Ch5: Distributed Forces: Centroids and Centers of Gravity

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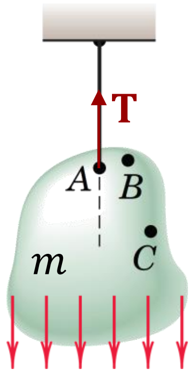
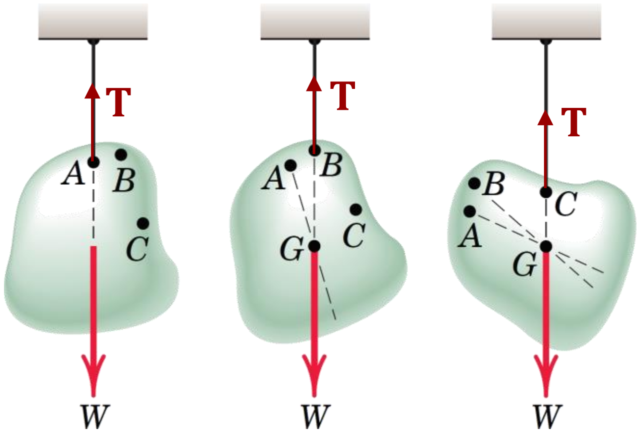
Centroid of Two-Dimensional Bodies

Center of Gravity

If we suspend the body from any point such as A, B, or C, the resultant $\mathbf{W}$ of the gravitational forces acting on all particles of the body is collinear with the cord (or tension $\mathbf{T}$). For all these points, the line of action of $\mathbf{W}$ will be concurrent at a single point G, which is called the **center of gravity** of the body.



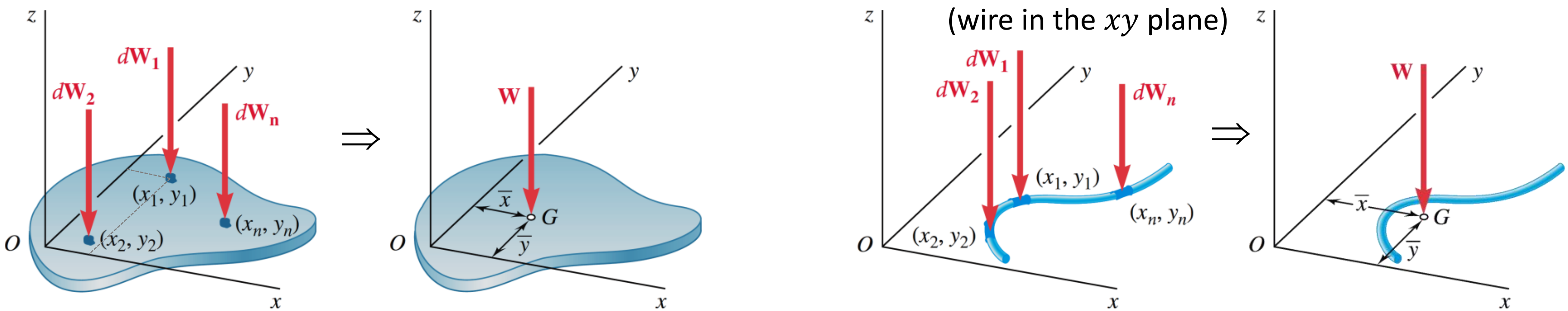
A large number of small forces distributed over the entire body



$$W = mg$$

Center of Gravity of a Plate or Wire

Consider a flat plate or wire divided into infinitesimal elements. The resultant of the forces exerted by earth (i.e., weight) on the elements is a single force **W** in the same direction.



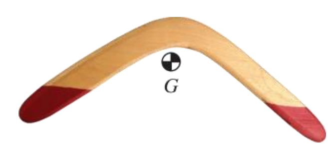
$$\Sigma F_z: \quad W = dW_1 + dW_2 + \cdots \quad \Rightarrow \quad W = \int dW$$

$$\Sigma M_y: \quad \bar{x}W = x_1 dW_1 + x_2 dW_2 + \cdots \Rightarrow \bar{x} = \frac{\int x dW}{W}$$

$$\Sigma M_x: \quad \bar{y}W = y_1 dW_1 + y_2 dW_2 + \cdots \Rightarrow \bar{y} = \frac{\int y dW}{W}$$

Coordinate of point G where **W** is applied, i.e., **Center of Gravity**

Note: The center of gravity G of a body may not located on the body.

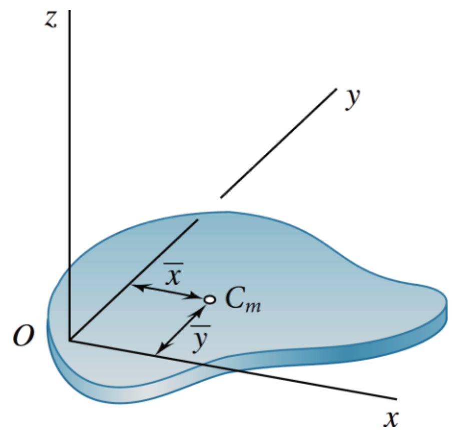


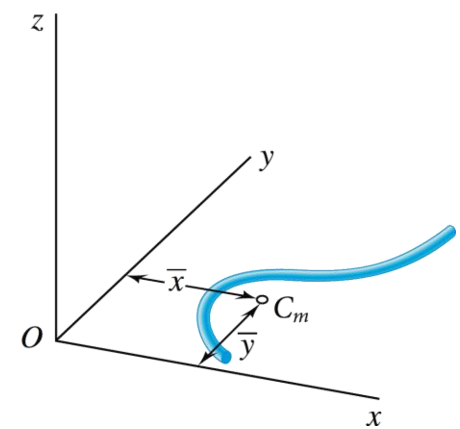
Center of Mass of a Plate or Wire

By substituting $W = mg$ and $dW = gdm$ into the coordinate of the Center of Gravity, the coordinate of the body's **Center of Mass** C_m is determined:

$$\bar{x} = \frac{\int xdm}{m} \qquad \bar{y} = \frac{\int ydm}{m}$$

$$m = \int dm$$





Center of Mass vs Center of Gravity:

- Center of mass **coincides** with the center of gravity as long as the gravity field is treated as uniform and parallel ($C_m \equiv G$).
- Center of mass is a function solely of the distribution of mass, thus, in the absence of gravitational field, the body still have its unique center of mass.

Centroid of an Area

Now, if the flat plate is **homogeneous** with uniform thickness:

$$m = \rho V = \rho t A$$
$$dm = \rho t \, dA$$

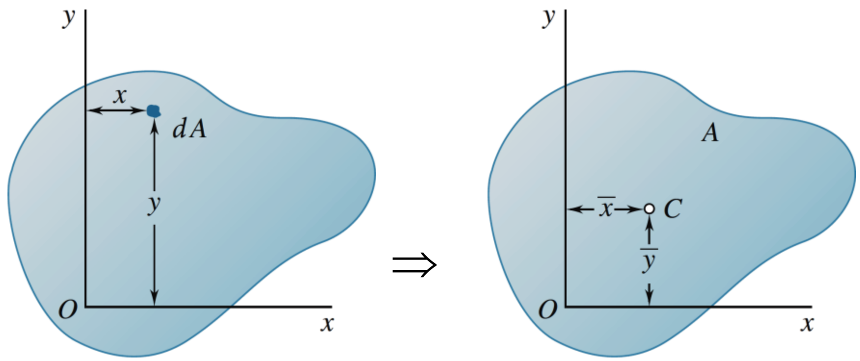
ρ: density, A: area of the plate, t: plate thickness

$$\bar{x} = \frac{\int x dm}{m} \longrightarrow \boxed{\bar{x} = \frac{\int x dA}{A}}$$

$$\bar{y} = \frac{\int y dm}{m} \longrightarrow \boxed{\bar{y} = \frac{\int y dA}{A}}$$

$$m = \int dm \longrightarrow \boxed{A = \int dA}$$

Coordinate of the **Centroid** C of the area A (or Center of Gravity G or Center of Mass C_m of the plate).



Note: If the plate is **not homogeneous**, the center of gravity G **does not coincides** with the centroid C.

Centroid of a Wire

Now, if the flat wire is **homogeneous** with uniform cross section : $m = \rho V = \rho a L$

$dm = \rho a \, dL$

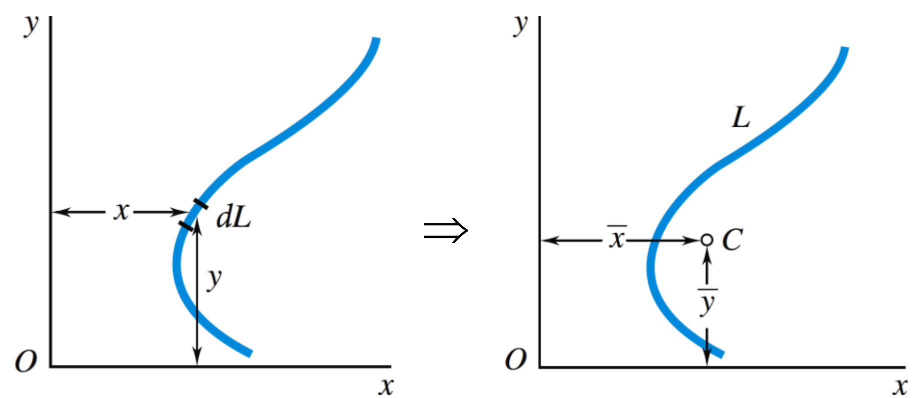
ρ: density, a: cross-sectional area, L: wire length

$$\bar{x} = \frac{\int x dm}{m} \longrightarrow \boxed{\bar{x} = \frac{\int x dL}{L}}$$

$$\bar{y} = \frac{\int y dm}{m} \longrightarrow \boxed{\bar{y} = \frac{\int y dL}{L}}$$

$$m = \int dm \longrightarrow \boxed{L = \int dL}$$

Coordinate of the **Centroid** C of the line L (or Center of Gravity G or Center of Mass C_m of the wire).



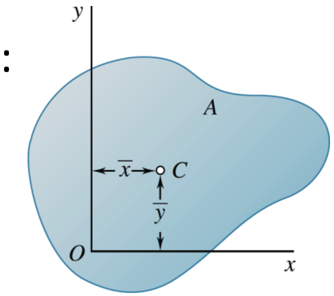
Note: If the wire is **not homogeneous**, the center of gravity G **does not coincides** with the **centroid** C.

First Moments of Areas and Lines

The **first moment of the area** A with respect to x and y axes are defined as:

$$Q_x = \bar{y}A = \int y dA$$

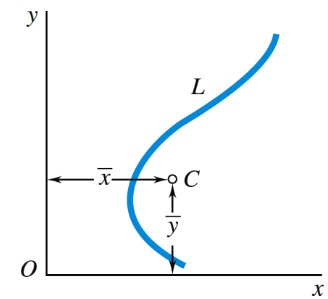
$$Q_y = \bar{x}A = \int x dA$$



The **first moment of the line** L with respect to x and y axes are defined as:

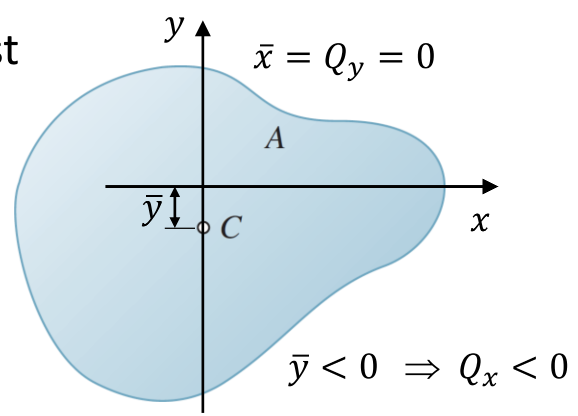
$$Q_x = \bar{y}L = \int y dL$$

$$Q_y = \bar{x}L = \int x dL$$



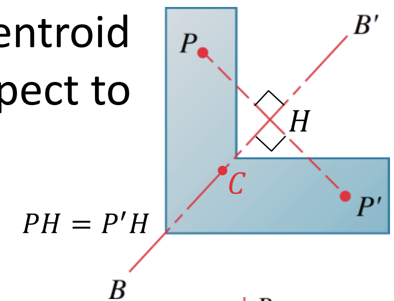
Note 1: If the centroid of an area is **located on an axis**, the first moment of the area with respect to that axis is **zero**.

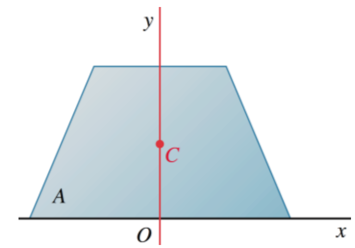
Note 2: First moments of areas, like moments of forces, can be **positive or negative**.



Determination of Centroids: Quick Method

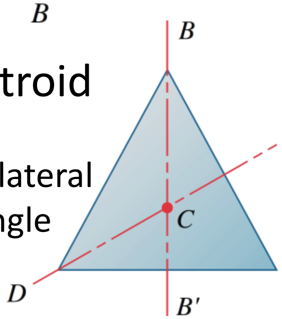
- If an area A or a line L has **an axis of symmetry**, its centroid C is located on that axis (and its first moment with respect to that axis is **zero**).

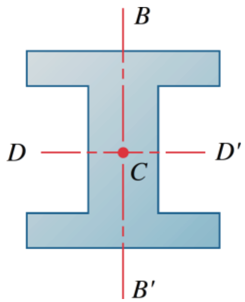




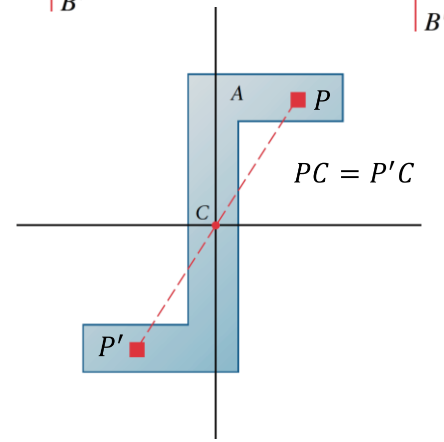
- If an area A or a line L has **two axes of symmetry**, its centroid C is located at the intersection of the two axes.

Equilateral Triangle





- If an area A or a line L has a **center of symmetry**, its centroid C is located at that center of symmetry.



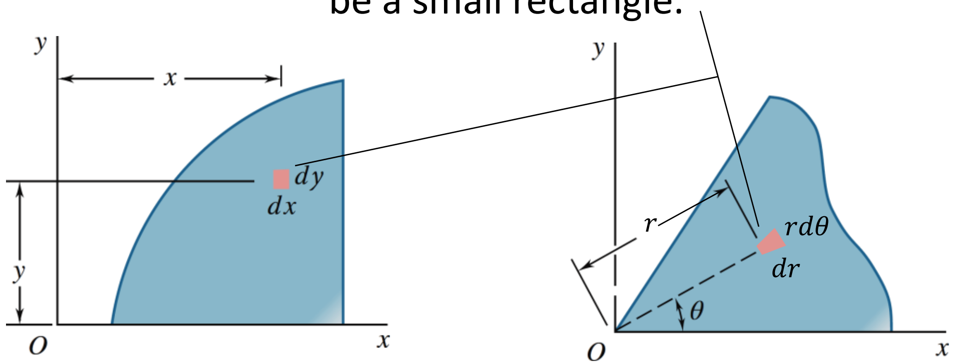
Determination of Centroids: Integration Method

Centroid of Areas: Double Integration Method

$$\bar{x} = \frac{\int x dA}{\int dA}$$

$$\bar{y} = \frac{\int y dA}{\int dA}$$

Element dA is chosen to be a small rectangle.



(Rectangular Coordinate)

$$dA = dxdy$$

(Polar Coordinate)

$$dA = rd\theta dr$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Determination of Centroids: Integration Method

Centroid of Areas: Single Integration Method (preferred method)

$$\bar{x} = \frac{\int x dA}{\int dA}$$

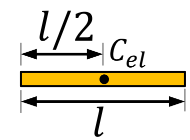
$$\bar{y} = \frac{\int y dA}{\int dA}$$

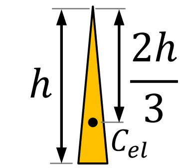
By choosing element dA to be a thin **rectangle** (horizontal or vertical in rectangular coordinate) or **sector/triangle** (in polar coordinate).

$$\bar{x} = \frac{\int \bar{x}_{el} dA}{\int dA}$$

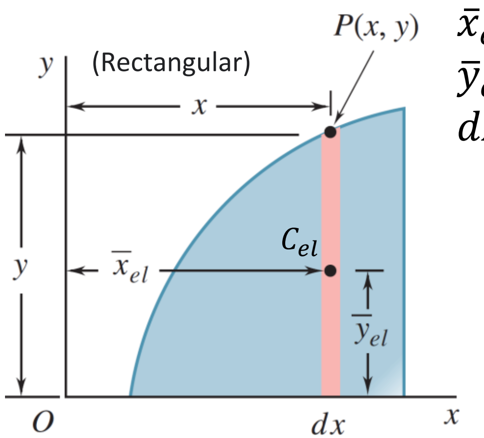
$$\bar{y} = \frac{\int \bar{y}_{el} dA}{\int dA}$$

$(\bar{x}_{el}, \bar{y}_{el})$ is centroid C_{el} of the element dA, which should be expressed in terms of x and y (or r and θ).

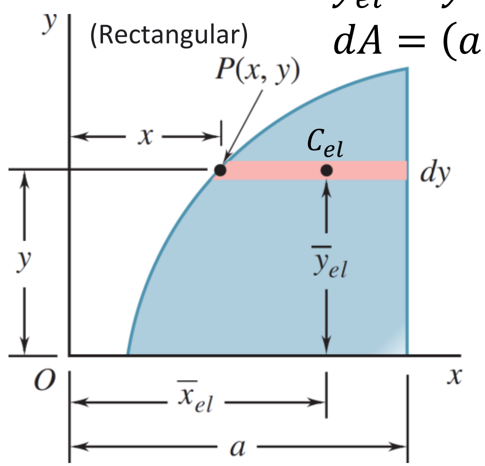




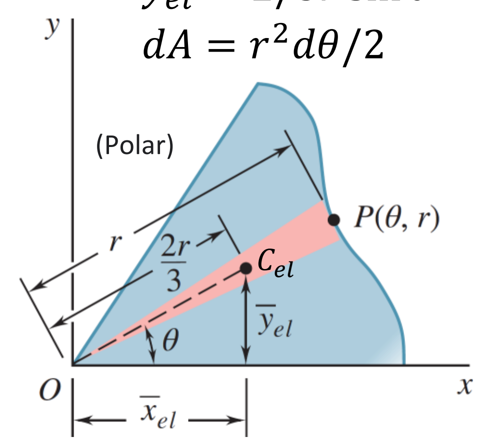
* Examples of choosing element dA for integration:



$\bar{x}_{el} = x$
$\bar{y}_{el} = y/2$
$dA = ydx$

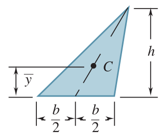
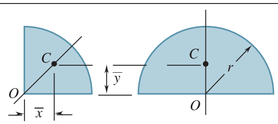
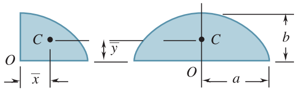
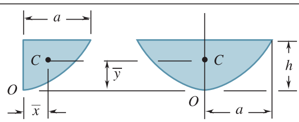
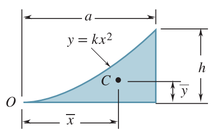
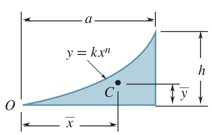
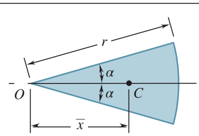


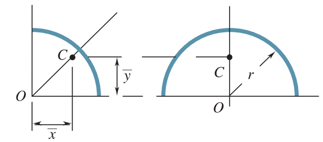
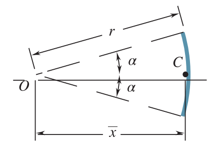
$\bar{x}_{el} = (a + x)/2$
$\bar{y}_{el} = y$
$dA = (a - x)dy$



$\bar{x}_{el} = 2/3r\cos\theta$
$\bar{y}_{el} = 2/3r\sin\theta$
$dA = r^2 d\theta/2$

Centroids of Common Shapes

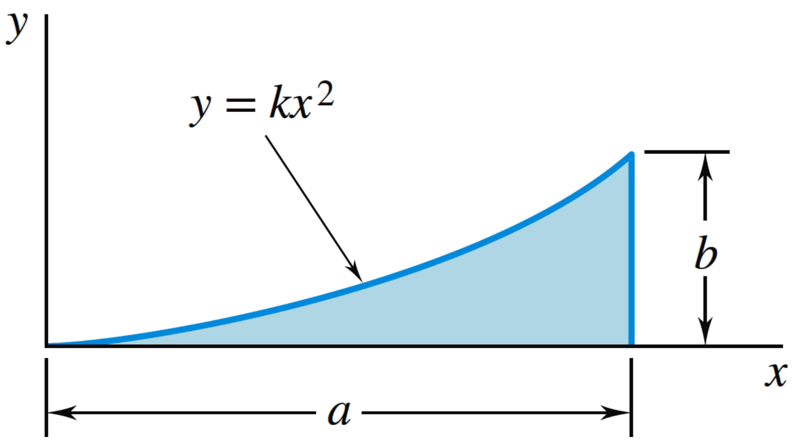
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

Note: The angle α must always be expressed in radians.

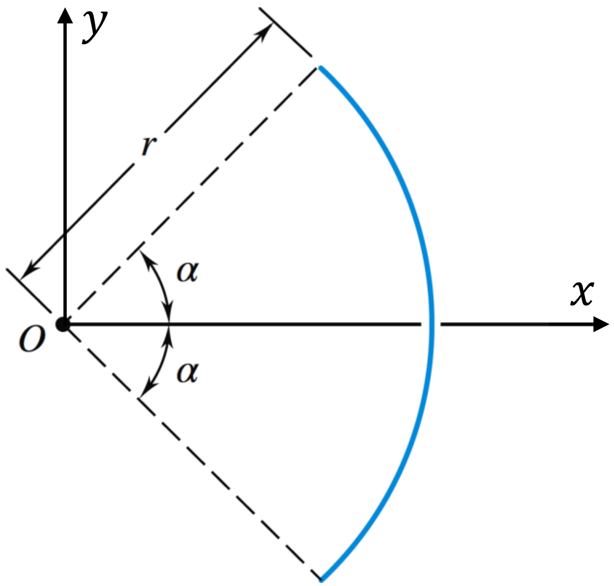
Sample Problem 5.4

Determine the location of the centroid of a parabolic spandrel by direct integration.



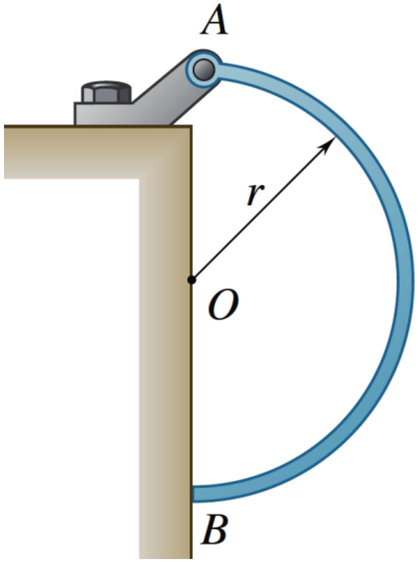
Sample Problem 5.5

Determine the location of the centroid of the circular arc shown.



Sample Problem 5.3

A uniform semicircular rod of weight W and radius r is attached to a pin at A and rests against a frictionless surface at B. Determine the reactions at A and B.



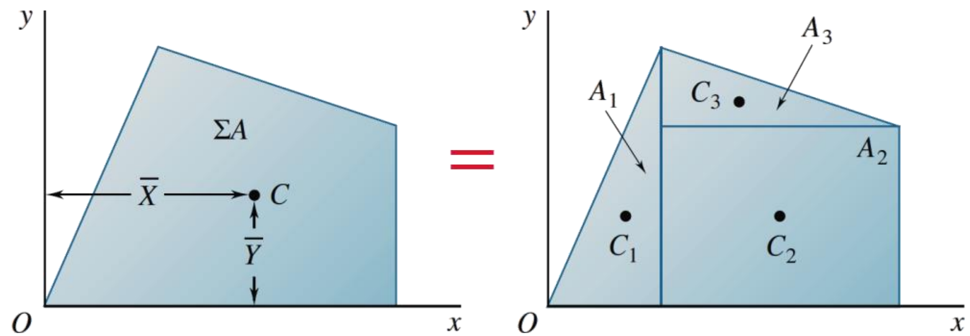
Centroid of Two-Dimensional Composite Bodies

Centroid of Composite Plates and Wires

In many instances, we can **divide** a flat plate or wire **into the common shapes** given in the Tables. We can find the location of the centroid of the composite plate or wire (area or line) from the centroids of its component.

$$\bar{X} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \cdots}{A_1 + A_2 + \cdots}, \qquad Q_y = \bar{X}\Sigma A = \bar{X}(A_1 + A_2 + \cdots)$$

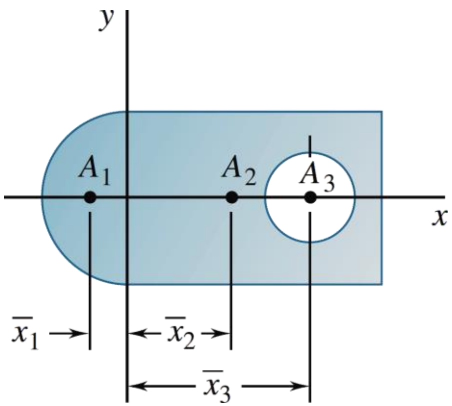
$$\bar{Y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \cdots}{A_1 + A_2 + \cdots}, \qquad Q_x = \bar{Y}\Sigma A = \bar{Y}(A_1 + A_2 + \cdots)$$



Centroid of Composite Plates and Wires

If a composite body has a **hole**, consider it without the hole, and consider the hole as an additional composite part having **negative area** (or **weight** or **mass**).

For example:



	$\bar{x}$	A	$\bar{x}A$
A_1 Semicircle	$-$	$+$	$-$
A_2 Full rectangle	$+$	$+$	$+$
A_3 Circular hole	$+$	$-$	$-$

Note the signs of $\bar{x}$ and A!

$$\bar{X} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3}{A_1 + A_2 + A_3}$$

$$\bar{Y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3}{A_1 + A_2 + A_3}$$

- It is convenient to construct a table listing the areas (or lengths for composite wires) and the respective coordinates of the centroids to find the centroid of composite plates/wires:

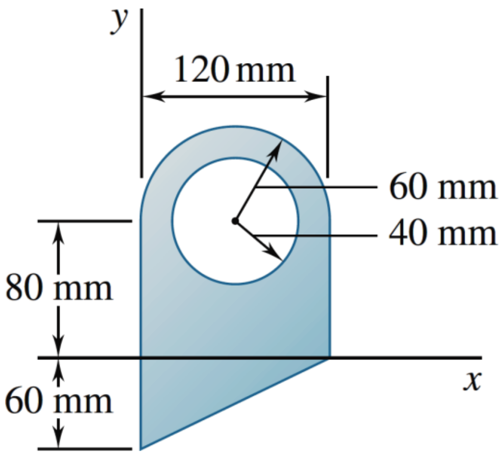
Component	A	$\bar{x}$	$\bar{y}$	$\bar{x}A$	$\bar{y}A$
A_1 Semicircle	…	…	…	…	…
A_2 Full Rectangle	…	…	…	…	…
A_3 Circular Hole	…	…	…	…	…
	$\Sigma A = \cdots$			$\Sigma(\bar{x}A) = \cdots$	$\Sigma(\bar{y}A) = \cdots$

$$\bar{X} = \frac{\Sigma(\bar{x}A)}{\Sigma A}$$

$$\bar{Y} = \frac{\Sigma(\bar{y}A)}{\Sigma A}$$

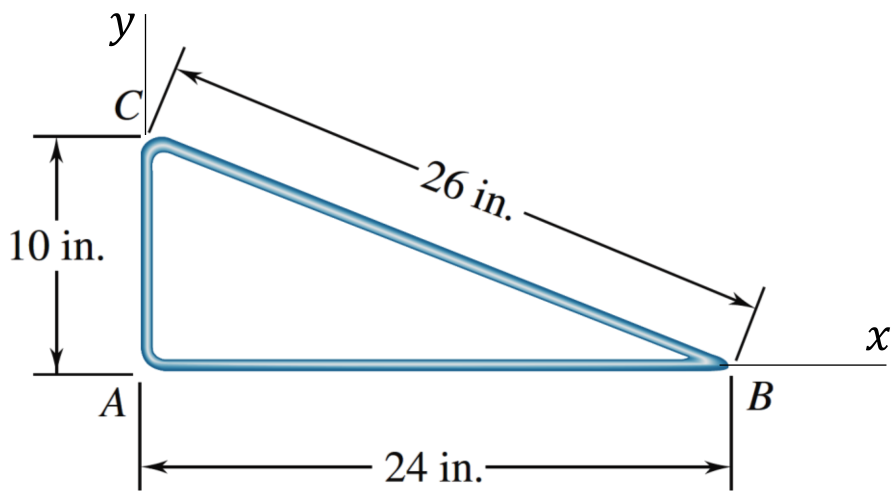
Sample Problem 5.1

For the plane area shown, determine (a) the first moments with respect to the x and y axes; (b) the location of the centroid.



Sample Problem 5.2

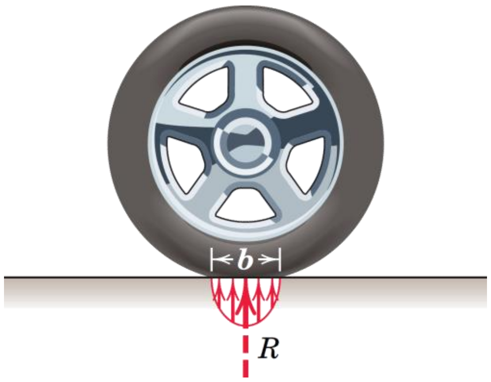
The figure shown is made from a piece of thin, homogeneous wire. Determine the location of its center of gravity.

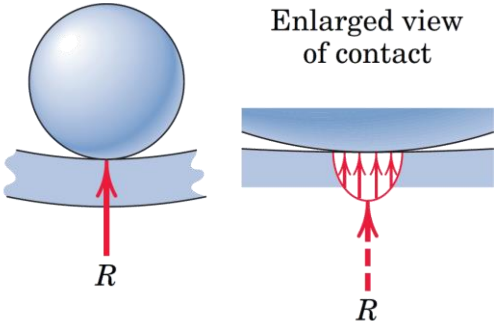


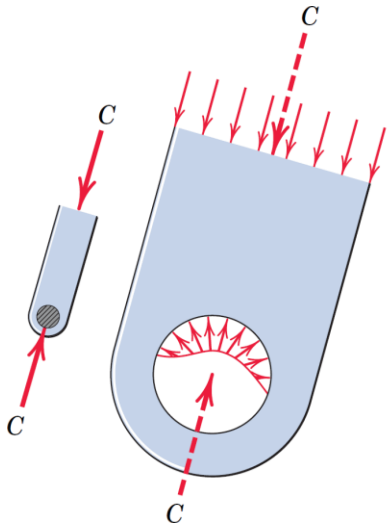
Application of Centroids

Concentrated Load vs Distributed Loads

Any force applied to a body is actually a **distributed load** over a line, area, or volume. If this line, area, or volume is negligible, the force can be modeled as a **concentrated load** at a single point when analyzing its external effect on the body. If not, the intensity of the force *w* at any location must be considered.

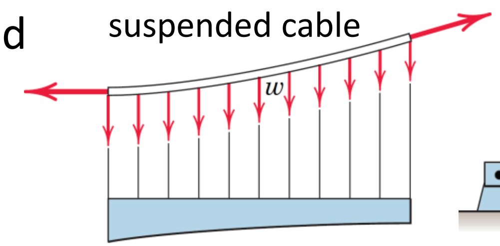


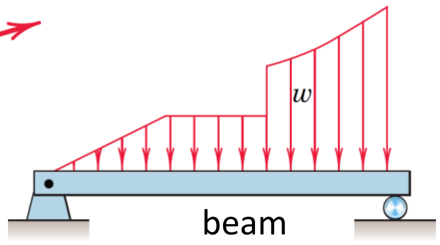




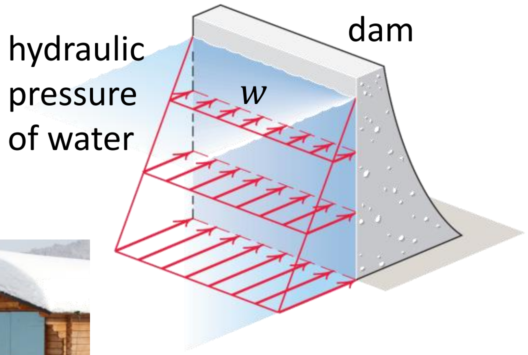
Categories of Load Distribution

(1) **Line Distribution**: when a force is distributed along a **line**. The intensity w of the loading is expressed as force per unit length of line.
Unit: N/m or lb/ft.

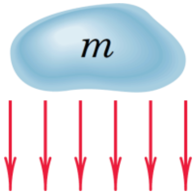




(2) **Area Distribution**: when a force is distributed over an **area**. The intensity w is expressed as force per unit area. This intensity is called pressure for the action of fluid forces and stress for the internal distribution of forces in solids.
Unit: N/m^2 (or Pa)) or lb/ft^2.

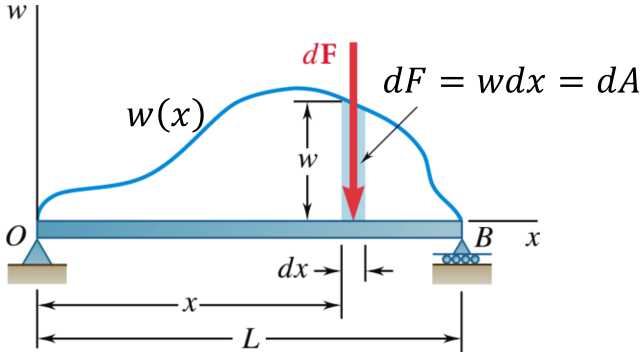


(3) **Volume Distribution**: when a force is distributed over the volume of a body. The most common body force is the force of gravitational attraction, which acts on all elements of mass in a body. The intensity of gravitational force is weight per unit volume or specific weight γ or ρg.

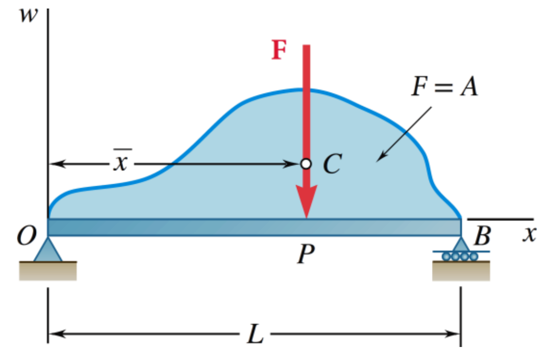


Distributed Loads on Beams

Consider a beam supporting a **distributed load** (line distribution). We want to determine the equivalent concentrated load $\mathbf{F}$:



$\equiv$



Magnitude of Resultant Force: $F = \int_L w(x)dx = \int_A dA = A \rightarrow$ F is equal to the area under the loading diagram $w(x)$.

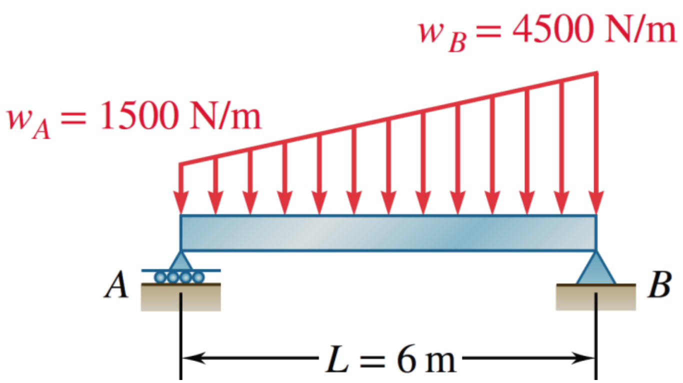
Location of Resultant Force: $\Sigma M_O \rightarrow \bar{x}F = \int_L xw(x)dx \rightarrow \bar{x}A = \int_L xdA$

$\rightarrow$ The line of action of F passes through the centroid C of the area under the loading diagram $w(x)$.

Note: This concentrated load can be used to determine reactions, but should **not** be used to compute internal forces and deflections.

Sample Problem 5.9

A beam supports a distributed load as shown. (a) Determine the equivalent concentrated load. (b) Determine the reactions at the supports.



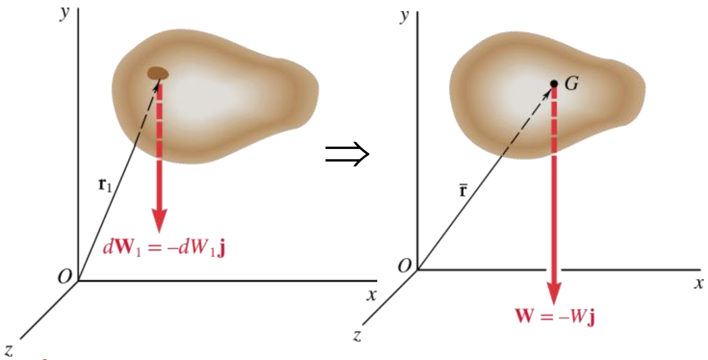
Centroid of Three-Dimensional Bodies

Center of Gravity of a 3D Body

Consider a three-dimensional body divided into infinitesimal elements. The resultant of the forces exerted by earth (i.e., weight) by the elements is a single force **W** in the same direction.

$$\Sigma\mathbf{F}:\quad -W\mathbf{j} = -dW_1\mathbf{j} - dW_2\mathbf{j} - \cdots \quad\Rightarrow\quad W = \int dW$$

$$\Sigma\mathbf{M}_O:\quad \bar{\mathbf{r}} \times (-W\mathbf{j}) = \mathbf{r}_1 \times (-dW_1\mathbf{j}) + \cdots \quad\Rightarrow\quad \bar{\mathbf{r}} = \frac{\int \mathbf{r}\,dW}{W}$$



$$\Rightarrow\quad \bar{x} = \frac{\int x\,dW}{W} \qquad \bar{y} = \frac{\int y\,dW}{W} \qquad \bar{z} = \frac{\int z\,dW}{W}$$

Coordinate of point G where **W** is applied, i.e., **Center of Gravity**

- By substituting $W = mg$ and $dW = g\,dm$: $\quad \bar{\mathbf{r}} = \dfrac{\int \mathbf{r}\,dm}{m}$

$$\Rightarrow\quad \bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$

Coordinate of the body's **Center of Mass**

Note: The center of gravity G of a body may not located on the body.

Centroid of a Volume

Now, if the body is made of a **homogeneous** material:

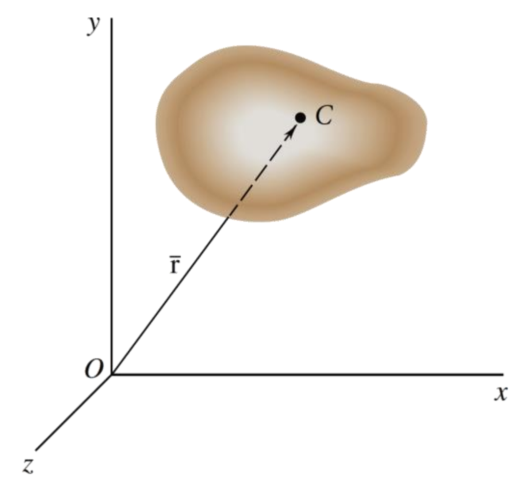
$$m = \rho V$$
$$dm = \rho dV$$

ρ: density, V: volume of the body

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m} \rightarrow \bar{\mathbf{r}} = \frac{\int \mathbf{r} dV}{V} \rightarrow$$

$$\bar{x} = \frac{\int x dV}{V}$$
$$\bar{y} = \frac{\int y dV}{V}$$
$$\bar{z} = \frac{\int z dV}{V}$$

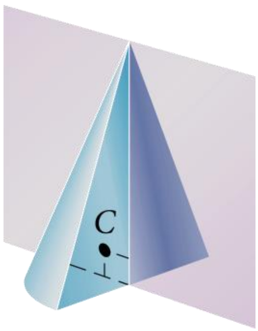
Coordinate of the **Centroid** C of the volume V (or Center of Gravity G or Center of Mass C_m of the body).



$$m = \int dm \longrightarrow V = \int dV$$

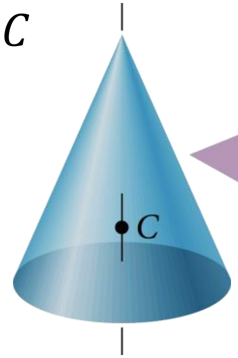
Note: If the body is **not homogeneous**, the center of gravity G **does not coincides** with the **centroid** C.

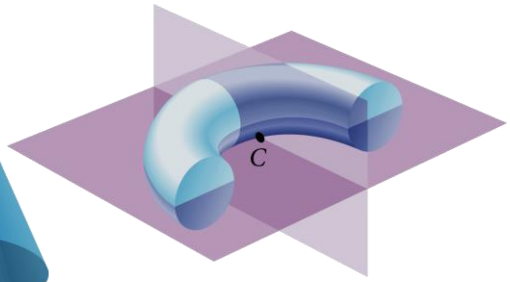
Determination of Centroids: Quick Method



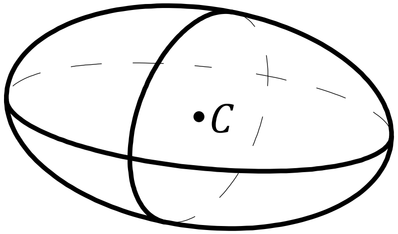
- If a volume V has **a plane of symmetry**, its centroid C is located in the plane (and its first moment with respect to that is **zero**).

- If a volume V has **two planes of symmetry**, its centroid C is located on the line of intersection of the two planes.





- If a volume V has **three planes of symmetry** that intersect at a point (i.e., not along a common line), its centroid C is located on the point of intersection.

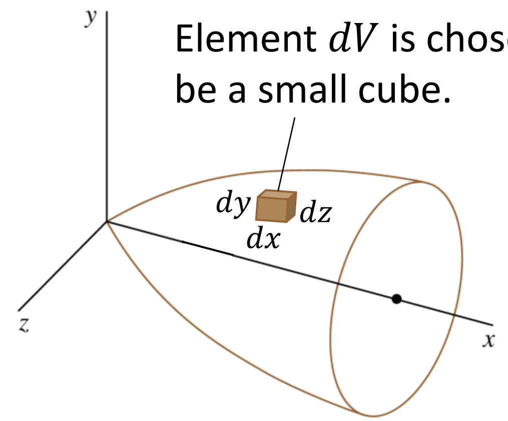


Determination of Centroids: Integration Method

- Triple Integration Method:

$$\bar{x} = \frac{\int x dV}{\int dV} \qquad \bar{y} = \frac{\int y dV}{\int dV} \qquad \bar{z} = \frac{\int z dV}{\int dV} \qquad dV = dxdydz$$

Element dV is chosen to be a small cube.



- Single Integration Method (preferred method):

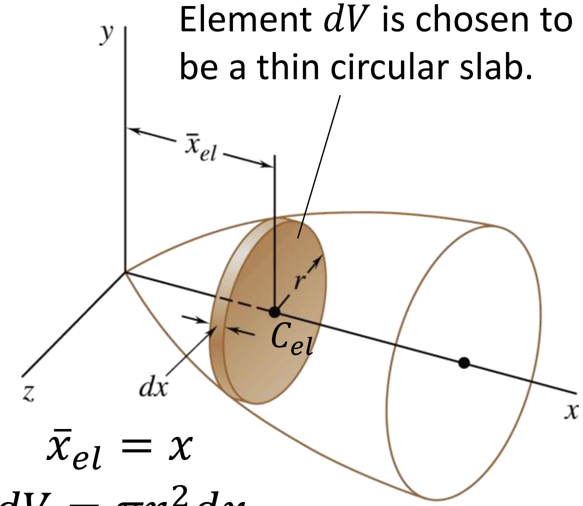
$$\bar{x} = \frac{\int x dV}{\int dV} \qquad \bar{y} = \frac{\int y dV}{\int dV} \qquad \bar{z} = \frac{\int z dV}{\int dV}$$

Choose element dV to be a thin slab **parallel** to one of the planes.

$$\bar{x} = \frac{\int \bar{x}_{el} dV}{\int dV} \qquad \bar{y} = \frac{\int \bar{y}_{el} dV}{\int dV} \qquad \bar{z} = \frac{\int \bar{z}_{el} dV}{\int dV}$$

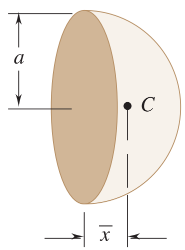
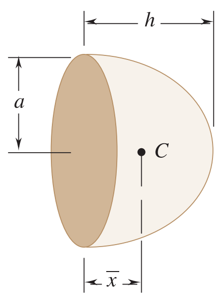
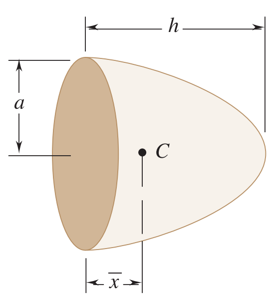
$(\bar{x}_{el}, \bar{y}_{el}, \bar{z}_{el})$ is centroid C_{el} of the element dV, which should be expressed in terms of x, y and z.

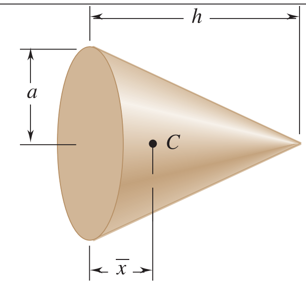
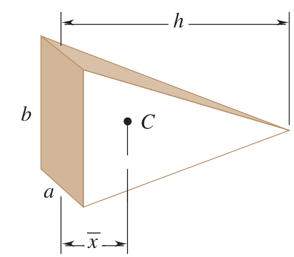
Element dV is chosen to be a thin circular slab.



$$\bar{x}_{el} = x$$
$$dV = \pi r^2 dx$$

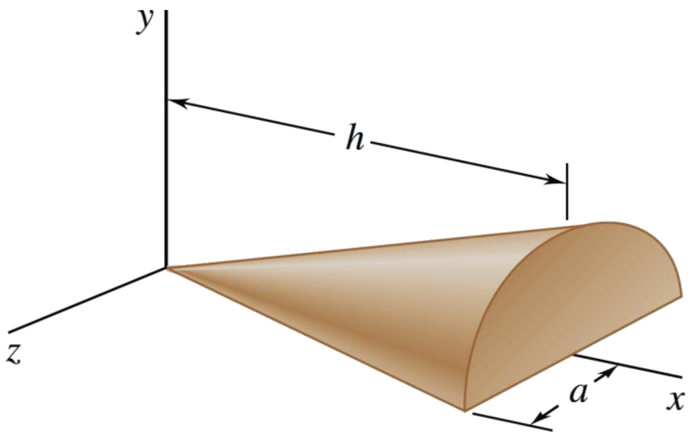
Centroids of Common Volumes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$

Shape		$\bar{x}$	Volume
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Sample Problem 5.11

Determine the location of the centroid of the half right circular cone shown.



Centroid of Three-Dimensional Composite Bodies

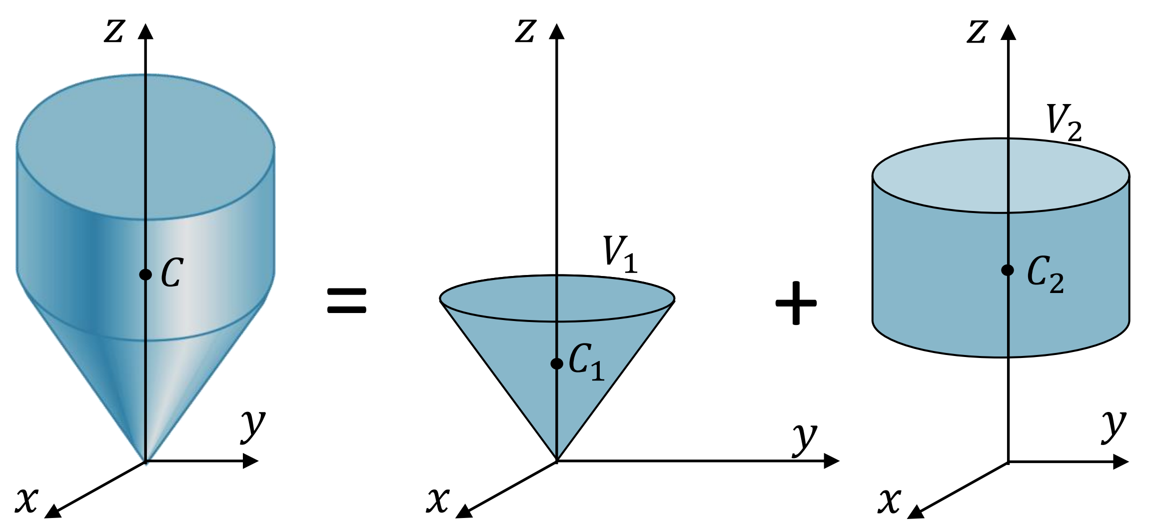
Centroid of Composite Bodies

In many instances, we can **divide** a body **into the common shapes** given in the Tables. We can find the location of the centroid of the composite body (volume) from the centroids of its component.

$$\bar{X} = \frac{\bar{x}_1 V_1 + \bar{x}_2 V_2 + \cdots}{V_1 + V_2 + \cdots}$$

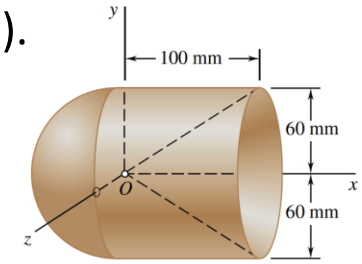
$$\bar{Y} = \frac{\bar{y}_1 V_1 + \bar{y}_2 V_2 + \cdots}{V_1 + V_2 + \cdots}$$

$$\bar{Z} = \frac{\bar{z}_1 V_1 + \bar{z}_2 V_2 + \cdots}{V_1 + V_2 + \cdots}$$



Centroid of Composite Bodies

If a composite body has a **hole**, consider it without the hole and consider the hole as an additional composite part having **negative volume** (or **weight** or **mass**).

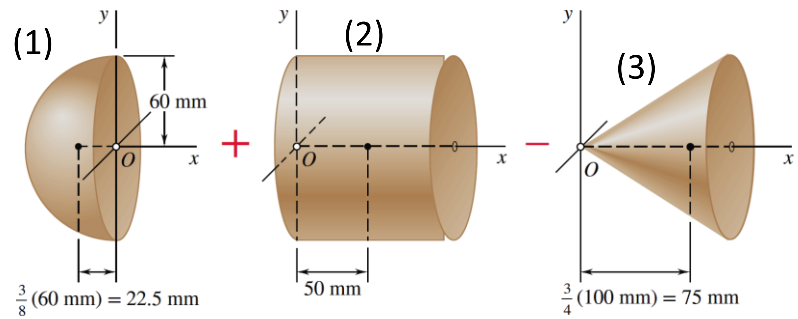


For example:

$$\bar{X} = \frac{\bar{x}_1 V_1 + \bar{x}_2 V_2 + \bar{x}_3 V_3}{V_1 + V_2 + V_3}$$

$$\bar{Y} = \frac{\bar{y}_1 V_1 + \bar{y}_2 V_2 + \bar{y}_3 V_3}{V_1 + V_2 + V_3}$$

$$\bar{Z} = \frac{\bar{z}_1 V_1 + \bar{z}_2 V_2 + \bar{z}_3 V_3}{V_1 + V_2 + V_3}$$

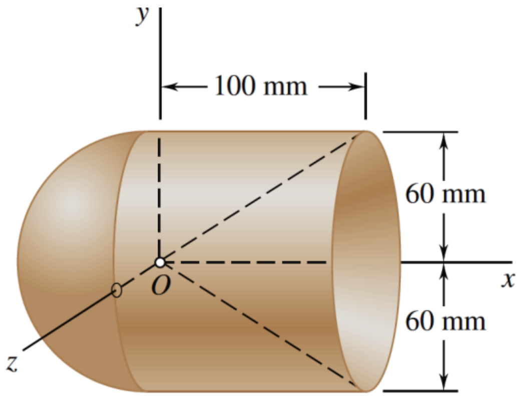


- It is more convenient to construct a table listing the volumes and the respective coordinates of the centroids to find the centroid of composite volume:

Component	Volume, mm^3	$\bar{x}$, mm	$\bar{x}V$, mm^4
Hemisphere	$\frac{1}{2}\frac{4\pi}{3}(60)^3 = 0.4524 \times 10^6$	–22.5	-10.18×10^6
Cylinder	$\pi(60)^2(100) = 1.1310 \times 10^6$	+50	$+56.55 \times 10^6$
Cone	$-\frac{\pi}{3}(60)^2(100) = -0.3770 \times 10^6$	+75	-28.28×10^6
	$\Sigma V = 1.206 \times 10^6$		$\Sigma \bar{x}V = +18.09 \times 10^6$

Sample Problem 5.12

Determine the location of the center of gravity of the homogeneous body of revolution shown that was obtained by joining a hemisphere and a cylinder and carving out a cone.



Sample Problem 5.13

Locate the center of gravity of the steel machine part shown. The diameter of each hole is 1 in.