# Ch4: Rigid-Body Motions – Part 2 (Transformation)

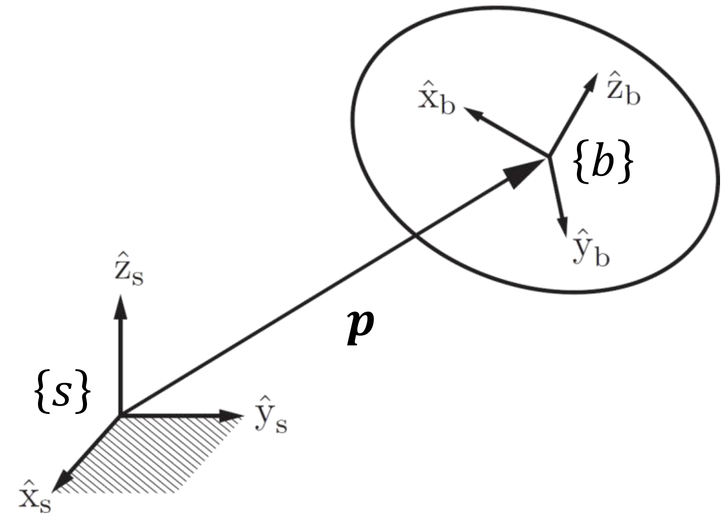
# Transformation Matrices

# Homogeneous Transformation Matrices

Rigid-body configuration can be represented by the pair $(\boldsymbol{R}, \boldsymbol{p})$ ($\boldsymbol{R} \in SO(3)$, $\boldsymbol{p} \in \mathbb{R}^3$). We can package $(\boldsymbol{R}, \boldsymbol{p})$ into a single $4 \times 4$ matrix as

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Transformation Matrix




This is an implicit representation of the C-space.

$\boldsymbol{p} := \boldsymbol{p}_s = \boldsymbol{p}_s^b = \boldsymbol{p}_s^{sb}$ Another notation for $\boldsymbol{p}_s^{sb}$: ${}^s\boldsymbol{p}_{sb}$

$\boldsymbol{R} := \boldsymbol{R}_{sb}$ Another notation for $\boldsymbol{R}_{sb}$: ${}^s\boldsymbol{R}_b$

$\boldsymbol{T} := \boldsymbol{T}_{sb}$ Another notation for $\boldsymbol{T}_{sb}$: ${}^s\boldsymbol{T}_b$

# Special Euclidean Group $SE(3)$

The **Special Euclidean Group** $SE(3)$, also known as the **group of rigid-body motions** or **homogeneous transformation matrices**, is the set of all $4 \times 4$ real matrices $\boldsymbol{T}$ of the form

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{T} \in SE(3)$
$\boldsymbol{R} \in SO(3)$
$\boldsymbol{p} \in \mathbb{R}^3$

$$SE(3) = \left\{ \boldsymbol{T} \in \mathbb{R}^{4\times4} \mid \boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}, \boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3 \right\}$$

The **special Euclidean group** $SE(2)$ is the set of all $3 \times 3$ real matrices $\boldsymbol{T}$ of the form

$$T = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{T} \in SE(2)$
$\boldsymbol{R} \in SO(2)$
$\boldsymbol{p} \in \mathbb{R}^2$
$\theta \in [0,2\pi)$

- $SE(2)$ is a subgroup of $SE(3)$: $\quad SE(2) \subset SE(3)$

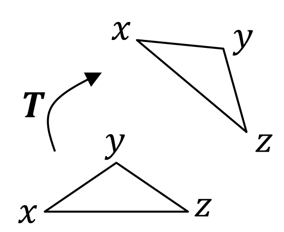
# Properties of Transformation Matrices

$SE(3)$ (or $SE(2)$) is a **matrix (Lie) group** (and the group operation • is matrix multiplication).

**Closure**: $\boldsymbol{T}_1\boldsymbol{T}_2 \in SE(3)$
**Associative**: $(\boldsymbol{T}_1\boldsymbol{T}_2)\boldsymbol{T}_3 = \boldsymbol{T}_1(\boldsymbol{T}_2\boldsymbol{T}_3)$ (but generally not commutative, $\boldsymbol{T}_1\boldsymbol{T}_2 \neq \boldsymbol{T}_2\boldsymbol{T}_1$)
**Identity**: $\exists \boldsymbol{I}_4 \in SE(3)$ such that $\boldsymbol{T}\boldsymbol{I}_4 = \boldsymbol{I}_4\boldsymbol{T} = \boldsymbol{T}$
**Inverse**: $\exists\ \boldsymbol{T}^{-1} \in SE(3)$ such that $\boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{I}_4$

$$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

**Note**: $\boldsymbol{T}$ preserves both distances and angles.

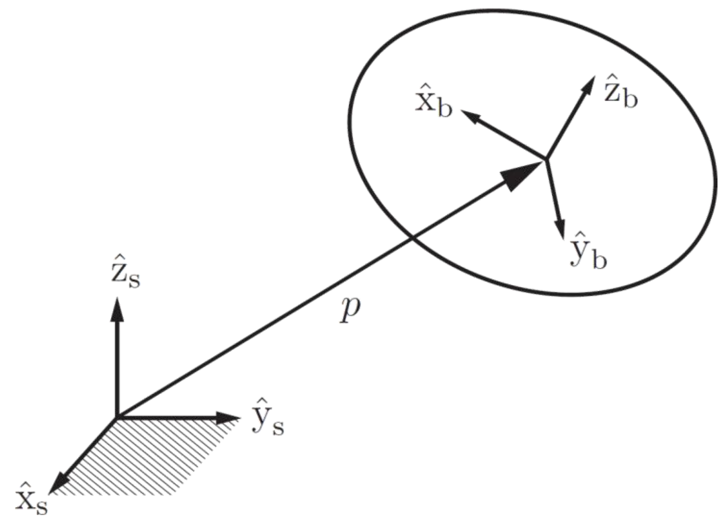


# Uses of Transformation Matrices (1)

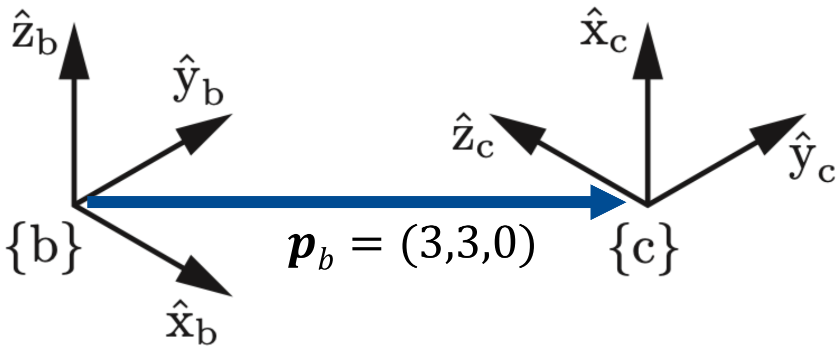
**(1)** Representing configuration (position and orientation) of a frame relative to another frame.

<u>Notation</u>: $\boldsymbol{T}_{sb}$ is the configuration of $\{b\}$ relative to $\{s\}$.

$$\boldsymbol{T}_{sb} = \begin{bmatrix} \boldsymbol{R}_{sb} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$



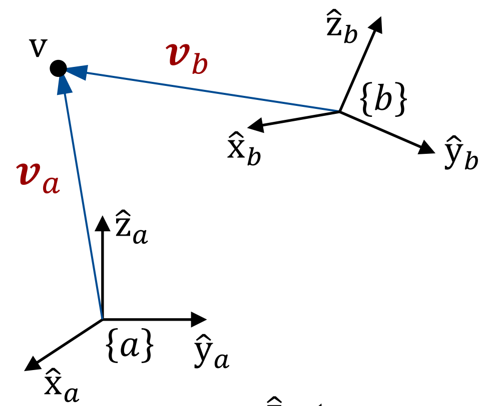
# Example




$\boldsymbol{T}_{bc}$ ?

# Uses of Transformation Matrices (2)

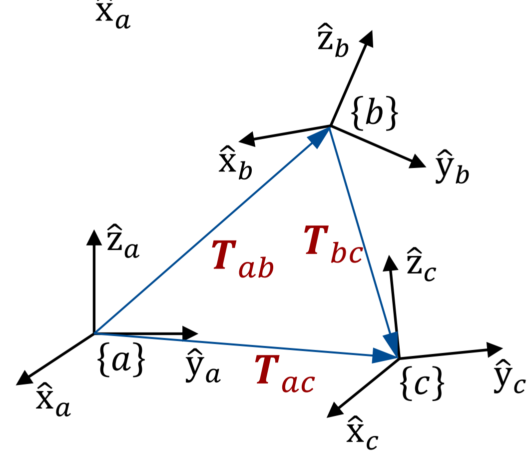
**(2)** Changing the reference frame of a vector or frame.




Subscript Cancellation Rule:

$$\boldsymbol{T}_{ab}\boldsymbol{v}_b = \boldsymbol{T}_{a\cancel{b}}\boldsymbol{v}_{\cancel{b}} = \boldsymbol{v}_a$$

$$\boldsymbol{T}_{ab}\boldsymbol{T}_{bc} = \boldsymbol{T}_{a\cancel{b}}\boldsymbol{T}_{\cancel{b}c} = \boldsymbol{T}_{ac}$$




$\boldsymbol{T}_{ab}$ can be viewed as a **mathematical operator** that changes the reference frame from $\{b\}$ to $\{a\}$.

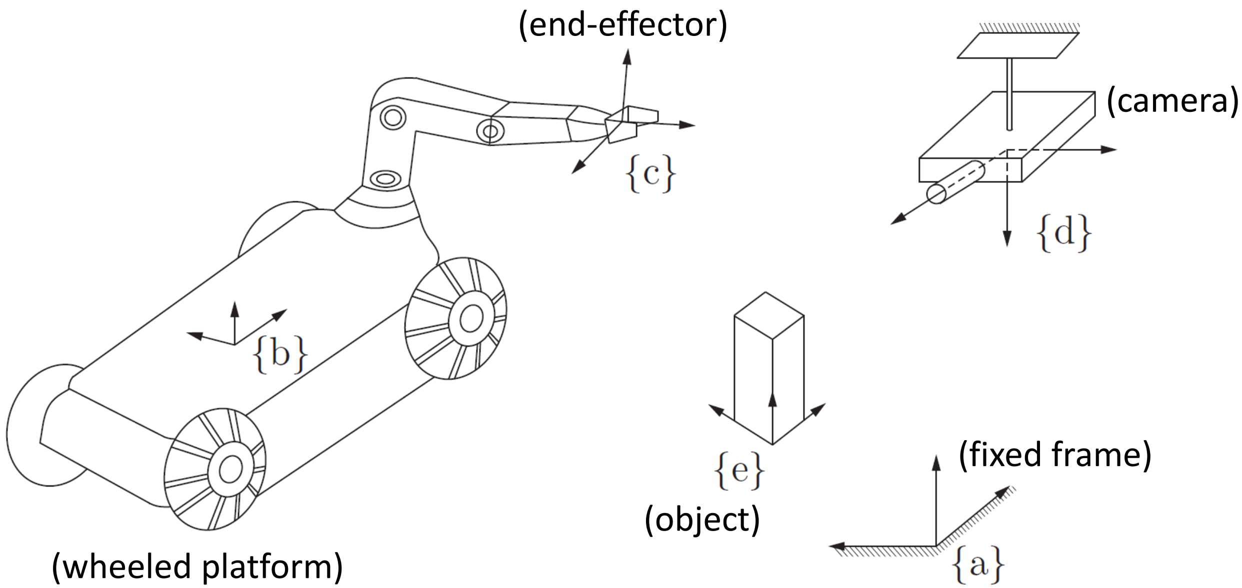
**Note**: $\boldsymbol{T}_{bc}\boldsymbol{T}_{cb} = \boldsymbol{I}_4$ or $\boldsymbol{T}_{bc} = \boldsymbol{T}_{cb}^{-1} = \begin{bmatrix} \boldsymbol{R}_{cb}^T & -\boldsymbol{R}_{cb}^T\boldsymbol{p}_c^{cb} \\ \boldsymbol{0} & 1 \end{bmatrix}$

**Note**: To calculate $\boldsymbol{T}\boldsymbol{v}$, we append a “1” to $\boldsymbol{v}$ and it is called **homogeneous coordinates** representation of $\boldsymbol{v}$.

$$\boldsymbol{v} = [v_1 \;\; v_2 \;\; v_3 \;\; 1]^T$$

# Example

A robot arm mounted on a wheeled mobile platform moving in a room, and a camera fixed to the ceiling. The robot must pick up an object with body frame $\{e\}$. What is the configuration of the object relative to the robot hand, $\boldsymbol{T}_{ce}$, given $\boldsymbol{T}_{db}$, $\boldsymbol{T}_{de}$, $\boldsymbol{T}_{bc}$, and $\boldsymbol{T}_{ad}$?



# Uses of Transformation Matrices (3)

**(3)** Displacing (rotating and translating) a <u>vector</u> or <u>frame</u>.

$$\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) = (\mathrm{Rot}(\widehat{\boldsymbol{\omega}}, \theta), \boldsymbol{p}) = \mathrm{Trans}(\boldsymbol{p})\overline{\mathrm{Rot}}(\widehat{\boldsymbol{\omega}}, \theta)$$

$$\overline{\mathrm{Rot}}(\widehat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} \mathrm{Rot}(\widehat{\boldsymbol{\omega}}, \theta) & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\mathrm{Trans}(\boldsymbol{p}) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

$\boldsymbol{T}$ can be viewed as a <u>**mathematical operator**</u> that rotates a frame or vector about a unit axis $\widehat{\boldsymbol{\omega}} = (\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3)$ by an amount $\theta$ + translating it by $\boldsymbol{p}$.

# Uses of Transformation Matrices (3) (cont.)

- Rotation of vector $\boldsymbol{v}$ about a unit axis $\hat{\boldsymbol{\omega}}$ (expressed in the same frame) by an amount $\theta$ and translation of it by $\boldsymbol{p}$ (expressed in the same frame) is vector $\boldsymbol{v}'$ expressed in the same frame:

$$\boldsymbol{v}'' = \boldsymbol{T}\boldsymbol{v} = \underbrace{\mathrm{Trans}(\boldsymbol{p})\overline{\mathrm{Rot}}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v}}_{\xleftarrow{\text{Interpretation}}} \equiv \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v} + \boldsymbol{p}$$

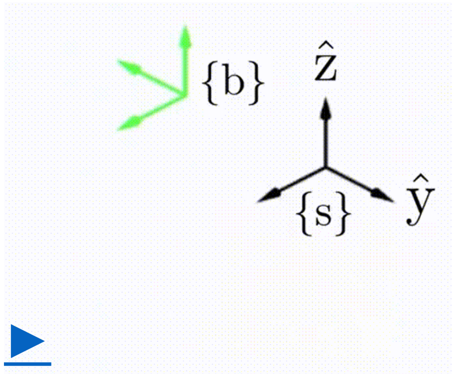
- **Fixed-frame Transformation**:

**2**. Translating $\{b'\}$ by $\boldsymbol{p}$ in $\{s\}$ to get $\{b''\}$

**1**. Rotating $\{b\}$ by $\theta$ about $\hat{\boldsymbol{\omega}}$ in $\{s\}$ (this can move $\{b\}$ origin) to get $\{b'\}$

$$\boldsymbol{T}_{sb''} = \boldsymbol{T}\boldsymbol{T}_{sb} = \mathrm{Trans}(\boldsymbol{p})\underbrace{\overline{\mathrm{Rot}}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{T}_{sb}}_{\xleftarrow{\text{Interpretation}}}$$

(pre-multiplication)



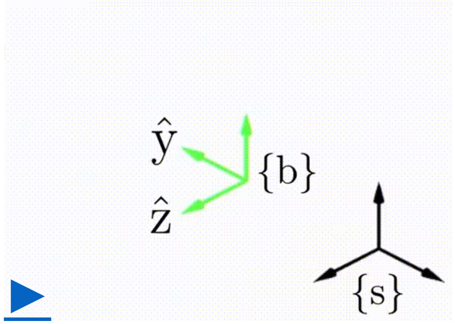

- **Body-frame Transformation**:

**1**. Translating $\{b\}$ by $\boldsymbol{p}$ in $\{b\}$ to get $\{b'\}$

**2**. Rotating $\{b'\}$ by $\theta$ about $\hat{\boldsymbol{\omega}}$ in $\{b'\}$ to get $\{b''\}$

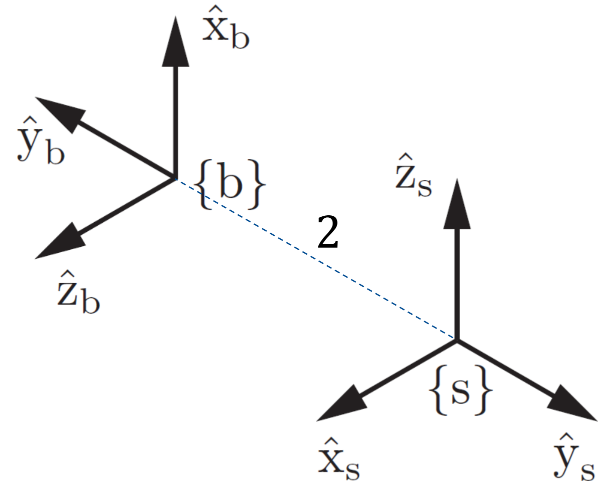
$$\boldsymbol{T}_{sb''} = \boldsymbol{T}_{sb}\boldsymbol{T} = \boldsymbol{T}_{sb}\underbrace{\mathrm{Trans}(\boldsymbol{p})\overline{\mathrm{Rot}}}_{\xrightarrow{\text{Interpretation}}}(\hat{\boldsymbol{\omega}}, \theta)$$

(post-multiplication)



# Example

Find fixed-frame and body-frame transformations corresponding to $\hat{\boldsymbol{\omega}} = (0,0,1)$, $\theta = 90°$, and $\boldsymbol{p} = (0,2,0)$.



# Twist

# Lie Algebra $se(3)$

- The set of all $4 \times 4$ matrices of the form

$$\begin{bmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

where $[\boldsymbol{\omega}] \in so(3)$ and $\boldsymbol{v} \in \mathbb{R}^3$ is called $se(3)$.

- $se(3)$ is the matrix representation of $6 \times 1$ vectors $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} \in \mathbb{R}^6$. Thus,

$$[\boldsymbol{\mathcal{V}}] = \begin{bmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$$

- $se(3)$ is called the Lie algebra of the Lie group $SE(3)$.

**Notations**:

- From $6 \times 1$ vector to $4 \times 4$ matrix representation: $[\boldsymbol{\mathcal{V}}]$ or $[\boldsymbol{\mathcal{V}}]_\times$ (Bracket notation), $\widehat{\boldsymbol{\mathcal{V}}}$ ($\hat{\cdot}$ hat notation), or $\boldsymbol{\mathcal{V}}^\wedge$.
- From $4 \times 4$ matrix representation to $6 \times 1$ vector: $[\boldsymbol{\mathcal{V}}]^\vee$ ($(\cdot)^\vee$ vee notation or $\check{\cdot}$)

# Spatial Velocity or Twist

A rigid body's **Spatial Velocity** or **Twist** can be represented as a point in $\mathbb{R}^6$ and defined as

$$\underbrace{\boldsymbol{\mathcal{V}}_x}_{\text{expressed in } \{x\}} = \begin{bmatrix} \text{angular velocity of body expressed in frame } \{x\} \\ \text{linear velocity of origin of frame } \{x\} \text{ on body (or its extention) expressed in frame } \{x\} \end{bmatrix} \in \mathbb{R}^6$$

A general form:

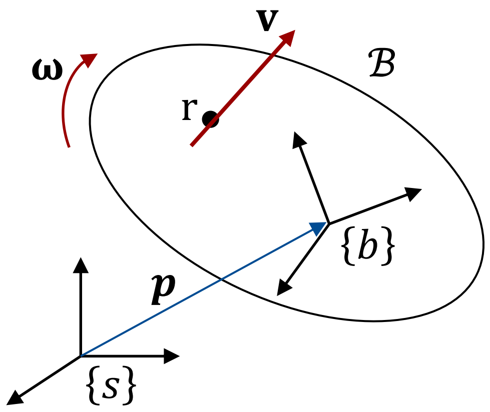
point where velocity is computed

$$\boldsymbol{\mathcal{V}}_x^{\mathcal{B}_r} = \begin{bmatrix} \text{angular velocity of body } \mathcal{B} \text{ expressed in frame } \{x\} \\ \text{linear velocity of point r on body } \mathcal{B} \text{ (or its extention) expressed in frame } \{x\} \end{bmatrix} \in \mathbb{R}^6$$

expressed in $\{x\}$

Let's find the twist $\boldsymbol{\mathcal{V}} \in \mathbb{R}^6$ of a moving body (or body frame $\{b\}$) in terms of $\boldsymbol{T}_{sb} = \boldsymbol{T}(t)$. Body Frame $\{b\}$ is instantaneously coincident with the body-attached frame.

$$\boldsymbol{T}(t) = \begin{bmatrix} \boldsymbol{R}(t) & \boldsymbol{p}(t) \\ \boldsymbol{0} & 1 \end{bmatrix}$$



# Body Twist $\mathcal{V}_b$

Similar to $\boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b]$, let's compute $\boldsymbol{T}^{-1}\dot{\boldsymbol{T}}$: $(\boldsymbol{R} := \boldsymbol{R}_{sb}\,, \boldsymbol{T} := \boldsymbol{T}_{sb})$

$$\boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{R}} & \dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^T\dot{\boldsymbol{R}} & \boldsymbol{R}^T\dot{\boldsymbol{p}} \\ \boldsymbol{0} & 0 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{v}_b \\ \boldsymbol{0} & 0 \end{bmatrix}$$

$$\xrightarrow[\;]{\boldsymbol{v}_b \in \mathbb{R}^3,\ [\boldsymbol{\omega}_b] \in so(3)} \quad \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{\mathcal{V}}_b] = \begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{v}_b \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$$

$$\boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} \in \mathbb{R}^6$$

$\boldsymbol{\mathcal{V}}_b$ is defined as **Body Twist**
(or spatial velocity in the body frame)

- $[\boldsymbol{\mathcal{V}}_b] \in se(3)$ is the matrix representations of the **body twists** $\boldsymbol{\mathcal{V}}_b \in \mathbb{R}^6$ associated with the rigid-body configuration $\boldsymbol{T} \in SE(3)$.
- $\boldsymbol{\mathcal{V}}_b$ does not depend on the choice of the fixed frame $\{s\}$,

# Spatial Twist $\mathcal{V}_s$

Similar to $\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s]$, let's compute $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$: $(\boldsymbol{R} = \boldsymbol{R}_{sb}\,, \boldsymbol{T} = \boldsymbol{T}_{sb})$

$$\begin{aligned}\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} &= \begin{bmatrix}\dot{\boldsymbol{R}} & \dot{\boldsymbol{p}}\\ \boldsymbol{0} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{R}^{\mathrm{T}} & -\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p}\\ \boldsymbol{0} & 1\end{bmatrix}\\ &= \begin{bmatrix}\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}} & \dot{\boldsymbol{p}} - \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{p}\\ \boldsymbol{0} & 0\end{bmatrix}\\ &= \begin{bmatrix}[\boldsymbol{\omega}_s] & \boldsymbol{v}_s\\ \boldsymbol{0} & 0\end{bmatrix}\end{aligned}$$

$$\xrightarrow[\quad]{\boldsymbol{v}_s \in \mathbb{R}^3,\ [\boldsymbol{\omega}_s] \in so(3)} \quad \dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\mathcal{V}_s] = \begin{bmatrix}[\boldsymbol{\omega}_s] & \boldsymbol{v}_s\\ \boldsymbol{0} & 0\end{bmatrix} \in se(3)$$

$$\mathcal{V}_s = \begin{bmatrix}\boldsymbol{\omega}_s\\ \boldsymbol{v}_s\end{bmatrix} \in \mathbb{R}^6$$

$\mathcal{V}_s$ is defined as **Spatial Twist**
(or spatial velocity in the space frame)

- $[\mathcal{V}_s] \in se(3)$ is the matrix representations of the **spatial twists** $\mathcal{V}_s \in \mathbb{R}^6$ associated with the rigid-body configuration $\boldsymbol{T} \in SE(3)$.
- $\mathcal{V}_s$ does not depend on the choice of the body frame $\{b\}$.

# Adjoint Map

$$\cancel{\mathcal{V}_s = \underbrace{T_{sb}}_{4\times 4}\mathcal{V}_b}$$

$6 \times 6$

$$[\mathcal{V}_b] = T^{-1}\dot{T}$$
$$[\mathcal{V}_s] = \dot{T}T^{-1}$$

$$\longrightarrow \quad [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1} \quad \longrightarrow$$

$$[\mathcal{V}_s] = \begin{bmatrix} R[\omega_b]R^{\mathrm{T}} & -R[\omega_b]R^{\mathrm{T}}p + Rv_b \\ \mathbf{0} & 0 \end{bmatrix} \xrightarrow[\substack{[\omega]p = -[p]\omega \\ p,\,\omega \in \mathbb{R}^3}]{R[\omega]R^{\mathrm{T}} = [R\omega]} = \begin{bmatrix} [R\omega_b] & [p]R\omega_b + Rv_b \\ \mathbf{0} & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R\omega_b \\ [p]R\omega_b + Rv_b \end{bmatrix} = \begin{bmatrix} R & \mathbf{0} \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_T]\mathcal{V}_b$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & \mathbf{0} \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

**Adjoint Map** associated with $T$
or Adjoint Representation of $T$

- Therefore, $\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$

Similarly, $\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s = \mathrm{Ad}_{T_{bs}}(\mathcal{V}_s)$

# Adjoint Map Properties

- Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in SE(3)$ and $\boldsymbol{\mathcal{V}} = (\boldsymbol{\omega}, \boldsymbol{v}) \in \mathbb{R}^6$. Then,

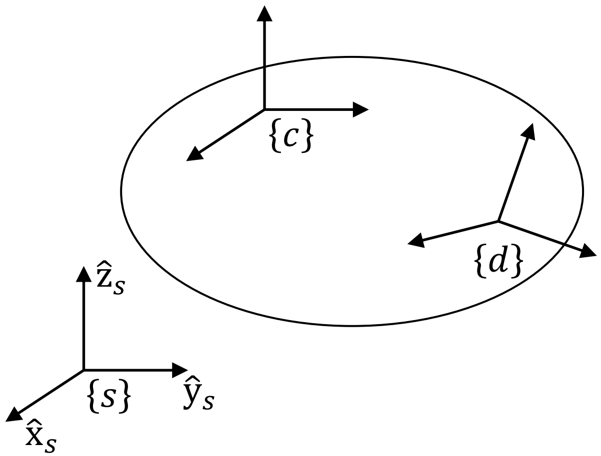
$$[\mathrm{Ad}_{\boldsymbol{T}_1}][\mathrm{Ad}_{\boldsymbol{T}_2}]\boldsymbol{\mathcal{V}} = [\mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]\boldsymbol{\mathcal{V}} \quad \text{or} \quad \mathrm{Ad}_{\boldsymbol{T}_1}\left(\mathrm{Ad}_{\boldsymbol{T}_2}(\boldsymbol{\mathcal{V}})\right) = \mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}(\boldsymbol{\mathcal{V}})$$

- For any $\boldsymbol{T} \in SE(3)$, $[\mathrm{Ad}_{\boldsymbol{T}}]^{-1} = [\mathrm{Ad}_{\boldsymbol{T}^{-1}}]$. Note that $[\mathrm{Ad}_{\boldsymbol{T}}]$ is always invertible.

- For any two frames $\{c\}$ and $\{d\}$, a twist represented in $\{c\}$ as $\boldsymbol{\mathcal{V}}_c$ is related to its representation in $\{d\}$ as $\boldsymbol{\mathcal{V}}_d$ by

$$\boldsymbol{\mathcal{V}}_c = [\mathrm{Ad}_{\boldsymbol{T}_{cd}}]\boldsymbol{\mathcal{V}}_d \qquad \boldsymbol{\mathcal{V}}_d = [\mathrm{Ad}_{\boldsymbol{T}_{dc}}]\boldsymbol{\mathcal{V}}_c$$

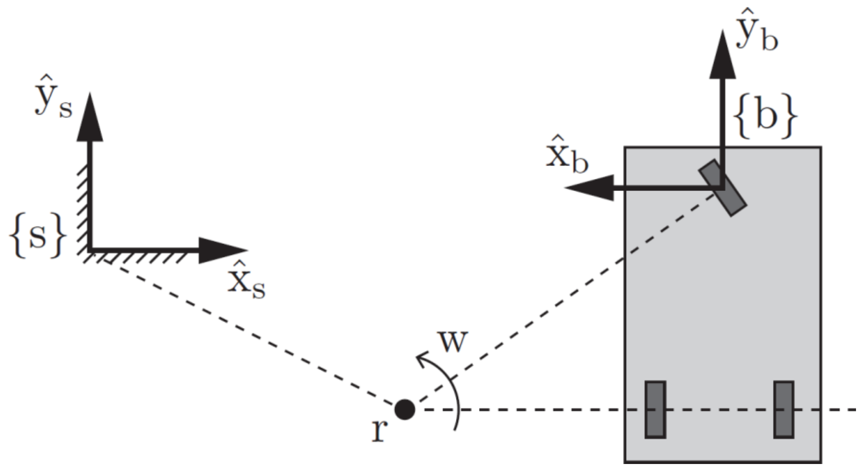
(changing the reference frame of a twist)



# Example

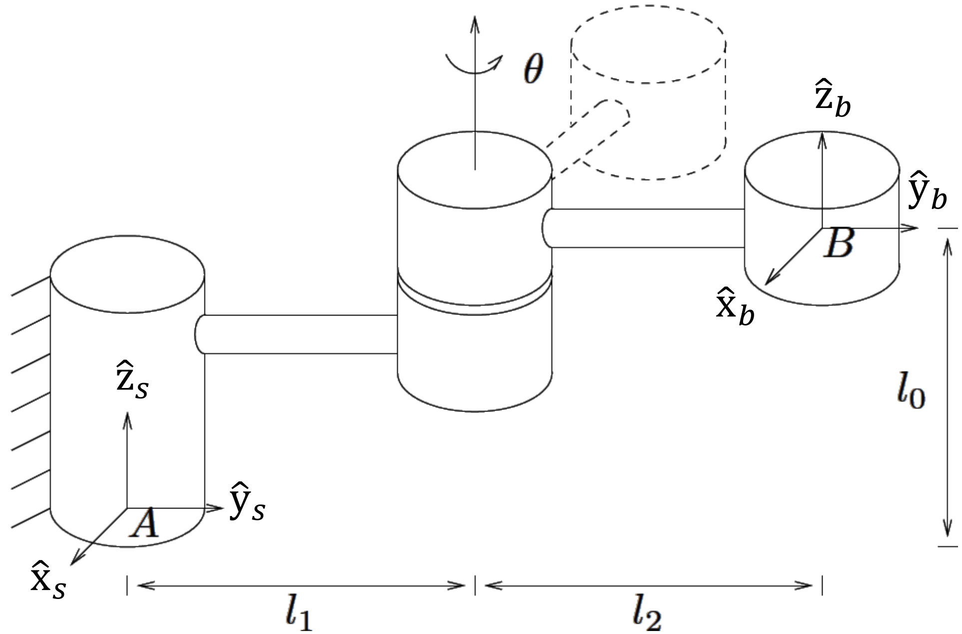
Consider a three-wheeled car with a single steerable front wheel, driving on a plane. The angle of the front wheel of the car causes the car's motion to be a pure angular velocity 2 rad/s about an axis out of the page at the point r in the plane. Find $\mathcal{V}_s$ and $\mathcal{V}_b$.

$$\boldsymbol{r}_s = (2, -1, 0)$$
$$\boldsymbol{r}_b = (2, -1.4, 0)$$



# Example

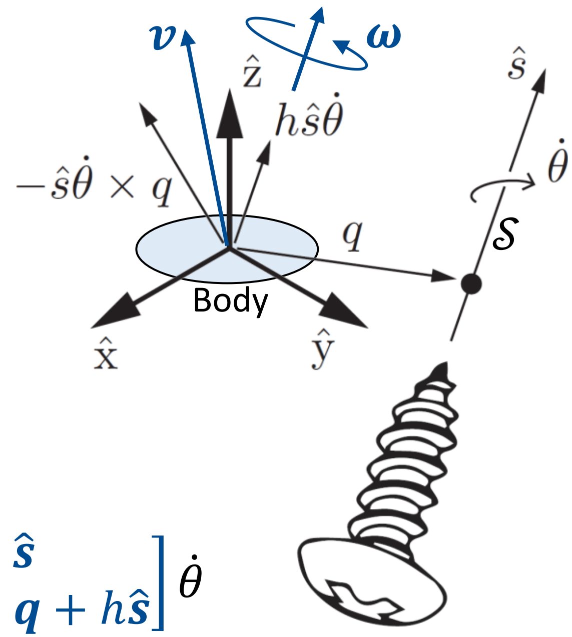
Find $\mathcal{V}_s$ and $\mathcal{V}_b$ for the shown one degree of freedom manipulator.



# Screw Interpretation of a Twist

Any rigid-body velocity or twist $\boldsymbol{\mathcal{V}}$ is equivalent to the instantaneous velocity $\dot{\theta}$ about some screw axis $\mathcal{S}$ (i.e., rotating about the axis while also translating along the axis).

A screw axis $\mathcal{S}$ is represented by an arbitrary point $\boldsymbol{q} \in \mathbb{R}^3$ on the axis, a unit vector $\hat{\boldsymbol{s}} \in S^2$ in the direction of the axis (or angular velocity $\boldsymbol{\omega}$), and a pitch $h \in \mathbb{R}_+$ (which is linear velocity along the axis divided by angular velocity $\dot{\theta}$ about the axis) as $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$. It also can be uniquely represented by **Plücker Coordinates** as $\{\boldsymbol{m}, \hat{\boldsymbol{s}}, h\}$ where $\boldsymbol{m} = \boldsymbol{q} \times \hat{\boldsymbol{s}}$.




Thus, twist $\boldsymbol{\mathcal{V}}$ can be represented as

$$\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{\omega} \times (-\boldsymbol{q}) + h\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}}\dot{\theta} \\ -\hat{\boldsymbol{s}}\dot{\theta} \times \boldsymbol{q} + h\dot{\theta}\hat{\boldsymbol{s}} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}} \\ -\hat{\boldsymbol{s}} \times \boldsymbol{q} + h\hat{\boldsymbol{s}} \end{bmatrix} \dot{\theta}$$

Due to rotation about $\mathcal{S}$ (which is in the plane orthogonal to $\hat{\boldsymbol{s}}$)

Due to translation along $\mathcal{S}$ (which is in the direction of $\hat{\boldsymbol{s}}$)

# Representation of Screw Axis

Now, instead of representing the screw axis $\mathcal{S}$ as $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$ (where $\boldsymbol{q}$ is not unique), we represent a "unit" screw axis (uniquely) as a vector as

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6 \quad \text{where} \quad \boldsymbol{\mathcal{V}} = \boldsymbol{S}\dot{\theta} \in \mathbb{R}^6 \qquad \boldsymbol{S}_\omega, \boldsymbol{S}_v \in \mathbb{R}^3$$

- Finding $\boldsymbol{S}$ and $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$ by having $\boldsymbol{\mathcal{V}}$:

**(a)** If $\|\boldsymbol{\omega}\| \neq 0$ ($\equiv$ rotation with/without translation along $\hat{\boldsymbol{s}}$):

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \boldsymbol{\mathcal{V}}/\|\boldsymbol{\omega}\| = \begin{bmatrix} \boldsymbol{\omega}/\|\boldsymbol{\omega}\| \\ \boldsymbol{v}/\|\boldsymbol{\omega}\| \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}} \\ -\hat{\boldsymbol{s}} \times \boldsymbol{q} + h\hat{\boldsymbol{s}} \end{bmatrix}$$

$$= \begin{bmatrix} \text{angular velocity when } \dot{\theta} = 1 \\ \text{linear velocity of origin when } \dot{\theta} = 1 \end{bmatrix}$$

Pitch $h$ is finite ($h = 0$ for pure rotation).

$h = \boldsymbol{S}_\omega^T \boldsymbol{S}_v = \boldsymbol{\omega}^T \boldsymbol{v}/\|\boldsymbol{\omega}\|^2$

$\hat{\boldsymbol{s}} = \boldsymbol{S}_\omega = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|, \quad \|\boldsymbol{S}_\omega\| = 1$

$\dot{\theta} = \|\boldsymbol{\omega}\|$ is interpreted as angular velocity about $\hat{\boldsymbol{s}}$

To find $\boldsymbol{q}$, use $\boldsymbol{v} - h\boldsymbol{\omega} = -\boldsymbol{\omega} \times \boldsymbol{q}$
or $(\boldsymbol{S}_v - h\boldsymbol{S}_\omega = -\boldsymbol{S}_\omega \times \boldsymbol{q})$

**(b)** If $\|\boldsymbol{\omega}\| = 0$ ($\equiv$ pure translation along $\hat{\boldsymbol{s}}$):

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \boldsymbol{\mathcal{V}}/\|\boldsymbol{v}\| = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{v}/\|\boldsymbol{v}\| \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \hat{\boldsymbol{s}} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{0} \\ \text{normalized linear velocity of origin} \end{bmatrix}$$

Pitch $h$ is infinite, $\quad \|\boldsymbol{S}_\omega\| = 0$

$\hat{\boldsymbol{s}} = \boldsymbol{S}_v = \boldsymbol{v}/\|\boldsymbol{v}\|, \quad \|\boldsymbol{S}_v\| = 1$

$\dot{\theta} = \|\boldsymbol{v}\|$ is interpreted as linear velocity along $\hat{\boldsymbol{s}}$

# Screw Axis Properties

- Since a screw axis $\boldsymbol{S}$ is just a normalized twist, the $4 \times 4$ matrix representation $[\boldsymbol{S}]$ of $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v) \in \mathbb{R}^6$ is

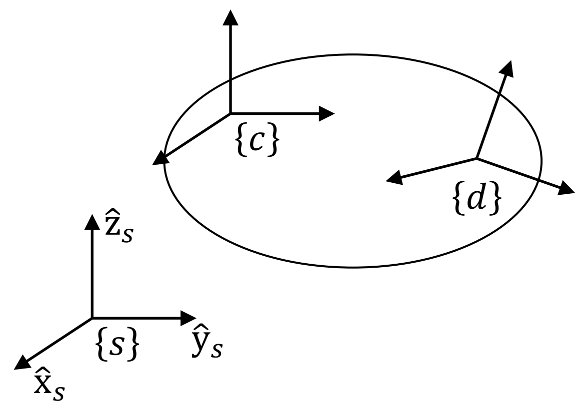
$$[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$$

$$\boldsymbol{\mathcal{V}} = \boldsymbol{S}\dot{\theta} \in \mathbb{R}^6 \quad \Rightarrow \quad [\boldsymbol{\mathcal{V}}] = [\boldsymbol{S}]\dot{\theta} \in se(3)$$

- Like twist $\boldsymbol{\mathcal{V}}$, the screw axis $\boldsymbol{S}$ is represented in a frame (e.g., $\{b\}$ or $\{s\}$). Therefore, for any two frames $\{c\}$ and $\{d\}$, a screw axis represented in $\{c\}$ as $\boldsymbol{S}_c$ is related to its representation in $\{d\}$ as $\boldsymbol{S}_d$ by:

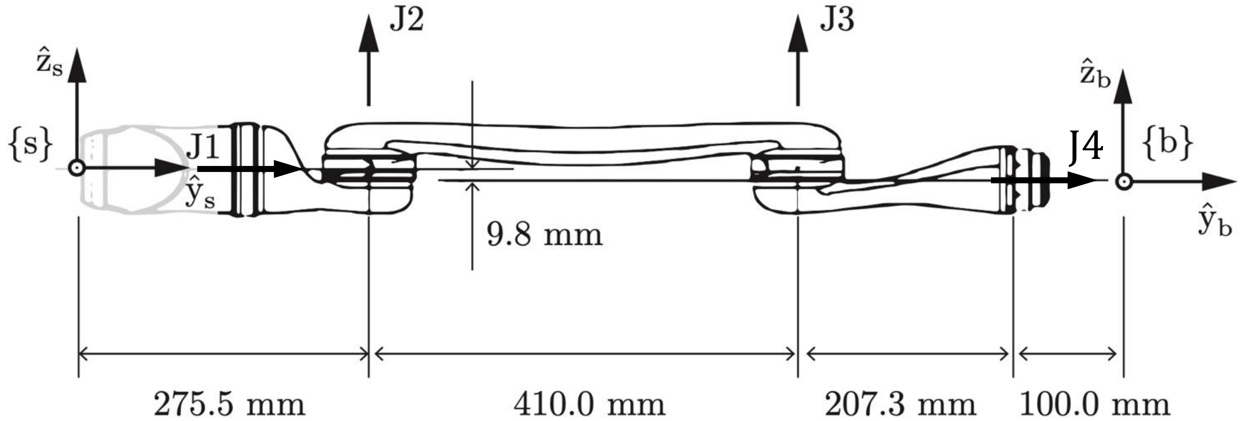
$$\boldsymbol{S}_c = [\mathrm{Ad}_{\boldsymbol{T}_{cd}}]\boldsymbol{S}_d \qquad \boldsymbol{S}_d = [\mathrm{Ad}_{\boldsymbol{T}_{dc}}]\boldsymbol{S}_c$$

(changing the reference frame of a screw axis)



# Example

What are the screw axis $\boldsymbol{S}_b$ and $\boldsymbol{S}_s$ for J4 and J2 for the shown Kinova 4-DOF arm?



# Exponential Coordinate Representation of Rigid-Body Motion

# Screw Motion

Instead of viewing a displacement as a rotation followed by a translation, both rotation and translation can be performed simultaneously.

Planar example of a screw motion:
The displacement in Figure 1 (rotation ❶ + translation ❷) can be viewed as a pure rotation of $\beta = 90°$ about a fixed-point $s$ as shown in Figure 2.

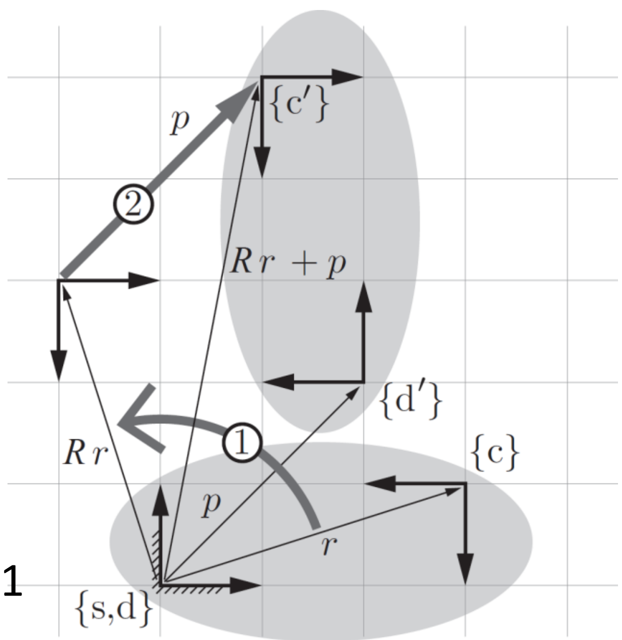



Figure 1

$\equiv$

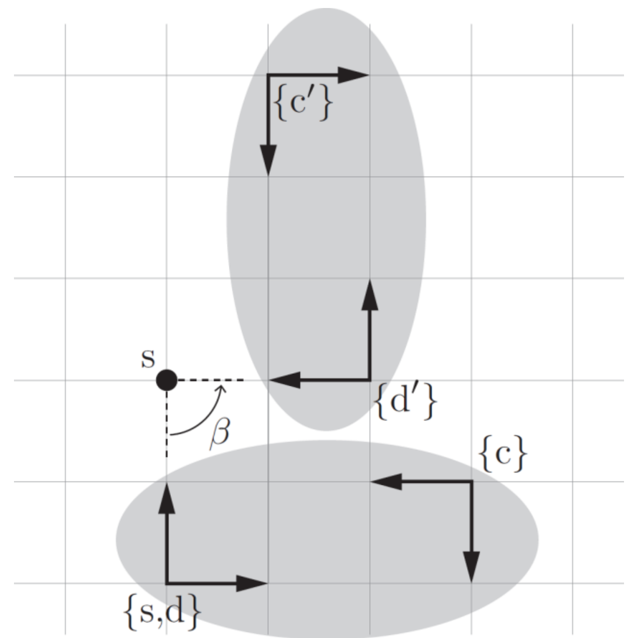



Figure 2

# Exponential Coordinates of Rigid-Body Motions

Chasles–Mozzi theorem states that every rigid-body displacement can be expressed as a finite rotation $\theta$ and translation $d$ about/along a fixed screw axis in space.

This theorem motivates a six-parameter representation of a configuration (or a homogeneous transformation $\boldsymbol{T} \in SE(3)$) called the **exponential coordinates** as $\boldsymbol{S}\theta \in \mathbb{R}^6$, where $\boldsymbol{S}$ is the screw axis and $\theta$ is the distance that must be traveled along the screw axis to take a frame from the origin $\boldsymbol{I}_4$ to $\boldsymbol{T}$.

**Note**: $\boldsymbol{T}$ is equivalent to the displacement obtained by rotating a frame from $\boldsymbol{I}_4$ about $\boldsymbol{S}$

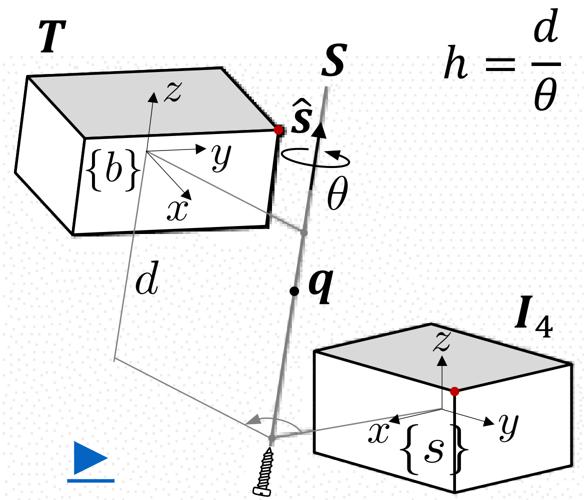
- by an angle $\theta$, or
- at a speed $\dot{\theta} = 1$ rad/s for $\theta$s, or
- at a speed $\dot{\theta} = \theta$ for 1s, or
- by constant twist $\boldsymbol{\mathcal{V}}$ for 1s.

$(\boldsymbol{\mathcal{V}}t = \boldsymbol{S}\theta)$

**Constant Screw Motion**: A rotation $\theta$ + a translation $d$ about/along a fixed screw axis $\boldsymbol{S}$.

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \boldsymbol{q} + h\hat{\mathbf{s}} \end{bmatrix}$$ (for rotation with/without translation along $\hat{\mathbf{s}}$)

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{s}} \end{bmatrix}$$ (for pure translation along $\hat{\mathbf{s}}$)



# Exponential Coordinates of Rigid-Body Motions

As with rotations, we can define a matrix exponential (exp) and matrix logarithm (log).
For any transformation matrix $\boldsymbol{T} \in SE(3)$, we can always find a screw axis $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v) \in \mathbb{R}^6$ (where $\|\boldsymbol{S}_\omega\| = 1$ for rotation with/without translation or $\boldsymbol{S}_\omega = \boldsymbol{0}$ and $\|\boldsymbol{S}_v\| = 1$ for pure translation) and scalar $\theta \in \mathbb{R}$ such that $\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$.

$$
\begin{array}{llcll}
\text{exp:} & [\boldsymbol{S}]\theta \in se(3) & \rightarrow & \boldsymbol{T} \in SE(3) & : \quad e^{[\boldsymbol{S}]\theta} = \boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p}) \\
\text{log:} & \boldsymbol{T} \in SE(3) & \rightarrow & [\boldsymbol{S}]\theta \in se(3) & : \quad \log(\boldsymbol{T}) = [\boldsymbol{S}]\theta
\end{array}
$$

$\boldsymbol{S}\theta \in \mathbb{R}^6$ : Exponential coordinates of $\boldsymbol{T} \in SE(3)$
$[\boldsymbol{S}]\theta = [\boldsymbol{S}\theta] \in se(3)$ : Matrix logarithm of $\boldsymbol{T}$ (inverse of the matrix exponential)

**Note**: $\boldsymbol{T}$ and $\boldsymbol{S}$ have the same base.

# Matrix Exponential

$$\text{exp:} \quad [\boldsymbol{S}]\theta \in se(3) \;\to\; \boldsymbol{T} \in SE(3) \quad : \quad e^{[\boldsymbol{S}]\theta} = \boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$$

- Finding $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$ by having $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ and $\theta$:

**(a)** If $\boldsymbol{S}_\omega \neq \boldsymbol{0}$ (and $\|\boldsymbol{S}_\omega\| = 1$) (i.e., rotation with/without translation):

$$e^{[\boldsymbol{S}]\theta} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Using Taylor expansion

Use Rodrigues Formula

$$\boldsymbol{G}(\theta) = \boldsymbol{I}_3\theta + (1 - \cos\theta)[\boldsymbol{S}_\omega] + (\theta - \sin\theta)[\boldsymbol{S}_\omega]^2 \in \mathbb{R}^{3\times 3}$$

**(b)** If $\boldsymbol{S}_\omega = \boldsymbol{0}$ (and $\|\boldsymbol{S}_v\| = 1$) (i.e., pure translation):

$$e^{[\boldsymbol{S}]\theta} = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{S}_v\theta \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

# Matrix Exponential: Remark

- For a given transformation matrix $\boldsymbol{T}_{sb}$:

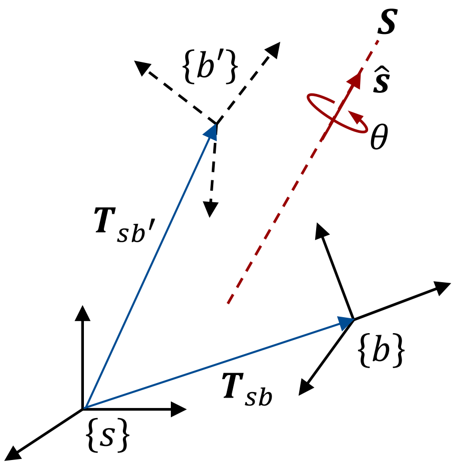
**Fixed-frame Displacement** is rotation by $\theta$ about/along a screw axis $\boldsymbol{S}_s$, expressed in fixed frame $\{s\}$ as:

$$\boldsymbol{T}_{sb'} = e^{[\boldsymbol{S}_s]\theta}\boldsymbol{T}_{sb}$$

**Body-frame Displacement** is rotation by $\theta$ about/along a screw axis $\boldsymbol{S}_b$, expressed in body frame $\{b\}$ as:

$$\boldsymbol{T}_{sb'} = \boldsymbol{T}_{sb}e^{[\boldsymbol{S}_b]\theta}$$

$(\boldsymbol{S}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{S}_b)$



# Matrix Logarithm

$$\log: \quad \boldsymbol{T} \in SE(3) \quad \to \quad [\boldsymbol{S}]\theta \in se(3) \quad : \quad \log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$$

❖ Finding $\boldsymbol{S} = (\boldsymbol{S}_\omega, \boldsymbol{S}_v)$ and $\theta \in [0, \pi]$ by having $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{p})$:

**(a)** If $\mathrm{tr}\boldsymbol{R} = 3$ (or $\boldsymbol{R} = \boldsymbol{I}_3$), then set $\boldsymbol{S}_\omega = \boldsymbol{0}$, $\boldsymbol{S}_v = \boldsymbol{p}/\|\boldsymbol{p}\|$, and $\theta = \|\boldsymbol{p}\|$.

**(b)** Otherwise, use the matrix logarithm $\log(\boldsymbol{R}) = [\boldsymbol{S}_\omega]\theta$ to determine $\boldsymbol{S}_\omega$ (this is $\hat{\boldsymbol{\omega}}$ in the $SO(3)$ algorithm) and $\theta \in [0, \pi]$. Then, $\boldsymbol{S}_v$ is calculated as

$$\boldsymbol{S}_v = \boldsymbol{G}^{-1}(\theta)\boldsymbol{p}$$

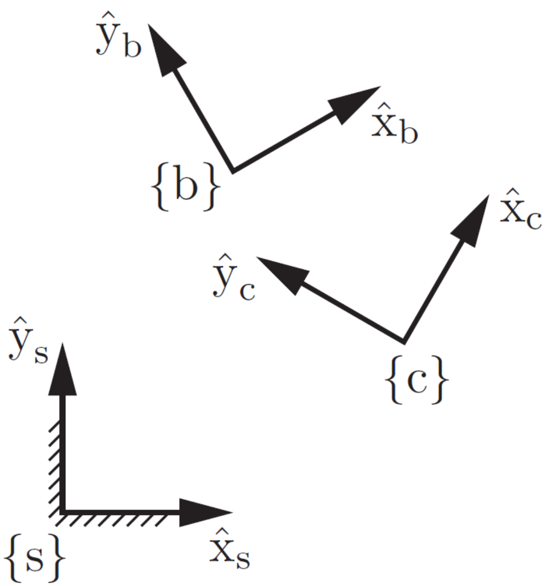
where $$\boldsymbol{G}^{-1}(\theta) = \frac{1}{\theta}\boldsymbol{I}_3 - \frac{1}{2}[\boldsymbol{S}_\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\boldsymbol{S}_\omega]^2 \in \mathbb{R}^{3\times 3}$$

($\theta$ is in radian)

# Example

The initial frame $\{b\}$ and final frame $\{c\}$ are given. Find the screw motion expressed in $\{s\}$ $(\boldsymbol{S}_s, \theta)$ that displaces the frame at $T_{sb}$ to $T_{sc}$.

$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




$$T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Unit Dual Quaternions

# Dual Quaternion

In general, dual numbers are defined as $d = a + \epsilon b$ where $a$ and $b$ are elements of an algebraic field, and $\epsilon$ is a dual unit with $\epsilon^2 = 0$, $\epsilon \neq 0$. Similarly, a dual quaternion $\boldsymbol{D}$ is defined as $\boldsymbol{D} = \boldsymbol{p} + \epsilon \boldsymbol{q}$ where $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{H}$ are quaternions.

- Addition and multiplication of two dual quaternions $\boldsymbol{D}_1 = \boldsymbol{p}_1 + \epsilon \boldsymbol{q}_1$ and $\boldsymbol{D}_2 = \boldsymbol{p}_2 + \epsilon \boldsymbol{q}_2$:

$$\boldsymbol{D}_1 + \boldsymbol{D}_2 = (\boldsymbol{p}_1 + \boldsymbol{p}_2) + \epsilon(\boldsymbol{q}_1 + \boldsymbol{q}_2)$$

$$\boldsymbol{D}_1 \boldsymbol{D}_2 = (\boldsymbol{p}_1 \boldsymbol{p}_2) + \epsilon(\boldsymbol{p}_1 \boldsymbol{q}_2 + \boldsymbol{q}_1 \boldsymbol{p}_2) \neq \boldsymbol{D}_2 \boldsymbol{D}_1 \quad \text{(not commutative)}$$

- Conjugate of $\boldsymbol{D}$: $\boldsymbol{D}^* = \boldsymbol{p}^* + \epsilon \boldsymbol{q}^*$ or $\boldsymbol{D}^\dagger = \boldsymbol{p}^* - \epsilon \boldsymbol{q}^*$

- Norm of $\boldsymbol{D}$: $\|\boldsymbol{D}\| = \sqrt{\boldsymbol{D}\boldsymbol{D}^*} = \sqrt{\boldsymbol{p}\boldsymbol{p}^* + \epsilon(\boldsymbol{p}\boldsymbol{q}^* + \boldsymbol{q}\boldsymbol{p}^*)}$

- Inverse of $\boldsymbol{D}$: $\boldsymbol{D}^{-1} = \dfrac{\boldsymbol{D}^*}{\|\boldsymbol{D}\|^2}$

The dual quaternion $\boldsymbol{D} = \boldsymbol{p} + \epsilon \boldsymbol{q}$ is a **Unit Dual Quaternion** if $\|\boldsymbol{D}\| = 1$, i.e., $\sqrt{\boldsymbol{p}\boldsymbol{p}^*} = \|\boldsymbol{p}\| = 1$ ($\boldsymbol{p}$ is unit quaternion) and $\boldsymbol{p}\boldsymbol{q}^* + \boldsymbol{q}\boldsymbol{p}^* = \boldsymbol{0}$. Consequently, $\boldsymbol{D}^{-1} = \boldsymbol{D}^*$ and $\boldsymbol{p} \cdot \boldsymbol{q} = 0$.

# Unit Dual Quaternion

The homogeneous transformation $\boldsymbol{T} \in SE(3)$ (i.e., the rotation $\boldsymbol{R}$ followed by the translation $\boldsymbol{p}$) can be also represented by a unit dual quaternion as

$$\boldsymbol{D}_T = \boldsymbol{q}_R + \epsilon \boldsymbol{q}_d = \boldsymbol{q}_R + \frac{\epsilon}{2} \boldsymbol{q}_p \boldsymbol{q}_R$$

where $\boldsymbol{q}_R = \left(\cos\left(\frac{\theta}{2}\right), \hat{\boldsymbol{\omega}} \sin\left(\frac{\theta}{2}\right)\right) \in S^3$ is a unit quaternion representing rotation $\boldsymbol{R}$, $\boldsymbol{q}_d \in \mathbb{H}$ is a quaternion encoding translation, $\boldsymbol{q}_p = (0, \boldsymbol{p}) \in \mathbb{H}$, and $\boldsymbol{q}_R \cdot \boldsymbol{q}_d = 0$.

**Note**: For pure rotation $\boldsymbol{D}_T = \boldsymbol{q}_R + \epsilon \boldsymbol{0}$ (or $\boldsymbol{D}_T = \boldsymbol{q}_R + \epsilon(0,0,0,0)$) and pure translation $\boldsymbol{D}_T = \boldsymbol{1} + \frac{\epsilon}{2} \boldsymbol{q}_p$ (or $\boldsymbol{D}_T = (1,0,0,0) + \frac{\epsilon}{2} \boldsymbol{q}_p$).

**Note**: If we are given a unit dual quaternion $\boldsymbol{D}_T$, to convert it into the transformation matrix $\boldsymbol{T} \in SE(3)$, we convert the unit quaternion $\boldsymbol{q}_R$ into a rotation matrix $\boldsymbol{R} \in SO(3)$ and the translation $\boldsymbol{p} \in \mathbb{R}^3$ is obtained from $2\boldsymbol{q}_d \boldsymbol{q}_R^* = \boldsymbol{q}_p = (0, \boldsymbol{p})$.

# Unit Dual Quaternion

- The transformation of a point or vector $\boldsymbol{p} \in \mathbb{R}^3$ using unit dual quaternion $\boldsymbol{D}_T$ is determined as

$$\boldsymbol{D}_{p'} = \boldsymbol{D}_T(\boldsymbol{1} + \epsilon \boldsymbol{q}_p)\boldsymbol{D}_T^{\dagger} = \boldsymbol{1} + \epsilon(\boldsymbol{q}_R \boldsymbol{q}_p \boldsymbol{q}_R^{-1} + \boldsymbol{q}_p) \quad \leftrightarrow \quad \boldsymbol{p}' = \boldsymbol{T}\boldsymbol{p}$$

- The screw displacements $\{\boldsymbol{m}, \hat{\boldsymbol{s}}, h = d/\theta\}$ can be expressed by the dual quaternions as

$$\boldsymbol{D}_T = \cos\frac{\Phi}{2} + \boldsymbol{L}\sin\frac{\Phi}{2} = \left(\cos\frac{\theta}{2}, \hat{\boldsymbol{s}}\sin\frac{\theta}{2}\right) + \epsilon\left(-\frac{d}{2}\sin\frac{\theta}{2}, \frac{d}{2}\cos\frac{\theta}{2}\hat{\boldsymbol{s}} + \sin\frac{\theta}{2}\boldsymbol{m}\right) \quad \leftrightarrow \quad \boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$$

$\Phi = \theta + \epsilon d$ (dual number)

$\boldsymbol{L} = \hat{\boldsymbol{s}} + \epsilon \boldsymbol{m}$ (dual vector)

**Note**: $\theta = 0, \pi$ corresponds to pure translation. In this case, $\boldsymbol{L} = \hat{\boldsymbol{s}} + \epsilon \boldsymbol{0}$ where $\hat{\boldsymbol{s}}$ is the unit vector along the axis of translation.

- A power of the unit dual quaternion $\boldsymbol{D}_T$ is defined as

$$\boldsymbol{D}_T^{\tau} = \cos\frac{\tau\Phi}{2} + \boldsymbol{L}\sin\frac{\tau\Phi}{2} = \left(\cos\frac{\tau\theta}{2}, \hat{\boldsymbol{s}}\sin\frac{\tau\theta}{2}\right) + \epsilon\left(-\frac{\tau d}{2}\sin\frac{\tau\theta}{2}, \frac{\tau d}{2}\cos\frac{\tau\theta}{2}\hat{\boldsymbol{s}} + \sin\frac{\tau\theta}{2}\boldsymbol{m}\right)$$

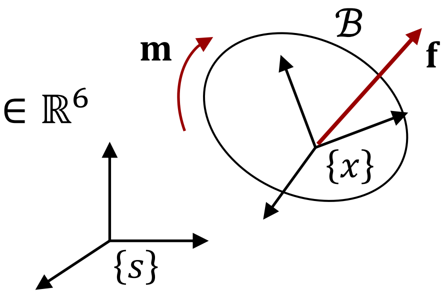
# Wrench

# Spatial Force or Wrench

A rigid body's **Spatial Force** or **Wrench** can be represented as a point in $\mathbb{R}^6$ and defined as

expressed in $\{x\}$

$$\boldsymbol{\mathcal{F}}_x = \begin{bmatrix} \text{moment applied to body expressed in frame } \{x\} \\ \text{force applied to origin of frame } \{x\} \text{ on body expressed in frame } \{x\} \end{bmatrix} \in \mathbb{R}^6$$

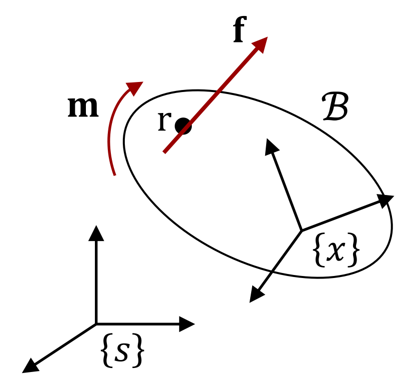



A general form:

point where force is applied

$$\boldsymbol{\mathcal{F}}_x^{\mathcal{B}_r} = \begin{bmatrix} \text{moment applied to body } \mathcal{B} + \text{ moment of force applied to point r on body } \mathcal{B} \text{ in } \{x\} \\ \text{force applied to point r on body } \mathcal{B} \text{ expressed in frame } \{x\} \end{bmatrix} \in \mathbb{R}^6$$

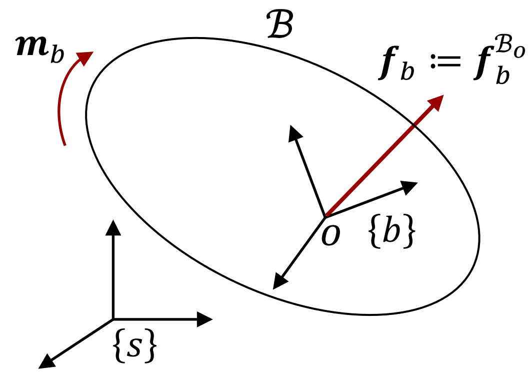
expressed in $\{x\}$



# Body Wrench $\mathcal{F}_b$

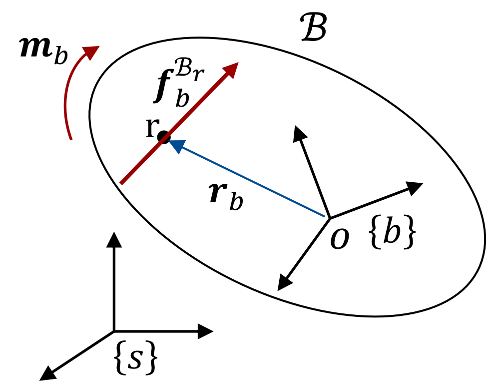
Let $\boldsymbol{m}_b \in \mathbb{R}^3$ be a moment applied to the body expressed in $\{b\}$ and $\boldsymbol{f}_b \in \mathbb{R}^3$ be a force applied to the body at the origin of frame $\{b\}$ and expressed in $\{b\}$. **Body Wrench $\mathcal{F}_b$** is defined as

$$\mathcal{F}_b = \begin{bmatrix} \boldsymbol{m}_b \\ \boldsymbol{f}_b \end{bmatrix} \in \mathbb{R}^6$$




**General Case**: If force $\mathbf{f}$ is applied at the point $\mathrm{r}$ of body $\mathcal{B}$, the body wrench in $\{b\}$ will be:

$$\mathcal{F}_b^{\mathcal{B}_r} = \begin{bmatrix} \boldsymbol{m}_b + \boldsymbol{r}_b \times \boldsymbol{f}_b^{\mathcal{B}_r} \\ \boldsymbol{f}_b^{\mathcal{B}_r} \end{bmatrix} \in \mathbb{R}^6$$




where $\boldsymbol{r}_b \in \mathbb{R}^3$ is the position vector of point $\mathrm{r}$ in $\{b\}$ and $\boldsymbol{r}_b \times \boldsymbol{f}_b^{\mathcal{B}_r}$ is the moment created by force $\boldsymbol{f}_b^{\mathcal{B}_r}$ about the origin of $\{b\}$.

# Spatial Wrench $\mathcal{F}_s$

The **power** is a coordinate-independent quantity, i.e., the power generated (or dissipated) by a wrench $\mathcal{F}$ and twist $\mathcal{V}$ pair must be the same regardless of the frame in which it is represented:

$(\mathcal{V} \cdot \mathcal{F} = \text{power})$

$$\mathcal{V}_s^T \mathcal{F}_s = \mathcal{V}_b^T \mathcal{F}_b = \text{power}$$

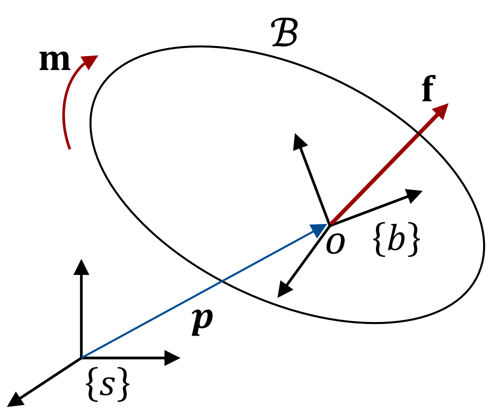
$(\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s)$

$$\mathcal{V}_s^T \mathcal{F}_s = \left([\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s\right)^T \mathcal{F}_b$$

$$= \mathcal{V}_s^{\mathrm{T}} [\mathrm{Ad}_{T_{bs}}]^T \mathcal{F}_b$$

Since this must hold for all $\mathcal{V}_s$

$$\underbrace{\mathcal{F}_s}_{\text{spatial wrench}} = [\mathrm{Ad}_{T_{bs}}]^T \underbrace{\mathcal{F}_b}_{\text{body wrench}}$$

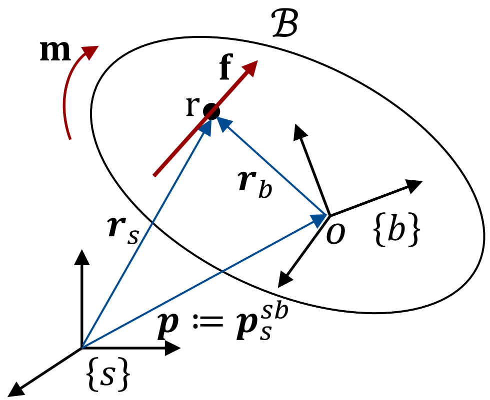



Therefore: $$\mathcal{F}_s = [\mathrm{Ad}_{T_{bs}}]^T \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} m_s + p \times f_s \\ f_s \end{bmatrix}$$

# Spatial Wrench $\mathcal{F}_s$: General Case

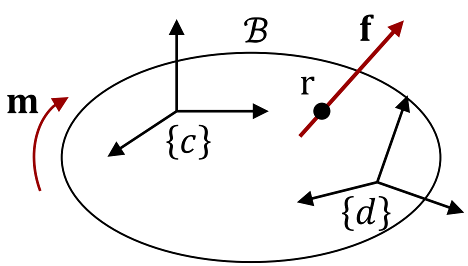
$$\boldsymbol{\mathcal{F}}_s^{\mathcal{B}_r} = [\mathrm{Ad}_{\boldsymbol{T}_{bs}}]^T \boldsymbol{\mathcal{F}}_b^{\mathcal{B}_r} = \begin{bmatrix} \boldsymbol{R}_{sb} & -\boldsymbol{R}_{sb}[\boldsymbol{p}_b^{bs}] \\ \boldsymbol{0} & \boldsymbol{R}_{sb} \end{bmatrix} \begin{bmatrix} \boldsymbol{m}_b^{\mathcal{B}} + \boldsymbol{r}_b \times \boldsymbol{f}_b^{\mathcal{B}_r} \\ \boldsymbol{f}_b^{\mathcal{B}_r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{sb}\boldsymbol{m}_b^{\mathcal{B}} + \boldsymbol{R}_{sb}\left((\boldsymbol{r}_b - \boldsymbol{p}_b^{bs}) \times \boldsymbol{f}_b^{\mathcal{B}_r}\right) \\ \boldsymbol{R}_{sb}\boldsymbol{f}_b^{\mathcal{B}_r} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{R}_{sb}\boldsymbol{m}_b^{\mathcal{B}} + \boldsymbol{R}_{sb}\boldsymbol{p}_b^{sr} \times \boldsymbol{R}_{sb}\boldsymbol{f}_b^{\mathcal{B}_r} \\ \boldsymbol{R}_{sb}\boldsymbol{f}_b^{\mathcal{B}_r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{m}_s^{\mathcal{B}} + \boldsymbol{r}_s \times \boldsymbol{f}_s^{\mathcal{B}_r} \\ \boldsymbol{f}_s^{\mathcal{B}_r} \end{bmatrix} \in \mathbb{R}^6$$



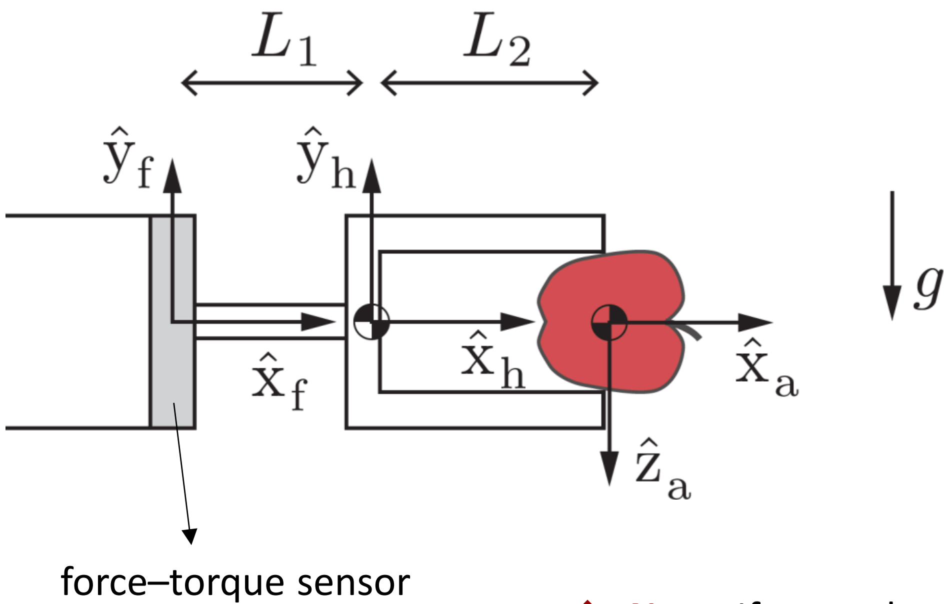

- In general, if we have the wrench in frame $\{d\}$, we can express it in another frame $\{d\}$ as:

$$\boldsymbol{\mathcal{F}}_c^{\mathcal{B}_r} = [\mathrm{Ad}_{\boldsymbol{T}_{dc}}]^T \boldsymbol{\mathcal{F}}_d^{\mathcal{B}_r}$$



# Example

The robot hand shown is holding an apple with a mass of 0.1 kg in a gravitational field $g$=10 m/s$^2$. The mass of the hand is 0.5 kg, $L_1$=10 cm, and $L_2$=15 cm. What is the force and torque measured by the six-axis force–torque sensor between the hand and the robot arm?




❖ **Note**: If more than one wrench acts on a rigid body, the total wrench on the body is simply the vector sum of the individual wrenches, provided that the wrenches are expressed in the same frame.

# Review

# Rigid-Body Motions

| Rotations | Transformations |
|---|---|
| $\boldsymbol{R} \in SO(3)$: $3 \times 3$ matrices<br>$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}_3$, $\det(\boldsymbol{R}) = 1$ | $\boldsymbol{T} \in SE(3)$: $4 \times 4$ matrices<br>$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$,<br>where $\boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3$ |
| $\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$ | $\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$ |
| Change of coordinate frame:<br>$\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{ac}$, $\boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$<br>$\left(\boldsymbol{R}_{ab} = \boldsymbol{R}_{ba}^{-1} = \boldsymbol{R}_{ba}^T\right)$ | Change of coordinate frame:<br>$\boldsymbol{T}_{ab}\boldsymbol{T}_{bc} = \boldsymbol{T}_{ac}$, $\boldsymbol{T}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$<br>$\left(\boldsymbol{T}_{ab} = \boldsymbol{T}_{ba}^{-1}\right)$ |

# Rigid-Body Motions

| Rotations | Transformations |
|---|---|
| Rotating a frame $\{b\}$: $$\boldsymbol{R} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$ $\boldsymbol{R}_{sb'} = \boldsymbol{R}\boldsymbol{R}_{sb}$: rotate $\theta$ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ <br> $\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\boldsymbol{R}$: rotate $\theta$ about $\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}$ | Displacing a frame $\{b\}$: $$\boldsymbol{T} = \begin{bmatrix} \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$ $\boldsymbol{T}_{sb'} = \boldsymbol{T}\boldsymbol{T}_{sb}$: rotate $\theta$ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ (moves $\{b\}$ origin), translate $\boldsymbol{p}$ in $\{s\}$ <br> $\boldsymbol{T}_{sb'} = \boldsymbol{T}_{sb}\boldsymbol{T}$: translate $\boldsymbol{p}$ in $\{b\}$, rotate $\theta$ about $\hat{\boldsymbol{\omega}}$ in new body frame |
| Unit rotation axis is $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$, where $\|\hat{\boldsymbol{\omega}}\| = 1$ | "Unit" screw axis is $\boldsymbol{\mathcal{S}} = \begin{bmatrix} \boldsymbol{\mathcal{S}}_\omega \\ \boldsymbol{\mathcal{S}}_v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\|\boldsymbol{\mathcal{S}}_\omega\| = 1$ or (ii) $\|\boldsymbol{\mathcal{S}}_\omega\| = 0$, $\|\boldsymbol{\mathcal{S}}_v\| = 1$ |
| | For a screw axis $\{\boldsymbol{q}, \hat{\boldsymbol{s}}, h\}$ with finite $h$, $$\boldsymbol{\mathcal{S}} = \begin{bmatrix} \boldsymbol{\mathcal{S}}_\omega \\ \boldsymbol{\mathcal{S}}_v \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}} \\ -\hat{\boldsymbol{s}} \times \boldsymbol{q} + h\hat{\boldsymbol{s}} \end{bmatrix}$$ |
| Angular velocity is $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}$ | Twist is $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{\mathcal{S}}\dot{\theta}$ |

# Rigid-Body Motions

| Rotations | Transformations |
|---|---|
| For any $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, $$[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$ **Properties**: For any $\boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)$: $$[\boldsymbol{\omega}] = -[\boldsymbol{\omega}]^{\mathrm{T}}, [\boldsymbol{\omega}]\boldsymbol{x} = -[\boldsymbol{x}]\boldsymbol{\omega},$$ $$[\boldsymbol{\omega}][\boldsymbol{x}] = ([\boldsymbol{x}][\boldsymbol{\omega}])^T, \boldsymbol{R}[\boldsymbol{\omega}]\boldsymbol{R}^{\mathrm{T}} = [\boldsymbol{R}\boldsymbol{\omega}]$$ | For any $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} \in \mathbb{R}^6$ or $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$, $$[\boldsymbol{\mathcal{V}}] = \begin{bmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3),$$ $$[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \boldsymbol{0} & 0 \end{bmatrix} \in se(3)$$ |
| $\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s],\ \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b] \quad (\boldsymbol{R} \coloneqq \boldsymbol{R}_{sb})$ | $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\boldsymbol{\mathcal{V}}_s],\ \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{\mathcal{V}}_b] \quad (\boldsymbol{T} \coloneqq \boldsymbol{T}_{sb})$ |
| | $$[\mathrm{Ad}_{\boldsymbol{T}}] = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ [\boldsymbol{p}]\boldsymbol{R} & \boldsymbol{R} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$ **Properties**: $[\mathrm{Ad}_{\boldsymbol{T}}]^{-1} = [\mathrm{Ad}_{\boldsymbol{T}^{-1}}]$, $$[\mathrm{Ad}_{\boldsymbol{T}_1}][\mathrm{Ad}_{\boldsymbol{T}_2}] = [\mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]$$ |
| Change of coordinate frame: $$\hat{\boldsymbol{\omega}}_a = \boldsymbol{R}_{ab}\hat{\boldsymbol{\omega}}_b, \quad \boldsymbol{\omega}_a = \boldsymbol{R}_{ab}\boldsymbol{\omega}_b$$ | Change of coordinate frame: $$\boldsymbol{S}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b, \quad \boldsymbol{\mathcal{V}}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{\mathcal{V}}_b$$ |

# Rigid-Body Motions

| Rotations | Transformations |
|---|---|
| $\hat{\boldsymbol{\omega}}_s = \boldsymbol{R}_{sb}\hat{\boldsymbol{\omega}}_b$ | $\boldsymbol{S}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{S}_b, \boldsymbol{\mathcal{V}}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{\mathcal{V}}_b$ |
| Exponential coordinate for $\boldsymbol{R} \in SO(3)$:<br>$\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ | Exponential coordinate for $\boldsymbol{T} \in SE(3)$:<br>$\boldsymbol{S}\theta \in \mathbb{R}^6$ |
| $\exp: [\hat{\boldsymbol{\omega}}]\theta \in so(3) \rightarrow \boldsymbol{R} \in SO(3)$<br>$\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = e^{[\hat{\boldsymbol{\omega}}]\theta}$<br>$\boldsymbol{R} = \boldsymbol{I}_3 + \sin\theta[\hat{\boldsymbol{\omega}}] + (1 - \cos\theta)[\hat{\boldsymbol{\omega}}]^2$<br>(Rodrigues' formula for rotations) | $\exp: [\boldsymbol{S}]\theta \in se(3) \rightarrow \boldsymbol{T} \in SE(3)$<br>$\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$<br>$\boldsymbol{T} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix}$<br>$\boldsymbol{G}(\theta) = \boldsymbol{I}_3\theta + (1 - \cos\theta)[\boldsymbol{S}_\omega] + (\theta - \sin\theta)[\boldsymbol{S}_\omega]^2$ |
| $\log: \boldsymbol{R} \in SO(3) \rightarrow [\hat{\boldsymbol{\omega}}]\theta \in so(3)$<br>$\log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta$ | $\log: \boldsymbol{T} \in SE(3) \rightarrow [\boldsymbol{S}]\theta \in se(3)$<br>$\log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$ |
| Moment change of coordinate frame:<br>$\boldsymbol{m}_a = \boldsymbol{R}_{ab}\boldsymbol{m}_b$ | Wrench change of coordinate frame:<br>$\boldsymbol{\mathcal{F}}_a = \begin{bmatrix} \boldsymbol{m}_a \\ \boldsymbol{f}_a \end{bmatrix} = [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{\mathrm{T}}\boldsymbol{\mathcal{F}}_b$ |