Ch3: Rigid-Body Motions – Part 2

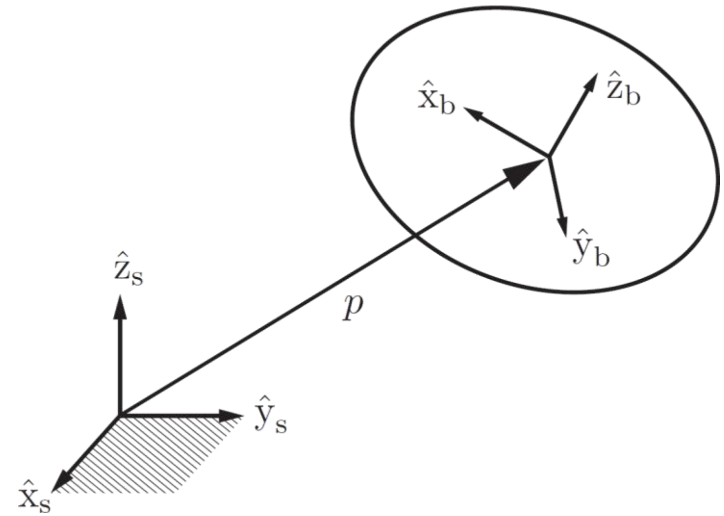
Rigid-Body Motions

Homogeneous Transformation Matrices

Rigid-body configuration can be represented by the pair $(\boldsymbol{R}, \boldsymbol{p})$ ($\boldsymbol{R} \in SO(3)$, $\boldsymbol{p} \in \mathbb{R}^3$). We can package $(\boldsymbol{R}, \boldsymbol{p})$ into a single 4×4 matrix as

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

(an implicit representation of the C-space)



Special Euclidean Group $SE(n)$

The **Special Euclidean Group** $SE(3)$, also known as the **group of rigid-body motions** or **homogeneous transformation matrices** in $\mathbb{R}^3$, is the set of all 4×4 real matrices $\boldsymbol{T}$ of the form

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T} \in SE(3) \quad \boldsymbol{R} \in SO(3) \quad \boldsymbol{p} \in \mathbb{R}^3$$

The **special Euclidean group** $SE(2)$ is the set of all 3×3 real matrices $\boldsymbol{T}$ of the form

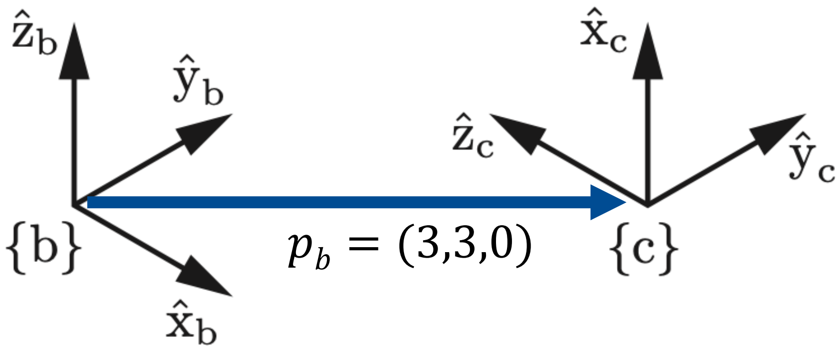
$$T = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T} \in SE(2) \quad \boldsymbol{R} \in SO(2) \quad \boldsymbol{p} \in \mathbb{R}^2 \quad \theta \in [0,2\pi)$$

$SE(2)$ is a subgroup of $SE(3)$: $\quad SE(2) \subset SE(3)$

$$\boldsymbol{T} = (\boldsymbol{R},\boldsymbol{p}) \in SE(3) \qquad SE(3) = \{(\boldsymbol{R},\boldsymbol{p}) \mid \boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3\}$$

Example



T_{bc}?

Properties of Transformation Matrices

$SE(3)$ (or $SE(2)$) is a **matrix (Lie) group** (and the group operation • is matrix multiplication).

Closure: $\boldsymbol{T}_1\boldsymbol{T}_2 \in SE(3)$

Associative: $(\boldsymbol{T}_1\boldsymbol{T}_2)\boldsymbol{T}_3 = \boldsymbol{T}_1(\boldsymbol{T}_2\boldsymbol{T}_3)$ (but generally not commutative, $\boldsymbol{T}_1\boldsymbol{T}_2 \neq \boldsymbol{T}_2\boldsymbol{T}_1$)

Identity: $\exists\ \boldsymbol{I} \in SE(3)$ such that $\boldsymbol{TI} = \boldsymbol{IT} = \boldsymbol{T}$

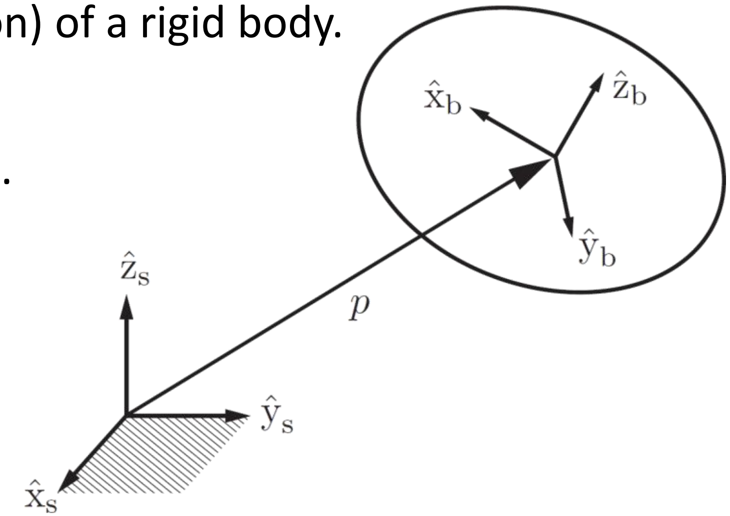
Inverse: $\exists\ \boldsymbol{T}^{-1} \in SE(3)$ such that $\boldsymbol{TT}^{-1} = \boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{I}$

$$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Note: $\boldsymbol{T}$ preserves both distances and angles.

Uses of Transformation Matrices (1)

(1) Representing configuration (position and orientation) of a rigid body.



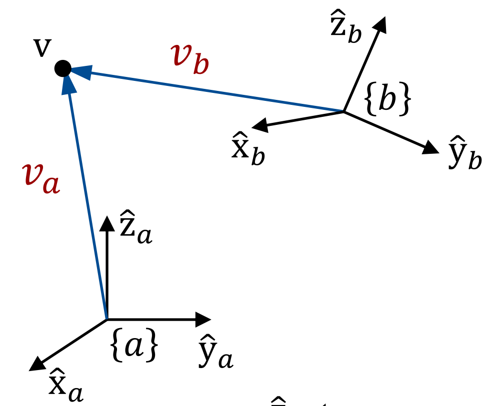
Notation: $\boldsymbol{T}_{sb}$ is the configuration of $\{b\}$ relative to $\{s\}$.

$$\boldsymbol{T}_{sb} = \begin{bmatrix} \boldsymbol{R}_{sb} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

$$\boldsymbol{T}_{sb}\boldsymbol{T}_{bs} = \boldsymbol{I} \quad \text{or} \quad \boldsymbol{T}_{bs} = \boldsymbol{T}_{sb}^{-1} = \begin{bmatrix} \boldsymbol{R}_{sb}^T & -\boldsymbol{R}_{sb}^T\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Uses of Transformation Matrices (2)

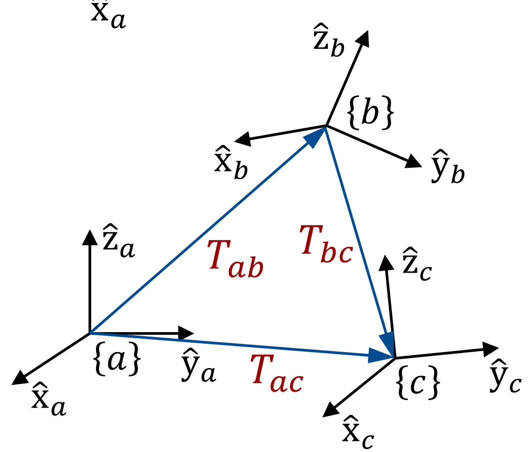
(2) Changing the reference frame of a vector or frame.



Subscript Cancellation Rule:

$$\boldsymbol{T}_{ab}\boldsymbol{v}_b = \boldsymbol{T}_{a\cancel{b}}\boldsymbol{v}_{\cancel{b}} = \boldsymbol{v}_a$$

$$\boldsymbol{T}_{ab}\boldsymbol{T}_{bc} = \boldsymbol{T}_{a\cancel{b}}\boldsymbol{T}_{\cancel{b}c} = \boldsymbol{T}_{ac}$$



$\boldsymbol{T}_{ab}$ can be viewed as a **mathematical operator** that changes the reference frame from $\{b\}$ to $\{a\}$.

Note: To calculate $\boldsymbol{Tv}$, we append a “1” to $\boldsymbol{v}$ and it is called **homogeneous coordinates** representation of $\boldsymbol{v}$.

$$\boldsymbol{v} = [v_1\ \ v_2\ \ v_3\ \ 1]^T$$

Uses of Transformation Matrices (3)

(3) Displacing (rotating and translating) a vector or frame.

$$\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) = (\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta), \boldsymbol{p}) = \mathrm{Trans}(\boldsymbol{p})\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

$$\mathrm{Trans}(\boldsymbol{p}) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

$\boldsymbol{T}$ can be viewed as a **mathematical operator** that rotates a frame or vector about a unit axis $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$ by an amount θ + translating it by $\boldsymbol{p}$.

Uses of Transformation Matrices (3) (cont.)

- Rotation of vector $\boldsymbol{v}$ about a unit axis $\hat{\boldsymbol{\omega}}$ (expressed in the same frame) by an amount θ and translation of it by $\boldsymbol{p}$ (expressed in the same frame) is vector $\boldsymbol{v}'$ expressed in the same frame:

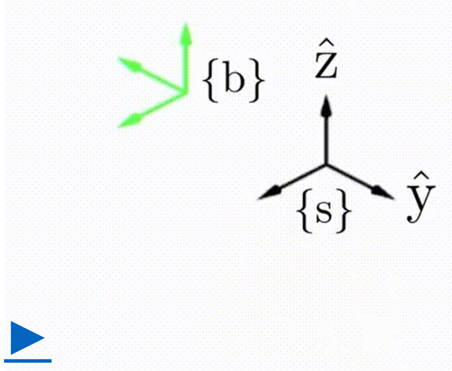
$$\boldsymbol{v}' = \boldsymbol{T}\boldsymbol{v} = \underbrace{\text{Trans}(\boldsymbol{p})\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v}}_{\xleftarrow{\text{Interpretation}}}$$

- **Fixed-frame Transformation**:

2. Translating it by $\boldsymbol{p}$ in $\{s\}$ to get $\{b'\}$

1. Rotating $\{b\}$ by θ about $\hat{\boldsymbol{\omega}}$ in $\{s\}$ (this moves $\{b\}$ origin)

$$\boldsymbol{T}_{sb'} = \boldsymbol{T}\boldsymbol{T}_{sb} = \underbrace{\text{Trans}(\boldsymbol{p})\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)}_{\xleftarrow{\text{Interpretation}}}\boldsymbol{T}_{sb}$$

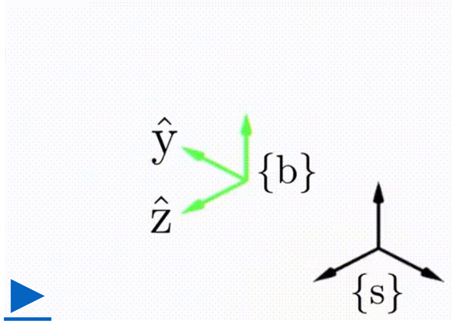


- **Body-frame Transformation**:

1. Translating $\{b\}$ by $\boldsymbol{p}$ in $\{b\}$

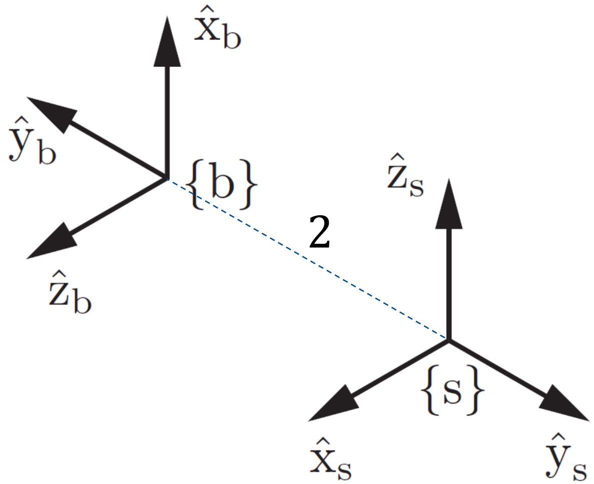
2. Rotating it by θ about $\hat{\boldsymbol{\omega}}$ in the new body frame to get $\{b"\}$

$$\boldsymbol{T}_{sb''} = \boldsymbol{T}_{sb}\boldsymbol{T} = \boldsymbol{T}_{sb}\underbrace{\text{Trans}(\boldsymbol{p})\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)}_{\xrightarrow{\text{Interpretation}}}$$



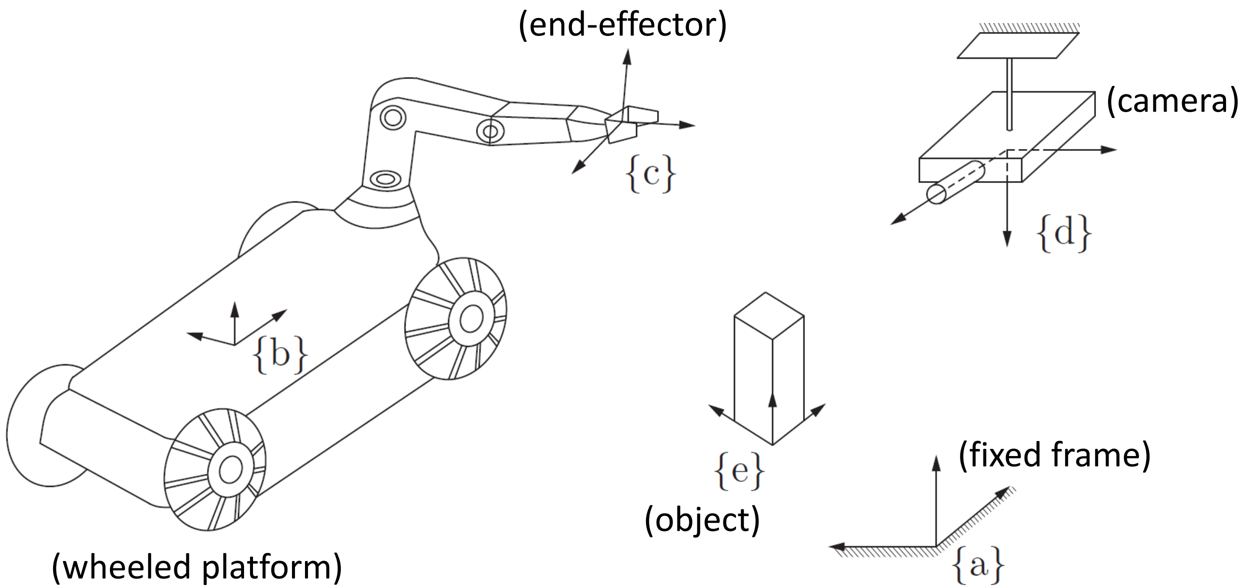
Example

Fixed-frame and body-frame transformations corresponding to $\hat{\boldsymbol{\omega}} = (0,0,1)$, $\theta = 90°$, and $\boldsymbol{p} = (0,2,0)$.



Example

A robot arm mounted on a wheeled mobile platform moving in a room, and a camera fixed to the ceiling. The robot must pick up an object with body frame $\{e\}$. What is the configuration of the object relative to the robot hand, $\boldsymbol{T}_{ce}$, given $\boldsymbol{T}_{db}$, $\boldsymbol{T}_{de}$, $\boldsymbol{T}_{bc}$, and $\boldsymbol{T}_{ad}$?

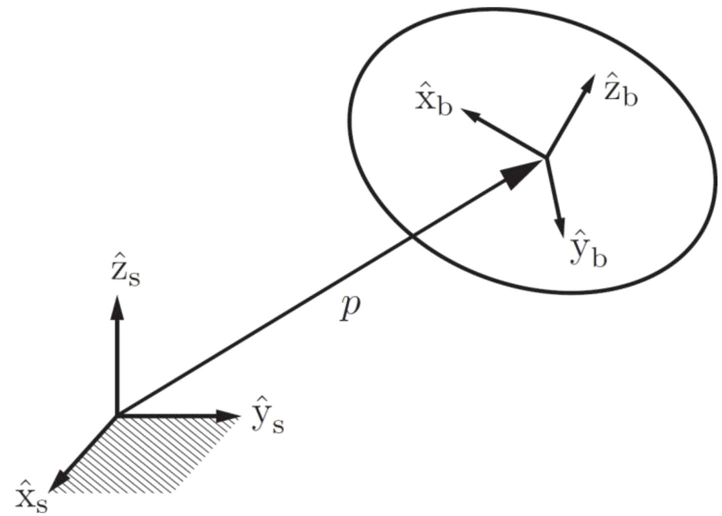


Twists

Spatial Velocity or Twist

Finding both the linear and angular velocity of frame $\{b\}$ attached to a moving body.

$$\boldsymbol{T}(t) = \begin{bmatrix} \boldsymbol{R}(t) & \boldsymbol{p}(t) \\ \boldsymbol{0} & 1 \end{bmatrix} : \ \boldsymbol{T}_{sb} \text{ at time } t$$



Body Twist

Similar to $\boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b]$, lets compute $\boldsymbol{T}^{-1}\dot{\boldsymbol{T}}$:

$$\boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{R}} & \dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}} & \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{v}_b \\ \boldsymbol{0} & 0 \end{bmatrix} \xrightarrow[{[\boldsymbol{\omega}_b] \in so(3)}]{\boldsymbol{v}_b \in \mathbb{R}^3} \boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} \in \mathbb{R}^6$$

$\boldsymbol{\mathcal{V}}_b$ is defined as **Body Twist**
(or spatial velocity in the body frame)

$$\boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{\mathcal{V}}_b] = \begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{v}_b \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$$

The set of all 4×4 matrices of this form is called $se(3)$ and comprises the 4×4 matrix representations of the **body twists** associated with the rigid-body configurations $SE(3)$.

($se(3)$ is called the Lie algebra of the Lie group $SE(3)$)

Spatial Twist

Similar to $\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s]$, lets compute $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$:

$$\begin{aligned}\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} &= \begin{bmatrix}\dot{\boldsymbol{R}} & \dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1\end{bmatrix} \\ &= \begin{bmatrix}\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}} & \dot{\boldsymbol{p}} - \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 0\end{bmatrix} \\ &= \begin{bmatrix}[\boldsymbol{\omega}_s] & \boldsymbol{v}_s \\ \boldsymbol{0} & 0\end{bmatrix}\end{aligned}$$

$$\xrightarrow[]{\substack{\boldsymbol{v}_s \in \mathbb{R}^3 \\ [\boldsymbol{\omega}_s] \in so(3)}} \quad \boldsymbol{\mathcal{V}}_s = \begin{bmatrix}\boldsymbol{\omega}_s \\ \boldsymbol{v}_s\end{bmatrix} \in \mathbb{R}^6$$

$\boldsymbol{\mathcal{V}}_s$ is defined as **Spatial Twist**
(or spatial velocity in the space frame)

$$\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\boldsymbol{\mathcal{V}}_s] = \begin{bmatrix}[\boldsymbol{\omega}_s] & \boldsymbol{v}_s \\ \boldsymbol{0} & 0\end{bmatrix} \in se(3)$$

The set of all 4×4 matrices of this form is called $se(3)$ and comprises the 4×4 matrix representations of the **spatial twists** associated with the rigid-body configurations $SE(3)$.

Adjoint Map

$$\cancel{\mathcal{V}_s = T_{sb}\mathcal{V}_b}$$

4×4

6×6

$$[\mathcal{V}_b] = T^{-1}\dot{T}$$
$$[\mathcal{V}_s] = \dot{T}T^{-1}$$

$$\longrightarrow [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1} \longrightarrow$$

$$[\mathcal{V}_s] = \begin{bmatrix} R[\omega_b]R^{\mathrm{T}} & -R[\omega_b]R^{\mathrm{T}}p + Rv_b \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow[{[\omega]p = -[p]\omega \quad p, \omega \in \mathbb{R}^3}]{R[\omega]R^{\mathrm{T}} = [R\omega]}$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Adjoint Map associated with T
or Adjoint Representation of T

$$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$$

Similarly, $\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s = \mathrm{Ad}_{T_{bs}}(\mathcal{V}_s)$

Adjoint Map Properties

- Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in SE(3)$ and $\boldsymbol{\mathcal{V}} = (\boldsymbol{\omega}, \boldsymbol{v})$. Then,

$$[\mathrm{Ad}_{\boldsymbol{T}_1}][\mathrm{Ad}_{\boldsymbol{T}_2}]\boldsymbol{\mathcal{V}} = [\mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]\boldsymbol{\mathcal{V}} \qquad \text{or} \qquad \mathrm{Ad}_{\boldsymbol{T}_1}\big(\mathrm{Ad}_{\boldsymbol{T}_2}(\boldsymbol{\mathcal{V}})\big) = \mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}(\boldsymbol{\mathcal{V}})$$

- For any $\boldsymbol{T} \in SE(3)$, $[\mathrm{Ad}_{\boldsymbol{T}}]^{-1} = [\mathrm{Ad}_{\boldsymbol{T}^{-1}}]$

- For any two frames $\{c\}$ and $\{d\}$, a twist represented as $\boldsymbol{\mathcal{V}}_c$ in $\{c\}$ is related to its representation $\boldsymbol{\mathcal{V}}_d$ in $\{d\}$ by

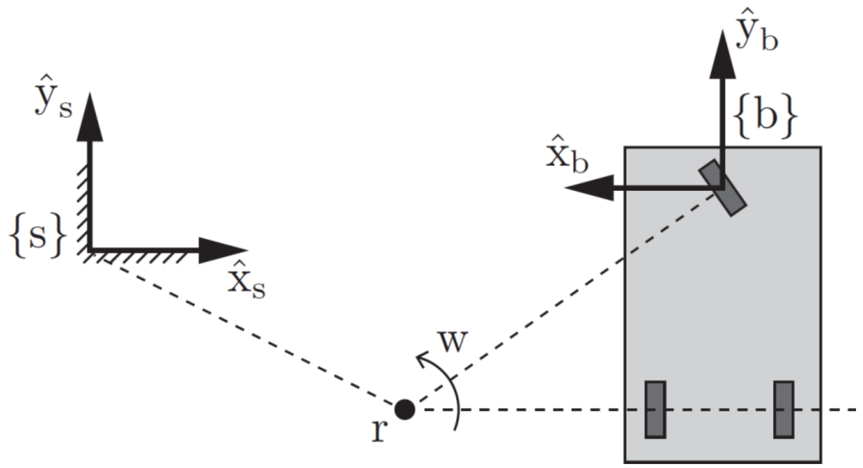
$$\boldsymbol{\mathcal{V}}_c = [\mathrm{Ad}_{\boldsymbol{T}_{cd}}]\boldsymbol{\mathcal{V}}_d \qquad\qquad \boldsymbol{\mathcal{V}}_d = [\mathrm{Ad}_{\boldsymbol{T}_{dc}}]\boldsymbol{\mathcal{V}}_c$$

(changing the reference frame of a twist)

Example

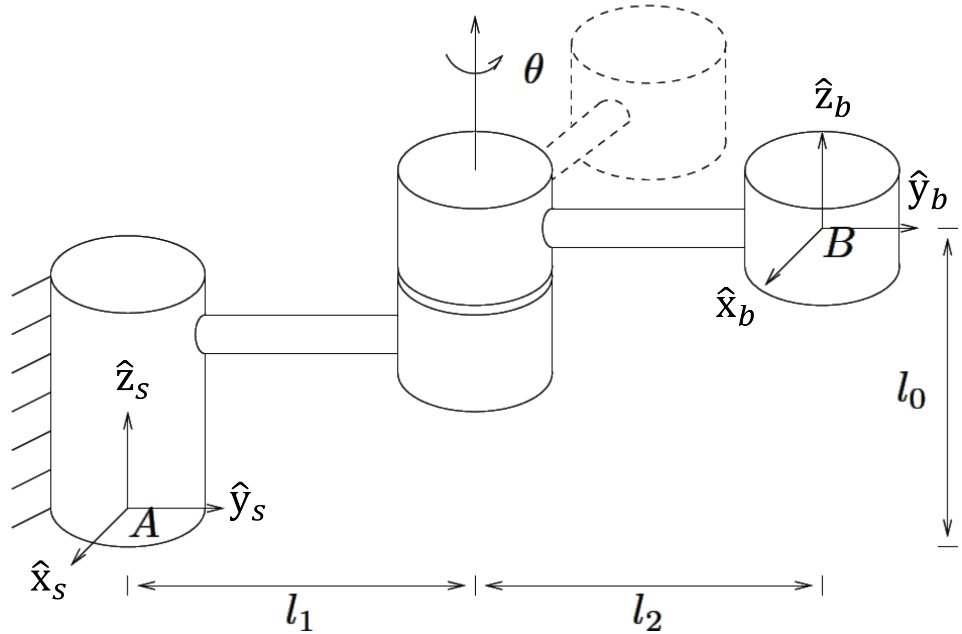
Consider a three-wheeled car with a single steerable front wheel, driving on a plane. The angle of the front wheel of the car causes the car's motion to be a pure angular velocity 2 rad/s about an axis out of the page at the point r in the plane. Find $\mathcal{V}_s$ and $\mathcal{V}_b$ when

$$r_s = (2, -1, 0)$$
$$r_b = (2, -1.4, 0)$$



Example

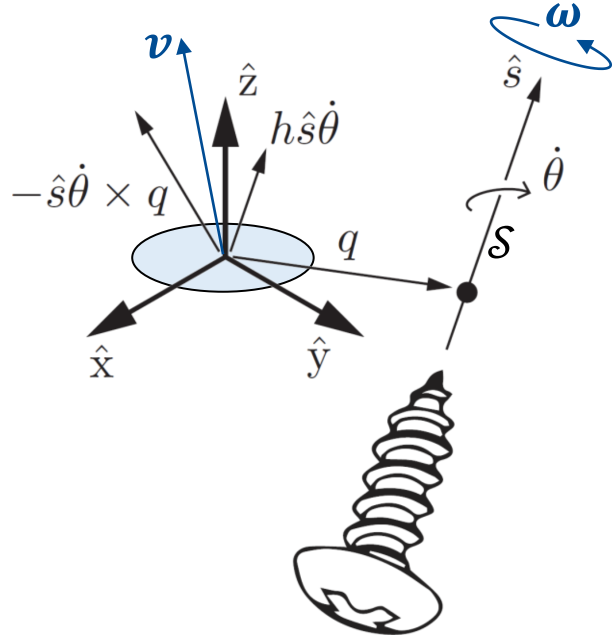
Find $\mathcal{V}_s$ and $\mathcal{V}_b$ for the shown one degree of freedom manipulator.



Screw Interpretation of a Twist

Any rigid-body velocity or twist $\mathcal{V}$ is equivalent to the <u>instantaneous</u> velocity $\dot{\theta}$ about some screw axis $\mathcal{S}$ (i.e., rotating about the axis while also translating along the axis).

A screw axis $\mathcal{S}$ represented by a point $\boldsymbol{q} \in \mathbb{R}^3$ on the axis, a unit vector $\hat{\boldsymbol{s}} \in S^2$ in the direction of the axis, and a pitch $h \in \mathbb{R}$ (linear velocity along the axis / angular velocity $\dot{\theta}$ about the axis) as $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$.



Thus, twist $\mathcal{V}$ can be represented as

$$\mathcal{V} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}}\dot{\theta} \\ -\hat{\boldsymbol{s}}\dot{\theta} \times \boldsymbol{q} + h\dot{\theta}\hat{\boldsymbol{s}} \end{bmatrix}$$

Due to rotation about $\mathcal{S}$ (which is in the plane orthogonal to $\hat{\boldsymbol{s}}$)

Due to translation along $\mathcal{S}$ (which is in the direction of $\hat{\boldsymbol{s}}$)

Screw Interpretation of a Twist

Instead of representing the screw axis $\mathcal{S}$ as $\{\boldsymbol{q}, \hat{\mathbf{s}}, h\}$ (with the non-uniqueness of $\boldsymbol{q}$), we represent a "unit" screw axis as a vector as

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6 \quad \text{where} \quad \boldsymbol{\mathcal{V}} = \boldsymbol{S}\dot{\theta} \in \mathbb{R}^6 \qquad \boldsymbol{S}_\omega, \boldsymbol{S}_v \in \mathbb{R}^3$$

Finding $\boldsymbol{S}$ by having $\boldsymbol{\mathcal{V}}$:

(a) If $||\boldsymbol{\omega}|| \neq 0$ ($\equiv$ rotation with/without translation along $\hat{\mathbf{s}}$):

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \boldsymbol{\mathcal{V}}/||\boldsymbol{\omega}|| = \begin{bmatrix} \boldsymbol{\omega}/||\boldsymbol{\omega}|| \\ \boldsymbol{v}/||\boldsymbol{\omega}|| \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \boldsymbol{q} + h\hat{\mathbf{s}} \end{bmatrix}$$

$$= \begin{bmatrix} \text{angular velocity when } \dot{\theta} = 1 \\ \text{linear velocity of origin when } \dot{\theta} = 1 \end{bmatrix}$$

Pitch h is finite, $h = \boldsymbol{\omega}^T \boldsymbol{v}/||\boldsymbol{\omega}||^2$

$\hat{\mathbf{s}} = \boldsymbol{\omega}/||\boldsymbol{\omega}||, \qquad ||\boldsymbol{S}_\omega|| = 1$

$\dot{\theta} = ||\boldsymbol{\omega}||$ is interpreted as angular velocity about $\hat{\mathbf{s}}$

(b) If $||\boldsymbol{\omega}|| = 0$ ($\equiv$ pure translation along $\hat{\mathbf{s}}$):

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \boldsymbol{\mathcal{V}}/||\boldsymbol{v}|| = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v}/||\boldsymbol{v}|| \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{s}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} \\ \text{normalized linear velocity of origin} \end{bmatrix}$$

Pitch h is infinite, $\quad ||\boldsymbol{S}_\omega|| = 0$

$\hat{\mathbf{s}} = \boldsymbol{v}/||\boldsymbol{v}||, \qquad ||\boldsymbol{S}_v|| = 1$

$\dot{\theta} = ||\boldsymbol{v}||$ is interpreted as linear velocity along $\hat{\mathbf{s}}$

Screw Interpretation of a Twist

- Since a screw axis $\boldsymbol{S}$ is just a normalized twist, the 4×4 matrix representation $[\boldsymbol{S}]$ of $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ is

$$[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3) \qquad [\boldsymbol{\mathcal{V}}] = [\boldsymbol{S}]\dot{\theta} \in se(3)$$

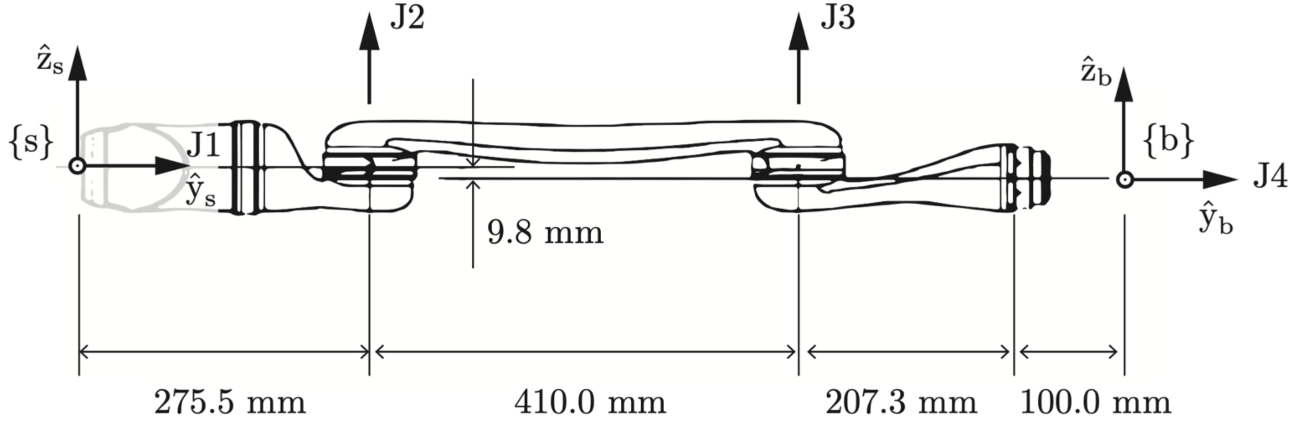
- Relation between a screw axis represented as $\boldsymbol{S}_a$ in a frame $\{a\}$ and $\boldsymbol{S}_b$ in a frame $\{b\}$:

$$\boldsymbol{S}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b \qquad \boldsymbol{S}_b = [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]\boldsymbol{S}_a$$

(changing the reference frame of a screw axis)

Example

Kinova lightweight 4-dof arm:



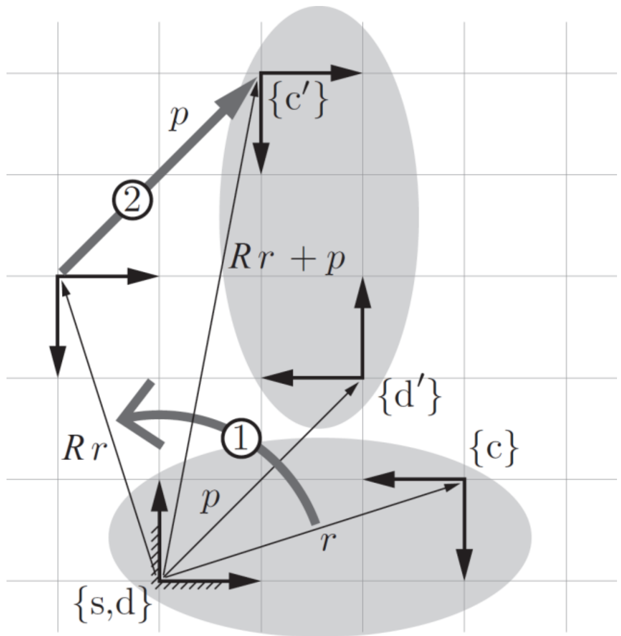
What are the screw axis $\boldsymbol{S}_b$ and $\boldsymbol{S}_s$ for J4 and J2?

Exponential Coordinate Representation of Rigid-Body Motion

Screw Motion

Instead of viewing a displacement as a rotation followed by a translation, both rotation and translation can be performed simultaneously.

Planar example of a screw motion:
The displacement can be viewed as a rotation of $\beta = 90°$ about a fixed point s.



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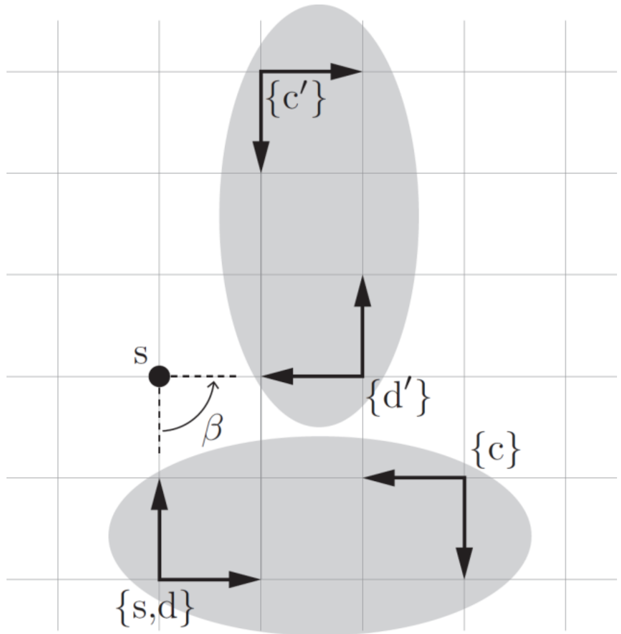


Exponential Coordinates of Rigid-Body Motions

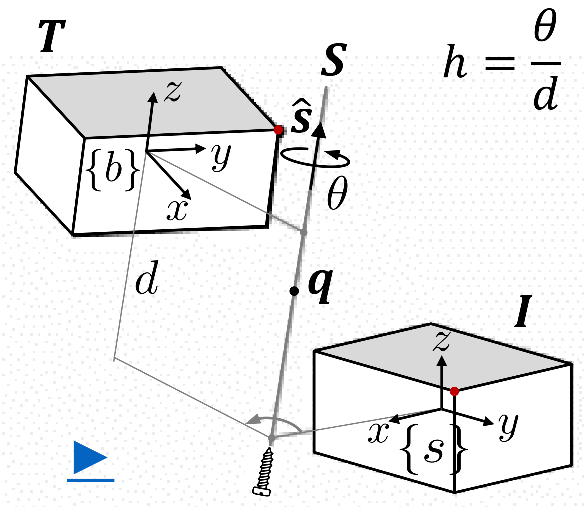
Chasles–Mozzi theorem states that every rigid-body displacement can be expressed as a finite rotation and translation about a fixed screw axis in space.

This theorem motivates a six-parameter representation of a configuration (or a homogeneous transformation $\boldsymbol{T} \in SE(3)$) called the **exponential coordinates** as $\boldsymbol{S}\theta \in \mathbb{R}^6$, where $\boldsymbol{S}$ is the screw axis and θ is the distance that must be traveled along the screw axis to take a frame from the origin $\boldsymbol{I}$ to $\boldsymbol{T}$.

Note: $\boldsymbol{T}$ is equivalent to the displacement obtained by rotating a frame from $\boldsymbol{I}$ about $\boldsymbol{S}$

- by an angle θ, or
- at a speed $\dot{\theta} = 1$ rad/s for θs, or
- at a speed $\dot{\theta} = \theta$ for unit time , or
- by twist $\boldsymbol{\mathcal{V}}$ for unit time.

Constant Screw Motion: A rotation θ + a translation d about/along a fixed screw axis $\boldsymbol{S}$.



$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \boldsymbol{q} + h\hat{\mathbf{s}} \end{bmatrix}$$ (for rotation with/without translation along $\hat{\mathbf{s}}$)

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{s}} \end{bmatrix}$$ (for pure translation along $\hat{\mathbf{s}}$)

Exponential Coordinates of Rigid-Body Motions

As with rotations, we can define a matrix exponential (exp) and matrix logarithm (log).
For any transformation matrix $\boldsymbol{T} \in SE(3)$, we can always find a screw axis $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v) \in \mathbb{R}^6$ ($\|\boldsymbol{S}_\omega\| = 1$ or $\boldsymbol{S}_\omega = \boldsymbol{0}$, $\|\boldsymbol{S}_v\| = 1$) and scalar $\theta \in \mathbb{R}$ such that $\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$.

$$\begin{array}{llll} \text{exp:} & [\boldsymbol{S}]\theta \in se(3) \rightarrow \boldsymbol{T} \in SE(3) & : & e^{[\boldsymbol{S}]\theta} = \boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) \\ \text{log:} & \boldsymbol{T} \in SE(3) \rightarrow [\boldsymbol{S}]\theta \in se(3) & : & \log(\boldsymbol{T}) = [\boldsymbol{S}]\theta \end{array}$$

$\boldsymbol{S}\theta \in \mathbb{R}^6$: Exponential coordinates of $\boldsymbol{T} \in SE(3)$
$[\boldsymbol{S}]\theta = [\boldsymbol{S}\theta] \in se(3)$: Matrix logarithm of $\boldsymbol{T}$ (inverse of the matrix exponential)

Matrix Exponential

$$\exp: \quad [\boldsymbol{S}]\theta \in se(3) \rightarrow \boldsymbol{T} \in SE(3) \quad : \quad e^{[\boldsymbol{S}]\theta} = \boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$$

❖ Finding $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$ by having $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ and θ:

$$e^{[\boldsymbol{S}]\theta} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

$$\boldsymbol{G}(\theta) = \boldsymbol{I}\theta + (1 - \cos\theta)[\boldsymbol{S}_\omega] + (\theta - \sin\theta)[\boldsymbol{S}_\omega]^2 \in \mathbb{R}^{3\times 3}$$

Matrix Exponential: Remark

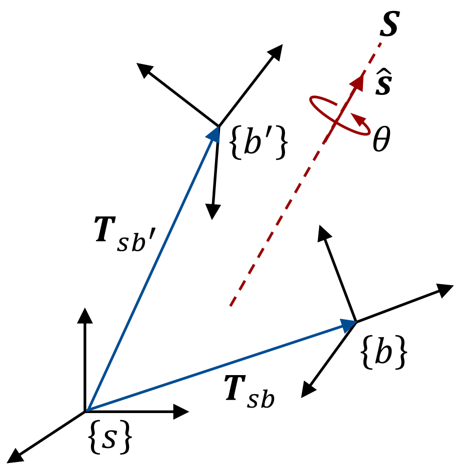
For a given $\boldsymbol{S}$: $(\boldsymbol{S}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b)$

$\boldsymbol{S}$ is expressed in $\{b\}$

Body-frame displacement: $\boldsymbol{T}_{sb'} = \boldsymbol{T}_{sb}e^{[\boldsymbol{S}_b]\theta}$

Fixed-frame displacement: $\boldsymbol{T}_{sb'} = e^{[\boldsymbol{S}_s]\theta}\boldsymbol{T}_{sb}$

$\boldsymbol{S}$ is expressed in $\{s\}$



Matrix Logarithm

$$\log: \quad \boldsymbol{T} \in SE(3) \quad \to [\boldsymbol{S}]\theta \in se(3) \quad : \quad \log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$$

❖ Finding $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ and $\theta \in [0, \pi]$ by having $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$:

(a) If $\boldsymbol{R} = \boldsymbol{I}$, then set $\boldsymbol{S}_\omega = \mathbf{0}$, $\boldsymbol{S}_v = \boldsymbol{p}/\|\boldsymbol{p}\|$, and $\theta = \|\boldsymbol{p}\|$.

(b) Otherwise, use the matrix logarithm $\log(\boldsymbol{R}) = [\boldsymbol{S}_\omega]\theta$ to determine $\boldsymbol{S}_\omega$ ($\hat{\boldsymbol{\omega}}$ in the $SO(3)$ algorithm) and $\theta \in [0, \pi]$. Then, $\boldsymbol{S}_v$ is calculated as

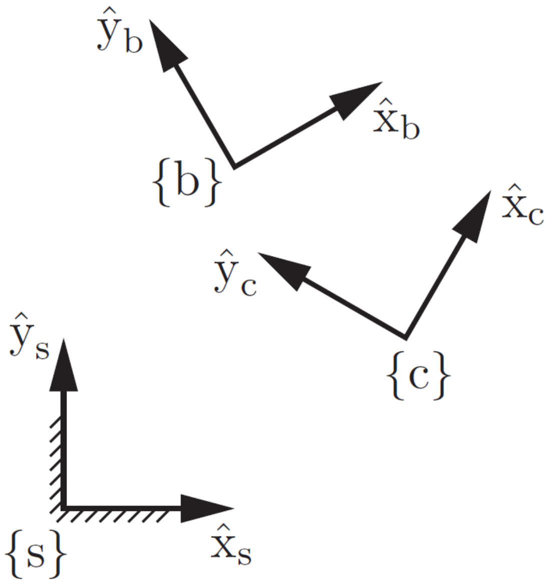
$$\boldsymbol{S}_v = \boldsymbol{G}^{-1}(\theta)\boldsymbol{p}$$

$$\boldsymbol{G}^{-1}(\theta) = \frac{1}{\theta}\boldsymbol{I} - \frac{1}{2}[\boldsymbol{S}_\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\boldsymbol{S}_\omega]^2 \in \mathbb{R}^{3\times 3}$$

Example

The initial frame $\{b\}$ and final frame $\{c\}$ are given. Find the screw motion that displaces the frame at T_{sb} to T_{sc}.

$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Wrenches

Wrench

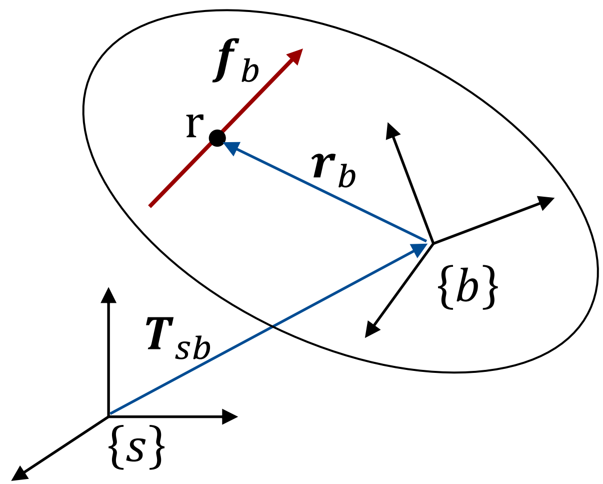
Consider a linear force $\mathbf{f}$ acting on a rigid body at a point r. Both $\boldsymbol{f}_b \in \mathbb{R}^3$ and $\boldsymbol{r}_b \in \mathbb{R}^3$ are represented in $\{b\}$. This force creates a torque or moment $\boldsymbol{m}_b \in \mathbb{R}^3$ in $\{b\}$ as

$$\boldsymbol{m}_b = \boldsymbol{r}_b \times \boldsymbol{f}_b$$

We can package the moment and force together in a single six-dimensional vector called **wrench** (or **spatial force**) in $\{b\}$ as

$$\boldsymbol{\mathcal{F}}_b = \begin{bmatrix} \boldsymbol{m}_b \\ \boldsymbol{f}_b \end{bmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{\mathcal{F}}_s = ?$$



Wrench

The **power** is a coordinate-independent quantity, i.e., the power generated (or dissipated) by an ($\boldsymbol{\mathcal{F}}$,$\boldsymbol{\mathcal{V}}$) pair (wrench and twist) must be the same regardless of the frame in which it is represented:

$$(\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{F}} = \text{power}) \qquad \boldsymbol{\mathcal{V}}_s^{\mathrm{T}} \boldsymbol{\mathcal{F}}_s = \boldsymbol{\mathcal{V}}_b^{\mathrm{T}} \boldsymbol{\mathcal{F}}_b = \text{power} \qquad (\boldsymbol{\mathcal{V}}_b = [\mathrm{Ad}_{T_{bs}}]\boldsymbol{\mathcal{V}}_s)$$

$$\begin{aligned} \boldsymbol{\mathcal{V}}_s^{\mathrm{T}} \boldsymbol{\mathcal{F}}_s &= \left([\mathrm{Ad}_{T_{bs}}]\boldsymbol{\mathcal{V}}_s\right)^{\mathrm{T}} \boldsymbol{\mathcal{F}}_b \\ &= \boldsymbol{\mathcal{V}}_s^{\mathrm{T}} [\mathrm{Ad}_{T_{bs}}]^{\mathrm{T}} \boldsymbol{\mathcal{F}}_b \end{aligned}$$

Since this must hold for all $\boldsymbol{\mathcal{V}}_s$

$$\boldsymbol{\mathcal{F}}_s = [\mathrm{Ad}_{T_{bs}}]^{\mathrm{T}} \boldsymbol{\mathcal{F}}_b$$

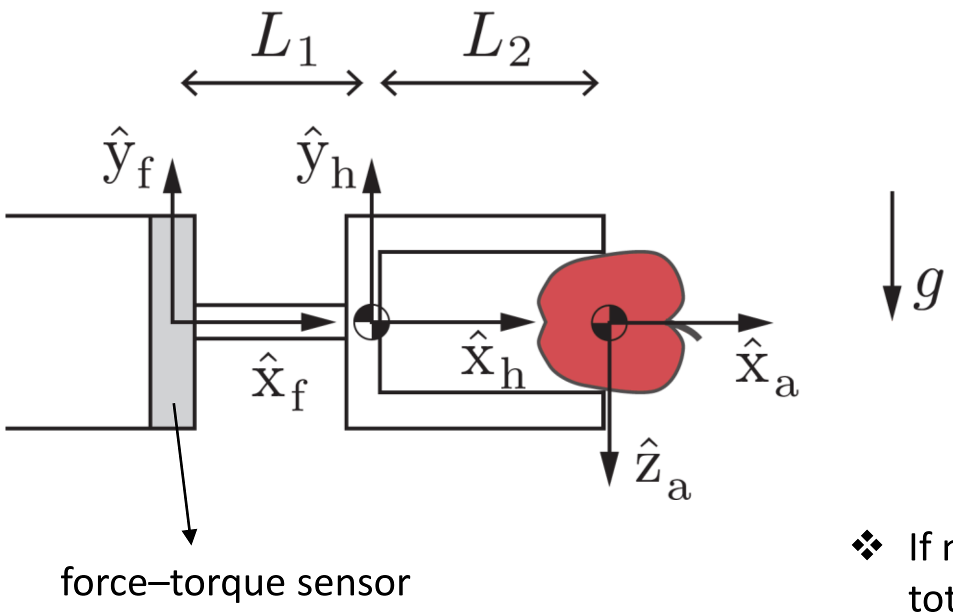
spatial wrench body wrench

$$\boldsymbol{\mathcal{F}}_a = [\mathrm{Ad}_{T_{ba}}]^{T} \boldsymbol{\mathcal{F}}_b \qquad \boldsymbol{\mathcal{F}}_b = [\mathrm{Ad}_{T_{ab}}]^{T} \boldsymbol{\mathcal{F}}_a$$

(changing the reference frame of a twist)

Example

The robot hand shown is holding an apple with a mass of 0.1 kg in a gravitational field g=10 m/s^2. The mass of the hand is 0.5 kg, L_1=10 cm, and L_2=15 cm. What is the force and torque measured by the six-axis force–torque sensor between the hand and the robot arm?



❖ If more than one wrench acts on a rigid body, the total wrench on the body is simply the vector sum of the individual wrenches, provided that the wrenches are expressed in the same frame.

Review

Review

Rotations	Rigid-Body Motions
$\boldsymbol{R} \in SO(3)$: 3×3 matrices $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}, \det(\boldsymbol{R}) = 1$	$\boldsymbol{T} \in SE(3)$: 4×4 matrices $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix},$ where $\boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3$
$\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$	$\boldsymbol{R}^{-1} = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$
Change of coordinate frame: $\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{ac}, \ \boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$	Change of coordinate frame: $\boldsymbol{T}_{ab}\boldsymbol{T}_{bc} = \boldsymbol{T}_{ac}, \ \boldsymbol{T}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$

Review

Rotations	Rigid-Body Motions
Rotating a frame {b}: $$\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$ $\boldsymbol{R}_{sb'} = \boldsymbol{R}\boldsymbol{R}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ $\boldsymbol{R}_{sb''} = \boldsymbol{R}_{sb}\boldsymbol{R}$: rotate θ about $\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}$	Displacing a frame {b}: $$\boldsymbol{T} = \begin{bmatrix} \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$ $\boldsymbol{T}_{sb'} = \boldsymbol{T}\boldsymbol{T}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ (moves {b} origin), translate $\boldsymbol{p}$ in {s} $\boldsymbol{T}_{sb''} = \boldsymbol{T}_{sb}\boldsymbol{T}$: translate $\boldsymbol{p}$ in {b}, rotate θ about $\hat{\boldsymbol{\omega}}$ in new body frame
Unit rotation axis is $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$, where $\lVert\hat{\boldsymbol{\omega}}\rVert = 1$	"Unit" screw axis is $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\lVert\boldsymbol{S}_\omega\rVert = 1$ or (ii) $\lVert\boldsymbol{S}_\omega\rVert = 0$, $\lVert\boldsymbol{S}_v\rVert = 1$
	For a screw axis $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$ with finite h, $$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}} \\ -\hat{\boldsymbol{s}} \times \boldsymbol{q} + h\hat{\boldsymbol{s}} \end{bmatrix}$$
Angular velocity is $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}$	Twist is $\boldsymbol{\mathcal{V}} = \boldsymbol{S}\dot{\theta}$

Review

Rotations	Rigid-Body Motions
Rotating a frame {b}: $\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$ $\boldsymbol{R}_{sb'} = \boldsymbol{R}\boldsymbol{R}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ $\boldsymbol{R}_{sb''} = \boldsymbol{R}_{sb}\boldsymbol{R}$: rotate θ about $\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}$	Displacing a frame {b}: $\boldsymbol{T} = \begin{bmatrix} \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$ $\boldsymbol{T}_{sb'} = \boldsymbol{T}\boldsymbol{T}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ (moves {b} origin), translate $\boldsymbol{p}$ in {s} $\boldsymbol{T}_{sb''} = \boldsymbol{T}_{sb}\boldsymbol{T}$: translate $\boldsymbol{p}$ in {b}, rotate θ about $\hat{\boldsymbol{\omega}}$ in new body frame
Unit rotation axis is $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$, where $\lVert\hat{\boldsymbol{\omega}}\rVert = 1$	"Unit" screw axis is $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\lVert\boldsymbol{S}_\omega\rVert = 1$ or (ii) $\lVert\boldsymbol{S}_\omega\rVert = 0$, $\lVert\boldsymbol{S}_v\rVert = 1$
	For a screw axis $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$ with finite h, $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}} \\ -\hat{\boldsymbol{s}} \times \boldsymbol{q} + h\hat{\boldsymbol{s}} \end{bmatrix}$
Angular velocity is $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}$	Twist is $\boldsymbol{\mathcal{V}} = \boldsymbol{S}\dot{\theta}$

Review

Rotations	Rigid-Body Motions
Exponential coordinate for $\boldsymbol{R} \in SO(3)$: $\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$	Exponential coordinate for $\boldsymbol{T} \in SE(3)$: $\boldsymbol{S}\theta \in \mathbb{R}^6$
$\exp: [\hat{\boldsymbol{\omega}}]\theta \in so(3) \to \boldsymbol{R} \in SO(3)$ $\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = e^{[\hat{\boldsymbol{\omega}}]\theta}$ $\boldsymbol{R} = \boldsymbol{I} + \sin\theta[\hat{\boldsymbol{\omega}}] + (1-\cos\theta)[\hat{\boldsymbol{\omega}}]^2$ (Rodrigues' formula for rotations)	$\exp: [\boldsymbol{S}]\theta \in se(3) \to \boldsymbol{T} \in SE(3)$ $\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$ $\boldsymbol{T} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix}$ $\boldsymbol{G}(\theta)$ $= \boldsymbol{I}\theta + (1-\cos\theta)[\boldsymbol{S}_\omega] + (\theta - \sin\theta)[\boldsymbol{S}_\omega]^2$
$\log: \boldsymbol{R} \in SO(3) \to [\hat{\boldsymbol{\omega}}]\theta \in so(3)$ $\log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta$	$\log: \boldsymbol{T} \in SE(3) \to [\boldsymbol{S}]\theta \in se(3)$ $\log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$
Moment change of coordinate frame: $\boldsymbol{m}_a = \boldsymbol{R}_{ab}\boldsymbol{m}_b$	Wrench change of coordinate frame: $\boldsymbol{\mathcal{F}}_a = (\boldsymbol{m}_a, \boldsymbol{f}_a) = \left[\mathrm{Ad}_{\boldsymbol{T}_{ba}}\right]^{\mathrm{T}} \boldsymbol{\mathcal{F}}_b$