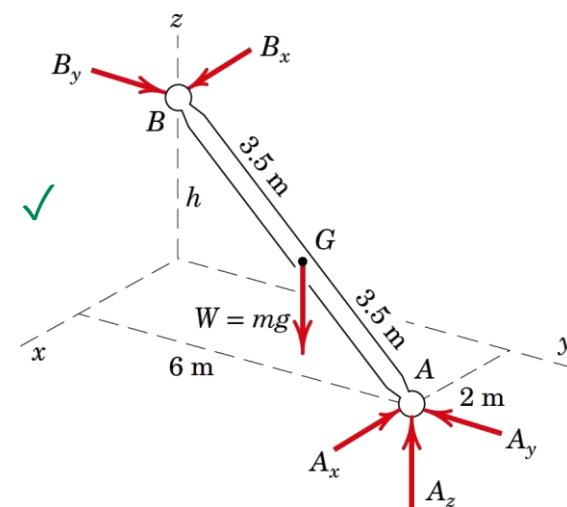
Note: When the sense of an unknown force or couple is unknown, arbitrarily assume the sense. If the sign of the answer yields a **negative** scalar for a force or couple moment magnitude, it indicates that the correct sense is **opposite** to that assumed on the free-body diagram.

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   **Equilibrium in 3D**   Statical Determinacy & Constraints

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Equations of Equilibrium

(2) Sum moments **about an axis** that it intersects the lines of action of as many unknown forces as possible (say all but one of the unknown reactions).

Finding tension T:

$$\Sigma \mathbf{M}_{AB} = \mathbf{0} \rightarrow T \quad \checkmark$$

vs

$$\Sigma \mathbf{M}_{A} = \mathbf{0} \rightarrow B_x, B_z, T \quad \checkmark$$

In this way, we can eliminate the reactions at two points A and B by writing $\Sigma \mathbf{M}_{AB} = \mathbf{0}$, which involves the computation of the moments of the forces about an axis AB joining points A and B.

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   Equilibrium in 3D   Statical Determinacy & Constraints

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Sample Problem 4.7

A 20-kg ladder used to reach high shelves in a storeroom is supported by two flanged wheels A and B mounted on a rail and by a flangeless wheel C resting against a rail fixed to the wall. (The ladder itself is symmetric, with wheel C located on the plane of symmetry.) An 80-kg man stands on the ladder and leans to the right. The line of action of the combined weight $\mathbf{W}$ of the man and ladder intersects the floor at point D. Determine the reactions at A, B, and C.

Equilibrium & Free-Body Diagrams

Equilibrium in 2D

Two Special Cases

Equilibrium in 3D

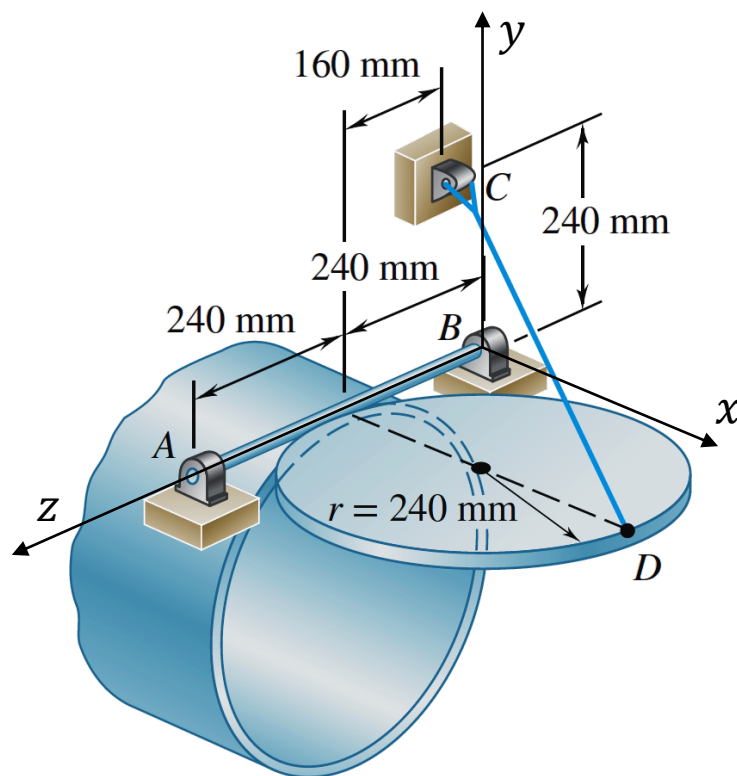Statical Determinacy & Constraints

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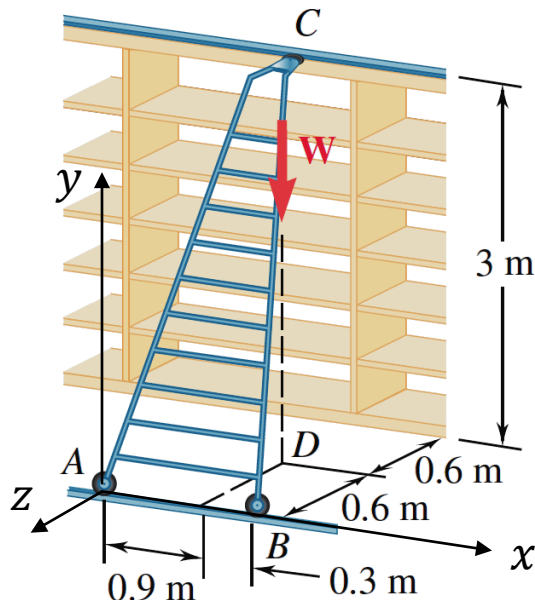
Sample Problem 4.8

A 5 × 8-ft sign of uniform density weighs 270 lb and is supported by a ball-and-socket joint at A and by two cables. Determine the tension in each cable and the reaction at A.

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   **Equilibrium in 3D**   Statical Determinacy & Constraints

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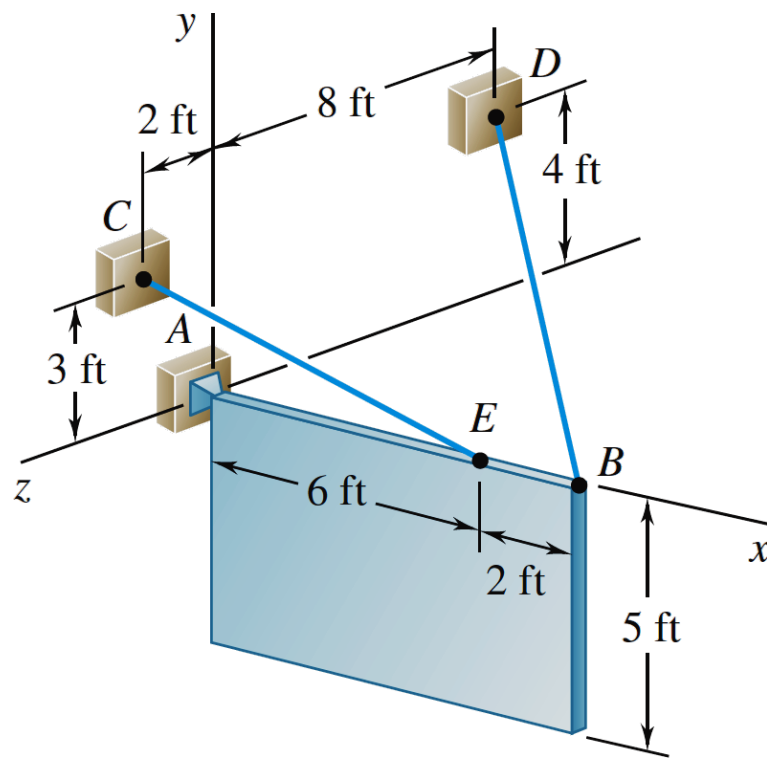
Sample Problem 4.9

A uniform pipe cover of radius $r = 240$ mm and mass $m = 30$ kg is held in a horizontal position by the cable CD. Assuming that only the bearing at B does not exert any axial thrust, determine the tension in the cable and the reactions at A and B.
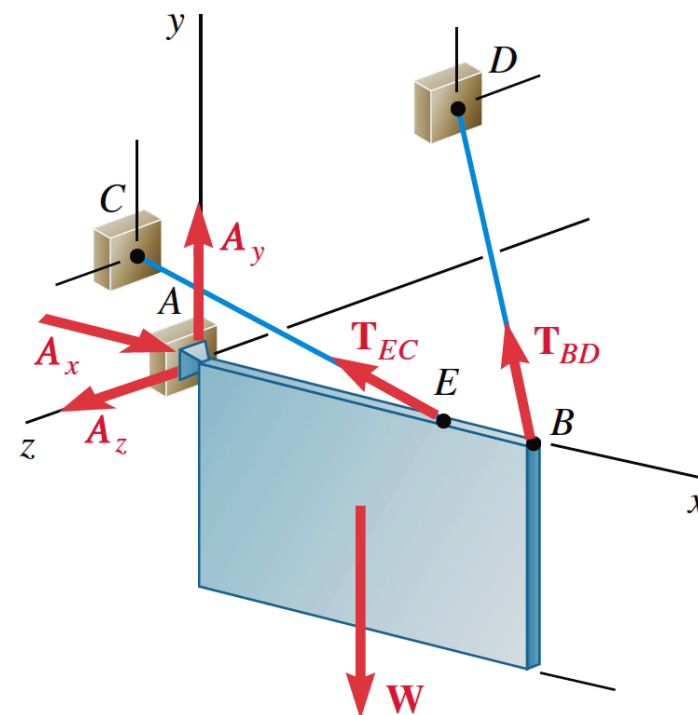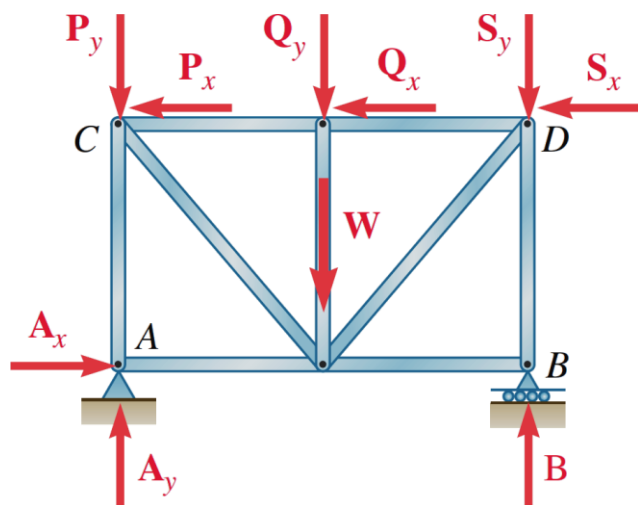
Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   Equilibrium in 3D   **Statical Determinacy & Constraints**

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Statical Determinacy and Constraints

Equilibrium & Free-Body Diagrams Equilibrium in 2D Two Special Cases Equilibrium in 3D Statical Determinacy & Constraints

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Statical Determinacy

A rigid body which has **the minimum number of supports/constraints** necessary to maintain an equilibrium configuration <u>under given loads</u> is called **statically determinate**. For such bodies,

- The equilibrium equations are **sufficient** to determine all the unknown reactions (up to three unknowns in 2D or six unknowns in 3D).

Equilibrium & Free-Body Diagrams   Equilibrium in 2D   Two Special Cases   Equilibrium in 3D   **Statical Determinacy & Constraints**

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Statical Determinacy

A rigid body which has **more supports/constraints than are necessary** to maintain an equilibrium configuration <u>under given loads</u> is called **statically indeterminate**. For such bodies

- The equilibrium equations are **insufficient** to determine all the unknown reactions (e.g., **more than** three unknowns in 2D or six unknowns in 3D).

- You may be able to calculate **a few of reactions**. The remaining reactions can be determined by additional equations obtained from the deformation conditions at the points of support (MEC 363 [Mechanics of Solids]).