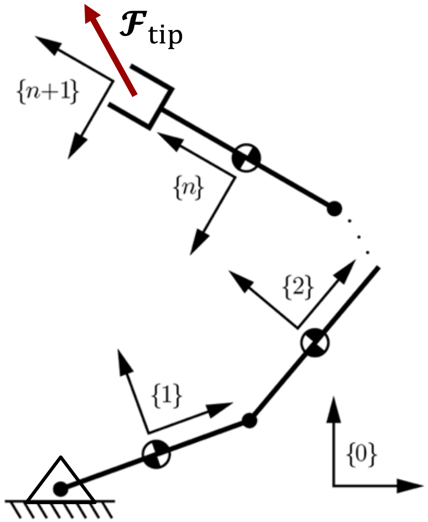
Ch2: Robot Dynamics – Part 2

Inverse Dynamics

Inverse Dynamic Equations in Closed Form

Inverse dynamic equations of an open-chain manipulator (finding $\boldsymbol{\tau}$ given $\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}, \boldsymbol{\mathcal{F}}_{\text{tip}}$) can be organized into a closed-form as

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}\end{aligned}$$



$\boldsymbol{\theta} \in \mathbb{R}^n$: Joint Variables

$\boldsymbol{\tau} \in \mathbb{R}^n$: Joint Torques/Forces

$\boldsymbol{M}(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$: Mass Matrix

$\boldsymbol{g}(\boldsymbol{\theta}) \in \mathbb{R}^n$: Gravitational Terms

$\boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$: Coriolis and Centripetal, and Gravitational Terms

$\boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) \in \mathbb{R}^n$: Coriolis and Centripetal Terms (velocity-product term or quadratic velocity term)

$\boldsymbol{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n\times n}$: Coriolis Matrix

$\boldsymbol{\Gamma}(\boldsymbol{\theta})$: $n \times n \times n$ matrix of Christoffel symbols of the first kind

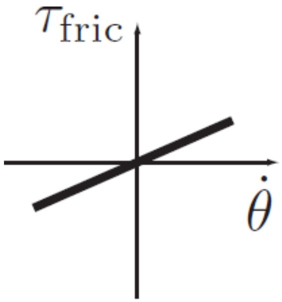
$\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{n\times 6}$: Jacobian in the same frame as $\boldsymbol{\mathcal{F}}_{\text{tip}}$

$\boldsymbol{\mathcal{F}}_{\text{tip}} \in \mathbb{R}^6$: Wrench applied to the environment by end-effector in the same frame as $\boldsymbol{J}(\boldsymbol{\theta})$

Friction Torques/Forces at Joints

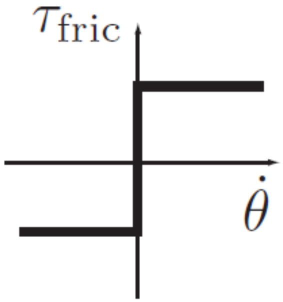
The Lagrangian and Newton–Euler dynamics do not account for friction at the joints. However, the friction torques/forces in gearheads and bearings may be significant.

Friction models often include a static friction term and a velocity-dependent viscous friction term.



Viscous friction

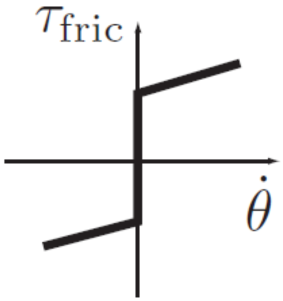
$\tau_{\text{fric}} = b_{\text{vis}}\,\dot{\theta}$



Static or Coulomb friction

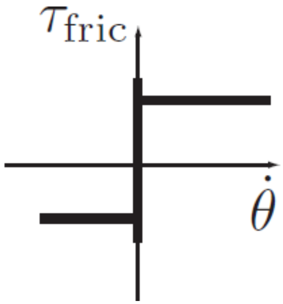
$\tau_{\text{fric}} = \mu_{\text{s}}\,\text{sgn}(\dot{\theta})$

$\tau_{\text{fric}} \in [-\mu_{\text{s}}, \mu_{\text{s}}]$ when $\dot{\theta} = 0$



Static and viscous friction

$\tau_{\text{fric}} = \mu_{\text{s}}\,\text{sgn}(\dot{\theta}) + b_{\text{vis}}\,\dot{\theta}$

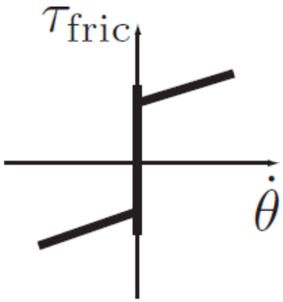


Static and kinetic friction

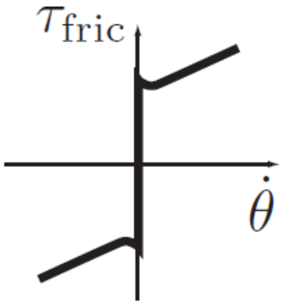
$\tau_{\text{fric}} \geq |\mu_{\text{s}}|$ to initiate motion

$\tau_{\text{fric}} = \mu_{\text{k}}\,\text{sgn}(\dot{\theta})$ during motion

$\mu_{\text{s}} > \mu_{\text{k}}$



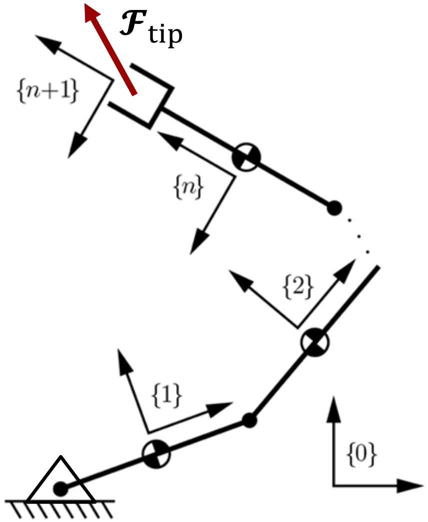
Static, kinetic, and viscous friction



A friction law exhibiting the Stribeck effect

Inverse Dynamic Equations in Closed Form

In the presence of the viscous and static friction torques/forces at the joints:



$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{f}_v(\dot{\boldsymbol{\theta}}) + \boldsymbol{f}_s(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \underbrace{\boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s\,\mathbf{sgn}(\dot{\boldsymbol{\theta}})}_{\text{simplified models}} + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}\end{aligned}$$

$\boldsymbol{F}_v \in \mathbb{R}^{n\times n}$: Diagonal matrix of viscous friction coefficients

$\boldsymbol{F}_s \in \mathbb{R}^{n\times n}$: Diagonal matrix of Coulomb friction coefficients

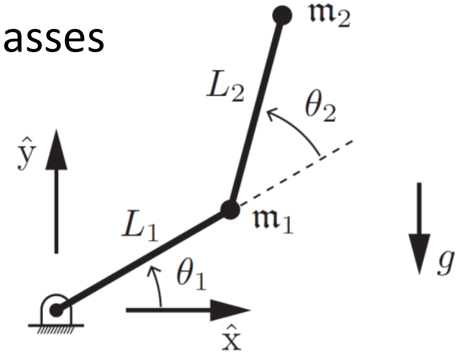
$\mathbf{sgn}(\dot{\boldsymbol{\theta}}) \in \mathbb{R}^{n\times 1}$: A vector whose components are the sign functions of $\dot{\theta}_i$

We can also add a disturbance $\boldsymbol{\tau}_{\text{dist}}$ to represent inaccurately modeled dynamics, etc.

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s\mathbf{sg\,n}(\dot{\boldsymbol{\theta}}) + \boldsymbol{\tau}_{\text{dist}} + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$$

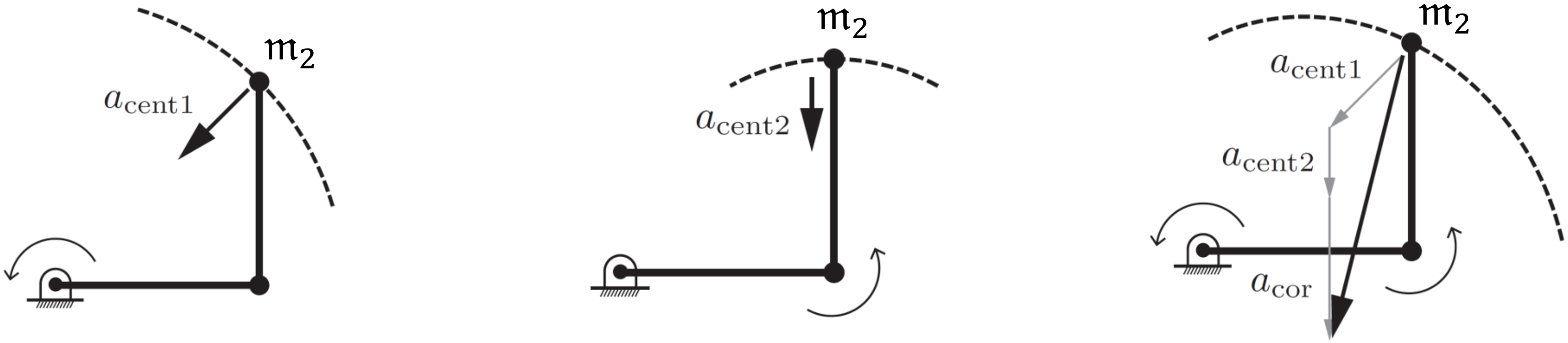
Understanding Centripetal and Coriolis Terms

Consider a planar 2R open chain whose links are modeled as point masses concentrated at the ends of each link:



Accelerations of $\mathfrak{m}_2$ when $\boldsymbol{\theta} = (0, \pi/2)$ and $\ddot{\boldsymbol{\theta}} = \mathbf{0}$:

$$\begin{bmatrix}\ddot{x}_2\\ \ddot{y}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-L_1\dot{\theta}_1^2\\ -L_2\dot{\theta}_1^2 - L_2\dot{\theta}_2^2\end{bmatrix}}_{\text{centripetal terms}} + \underbrace{\begin{bmatrix}0\\ -2L_2\dot{\theta}_1\dot{\theta}_2\end{bmatrix}}_{\text{Coriolis terms}}$$



$\boldsymbol{a}_{\text{cent1}} = (-L_1\dot{\theta}_1^2, -L_2\dot{\theta}_1^2)$

$\dot{\theta}_1 > 0, \quad \dot{\theta}_2 = 0$

$\boldsymbol{a}_{\text{cent2}} = (0, -L_2\dot{\theta}_2^2)$

$\dot{\theta}_1 = 0, \quad \dot{\theta}_2 > 0$

$\boldsymbol{a}_{\text{cor}} = (0, -2L_2\dot{\theta}_1\dot{\theta}_2)$

$\dot{\theta}_1, \dot{\theta}_2 > 0$

Understanding Mass Matrix

The total kinetic energy $\mathcal{K}$ of a robot can be expressed as the sum of the kinetic energies of each link:

$$\mathcal{K} = \frac{1}{2}\sum_{i=1}^{n} \mathcal{V}_i^{\mathrm{T}} \mathcal{G}_i \mathcal{V}_i$$

twist of link frame $\{i\}$ in $\{i\}$

spatial inertia matrix of link i in $\{i\}$

Let define $\boldsymbol{J}_{ib}(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$ as body Jacobian of link frame $\{i\}$ such that $\boldsymbol{\mathcal{V}}_i = \boldsymbol{J}_{ib}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$, $i = 1, \dots, n$, thus:

$$[\mathcal{V}_i] = \boldsymbol{T}_{0i}^{-1}\dot{\boldsymbol{T}}_{0i}$$

$$\mathcal{K} = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}} \underbrace{\left(\sum_{i=1}^{n} \boldsymbol{J}_{ib}^{\mathrm{T}}(\boldsymbol{\theta})\mathcal{G}_i \boldsymbol{J}_{ib}(\boldsymbol{\theta})\right)}_{\boldsymbol{M}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{J}_{ib}^{\mathrm{T}}(\boldsymbol{\theta})\mathcal{G}_i \boldsymbol{J}_{ib}(\boldsymbol{\theta})} \dot{\boldsymbol{\theta}}$$

This is the mass matrix

$$\Rightarrow \quad \mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}} \boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

kinetic energy of an open-chain robot

Understanding Mass Matrix (cont.)

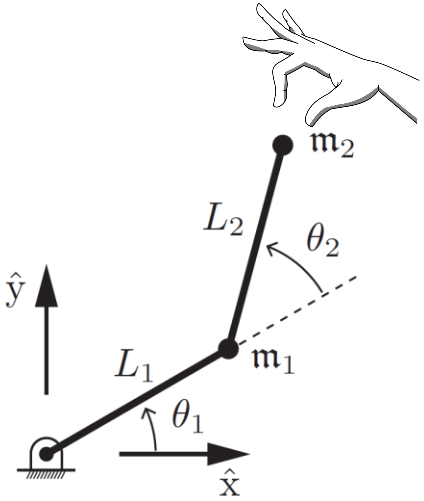
A mass matrix $\boldsymbol{M}(\boldsymbol{\theta})$ presents a different effective mass in different acceleration directions. For better understanding, let represent $\boldsymbol{M}(\boldsymbol{\theta})$ as an effective (or apparent) mass of the end-effector as $\boldsymbol{M}_C(\boldsymbol{\theta})$, because it is possible to feel this mass directly by grabbing and moving the end-effector.

If $\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is the end-effector twist and $\boldsymbol{J}(\boldsymbol{\theta})$ is invertible,

$$\mathcal{K} = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \frac{1}{2}\boldsymbol{\mathcal{V}}^{\mathrm{T}}\boldsymbol{M}_C(\boldsymbol{\theta})\boldsymbol{\mathcal{V}}$$

Kinetic energy of the robot regardless of the coordinates.

$$\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{M}_C(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$



If $\boldsymbol{J}$ is invertible: $\boldsymbol{M}_C(\boldsymbol{\theta}) = \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{J}^{-1}(\boldsymbol{\theta})$

If $\boldsymbol{J}$ is not invertible:

A general expression that applies to both redundant and nonredundant manipulators:

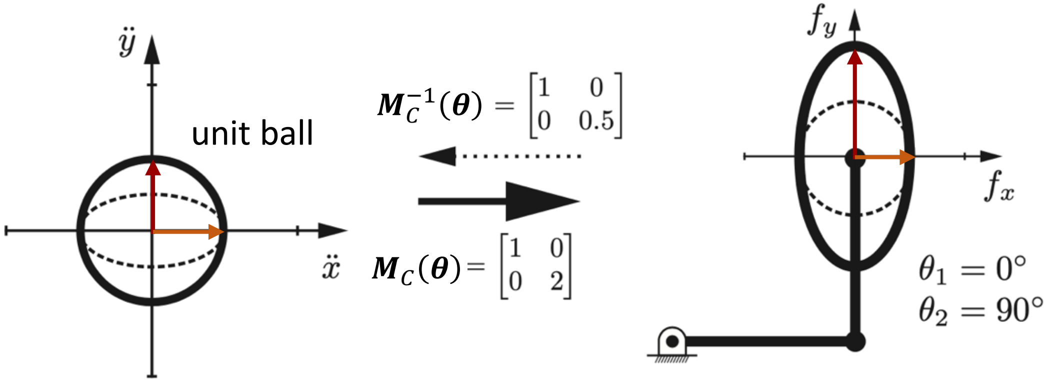
$$\boldsymbol{M}_C(\boldsymbol{\theta}) = \left(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})^{-1}\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\right)^{-1}$$

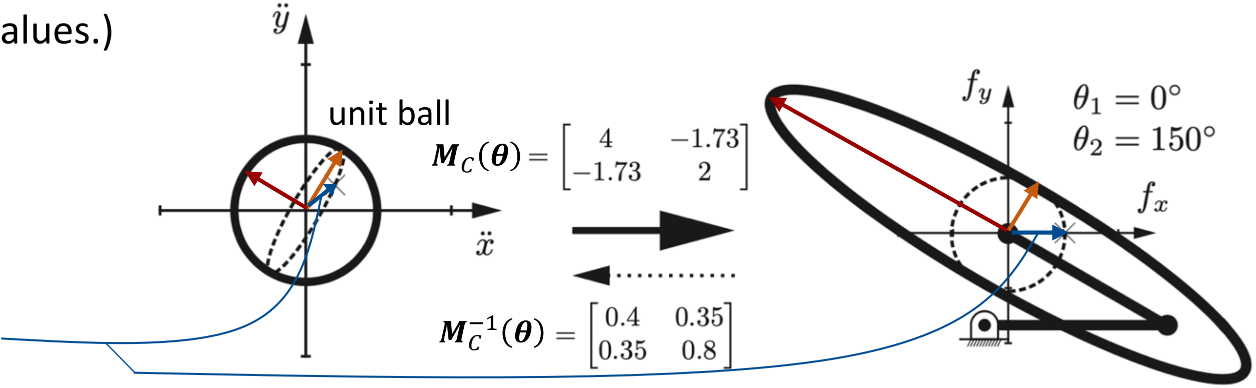
Understanding Mass Matrix (cont.)

Consider the 2R robot with $L_1 = L_2 = \mathfrak{m}_1 = \mathfrak{m}_2 = 1$. When the robot is at rest ($\dot{\boldsymbol{\theta}} = \mathbf{0}$) and $g = 0$, $\boldsymbol{M}_C(\boldsymbol{\theta})$ maps the endpoint acceleration $(\ddot{x}, \ddot{y})$ to (f_x, f_y), i.e., $(f_x, f_y) = \boldsymbol{M}_C(\boldsymbol{\theta})(\ddot{x}, \ddot{y})$.

Force and acceleration are only parallel along principal axes.

(Principal-axis directions given by the eigenvectors of $\boldsymbol{M}_C(\boldsymbol{\theta})$ and principal s axis lengths given by its eigenvalues.)





An example where force and acceleration are not parallel.

Finding Dynamic Terms

Finding Dynamic Terms Using Lagrangian Formulation

$$\boldsymbol{\tau} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}{\partial \dot{\boldsymbol{\theta}}}\right] - \frac{\partial \mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}}, \qquad \mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \mathcal{P}(\boldsymbol{\theta}), \qquad \mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\theta}}} = \frac{\partial \mathcal{K}}{\partial \dot{\boldsymbol{\theta}}} = \boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\theta}}} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}, \qquad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\theta}}\left(\dot{\boldsymbol{\theta}}^{T}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right) - \frac{\partial \mathcal{P}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$

⇨ $$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\theta}}\left[\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right] + \frac{\partial \mathcal{P}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \xrightarrow{\text{Comparing with}} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta})$$

⇨ $$\begin{cases} \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} - \dfrac{1}{2}\dfrac{\partial}{\partial \boldsymbol{\theta}}\left[\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right] = \dot{\boldsymbol{M}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} - \dfrac{\partial \mathcal{K}}{\partial \boldsymbol{\theta}}, \\ \boldsymbol{g}(\boldsymbol{\theta}) = \dfrac{\partial \mathcal{P}}{\partial \boldsymbol{\theta}} \end{cases}$$

Finding Dynamic Terms Using Lagrangian Formulation

(cont.)

Componentwise Analysis: $\tau_k = \frac{d}{dt}\frac{\partial \mathcal{L}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}{\partial \dot{\theta}_k} - \frac{\partial \mathcal{L}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})}{\partial \theta_k} \qquad k = 1, \dots, n$

- $\frac{\partial \mathcal{L}}{\partial \dot{\theta}_k} = \sum_{j=1}^{n} m_{kj}(\theta)\dot{\theta}_j$
- $\frac{\partial \mathcal{L}}{\partial \theta_k} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial m_{ij}}{\partial \theta_k}\dot{\theta}_j\dot{\theta}_i - \frac{\partial P}{\partial \theta_k}$
- $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_k} = \sum_{j=1}^{n}\left(m_{kj}(\theta)\ddot{\theta}_j + \left[\frac{d}{dt}m_{kj}(\theta)\right]\dot{\theta}_j\right)$

$$= \sum_{j=1}^{n} m_{kj}\ddot{\theta}_j + \underbrace{\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{kj}}{\partial \theta_i}\dot{\theta}_i\dot{\theta}_j}$$

(due to symmetry)

$$\boxed{\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{kj}}{\partial \theta_i}\dot{\theta}_i\dot{\theta}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j}\right]\dot{\theta}_j\dot{\theta}_i}$$

$$\Rightarrow \quad \tau_k = \sum_{j=1}^{n} m_{kj}\ddot{\theta}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{\frac{1}{2}\left[\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right]}_{\Gamma_{ijk}(\boldsymbol{\theta})}\dot{\theta}_j\dot{\theta}_i + \frac{\partial P}{\partial \theta_k}, \qquad k = 1, \dots, n$$

$\Gamma_{ijk}(\boldsymbol{\theta})$ is a $n \times n \times n$ matrix known as Christoffel symbols of the first kind.

Finding Dynamic Terms Using Lagrangian Formulation

(cont.)

Thus, we can write the components of $\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ as

$$c_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma_{ijk}(\boldsymbol{\theta})\dot{\theta}_j\dot{\theta}_i$$

$$\Rightarrow \quad \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \underbrace{\boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}\,\dot{\boldsymbol{\theta}} = \underbrace{\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}}$$

$$C_{kj}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \sum_{i=1}^{n} \Gamma_{ijk}(\boldsymbol{\theta})\dot{\theta}_i$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left[\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right]\dot{\theta}_i$$

$$\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \equiv \begin{bmatrix} \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}_1(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}_2(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \\ \vdots \\ \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}_n(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \end{bmatrix}$$

$\boldsymbol{\Gamma}_i(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{\Gamma}_i(\boldsymbol{\theta}) = \boldsymbol{\Gamma}_i^T(\boldsymbol{\theta})$

(j, k)th element of $\Gamma_{ijk}(\boldsymbol{\theta})$

Finding Dynamic Terms Using Newton–Euler Formulation (Method 1: Closed Form)

Using the closed form of dynamic equations, we can write

$$
\begin{aligned}
\boldsymbol{\mathcal{V}} &= \boldsymbol{\mathcal{L}}(\boldsymbol{\theta})\big(\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}} + \boldsymbol{\mathcal{V}}_{\text{base}}\big)\\
\dot{\boldsymbol{\mathcal{V}}} &= \boldsymbol{\mathcal{L}}(\boldsymbol{\theta})\big(\boldsymbol{\mathcal{A}}\ddot{\boldsymbol{\theta}} - [\text{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}](\boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\boldsymbol{\mathcal{V}} + \boldsymbol{\mathcal{V}}_{\text{base}}) + \dot{\boldsymbol{\mathcal{V}}}_{\text{base}}\big)\\
\boldsymbol{\mathcal{F}} &= \boldsymbol{\mathcal{L}}^{T}(\boldsymbol{\theta})\big(\boldsymbol{\mathcal{G}}\dot{\boldsymbol{\mathcal{V}}} - [\text{ad}_{\boldsymbol{\mathcal{V}}}]^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{V}} + \overline{\boldsymbol{\mathcal{F}}}_{\text{tip}}\big),\\
\boldsymbol{\tau} &= \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{F}}
\end{aligned}
$$

$$\Big\downarrow \quad \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}\big(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\big) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$$

For a fixed based manipulator where $\boldsymbol{\mathcal{V}}_0 = \mathbf{0}$.

$$
\begin{aligned}
\boldsymbol{M}(\boldsymbol{\theta}) &= \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}^{T}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}(\boldsymbol{\theta})\boldsymbol{\mathcal{A}}\\
\boldsymbol{c}\big(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\big) &= -\boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}^{T}(\boldsymbol{\theta})\big(\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}(\boldsymbol{\theta})[\text{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}]\boldsymbol{\mathcal{W}}(\boldsymbol{\theta}) + [\text{ad}_{\boldsymbol{\mathcal{V}}}]^{\mathrm{T}}\boldsymbol{\mathcal{G}}\big)\boldsymbol{\mathcal{L}}(\boldsymbol{\theta})\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}\\
\boldsymbol{g}(\boldsymbol{\theta}) &= \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}^{T}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}(\boldsymbol{\theta})\dot{\boldsymbol{\mathcal{V}}}_{\text{base}}\\
\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}} &= \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}^{T}(\boldsymbol{\theta})\overline{\boldsymbol{\mathcal{F}}}_{\text{tip}}
\end{aligned}
$$

Note: $\dot{\boldsymbol{M}}$ can be written explicitly as

$$\dot{\boldsymbol{M}} = -\boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}^{\mathrm{T}}\boldsymbol{\mathcal{W}}^{\mathrm{T}}\big[\text{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}\big]^{\mathrm{T}}\boldsymbol{\mathcal{L}}^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{L}}\big[\text{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}\big]\boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{A}}$$

Finding Dynamic Terms Using Newton–Euler Formulation (Method 2)

We know that using the recursive Newton-Euler inverse dynamics algorithm we can find $\boldsymbol{\tau}$. Thus,

- Term $\boldsymbol{g}(\boldsymbol{\theta})$ is computed by finding $\boldsymbol{\tau}|_{\dot{\boldsymbol{\theta}}=\ddot{\boldsymbol{\theta}}=\mathbf{0},\, \boldsymbol{\mathcal{F}}_{\text{tip}}=\mathbf{0}}$

- Term $\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is computed by finding $\boldsymbol{\tau}|_{\ddot{\boldsymbol{\theta}}=\mathbf{0},\, \boldsymbol{\mathcal{F}}_{\text{tip}}=\mathbf{0},\, \mathfrak{g}=\mathbf{0}}$

- Term $\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$ is computed by finding $\boldsymbol{\tau}|_{\dot{\boldsymbol{\theta}}=\ddot{\boldsymbol{\theta}}=\mathbf{0},\, \mathfrak{g}=\mathbf{0}}$

- Term $\boldsymbol{M}(\boldsymbol{\theta}) = [\boldsymbol{M}_1(\boldsymbol{\theta}), \ldots, \boldsymbol{M}_n(\boldsymbol{\theta})]$ is computed by

$$\boldsymbol{M}_i(\boldsymbol{\theta}) = \boldsymbol{\tau}\Big|_{\dot{\boldsymbol{\theta}}=\mathbf{0},\, \boldsymbol{\mathcal{F}}_{\text{tip}}=\mathbf{0},\, \mathfrak{g}=\mathbf{0},\, \ddot{\theta}_i=1,\, \ddot{\theta}_j=0,\, \forall j\neq i}$$

(Alternatively, we can use: $\boldsymbol{M}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{J}_{ib}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}_i\boldsymbol{J}_{ib}(\boldsymbol{\theta})$)

- Term $\boldsymbol{b}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\mathcal{F}}_{\text{tip}}) = \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$ is computed by finding $\boldsymbol{\tau}|_{\ddot{\boldsymbol{\theta}}=\mathbf{0}}$

Forward Dynamics

Forward Dynamics

Finding $\ddot{\boldsymbol{\theta}}$ given the $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\boldsymbol{\mathcal{F}}_{\text{tip}}$, $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$$

After computing $\boldsymbol{b}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\boldsymbol{\mathcal{F}}_{\text{tip}}) = \boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}_{\text{tip}}$ and $\boldsymbol{M}(\boldsymbol{\theta})$, we can use any efficient algorithm to solve $\boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \boldsymbol{\tau} - \boldsymbol{b}$ for $\ddot{\boldsymbol{\theta}}$.

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}(\boldsymbol{\theta})\left(\boldsymbol{\tau} - \boldsymbol{b}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\boldsymbol{\mathcal{F}}_{\text{tip}})\right)$$

or $\ddot{\boldsymbol{\theta}} = \boldsymbol{M}(\boldsymbol{\theta})\backslash\left(\boldsymbol{\tau} - \boldsymbol{b}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\boldsymbol{\mathcal{F}}_{\text{tip}})\right)$ in MATLAB

Numerical Simulation of Robot Motion

Forward dynamics can be used to **simulate the motion of the robot** for $t \in [0, t_f]$ given $\boldsymbol{\tau}(t), \boldsymbol{\mathcal{F}}_{\text{tip}}(t)$, and its initial state $\boldsymbol{\theta}(0), \dot{\boldsymbol{\theta}}(0)$. These equations are coupled, non-linear ODEs, and they can be solved using numerical integration.

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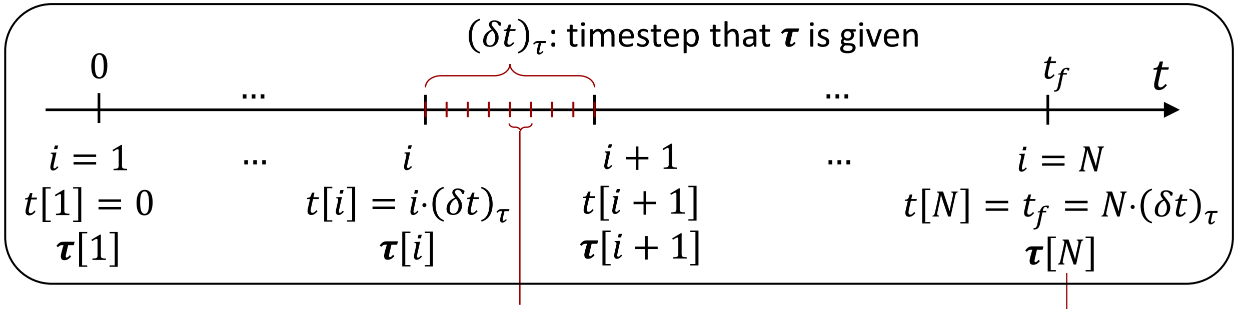
Given $\boldsymbol{\tau}[i], \boldsymbol{\mathcal{F}}_{\text{tip}}[i]$ $(i = 1, \dots, N)$
Set $\boldsymbol{\theta}[1] = \boldsymbol{\theta}(0),\ \dot{\boldsymbol{\theta}}[1] = \dot{\boldsymbol{\theta}}(0)$
Set $\bar{\boldsymbol{\theta}} = \boldsymbol{\theta}[1], \dot{\bar{\boldsymbol{\theta}}} = \dot{\boldsymbol{\theta}}[1]$
For $i = 1$ to $N - 1$
    For $j = 1$ to $n_{\text{res}}$
        $\ddot{\boldsymbol{\theta}} = \text{ForwardDynamics}\left(\bar{\boldsymbol{\theta}}, \dot{\bar{\boldsymbol{\theta}}}, \boldsymbol{\tau}[i], \boldsymbol{\mathcal{F}}_{\text{tip}}[i]\right)$
        $\bar{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}} + \dot{\bar{\boldsymbol{\theta}}} \cdot (\delta t)_{\ddot{\theta}}$
        $\dot{\bar{\boldsymbol{\theta}}} = \dot{\bar{\boldsymbol{\theta}}} + \ddot{\boldsymbol{\theta}} \cdot (\delta t)_{\ddot{\theta}}$
    end
$\boldsymbol{\theta}[i+1] = \bar{\boldsymbol{\theta}}$
$\dot{\boldsymbol{\theta}}[i+1] = \dot{\bar{\boldsymbol{\theta}}}$
end
```



$(\delta t)_{\ddot{\theta}} = (\delta t)_{\tau} / n_{\text{res}}$: timestep for motion simulation and computing $\ddot{\boldsymbol{\theta}}$.
n_{res}: Integration resolution.

N: Number of samples

First-order Euler Integration
(we can also use any other ODE solver like **ode45** which is based on an explicit **Runge-Kutta** (4,5) formula)

$(\delta t)_{\tau}, (\delta t)_{\ddot{\theta}} \in \mathbb{R}_{+}$

Properties of Dynamic Parameters

Properties of Robot Dynamic Equations

Fundamental properties of the dynamic model of n-DOF open-chain robots are of particular importance in the study of control systems for robot manipulators.

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta}) \\ &= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta})\end{aligned}$$

$\boldsymbol{\theta} \in \mathbb{R}^n$: Joint Variables

$\boldsymbol{\tau} \in \mathbb{R}^n$: Joint Torques/Forces

$\boldsymbol{M}(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$: Mass Matrix

$\boldsymbol{g}(\boldsymbol{\theta}) \in \mathbb{R}^n$: Gravitational Terms

$\boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^{n\times n}$: Coriolis Matrix

$\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$: Coriolis and Centripetal Terms (velocity-product term or quadratic velocity term)

Properties of Mass or Inertia Matrix $\boldsymbol{M}(\boldsymbol{\theta})$

- The total kinetic energy $\mathcal{K} \in \mathbb{R}_+$ of an open-chain robot: $\mathcal{K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

- $\boldsymbol{M}(\boldsymbol{\theta})$ **depends** only on $\boldsymbol{\theta}$.
- $\boldsymbol{M}(\boldsymbol{\theta})$ is always **symmetric** and **positive-definite** ($\boldsymbol{x}^T\boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \neq \boldsymbol{0}$).
- $\boldsymbol{M}^{-1}(\boldsymbol{\theta})$ always exist and is positive definite.
- $\boldsymbol{M}(\boldsymbol{\theta})$ is bounded above and below:

$$\mu_1\boldsymbol{I}_n \leq \boldsymbol{M}(\boldsymbol{\theta}) \leq \mu_2\boldsymbol{I}_n \qquad \forall\boldsymbol{\theta} \in \mathbb{R}^n, \mu_1, \mu_2 \in \mathbb{R}_{++}$$

$$\frac{1}{\mu_1}\boldsymbol{I}_n \geq \boldsymbol{M}^{-1}(\boldsymbol{\theta}) \geq \frac{1}{\mu_2}\boldsymbol{I}_n \qquad \boldsymbol{I}_n = \mathrm{diag}(1) \in \mathbb{R}^n$$

- This property can also be expressed as

$$0 < m_1 \leq \|\boldsymbol{M}(\boldsymbol{\theta})\| \leq m_2 < \infty \qquad \forall\boldsymbol{\theta} \in \mathbb{R}^n$$

$$\frac{1}{m_1} \geq \|\boldsymbol{M}^{-1}(\boldsymbol{\theta})\| \geq \frac{1}{m_2} \qquad m_1, m_2 \in \mathbb{R}_{++}$$

$\|\cdot\|$ is any matrix norm.

- If the arm is revolute, μ_1, μ_2, m_1, m_2 are constants, and if the arm has prismatic joints, μ_1, μ_2, m_1, m_2 may depend on $\boldsymbol{\theta}$.

Properties of Coriolis & Centripetal Terms

- $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is quadratic in $\dot{\boldsymbol{\theta}}$.
- $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})|_{\dot{\boldsymbol{\theta}}=\mathbf{0}} = \mathbf{0}$.
- $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ can be bounded above by a quadratic function of $\dot{\boldsymbol{\theta}}$: $\|\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\| \le c_b\|\dot{\boldsymbol{\theta}}\|^2$

$\|\cdot\|$ is any vector norm, $c_b \in \mathbb{R}_+$, $\forall\boldsymbol{\theta},\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$

- If the arm is revolute, c_b is constant, and if the arm has prismatic joints, c_b may depend on $\boldsymbol{\theta}$.

- If $\|\cdot\|$ is 2-norm: $c_b = n^2\left(\max_{k,i,j,\boldsymbol{\theta}}\left|\Gamma_{k_{ij}}(\boldsymbol{\theta})\right|\right)$

- Matrix $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ may not be unique, but the vector $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$ is indeed unique.

- In general, $\dot{\boldsymbol{\theta}}^T(\dot{\boldsymbol{M}} - 2\boldsymbol{C})\dot{\boldsymbol{\theta}} = \mathbf{0}$.

- We can always find the standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ that $\boldsymbol{S}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{M}} - 2\boldsymbol{C} \in \mathbb{R}^{n\times n}$ is **skew symmetric**, i.e., $\boldsymbol{x}^T(\dot{\boldsymbol{M}} - 2\boldsymbol{C})\boldsymbol{x} = \mathbf{0}$, $\forall\boldsymbol{x} \in \mathbb{R}^n$. (Passivity Property)

- For a standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$, $\dot{\boldsymbol{M}} = \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})^T$.

Properties of Coriolis & Centripetal Terms

- We can find the standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ as $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = 1/2(\dot{\boldsymbol{M}} + \boldsymbol{U}^T - \boldsymbol{U})$

$$\dot{\boldsymbol{M}}(\boldsymbol{\theta}\ ) = (\dot{\boldsymbol{\theta}}^T \otimes \boldsymbol{I}_n)\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{\theta}} \in \mathbb{R}^{n\times n}, \quad \boldsymbol{U}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = (\boldsymbol{I}_n \otimes \dot{\boldsymbol{\theta}}^T)\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{\theta}} \in \mathbb{R}^{n\times n}, \quad \boldsymbol{I}_n = \mathrm{diag}(1) \in \mathbb{R}^n$$

- We can find 2 other (non-standard) choices of $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ as $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{M}} - 1/2\boldsymbol{U}$

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{U}^T - 1/2\boldsymbol{U}$$

Let define **Kronecker Product** of two metrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{p\times q}$ as

$$\boldsymbol{A} \otimes \boldsymbol{B} = [a_{ij}\boldsymbol{B}] \in \mathbb{R}^{mp\times nq}$$

For instance, for $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$: $\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & a_{13}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & a_{23}\boldsymbol{B} \\ a_{31}\boldsymbol{B} & a_{32}\boldsymbol{B} & a_{33}\boldsymbol{B} \end{bmatrix}$

Also, for $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{R}^{m\times n}$, $\boldsymbol{x} \in \mathbb{R}^p$, let define the matrix derivative as $\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial \boldsymbol{A}}{\partial x_1} \\ \vdots \\ \frac{\partial \boldsymbol{A}}{\partial x_p} \end{bmatrix} \in \mathbb{R}^{mp\times n}$

Properties of Gravitational Terms $\boldsymbol{g}(\boldsymbol{q})$

- Let $\mathcal{P} \in \mathbb{R}_+$ be the total gravitational potential energy of an open-chain robot. Then,

$$\boldsymbol{g}(\boldsymbol{\theta}) = \frac{\partial \mathcal{P}}{\partial \boldsymbol{\theta}}$$

- $\boldsymbol{g}(\boldsymbol{\theta})$ **depends** only on $\boldsymbol{\theta}$.
- $\boldsymbol{g}(\boldsymbol{\theta})$ is bounded above: $\|\boldsymbol{g}(\boldsymbol{\theta})\| \leq g_b \qquad \forall \boldsymbol{\theta} \in \mathbb{R}^n$

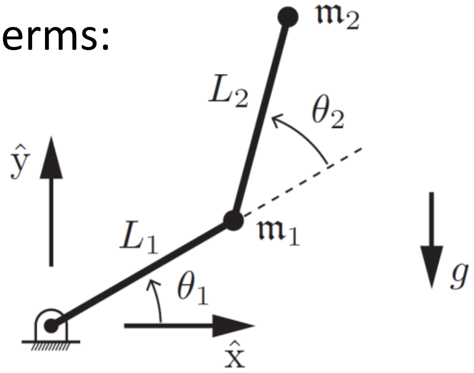
  $\|\cdot\|$ is any vector norm, $g_b \in \mathbb{R}_+$

  - If the arm is revolute, g_b is constant, and if the arm has prismatic joints, g_b may depend on $\boldsymbol{\theta}$.

- $$\int_0^{t_f} \boldsymbol{g}\big(\boldsymbol{\theta}(t)\big)^T \dot{\boldsymbol{\theta}}(t)dt = \mathcal{P}\left(\boldsymbol{\theta}(t_f)\right) - \mathcal{P}\big(\boldsymbol{\theta}(0)\big)$$

Example

Dynamic equations of a planar 2R open-chain in absence of friction terms:



$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) + \boldsymbol{g}(\boldsymbol{\theta})$$

$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix}$$

$$\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} \qquad \boldsymbol{g}(\boldsymbol{\theta}) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2 \cos(\theta_1 + \theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Find the bounds on the $\boldsymbol{M}(\boldsymbol{\theta})$, $\boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)$, $\boldsymbol{g}(\boldsymbol{\theta})$. Suppose that the joint angles θ_1 and θ_2 are limited by $\pm\pi/2$.

Note: The selection of a suitable norm is not always straightforward. This choice often depends simply on which norm makes it possible to evaluate the bounds. Usually, 1-norm is an easy choice.

Example (cont.)

- The induced 1-norm for $\boldsymbol{M}(\boldsymbol{\theta})$:

$$\|\boldsymbol{M}(\boldsymbol{\theta})\|_1 = \left|\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right| + \left|\mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\right|$$

$$m_1 \le \|\boldsymbol{M}(\boldsymbol{\theta})\|_1 \le m_2$$

$$m_2 = (\mathfrak{m}_1 + \mathfrak{m}_2)L_1^2 + 2\mathfrak{m}_2L_2^2 + 3\mathfrak{m}_2L_1L_2$$

$$m_1 = (\mathfrak{m}_1 + \mathfrak{m}_2)L_1^2 + 2\mathfrak{m}_2L_2^2$$

- The 1-norm of $\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$: $\left\|\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\right\|_1 = \left|\mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\right| + \left|\mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2\right|$

$$\le \mathfrak{m}_2L_1L_2\left(|\dot{\theta}_1| + |\dot{\theta}_2|\right)^2 \equiv c_b\left\|\dot{\boldsymbol{\theta}}\right\|_1^2$$

$$\left\|\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\right\| \le c_b\left\|\dot{\boldsymbol{\theta}}\right\|^{\mathbf{2}}$$

$$c_b = \mathfrak{m}_2L_1L_2$$

- The 1-norm of $\boldsymbol{g}(\boldsymbol{\theta})$:

$$\|\boldsymbol{g}(\boldsymbol{\theta})\|_1 = |(\mathfrak{m}_1 + \mathfrak{m}_2)L_1g\cos\theta_1 + \mathfrak{m}_2gL_2\cos(\theta_1 + \theta_2)|$$

$$+|\mathfrak{m}_2gL_2\cos(\theta_1 + \theta_2)|$$

$$\le (\mathfrak{m}_1 + \mathfrak{m}_2)gL_1 + 2\mathfrak{m}_2gL_2 \equiv g_b$$

$$\|\boldsymbol{g}(\boldsymbol{\theta})\| \le g_b$$

Example (cont.)

- We can find the standard $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ where $\boldsymbol{c}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$ as:

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=1/2(\dot{\boldsymbol{M}}+\boldsymbol{U}^T-\boldsymbol{U})=\begin{bmatrix}-\dot{\theta}_2\mathfrak{m}_2L_1L_2\sin\theta_2 & -(\dot{\theta}_1+\dot{\theta}_2)\mathfrak{m}_2L_1L_2\sin\theta_2\\ \dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2 & 0\end{bmatrix}$$

where $\boldsymbol{U}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=(\boldsymbol{I}_n\otimes\dot{\boldsymbol{\theta}}^T)\dfrac{\partial\boldsymbol{M}}{\partial\boldsymbol{\theta}}=\begin{bmatrix}0 & 0\\ -(2\dot{\theta}_1+\dot{\theta}_2)\mathfrak{m}_2L_1L_2\sin\theta_2 & -\dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2\end{bmatrix}.$

- Two other choices of $\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ are

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=\dot{\boldsymbol{M}}-1/2\boldsymbol{U}=\begin{bmatrix}-2\dot{\theta}_2\mathfrak{m}_2L_1L_2\sin\theta_2 & -\dot{\theta}_2\mathfrak{m}_2L_1L_2\sin\theta_2\\ (\dot{\theta}_1-1/2\dot{\theta}_2)\mathfrak{m}_2L_1L_2\sin\theta_2 & 1/2\dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2\end{bmatrix}$$

$$\boldsymbol{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=\boldsymbol{U}^T-1/2\boldsymbol{U}=\begin{bmatrix}0 & -(2\dot{\theta}_1+\dot{\theta}_2)\mathfrak{m}_2L_1L_2\sin\theta_2\\ (\dot{\theta}_1+1/2\dot{\theta}_2)\mathfrak{m}_2L_1L_2\sin\theta_2 & 1/2\dot{\theta}_1\mathfrak{m}_2L_1L_2\sin\theta_2\end{bmatrix}$$

Example (cont.)

Matrix of Christoffel symbols of the first kind $\boldsymbol{\Gamma}(\boldsymbol{\theta})$:

$$\boldsymbol{\Gamma}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\Gamma}_1(\boldsymbol{\theta}) \\ \boldsymbol{\Gamma}_2(\boldsymbol{\theta}) \end{bmatrix} \qquad \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \dot{\boldsymbol{\theta}}^{\mathrm{T}} \boldsymbol{\Gamma}(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}} = \begin{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \overbrace{\begin{bmatrix} 0 & -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 \\ -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 & -\mathfrak{m}_2 L_1 L_2 \sin\theta_2 \end{bmatrix}}^{\boldsymbol{\Gamma}_1(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \underbrace{\begin{bmatrix} \mathfrak{m}_2 L_1 L_2 \sin\theta_2 & 0 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\Gamma}_2(\boldsymbol{\theta})} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \end{bmatrix}$$

Using $\boldsymbol{\Gamma}_1(\boldsymbol{\theta})$ and $\boldsymbol{\Gamma}_2(\boldsymbol{\theta})$, we can find c_b in $\left\|\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\right\| \leq c_b \left\|\dot{\boldsymbol{\theta}}\right\|^{\mathbf{2}}$ when $\|\cdot\|$ is 2-norm by

$$c_b = n^2 \left(\max_{k,i,j,\boldsymbol{\theta}} \left| \Gamma_{k_{ij}}(\boldsymbol{\theta}) \right| \right)$$

$$\begin{aligned} &\max_{\theta} \left| \Gamma_{1_{11}}(\boldsymbol{\theta}) \right| = 0\,, && \max_{\theta} \left| \Gamma_{2_{11}}(\boldsymbol{\theta}) \right| = \mathfrak{m}_2 L_1 L_2 \\ &\max_{\theta} \left| \Gamma_{1_{12}}(\boldsymbol{\theta}) \right| = \mathfrak{m}_2 L_1 L_2\,, && \max_{\theta} \left| \Gamma_{2_{12}}(\boldsymbol{\theta}) \right| = 0 \\ &\max_{\theta} \left| \Gamma_{1_{21}}(\boldsymbol{\theta}) \right| = \mathfrak{m}_2 L_1 L_2\,, && \max_{\theta} \left| \Gamma_{2_{21}}(\boldsymbol{\theta}) \right| = 0 \\ &\max_{\theta} \left| \Gamma_{1_{22}}(\boldsymbol{\theta}) \right| = \mathfrak{m}_2 L_1 L_2\,, && \max_{\theta} \left| \Gamma_{2_{22}}(\boldsymbol{\theta}) \right| = 0. \end{aligned}$$

$$\Rightarrow c_b = 4\mathfrak{m}_2 L_1 L_2$$

Linearity in Dynamic Model

Linearity in Dynamic Model

An important property of the dynamic model of an open-chain manipulator is the linearity with respect to a suitable set of parameters $\boldsymbol{\pi} \in \mathbb{R}^p$, including dynamic parameters (e.g., mass m_i, first moment of inertia $m_i l_{C_x,i}, m_i l_{C_y,i}, m_i l_{C_z,i}$, and the six components of inertia matrix $\boldsymbol{I}_{b,i}$) and friction parameters (e.g., $F_{v,i}$ and $F_{s,i}$) as

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) + \boldsymbol{F}_v\dot{\boldsymbol{\theta}} + \boldsymbol{F}_s \,\mathbf{sgn}(\dot{\boldsymbol{\theta}}) = \boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})\boldsymbol{\pi}$$

$\boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \in \mathbb{R}^{n \times p}$ is called **regressor**.

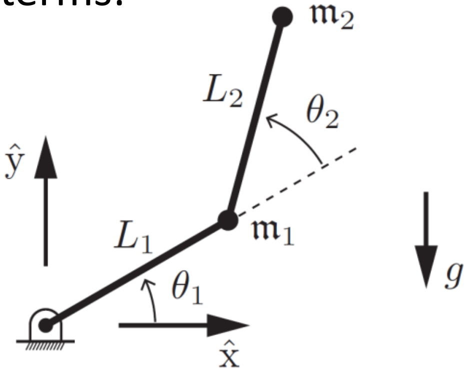
- This property is useful in **Adaptive Control**, where some or all the parameters maybe unknown.
- Note that $p \leq 12n$, since not all the dynamic/friction parameters appear in dynamic equations or are unknown.

Example

Dynamic equations of a planar 2R open chain in presence of friction terms:

$$\tau_1 = \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right)\ddot{\theta}_1$$
$$+\mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_2 - \mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)$$
$$+(\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) + F_{v,1}\dot{\theta}_1 + F_{s,1}\,\mathrm{sgn}\,\dot{\theta}_1,$$

$$\tau_2 = \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_1 + \mathfrak{m}_2L_2^2\ddot{\theta}_2 + \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2$$
$$+\mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) + F_{v,2}\dot{\theta}_2 + F_{s,2}\,\mathrm{sgn}\,\dot{\theta}_2.$$



If the set of unknown parameters $\boldsymbol{\pi}$ is defined as $\boldsymbol{\pi} = \left[\mathfrak{m}_1, \mathfrak{m}_2, F_{s,1}, F_{v,1}, F_{s,2}, F_{v,2}\right]^T$,

find $\boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$ where $\boldsymbol{\tau} = \boldsymbol{Y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})\boldsymbol{\pi}$.

Example

We can find $\boldsymbol{Y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}})$ as

$$\boldsymbol{Y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}}) = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & 0 & 0 \\ 0 & Y_{22} & 0 & 0 & Y_{25} & Y_{26} \end{bmatrix}$$

$$\begin{aligned}
Y_{11} &= L_1^2\ddot{\theta}_1 + gL_1\cos\theta_1 \\
Y_{12} &= [L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2]\ddot{\theta}_1 + [L_2^2 + L_1L_2\cos\theta_2]\ddot{\theta}_2 \\
&\quad -L_1L_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\sin\theta_2 + gL_1\cos\theta_1 + gL_2\cos(\theta_1+\theta_2) \\
Y_{13} &= \operatorname{sgn}\left(\dot{\theta}_1\right) \\
Y_{14} &= \dot{\theta}_1 \\
Y_{22} &= [L_2^2 + L_1L_2\cos\theta_2]\ddot{\theta}_1 + L_2^2\ddot{\theta}_2 + L_1L_2\dot{\theta}_1^2\sin\theta_2 + gL_2\cos(\theta_1+\theta_2) \\
Y_{25} &= \operatorname{sgn}\left(\dot{\theta}_2\right) \\
Y_{26} &= \dot{\theta}_2
\end{aligned}$$