Ch4: Rigid-Body Motion – Part 1 (Rotation)

Reference Frames

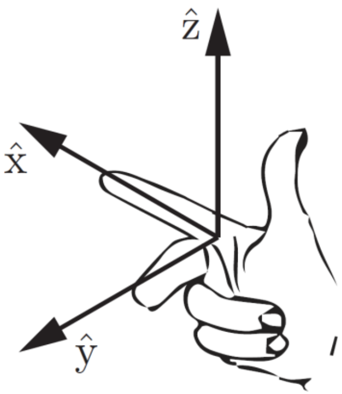
Reference Frames

- **Fixed Space Frame** $\{s\}$: A stationary, inertial frame and there is only one.
- **Body-attached Frame**: A frame fixed to a body and moves with it.
- **Body Frame** $\{b\}$: A stationary, inertial frame that is instantaneously coincident with the body-attached frame.

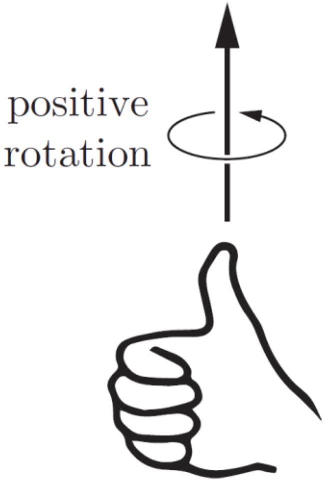
In this course, all frames are instantaneously stationary.

Reference Frames

All reference frames are **right-handed**.



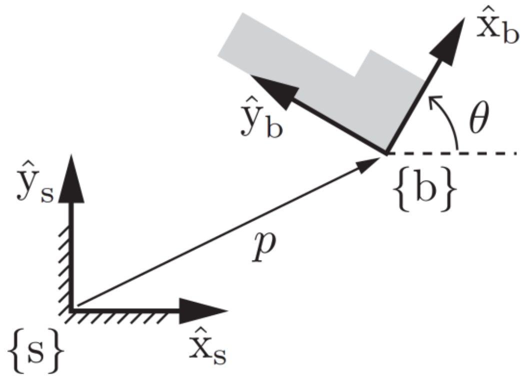
A **positive rotation** about an axis is defined as the direction in which the fingers of the right-hand curl when the thumb is pointed along the axis.



Rotation Matrices

Rotation in 2D Space

In 2D, the simplest way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by specifying the angle θ.



Another way is to express the unit axes $\hat{\boldsymbol{x}}_b$ and $\hat{\boldsymbol{y}}_b$ of frame $\{b\}$ in frame $\{s\}$.

$$\Rightarrow \quad \boldsymbol{R} = [\hat{\boldsymbol{x}}_b \quad \hat{\boldsymbol{y}}_b] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \in [0,2\pi)$$

Rotation Matrix

$\boldsymbol{R} := \boldsymbol{R}_{sb}$

Another notation for $\boldsymbol{R}_{sb}$: ${}^{s}\boldsymbol{R}_{b}$

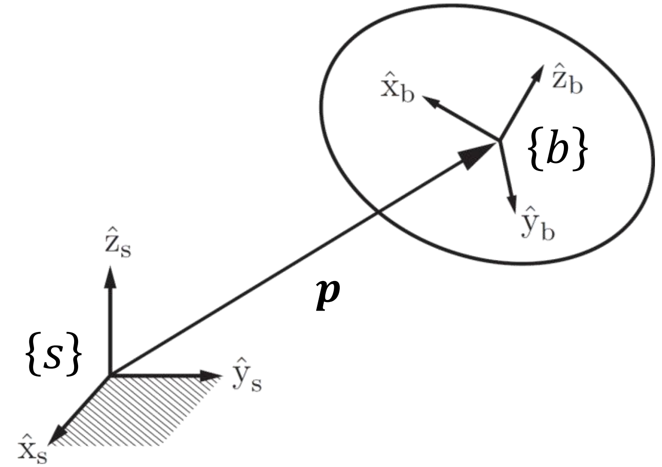
Rotation in 3D Space

In 3D, a way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by expressing the unit axes $\hat{\boldsymbol{x}}_b$, $\hat{\boldsymbol{y}}_b$, and $\hat{\boldsymbol{z}}_b$ of frame $\{b\}$ in frame $\{s\}$.

$$\boldsymbol{R} = [\hat{\boldsymbol{x}}_b \quad \hat{\boldsymbol{y}}_b \quad \hat{\boldsymbol{z}}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Rotation Matrix

This is as implicit representation the C-space.



$\boldsymbol{R} := \boldsymbol{R}_{sb}$

Another notation for $\boldsymbol{R}_{sb}$: ${}^{s}\boldsymbol{R}_b$

Constraints on Rotation Matrix

1- The unit norm condition: $\hat{\boldsymbol{x}}_b$, $\hat{\boldsymbol{y}}_b$, and $\hat{\boldsymbol{z}}_b$ are all unit vectors.

2- The orthogonality condition: $\hat{\boldsymbol{x}}_b.\hat{\boldsymbol{y}}_b = \hat{\boldsymbol{x}}_b.\hat{\boldsymbol{z}}_b = \hat{\boldsymbol{y}}_b.\hat{\boldsymbol{z}}_b = 0$

Compact form of all these six constraints: $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3$

For right-handed frames: $\det(\boldsymbol{R}) = 1$

Definition of a Group

A **group** is a set of elements $G = \{a, b, c, \ldots\}$ and a binary operation $\bullet$ on any two elements satisfying

- **Closure**: $a \bullet b \in G \quad \forall a, b \in G$
- **Associativity**: $(a \bullet b) \bullet c = a \bullet (b \bullet c) \quad \forall a, b, c \in G$
- **Identity Element Existence**: $\exists I \in G$ such that $a \bullet I = I \bullet a = a \quad \forall a \in G$
- **Inverse Element Existence**: $\forall a \in G, \exists a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = I$

Special Orthogonal Group $SO(3)$

The **special orthogonal group** $SO(3)$, also known as the (Lie) group of rotation matrices, is the set of all 3×3 real matrices $\boldsymbol{R}$ that satisfy **(i)** $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3$ and **(ii)** $\det(\boldsymbol{R}) = 1$.

orthogonal

special

$SO(2)$ is a subgroup of $SO(3)$: $SO(2) \subset SO(3)$

$\boldsymbol{R} \in SO(3)$

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times3} | \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}_3, \det(\boldsymbol{R}) = 1\}$$

Properties of Rotation Matrices

$SO(3)$ (or $SO(2)$) is a **matrix (Lie) group** (and the group operation • is matrix multiplication).

- **Closure**: $\boldsymbol{R}_1\boldsymbol{R}_2 \in SO(3)$
- **Associative**: $(\boldsymbol{R}_1\boldsymbol{R}_2)\boldsymbol{R}_3 = \boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{R}_3)$ (but generally not commutative, $\boldsymbol{R}_1\boldsymbol{R}_2 \neq \boldsymbol{R}_2\boldsymbol{R}_1$)
- **Identity**: $\exists \boldsymbol{I}_3 \in SO(3)$ such that $\boldsymbol{R}\boldsymbol{I}_3 = \boldsymbol{I}_3\boldsymbol{R} = \boldsymbol{R}$
- **Inverse**: $\exists\ \boldsymbol{R}^{-1} \in SO(3)$ such that $\boldsymbol{R}\boldsymbol{R}^{-1} = \boldsymbol{R}^{-1}\boldsymbol{R} = \boldsymbol{I}_3$ $\quad(\Rightarrow \boldsymbol{R}^{-1} = \boldsymbol{R}^T)$

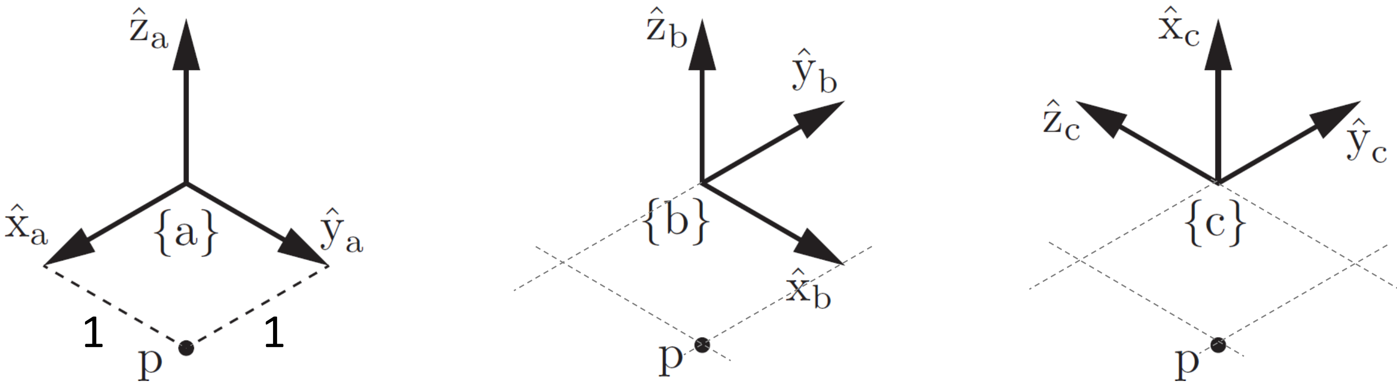
* For any vector $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{R} \in \mathrm{SO}(3)$, the vector $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x}$ has the same length as $\boldsymbol{x}$ ($\|\boldsymbol{x}\| = \|\boldsymbol{R}\boldsymbol{x}\|$).

Uses of Rotation Matrices (1)

(1) Representing orientation of a frame relative to another frame.

Notation: $\boldsymbol{R}_{bc}$ is the orientation of $\{c\}$ relative to $\{b\}$.

Example:
Find $\boldsymbol{R}_{ab}$ and $\boldsymbol{R}_{ac}$.
(All frames have the same origin)



Uses of Rotation Matrices (2)

(2) Changing the reference frame of a vector or frame.

Subscript Cancellation Rule:

$$\boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{p}_{\cancel{b}} = \boldsymbol{p}_a$$

$$\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{R}_{\cancel{b}c} = \boldsymbol{R}_{ac}$$

$\boldsymbol{R}_{ab}$ can be viewed as a **mathematical operator** that changes the reference frame from $\{b\}$ to $\{a\}$.

Note: $\boldsymbol{R}_{bc}\boldsymbol{R}_{cb} = \boldsymbol{I}_3$ or $\boldsymbol{R}_{bc} = \boldsymbol{R}_{cb}^T = \boldsymbol{R}_{cb}^{-1}$

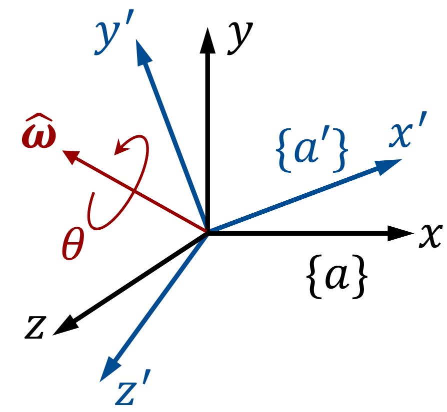
Example

Given $\boldsymbol{R}_1 = \boldsymbol{R}_{ab}$, $\boldsymbol{R}_2 = \boldsymbol{R}_{bc}$, and $\boldsymbol{R}_3 = \boldsymbol{R}_{ad}$, write $\boldsymbol{R}_{dc}$ in terms of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{R}_3$.

Given $\boldsymbol{p}_b$, what is $\boldsymbol{p}_d$ in terms of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{R}_3$?

Uses of Rotation Matrices (3)

(3) Rotating a <u>vector</u> or <u>frame</u> (about a unit axis $\hat{\boldsymbol{\omega}}$ by an amount θ).



$$\boldsymbol{R} = \boldsymbol{R}_{aa'} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

$\boldsymbol{R}$ can be viewed as a **<u>mathematical operator</u>** that rotates $\{a\}$ about a unit axis $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$ (expressed in $\{a\}$) by an amount θ to obtain $\{a'\}$.

$$\mathrm{Rot}(\hat{\boldsymbol{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \mathrm{Rot}(\hat{\boldsymbol{y}}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \mathrm{Rot}(\hat{\boldsymbol{z}}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta, c_\theta = \cos\theta$$

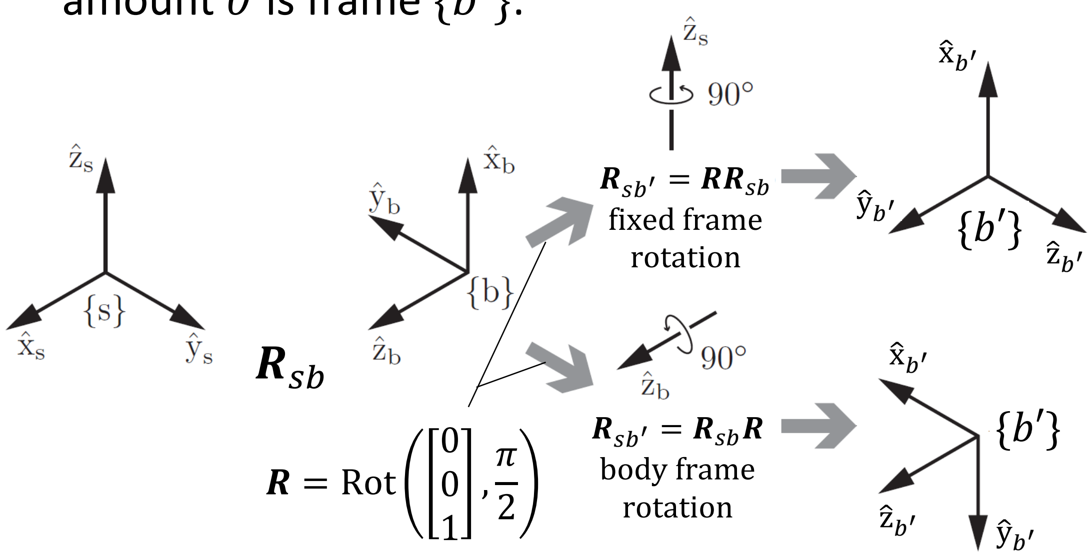
- $\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \mathrm{Rot}(-\hat{\boldsymbol{\omega}}, -\theta)$

Uses of Rotation Matrices (3) (cont.)

- Rotation of vector $\boldsymbol{v}$ about a unit axis $\hat{\boldsymbol{\omega}}$ (expressed in the same frame) by an amount θ is vector $\boldsymbol{v}'$ expressed in the same frame: $$\boldsymbol{v}' = \boldsymbol{R}\boldsymbol{v} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v}$$

- **Fixed-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{s\}$ by an amount θ is frame $\{b'\}$: $$\boldsymbol{R}_{sb'} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{R}_{sb}$$
  (pre-multiplication)
  Interpretation

- **Body-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{b\}$ by an amount θ is frame $\{b'\}$: $$\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$
  Interpretation
  (post-multiplication)



Angular Velocity

Set of Skew-Symmetric Matrices $so(3)$

The set of all 3×3 real skew-symmetric matrices is called $so(3)$ (which is the Lie algebra of the Lie group $SO(3)$).

$$so(3) = \{\mathbf{S} \in \mathbb{R}^{3\times3} | \mathbf{S}^T = -\mathbf{S}\} \qquad so(3) \subset \mathbb{R}^{3\times3}$$

$$\boldsymbol{x} \in \mathbb{R}^3 \qquad [\boldsymbol{x}] \in so(3)$$

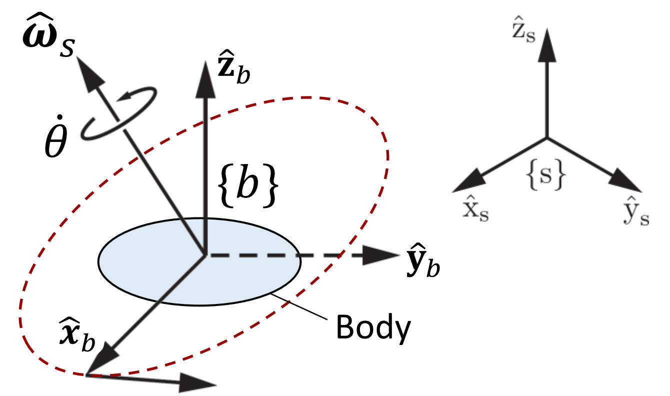
- Given any $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{R} \in SO(3)$, $\boldsymbol{R}[\boldsymbol{x}]\boldsymbol{R}^T = [\boldsymbol{R}\boldsymbol{x}]$.

- Given $[\boldsymbol{x}] \in so(3)$, $[\boldsymbol{x}]^2 = \boldsymbol{x}\boldsymbol{x}^T - \|\boldsymbol{x}\|^2 \boldsymbol{I}_3$ and $[\boldsymbol{x}]^3 = -\|\boldsymbol{x}\|^2[\boldsymbol{x}]$, and and higher powers of $[\boldsymbol{x}]$ can be calculated recursively.

Fixed-Frame Angular Velocity $\boldsymbol{\omega}_s$

Let's find the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of a rotating body (or body frame $\{b\}$) in terms of $\boldsymbol{R}_{sb} = \boldsymbol{R}(t)$. Body Frame $\{b\}$ is instantaneously coincident with the body-attached frame.

- If $\boldsymbol{\omega}$ is expressed in $\{s\}$: $\boldsymbol{\omega} = \boldsymbol{\omega}_s = \dot{\theta}\hat{\boldsymbol{\omega}}_s$ (1)



$\boldsymbol{R}(t) = [\hat{\boldsymbol{x}}_b \quad \hat{\mathbf{y}}_b \quad \hat{\mathbf{z}}_b]$: $\boldsymbol{R}_{sb}$ at time t

$\dot{\boldsymbol{R}}(t) = [\dot{\hat{\mathbf{x}}}_b \quad \dot{\hat{\mathbf{y}}}_b \quad \dot{\hat{\mathbf{z}}}_b]$: Time rate of change of $\boldsymbol{R}_{sb}$ at time t

$$\dot{\hat{\mathbf{x}}}_b = \boldsymbol{\omega}_s \times \hat{\boldsymbol{x}}_b$$
$$\dot{\hat{\mathbf{y}}}_b = \boldsymbol{\omega}_s \times \hat{\mathbf{y}}_b$$
$$\dot{\hat{\mathbf{z}}}_b = \boldsymbol{\omega}_s \times \hat{\mathbf{z}}_b$$

$\dot{\hat{\mathbf{x}}}_b = \boldsymbol{\omega}_s \times \hat{\boldsymbol{x}}_b$ (linear velocity of tip of vector)

$\boldsymbol{\omega}_s$: Fixed-frame angular velocity (angular velocity expressed in $\{s\}$)

$$\dot{\boldsymbol{R}} = [[\boldsymbol{\omega}_s]\hat{\boldsymbol{x}}_b \quad [\boldsymbol{\omega}_s]\hat{\mathbf{y}}_b \quad [\boldsymbol{\omega}_s]\hat{\mathbf{z}}_b] = [\boldsymbol{\omega}_s]\boldsymbol{R} \quad \Rightarrow \quad \boxed{[\boldsymbol{\omega}_s] = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = \dot{\boldsymbol{R}}\boldsymbol{R}^T}$$

Note: $\boldsymbol{\omega}_s$ does not depend on the choice of body frame $\{b\}$.

Body-Frame Angular Velocity $\boldsymbol{\omega}_b$

- If $\boldsymbol{\omega}$ is expressed in $\{b\}$: $\boldsymbol{\omega} = \boldsymbol{\omega}_b = \dot{\theta}\hat{\boldsymbol{\omega}}_b$ [(1)]

$\boldsymbol{\omega}_b$: Body-frame angular velocity (angular velocity expressed in $\{b\}$)

$$\boldsymbol{\omega}_s = \boldsymbol{R}\boldsymbol{\omega}_b$$

$$\boldsymbol{\omega}_b = \boldsymbol{R}^{-1}\boldsymbol{\omega}_s = \boldsymbol{R}^T\boldsymbol{\omega}_s$$

$$\begin{aligned}[\boldsymbol{\omega}_b] &= [\boldsymbol{R}^T\boldsymbol{\omega}_s] \\ &= \boldsymbol{R}^T[\boldsymbol{\omega}_s]\boldsymbol{R} \\ &= \boldsymbol{R}^T\left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right)\boldsymbol{R} \\ &= \boldsymbol{R}^T\dot{\boldsymbol{R}} = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\end{aligned}$$

Recall: $\boldsymbol{R}[\boldsymbol{x}]\boldsymbol{R}^T = [\boldsymbol{R}\boldsymbol{x}]$.

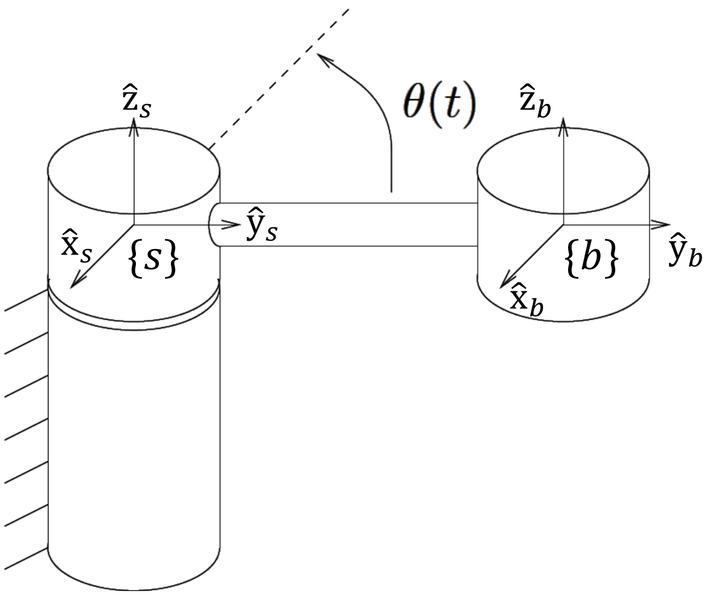
$$\Rightarrow \quad \boxed{[\boldsymbol{\omega}_b] = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = \boldsymbol{R}^T\dot{\boldsymbol{R}}}$$

Note: $\boldsymbol{\omega}_b$ does not depend on the choice of fixed frame $\{s\}$.

[(1)] Any angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ can be represented by product of a unit instantaneous axis of rotation ($\hat{\boldsymbol{\omega}} \in S^2$) and the speed of rotation $\dot{\theta} \in \mathbb{R}$ about it, i.e., $\boldsymbol{\omega} = \|\boldsymbol{\omega}\|\ \boldsymbol{\omega}/\|\boldsymbol{\omega}\| = \dot{\theta}\hat{\boldsymbol{\omega}}$.

Example

Find $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_b$ for rotational motion of a one degree of freedom manipulator.



Exponential Coordinate Representation of Rotation

Matrix Exponential

Scalar Linear ODE:

$$\dot{x}(t) = ax(t) \xrightarrow[x(0) = x_0]{x(t) \in \mathbb{R},\, a \in \mathbb{R} \text{ is constant}} x(t) = e^{at}x_0$$

Vector Linear ODE:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \xrightarrow[\boldsymbol{x}(0) = \boldsymbol{x}_0]{\boldsymbol{x}(t) \in \mathbb{R}^n,\, \boldsymbol{A} \in \mathbb{R}^{n\times n} \text{ is constant}} \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$$

Properties of Matrix Exponential $e^{\boldsymbol{A}t}$

Taylor expansion of $e^{\boldsymbol{A}t}$: $$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{(\boldsymbol{A}t)^2}{2!} + \frac{(\boldsymbol{A}t)^3}{3!} + \cdots$$

$\forall \boldsymbol{A} \in \mathbb{R}^{n\times n}, t \in \mathbb{R}$: $$d\left(e^{\boldsymbol{A}t}\right)/dt = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$$

If $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ for some $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ and invertible $\boldsymbol{P} \in \mathbb{R}^{n\times n}$: $$e^{\boldsymbol{A}t} = \boldsymbol{P}e^{\boldsymbol{D}t}\boldsymbol{P}^{-1}$$

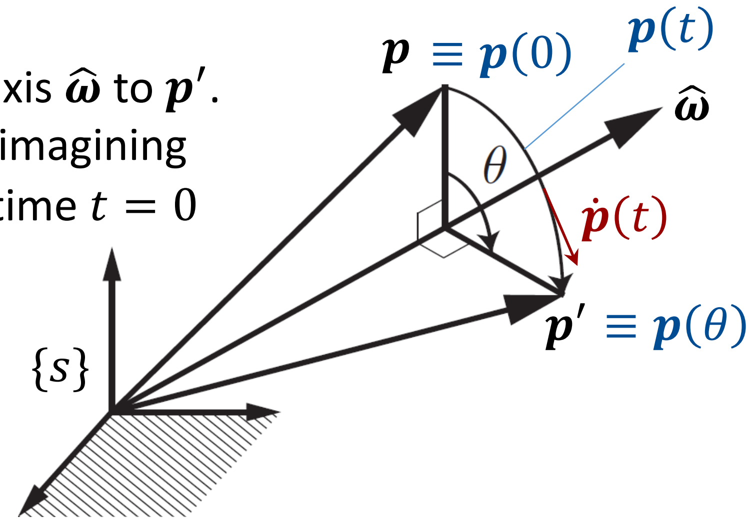
If $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ is diagonal, i.e., $\boldsymbol{D} = \mathrm{diag}\{d_1, d_2, \dots, d_n\}$: $$e^{\boldsymbol{D}t} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}$$

If $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$, then $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}}$.

$$\left(e^{\boldsymbol{A}}\right)^{-1} = e^{-\boldsymbol{A}}$$

Exponential Coordinates of Rotations

The vector $\boldsymbol{p}$ is rotated by an angle θ about the unit axis $\hat{\boldsymbol{\omega}}$ to $\boldsymbol{p}'$. Thus, $\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}$. This rotation can be also achieved by imagining that $\boldsymbol{p}$ rotates at a constant rate of $\dot{\theta} = 1$ rad/s from time $t = 0$ to $t = \theta$ (all vectors are expressed in $\{s\}$).



$$\dot{\boldsymbol{p}} = \dot{\theta}\hat{\boldsymbol{\omega}} \times \boldsymbol{p}(t) = [\hat{\boldsymbol{\omega}}]\boldsymbol{p}(t) \qquad (\|\hat{\boldsymbol{\omega}}\| = 1, \dot{\theta} = 1 \text{ rad/s})$$

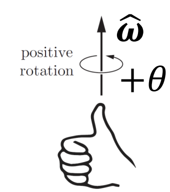
$$\boldsymbol{p}(t) = e^{[\hat{\boldsymbol{\omega}}]t}\boldsymbol{p}(0)$$

at $t = \theta$

$$\boldsymbol{p}(\theta) = e^{[\hat{\boldsymbol{\omega}}]\theta}\boldsymbol{p}(0) \xrightarrow{\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}} \boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) \in SO(3) \qquad [\hat{\boldsymbol{\omega}}]\theta = [\hat{\boldsymbol{\omega}}\theta] \in so(3)$$

$\Rightarrow$ Any rotation matrix $\boldsymbol{R} \in SO(3)$ can be obtained by rotating from the identity matrix $\boldsymbol{I}_3$ about a unit rotation axis $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{\omega}}\| = 1$) by an angle of rotation $\theta \in \mathbb{R}$ about that axis. This motivates a three-parameter representation of a rotation $\boldsymbol{R}$ called the **exponential coordinates** as $\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ (equivalently, $\hat{\boldsymbol{\omega}}$ and θ can be written individually as the **axis-angle representation** of a rotation).

Note: The angle θ is taken to be positive if the rotation is made based on right hand rule:



Remarks: Minimal Representation

- **Rotation Matrix** give an implicit redundant description of orientation; in fact, they are characterized by 9 elements which are not independent but related by 6 constraints due to the orthogonality conditions. This implies that three parameters are sufficient to describe orientation of a rigid body in space.

- A representation of orientation in terms of 3 independent parameters is called an **Explicit** or **Minimal Representation**. This involves **Exponential Coordinates** and **Euler Angles**. This three-parameter representation is prone to **representation singularities**.

- A major advantage of using rotation matrix is that it is **singularity-free** and simplifies the use of **linear algebra operations**.

Exponential Coordinates of Rotations

For any rotation matrix $\boldsymbol{R} \in SO(3)$, we can always find a unit rotation axis $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{\omega}}\| = 1$) and scalar $\theta \in \mathbb{R}$ such that $\boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta}$.

$$
\begin{array}{llcllcl}
\exp: & [\hat{\boldsymbol{\omega}}]\theta \in so(3) & \rightarrow & \boldsymbol{R} \in SO(3) & : & e^{[\hat{\boldsymbol{\omega}}]\theta} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R} \\
\log: & \boldsymbol{R} \in SO(3) & \rightarrow & [\hat{\boldsymbol{\omega}}]\theta \in so(3) & : & \log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta
\end{array}
$$

$\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$: Exponential coordinates of $\boldsymbol{R} \in SO(3)$

$[\hat{\boldsymbol{\omega}}]\theta = [\hat{\boldsymbol{\omega}}\theta] \in so(3)$: Matrix logarithm of $\boldsymbol{R}$ (inverse of the matrix exponential)

Note: $\boldsymbol{R}$ and $\hat{\boldsymbol{\omega}}$ have the same base.

Matrix Exponential

$$\text{exp:} \quad [\hat{\boldsymbol{\omega}}]\theta \in so(3) \; \rightarrow \; \boldsymbol{R} \in SO(3) \qquad : e^{[\hat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R}$$

❖ Finding $\boldsymbol{R}$ by having $\hat{\boldsymbol{\omega}}$ and θ:

$$e^{[\hat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + [\hat{\boldsymbol{\omega}}]\theta + [\hat{\boldsymbol{\omega}}]^2\frac{\theta^2}{2!} + [\hat{\boldsymbol{\omega}}]^3\frac{\theta^3}{3!} + \cdots$$

$$= \boldsymbol{I} + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta} [\hat{\boldsymbol{\omega}}] + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}_{1-\cos\theta} [\hat{\boldsymbol{\omega}}]^2$$

$$\text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = e^{[\hat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + \sin\theta[\hat{\boldsymbol{\omega}}] + (1-\cos\theta)[\hat{\boldsymbol{\omega}}]^2$$ (Rodrigues' formula for rotations)

$$\text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta,\; c_\theta = \cos\theta, \quad \hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$$

Matrix Exponential: Remarks

- Since $\text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \text{Rot}(-\hat{\boldsymbol{\omega}}, -\theta)$, a rotation by $-\theta$ about $-\hat{\boldsymbol{\omega}}$ cannot be distinguished from a rotation by θ about $\hat{\boldsymbol{\omega}}$; hence, Exponential Coordinate representation is not unique.

- The inverse (or transpose) of a rotation matrix $\boldsymbol{R} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$ corresponds to a rotation by the negative of the original angle, but the same axis of rotation, i.e., $\boldsymbol{R}^T = \boldsymbol{R}^{-1} = \text{Rot}(\hat{\boldsymbol{\omega}}, -\theta)$.

- If $\boldsymbol{r}$ and $\hat{\boldsymbol{\omega}}$ are aligned and $\boldsymbol{R} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$, then, $\boldsymbol{R}\boldsymbol{r} = \boldsymbol{R}^T\boldsymbol{r} = \boldsymbol{r}$.

- For a given rotation matrix $\boldsymbol{R}_{sb}$:

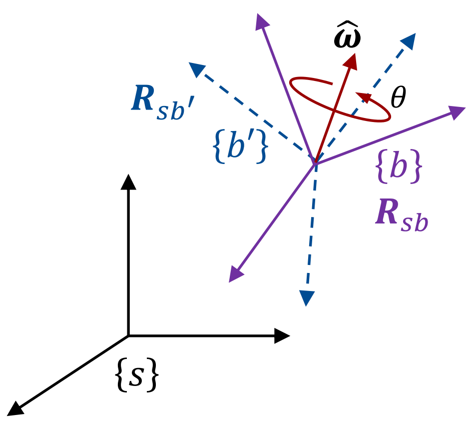
Fixed-frame Rotation is rotation by θ about a unit axis $\hat{\boldsymbol{\omega}}_s$, expressed in fixed frame $\{s\}$ as:

$$\boldsymbol{R}_{sb'} = \text{Rot}(\hat{\boldsymbol{\omega}}_s, \theta)\boldsymbol{R}_{sb} = e^{[\hat{\boldsymbol{\omega}}_s]\theta}\boldsymbol{R}_{sb}$$

Body-frame Rotation is rotation by θ about a unit axis $\hat{\boldsymbol{\omega}}_b$, expressed in body frame $\{b\}$ as:

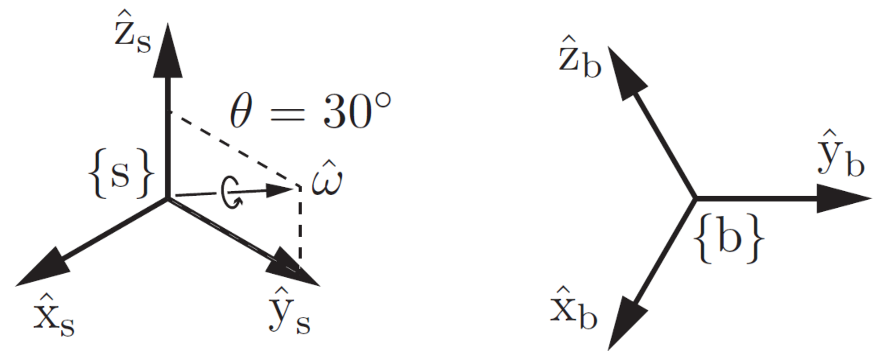
$$\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\text{Rot}(\hat{\boldsymbol{\omega}}_b, \theta) = \boldsymbol{R}_{sb}e^{[\hat{\boldsymbol{\omega}}_b]\theta}$$

$(\hat{\boldsymbol{\omega}}_s = \boldsymbol{R}_{sb}\hat{\boldsymbol{\omega}}_b)$



Example

The frame $\{b\}$ is obtained by a rotation from $\{s\}$ by $\theta_1 = 30°$ about $\hat{\boldsymbol{\omega}}_1 = (0, 0.866, 0.5)$. Find the rotation matrix representation of $\{b\}$.



Find the new rotation matrix if $\{b\}$ is then rotated by θ_2 about
(a) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{s\}$.
(b) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{b\}$.

Matrix Logarithm

$$\log: \quad \boldsymbol{R} \in SO(3) \quad \to \quad [\hat{\boldsymbol{\omega}}]\theta \in so(3) \quad : \quad \log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta$$

❖ Finding $\hat{\boldsymbol{\omega}}$ and $\theta \in [0, \pi]$ by having $\boldsymbol{R}$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

By inspection:

- $\operatorname{tr}\boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$
- $\frac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = [\hat{\boldsymbol{\omega}}]$
- $\boldsymbol{R}|_{\theta=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}_3$
- $\boldsymbol{R}|_{\theta=\pi} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -1+2\hat{\omega}_1^2 & 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_1\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & -1+2\hat{\omega}_2^2 & 2\hat{\omega}_2\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_3 & 2\hat{\omega}_2\hat{\omega}_3 & -1+2\hat{\omega}_3^2 \end{bmatrix}$

Therefore, we can propose an algorithm to determine $\hat{\boldsymbol{\omega}}$ and θ.

Matrix Logarithm: Algorithm

(a) If $\operatorname{tr}\boldsymbol{R} = 3$ or $\boldsymbol{R} = \boldsymbol{I}_3$ (null rotation), then $\theta = 0$ and $\hat{\boldsymbol{\omega}}$ is undefined/arbitrary (singularity).

(b) If $\operatorname{tr}\boldsymbol{R} = -1$, then $\theta = \pi$ and $\hat{\boldsymbol{\omega}}$ is equal to any of the three vectors that is a feasible solution:

$$\hat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \quad \text{or} \quad \hat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix} \quad \text{or} \quad \hat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}$$

(Note: In this case, if $\hat{\boldsymbol{\omega}}$ is a solution, then so is $-\hat{\boldsymbol{\omega}}$)

(c) Otherwise,

$$\theta = \cos^{-1}\left(\frac{1}{2}(\operatorname{tr}\boldsymbol{R} - 1)\right) = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \in (0, \pi)$$

$$[\hat{\boldsymbol{\omega}}] = \frac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) \quad \Rightarrow \quad \hat{\boldsymbol{\omega}} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

Remark

Since the formula for finding θ provides only values in $(0, \pi)$ (never negative angles) and numerical accuracy with $\cos^{-1}(\cdot)$ for $\theta \to 0$ is low, we can use another formula based on atan2.

- $\operatorname{tr} \boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$
- $\frac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & -\widehat{\omega}_3 & \widehat{\omega}_2 \\ \widehat{\omega}_3 & 0 & -\widehat{\omega}_1 \\ -\widehat{\omega}_2 & \widehat{\omega}_1 & 0 \end{bmatrix} = [\widehat{\boldsymbol{\omega}}]$
- $\|\widehat{\boldsymbol{\omega}}\|_2 = 1$

$$\theta = \operatorname{atan2}\left\{\pm\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1\right\}$$

Note: Presence of division by $\sin\theta$ express the singularity in this three-parameter representation.

Euler Angles

Euler Angles

Another minimal representation of orientation can be obtained by using a set of three angles (α, β, γ). It composes a suitable sequence of three elementary rotations, each about one of the coordinate axes of fixed frame $\{s\}$ or body/current frame $\{b\}$, while guaranteeing that two successive rotations are not made about parallel axes.

This implies that 12 (3x2x2) distinct sets (triplet) of Euler angles are allowed out of all 27 (3x3x3) possible combinations.

Z-X-Z, X-Y-X, Y-Z-Y, Z-Y-Z, X-Z-X, Y-X-Y, X-Y-Z, Y-Z-X, Z-X-Y, X-Z-Y, Z-Y-X, Y-X-Z

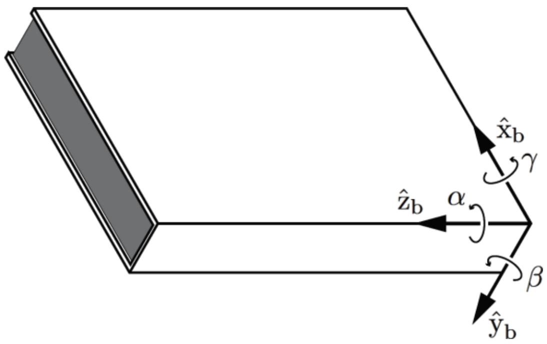
Two Examples:

- ZYX Euler angles (with rotations about the body/current frame $\{b\}$).
- XYZ Euler angles (with rotations about the fixed frame $\{s\}$). This is also called **Roll–Pitch–Yaw** (**RPY**) angles.

Euler Angles ZYX
(about the body/current frame)

ZYX Euler angles (with rotations about the body/current frame $\{b\}$):

- Rotation by α about the body $\hat{\boldsymbol{z}}_b$,
- then by β about the body $\hat{\boldsymbol{y}}'_b$, and
- finally, by γ about the body $\hat{\boldsymbol{x}}''_b$.

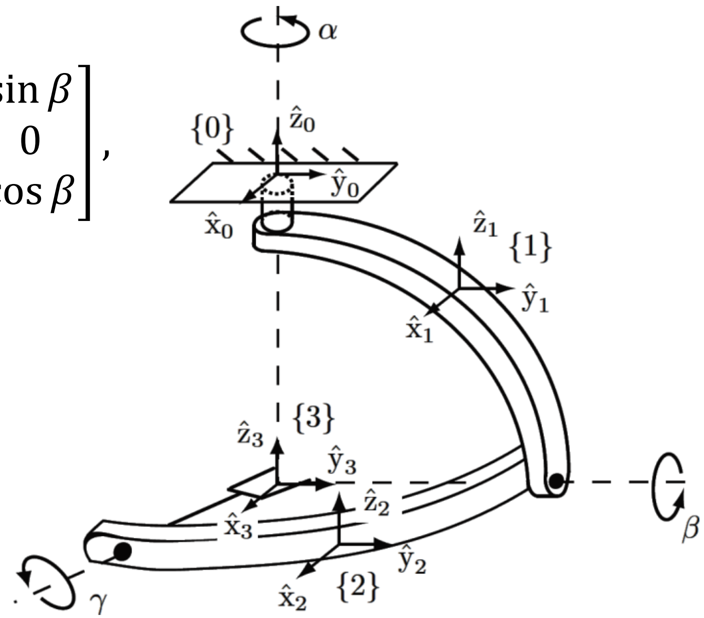


$$\boldsymbol{R}(\alpha,\beta,\gamma) = \boldsymbol{I}_3 \mathrm{Rot}(\hat{\boldsymbol{z}}_b,\alpha)\mathrm{Rot}(\hat{\boldsymbol{y}}'_b,\beta)\mathrm{Rot}(\hat{\boldsymbol{x}}''_b,\gamma)$$

$$\mathrm{Rot}(\hat{\boldsymbol{x}},\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}, \mathrm{Rot}(\hat{\boldsymbol{y}},\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\boldsymbol{z}},\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{R}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$



Euler Angles ZYX
(about the body/current frame)

Finding (α, β, γ) for any given rotation matrix $\boldsymbol{R} \in SO(3)$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

- If $r_{31} \neq \pm 1$ (i.e., when $\beta \in (-\pi/2\,, \pi/2)$):

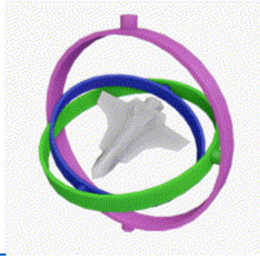
$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\alpha = \operatorname{atan2}(r_{21}, r_{11})$$

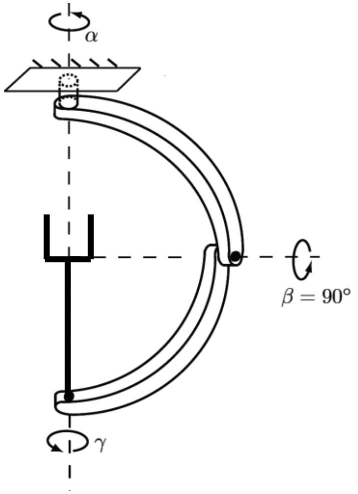
$$\gamma = \operatorname{atan2}(r_{32}, r_{33})$$

- If $r_{31} = -1$, then $\beta = \pi/2$, and if $r_{31} = 1$, then $\beta = -\pi/2$. In these singular cases, $\hat{\boldsymbol{z}}_b$ and $\hat{\boldsymbol{x}}_b$ axes are parallel, and it is possible to determine only the sum or difference of α and γ.

Singularity of the Euler angles:



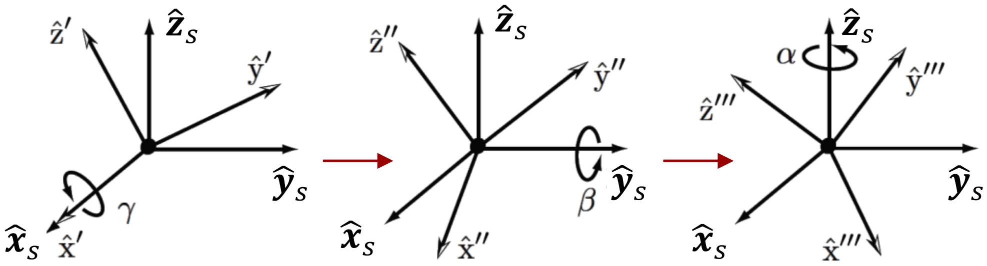
(Gimbal lock)



Roll–Pitch–Yaw or RPY Angles (XYZ)
(about the fixed frame)

XYZ Euler angles (with rotations about the fixed frame $\{s\}$):

- Rotation by γ about the fixed $\hat{\boldsymbol{x}}_s$,
- then by β about the fixed $\hat{\boldsymbol{y}}_s$, and
- finally, by α about the fixed $\hat{\boldsymbol{z}}_s$.



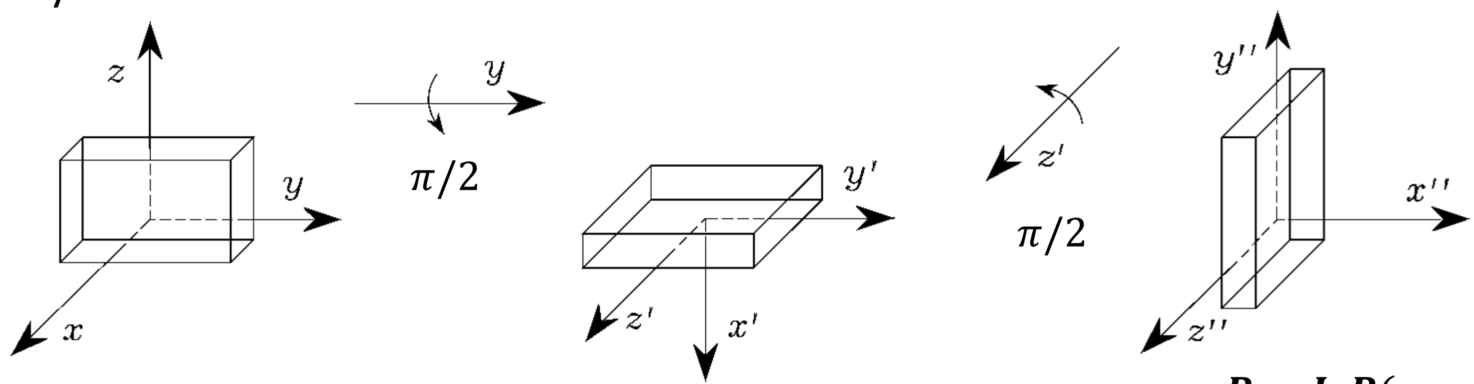
$$\boldsymbol{R}(\alpha,\beta,\gamma) = \mathrm{Rot}(\hat{\boldsymbol{z}}_s,\alpha)\,\mathrm{Rot}(\hat{\boldsymbol{y}}_s,\beta)\,\mathrm{Rot}(\hat{\boldsymbol{x}}_s,\gamma)\,\boldsymbol{I}_3$$

$$\boldsymbol{R}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

This product of three rotations XYZ Euler angles with rotations about the fixed frame $\{s\}$ is the same as that for the ZYX Euler angles with rotations about the body/current frame $\{b\}$, i.e., <u>the same product of three rotations admits two different physical interpretations</u>.

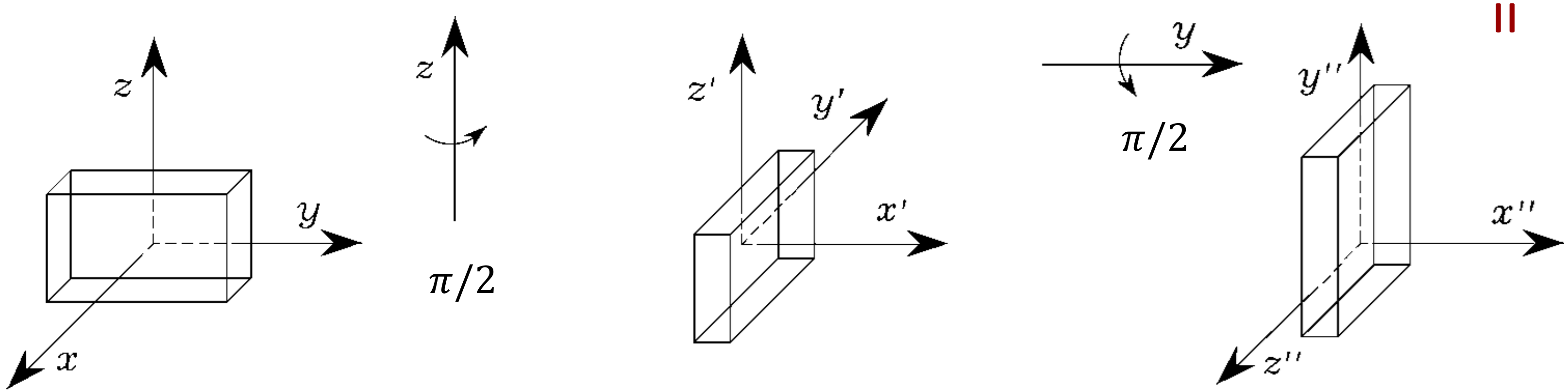
Successive Rotations about Axes of Fixed & Body/Current Frames

Body-frame rotation:



$$\boldsymbol{R} = \boldsymbol{I}_3 \boldsymbol{R}(y, \pi/2) \boldsymbol{R}(z', \pi/2)$$

Fixed-frame rotation:



$$\boldsymbol{R} = \boldsymbol{R}(y, \pi/2) \boldsymbol{R}(z, \pi/2) \boldsymbol{I}_3$$

Unit Quaternions

Quaternions

The **Quaternions** are the set of hypercomplex numbers. A quaternion $\boldsymbol{q} \in \mathbb{H}$ can be represented as $\boldsymbol{q} = q_0 + \boldsymbol{q}_c = q_0 + q_1 i + q_2 j + q_3 k$ or a 4-tuple $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3)$ where $\mathbb{H}$ is set of all quaternions, $q_0 \in \mathbb{R}$ is the real scalar part, $\boldsymbol{q}_c$ is the imaginary vector part, $\boldsymbol{q}_v = (q_1, q_2, q_3) \in \mathbb{R}^3$ is a real vector, and i, j, k are quaternion units or imaginary orthogonal axes that obey Hamilton's rule $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$.

- Addition and multiplication of two quaternions $\boldsymbol{p} = (p_0, \boldsymbol{p}_v)$ and $\boldsymbol{q} = (q_0, \boldsymbol{q}_v)$:

$$\boldsymbol{p} + \boldsymbol{q} = (p_0 + q_0, \boldsymbol{p}_v + \boldsymbol{q}_v)$$

$$\boldsymbol{pq} = (p_0 q_0 - \boldsymbol{p}_v \cdot \boldsymbol{q}_v \,, p_0 \boldsymbol{q}_v + q_0 \boldsymbol{p}_v + \boldsymbol{p}_v \times \boldsymbol{q}_v) \neq \boldsymbol{qp} \quad \text{(not commutative)}$$

- Conjugate of $\boldsymbol{q}$: $\boldsymbol{q}^* = (q_0, -\boldsymbol{q}_v)$
- Norm of $\boldsymbol{q}$: $\|\boldsymbol{q}\| = \sqrt{\boldsymbol{q}^* \boldsymbol{q}} = \sqrt{\boldsymbol{q}\boldsymbol{q}^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$
- Inverse of $\boldsymbol{q}$: $\boldsymbol{q}^{-1} = \dfrac{\boldsymbol{q}^*}{\|\boldsymbol{q}\|^2}$
- $\boldsymbol{p} \cdot \boldsymbol{q} = \mathrm{Re}\left(\frac{1}{2}(\boldsymbol{pq}^* + \boldsymbol{qp}^*)\right) = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3$

The quaternion $\boldsymbol{q}$ is a **Unit Quaternion** if $\|\boldsymbol{q}\| = 1$, and consequently, $\boldsymbol{q}^{-1} = \boldsymbol{q}^*$.

Unit Quaternion

Unit Quaternion (a.k.a. **Euler Parameters**) is a nonminimal four-parameter representation of rotation that alleviates the exponential coordinates singularity (division by $\sin\theta$) and Euler angle singularity (Gimbal lock), but at the cost of four variables subject to one constraint in the representation.

Let $\boldsymbol{R} \in SO(3)$ have the exponential coordinate representation $\hat{\boldsymbol{\omega}}\theta$, i.e., $\boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta}$, where $\|\hat{\boldsymbol{\omega}}\| = 1$ and $\theta = [0, \pi]$. The Unit Quaternion is defined as

$$\boldsymbol{q} = \begin{bmatrix} \overbrace{q_0}^{\text{scalar part}} \\ \underbrace{\boldsymbol{q}_v}_{\text{vector part}} \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \hat{\boldsymbol{\omega}}\sin(\theta/2) \end{bmatrix} \in S^3 \qquad \|\boldsymbol{q}\| = 1$$

- $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3)$ is interpreted as a rotation about the unit axis, in the direction of (q_1, q_2, q_3) by an angle $\theta = 2\cos^{-1} q_0$.
- If $-\pi \leq \theta \leq \pi$, then $q_0 \geq 0$, and if $\pi \leq \theta \leq 3\pi$, then $q_0 \leq 0$.

Unit Quaternion: Finding $\boldsymbol{R}$ by Having $\boldsymbol{q}$

The rotation matrix $\boldsymbol{R}$ corresponding to a given unit quaternion $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3) = \left(\cos(\theta/2), \hat{\boldsymbol{\omega}} \sin\left(\frac{\theta}{2}\right)\right)$ where $\|\boldsymbol{q}\| = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$:

$$\boldsymbol{R} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2({q_0}^2 + q_1^2) - 1 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & 2({q_0}^2 + q_2^2) - 1 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & 2({q_0}^2 + q_3^2) - 1 \end{bmatrix}$$

$$= (2{q_0}^2 - 1)\boldsymbol{I}_3 + 2\boldsymbol{q}_v{\boldsymbol{q}_v}^T + 2q_0[\boldsymbol{q}_v]$$

Note: Finding $\hat{\boldsymbol{\omega}}$ and θ by having $\boldsymbol{q} = (q_0, \boldsymbol{q}_v)$: $\quad \hat{\boldsymbol{\omega}} = \dfrac{\boldsymbol{q}_v}{\|\boldsymbol{q}_v\|}, \quad \theta = 2\mathrm{atan2}(\|\boldsymbol{q}_v\|, q_0)$

Unit Quaternion: Finding q by Having R

The unit quaternion $\boldsymbol{q}$ corresponding to a given rotation matrix $\boldsymbol{R}$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} = \frac{1}{2}\sqrt{1 + \operatorname{tr}\boldsymbol{R}}$$

$$\boldsymbol{q}_v = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

Notes:

- It has been implicitly assumed $q_0 \geq 0$; this corresponds to an angle $\theta \in [-\pi, \pi]$, and thus, any rotation can be described.
- For every rotation matrix $\boldsymbol{R}$ there exists two unit-quaternion representations $\pm\boldsymbol{q}$ that are **antipodal** to each other, i.e., $+\boldsymbol{q}$ and $-\boldsymbol{q}$ represent the same rotation $\boldsymbol{R}$.
- Unlike the exponential coordinate representation, no singularity occurs in inverse solution.
- A rotation by $-\theta$ about $-\hat{\boldsymbol{\omega}}$ gives the same quaternion $\boldsymbol{q}$ as that associated with a rotation by θ about $\hat{\boldsymbol{\omega}}$.
- If the unit quaternion $\boldsymbol{q} = (q_0, \boldsymbol{q}_v)$ corresponds to a rotation matrix $\boldsymbol{R}$, the unit quaternion extracted from $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$ is denoted as $\boldsymbol{q}^{-1}$, and can be computed as $\boldsymbol{q}^{-1} = (q_0, -\boldsymbol{q}_v)$.

Unit Quaternion: Remarks

- Let $\boldsymbol{p} = (p_0, \boldsymbol{p}_v) = (p_0, p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3)$ denote the quaternions corresponding to the rotation matrices $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, respectively. The quaternion corresponding to the product $\boldsymbol{R}_n = \boldsymbol{R}_p \boldsymbol{R}_q$ is given by

$$\boldsymbol{n} = \boldsymbol{p}\boldsymbol{q} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 \\ p_1 q_0 + p_0 q_1 - p_3 q_2 + p_2 q_3 \\ p_2 q_0 + p_0 q_2 + p_3 q_1 - p_1 q_3 \\ p_3 q_0 + p_0 q_3 - p_2 q_1 + p_1 q_2 \end{bmatrix}$$

quaternion product operator

$$= (p_0 q_0 - \boldsymbol{q}_v \cdot \boldsymbol{q}_v, p_0 \boldsymbol{q}_v + q_0 \boldsymbol{p}_v + \boldsymbol{p}_v \times \boldsymbol{q}_v)$$

- $\boldsymbol{q}\boldsymbol{q}^{-1} = (1, \boldsymbol{0})$ which is the identity element.

- The rotation of a point or vector $\boldsymbol{p} \in \mathbb{R}^3$ by the angle θ about a unit axis $\widehat{\boldsymbol{\omega}}$ is $\boldsymbol{p}' = \mathrm{Rot}(\widehat{\boldsymbol{\omega}}, \theta)\, \boldsymbol{p}$ which can be also determined using unit quaternions as $\boldsymbol{q}_{p'} = \boldsymbol{q}\boldsymbol{q}_p\boldsymbol{q}^{-1}$ where $\boldsymbol{q} = (q_0, \boldsymbol{q}_v)$ is unit quaternion representation of $\mathrm{Rot}(\widehat{\boldsymbol{\omega}}, \theta)$, $\boldsymbol{q}^{-1} = (q_0, -\boldsymbol{q}_v)$, $\boldsymbol{q}_p = (0, \boldsymbol{p})$, and $\boldsymbol{q}_{p'} = (0, \boldsymbol{p}')$.

$$\boldsymbol{p}' = \mathrm{Rot}(\widehat{\boldsymbol{\omega}}, \theta)\, \boldsymbol{p} \quad \leftrightarrow \quad \boldsymbol{q}_{p'} = \boldsymbol{q}\boldsymbol{q}_p\boldsymbol{q}^{-1}$$