MEC529: Introduction to Robotics (Theory and Applications)

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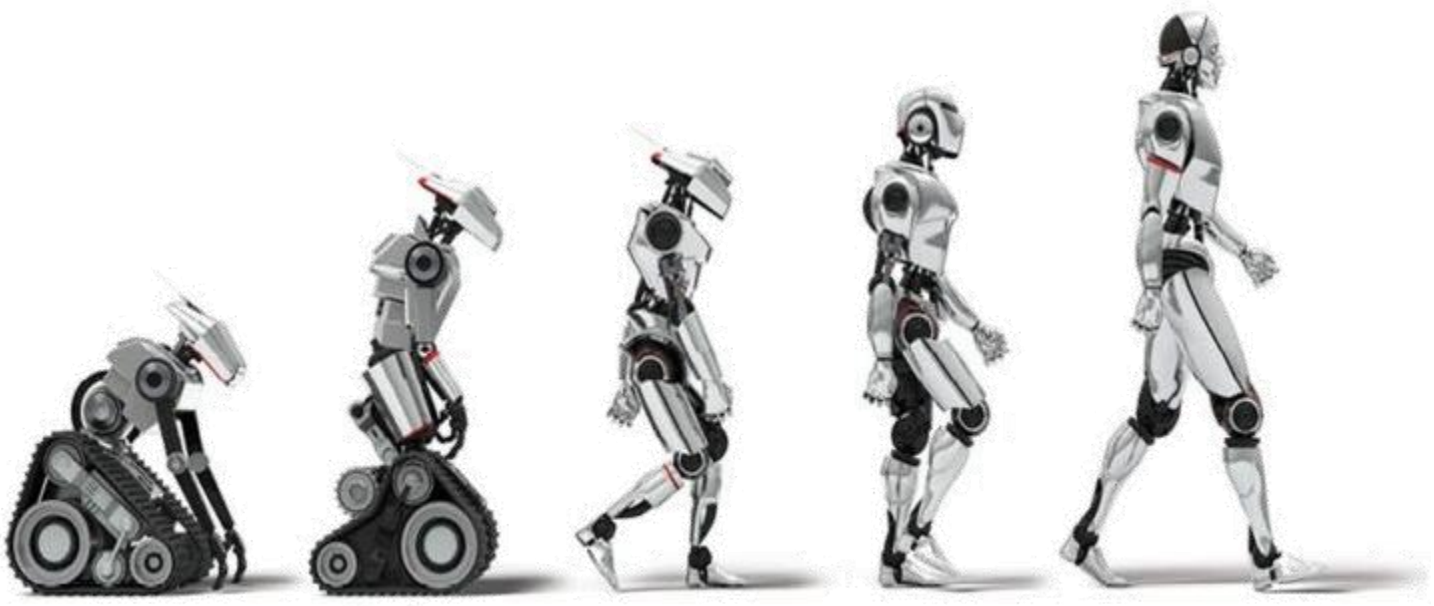
Ch1: A Review of Linear Algebra

Introduction

Robotics

Robotics is an interdisciplinary field which integrates computer science, mechanical engineering, electrical engineering, information engineering, bioengineering, computer engineering, control engineering, software engineering, mathematics, etc.

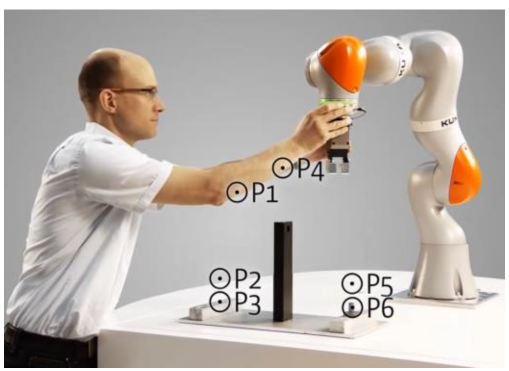
The goal of robotics is to design machines that can assist humans or replicate human actions.

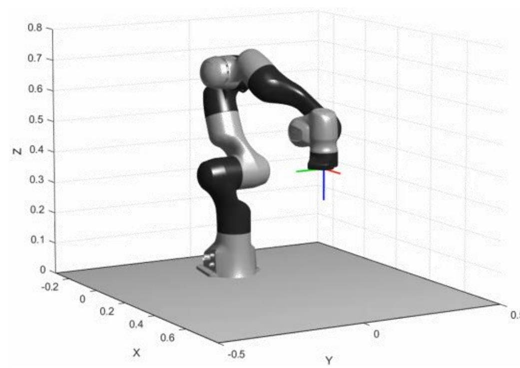


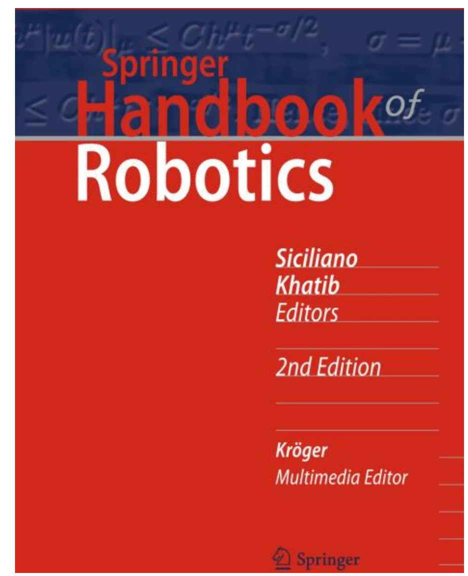
Robotics

- Classifications of robots based on structure:

- **Fixed-Base Robots** (e.g., Serial or Open-Chain Manipulators, Parallel Manipulators)
- **Mobile Robots**
  - **Ground Robots** (e.g., Wheeled Robots, Legged Robots)
  - **Submarine Robots**
  - **Aerial Robots**







https://link.springer.com/book/10.1007/978-3-319-32552-1

Vectors

Basic Notation

$\forall$

$\exists$

$\in$

$\implies$

$\iff$

$:=$

$\dot{x}$

$f : \mathcal{D} \to \mathcal{R}$

$\mathbb{R}$

$\mathbb{R}_+, \mathbb{R}_{++}$

$|x|$

Coordinate-Free Vector and Point

A **coordinate-free vector** v is a geometric quantity with a length and a direction.

Given a reference frame, vector v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then, the vector v can be represented by its coordinates $\boldsymbol{v}$ in the reference frame.

- v refers to a physical quantity in the underlying space.
- $\boldsymbol{v}$ is a representation of v that depends on the choice of coordinate frame.

- A **point** p denotes a point in the physical space.
- A point p can be represented by as a **vector** from frame origin to p.
- $\boldsymbol{p}$ denotes the coordinate of a point p , which depends on the choice of reference frame.

Vector

$\boldsymbol{x} \in \mathbb{R}^n$: (an n-dimensional real vector in the column format)

$\mathbb{R}^n$: n-dimensional real space (Euclidian Space)

$\boldsymbol{x}^T$:

Vector Norm

General Definition: Given $\boldsymbol{x} \in \mathbb{R}^n$, vector norm $\|\boldsymbol{x}\| \in \mathbb{R}_+$ is defined such that

- $\|\boldsymbol{x}\| > 0$ when $\boldsymbol{x} \neq \boldsymbol{0}$ and $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$.
- $\|k\boldsymbol{x}\| = |k|\|\boldsymbol{x}\|, \ \forall k \in \mathbb{R}$.
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|, \ \forall \boldsymbol{y} \in \mathbb{R}^n$.

❖ The p-norm (or ℓ_p-norm) of $\boldsymbol{x}$ for $p \in \mathbb{R}, p \geq 1$ is defined as $\|\boldsymbol{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

e.g. $\|\boldsymbol{x}\|_2 = \|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$ (Euclidean Norm)

Special case: $\|\boldsymbol{x}\|_\infty := \max_i |x_i|$

Schwartz Inequality: $|\boldsymbol{x}^T \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

Unit Vector: $\|\hat{\boldsymbol{x}}\|_2 = \hat{\boldsymbol{x}}^T \hat{\boldsymbol{x}} = 1, \quad \hat{\boldsymbol{x}} = \boldsymbol{x}/\|\boldsymbol{x}\|_2$

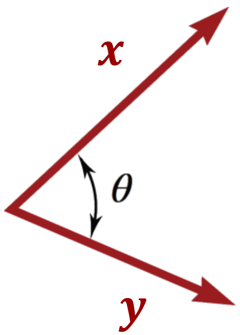
Dot Product or Scalar Product or Inner Product

Dot Product or Scalar Product or Inner Product of two vectors $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^n$ is a scalar defined as

(Algebraic Definition) $$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i = \boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{x}$$

(Geometric Definition) $$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \cos\theta$$

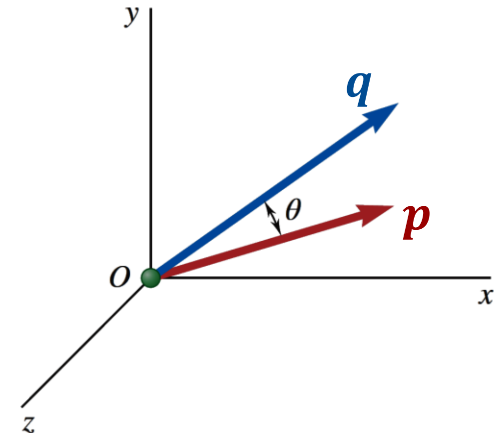
$$(0 \le \theta \le \pi)$$



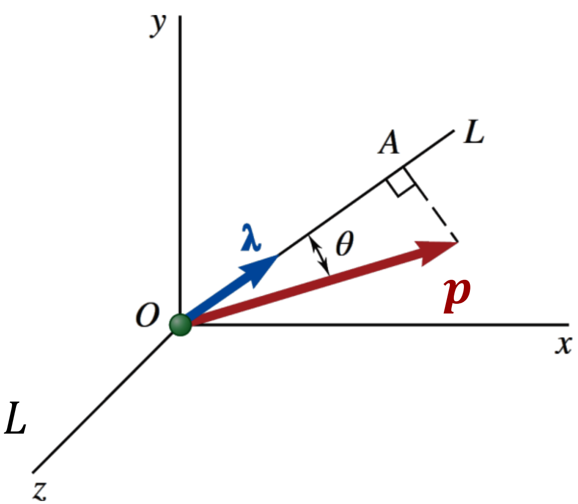
Orthogonal Vectors:

Applications of Dot Product

(1) Finding angle formed between two given vectors $\boldsymbol{p} \in \mathbb{R}^n$, $\boldsymbol{q} \in \mathbb{R}^n$ (or intersecting lines):



(2) Finding projection of a vector $\boldsymbol{p} \in \mathbb{R}^n$ on a given axis or directed line:

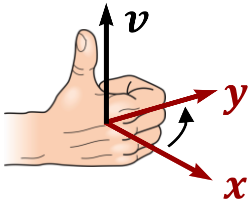


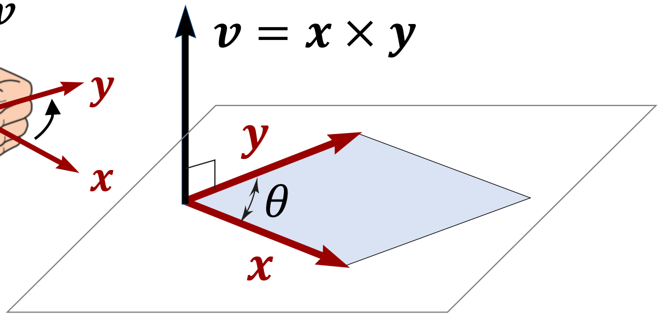
$\boldsymbol{\lambda}$: unit vector of line L

Cross Product or Vector Product

Cross product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ (in the Euclidean space) is defined as a vector $\boldsymbol{v} = \boldsymbol{x} \times \boldsymbol{y} \in \mathbb{R}^3$ that is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$ ($\boldsymbol{v} \perp \boldsymbol{x}$, $\boldsymbol{v} \perp \boldsymbol{y}$), with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

$$\|\boldsymbol{v}\|_2 = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \sin\theta \qquad (0 \le \theta \le \pi)$$





$$\boldsymbol{v} = \boldsymbol{x} \times \boldsymbol{y} = (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) \times (y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})$$

Coordinate notation

$$\boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3)$$

$$\boldsymbol{v} = \boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Matrix notation

Matrices

Matrix

$\boldsymbol{A} \in \mathbb{R}^{m \times n}$ (an m by n dimensional real matrix)

$\boldsymbol{A}^T$

Tall, Fat, and Square Matrices:

Matrix-vector multiplication $\boldsymbol{Ax}$ as linear combination of columns of $\boldsymbol{A}$:

Particular Matrices

Square Matrix:

- Upper Triangular
- Lower Triangular
- Diagonal
    - Identity Matrix
- Null Matrix

Symmetric Matrix:

Skew-symmetric Matrix:

Partitioned Matrix: A matrix whose elements are matrices (blocks) of proper dimensions.

Matrix Operations

Trace of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$: $\mathrm{tr}(\boldsymbol{A})$

Sum of matrices: $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$

Symmetric and skew-symmetric part of a square matrix $\boldsymbol{A}$:

Product of matrices: $\boldsymbol{C} = \boldsymbol{AB}$

Determinant of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$: $\det(\boldsymbol{A})$

Principle minors of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$:

Singular and Nonsingular Matrices:

Matrix Operations (cont.)

Linearly Independent Vectors $\boldsymbol{x}_i \in \mathbb{R}^m, i = 1, \ldots, n$

Rank of a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$: $\mathrm{rank}(\boldsymbol{A})$

Inverse of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$: $\boldsymbol{A}^{-1}$

Orthogonal Matrix:

Derivative of $\boldsymbol{A}(t) \in \mathbb{R}^{m\times n}$: $\frac{d}{dt}\boldsymbol{A}(t) = \dot{\boldsymbol{A}}(t)$

Derivative of $\boldsymbol{A}^{-1}(t) \in \mathbb{R}^{n\times n}$:

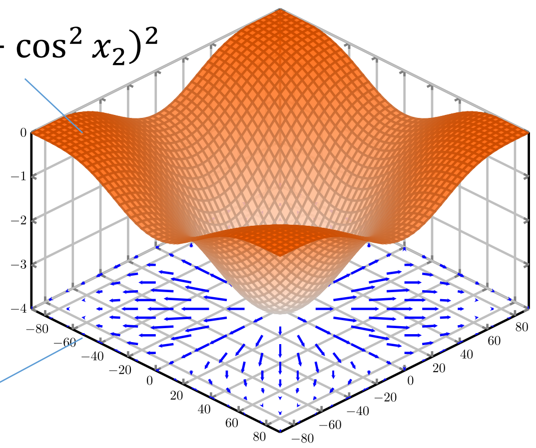
Gradient

For a **scalar function** $f: \mathbb{R}^n \to \mathbb{R}$ which is differentiable with respect to the elements x_i of $\boldsymbol{x} \in \mathbb{R}^n$, its **gradient** with respect to $\boldsymbol{x}$ is an n-dimensional column vector $\boldsymbol{\nabla}_{\boldsymbol{x}} f \in \mathbb{R}^n$ as:

(nabla symbol and pronounced "del")

$$\boldsymbol{\nabla}_{\boldsymbol{x}} f(\boldsymbol{x}) = \left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^T = \begin{bmatrix} \dfrac{\partial f(\boldsymbol{x})}{\partial x_1} \\ \vdots \\ \dfrac{\partial f(\boldsymbol{x})}{\partial x_n} \end{bmatrix}$$

$f(x_1, x_2) = -(\cos^2 x_1 + \cos^2 x_2)^2$



The gradient depicted as a projected vector field.

- If $\boldsymbol{x}(t)$ is a differentiable function with respect to t:

$$\dot{f}(\boldsymbol{x}) = \frac{d}{dt} f\big(\boldsymbol{x}(t)\big) = \frac{\partial f}{\partial \boldsymbol{x}} \frac{d\boldsymbol{x}}{dt} = \frac{\partial f}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = \boldsymbol{\nabla}_{\boldsymbol{x}}^T f(\boldsymbol{x}) \dot{\boldsymbol{x}} \qquad \text{(Chain Rule)}$$

Jacobian

For a **vector function** $\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m$ whose elements f_i are differentiable with respect to the elements x_i of $\boldsymbol{x} \in \mathbb{R}^n$, its **Jacobian** with respect to $\boldsymbol{x}$ is matrix $\boldsymbol{J_f} \in \mathbb{R}^{m \times n}$ as:

$$\boldsymbol{J_f}(\boldsymbol{x}) = \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{\partial f_1(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \dfrac{\partial f_2(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \\ \dfrac{\partial f_m(\boldsymbol{x})}{\partial \boldsymbol{x}} \end{bmatrix}$$

- If $\boldsymbol{x}(t)$ is a differentiable function with respect to t:

$$\dot{\boldsymbol{f}}(\boldsymbol{x}) = \frac{d}{dt}\boldsymbol{f}\big(\boldsymbol{x}(t)\big) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \boldsymbol{J_f}(\boldsymbol{x})\dot{\boldsymbol{x}} \qquad \text{(Chain Rule)}$$

Cross Product as a Matrix-Vector Multiplication

Cross product $\boldsymbol{x} \times \boldsymbol{y}$ ($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$) can be thought of as a multiplication of a vector by a 3×3 skew-symmetric matrix as

$$\boldsymbol{x} \times \boldsymbol{y} = \underbrace{\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}}_{[\boldsymbol{x}]} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [\boldsymbol{x}]\boldsymbol{y} = -[\boldsymbol{y}]\boldsymbol{x}$$

The matrix $[\boldsymbol{x}] \in \mathbb{R}^{3\times 3}$ is a <u>skew-symmetric matrix representation</u> of $\boldsymbol{x}$. $[\boldsymbol{x}] = -[\boldsymbol{x}]^T$