Ch4: Rigid-Body Motion – Part 1 (Rotation)

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Reference Frames

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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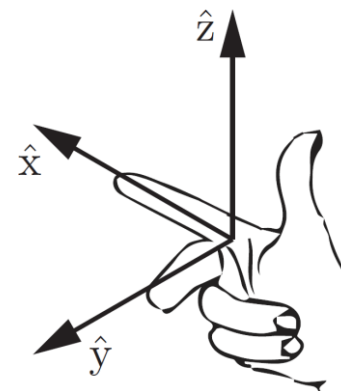Unit Quaternions
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Reference Frames

- **Fixed Space Frame** $\{s\}$: A <u>stationary</u>, inertial frame and there is only one.

- **Body-attached Frame**: A frame fixed to a body and moves with it.

- **Body Frame** $\{b\}$: A <u>stationary</u>, inertial frame that is instantaneously coincident with the body-attached frame.
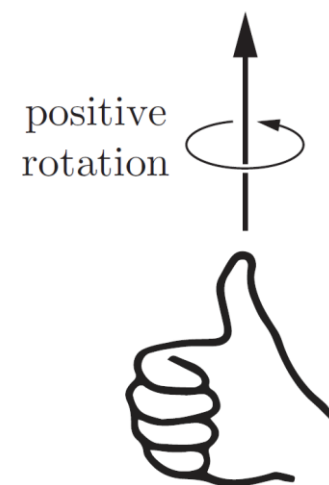
In this course, all frames are instantaneously stationary.

Reference Frames   Rotation Matrices   Angular Velocity   Exponential Coordinate Representation   Euler Angles   Unit Quaternions

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Reference Frames

All reference frames are **right-handed**.

A **positive rotation** about an axis is defined as the direction in which the fingers of the right-hand curl when the thumb is pointed along the axis.

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Rotation Matrices

Rotation in 2D Space

In 2D, the simplest way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by specifying the angle θ.
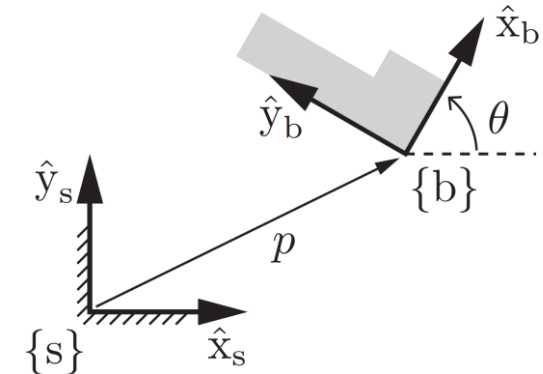
Another way is to express the unit axes $\widehat{\boldsymbol{x}}_b$ and $\widehat{\boldsymbol{y}}_b$ of frame $\{b\}$ in frame $\{s\}$.

$$\Rightarrow \quad \boldsymbol{R} = [\widehat{\boldsymbol{x}}_b \quad \widehat{\boldsymbol{y}}_b] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \in [0, 2\pi)$$

Rotation Matrix

$$\boldsymbol{R} := \boldsymbol{R}_{sb}$$

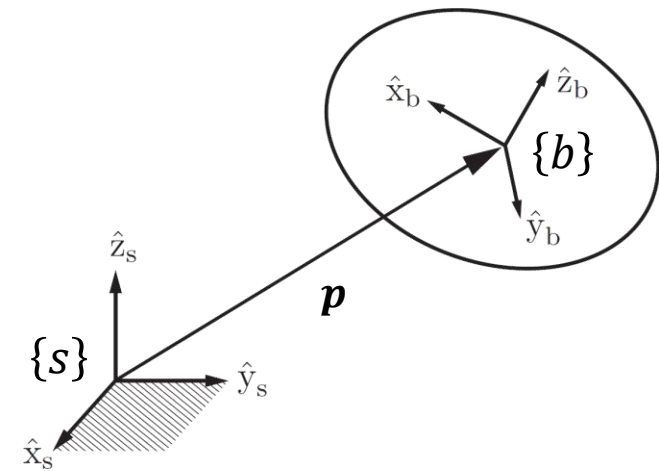Another notation for $\boldsymbol{R}_{sb}$: ${}^{s}\boldsymbol{R}_b$

Rotation in 3D Space

In 3D, a way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by expressing the unit axes $\hat{\boldsymbol{x}}_b$, $\hat{\boldsymbol{y}}_b$, and $\hat{\boldsymbol{z}}_b$ of frame $\{b\}$ in frame $\{s\}$.

$$\boldsymbol{R} = [\hat{\boldsymbol{x}}_b \quad \hat{\boldsymbol{y}}_b \quad \hat{\boldsymbol{z}}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in \mathbb{R}^{3\times3}$$

Rotation Matrix

This is as implicit representation the C-space.

$\boldsymbol{R} := \boldsymbol{R}_{sb}$

Another notation for $\boldsymbol{R}_{sb}$: ${}^s\boldsymbol{R}_b$

Constraints on Rotation Matrix

1- The unit norm condition: $\hat{\boldsymbol{x}}_b$, $\hat{\boldsymbol{y}}_b$, and $\hat{\boldsymbol{z}}_b$ are all unit vectors.

2- The orthogonality condition: $\hat{\boldsymbol{x}}_b . \hat{\boldsymbol{y}}_b = \hat{\boldsymbol{x}}_b . \hat{\boldsymbol{z}}_b = \hat{\boldsymbol{y}}_b . \hat{\boldsymbol{z}}_b = 0$

Compact form of all these six constraints: $\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}_3$

For right-handed frames: $\det(\boldsymbol{R}) = 1$

Definition of a Group

A **group** is a set of elements $G = \{a, b, c, \dots\}$ and a binary operation • on any two elements satisfying

- **Closure**: $a \cdot b \in G \quad \forall a, b \in G$
- **Associativity**: $(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad \forall a, b, c \in G$
- **Identity Element Existence**: $\exists I \in G$ such that $a \cdot I = I \cdot a = a \qquad \forall a \in G$
- **Inverse Element Existence**: $\forall a \in G, \exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = I$

Reference Frames   **Rotation Matrices**   Angular Velocity   Exponential Coordinate Representation   Euler Angles   Unit Quaternions

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Special Orthogonal Group $SO(3)$

The **special orthogonal group** $SO(3)$, also known as the (Lie) group of rotation matrices, is the set of all 3×3 real matrices $\boldsymbol{R}$ that satisfy **(i)** $\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}_3$ and **(ii)** $\det(\boldsymbol{R}) = 1$.

orthogonal   *special*

$SO(2)$ is a subgroup of $SO(3)$: $SO(2) \subset SO(3)$

$\boldsymbol{R} \in SO(3)$ $SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times3} | \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}_3, \det(\boldsymbol{R}) = 1\}$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Properties of Rotation Matrices

$SO(3)$ (or $SO(2)$) is a **matrix (Lie) group** (and the group operation • is matrix multiplication).

- **Closure**: $R_1 R_2 \in SO(3)$
- **Associative**: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$ (but generally not commutative, $R_1 R_2 \neq R_2 R_1$)
- **Identity**: $\exists I_3 \in SO(3)$ such that $R I_3 = I_3 R = R$
- **Inverse**: $\exists\, R^{-1} \in SO(3)$ such that $R R^{-1} = R^{-1} R = I_3$ ($\Rightarrow R^{-1} = R^T$)

* For any vector $x \in \mathbb{R}^3$ and $R \in SO(3)$, the vector $y = Rx$ has the same length as x ($\|x\| = \|Rx\|$).

Reference Frames   **Rotation Matrices**   Angular Velocity   Exponential Coordinate Representation   Euler Angles   Unit Quaternions

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Uses of Rotation Matrices (1)

(1) Representing orientation of a frame relative to another frame.

<u>**Notation**</u>: $\boldsymbol{R}_{bc}$ is the orientation of $\{c\}$ relative to $\{b\}$.

Example:
Find $\boldsymbol{R}_{ab}$ and $\boldsymbol{R}_{ac}$.
(All frames have the same origin)

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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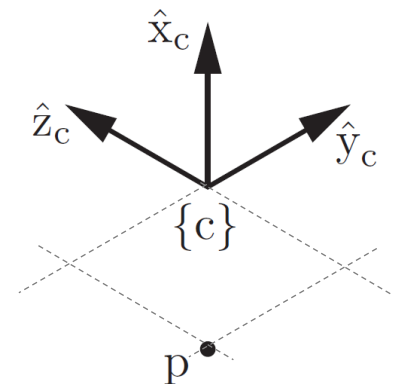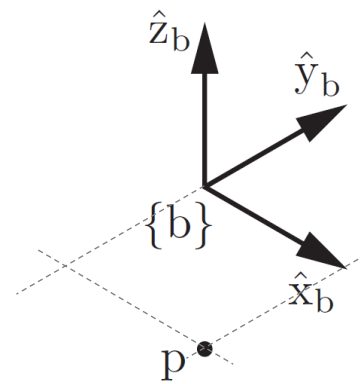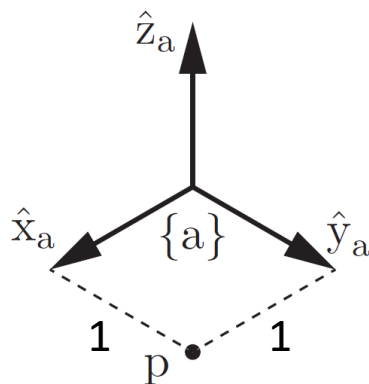
Uses of Rotation Matrices (2)

(2) Changing the reference frame of a <u>vector</u> or <u>frame</u>.

Subscript Cancellation Rule:

$$\boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{p}_{\cancel{b}} = \boldsymbol{p}_a$$

$$\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{R}_{\cancel{b}c} = \boldsymbol{R}_{ac}$$

$\boldsymbol{R}_{ab}$ can be viewed as a **<u>mathematical operator</u>** that changes the reference frame from $\{b\}$ to $\{a\}$.

Note: $\boldsymbol{R}_{bc}\boldsymbol{R}_{cb} = \boldsymbol{I}_3$ or $\boldsymbol{R}_{bc} = \boldsymbol{R}_{cb}^T = \boldsymbol{R}_{cb}^{-1}$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Example

Given $R_1 = R_{ab}$, $R_2 = R_{bc}$, and $R_3 = R_{ad}$, write R_{dc} in terms of R_1 , R_2 , and R_3.

Given p_b, what is p_d in terms of R_1 , R_2 , and R_3?

Uses of Rotation Matrices (3)

(3) Rotating a <u>vector</u> or <u>frame</u> (about a unit axis $\widehat{\boldsymbol{\omega}}$ by an amount θ).

$$\boldsymbol{R} = \boldsymbol{R}_{aa'} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)$$

$\boldsymbol{R}$ can be viewed as a **mathematical operator** that rotates $\{a\}$ about a unit axis $\widehat{\boldsymbol{\omega}} = (\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3)$ (expressed in $\{a\}$) by an amount θ to obtain $\{a'\}$.
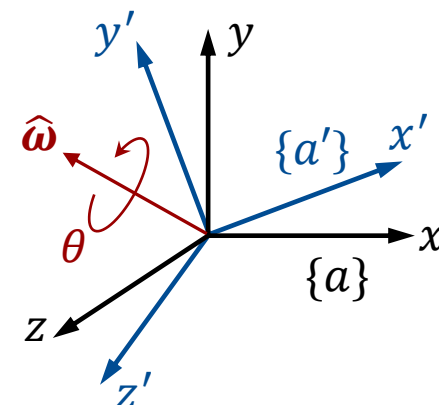
$$\text{Rot}(\widehat{\boldsymbol{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \text{Rot}(\widehat{\boldsymbol{y}}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \text{Rot}(\widehat{\boldsymbol{z}}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1 - c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1 - c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta, c_\theta = \cos\theta$$

- $\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \text{Rot}(-\widehat{\boldsymbol{\omega}}, -\theta)$

Uses of Rotation Matrices (3) (cont.)

- Rotation of vector $\boldsymbol{v}$ about a unit axis $\hat{\boldsymbol{\omega}}$ (expressed in the same frame) by an amount θ is vector $\boldsymbol{v}'$ expressed in the same frame:

$$\boldsymbol{v}' = \boldsymbol{R}\boldsymbol{v} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v}$$

- **Fixed-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{s\}$ by an amount θ is frame $\{b'\}$:

  (pre-multiplication)

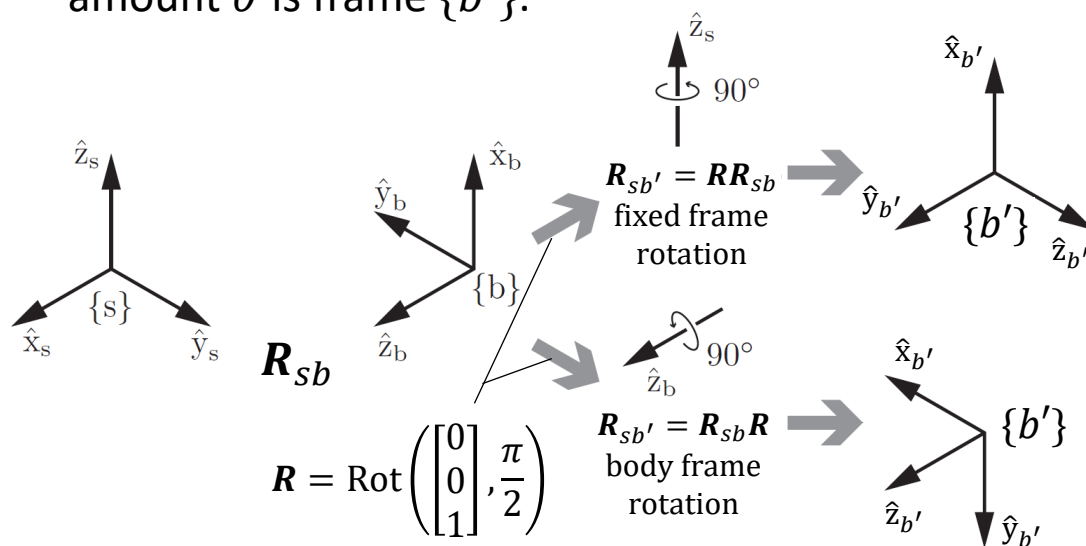$$\boldsymbol{R}_{sb'} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{R}_{sb}$$

  Interpretation

- **Body-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{b\}$ by an amount θ is frame $\{b'\}$:

$$\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

  Interpretation

  (post-multiplication)

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Angular Velocity

Reference Frames   Rotation Matrices   **Angular Velocity**   Exponential Coordinate Representation   Euler Angles   Unit Quaternions

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Set of Skew-Symmetric Matrices $so(3)$

The set of all 3×3 real skew-symmetric matrices is called $so(3)$ (which is the Lie algebra of the Lie group $SO(3)$).

$$so(3) = \{\mathbf{S} \in \mathbb{R}^{3 \times 3} | \mathbf{S}^T = -\mathbf{S}\} \qquad so(3) \subset \mathbb{R}^{3 \times 3}$$

$$\boldsymbol{x} \in \mathbb{R}^3 \qquad [\boldsymbol{x}] \in so(3)$$

- Given any $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{R} \in SO(3)$, $\boldsymbol{R}[\boldsymbol{x}]\boldsymbol{R}^T = [\boldsymbol{Rx}]$.

- Given $[\boldsymbol{x}] \in so(3)$, $[\boldsymbol{x}]^2 = \boldsymbol{xx}^T - \|\boldsymbol{x}\|^2 \boldsymbol{I}_3$ and $[\boldsymbol{x}]^3 = -\|\boldsymbol{x}\|^2[\boldsymbol{x}]$, and and higher powers of $[\boldsymbol{x}]$ can be calculated recursively.

Fixed-Frame Angular Velocity ω_s

Let's find the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of a rotating body (or body frame $\{b\}$) in terms of $\boldsymbol{R}_{sb} = \boldsymbol{R}(t)$. Body Frame $\{b\}$ is instantaneously coincident with the body-attached frame.

- If $\boldsymbol{\omega}$ is expressed in $\{s\}$: $\boldsymbol{\omega} = \boldsymbol{\omega}_s = \dot{\theta}\hat{\boldsymbol{\omega}}_s$ [(1)]

$\boldsymbol{R}(t) = [\hat{\boldsymbol{x}}_b \quad \hat{\boldsymbol{y}}_b \quad \hat{\boldsymbol{z}}_b]$: $\boldsymbol{R}_{sb}$ at time t

$\dot{\boldsymbol{R}}(t) = [\dot{\hat{\boldsymbol{x}}}_b \quad \dot{\hat{\boldsymbol{y}}}_b \quad \dot{\hat{\boldsymbol{z}}}_b]$: Time rate of change of $\boldsymbol{R}_{sb}$ at time t

$$\dot{\hat{\boldsymbol{x}}}_b = \boldsymbol{\omega}_s \times \hat{\boldsymbol{x}}_b$$
$$\dot{\hat{\boldsymbol{y}}}_b = \boldsymbol{\omega}_s \times \hat{\boldsymbol{y}}_b$$
$$\dot{\hat{\boldsymbol{z}}}_b = \boldsymbol{\omega}_s \times \hat{\boldsymbol{z}}_b$$

$\dot{\hat{\boldsymbol{x}}}_b = \boldsymbol{\omega}_s \times \hat{\boldsymbol{x}}_b$ (linear velocity of tip of vector)

$\boldsymbol{\omega}_s$: Fixed-frame angular velocity
(angular velocity expressed in $\{s\}$)

$\dot{\boldsymbol{R}} = [[\boldsymbol{\omega}_s]\hat{\boldsymbol{x}}_b \quad [\boldsymbol{\omega}_s]\hat{\boldsymbol{y}}_b \quad [\boldsymbol{\omega}_s]\hat{\boldsymbol{z}}_b] = [\boldsymbol{\omega}_s]\boldsymbol{R} \qquad \Rightarrow \qquad \boxed{[\boldsymbol{\omega}_s] = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = \dot{\boldsymbol{R}}\boldsymbol{R}^T}$

Note: $\boldsymbol{\omega}_s$ does not depend on the choice of body frame $\{b\}$.

Reference Frames ○○
Rotation Matrices ○○○○○○○○○○○
Angular Velocity ○○○●○
Exponential Coordinate Representation ○○○○○○○○○○○○○
Euler Angles ○○○○○
Unit Quaternions ○○○○○

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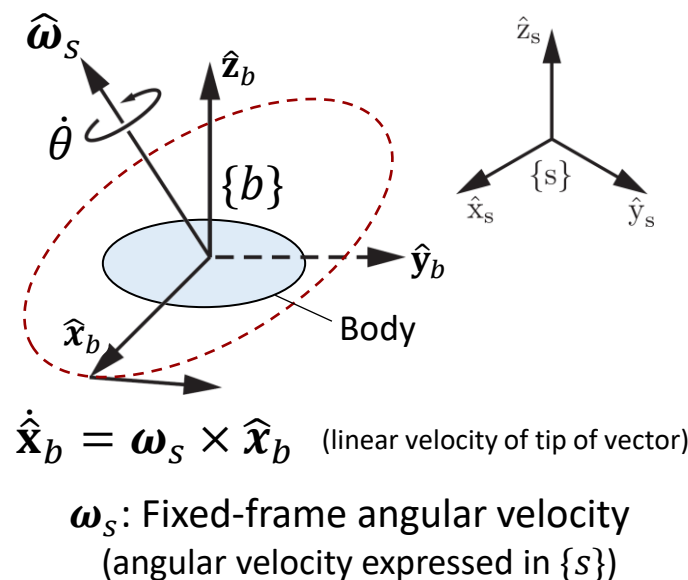
Body-Frame Angular Velocity ω_b

- If $\boldsymbol{\omega}$ is expressed in $\{b\}$: $\quad \boldsymbol{\omega} = \boldsymbol{\omega}_b = \dot{\theta}\hat{\boldsymbol{\omega}}_b$ [1]

ω_b: Body-frame angular velocity
(angular velocity expressed in $\{b\}$)

$$\boldsymbol{\omega}_s = \boldsymbol{R}\boldsymbol{\omega}_b$$

$$\boldsymbol{\omega}_b = \boldsymbol{R}^{-1}\boldsymbol{\omega}_s = \boldsymbol{R}^T\boldsymbol{\omega}_s$$

$$\begin{aligned}[\boldsymbol{\omega}_b] &= [\boldsymbol{R}^T\boldsymbol{\omega}_s] \\ &= \boldsymbol{R}^T[\boldsymbol{\omega}_s]\boldsymbol{R} \\ &= \boldsymbol{R}^T(\dot{\boldsymbol{R}}\boldsymbol{R}^T)\boldsymbol{R} \\ &= \boldsymbol{R}^T\dot{\boldsymbol{R}} = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\end{aligned}$$

Recall: $\boldsymbol{R}[\boldsymbol{x}]\boldsymbol{R}^T = [\boldsymbol{R}\boldsymbol{x}]$.

$$\Rightarrow \quad \boxed{[\boldsymbol{\omega}_b] = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = \boldsymbol{R}^T\dot{\boldsymbol{R}}}$$

Note: ω_b does not depend on the choice of fixed frame $\{s\}$.

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[1] Any angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ can be represented by product of a unit instantaneous axis of rotation ($\hat{\boldsymbol{\omega}} \in S^2$) and the speed of rotation $\dot{\theta} \in \mathbb{R}$ about it, i.e., $\boldsymbol{\omega} = \|\boldsymbol{\omega}\| \, \boldsymbol{\omega}/\|\boldsymbol{\omega}\| = \dot{\theta}\hat{\boldsymbol{\omega}}$.

Example

Find $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_b$ for rotational motion of a one degree of freedom manipulator.

Exponential Coordinate Representation of Rotation

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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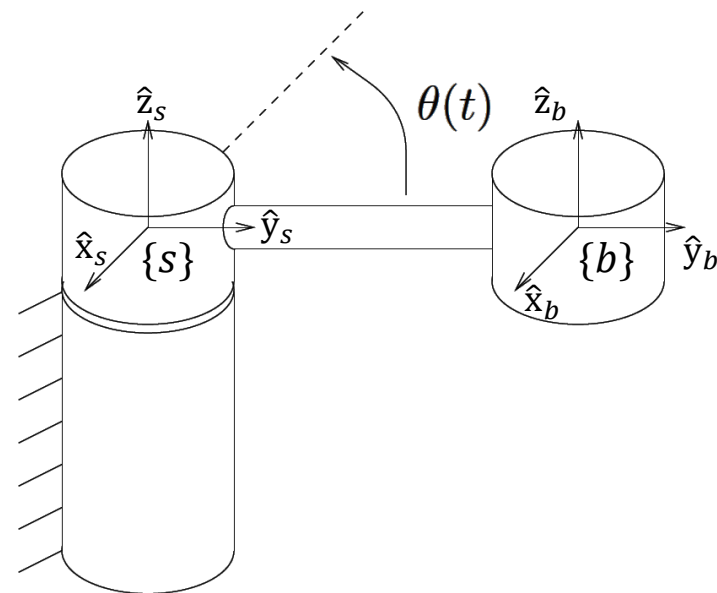
Matrix Exponential

Scalar Linear ODE:

$$\dot{x}(t) = ax(t)$$

$x(t) \in \mathbb{R}, a \in \mathbb{R}$ is constant

$x(0) = x_0$

$$x(t) = e^{at}x_0$$

Vector Linear ODE:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

$\boldsymbol{x}(t) \in \mathbb{R}^n, \boldsymbol{A} \in \mathbb{R}^{n \times n}$ is constant

$\boldsymbol{x}(0) = \boldsymbol{x}_0$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Properties of Matrix Exponential e^{At}

Taylor expansion of e^{At}: $e^{At} = I + At + \dfrac{(At)^2}{2!} + \dfrac{(At)^3}{3!} + \cdots$

$\forall A \in \mathbb{R}^{n \times n}, t \in \mathbb{R}$: $d(e^{At})/dt = Ae^{At} = e^{At}A$

If $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n \times n}$ and invertible $P \in \mathbb{R}^{n \times n}$: $e^{At} = Pe^{Dt}P^{-1}$

If $D \in \mathbb{R}^{n \times n}$ is diagonal, i.e., $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$: $e^{Dt} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}$

If $AB = BA$, then $e^A e^B = e^{A+B}$.

$\left(e^A\right)^{-1} = e^{-A}$

Reference Frames　　Rotation Matrices　　Angular Velocity　　**Exponential Coordinate Representation**　　Euler Angles　　Unit Quaternions

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Exponential Coordinates of Rotations

The vector $\boldsymbol{p}$ is rotated by an angle θ about the unit axis $\widehat{\boldsymbol{\omega}}$ to $\boldsymbol{p}'$. Thus, $\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}$. This rotation can be also achieved by imagining that $\boldsymbol{p}$ rotates at a constant rate of $\dot{\theta} = 1$ rad/s from time $t = 0$ to $t = \theta$ (all vectors are expressed in $\{s\}$).

$\boldsymbol{p} \equiv \boldsymbol{p}(0)$　　$\boldsymbol{p}(t)$

$\widehat{\boldsymbol{\omega}}$

θ

$\dot{\boldsymbol{p}}(t)$

$\boldsymbol{p}' \equiv \boldsymbol{p}(\theta)$

$\{s\}$
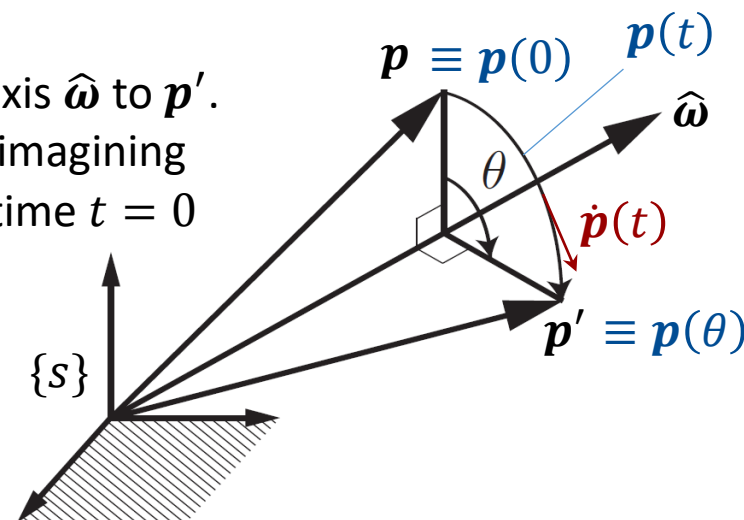
$$\dot{\boldsymbol{p}} = \dot{\theta}\widehat{\boldsymbol{\omega}} \times \boldsymbol{p}(t) = [\widehat{\boldsymbol{\omega}}]\boldsymbol{p}(t) \quad (\|\widehat{\boldsymbol{\omega}}\| = 1, \dot{\theta} = 1 \text{ rad/s})$$

$$\boldsymbol{p}(t) = e^{[\widehat{\boldsymbol{\omega}}]t}\boldsymbol{p}(0)$$

at $t = \theta$

$$\boldsymbol{p}(\theta) = e^{[\widehat{\boldsymbol{\omega}}]\theta}\boldsymbol{p}(0) \xrightarrow{\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}} \boldsymbol{R} = e^{[\widehat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) \in SO(3) \quad [\widehat{\boldsymbol{\omega}}]\theta = [\widehat{\boldsymbol{\omega}}\theta] \in so(3)$$

$\Rightarrow$ Any rotation matrix $\boldsymbol{R} \in SO(3)$ can be obtained by rotating from the identity matrix $\boldsymbol{I}_3$ about a unit rotation axis $\widehat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\widehat{\boldsymbol{\omega}}\| = 1$) by an angle of rotation $\theta \in \mathbb{R}$ about that axis. This motivates a three-parameter representation of a rotation $\boldsymbol{R}$ called the **exponential coordinates** as $\widehat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ (equivalently, $\widehat{\boldsymbol{\omega}}$ and θ can be written individually as the **axis-angle representation** of a rotation).

Note: The angle θ is taken to be <u>positive</u> if the rotation is made based on right hand rule:

positive rotation　$\widehat{\boldsymbol{\omega}}$　$+\theta$
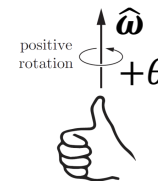
Remarks: Minimal Representation

❖ **Rotation Matrix** give an implicit redundant description of orientation; in fact, they are characterized by 9 elements which are not independent but related by 6 constraints due to the orthogonality conditions. This implies that three parameters are sufficient to describe orientation of a rigid body in space.

❖ A representation of orientation in terms of 3 independent parameters is called an **Explicit** or **Minimal Representation**. This involves **Exponential Coordinates** and **Euler Angles**. This three-parameter representation is prone to **representation singularities**.

❖ A major advantage of using rotation matrix is that it is **singularity-free** and simplifies the use of **linear algebra operations**.

Exponential Coordinates of Rotations

For any rotation matrix $\boldsymbol{R} \in SO(3)$, we can always find a unit rotation axis $\widehat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\widehat{\boldsymbol{\omega}}\| = 1$) and scalar $\theta \in \mathbb{R}$ such that $\boldsymbol{R} = e^{[\widehat{\boldsymbol{\omega}}]\theta}$.

$$\text{exp:} \quad [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \;\rightarrow\; \boldsymbol{R} \in SO(3) \quad : \quad e^{[\widehat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R}$$

$$\text{log:} \quad \boldsymbol{R} \in SO(3) \quad\;\; \rightarrow\; [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \quad : \quad \log(\boldsymbol{R}) = [\widehat{\boldsymbol{\omega}}]\theta$$

$\widehat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$: Exponential coordinates of $\boldsymbol{R} \in SO(3)$

$[\widehat{\boldsymbol{\omega}}]\theta = [\widehat{\boldsymbol{\omega}}\theta] \in so(3)$: Matrix logarithm of $\boldsymbol{R}$ (inverse of the matrix exponential)

Note: $\boldsymbol{R}$ and $\widehat{\boldsymbol{\omega}}$ have the same base.

Matrix Exponential

$$\boxed{\text{exp:} \quad [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \;\rightarrow\; \boldsymbol{R} \in SO(3) \quad : e^{[\widehat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R}}$$

❖ Finding $\boldsymbol{R}$ by having $\widehat{\boldsymbol{\omega}}$ and θ:

$$e^{[\widehat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + [\widehat{\boldsymbol{\omega}}]\theta + [\widehat{\boldsymbol{\omega}}]^2 \frac{\theta^2}{2!} + [\widehat{\boldsymbol{\omega}}]^3 \frac{\theta^3}{3!} + \cdots$$

$$= \boldsymbol{I} + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta} [\widehat{\boldsymbol{\omega}}] + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}_{1-\cos\theta} [\widehat{\boldsymbol{\omega}}]^2$$

$$\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = e^{[\widehat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + \sin\theta[\widehat{\boldsymbol{\omega}}] + (1 - \cos\theta)[\widehat{\boldsymbol{\omega}}]^2 \quad \textcolor{red}{\text{(Rodrigues' formula for rotations)}}$$

$$\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1-c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1-c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1-c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1-c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1-c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1-c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1-c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1-c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta, \, c_\theta = \cos\theta, \quad \widehat{\boldsymbol{\omega}} = (\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3)$$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Matrix Exponential: Remarks

- Since $\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \text{Rot}(-\widehat{\boldsymbol{\omega}}, -\theta)$, a rotation by $-\theta$ about $-\widehat{\boldsymbol{\omega}}$ cannot be distinguished from a rotation by θ about $\widehat{\boldsymbol{\omega}}$; hence, <u>Exponential Coordinate representation is not unique</u>.

- The inverse (or transpose) of a rotation matrix $\boldsymbol{R} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)$ corresponds to a rotation by the negative of the original angle, but the same axis of rotation, i.e., $\boldsymbol{R}^T = \boldsymbol{R}^{-1} = \text{Rot}(\widehat{\boldsymbol{\omega}}, -\theta)$.

- If $\boldsymbol{r}$ and $\widehat{\boldsymbol{\omega}}$ are aligned and $\boldsymbol{R} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)$, then, $\boldsymbol{Rr} = \boldsymbol{R}^T\boldsymbol{r} = \boldsymbol{r}$.

- For a given rotation matrix $\boldsymbol{R}_{sb}$:

Fixed-frame Rotation is rotation by θ about a unit axis $\widehat{\boldsymbol{\omega}}_s$, expressed in fixed frame $\{s\}$ as:

$$\boldsymbol{R}_{sb'} = \text{Rot}(\widehat{\boldsymbol{\omega}}_s, \theta)\boldsymbol{R}_{sb} = e^{[\widehat{\boldsymbol{\omega}}_s]\theta}\boldsymbol{R}_{sb}$$

Body-frame Rotation is rotation by θ about a unit axis $\widehat{\boldsymbol{\omega}}_b$, expressed in body frame $\{b\}$ as:

$$\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\text{Rot}(\widehat{\boldsymbol{\omega}}_b, \theta) = \boldsymbol{R}_{sb}e^{[\widehat{\boldsymbol{\omega}}_b]\theta}$$

$(\widehat{\boldsymbol{\omega}}_s = \boldsymbol{R}_{sb}\widehat{\boldsymbol{\omega}}_b)$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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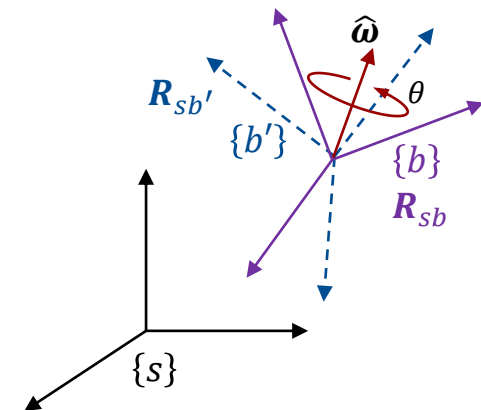
Example

The frame $\{b\}$ is obtained by a rotation from $\{s\}$ by $\theta_1 = 30°$ about $\hat{\boldsymbol{\omega}}_1 = (0, 0.866, 0.5)$.
Find the rotation matrix representation of $\{b\}$.
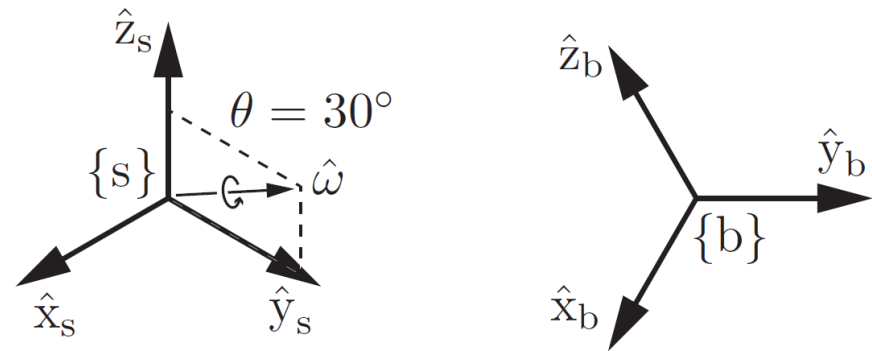
Find the new rotation matrix if $\{b\}$ is then rotated by θ_2 about
(a) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{s\}$.
(b) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{b\}$.

Matrix Logarithm

$$\log: \qquad \boldsymbol{R} \in SO(3) \qquad \rightarrow \qquad [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \quad : \qquad \log(\boldsymbol{R}) = [\widehat{\boldsymbol{\omega}}]\theta$$

❖ Finding $\widehat{\boldsymbol{\omega}}$ and $\theta \in [0, \pi]$ by having $\boldsymbol{R}$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1 - c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1 - c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

By inspection:

- $\operatorname{tr} \boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$

- $\dfrac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & -\widehat{\omega}_3 & \widehat{\omega}_2 \\ \widehat{\omega}_3 & 0 & -\widehat{\omega}_1 \\ -\widehat{\omega}_2 & \widehat{\omega}_1 & 0 \end{bmatrix} = [\widehat{\boldsymbol{\omega}}]$

- $\boldsymbol{R}|_{\theta=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}_3$ • $\boldsymbol{R}|_{\theta=\pi} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -1 + 2\widehat{\omega}_1^2 & 2\widehat{\omega}_1\widehat{\omega}_2 & 2\widehat{\omega}_1\widehat{\omega}_3 \\ 2\widehat{\omega}_1\widehat{\omega}_2 & -1 + 2\widehat{\omega}_2^2 & 2\widehat{\omega}_2\widehat{\omega}_3 \\ 2\widehat{\omega}_1\widehat{\omega}_3 & 2\widehat{\omega}_2\widehat{\omega}_3 & -1 + 2\widehat{\omega}_3^2 \end{bmatrix}$

Therefore, we can propose an algorithm to determine $\widehat{\boldsymbol{\omega}}$ and θ.

Reference Frames　　Rotation Matrices　　Angular Velocity　　**Exponential Coordinate Representation**　　Euler Angles　　Unit Quaternions

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Matrix Logarithm: Algorithm

(a) If $\text{tr}\,\boldsymbol{R} = 3$ or $\boldsymbol{R} = \boldsymbol{I}_3$ (null rotation), then $\theta = 0$ and $\widehat{\boldsymbol{\omega}}$ is undefined/arbitrary (singularity).

(b) If $\text{tr}\,\boldsymbol{R} = -1$, then $\theta = \pi$ and $\widehat{\boldsymbol{\omega}}$ is equal to any of the three vectors that is a feasible solution:

$$\widehat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \quad \text{or} \quad \widehat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix} \quad \text{or} \quad \widehat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}$$

(Note: In this case, if $\widehat{\boldsymbol{\omega}}$ is a solution, then so is $-\widehat{\boldsymbol{\omega}}$)

(c) Otherwise,　　$$\theta = \cos^{-1}\left(\frac{1}{2}(\text{tr}\,\boldsymbol{R} - 1)\right) = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \in (0, \pi)$$

$$[\widehat{\boldsymbol{\omega}}] = \frac{1}{2\sin\theta}(\boldsymbol{R} - \boldsymbol{R}^{\text{T}}) \quad \Rightarrow \quad \widehat{\boldsymbol{\omega}} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

Reference Frames ○○ | Rotation Matrices ○○○○○○○○○○○○ | Angular Velocity ○○○○ | **Exponential Coordinate Representation** ○○○○○○○○○○● | Euler Angles ○○○○○ | Unit Quaternions ○○○○○
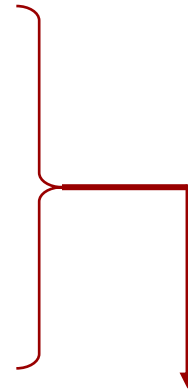
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Remark

Since the formula for finding θ provides only values in $(0, \pi)$ (never negative angles) and numerical accuracy with $\cos^{-1}(\cdot)$ for $\theta \to 0$ is low, we can use another formula based on atan2.

- $\text{tr } \boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$

- $\dfrac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & -\widehat{\omega}_3 & \widehat{\omega}_2 \\ \widehat{\omega}_3 & 0 & -\widehat{\omega}_1 \\ -\widehat{\omega}_2 & \widehat{\omega}_1 & 0 \end{bmatrix} = [\widehat{\boldsymbol{\omega}}]$

- $\|\widehat{\boldsymbol{\omega}}\|_2 = 1$

$$\theta = \text{atan2}\left\{\pm\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1\right\}$$

Note: Presence of division by $\sin\theta$ express the singularity in this three-parameter representation.

Reference Frames
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Rotation Matrices
OOOOOOOOOOOO

Angular Velocity
OOOO

Exponential Coordinate Representation
OOOOOOOOOOOO

Euler Angles
OOOOO

Unit Quaternions
OOOOO

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Euler Angles

Euler Angles

Another minimal representation of orientation can be obtained by using a set of three angles (α, β, γ). It composes <u>a suitable sequence of three elementary rotations</u>, each about one of the coordinate axes of fixed frame $\{s\}$ or body/current frame $\{b\}$, while guaranteeing that <u>two successive rotations are not made about parallel axes</u>.

This implies that 12 (3x2x2) distinct sets (triplet) of Euler angles are allowed out of all 27 (3x3x3) possible combinations.

Z-X-Z, X-Y-X, Y-Z-Y, Z-Y-Z, X-Z-X, Y-X-Y, X-Y-Z, Y-Z-X, Z-X-Y, X-Z-Y, Z-Y-X, Y-X-Z

Two Examples:
- ZYX Euler angles (with rotations about the body/current frame $\{b\}$).
- XYZ Euler angles (with rotations about the fixed frame $\{s\}$). This is also called **Roll–Pitch–Yaw** (**RPY**) angles.

Euler Angles ZYX
(about the body/current frame)

ZYX Euler angles (with rotations about the body/current frame $\{b\}$):

- Rotation by α about the body $\hat{\boldsymbol{z}}_b$,
- then by β about the body $\hat{\boldsymbol{y}}_b'$, and
- finally, by γ about the body $\hat{\boldsymbol{x}}_b''$.
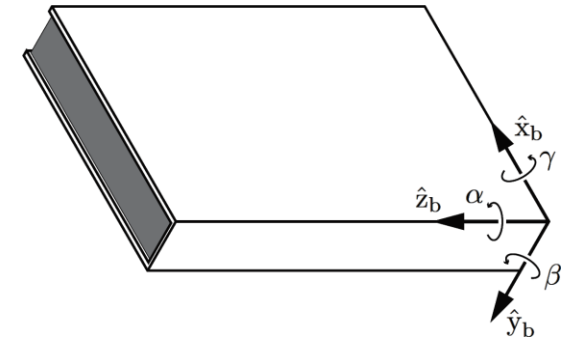
$$\boldsymbol{R}(\alpha, \beta, \gamma) = \boldsymbol{I}_3 \mathrm{Rot}(\hat{\boldsymbol{z}}_b, \alpha) \mathrm{Rot}(\hat{\boldsymbol{y}}_b', \beta) \mathrm{Rot}(\hat{\boldsymbol{x}}_b'', \gamma)$$

$$\mathrm{Rot}(\hat{\boldsymbol{x}}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}, \mathrm{Rot}(\hat{\boldsymbol{y}}, \beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\boldsymbol{z}}, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$
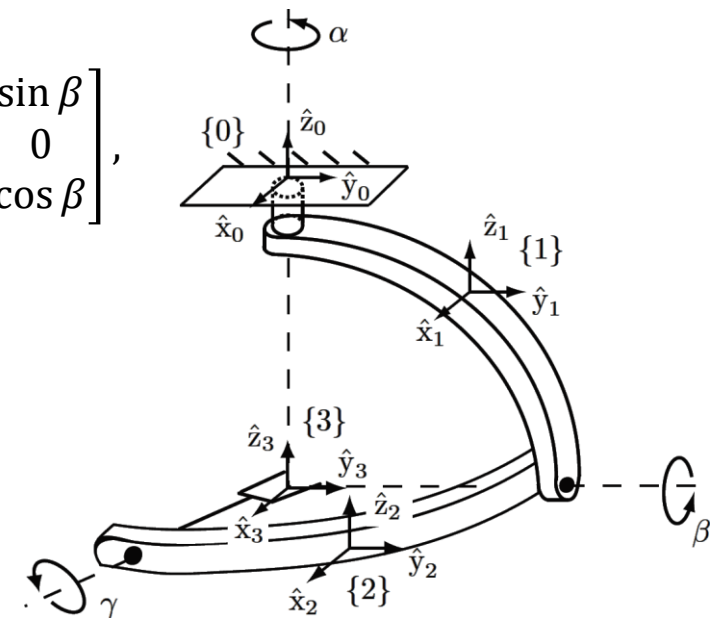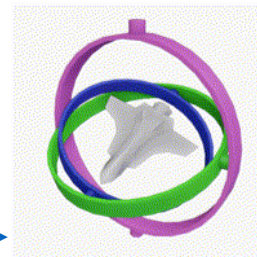
Euler Angles ZYX
(about the body/current frame)

Finding (α, β, γ) for any given rotation matrix $\boldsymbol{R} \in SO(3)$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

- If $r_{31} \neq \pm 1$ (i.e., when $\beta \in (-\pi/2, \pi/2)$):

$$\beta = \operatorname{atan}2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\alpha = \operatorname{atan}2(r_{21}, r_{11})$$

$$\gamma = \operatorname{atan}2(r_{32}, r_{33})$$

- If $r_{31} = -1$, then $\beta = \pi/2$, and if $r_{31} = 1$, then $\beta = -\pi/2$. In these singular cases, $\hat{\boldsymbol{z}}_b$ and $\hat{\boldsymbol{x}}_b$ axes are parallel, and it is possible to determine only the sum or difference of α and γ.

Singularity of the Euler angles:
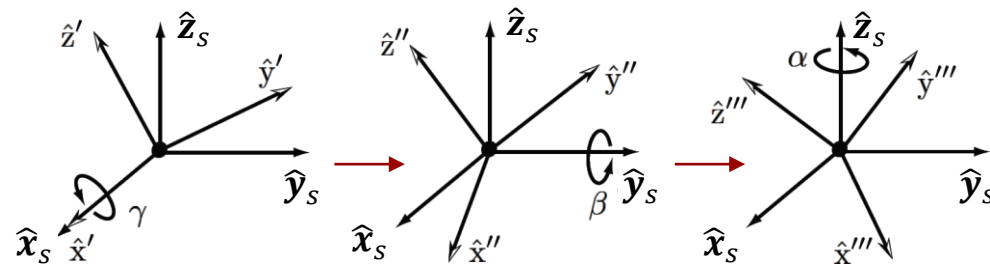
▶   ▶   (Gimbal lock)
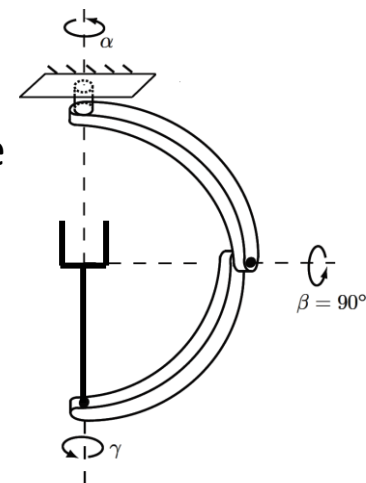
Roll–Pitch–Yaw or RPY Angles (XYZ)
(about the fixed frame)

XYZ Euler angles (with rotations about the fixed frame $\{s\}$):

- Rotation by γ about the fixed $\hat{\boldsymbol{x}}_s$,
- then by β about the fixed $\hat{\boldsymbol{y}}_s$, and
- finally, by α about the fixed $\hat{\boldsymbol{z}}_s$.

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \text{Rot}(\hat{\boldsymbol{z}}_s, \alpha)\, \text{Rot}(\hat{\boldsymbol{y}}_s, \beta)\, \text{Rot}(\hat{\boldsymbol{x}}_s, \gamma)\, \boldsymbol{I}_3$$

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$
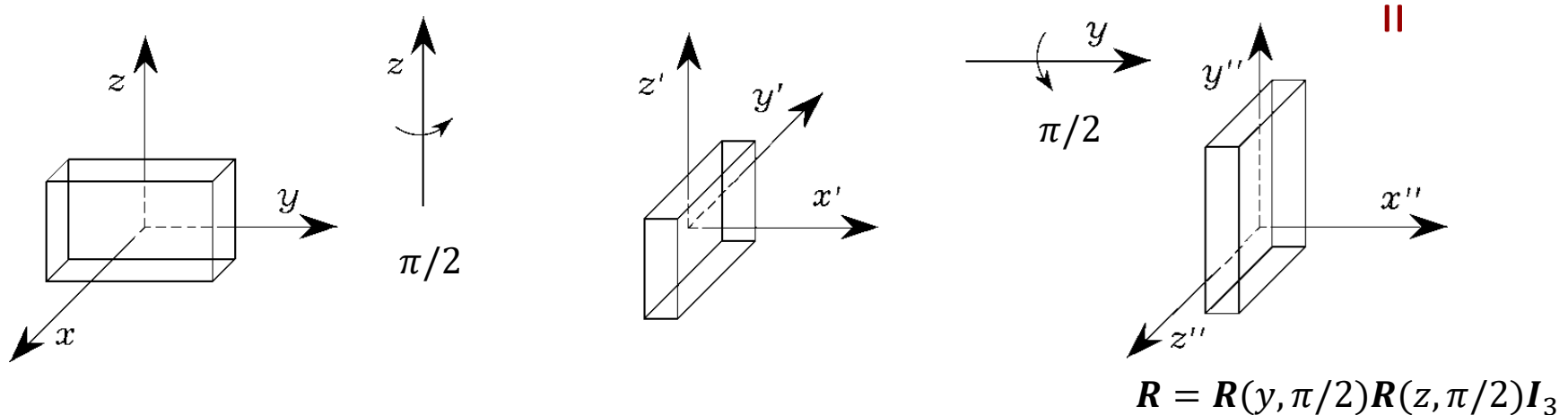
This product of three rotations XYZ Euler angles with rotations about the fixed frame $\{s\}$ is the same as that for the ZYX Euler angles with rotations about the body/current frame $\{b\}$, i.e., the same product of three rotations admits two different physical interpretations.

Successive Rotations about Axes of Fixed & Body/Current Frames

Body-frame rotation:

$$\boldsymbol{R} = \boldsymbol{I}_3 \boldsymbol{R}(y, \pi/2) \boldsymbol{R}(z', \pi/2)$$

Fixed-frame rotation:

$$\boldsymbol{R} = \boldsymbol{R}(y, \pi/2) \boldsymbol{R}(z, \pi/2) \boldsymbol{I}_3$$

Reference Frames
OO

Rotation Matrices
OOOOOOOOOOOO

Angular Velocity
OOOO

Exponential Coordinate Representation
OOOOOOOOOOOO

Euler Angles
OOOOO

Unit Quaternions
OOOOO

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Unit Quaternions

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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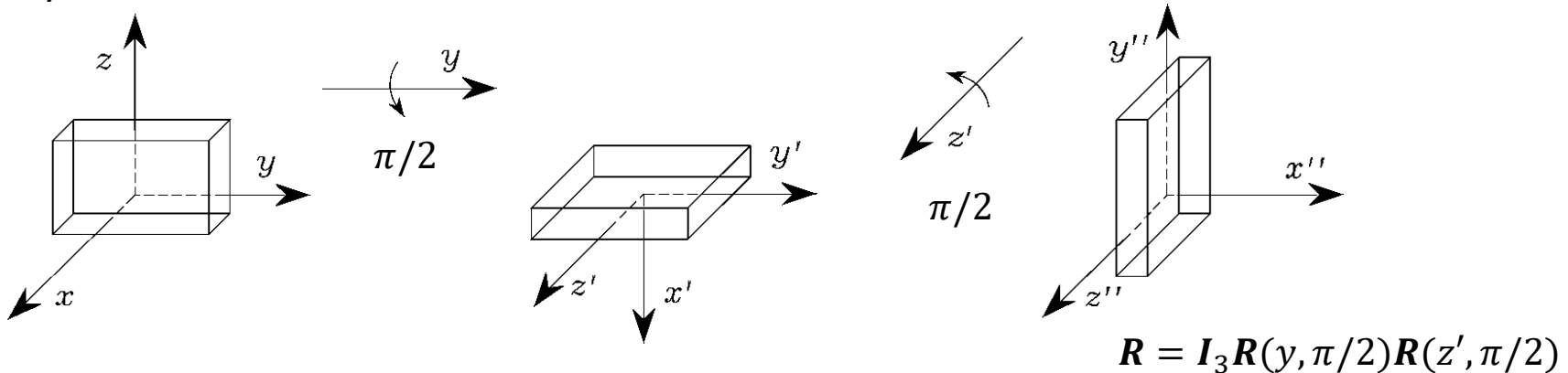
Quaternions

The **Quaternions** are the set of hypercomplex numbers. A quaternion $q \in \mathbb{H}$ can be represented as $q = q_0 + q_c = q_0 + q_1 i + q_2 j + q_3 k$ or a 4-tuple $q = (q_0, q_v) = (q_0, q_1, q_2, q_3)$ where $\mathbb{H}$ is set of all quaternions, $q_0 \in \mathbb{R}$ is the real scalar part, q_c is the imaginary vector part, $q_v = (q_1, q_2, q_3) \in \mathbb{R}^3$ is a real vector, and i, j, k are quaternion units or imaginary orthogonal axes that obey Hamilton's rule $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$.

- Addition and multiplication of two quaternions $p = (p_0, p_v)$ and $q = (q_0, q_v)$:
$$p + q = (p_0 + q_0, p_v + q_v)$$
$$pq = (p_0 q_0 - p_v \cdot q_v, p_0 q_v + q_0 p_v + p_v \times q_v) \neq qp \quad \text{(not commutative)}$$

- Conjugate of q: $\quad q^* = (q_0, -q_v)$

- Norm of q: $\quad \|q\| = \sqrt{q^* q} = \sqrt{q q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

- Inverse of q: $\quad q^{-1} = \dfrac{q^*}{\|q\|^2}$ $\quad$ • $p \cdot q = \text{Re}\left(\frac{1}{2}(pq^* + qp^*)\right) = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3$

The quaternion q is a **Unit Quaternion** if $\|q\| = 1$, and consequently, $q^{-1} = q^*$.

Reference Frames ○○ | Rotation Matrices ○○○○○○○○○○○○ | Angular Velocity ○○○○ | Exponential Coordinate Representation ○○○○○○○○○○○ | Euler Angles ○○○○○ | Unit Quaternions ○●○○○

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Unit Quaternion

Unit Quaternion (a.k.a. **Euler Parameters**) is a nonminimal four-parameter representation of rotation that alleviates the exponential coordinates singularity (division by $\sin\theta$) and Euler angle singularity (Gimbal lock), but at the cost of four variables subject to one constraint in the representation.

Let $\boldsymbol{R} \in SO(3)$ have the exponential coordinate representation $\hat{\boldsymbol{\omega}}\theta$, i.e., $\boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta}$, where $\|\hat{\boldsymbol{\omega}}\| = 1$ and $\theta = [0, \pi]$. The Unit Quaternion is defined as

$$\text{scalar part} \leftarrow \qquad \text{vector part} \leftarrow \qquad \boldsymbol{q} = \begin{bmatrix} q_0 \\ \boldsymbol{q}_v \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \hat{\boldsymbol{\omega}}\sin(\theta/2) \end{bmatrix} \in S^3 \qquad \|\boldsymbol{q}\| = 1$$

- $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3)$ is interpreted as a rotation about the unit axis, in the direction of (q_1, q_2, q_3) by an angle $\theta = 2\cos^{-1} q_0$.

- If $-\pi \leq \theta \leq \pi$, then $q_0 \geq 0$, and if $\pi \leq \theta \leq 3\pi$, then $q_0 \leq 0$.

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternions
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Unit Quaternion: Finding R by Having q

The rotation matrix R corresponding to a given unit quaternion $q = (q_0, q_v) = (q_0, q_1, q_2, q_3) = \left(\cos(\theta/2), \widehat{\omega}\sin\left(\frac{\theta}{2}\right)\right)$ where $\|q\| = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$:

$$R = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1 - c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1 - c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2(q_0 q_3 + q_1 q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}$$

$$= (2q_0^2 - 1)I_3 + 2q_v q_v^T + 2q_0[q_v]$$

Note: Finding $\widehat{\omega}$ and θ by having $q = (q_0, q_v)$: $\widehat{\omega} = \dfrac{q_v}{\|q_v\|}, \qquad \theta = 2\text{atan2}(\|q_v\|, q_0)$

Unit Quaternion: Finding q by Having R

The unit quaternion q corresponding to a given rotation matrix R: $\qquad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} = \frac{1}{2}\sqrt{1 + \operatorname{tr} R}$$

$$q_v = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

Notes:

- It has been implicitly assumed $q_0 \geq 0$; this corresponds to an angle $\theta \in [-\pi, \pi]$, and thus, any rotation can be described.
- For every rotation matrix R there exists two unit-quaternion representations $\pm q$ that are **antipodal** to each other, i.e., $+q$ and $-q$ represent the same rotation R.
- Unlike the exponential coordinate representation, no singularity occurs in inverse solution.
- A rotation by $-\theta$ about $-\hat{\omega}$ gives the same quaternion q as that associated with a rotation by θ about $\hat{\omega}$.
- If the unit quaternion $q = (q_0, q_v)$ corresponds to a rotation matrix R, the unit quaternion extracted from $R^{-1} = R^T$ is denoted as q^{-1}, and can be computed as $q^{-1} = (q_0, -q_v)$.

Reference Frames
OO

Rotation Matrices
OOOOOOOOOOO

Angular Velocity
OOOO

Exponential Coordinate Representation
OOOOOOOOOOO

Euler Angles
OOOOO

Unit Quaternions
OOOOO●

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Unit Quaternion: Remarks

- Let $\boldsymbol{p} = (p_0, \boldsymbol{p}_v) = (p_0, p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3)$ denote the quaternions corresponding to the rotation matrices $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, respectively. The quaternion corresponding to the product $\boldsymbol{R}_n = \boldsymbol{R}_p \boldsymbol{R}_q$ is given by

$$\boldsymbol{n} = \boldsymbol{p}\boldsymbol{q} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 \\ p_1 q_0 + p_0 q_1 - p_3 q_2 + p_2 q_3 \\ p_2 q_0 + p_0 q_2 + p_3 q_1 - p_1 q_3 \\ p_3 q_0 + p_0 q_3 - p_2 q_1 + p_1 q_2 \end{bmatrix}$$

quaternion
product operator

$$= (p_0 q_0 - \boldsymbol{q}_v \cdot \boldsymbol{q}_v, \; p_0 \boldsymbol{q}_v + q_0 \boldsymbol{p}_v + \boldsymbol{p}_v \times \boldsymbol{q}_v)$$

- $\boldsymbol{q}\boldsymbol{q}^{-1} = (1, \boldsymbol{0})$ which is the identity element.

- The rotation of a point or vector $\boldsymbol{p} \in \mathbb{R}^3$ by the angle θ about a unit axis $\widehat{\boldsymbol{\omega}}$ is $\boldsymbol{p}' = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)\, \boldsymbol{p}$ which can be also determined using unit quaternions as $\boldsymbol{q}_{p'} = \boldsymbol{q}\boldsymbol{q}_p\boldsymbol{q}^{-1}$ where $\boldsymbol{q} = (q_0, \boldsymbol{q}_v)$ is unit quaternion representation of $\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)$, $\boldsymbol{q}^{-1} = (q_0, -\boldsymbol{q}_v)$, $\boldsymbol{q}_p = (0, \boldsymbol{p})$, and $\boldsymbol{q}_{p'} = (0, \boldsymbol{p}')$.

$$\boldsymbol{p}' = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)\, \boldsymbol{p} \quad \leftrightarrow \quad \boldsymbol{q}_{p'} = \boldsymbol{q}\boldsymbol{q}_p\boldsymbol{q}^{-1}$$