MEC 549: Robot Dynamics and Control

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Ch1: A Review of Linear Algebra & Robot Kinematics

Linear Algebra: Matrices

Matrix Norm

General Definition: Given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, matrix norm $||\boldsymbol{A}|| \in \mathbb{R}_+$ is defined such that

- $||\boldsymbol{A}|| > 0$ when $\boldsymbol{A} \neq \boldsymbol{0}$ and $||\boldsymbol{A}|| = 0$ iff $\boldsymbol{A} = \boldsymbol{0}$.
- $||k\boldsymbol{A}|| = |k|||\boldsymbol{A}||, \ \forall k \in \mathbb{R}$.
- $||\boldsymbol{A}+\boldsymbol{B}|| \leq ||\boldsymbol{A}|| + ||\boldsymbol{B}||, \ \forall \boldsymbol{B} \in \mathbb{R}^{n\times n}$.
- $||\boldsymbol{A}\boldsymbol{B}|| \leq ||\boldsymbol{A}||||\boldsymbol{B}||, \ \forall \boldsymbol{B} \in \mathbb{R}^{n\times n}$.

The p-norm of $\boldsymbol{A}$ (induced by vector p-norms [1]) for $0 \leq p \leq \infty$ is defined as

$$||\boldsymbol{A}||_p = \sup_{||\boldsymbol{x}||_p=1} ||\boldsymbol{A}\boldsymbol{x}||_p = \sup_{\boldsymbol{x}\neq 0} \frac{||\boldsymbol{A}\boldsymbol{x}||_p}{||\boldsymbol{x}||_p} \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

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[1] The p-norm (or ℓ_p-norm) of $\boldsymbol{x}$ for $p \in \mathbb{R}, p \geq 1$ is defined as $||\boldsymbol{x}||_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

Matrix Norm (cont.)

In the special cases of $p = 1,2,\infty$, these norms can be computed/estimated by:

- $\|\boldsymbol{A}\|_1 = \max\limits_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$ (the max. absolute column sum of $\boldsymbol{A}$)
- $\|\boldsymbol{A}\|_2 = \sqrt{\lambda_{\max}(\boldsymbol{A}^T\boldsymbol{A})} = \sigma_{\max}(\boldsymbol{A})$ (Spectral Norm) $\xrightarrow{\text{If } \boldsymbol{A} = \boldsymbol{A}^T}$ $\begin{cases} \|\boldsymbol{A}\|_2 = \max\limits_i |\lambda_i(\boldsymbol{A})| \\ \|\boldsymbol{A}^{-1}\|_2 = 1/\min\limits_i |\lambda_i(\boldsymbol{A})| \end{cases}$

  (the square root of the maximum eigenvalue of $\boldsymbol{A}^T\boldsymbol{A}$, or the largest singular value of $\boldsymbol{A}$)
- $\|\boldsymbol{A}\|_\infty = \max\limits_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$ (the max. absolute row sum of $\boldsymbol{A}$)
- **Frobenius Norm**: $\|\boldsymbol{A}\|_F = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = \sqrt{\operatorname{tr}(\boldsymbol{A}^T\boldsymbol{A})}$

$\|\boldsymbol{Ax}\|_2 \le \|\boldsymbol{A}\|_2 \|\boldsymbol{x}\|_2$, $\quad \|\boldsymbol{A}\|_2 \le \|\boldsymbol{A}\|_F$

Bilinear Form

A **Bilinear Form** is a scalar function $B\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ such that:

$$B(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{A}^T \boldsymbol{x} \qquad \forall\, \boldsymbol{x}, \boldsymbol{y}$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_m)$, $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$, and $\boldsymbol{A} = \left[a_{ij}\right] \in \mathbb{R}^{m \times n}$.

- The gradient of the bilinear form with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$ are given by

$$\boldsymbol{\nabla}_{\boldsymbol{x}} B(\boldsymbol{x}, \boldsymbol{y}) = \left(\frac{\partial B(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{x}}\right)^T = \boldsymbol{A}\boldsymbol{y}$$

$$\boldsymbol{\nabla}_{\boldsymbol{y}} B(\boldsymbol{x}, \boldsymbol{y}) = \left(\frac{\partial B(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{y}}\right)^T = \boldsymbol{A}^T \boldsymbol{x}$$

Quadratic Form

A special case of bilinear form is the **Quadratic Form**. The quadratic form associated with a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is the scalar function $Q: \mathbb{R}^n \to \mathbb{R}$ such that:

$$Q(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \quad \forall \boldsymbol{x}$$

$$Q(x) = ax^2$$
$$Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$$
$$Q(x_1, x_2, x_3) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_2x_3 + ex_3^2 + fx_1x_3$$

- The quadratic function associated with a skew-symmetric matrix $\boldsymbol{A}_{ss}$ is always **zero**.

$$\boldsymbol{A}_{ss} \text{ is skew-symmetric} \quad \Leftrightarrow \quad \boldsymbol{x}^T \boldsymbol{A}_{ss} \boldsymbol{x} = 0 \quad (\forall \boldsymbol{x})$$

- Therefore, each quadratic function $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$ is always equal to a quadratic function with the symmetric part of matrix.

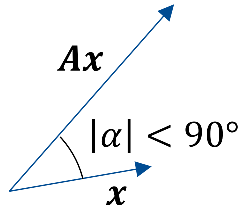
$$Q(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \boldsymbol{x}^T (\boldsymbol{A}_s + \boldsymbol{A}_{ss}) \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{A}_s \boldsymbol{x}$$

- If $\boldsymbol{A} = \boldsymbol{A}^T$: $\quad \nabla_x Q(\boldsymbol{x}) = \left(\frac{\partial Q(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T = 2\boldsymbol{A}\boldsymbol{x}\,, \qquad \dot{Q}(x) = \frac{d}{dt} Q(\boldsymbol{x}(t)) = 2\boldsymbol{x}^T \boldsymbol{A} \dot{\boldsymbol{x}} + \boldsymbol{x}^T \dot{\boldsymbol{A}} \boldsymbol{x}$

Definite and Semi-Definite Matrices

A square not necessarily symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is

- **Positive Definite** (PD or $\boldsymbol{A} > 0$) if $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} > 0$ for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$.
- **Positive Semi-Definite** (PSD or $\boldsymbol{A} \geq 0$) if $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
- **Negative Definite** (ND or $\boldsymbol{A} < 0$) if $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} < 0$ for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$.
- **Negative Semi-Definite** (NSD or $\boldsymbol{A} \leq 0$) if $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \leq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
- **Indefinite** if $\boldsymbol{A}$ neither positive semi-definite nor negative semi-definite.



Geometric Interpretation of the Positive Definiteness of $\boldsymbol{A}$.

- The quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is said to be PD iff matrix $\boldsymbol{A}$ is PD.

- A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is negative definite if $-\boldsymbol{A}$ is positive definite and it is negative semidefinite if $-\boldsymbol{A}$ is positive semidefinite.

- A **necessary** condition for $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ to be PD [or PSD] is that its diagonal elements be strictly positive [or nonnegative].

- Since $\boldsymbol{x}^T\boldsymbol{A}_{ss}\boldsymbol{x} = 0$, the test for the definiteness of $\boldsymbol{A}$ can be done by considering only its symmetric part.

Definite and Semi-Definite Matrices (cont.)

$\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is symmetric and PD [or PSD]. $\Leftrightarrow$ (Sylvester's theorem) Principal minors (i.e., a_{11}, $a_{11}a_{22} - a_{21}a_{12}$, ..., det $\boldsymbol{A}$) all are strictly positive [or nonnegative]. $\Leftrightarrow$ All its eigenvalues are strictly positive [or nonnegative].

- Any symmetric PD matrix $\boldsymbol{A} = \boldsymbol{A}^T > 0$ is always full-rank (nonsingular, invertible).

- Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a symmetric PD matrix and $\lambda_{\min}$, $\lambda_{\max}$ be the minimum and maximum eigenvalues of $\boldsymbol{A}$. For any $\boldsymbol{x} \in \mathbb{R}^n$,

$(\boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|_2^2)$

$$\lambda_{\min}\boldsymbol{x}^T\boldsymbol{x} \le \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le \lambda_{\max}\boldsymbol{x}^T\boldsymbol{x}$$ (Rayleigh–Ritz Theorem)

- Semi-definiteness of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ implies that $\text{rank}(\boldsymbol{A}) = r < n$, $\text{null}(\boldsymbol{A}) = n - r$, and thus r eigenvalues of $\boldsymbol{A}$ are positive/negative and $n - r$ are 0. For $\boldsymbol{x} \in \text{null}(\boldsymbol{A})$, $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$.

- A matrix inequality of the form $\boldsymbol{A}_1 > \boldsymbol{A}_2$, where $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{n \times n}$ means that $\boldsymbol{A}_1 - \boldsymbol{A}_2 > 0$, i.e., $\boldsymbol{A}_1 - \boldsymbol{A}_2$ is PD. Similar notations apply to the concepts of PSD, ND, NSD.

Rigid-Body Motions

Rigid-Body Motions

Rotations	Rigid-Body Motions
$\boldsymbol{R} \in SO(3)$: 3×3 matrices $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}_3$, $\det(\boldsymbol{R}) = 1$	$\boldsymbol{T} \in SE(3)$: 4×4 matrices $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$, where $\boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3$
$\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$	$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$
Change of coordinate frame: $\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{ac}$, $\boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$ $\left(\boldsymbol{R}_{ab} = \boldsymbol{R}_{ba}^{-1} = \boldsymbol{R}_{ba}^T\right)$	Change of coordinate frame: $\boldsymbol{T}_{ab}\boldsymbol{T}_{bc} = \boldsymbol{T}_{ac}$, $\boldsymbol{T}_{ab}\boldsymbol{p}_b = \boldsymbol{p}_a$ $\left(\boldsymbol{T}_{ab} = \boldsymbol{T}_{ba}^{-1}\right)$

Rigid-Body Motions

Rotations	Rigid-Body Motions
Rotating a frame $\{\mathrm{b}\}$: $$\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$ $\boldsymbol{R}_{sb'} = \boldsymbol{R}\boldsymbol{R}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ $\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\boldsymbol{R}$: rotate θ about $\hat{\boldsymbol{\omega}}_b = \hat{\boldsymbol{\omega}}$	Displacing a frame $\{\mathrm{b}\}$: $$\boldsymbol{T} = \begin{bmatrix} \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$ $\boldsymbol{T}_{sb'} = \boldsymbol{T}\boldsymbol{T}_{sb}$: rotate θ about $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{\omega}}$ (moves $\{\mathrm{b}\}$ origin), translate $\boldsymbol{p}$ in $\{\mathrm{s}\}$ $\boldsymbol{T}_{sb'} = \boldsymbol{T}_{sb}\boldsymbol{T}$: translate $\boldsymbol{p}$ in $\{\mathrm{b}\}$, rotate θ about $\hat{\boldsymbol{\omega}}$ in new body frame
Unit rotation axis is $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$, where $\|\hat{\boldsymbol{\omega}}\| = 1$	"Unit" screw axis is $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\|\boldsymbol{S}_\omega\| = 1$ or (ii) $\|\boldsymbol{S}_\omega\| = 0$, $\|\boldsymbol{S}_v\| = 1$
	For a screw axis $\{\boldsymbol{q}, \hat{\mathbf{s}}, h\}$ with finite h, $$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \\ -\hat{\mathbf{s}} \times \boldsymbol{q} + h\hat{\mathbf{s}} \end{bmatrix}$$
Angular velocity is $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\dot{\theta}$	Twist is $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{S}\dot{\theta}$

Rigid-Body Motions

Rotations	Rigid-Body Motions
For any $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, $[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ **Properties**: For any $\boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)$: $[\boldsymbol{\omega}] = -[\boldsymbol{\omega}]^{\mathrm{T}}, [\boldsymbol{\omega}]\boldsymbol{x} = -[\boldsymbol{x}]\boldsymbol{\omega}$, $[\boldsymbol{\omega}][\boldsymbol{x}] = ([\boldsymbol{x}][\boldsymbol{\omega}])^T, \boldsymbol{R}[\boldsymbol{\omega}]\boldsymbol{R}^{\mathrm{T}} = [\boldsymbol{R}\boldsymbol{\omega}]$	For any $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix} \in \mathbb{R}^6$ or $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_\omega \\ \boldsymbol{S}_v \end{bmatrix} \in \mathbb{R}^6$, $[\boldsymbol{\mathcal{V}}] = \begin{bmatrix} [\boldsymbol{\omega}] & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$, $[\boldsymbol{S}] = \begin{bmatrix} [\boldsymbol{S}_\omega] & \boldsymbol{S}_v \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$
$\dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = [\boldsymbol{\omega}_s],\ \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = [\boldsymbol{\omega}_b] \quad (\boldsymbol{R} \coloneqq \boldsymbol{R}_{sb})$	$\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} = [\boldsymbol{\mathcal{V}}_s],\ \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} = [\boldsymbol{\mathcal{V}}_b] \quad (\boldsymbol{T} \coloneqq \boldsymbol{T}_{sb})$
	$[\mathrm{Ad}_{\boldsymbol{T}}] = \begin{bmatrix} \boldsymbol{R} & \mathbf{0} \\ [\boldsymbol{p}]\boldsymbol{R} & \boldsymbol{R} \end{bmatrix} \in \mathbb{R}^{6\times 6}$ **Properties**: $[\mathrm{Ad}_{\boldsymbol{T}}]^{-1} = [\mathrm{Ad}_{\boldsymbol{T}^{-1}}]$, $[\mathrm{Ad}_{\boldsymbol{T}_1}][\mathrm{Ad}_{\boldsymbol{T}_2}] = [\mathrm{Ad}_{\boldsymbol{T}_1\boldsymbol{T}_2}]$
Change of coordinate frame: $\hat{\boldsymbol{\omega}}_a = \boldsymbol{R}_{ab}\hat{\boldsymbol{\omega}}_b, \quad \boldsymbol{\omega}_a = \boldsymbol{R}_{ab}\boldsymbol{\omega}_b$	Change of coordinate frame: $\boldsymbol{S}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{S}_b, \quad \boldsymbol{\mathcal{V}}_a = [\mathrm{Ad}_{\boldsymbol{T}_{ab}}]\boldsymbol{\mathcal{V}}_b$

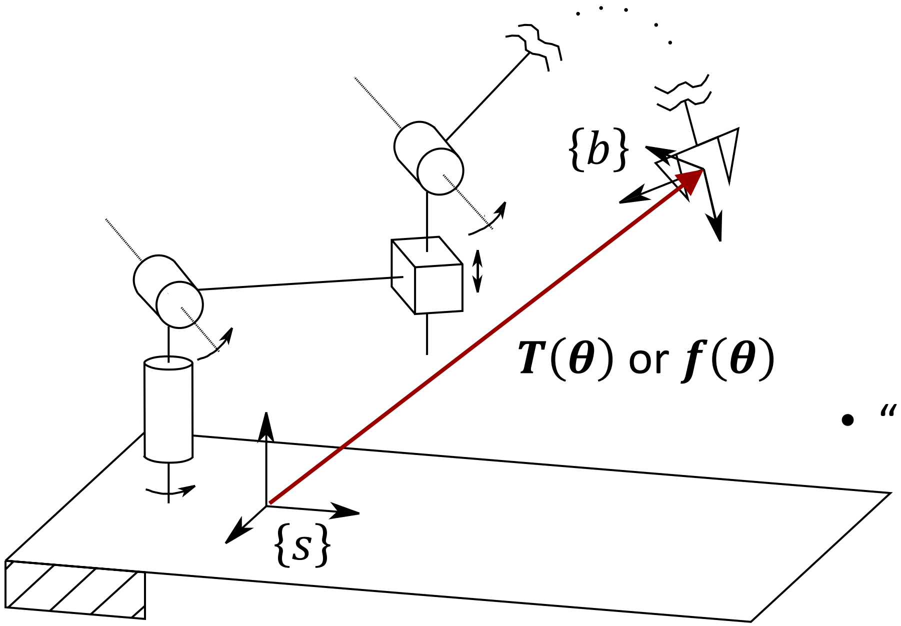
Rigid-Body Motions

Rotations	Rigid-Body Motions
$\hat{\boldsymbol{\omega}}_s = \boldsymbol{R}_{sb}\hat{\boldsymbol{\omega}}_b$	$\boldsymbol{S}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{S}_b, \boldsymbol{\mathcal{V}}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{\mathcal{V}}_b$
Exponential coordinate for $\boldsymbol{R} \in SO(3)$: $\hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$	Exponential coordinate for $\boldsymbol{T} \in SE(3)$: $\boldsymbol{S}\theta \in \mathbb{R}^6$
exp: $[\hat{\boldsymbol{\omega}}]\theta \in so(3) \rightarrow \boldsymbol{R} \in SO(3)$ $\boldsymbol{R} = \mathrm{Rot}(\hat{\boldsymbol{\omega}}, \theta) = e^{[\hat{\boldsymbol{\omega}}]\theta}$ $\boldsymbol{R} = \boldsymbol{I}_3 + \sin\theta[\hat{\boldsymbol{\omega}}] + (1-\cos\theta)[\hat{\boldsymbol{\omega}}]^2$ (Rodrigues' formula for rotations)	exp: $[\boldsymbol{S}]\theta \in se(3) \rightarrow \boldsymbol{T} \in SE(3)$ $\boldsymbol{T} = e^{[\boldsymbol{S}]\theta}$ $\boldsymbol{T} = \begin{bmatrix} e^{[\boldsymbol{S}_\omega]\theta} & \boldsymbol{G}(\theta)\boldsymbol{S}_v \\ \boldsymbol{0} & 1 \end{bmatrix}$ $\boldsymbol{G}(\theta) = \boldsymbol{I}_3\theta + (1-\cos\theta)[\boldsymbol{S}_\omega] + (\theta - \sin\theta)[\boldsymbol{S}_\omega]^2$
log: $\boldsymbol{R} \in SO(3) \rightarrow [\hat{\boldsymbol{\omega}}]\theta \in so(3)$ $\log(\boldsymbol{R}) = [\hat{\boldsymbol{\omega}}]\theta$	log: $\boldsymbol{T} \in SE(3) \rightarrow [\boldsymbol{S}]\theta \in se(3)$ $\log(\boldsymbol{T}) = [\boldsymbol{S}]\theta$
Moment change of coordinate frame: $\boldsymbol{m}_a = \boldsymbol{R}_{ab}\boldsymbol{m}_b$	Wrench change of coordinate frame: $\boldsymbol{\mathcal{F}}_a = \begin{bmatrix} \boldsymbol{m}_a \\ \boldsymbol{f}_a \end{bmatrix} = [\mathrm{Ad}_{\boldsymbol{T}_{ba}}]^{\mathrm{T}}\boldsymbol{\mathcal{F}}_b$

Forward/Velocity/Inverse Kinematics

Forward Kinematics

The forward kinematics of a robot refers to the calculation of the position and orientation (**pose**) of its end-effector frame from its joint positions $\boldsymbol{\theta}$.



- "Geometric" forward kinematics:

$$\text{Given } \boldsymbol{\theta} \in \mathbb{R}^n, \text{ Find } \boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$$

$$\boldsymbol{T}: \mathbb{R}^n \rightarrow SE(3)$$

(Using PoE or D-H Method)

$$\boldsymbol{T}(\boldsymbol{\theta}) = e^{[\boldsymbol{S}_1]\theta_1} \dots e^{[\boldsymbol{S}_{n-1}]\theta_{n-1}} e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M}$$

where $\boldsymbol{M} = \boldsymbol{T}_{sb}(\boldsymbol{0}) \in SE(3)$ and $\boldsymbol{S}_1, \dots, \boldsymbol{S}_n$ are screw axes expressed in $\{s\}$ when $\boldsymbol{\theta} = \boldsymbol{0}$.

- "Minimum-Coordinate" forward kinematics:

$$\text{Given } \boldsymbol{\theta} \in \mathbb{R}^n, \text{ Find } \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^r$$

$$\boldsymbol{f}: \mathbb{R}^n \rightarrow \mathbb{R}^r$$

Velocity Kinematics

- $\mathcal{V}_s = \begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\mathcal{V}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

$\underbrace{\hspace{10em}}_{\textbf{Geometric Jacobian}}$

- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{\phi}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\quad \boldsymbol{\phi} = (\alpha, \beta, \gamma)$
- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,q}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\quad \boldsymbol{q} = (q_0, q_1, q_2, q_3)$ $(\|\boldsymbol{q}\| = 1)$
- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\quad \boldsymbol{r} = \hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ $(\|\hat{\boldsymbol{\omega}}\| = 1, \theta \in [0, \pi])$

$\underbrace{\hspace{10em}}_{\textbf{Analytic Jacobian}}$

- $\boldsymbol{J}_s(\boldsymbol{\theta}) = [\boldsymbol{J}_{s1} \quad \boldsymbol{J}_{s2}(\boldsymbol{\theta}) \quad \cdots \quad \boldsymbol{J}_{sn}(\boldsymbol{\theta})]$, $\boldsymbol{J}_{si}(\boldsymbol{\theta}) = \left[\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}\right] \mathcal{S}_i$ $\quad i = 2, \dots, n,$ $\boldsymbol{J}_{s1} = \mathcal{S}_1$

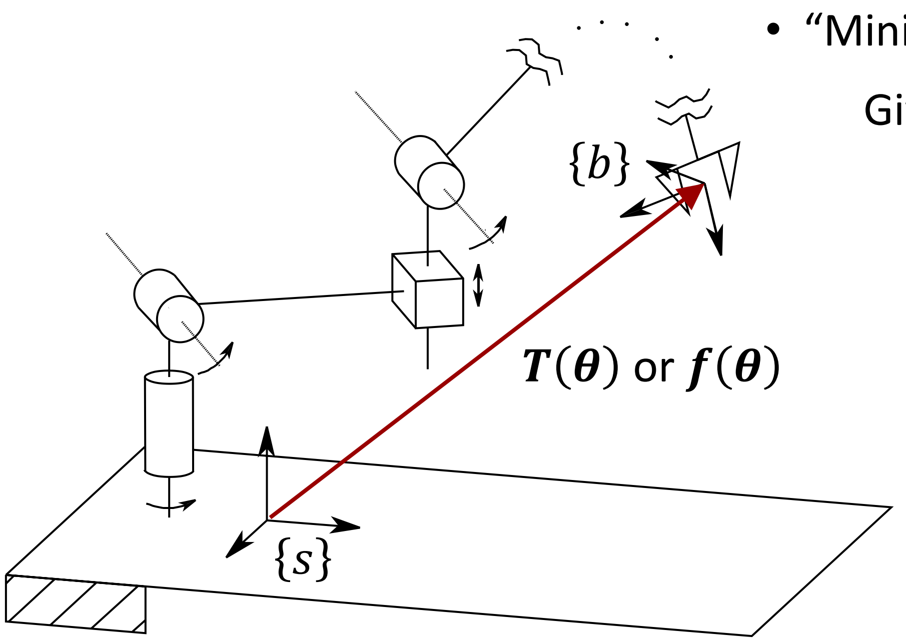
- $\boldsymbol{J}_b(\boldsymbol{\theta}) = [\mathrm{Ad}_{\boldsymbol{T}_{bs}}]\boldsymbol{J}_s(\boldsymbol{\theta})$

- Statics: $\boldsymbol{\tau} = \boldsymbol{J}_b^{\mathrm{T}}(\boldsymbol{\theta})\mathcal{F}_b$, $\boldsymbol{\tau} = \boldsymbol{J}_s^{\mathrm{T}}(\boldsymbol{\theta})\mathcal{F}_s$

- In singular configuration $\boldsymbol{\theta}^*$, $\boldsymbol{J}(\boldsymbol{\theta}^*) \in \mathbb{R}^{r \times n}$ is rank-deficient, i.e., $\mathrm{rank}\big(\boldsymbol{J}(\boldsymbol{\theta}^*)\big) < r$.

Inverse Kinematics

The inverse kinematics of a robot refers to the calculation of the joint coordinates $\boldsymbol{\theta}$ from the position and orientation (**pose**) of its end-effector frame.



- "Geometric" inverse kinematics:

  Given $\boldsymbol{T}_{sb} = \boldsymbol{T}(\boldsymbol{\theta}) \in SE(3)$, Find $\boldsymbol{\theta} \in \mathbb{R}^n$

- "Minimum-Coordinate" inverse kinematics:

  Given $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta}) \in \mathbb{R}^m$, Find $\boldsymbol{\theta} \in \mathbb{R}^n$

  - **Analytic Methods**: Finding closed-form solutions using algebraic or geometric intuition intuitions.
  - **Iterative Numerical Methods**: Using Newton–Raphson or Gradient Descent methods, respectively:

$$\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k + \lambda \boldsymbol{J}_a^{+}(\boldsymbol{\theta}^k)\left(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^k)\right)$$

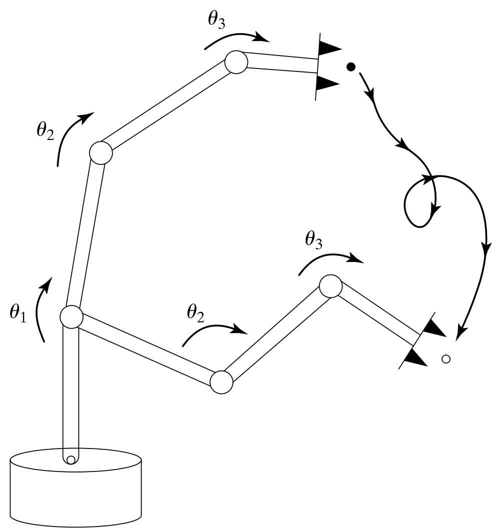
$$\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k + \lambda \boldsymbol{J}_a^{T}(\boldsymbol{\theta}^k)\left(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{\theta}^k)\right)$$

Trajectory Generation

Trajectory Generation: Path & Time Scaling

Trajectory $\mathcal{C}(s(t))$ or $\mathcal{C}(t)$ specifies the robot configuration as a function of time, i.e., the combination of a **path** $\mathcal{C}(s)$ and a **time scaling** $s(t)$.

$$\mathcal{C}: [0,1] \rightarrow \mathbb{C} \qquad\qquad s: [0, t_f] \rightarrow [0,1]$$



- Straight-Line Path in Joint Space: $\boldsymbol{\theta}(s) = \boldsymbol{\theta}_{\text{start}} + s(\boldsymbol{\theta}_{\text{end}} - \boldsymbol{\theta}_{\text{start}})$

- Straight-Line Path in Task Space:

(1) $\boldsymbol{x}(s) = \boldsymbol{x}_{\text{start}} + s(\boldsymbol{x}_{\text{end}} - \boldsymbol{x}_{\text{start}}) \in \mathbb{R}^m$

(2) $\boldsymbol{p}(s) = \boldsymbol{p}_{\text{start}} + s(\boldsymbol{p}_{\text{end}} - \boldsymbol{p}_{\text{start}}) \in \mathbb{R}^3$

$\boldsymbol{R}(s) = \boldsymbol{R}_{\text{start}} \exp\left(\log\left(\boldsymbol{R}_{\text{start}}^{\mathrm{T}} \boldsymbol{R}_{\text{end}}\right) s\right) \in SO(3)$

(3) $\boldsymbol{T}(s) = \boldsymbol{T}_{\text{start}} \exp(\log(\boldsymbol{T}_{\text{start}}^{-1} \boldsymbol{T}_{\text{end}}) s) \in SE(3)$

Examples of Time Scaling:

- 3rd-Order, 5th-Order Polynomial Position Profile $\begin{cases} s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \end{cases}$
- Trapezoidal/S-Curve Velocity Profile
- Polynomial Via Point Trajectories