Ch6: Velocity Kinematics – Part 2 (Inverse and Statics)

Analysis of Velocity Kinematics

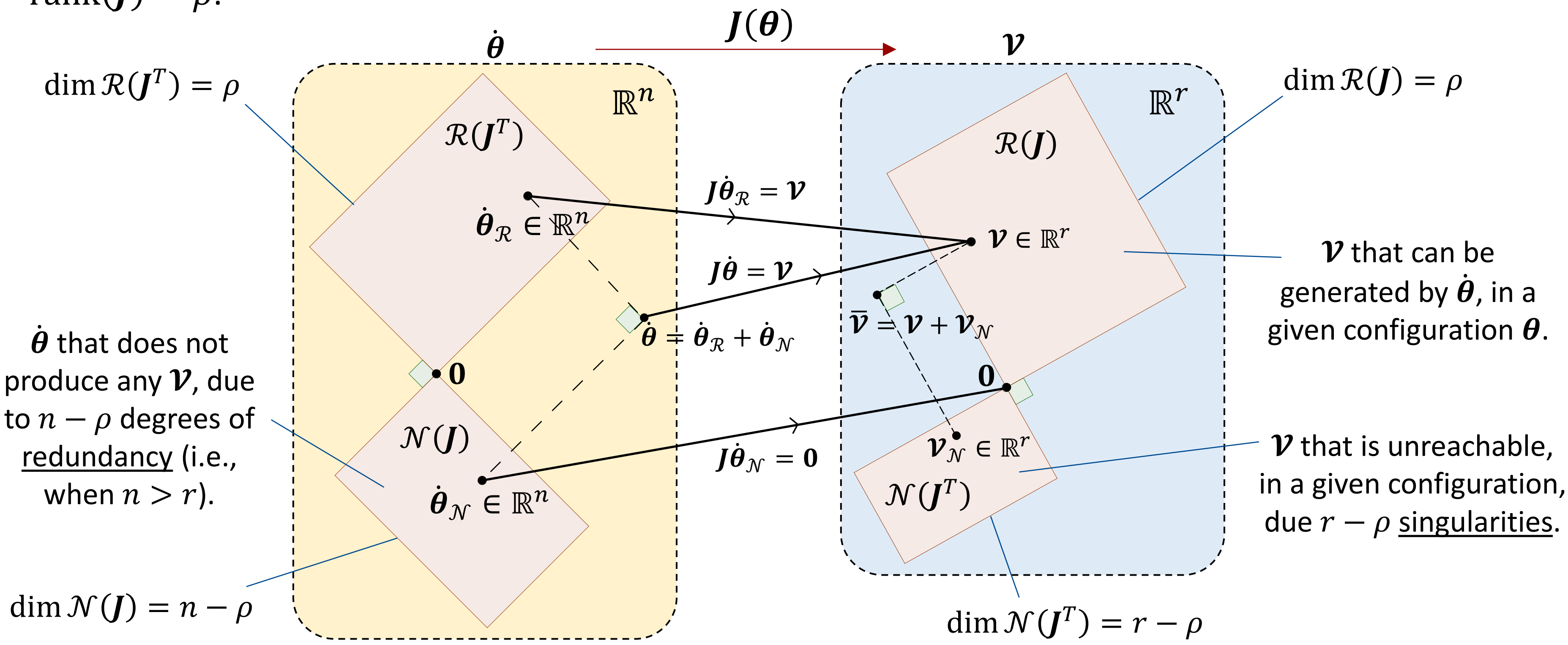
Analysis of Velocity Kinematics and Kinematic Redundancy

Assume a n-DOF robot that is not at a singular configuration and $\dim(\text{T−Space}) = r$,

- If $n < r$, then arbitrary twists $\boldsymbol{\mathcal{V}}$ cannot be achieved (the robot does not have enough joints).
- If $n = r$, then any arbitrary twists $\boldsymbol{\mathcal{V}}$ can be achieved (the robot have enough joints).
- If $n > r$, then not only any arbitrary twists $\boldsymbol{\mathcal{V}}$ can be achieved (not at singular configurations) but also the remaining $n - r$ degrees of freedoms (redundant DOFs) can generate **internal motions** at the joints of the robot that are not evident in the motion of the end-effector.

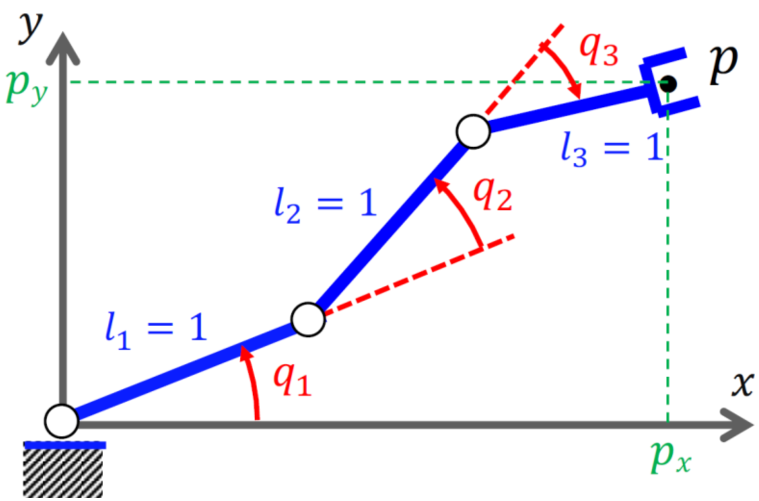
Analysis of Velocity Kinematics, Redundancy, and Singularity

Consider the velocity kinematics equation $\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ where $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$, $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$, $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$, and $\mathrm{rank}(\boldsymbol{J}) = \rho$.



- If the Jacobian is full-rank (robot is not at a singular configuration) and $n > r$ (robot is redundant): $\dim \mathcal{R}(\boldsymbol{J}) = r$, $\dim \mathcal{N}(\boldsymbol{J}) = n - r$, and $\dim \mathcal{N}(\boldsymbol{J}^T) = 0$.

Example: Analysis for a Planar 3R Robot



$$l_1 = l_2 = l_3 = 1 \qquad n = 3, r = 2$$

$$\boldsymbol{x}_e = \boldsymbol{p} = (p_x, p_y)$$

$$\text{FK:} \quad \boldsymbol{p} = \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix}$$

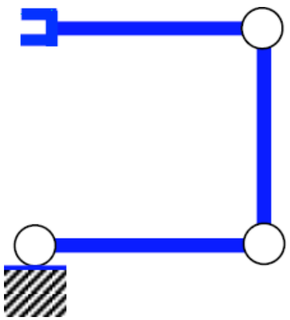
$$\text{Reachable Workspace} = \{\boldsymbol{p} \in \mathbb{R}^2 : \|\boldsymbol{p}\| \le 3\} \subset \mathbb{R}^2$$

$$\text{Dexterous Workspace} = \{\boldsymbol{p} \in \mathbb{R}^2 : \|\boldsymbol{p}\| \le 1\} \subset \mathbb{R}^2$$

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \begin{pmatrix} -s_1 - s_{12} - s_{123} & -s_{12} - s_{123} & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Let's consider two cases.

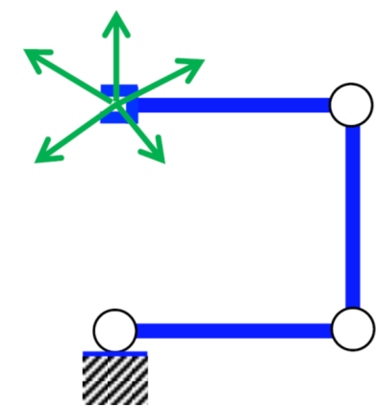
Example: Analysis for a Planar 3R Robot – Case 1



$$\boldsymbol{q} = (0, \pi/2, \pi/2) \qquad \boldsymbol{J} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \qquad \boldsymbol{J}^T = \begin{pmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{pmatrix}$$

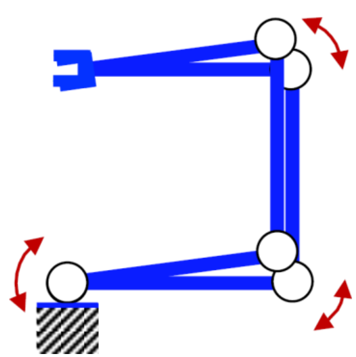
$$\operatorname{rank}(\boldsymbol{J}) = \operatorname{rank}(\boldsymbol{J}^T) = 2$$

$$\mathcal{R}(\boldsymbol{J}^T) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{R}(\boldsymbol{J}) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$$



$$\mathcal{N}(\boldsymbol{J}) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{N}(\boldsymbol{J}^T) = \emptyset$$

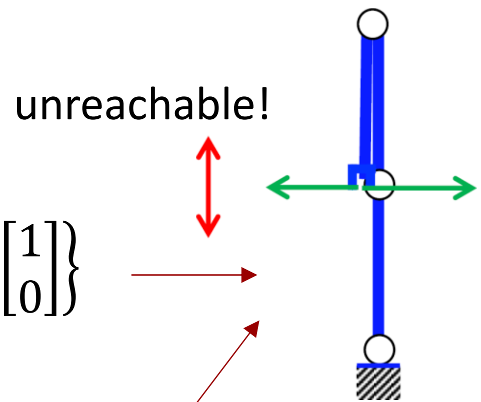
$$\dim \mathcal{N}(\boldsymbol{J}) = 1$$



Example: Analysis for a Planar 3R Robot – Case 2

$$\boldsymbol{q} = (\pi/2, 0, \pi) \qquad \boldsymbol{J} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{J}^T = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

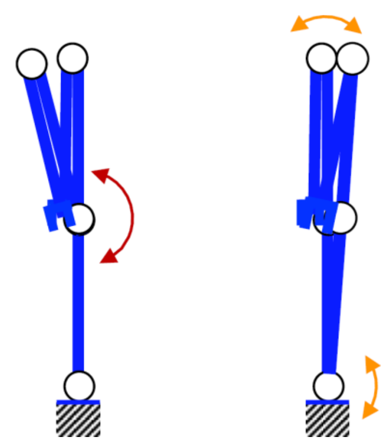
$$\operatorname{rank}(\boldsymbol{J}) = \operatorname{rank}(\boldsymbol{J}^T) = 1$$



$$\mathcal{R}(\boldsymbol{J}^T) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{R}(\boldsymbol{J}) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$\mathcal{N}(\boldsymbol{J}) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{N}(\boldsymbol{J}^T) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$\dim \mathcal{N}(\boldsymbol{J}) = 2 \qquad \dim \mathcal{N}(\boldsymbol{J}^T) = 1$$



Inverse Velocity Kinematics

Inverse Velocity Kinematics

Given a desired EE twist $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$, what joint velocities $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ are needed?

- If $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ is square ($n = r$) and full rank $\text{rank}(\boldsymbol{J}) = r = n$, (i.e., not at a singular configuration), then $\boldsymbol{J}(\boldsymbol{\theta})$ is invertible and there is a unique solution as $\dot{\boldsymbol{\theta}} = \boldsymbol{J}\,(\boldsymbol{\theta})^{-1}\boldsymbol{\mathcal{V}}$.
- If $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ is not square and $n > r$ (i.e., robot is redundant, $\boldsymbol{J}$ is a fat matrix, and $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$), and also $\boldsymbol{J}$ is full (row) rank, $\text{rank}(\boldsymbol{J}) = r$ (i.e., robot is not at a singular configuration and $\mathcal{N}(\boldsymbol{J}^T) = \emptyset$), then infinite exact solutions $\dot{\boldsymbol{\theta}}$ exist to $\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ as

$$\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^* + \boldsymbol{P}\dot{\boldsymbol{\theta}}_0 = \boldsymbol{J}\,(\boldsymbol{\theta})^{+}\boldsymbol{\mathcal{V}} + \big(\boldsymbol{I}_n - \boldsymbol{J}(\boldsymbol{\theta})^{+}\boldsymbol{J}(\boldsymbol{\theta})\big)\dot{\boldsymbol{\theta}}_0 \qquad \forall\dot{\boldsymbol{\theta}}_0 \in \mathbb{R}^n$$

where $\boldsymbol{J}^{+} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$ is the right pseudo-inverse as $\boldsymbol{J}\boldsymbol{J}^{+} = \boldsymbol{I}_r$.

- This solution is derived from this optimization problem:

  (a solution "biased" toward the joint velocity $\dot{\boldsymbol{\theta}}_0$, chosen to avoid obstacles, joint limits, etc.)

$$\begin{aligned} \min_{\dot{\boldsymbol{\theta}}}. \quad & \frac{1}{2}\left\|\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_0\right\|_2^2 \\ \text{subject to} \quad & \boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}} \end{aligned}$$

- The solution $\dot{\boldsymbol{\theta}}^* = \boldsymbol{J}^{+}\boldsymbol{\mathcal{V}}$ locally minimizes the norm of joint velocities $\dot{\boldsymbol{\theta}}$ (i.e., when $\dot{\boldsymbol{\theta}}_0 = \boldsymbol{0}$).
- The matrix $\boldsymbol{P} = \boldsymbol{I}_n - \boldsymbol{J}^{+}\boldsymbol{J}$ projects $\dot{\boldsymbol{\theta}}_0$ in $\mathcal{N}(\boldsymbol{J})$, so as not to violate the constraint $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$.

Remarks

- $\dot{\boldsymbol{\theta}}_0$ is a vector of arbitrary joint velocities that can generate **internal motions** and can be specified to satisfy an additional constraint due to the presence of redundant DOFs. The additional constraint has secondary priority with respect to the primary kinematic constraint $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$.
- The use of the pseudoinverse $\boldsymbol{J}^{+} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$ implicitly weights the cost of each joint velocity identically. We could instead give the joint velocities different weights; for example, the velocity at the first joint, which moves a lot of the robot's mass, could be weighted more heavily than the velocity at the last joint, which moves little of the robot's mass. Therefore, we can use the weighted right pseudo-inverse as

$$\boldsymbol{J}^{+} = \boldsymbol{W}_r^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{W}_r^{-1}\boldsymbol{J}^T)^{-1}$$

  $\boldsymbol{W}_r \in \mathbb{R}^{n\times n}$ is a positive definite matrix.

  For example, by using $\boldsymbol{W}_r = \boldsymbol{M}(\boldsymbol{\theta})$ where $\boldsymbol{M}(\boldsymbol{\theta})$ is the mass matrix of the robot, we can find the $\dot{\boldsymbol{\theta}}$ that minimizes the kinetic energy, while also satisfying $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$.
- **Inverse Velocity Kinematics** (IVK) and **Inverse Differential Kinematics** refer to the same concept.

Exploiting Redundant DOFs
(Redundancy Resolution)

A typical choice of $\dot{\boldsymbol{\theta}}_0$ for advantageously exploiting redundant DOFs is

$$\dot{\boldsymbol{\theta}}_0 = k_0 \boldsymbol{\nabla}_{\boldsymbol{\theta}} w(\boldsymbol{\theta}) = k_0 \left(\frac{\partial w(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \quad \text{where } k_0 \in \mathbb{R}_+$$

which is in the direction of the gradient of a (secondary) objective function $w(\boldsymbol{\theta})$ at a given $\boldsymbol{\theta}$ (i.e., in the direction at which function $w(\boldsymbol{\theta})$ increases the fastest). Thus, the solution $\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^* + \boldsymbol{P}\dot{\boldsymbol{\theta}}_0$ attempts to maximize $w(\boldsymbol{\theta})$ locally compatible to the primary objective $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$ (kinematic constraint).

❖ Three typical (secondary) objective functions $w(\boldsymbol{\theta})$:

1) **Manipulability measure**: $\quad w(\boldsymbol{\theta}) = \sqrt{\det(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})^T)}$

By maximizing $w(\boldsymbol{\theta})$, redundancy is exploited to <u>move away from singularities</u>. Note that $w(\boldsymbol{\theta})$ vanishes at a singular configuration.

Exploiting Redundant DOFs
(Redundancy Resolution)

2) **Distance from mechanical joint limits**:

$$w(\boldsymbol{\theta}) = -\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\theta_i - \theta_{i,\text{mid}}}{\theta_{i,\text{max}} - \theta_{i,\text{min}}}\right)^2$$

n: number of joints,
$\theta_{i,\text{max}}$, $\theta_{i,\text{min}}$: maximum, minimum joint limit,
$\theta_{i,\text{mid}}$: middle value of the joint range.

By maximizing $w(\boldsymbol{\theta})$, redundancy is exploited to keep the joint variables $\boldsymbol{\theta}$ as close as possible to the center of their ranges.

3) **Distance from an obstacle**:

$$w(\boldsymbol{\theta}) = \min_{\boldsymbol{p}_{\mathcal{B}},\boldsymbol{o}} \|\boldsymbol{p}_{\mathcal{B}}(\boldsymbol{\theta}) - \boldsymbol{o}\|$$

$\boldsymbol{o}$: position vector of a suitable point on the obstacle,
$\boldsymbol{p}_{\mathcal{B}}$: position vector of a generic point along the body $\mathcal{B}$ of the robot.

By maximizing $w(\boldsymbol{\theta})$, redundancy is exploited to avoid collision of the manipulator's body with an obstacle.

Higher-order Differential Inversion

Inversion of motion from task to joint space can be also performed at a higher differential level.

Acceleration Level: given $\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}$

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{J}^{+}(\boldsymbol{\theta})\left(\dot{\boldsymbol{\mathcal{V}}} - \dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right)$$

Jerk Level: given $\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}$

$$\dddot{\boldsymbol{\theta}} = \boldsymbol{J}^{+}(\boldsymbol{\theta})\left(\ddot{\boldsymbol{\mathcal{V}}} - 2\dot{\boldsymbol{J}}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} - \ddot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right)$$

Inverse Velocity Kinematics at Singularities

Inverse Velocity Kinematics at Kinematic Singularities

If $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ (square or non-square) is rank deficient (i.e., mathematically, $\operatorname{rank}(\boldsymbol{J}) < \min(r, n)$, $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$, and $\mathcal{N}\left(\boldsymbol{J}^{\mathrm{T}}\right) \neq \emptyset$, and physically, a redundant or non-redundant robot at a singular configuration), when $\boldsymbol{\mathcal{V}} \in \mathcal{R}(\boldsymbol{J})$, <u>infinite exact solutions</u> exist, and when $\boldsymbol{\mathcal{V}} \notin \mathcal{R}(\boldsymbol{J})$, <u>no exact solutions</u> but <u>infinite approximate solutions</u> exist. In both cases, the solutions are in the form

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}(\boldsymbol{\theta})^{+}\boldsymbol{\mathcal{V}} + \left(\boldsymbol{I}_n - \boldsymbol{J}(\boldsymbol{\theta})^{+}\boldsymbol{J}(\boldsymbol{\theta})\right)\dot{\boldsymbol{\theta}}_0 \qquad \forall \dot{\boldsymbol{\theta}}_0 \in \mathbb{R}^n$$

and $\boldsymbol{J}(\boldsymbol{\theta})^{+}$ is pseudo-inverse which can be computed using the **Singular Value Decomposition (SVD)** exactly or using **Damped Least Squares (DLS)** approximately.

Inverse Velocity Kinematics at Kinematic Singularities

- If $\boldsymbol{\mathcal{V}} \in \mathcal{R}(\boldsymbol{J})$, this means that the assigned $\boldsymbol{\mathcal{V}}$ is physically executable, even though it is at a singular configuration, and the exact solution $\boldsymbol{J}^{+}\boldsymbol{\mathcal{V}}$ minimizes $\|\dot{\boldsymbol{\theta}}\|_2$, i.e.,

$$\begin{aligned} \min_{\dot{\boldsymbol{\theta}}}. \quad & \frac{1}{2}\|\dot{\boldsymbol{\theta}}\|_2^2 \\ \text{subject to} \quad & \boldsymbol{J}\dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{V}} \end{aligned}$$

- If $\boldsymbol{\mathcal{V}} \notin \mathcal{R}(\boldsymbol{J})$, this means that the assigned $\boldsymbol{\mathcal{V}}$ is not physically executable at the singular configuration, and the approximate solution $\boldsymbol{J}^{+}\boldsymbol{\mathcal{V}}$ minimizes $\|\boldsymbol{J}\dot{\boldsymbol{\theta}} - \boldsymbol{\mathcal{V}}\|_2$, i.e.,

$$\min_{\dot{\boldsymbol{\theta}}}. \ \frac{1}{2}\|\boldsymbol{J}\dot{\boldsymbol{\theta}} - \boldsymbol{\mathcal{V}}\|_2^2$$

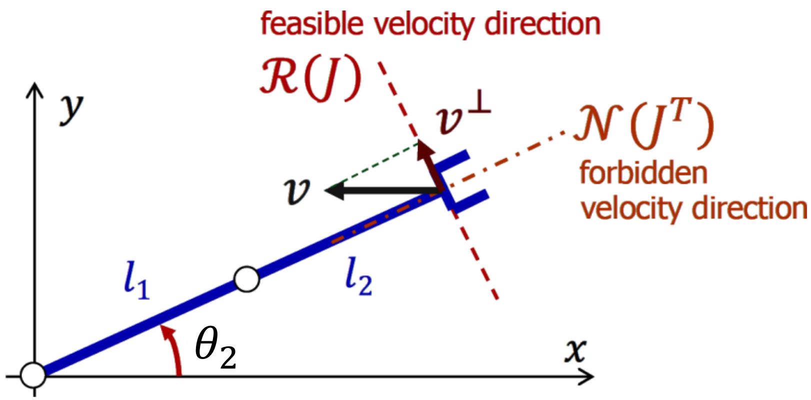
This solution satisfies $\boldsymbol{J}\dot{\boldsymbol{\theta}} = \boldsymbol{J}\boldsymbol{J}^{+}\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}^{\perp}$ where $\boldsymbol{J}\boldsymbol{J}^{+}$ is orthogonal projector onto $\mathcal{R}(\boldsymbol{J})$ and $\boldsymbol{J}\boldsymbol{J}^{+}\boldsymbol{\mathcal{V}}$ is the closest feasible twist that we can reach.

Example

Consider a planar 2R robot with $l_1 = l_2 = 1$ at the singular configuration $\theta_2 = 0$. We want to find $\dot{\boldsymbol{\theta}}$ for a given velocity of end-effector $\boldsymbol{\mathcal{V}} = (-0.5, 0)$ at two configurations of **Case (1)** $\theta_1 = \pi/6$ and **Case (2)** $\theta_1 = \pi/2$.

$$\boldsymbol{J} = \begin{pmatrix} -2s_1 & -s_1 \\ 2c_1 & c_1 \end{pmatrix} \qquad \boldsymbol{J}^+ = \frac{1}{5}\begin{pmatrix} -2s_1 & 2c_1 \\ -s_1 & c_1 \end{pmatrix}$$

$$\boldsymbol{J}\boldsymbol{J}^+ = \begin{pmatrix} {s_1}^2 & -s_1 c_1 \\ -s_1 c_1 & {c_1}^2 \end{pmatrix} \qquad \boldsymbol{J}^+\boldsymbol{J} = \begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{pmatrix}$$



Case (1) $\theta_1 = \pi/6$: $\boldsymbol{\mathcal{V}} = (-0.5, 0)$ is not completely feasible ($\boldsymbol{\mathcal{V}} \notin \mathcal{R}(\boldsymbol{J})$)! $\quad \dot{\boldsymbol{\theta}} = \boldsymbol{J}^+\boldsymbol{\mathcal{V}} = \begin{pmatrix} 0.1 \\ 0.05 \end{pmatrix}$

This solution satisfies $\boldsymbol{J}\dot{\boldsymbol{\theta}} = \boldsymbol{J}\boldsymbol{J}^+\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}^\perp = \begin{pmatrix} -1/8 \\ \sqrt{3}/8 \end{pmatrix}$ which is the closest feasible twist that we can reach.

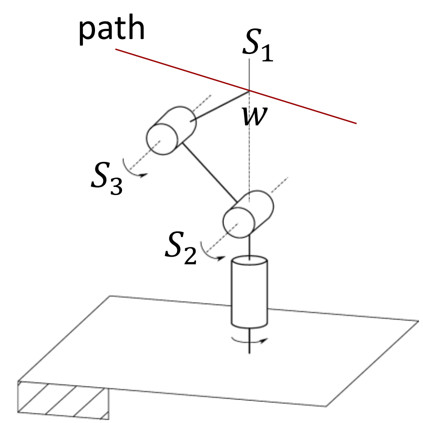
Case (2) $\theta_1 = \pi/2$: $\boldsymbol{\mathcal{V}} = (-0.5, 0)$ is completely feasible ($\boldsymbol{\mathcal{V}} \in \mathcal{R}(\boldsymbol{J})$)! $\quad \dot{\boldsymbol{\theta}} = \boldsymbol{J}^+\boldsymbol{\mathcal{V}} = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}$

This solution satisfies $\boldsymbol{J}\dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{V}} = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}$.

Inverse Velocity Kinematics at Kinematic Singularities

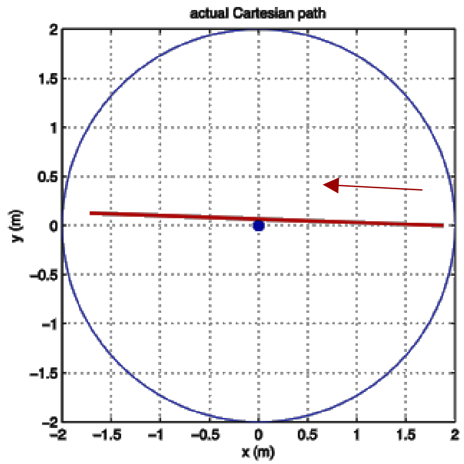
❖ In the neighborhood of a singularity, small velocities $\boldsymbol{\mathcal{V}}$ in the task space cause large velocities $\dot{\boldsymbol{\theta}}$ in the joint space.

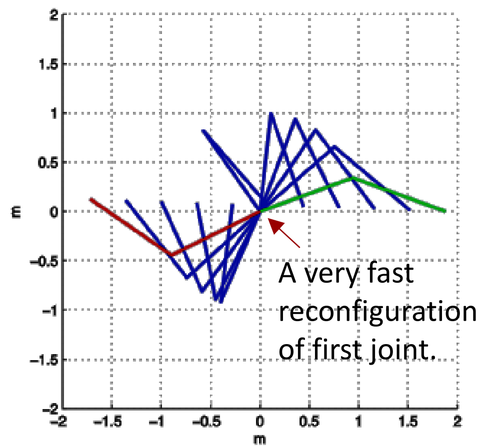
For example, consider the shoulder singularity for the anthropomorphic arm. If a path is assigned to the end-effector which passes nearby the axis S_1, while using $\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}$, the base joint is forced to make a rotation of about π in a relatively short time to allow the end-effector to keep tracking the imposed trajectory.

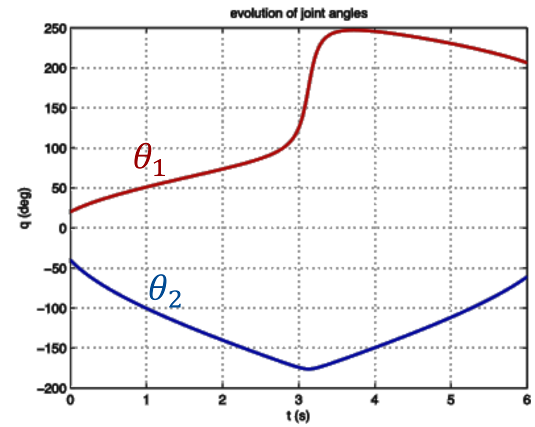


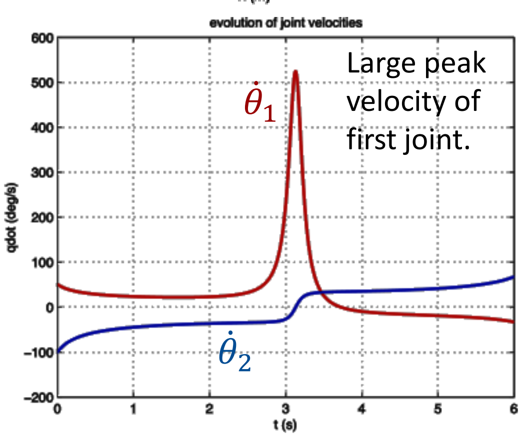
Example

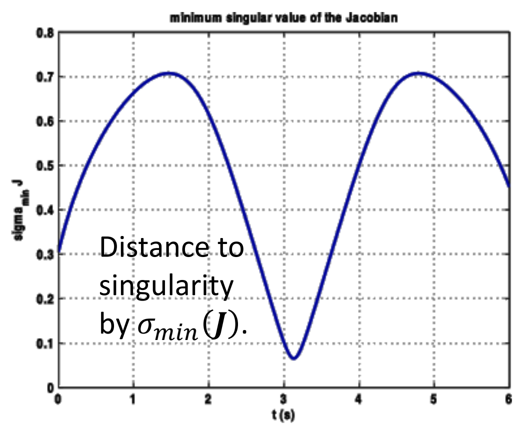
Consider a planar 2R robot with $l_1 = l_2$ moving a long a straight-line path that passes close to singularity at origin. Here are simulation results while using $\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}$:

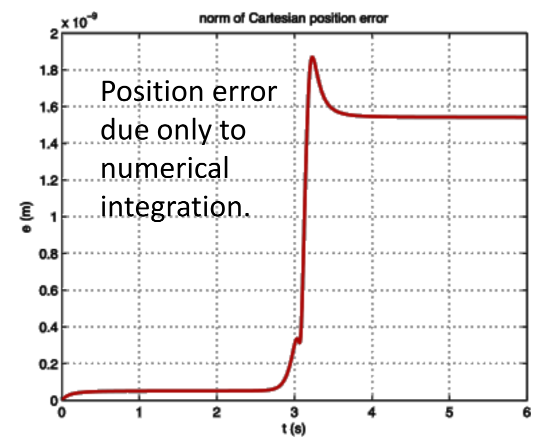












Inverse Velocity Kinematics at Kinematic Singularities

❖ The pseudo-inverse $\boldsymbol{J}(\boldsymbol{\theta})^+$ computed using the **Damped Least Squares (DLS)** method is more stable and computationally less expensive than the SVD method:

$$\boldsymbol{J}^+(\boldsymbol{\theta}) = \boldsymbol{J}(\boldsymbol{\theta})^T(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})^T + \lambda \boldsymbol{I}_r)^{-1}$$

The solution $\boldsymbol{J}^+\boldsymbol{\mathcal{V}}$ derived using DLS is an approximate solution which is equivalent to the optimization problem:

$$\min_{\dot{\boldsymbol{\theta}}}. \quad \frac{1}{2}\left\|\boldsymbol{J}\dot{\boldsymbol{\theta}} - \boldsymbol{\mathcal{V}}\right\|_2^2 + \frac{\lambda}{2}\left\|\dot{\boldsymbol{\theta}}\right\|_2^2$$

The damping factor $\lambda \in \mathbb{R}_+$ establishes the relative weight between the kinematic constraint $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$ and the minimum norm joint velocity requirement $\left\|\dot{\boldsymbol{\theta}}\right\|_2^2$. In the neighborhood of a singularity, λ is to be chosen large enough to render differential kinematics inversion well conditioned, whereas far from singularities, λ can be chosen small (even $\lambda = 0$) to guarantee accurate differential kinematics inversion $\boldsymbol{J}^{-1}$ or $\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$.

Inverse Velocity Kinematics with Constraints

Inverse Velocity Kinematics with Constraints

In the presence of constraints, we cannot find a closed-form for $\dot{\boldsymbol{\theta}}$ that achieves a given $\boldsymbol{\mathcal{V}}$, however, we can directly solve an <u>optimization problem</u> with constraints at each time step Δt to find $\dot{\boldsymbol{\theta}}$.

Example 1: Adding position, velocity, and acceleration constraints

By solving (1) at each time step Δt, we can find $\dot{\boldsymbol{\theta}}$ that achieve a given $\boldsymbol{\mathcal{V}}$ exactly or as closely as possible, while satisfying the constraints:

$$\begin{aligned} \min_{\dot{\boldsymbol{\theta}}}. \quad & \left\| \boldsymbol{J}(\boldsymbol{\theta}_c)\dot{\boldsymbol{\theta}} - \boldsymbol{\mathcal{V}} \right\|_2^2 \\ \text{subject to} \quad & \dot{\boldsymbol{\theta}}_{\min} \le \dot{\boldsymbol{\theta}} \le \dot{\boldsymbol{\theta}}_{\max} \\ & \boldsymbol{\theta}_{\min} \le \boldsymbol{\theta}_c + \Delta t \dot{\boldsymbol{\theta}} \le \boldsymbol{\theta}_{\max} \\ & \ddot{\boldsymbol{\theta}}_{\min} \le \frac{\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_c}{\Delta t} \le \ddot{\boldsymbol{\theta}}_{\max} \end{aligned} \tag{1}$$

The subscript c represent the current value.

Inverse Velocity Kinematics with Constraints

Example 2: Joint centering (toward $\boldsymbol{\theta}_{\text{mid}}$) with constraints

$$\begin{aligned} \min_{\dot{\boldsymbol{\theta}}}. \quad & \left\|\dot{\boldsymbol{\theta}} - \boldsymbol{K}(\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_c)\right\|_2^2 \\ \text{subject to} \quad & \boldsymbol{J}(\boldsymbol{\theta}_c)\dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{V}} \\ & + \text{other constraints} \end{aligned}$$

$\dot{\boldsymbol{\theta}} - \boldsymbol{K}(\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_c)$ choses a large $\dot{\boldsymbol{\theta}}$ when $\boldsymbol{\theta}_c$ is away from $\boldsymbol{\theta}_{\text{mid}}$ to take $\boldsymbol{\theta}_c$ close to $\boldsymbol{\theta}_{\text{mid}}$.

($\boldsymbol{\mathcal{V}}$ is achieved exactly)

$$\begin{aligned} \min_{\dot{\boldsymbol{\theta}}}. \quad & \left\|\boldsymbol{J}(\boldsymbol{\theta}_c)\dot{\boldsymbol{\theta}} - \boldsymbol{\mathcal{V}}\right\|_2^2 + \epsilon\left\|\dot{\boldsymbol{\theta}} - \boldsymbol{K}(\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_c)\right\|_2^2 \\ \text{subject to} \quad & \text{constraints} \end{aligned}$$

($\boldsymbol{\mathcal{V}}$ is achieved as closely as possible)

Statics of Open Chains

Statics of Open Chains

The goal of statics is to determine the relationship between wrench $\boldsymbol{\mathcal{F}}_{\text{ext}}$ applied to the end-effector and joint torques $\boldsymbol{\tau} \in \mathbb{R}^n$ exerted by the motors at the joints (forces for prismatic joints, torques for revolute joints) with the manipulator at a static equilibrium condition.

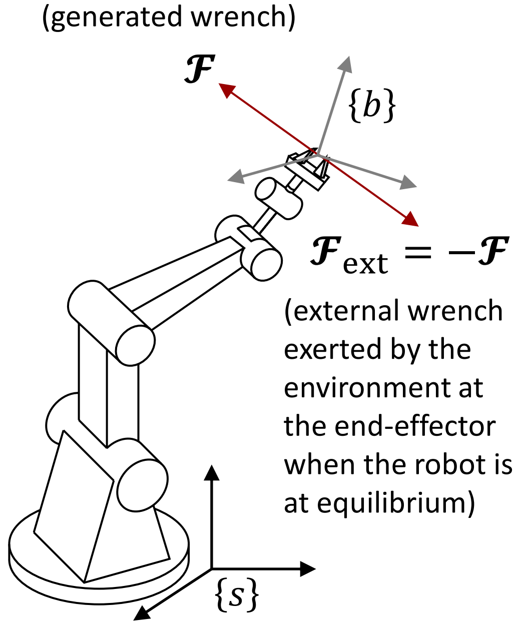
Principle of conservation of power:
power generated at the joints =
(power measured at the end-effector) + (power to move the robot)

At static equilibrium, no power is being used to move the robot, thus:

$$\boldsymbol{\tau}^T \dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{F}}^T \boldsymbol{\mathcal{V}} \qquad \dot{\boldsymbol{\theta}} \rightarrow \mathbf{0}$$

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$$



where $\boldsymbol{\mathcal{F}} \in \mathbb{R}^r$ is the equivalent wrench generated by the robot, $\boldsymbol{J} \in \mathbb{R}^{r\times n}$ is the corresponding geometric or analytic Jacobian matrix, and $\dim(\text{T–Space}) = r$ $(r \leq 6)$.

Statics and Kinematic Redundancy

Note: If the robot has to support itself against gravity to maintain static equilibrium, the torques $\boldsymbol{\tau}_{\mathcal{F}} = \boldsymbol{J}^T \boldsymbol{\mathcal{F}}$ must be added to the torques $\boldsymbol{\tau}_g$ that compensate gravity:

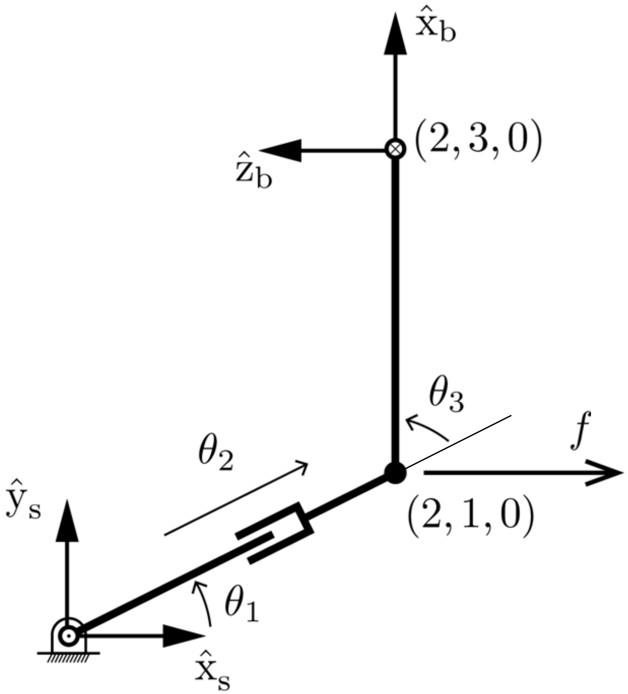
$$\boldsymbol{\tau}_{\text{joint}} = \boldsymbol{J}^T \boldsymbol{\mathcal{F}} + \boldsymbol{\tau}_g$$

Note: For an n-DOF robot that is not at a singular configuration and $\dim(\text{T−Space}) = r$,

- If $n = r$, fixing the end-effector will immobilize the robot.
- If $n > r$, then the robot is redundant, and even if the end-effector is fixed, the joint torques may cause internal motions of the links. The static equilibrium assumption is no longer satisfied, and we need to include dynamics to know what will happen to the robot.

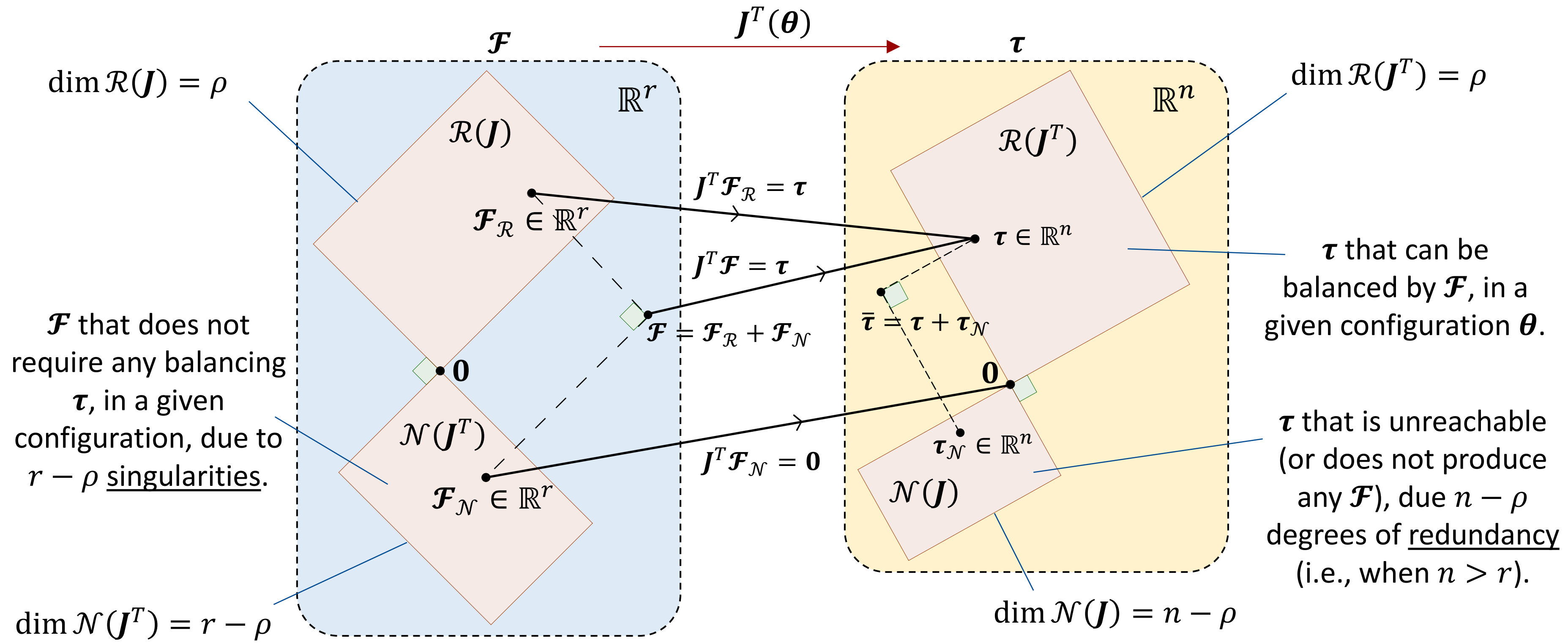
Example

A linear force f is applied to link 3 at the point shown. What are the joint toques needed to resist it?



Analysis of Statics, Redundancy, and Singularity

Consider static equation $\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$ where $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$, $\boldsymbol{\tau} \in \mathbb{R}^n$, $\boldsymbol{\mathcal{F}} \in \mathbb{R}^r$, and $\operatorname{rank}(\boldsymbol{J}) = \rho$.



- If the Jacobian is full-rank (robot is not at a singular configuration) and $n > r$ (robot is redundant): $\dim \mathcal{R}(\boldsymbol{J}) = r$, $\dim \mathcal{N}(\boldsymbol{J}) = n - r$, and $\dim \mathcal{N}(\boldsymbol{J}^T) = 0$.

Singularity Analysis

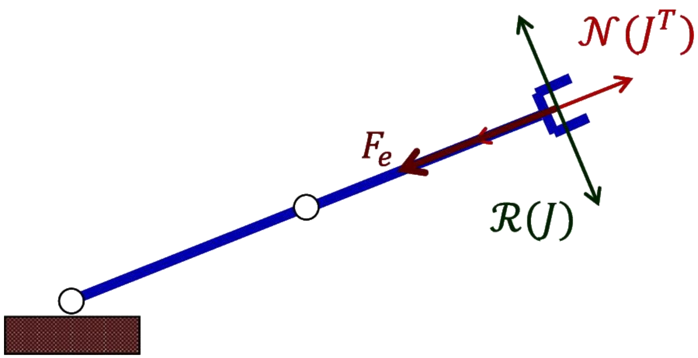
- Since $\text{rank}(\boldsymbol{J}) = \text{rank}(\boldsymbol{J}^T)$, the singular configurations for the velocity map and force map are the same.
- For a manipulator at a singular configuration, $\boldsymbol{\mathcal{F}}_{\mathcal{N}} \in \mathcal{N}(\boldsymbol{J}^T)$ are entirely absorbed by the structure in that the mechanical constraint reaction forces can balance them exactly.

Example: Consider a planar 2R robot with $l_1 = l_2$ at the singular configuration $\theta_2 = 0$. We want to find $\boldsymbol{\tau}$ that balances the external force $\boldsymbol{F}_{\text{ext}} = -F_e \begin{pmatrix} c_1 \\ s_1 \end{pmatrix}$ at the end-effector (along the stretched direction).

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix}, \quad \det(\boldsymbol{J}(\boldsymbol{\theta})) = l_1 l_2 s_2, \quad \boldsymbol{J}\Big|_{\theta_2=0} = \begin{pmatrix} -(l_1 + l_2)s_1 & -l_2 s_1 \\ (l_1 + l_2)c_1 & l_2 c_1 \end{pmatrix}$$

$$\mathcal{R}(\boldsymbol{J}) = \alpha \begin{pmatrix} -s_1 \\ c_1 \end{pmatrix}, \qquad \mathcal{R}(\boldsymbol{J}^T) = \gamma \begin{pmatrix} l_1 + l_2 \\ l_2 \end{pmatrix}$$

$$\mathcal{N}(\boldsymbol{J}^T) = \beta \begin{pmatrix} c_1 \\ s_1 \end{pmatrix}, \qquad \mathcal{N}(\boldsymbol{J}) = \delta \begin{pmatrix} l_2 \\ -(l_1 + l_2) \end{pmatrix}$$



Since $\boldsymbol{F}_{\text{ext}} \in \mathcal{N}(\boldsymbol{J}^T)$, then $\boldsymbol{\tau} = \boldsymbol{J}^T \boldsymbol{F}_{\text{ext}} = \boldsymbol{0}$.

i.e., no joint torques $\boldsymbol{\tau}$ is required to balance $\boldsymbol{F}_{\text{ext}}$.

Manipulability

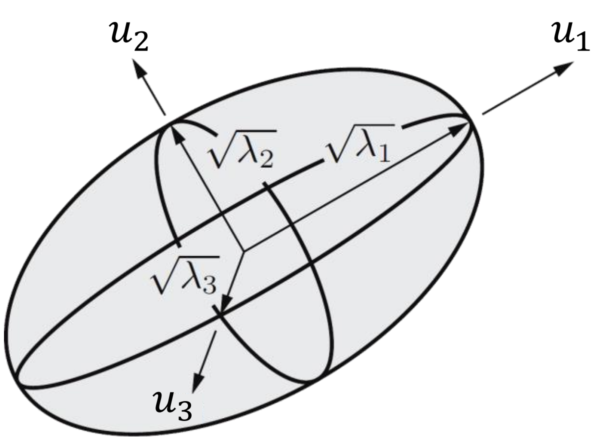
Preliminary: Ellipsoid Representation

For any symmetric positive-definite $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, the set of vectors $\boldsymbol{x} \in \mathbb{R}^{m}$ satisfying $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 1$ defines an ellipsoid (function of $\boldsymbol{x}$) in the m-dimensional space.

Assume that $\boldsymbol{u}_i \in \mathbb{R}^m$ are eigenvectors and $\lambda_i \in \mathbb{R}$ are eigenvalues of $\boldsymbol{A}^{-1}$ $(i = 1, \dots, m)$.

Therefore, for the ellipsoid,

- Directions of the principal axes are $\boldsymbol{u}_i$,
- Lengths of the principal semi-axes are $\sqrt{\lambda_i}$,
- Volume is proportional to $\sqrt{\lambda_1\lambda_2\cdots\lambda_m} = \sqrt{\det(\boldsymbol{A}^{-1})}$.



Velocity Manipulability Ellipsoid

The **velocity manipulability ellipsoid** corresponds to the end-effector twist $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$ for joint rates $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ satisfying unit norm $\|\dot{\boldsymbol{\theta}}\| = \dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = 1$. It shows how easily can the EE be moved in various T-space directions.

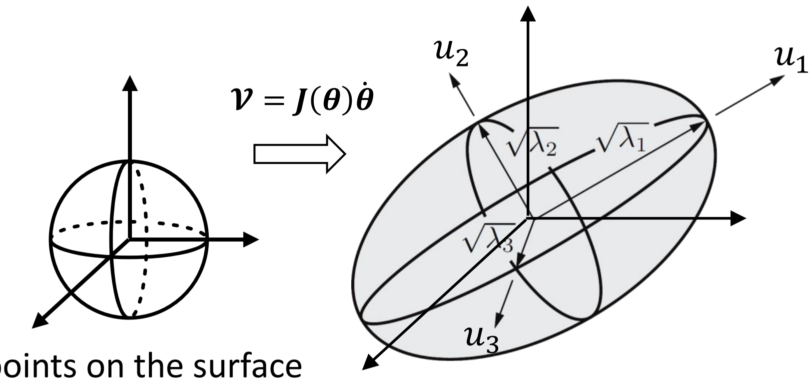
At a nonsingular configuration:

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \boldsymbol{\mathcal{V}} \in \mathbb{R}^r, \dot{\boldsymbol{\theta}} \in \mathbb{R}^n, \boldsymbol{J} \in \mathbb{R}^{r\times n}$$

$$\boldsymbol{J}^+ = \boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1}$$

$$n \geq r$$

$$\begin{aligned} 1 &= \dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} \\ &= (\boldsymbol{J}^+\boldsymbol{\mathcal{V}})^T(\boldsymbol{J}^+\boldsymbol{\mathcal{V}}) \\ &= \boldsymbol{\mathcal{V}}^T\boldsymbol{J}^{+T}\boldsymbol{J}^+\boldsymbol{\mathcal{V}} \\ &= \boldsymbol{\mathcal{V}}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}\boldsymbol{\mathcal{V}} \end{aligned}$$



(the points on the surface of a unit sphere)

(the points on the surface of an ellipsoid)

Note: $\boldsymbol{J}\boldsymbol{J}^T \in \mathbb{R}^{r\times r}$ is square, symmetric, and positive definite, as is $(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$.

Assume that $\boldsymbol{u}_i \in \mathbb{R}^r$ are eigenvectors and $\lambda_i \in \mathbb{R}$ are eigenvalues of $\boldsymbol{J}\boldsymbol{J}^T$ $(i = 1, \dots, r)$.

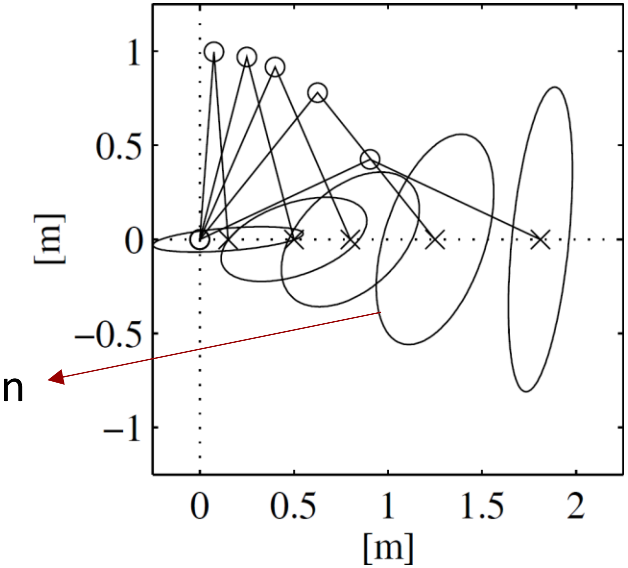
- Directions of the principal axes: $\boldsymbol{u}_i$
- Lengths of the principal semi-axes: $\sigma_i = \sqrt{\lambda_i}$ (σ_i are the singular values of $\boldsymbol{J}$)
- Volume is proportional to $\sqrt{\lambda_1\lambda_2\cdots\lambda_r} = \sqrt{\det(\boldsymbol{J}\boldsymbol{J}^T)} \xrightarrow{\text{if } n = r} = |\det(\boldsymbol{J})|$

Velocity Manipulability Ellipsoid

- Along the direction of the **major axis** of the ellipsoid, the end-effector can move at **large velocity**, while along the direction of the **minor axis small end-effector velocities** are obtained.
- The closer the ellipsoid is to a **sphere**, the better the end-effector can move isotropically along all directions of the task space.
- Velocity manipulability ellipsoid is used to visualize and characterize how close a nonsingular configuration of a robot is to being singular.

Velocity manipulability ellipses for a 2R planar arm (for $l_1 = l_2 = 1$):

$\theta_2 = \pi/2$: best posture for manipulation (similar to a human arm!)



Velocity Manipulability Measures

Velocity manipulability measures:

(1) The volume of the ellipsoid (proportional to $\sqrt{\lambda_1\lambda_2 \ldots}$):

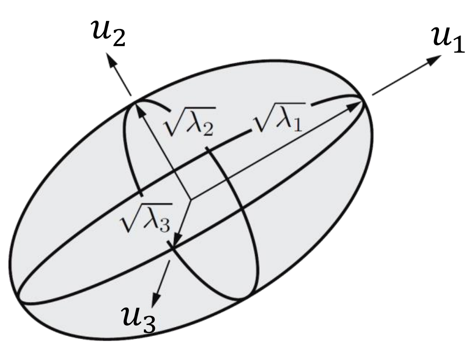
$$w_1(\boldsymbol{\theta}) = \sigma_1\sigma_2 \cdots = \sqrt{\lambda_1\lambda_2 \cdots} = \sqrt{\det(\boldsymbol{J}\boldsymbol{J}^T)} \geq 0 \xrightarrow[\text{(nonredundant)}]{\text{if } n=r} = |\det(\boldsymbol{J})|$$

- As the robot approaches a singularity, $w_1(\boldsymbol{\theta})$ goes to 0.

(2) The ratio of the largest to smallest principal semi-axes:

$$w_2(\boldsymbol{\theta}) = \frac{\sigma_{\max}}{\sigma_{\min}} = \frac{\sqrt{\lambda_{\max}(\boldsymbol{J}\boldsymbol{J}^T)}}{\sqrt{\lambda_{\min}(\boldsymbol{J}\boldsymbol{J}^T)}} = \sqrt{\frac{\lambda_{\max}(\boldsymbol{J}\boldsymbol{J}^T)}{\lambda_{\min}(\boldsymbol{J}\boldsymbol{J}^T)}} \geq 1$$

(w_2 is called condition number of $\boldsymbol{J}$)



(3) The ratio of the largest to smallest eigenvalues:

$$w_3(\boldsymbol{\theta}) = w_2(\boldsymbol{\theta})^2 = \frac{\lambda_{\max}(\boldsymbol{J}\boldsymbol{J}^T)}{\lambda_{\min}(\boldsymbol{J}\boldsymbol{J}^T)} \geq 1$$

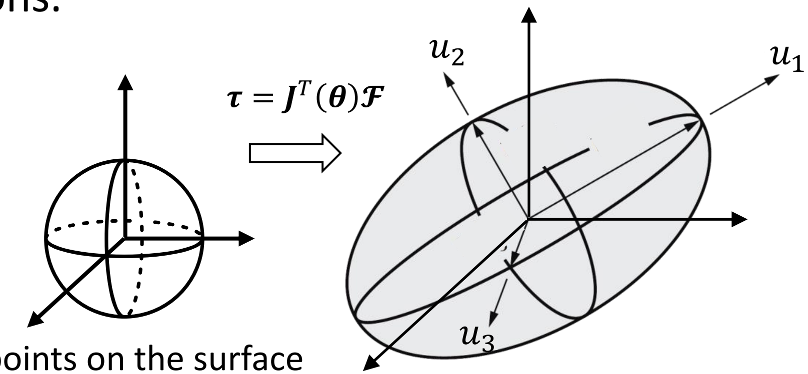
- When $w_2(\boldsymbol{\theta})$ or $w_3(\boldsymbol{\theta})$ is low (close to 1), the ellipsoid is nearly spherical or isotropic. As the robot approaches a singularity, $w_2(\boldsymbol{\theta})$ or $w_3(\boldsymbol{\theta})$ goes to infinity.

Force Manipulability Ellipsoid

The **force manipulability ellipsoid** corresponds to wrench $\boldsymbol{\mathcal{F}}$ generated at the end-effector by joint torques $\boldsymbol{\tau}$ satisfying $||\boldsymbol{\tau}|| = \boldsymbol{\tau}^T\boldsymbol{\tau} = 1$. It shows how easily can the EE apply wrenches (or balance applied ones) in various T-space directions.

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$$

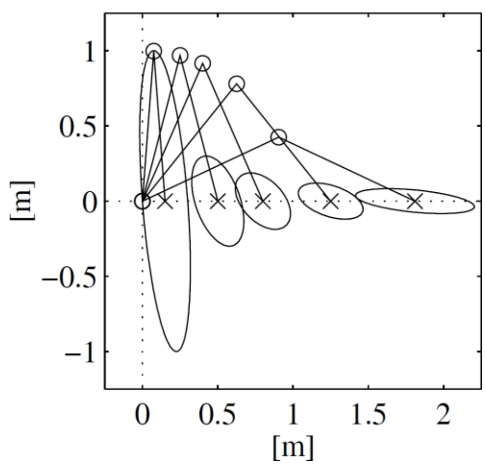
$$1 = \boldsymbol{\tau}^T\boldsymbol{\tau} = \boldsymbol{\mathcal{F}}^T\boldsymbol{J}\boldsymbol{J}^T\boldsymbol{\mathcal{F}}$$



(the points on the surface of a unit sphere)

(the points on the surface of an ellipsoid)

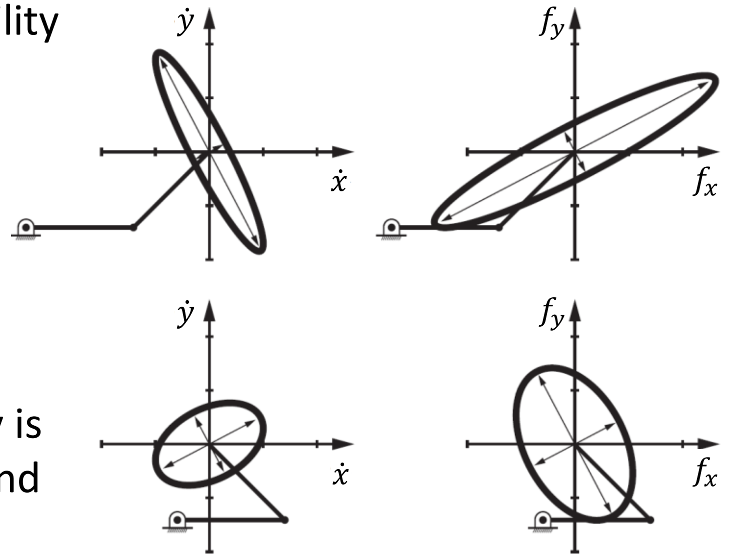
Force manipulability ellipses for a 2R planar arm (for $l_1 = l_2 = 1$):



Force/Velocity Duality

- The <u>principal axes</u> of the force manipulability ellipsoid <u>coincide</u> with the principal axes of the velocity manipulability ellipsoid.
- The <u>lengths</u> of the respective principal semi-axes are in <u>inverse proportion</u> ($\sqrt{\lambda_i}$ vs $1/\sqrt{\lambda_i}$).
- The <u>product of the volumes</u> of the velocity and force manipulability ellipsoids ($\propto \sqrt{\lambda_1 \lambda_2 \cdots}$ and $\propto 1/\sqrt{\lambda_1 \lambda_2 \cdots}$, respectively) is constant over $\boldsymbol{\theta}$.
- A direction along which it is easy to generate a tip velocity is a direction along which it is difficult to generate a force, and vice versa.

For a 2R Planar Robot:



- **At a singularity**,
  - the velocity manipulability ellipsoid collapses to a line segment (it loses dimension, and its area drops to zero). EE motion capability becomes zero in one (or more) direction(s),
  - the force manipulability ellipsoid becomes infinitely long in a direction orthogonal to the velocity manipulability ellipsoid line segment and skinny in the orthogonal direction (its area goes to infinity). EE can resist infinite force in one (or more) direction(s) without the need of any joint torque.

Visualizing Manipulability Ellipsoids

If it is desired to geometrically visualize the velocity (or force) manipulability ellipsoids in a space of dimension greater than 3, it is worth separating the components of linear velocity (or force) from those of angular velocity (or moment). This also resolves the issue regarding nonhomogeneous dimensions of the relevant quantities (e.g., m/s vs rad/s or N vs Nm).

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{J}_\omega(\boldsymbol{\theta}) \\ \boldsymbol{J}_v(\boldsymbol{\theta}) \end{bmatrix} \in \mathbb{R}^{6\times n}$$

$\boldsymbol{J}_\omega(\boldsymbol{\theta}) \in \mathbb{R}^{3\times n} \rightarrow$ angular velocity/moment ellipsoids

$\boldsymbol{J}_v(\boldsymbol{\theta}) \in \mathbb{R}^{3\times n} \rightarrow$ linear velocity/force ellipsoids

- When calculating the linear-velocity manipulability ellipsoid, it generally makes more sense to use the body Jacobian $\boldsymbol{J}_b$ or geometric Jacobian $\boldsymbol{J}_g$ instead of the space Jacobian $\boldsymbol{J}_s$, since we are usually interested in the linear velocity of a point at the origin of the end-effector frame rather than that of a point at the origin of the fixed-space frame.