Ch2: Robot Dynamics – Part 3

Dynamics in Task Space

Dynamics in Task Space
(Based on Geometric Jacobian)

Joint-space Dynamics: $\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}(\boldsymbol{\theta})^{\mathrm{T}}\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$

$$\begin{cases} \boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} & (\boldsymbol{\mathcal{V}}\text{: twist of end-effector}) \\ \dot{\boldsymbol{\mathcal{V}}} = \dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{J}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} & \\ \boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^{\mathrm{T}}\boldsymbol{\mathcal{F}} & (\boldsymbol{\mathcal{F}}\text{: wrench at end-effector}) \end{cases}$$

$$\boldsymbol{M}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}} = \ddot{\boldsymbol{\theta}} + \boldsymbol{M}^{-1}\boldsymbol{c} + \boldsymbol{M}^{-1}\boldsymbol{g} + \boldsymbol{M}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

$$\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}} = \boldsymbol{J}\ddot{\boldsymbol{\theta}} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{c} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{g} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

$$\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}} = \dot{\boldsymbol{\mathcal{V}}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{\theta}} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{c} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{g} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

$$\boldsymbol{\mathcal{F}} = \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{c}_C(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}_C(\boldsymbol{\theta}) + \boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$ (Task-space Dynamics)

$$\begin{cases} \boldsymbol{M}_C(\boldsymbol{\theta}) = \left(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{J}(\boldsymbol{\theta})^{\mathrm{T}}\right)^{-1} \\ \boldsymbol{c}_C(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{M}_C(\boldsymbol{\theta})\left(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right) \\ \boldsymbol{g}_C(\boldsymbol{\theta}) = \boldsymbol{M}_C(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{g}(\boldsymbol{\theta}) \end{cases}$$

Note: $\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^{\mathrm{T}}$ is invertible if and only if $\boldsymbol{J}$ is full-rank, i.e., in the absence of both kinematic and representation **singularities**.

Dynamics in Task Space
(Based on Geometric Jacobian)

For a nonredundant manipulator in a nonsingular configuration where $\boldsymbol{J}(\boldsymbol{\theta})$ is square and invertible:

$$\begin{cases} \dot{\boldsymbol{\theta}} = \boldsymbol{J}(\boldsymbol{\theta})^{-1}\boldsymbol{\mathcal{V}} \\ \ddot{\boldsymbol{\theta}} = \boldsymbol{J}(\boldsymbol{\theta})^{-1}\dot{\boldsymbol{\mathcal{V}}} - \boldsymbol{J}(\boldsymbol{\theta})^{-1}\dot{\boldsymbol{J}}(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})^{-1}\boldsymbol{\mathcal{V}} \\ \boldsymbol{\mathcal{F}} = \boldsymbol{J}(\boldsymbol{\theta})^{-\mathrm{T}}\boldsymbol{\tau} \end{cases}$$

$$\boxed{\boldsymbol{\mathcal{F}} = \boldsymbol{J}^{-\mathrm{T}}\boldsymbol{\tau} = \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{c}_C\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) + \boldsymbol{g}_C(\boldsymbol{\theta}) + \boldsymbol{\mathcal{F}}_{\mathrm{tip}}}$$

(Task-space Dynamics)

$$= \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{h}_C\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) + \boldsymbol{\mathcal{F}}_{\mathrm{tip}}$$

$$\begin{cases} \boldsymbol{M}_C(\boldsymbol{\theta}) = \boldsymbol{J}(\boldsymbol{\theta})^{-\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})^{-1} \\ \boldsymbol{c}_C\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) = \boldsymbol{J}(\boldsymbol{\theta})^{-\mathrm{T}}\boldsymbol{c}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) - \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \\ \boldsymbol{g}_C(\boldsymbol{\theta}) = \boldsymbol{J}(\boldsymbol{\theta})^{-\mathrm{T}}\boldsymbol{g}(\boldsymbol{\theta}) \end{cases}$$

Dynamics in Task Space
(Based on Geometric Jacobian) (cont.)

By considering $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$:

$$
\begin{aligned}
\boldsymbol{c}_C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) &= \boldsymbol{M}_C(\boldsymbol{\theta})\big(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) - \dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\big)\\
&= \boldsymbol{M}_C(\boldsymbol{\theta})\big(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\big)\\
&= \boldsymbol{M}_C(\boldsymbol{\theta})\left(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) - \dot{\boldsymbol{J}}(\boldsymbol{\theta})\right)\dot{\boldsymbol{\theta}}\\
&= \boldsymbol{C}_C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}
\end{aligned}
\quad \xrightarrow{\text{If } \boldsymbol{J}^{-1} \text{ exists}} \quad
\begin{aligned}
&= \big(\boldsymbol{J}^{-\mathrm{T}}\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) - \boldsymbol{M}_C(\boldsymbol{\theta})\dot{\boldsymbol{J}}\big)\boldsymbol{J}^{-1}\boldsymbol{\mathcal{V}}\\
&= \boldsymbol{C}_C(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\boldsymbol{\mathcal{V}}
\end{aligned}
$$

Note: In general, we cannot replace the dependence on $\boldsymbol{\theta}$ by a dependence on the end-effector configuration $\boldsymbol{T}$ because there may be multiple solutions to the inverse kinematics, and the dynamics depends on the specific joint configuration $\boldsymbol{\theta}$.

Note: For finding $\dot{\boldsymbol{J}}(\boldsymbol{\theta})$, let $\boldsymbol{J}_i(\boldsymbol{\theta})$ denote the ith column of $\boldsymbol{J}(\boldsymbol{\theta}) = [\boldsymbol{J}_1(\boldsymbol{\theta}), \dots, \boldsymbol{J}_n(\boldsymbol{\theta})]$, thus:

$$\dot{\boldsymbol{J}}(\boldsymbol{\theta}) = \frac{d}{dt}\boldsymbol{J}(\boldsymbol{\theta}) = \left[\frac{d}{dt}\boldsymbol{J}_1(\boldsymbol{\theta}), \dots, \frac{d}{dt}\boldsymbol{J}_n(\boldsymbol{\theta})\right] \quad \text{where} \quad \frac{d}{dt}\boldsymbol{J}_i(\boldsymbol{\theta}) = \sum_{j=1}^{n}\frac{\partial \boldsymbol{J}_i}{\partial \theta_j}\dot{\theta}_j$$

- If $\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{J}_s(\boldsymbol{\theta})$: $\dfrac{\partial \boldsymbol{J}_i}{\partial \theta_j} = \begin{cases} [\mathrm{ad}_{\boldsymbol{J}_j}]\,\boldsymbol{J}_i & i > j \\ \boldsymbol{0} & i \le j \end{cases}$

- If $\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{J}_b(\boldsymbol{\theta})$: $\dfrac{\partial \boldsymbol{J}_i}{\partial \theta_j} = \begin{cases} [\mathrm{ad}_{\boldsymbol{J}_i}]\,\boldsymbol{J}_j & i < j \\ \boldsymbol{0} & i \ge j \end{cases}$

Dynamics in Task Space
(Based on Analytic Jacobian)

In a similar way, by using $\boldsymbol{J}_a(\boldsymbol{\theta})$ where $\dot{\boldsymbol{x}} = \boldsymbol{J}_a(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and $\boldsymbol{\tau} = \boldsymbol{J}_a^T(\boldsymbol{\theta})\boldsymbol{F}$, dynamic equation in T-space can be written as

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{M}_C(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{c}_C\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) + \boldsymbol{g}_C(\boldsymbol{\theta}) + \boldsymbol{F}_{\text{tip}} \\ &= \boldsymbol{M}_C(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{C}_C\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + \boldsymbol{g}_C(\boldsymbol{\theta}) + \boldsymbol{F}_{\text{tip}}\end{aligned}$$

Note: All the properties of the J-space dynamic model carry over to the T-space dynamic model as long as $\boldsymbol{J}$ (or $\boldsymbol{J}_a$) is full-rank. For instance,

- $\boldsymbol{M}_C$ is symmetric and positive definite.
- For a revolute arm, $\boldsymbol{J}$ (or $\boldsymbol{J}_a$) is bounded, and $\boldsymbol{M}_C$ is bounded above and below.
- $\boldsymbol{S}_c = \dot{\boldsymbol{M}}_C - 2\boldsymbol{C}_C$ is skew-symmetric (if $\boldsymbol{C}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right)$ is in standard form).
- Property of linearity in the parameters: $\boldsymbol{F} = \boldsymbol{M}_C(\boldsymbol{\theta})\ddot{\boldsymbol{x}} + \boldsymbol{c}_C\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) + \boldsymbol{g}_C(\boldsymbol{\theta}) = \overbrace{\boldsymbol{J}^{-\mathrm{T}}\boldsymbol{Y}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}}\right)}^{\boldsymbol{Y}_C}\boldsymbol{\pi}$

Inverse & Forward Dynamics

Inverse Dynamics: Finding $\boldsymbol{\tau}$, given $\boldsymbol{T}$, $\boldsymbol{\mathcal{V}}$, $\dot{\boldsymbol{\mathcal{V}}}$, and $\boldsymbol{\mathcal{F}}_{\text{tip}}$ (or $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, $\ddot{\boldsymbol{x}}$, and $\boldsymbol{\mathcal{F}}_{\text{tip}}$).

$$\boldsymbol{T} = \text{FK}(\boldsymbol{\theta})$$
$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

or

$$\boldsymbol{x} = \text{FK}(\boldsymbol{\theta})$$
$$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\xrightarrow{\text{(Inverse Kinematics)}} \boldsymbol{\theta}, \dot{\boldsymbol{\theta}} \xrightarrow[\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}} = \boldsymbol{J}^{\mathrm{T}}\left(\boldsymbol{M}_C\ddot{\boldsymbol{x}} + \boldsymbol{c}_C + \boldsymbol{g}_C + \boldsymbol{F}_{\text{tip}}\right)]{\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}} = \boldsymbol{J}^{\mathrm{T}}\left(\boldsymbol{M}_C\dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{c}_C + \boldsymbol{g}_C + \boldsymbol{\mathcal{F}}_{\text{tip}}\right)} \boldsymbol{\tau}$$

For redundant manipulators, redundancy resolution is performed at **kinematic level**.

Forward Dynamics: Finding $\boldsymbol{T}$, $\boldsymbol{\mathcal{V}}$, $\dot{\boldsymbol{\mathcal{V}}}$ (or $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, $\ddot{\boldsymbol{x}}$) given the $\boldsymbol{\mathcal{F}}_{\text{tip}}$, $\boldsymbol{\tau}$.

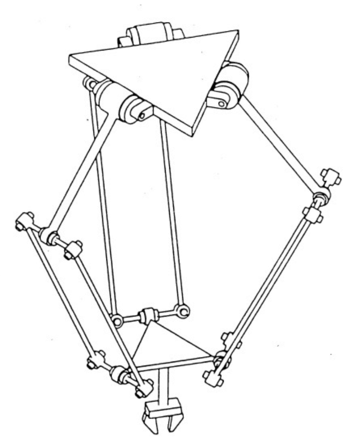
$$\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}} = \boldsymbol{J}^{\mathrm{T}}\left(\boldsymbol{M}_C\dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{c}_C + \boldsymbol{g}_C + \boldsymbol{\mathcal{F}}_{\text{tip}}\right)$$
$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$
$$\dot{\boldsymbol{\mathcal{V}}} = \dot{\boldsymbol{J}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{J}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}$$

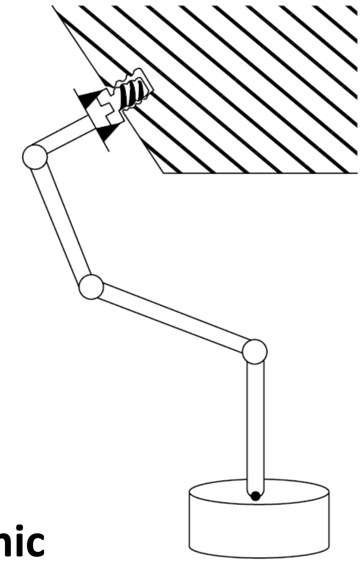
$$\longrightarrow \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}} \longrightarrow \boldsymbol{T}, \boldsymbol{\mathcal{V}}, \dot{\boldsymbol{\mathcal{V}}}$$

Constrained Dynamics

Constrained Dynamics

Sometimes robots are subject to a set of constraints on their motion.





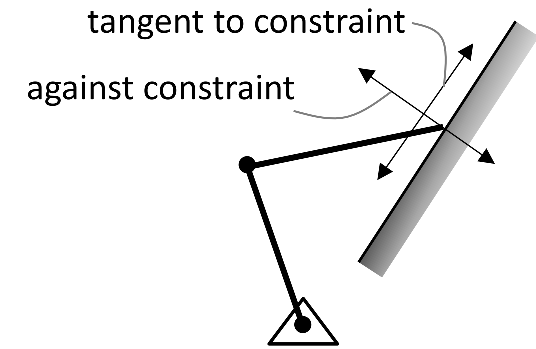
Consider the case where the n-joint robot is subject to a set of k **holonomic constraints** or **nonholonomic Pfaffian velocity constraints** of the form:

$$\boldsymbol{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0}, \qquad \boldsymbol{A}(\boldsymbol{\theta}) \in \mathbb{R}^{k\times n}$$

Assumption: These k **equality constraints** are **workless**, meaning that the forces that enforce these constraints do no work on the robot (e.g., frictionless contacts).

Constrained Dynamics

The space of joint torques/forces $\boldsymbol{\tau}$ is divided into two independent subspaces; (I) an $(n-k)$-dimensional subspace that affects the motion of the robot, but not the constraint force (this subspace is tangent to constraint) and (II) a k-dimensional subspace that affects the constraint force, but not the motion (this subspace is against the constraints).



$$\boldsymbol{\tau} = \underbrace{\boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}_{\text{Due to motion of the robot}} + \underbrace{\boldsymbol{\tau}_{\text{con}}}_{\text{Due to constraints}}$$

Workless Constraints Assumption: $\rightarrow$

$$\left.\begin{array}{l} \boldsymbol{\tau}_{\text{con}}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = 0 \\ \mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0} \end{array}\right\} \Rightarrow$$

$\boldsymbol{\tau}_{\text{con}}$ is a linear combination of the columns of $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})$ $\Rightarrow$

$$\boldsymbol{\tau}_{\text{con}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda}$$

$$\boldsymbol{\lambda} \in \mathbb{R}^k$$

($\boldsymbol{\lambda}$: Lagrange multipliers)

Constrained Equations of Motion

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda} & (1) \\ \mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0} \;\xrightarrow{\text{(or)}}\; \dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \mathbf{0} & (2) \end{cases}$$

$\Rightarrow$ $n+k$ equations, $n+k$ variables (for ID: $\boldsymbol{\lambda}, \boldsymbol{\tau}$, for FD: $\boldsymbol{\lambda}, \ddot{\boldsymbol{\theta}}$)

Constrained Dynamics

Thus, the robot has $n-k$ velocity freedoms and k force freedoms.

For finding $\boldsymbol{\lambda}$: (1) $\rightarrow$ $\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{h} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}\right)$ (3)

(2), (3) $\rightarrow$ $\boldsymbol{\lambda} = \left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{A}\boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{h}) + \dot{\boldsymbol{A}}\dot{\boldsymbol{\theta}}\right)$ (4)

By eliminating $\boldsymbol{\lambda}$ in (1) using (4), $n+k$ constrained equations of motion can be reduce to the dynamics projected to the $(n-k)$-dimensional space tangent to the constraints, i.e., (5):

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda} \\ \mathrm{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0} \quad \overset{\text{(or)}}{\rightarrow} \quad \dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \mathbf{0} \end{cases} \Rightarrow$$

(5) $\boldsymbol{P}\boldsymbol{\tau} = \boldsymbol{P}\left(\boldsymbol{M}\ddot{\boldsymbol{\theta}} + \boldsymbol{h}\right)$ where

$$\boldsymbol{P} = \boldsymbol{I}_n - \boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{A}\boldsymbol{M}^{-1} \in \mathbb{R}^{n\times n}$$

$\boldsymbol{I}_n$: $n \times n$ identity matrix, $\mathrm{rank}(\boldsymbol{P}) = n-k$

Note: If the constraint acts at the end-effector of an open-chain robot, then

$$\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\mathrm{tip}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda}$$

If $\boldsymbol{J}(\boldsymbol{\theta})$ is invertible: $\boldsymbol{\mathcal{F}}_{\mathrm{tip}} = \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda}$

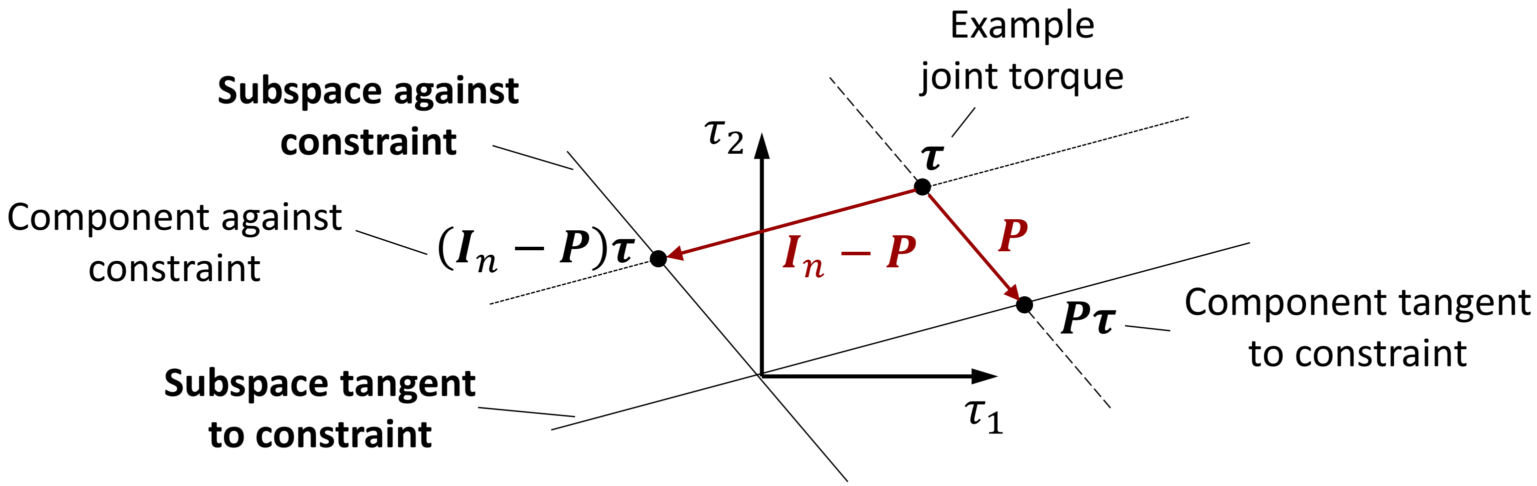
$\boldsymbol{\mathcal{F}}_{\mathrm{tip}}$: Wrench the end-effector applies against the constraint (and it does not have component tangent to constraint).

Constrained Dynamics

Therefore,

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\lambda} \\ \mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0} \end{cases} \equiv \qquad \boldsymbol{\tau} = \boldsymbol{P}(\theta)\boldsymbol{\tau} + \big(\boldsymbol{I}_n - \boldsymbol{P}(\boldsymbol{\theta})\big)\boldsymbol{\tau}$$

Matrix $\boldsymbol{P}(\theta)$ is a **projection matrix** that projects $\boldsymbol{\tau}$ to its component tangent to the constraint, the matrix $\boldsymbol{I}_n - \boldsymbol{P}(\boldsymbol{\theta})$ projects $\boldsymbol{\tau}$ to its component against the constraint.



Constrained Dynamics

Note: For numerical simulation (forward dynamics) of the constrained robot, we can use the following equations to find $\ddot{\boldsymbol{\theta}}$.

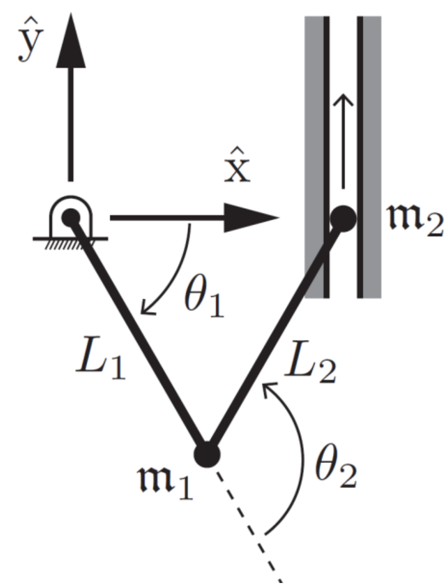
$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{h} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}\right) \quad \text{where}$$

$$\boldsymbol{\lambda} = \left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{A}\boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{h}) + \dot{\boldsymbol{A}}\dot{\boldsymbol{\theta}}\right)$$

Example

Consider a 2R robot whose tip is constrained to move in a frictionless linear channel at $x = 1$. The lengths of each link are $L_1 = L_2 = 1$, and the point masses at the ends of each link are $\mathfrak{m}_1 = \mathfrak{m}_2 = 1$. For simplicity, assume that $g = 0$. Consider the instance where $(\theta_1, \theta_2) = (-\pi/3, 2\pi/3)$ (as shown) and the tip is moving with the velocity $(\dot{x}, \dot{y}) = (0,1)$ m/s.

- Solve the constrained forward dynamics for $\ddot{\boldsymbol{\theta}} = (\ddot{\theta}_1, \ddot{\theta}_2)$ and λ when $\boldsymbol{\tau} = (\tau_1, \tau_2)$.
- Find the task-space constraint force $\boldsymbol{f}_{\text{tip}} = \left(f_x, f_y\right)$.
- Solve the constrained inverse dynamics for $\boldsymbol{\tau}$ given a $\ddot{\boldsymbol{\theta}}$ satisfying the constraint (i.e., $\dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \boldsymbol{0}$) and λ satisfying a desired force $\boldsymbol{f}_{\text{tip}} = \left(f_x, f_y\right) = (f, 0)$ against the channel.
- Find the projection $\boldsymbol{P}$.



$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix}$$

$$\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} \qquad \boldsymbol{g}(\boldsymbol{\theta}) = \boldsymbol{0} \qquad \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta})$$

Solution

There are $n = 2$ joint coordinates and $k = 1$ constraint.

If the tip of the robot is at (x, y), the robot's forward kinematics can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{bmatrix} \longrightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -s_1 - s_{12} & -s_{12} \\ c_1 + c_{12} & c_{12} \end{bmatrix}}_{\boldsymbol{J}(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \underbrace{\begin{bmatrix} -\dot{\theta}_1 c_1 - (\dot{\theta}_1 + \dot{\theta}_2)c_{12} & -(\dot{\theta}_1 + \dot{\theta}_2)c_{12} \\ -\dot{\theta}_1 s_1 - (\dot{\theta}_1 + \dot{\theta}_2)s_{12} & -(\dot{\theta}_1 + \dot{\theta}_2)s_{12} \end{bmatrix}}_{\dot{\boldsymbol{J}}(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The constraint is $c_1 + c_{12} = 1$. It is holonomic and can be written as

$$\underbrace{\begin{bmatrix} -s_1 - s_{12} & -s_{12} \end{bmatrix}}_{\boldsymbol{A}(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{\boldsymbol{A}}(\boldsymbol{\theta}) = \begin{bmatrix} -\dot{\theta}_1 c_1 - (\dot{\theta}_1 + \dot{\theta}_2)c_{12} & -(\dot{\theta}_1 + \dot{\theta}_2)c_{12} \end{bmatrix}$$

Solution

$(\theta_1, \theta_2) = (-\pi/3, 2\pi/3)$ and $(\dot{x}, \dot{y}) = (0,1)$ m/s imply that $(\dot{\theta}_1, \dot{\theta}_2) = (1,0)$ rad/s. Thus,

$$\boldsymbol{A}(\boldsymbol{\theta}) = [0 \quad -0.866] \qquad \dot{\boldsymbol{A}}(\boldsymbol{\theta}) = [-1 \quad -0.5]$$

$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \qquad \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} 0 \\ 0.866 \end{bmatrix}$$

Constrained forward dynamics:

$$\lambda = \left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{A}\boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{h}) + \dot{\boldsymbol{A}}\dot{\boldsymbol{\theta}}\right) \longrightarrow \lambda = 0.289\tau_1 - 1.155\tau_2 - 0.167$$

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{h} - \boldsymbol{A}^{\mathrm{T}}\lambda\right) \longrightarrow \begin{aligned} \ddot{\theta}_1 &= 0.5\tau_1 + 0.289 \\ \ddot{\theta}_2 &= -1.155 \end{aligned}$$

Solution

Task-space constraint forces: $\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{f}_{\mathrm{tip}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda$

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & -0.866 \\ 1 & 0.5 \end{bmatrix}$$

Since $\boldsymbol{J}\ (\boldsymbol{\theta})$ is invertible: $\boldsymbol{f}_{\mathrm{tip}} = \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda$

$$\boldsymbol{f}_{\mathrm{tip}} = \begin{bmatrix} 0.577 & -1.155 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -0.866 \end{bmatrix}\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda = \begin{bmatrix} 0.289\tau_1 - 1.155\tau_2 - 0.167 \\ 0 \end{bmatrix}$$

This agrees with our understanding that the robot can only apply forces against the constraint in the f_x-direction.

Note: If $\boldsymbol{\tau} = \mathbf{0}$, the task-space constraint force is $\boldsymbol{f}_{\mathrm{tip}} = (-0.167, 0)$, meaning that the robot's tip pushes to the left on the constraint while the constraint pushes back equally to the right to enforce the constraint. In the absence of the constraint, the acceleration of the tip of the robot would have a component to the left.

Solution

Constrained inverse dynamics:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda$$

$$\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{f}_{\mathrm{tip}} = \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta})\lambda \longrightarrow \begin{bmatrix} 0 & -0.866 \\ 1 & 0.5 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.866 \end{bmatrix} \lambda \longrightarrow \lambda = f$$

$$\dot{\boldsymbol{A}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{A}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \boldsymbol{0} \longrightarrow \ddot{\theta}_2 = -1.155, \ \forall \ddot{\theta}_1 \longrightarrow \ddot{\boldsymbol{\theta}} = (\ddot{\theta}_1, -1.155)$$

$$\Rightarrow \quad \begin{aligned} \tau_1 &= 2\ddot{\theta}_1 - 0.578 \\ \tau_2 &= 0.5\ddot{\theta}_1 - 0.866f - 0.289 \end{aligned}$$

Projection $\boldsymbol{P}$:

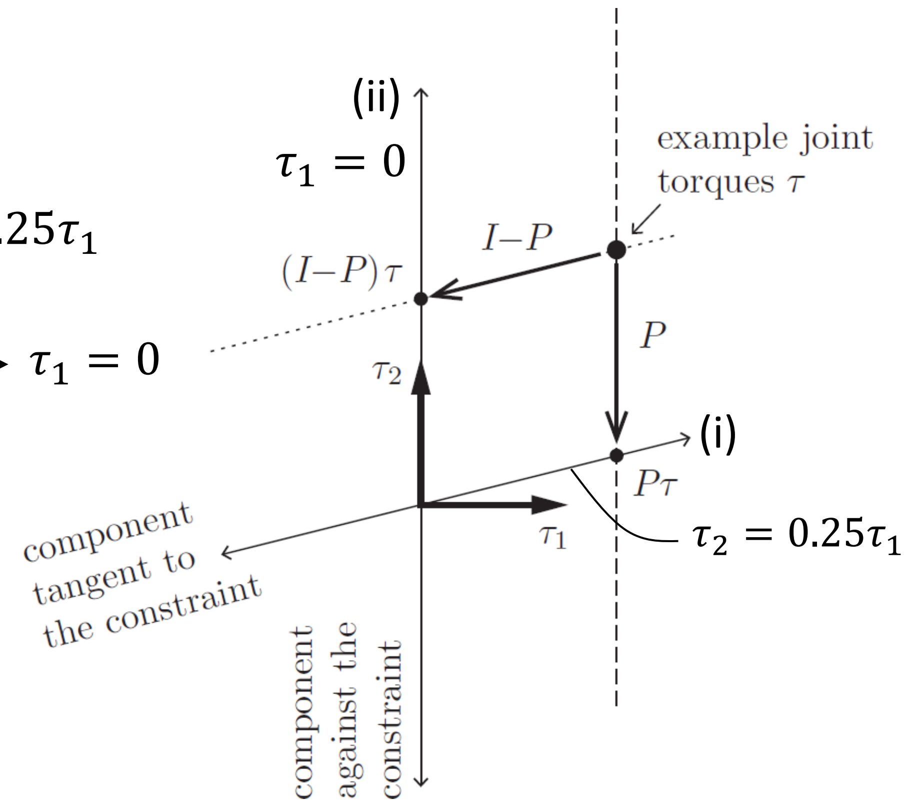
$$\boldsymbol{P} = \boldsymbol{I}_n - \boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}\boldsymbol{A}\boldsymbol{M}^{-1} \in \mathbb{R}^{n\times n} \longrightarrow \boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0.25 & 0 \end{bmatrix}, \qquad \boldsymbol{I}_2 - \boldsymbol{P} = \begin{bmatrix} 0 & 0 \\ -0.25 & 1 \end{bmatrix}$$

Solution

$$\boldsymbol{\tau} = \boldsymbol{P}(\theta)\boldsymbol{\tau} + \big(\boldsymbol{I}_2 - \boldsymbol{P}(\boldsymbol{\theta})\big)\boldsymbol{\tau}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0.25 & 0\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}\tau_1\\ 0.25\tau_1\end{bmatrix} \rightarrow \tau_2 = 0.25\tau_1$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ -0.25 & 1\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}0\\ -0.25\tau_1 + \tau_2\end{bmatrix} \rightarrow \tau_1 = 0$$



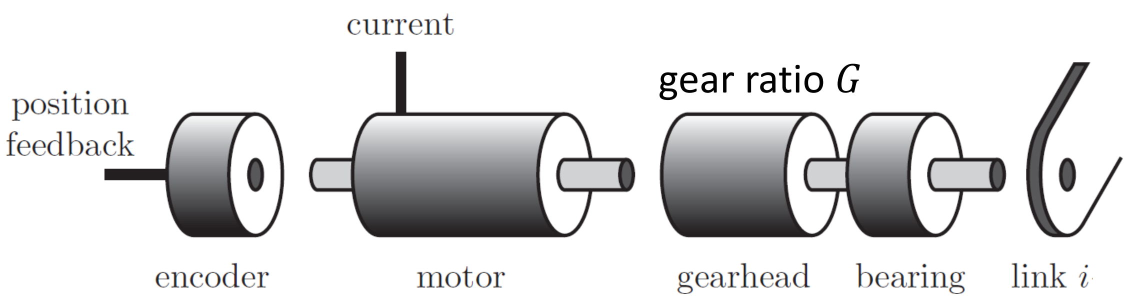
At the current state, (i) joint torques lying in the one-dimensional subspace $\tau_2 = 0.25\tau_1$ do not affect the constraint force, and (ii) joint torques lying in the one-dimensional subspace $\tau_1 = 0$ do not affect the motion of the robot.

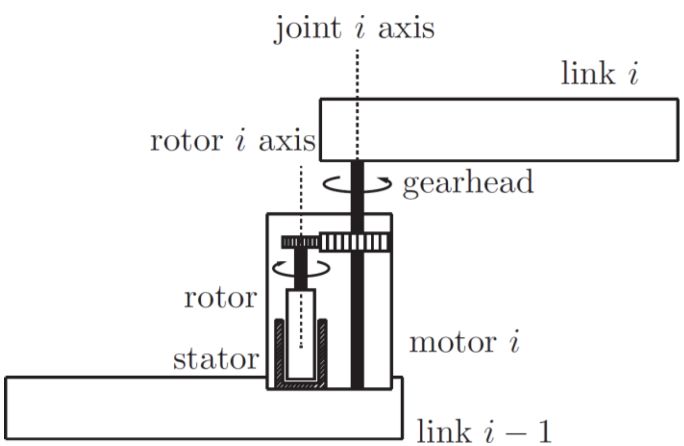
Actuation and Gearing

Actuation and Gearing

In practice, there are many types of **actuators** (e.g., electric, hydraulic, and pneumatic) and **mechanical power transformers** (e.g., gearheads). The actuators can be located at the joints themselves or remotely, with **mechanical power transmitted** by cables or timing belts.

Each combination of these has its own characteristics that can play a significant role in the "extended dynamics" mapping the actual control inputs to the motion of the robot.



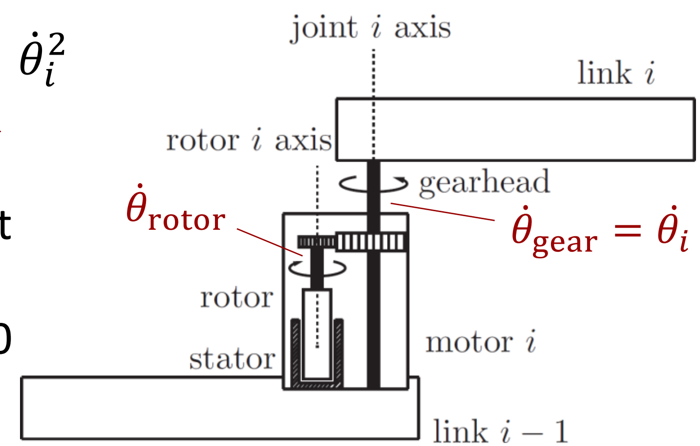


- The outer cases of the encoder, motor, gearhead, and bearing are all fixed to link $i-1$.
- The gearhead output shaft (rotor) is fixed in link i.

Apparent Motor Inertia

The **stator** of motor of link i is attached to link $i-1$ and its **rotor** is attached to link i, through a gearhead. Therefore, when calculating the contribution of a motor to the masses and inertias of the links, the mass and inertia of the stator must be assigned to link $i-1$ and the mass and inertia of the rotor must be assigned to link i. Typically, the mass and inertia of a motor's rotor ($m_{\text{rotor}}, \boldsymbol{I}_{\text{rotor}}$) are much less than the mass and inertia of link i ($m_i, \boldsymbol{I}_i$), so it's tempting to ignore the rotor's mass and inertia. However, the rotor spins G times faster than link i because of the gearhead, so the effect of the motor's inertia about the joint axis i could be significant.

$$\mathcal{K} = \frac{1}{2}\boldsymbol{I}_{\text{rotor}}\dot{\theta}_{\text{rotor}}^2 = \frac{1}{2}\boldsymbol{I}_{\text{rotor}}\left(\boldsymbol{G}\dot{\theta}_i\right)^2 = \frac{1}{2}\underbrace{\boldsymbol{G}^2\boldsymbol{I}_{\text{rotor}}}_{\text{apparent inertia}}\dot{\theta}_i^2$$

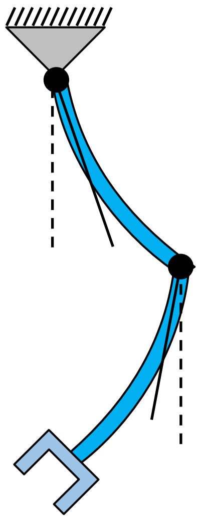


Therefore, the **apparent rotor inertia** $G^2\boldsymbol{I}_{\text{rotor}}$ ($\boldsymbol{I}_{\text{rotor}}$ as it appears at the output of the gearhead or joint axis i) may not be small, especially considering that gear ratios of 100 or more are common. Therefore, the rotor inertia should be also included in the dynamic analysis.

Joint and Link Flexibility

In practice, a robot's joints and links are likely to exhibit some flexibility and vibrations.

Flexible joints and links introduce extra states to the dynamics of the robot, significantly complicating the dynamics and control.



Dynamic Parameter Identification

Dynamic Parameter Identification

Using the dynamic equations of the manipulators for simulation and control purposes needs a good knowledge of dynamic parameters (e.g., m_i,$m_i l_{C_x,i}$, $m_i l_{C_y,i}$, $m_i l_{C_z,i}$, $\boldsymbol{I}_{b,i}$, $F_{v,i}$, $F_{s,i}$).

Different methods for identification of dynamic parameter:

- **Using CAD models** to compute the values of the inertial parameters of the various components (links, actuators, and transmissions) based on their **geometry** and **type of materials** employed ($\rightarrow$ inaccurate due to simplification typically introduced by geometric modelling, unable to model friction parameters)
- **Dismantling components** of the manipulator and perform measurements to find the dynamic parameters ($\rightarrow$ not easy to implement)
- **Using numerical identification techniques** which exploit the **property of linearity** of the dynamic model of open-chain manipulators ($\rightarrow$ accurate)

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s\,\mathbf{sgn}(\dot{\boldsymbol{\theta}}) = \boldsymbol{Y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}})\boldsymbol{\pi}$$

$$\boldsymbol{\pi} \in \mathbb{R}^p \qquad \boldsymbol{Y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}}) \in \mathbb{R}^{n\times p}$$

Dynamic Parameter Identification

In these techniques, we impose a suitable motion trajectory, and compute the parameter vector $\boldsymbol{\pi}$ from the measurements of joint torques $\boldsymbol{\tau}$ and evaluation of the matrix $\boldsymbol{Y}$ at N time instants $t_1, \dots, t_N$ along the trajectory (typically, $Nn \gg p$).

$$\bar{\boldsymbol{\tau}} = \begin{bmatrix} \boldsymbol{\tau}(t_1) \\ \vdots \\ \boldsymbol{\tau}(t_N) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}(t_1) \\ \vdots \\ \boldsymbol{Y}(t_N) \end{bmatrix} \boldsymbol{\pi} = \bar{\boldsymbol{Y}} \boldsymbol{\pi}$$

$\boldsymbol{\pi} \in \mathbb{R}^p$

$\bar{\boldsymbol{\tau}} \in \mathbb{R}^{Nn}$

$\bar{\boldsymbol{Y}} \in \mathbb{R}^{Nn \times p}$

(Tall Matrix)

By a least-squares technique: $$\boldsymbol{\pi} = \underbrace{(\bar{\boldsymbol{Y}}^T \bar{\boldsymbol{Y}})^{-1} \bar{\boldsymbol{Y}}^T}_{\text{left pseudo-inverse of } \bar{\boldsymbol{Y}}} \bar{\boldsymbol{\tau}}$$

Remarks:

- Assumption: Kinematic parameters in $\boldsymbol{Y}$ are known with good accuracy, e.g., after kinematic calibration.
- $\ddot{\boldsymbol{\theta}}$ needs to be calculated using measurements of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$.
- It is possible to identify only the dynamic parameters of the manipulator that contribute to the dynamic model.
- Some parameters can be identified in linear combinations whenever they do not appear isolated in the equations.
- Trajectories should not excite any unmodelled dynamic effects such as joint elasticity or link flexibility.
- The technique can be extended to the parameter identification of an unknown payload at the end-effector.

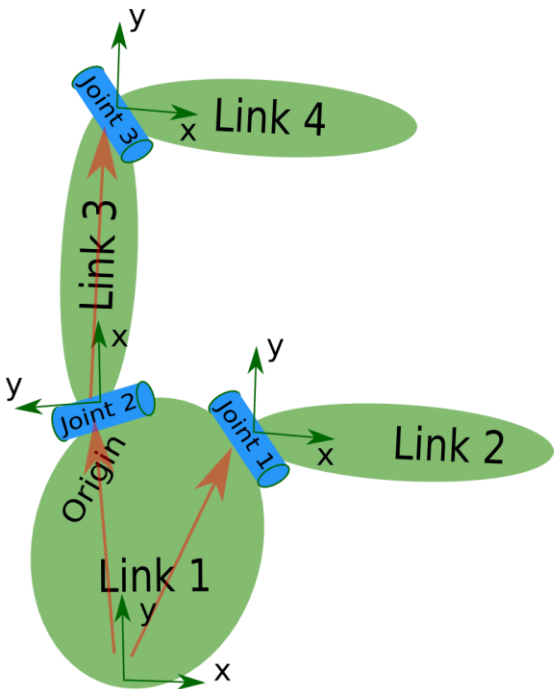
URDF

Universal Robot Description Format

The Universal Robot Description Format (URDF) is an XML (eXtensible Markup Language) file format used by the Robot Operating System (ROS) to describe the kinematics (in defining joints), inertial properties, and link geometry of robots (in defining links) of open-chain robots.

```
<joint name="joint1" type="continuous">
        <parent link="link1"/>
        <child link="link2"/>
        <origin xyz="0.5 0.3 0" rpy="0 0 0" />
        <axis xyz="-0.9 0.15 0" />
</joint>

<link name="link1">
        <inertial>
                <mass value="1"/>
                <origin rpy="0.1 0 0" xyz="0 0 0"/>
                <inertia ixx="0.004" ixy="0" ixz="0"
                        iyy="0.004" iyz="0" izz="0.007"/>
        </inertial>
</link>
```



Link

URDF: Defining Joints

Joints connect two links: a parent link and a child link.
The reference frame of each (child) link $\{L_i\}$ is located (at the bottom of the link) on the joint's axis.

(for example)

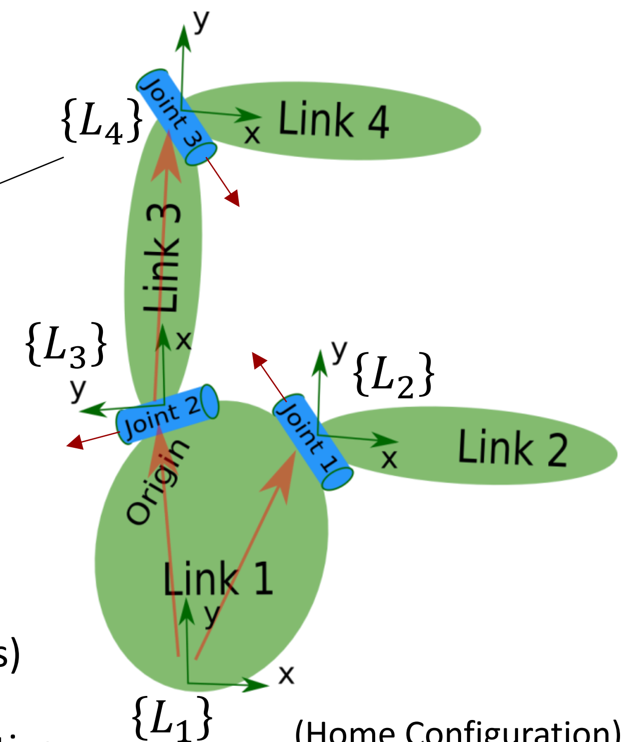
```
<joint name="joint3" type="continuous">
        <parent link="link3"/>
        <child link="link4"/>
        <origin xyz="0.5 0 0" rpy="0 0 -1.57" />
        <axis xyz="0.707 -0.707 0" />
</joint>
```

$\boxed{:\ \{L_4\}\ \text{w.r.t.}\ \{L_3\}}$

$\boxed{:\ \text{in}\ \{L_4\}}$

"rpy" Roll–Pitch–Yaw Angles (about fixed frames)

"origin" frame defines the pose of the child link frame relative to the parent link frame when the joint variable is zero.

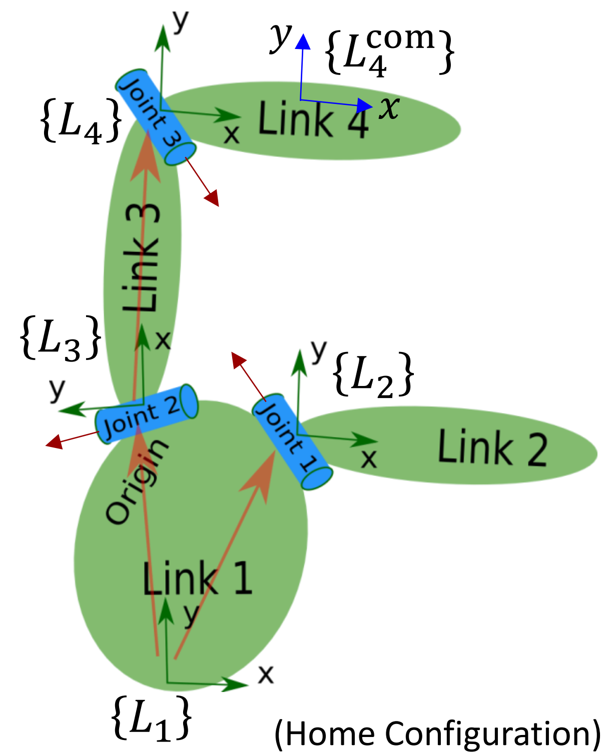


(Home Configuration)

"axis" defines the joint's axis, a <u>unit vector</u> expressed in the child link's frame, in the direction of <u>positive rotation</u> for a revolute joint or <u>positive translation</u> for a prismatic joint.

URDF: Defining Links

<link name=" **link4**">
    <inertial>
        <mass value="1"/>
        <origin xyz="0.1 0 0" rpy="0 0 0"/> : $\{L_4^{\text{com}}\}$ w.r.t. $\{L_4\}$
        <inertia ixx="0.004" ixy="0" ixz="0"
            iyy="0.004" iyz="0" izz="0.007"/> : in $\{L_4^{\text{com}}\}$
    </inertial>
    <visual>
        <geometry>
            <mesh filename=".../link1.stl" />
        </geometry>
        <material name="DarkGrey">
            <color rgba="0.3 0.3 0.3 1.0"/>
        </material>
    </visual>
</link>



(Home Configuration)

"inertia": six elements of inertia matrix relative to a frame at the link's center of mass.

"origin" frame defines the position and orientation of a frame at the link's center of mass relative to the link's frame at its joint.