Ch6: Velocity Kinematics – Part 1 (Jacobian and Singularity)

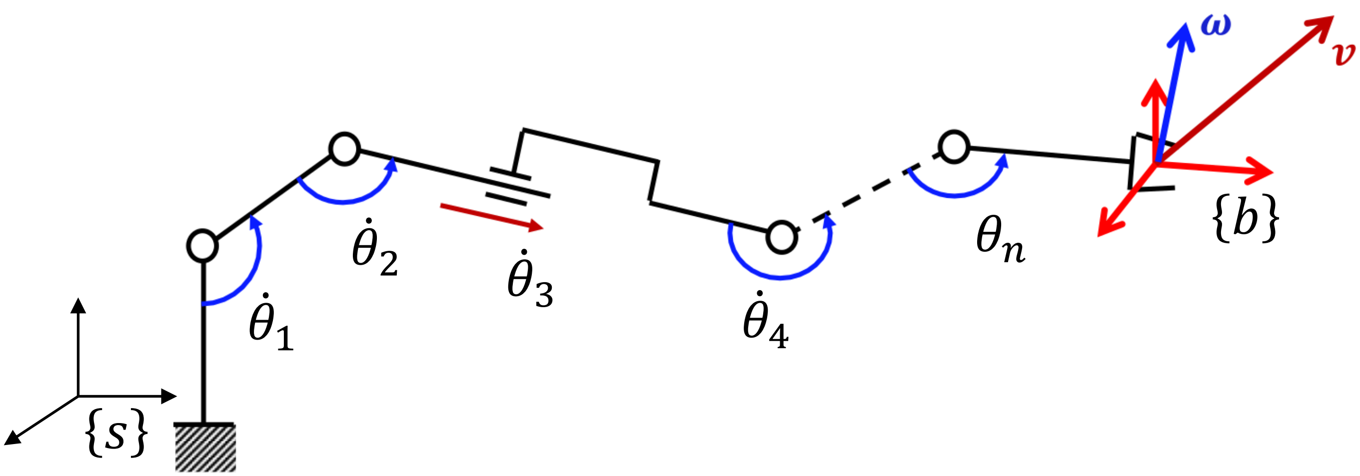
Manipulator Jacobian

Manipulator Jacobian

The **instantaneous velocity mapping** between velocity in joint space ($\dot{\boldsymbol{\theta}}$) and linear/angular velocity of the end-effector frame in task space can be obtained

- in a geometric way without derivatives (**Geometric Jacobian**) or
- through time differentiation of the forward kinematics (**Analytic Jacobian**).

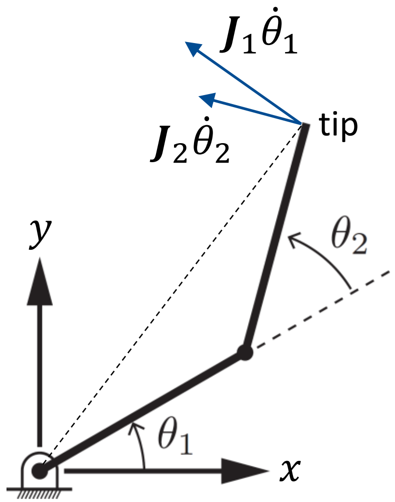
In both cases, the Jacobian matrix depends on the (current) configuration $\boldsymbol{\theta}$ of the robot.



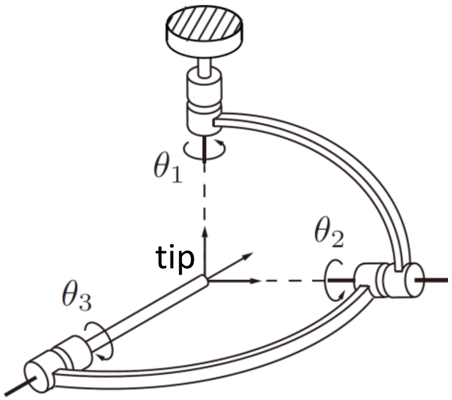
Manipulator Jacobian

In a 2R planar robot, we saw that $\boldsymbol{v}_{\text{tip}}$ is the linear velocity of the end-effector frame

$$\boldsymbol{v}_{\text{tip}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \boldsymbol{J}(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = [\boldsymbol{J}_1 \quad \boldsymbol{J}_2] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \boldsymbol{J}_1 \dot{\theta}_1 + \boldsymbol{J}_2 \dot{\theta}_2$$



In a pure orienting devices such as a wrist, $\boldsymbol{v}_{\text{tip}}$ is the angular velocity of the end-effector frame.



- Thus, $\boldsymbol{v}_{\text{tip}}$ determine the specific form of the Jacobian.

Geometric Jacobian

Space and Body Manipulator Jacobians

Let's assume that the configuration of the end-effector is expressed as $\boldsymbol{T}_{sb} = \boldsymbol{T} \in SE(3)$ and its velocity is expressed as a twist $\boldsymbol{\mathcal{V}} \in \mathbb{R}^6$ in the fixed base frame $\{s\}$ or the end-effector body frame $\{b\}$.

❖ The **Jacobian** is derived based on the following general idea:
Given the configuration $\boldsymbol{\theta} \in \mathbb{R}^n$ of the robot, $\boldsymbol{J}_i(\boldsymbol{\theta}) \in \mathbb{R}^6$, which is column i of $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$, is the twist $\boldsymbol{\mathcal{V}}$ when the robot is in an arbitrary configuration $\boldsymbol{\theta}$ (not in zero configuration), $\dot{\theta}_i = 1$, and all other joint velocities are zero.

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = [\boldsymbol{J}_1 \quad \boldsymbol{J}_2 \quad \cdots \quad \boldsymbol{J}_n]\dot{\boldsymbol{\theta}}$$

$\dot{\boldsymbol{\theta}}$: joint velocities

(Velocity or Differential Kinematics Equation is a linear map in a given configuration)

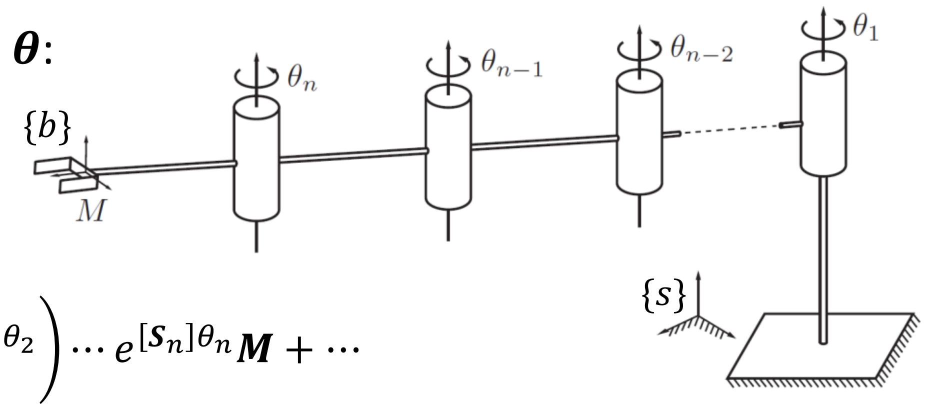
- If each column $\boldsymbol{J}_i(\boldsymbol{\theta})$ is expressed in the fixed space frame $\{s\}$: $\Rightarrow$ Space Jacobian $\quad \boldsymbol{\mathcal{V}}_s = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- If each column $\boldsymbol{J}_i(\boldsymbol{\theta})$ is expressed in the end-effector frame $\{b\}$: $\Rightarrow$ Body Jacobian $\quad \boldsymbol{\mathcal{V}}_b = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Geometric Jacobians

Space Jacobian

Consider an n-link open chain as configuration $\boldsymbol{\theta}$:

$$\boldsymbol{T}(\boldsymbol{\theta}) = e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M}$$ forward kinematics



$$[\boldsymbol{\mathcal{V}}_s] = \dot{\boldsymbol{T}}\boldsymbol{T}^{-1} \qquad (\boldsymbol{T} = \boldsymbol{T}_{sb})$$

$$\dot{\boldsymbol{T}} = \left(\frac{d}{dt} e^{[\boldsymbol{S}_1]\theta_1}\right) \cdots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + e^{[\boldsymbol{S}_1]\theta_1} \left(\frac{d}{dt} e^{[\boldsymbol{S}_2]\theta_2}\right) \cdots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + \cdots$$

$$= [\boldsymbol{S}_1]\dot{\theta}_1 e^{[\boldsymbol{S}_1]\theta_1} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + e^{[\boldsymbol{S}_1]\theta_1} [\boldsymbol{S}_2]\dot{\theta}_2 e^{[\boldsymbol{S}_2]\theta_2} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + \cdots$$

$$\boldsymbol{T}^{-1} = \boldsymbol{M}^{-1} e^{-[\boldsymbol{S}_n]\theta_n} \dots e^{-[\boldsymbol{S}_1]\theta_1}$$

$$[\boldsymbol{\mathcal{V}}_s] = [\boldsymbol{S}_1]\dot{\theta}_1 + e^{[\boldsymbol{S}_1]\theta_1} [\boldsymbol{S}_2] e^{-[\boldsymbol{S}_1]\theta_1} \dot{\theta}_2 + e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2} [\boldsymbol{S}_3] e^{-[\boldsymbol{S}_2]\theta_2} e^{-[\boldsymbol{S}_1]\theta_1} \dot{\theta}_3 + \cdots$$

$$\boldsymbol{A}[\boldsymbol{S}_i]\boldsymbol{A}^{-1} = [[\mathrm{Ad}_{\boldsymbol{A}}]\boldsymbol{S}_i] \qquad \boldsymbol{A} \in SE(3)$$

$$\boldsymbol{\mathcal{V}}_s = \underbrace{\boldsymbol{S}_1}_{\boldsymbol{J}_{s1}} \dot{\theta}_1 + \underbrace{\left[\mathrm{Ad}_{e^{[\boldsymbol{S}_1]\theta_1}}\right]\boldsymbol{S}_2}_{\boldsymbol{J}_{s2}} \dot{\theta}_2 + \underbrace{\left[\mathrm{Ad}_{e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2}}\right]\boldsymbol{S}_3}_{\boldsymbol{J}_{s3}} \dot{\theta}_3 + \cdots$$

$$\boldsymbol{\mathcal{V}}_s = \boldsymbol{J}_{s1}\dot{\theta}_1 + \boldsymbol{J}_{s2}(\theta_1)\dot{\theta}_2 + \cdots + \boldsymbol{J}_{sn}(\theta_1, \dots, \theta_{n-1})\dot{\theta}_n$$

Space Jacobian (cont.)

$$\boldsymbol{\mathcal{V}}_s = [\boldsymbol{J}_{s1} \quad \boldsymbol{J}_{s2}(\theta_1) \quad \cdots \quad \boldsymbol{J}_{sn}(\theta_1, \dots, \theta_{n-1})] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

The **space Jacobian** $\boldsymbol{J}_s(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ to the spatial twist $\boldsymbol{\mathcal{V}}_s$. The ith column of $\boldsymbol{J}_s(\boldsymbol{\theta})$ is

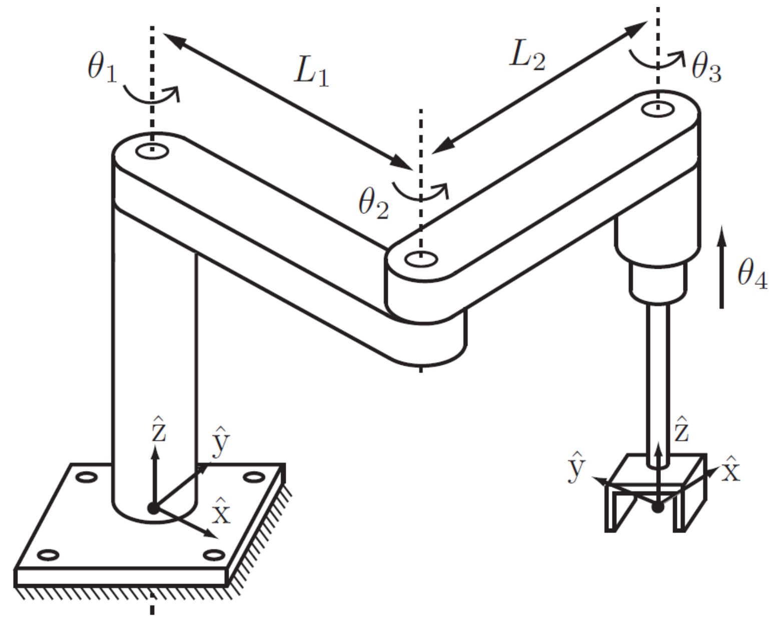
$$\boldsymbol{J}_{si}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{J}_{\omega_s,i}(\boldsymbol{\theta}) \\ \boldsymbol{J}_{v_s,i}(\boldsymbol{\theta}) \end{bmatrix} = \left[\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}\right] \boldsymbol{\mathcal{S}}_i \qquad \begin{array}{c} \boldsymbol{J}_{s1} = \boldsymbol{\mathcal{S}}_1 \\ i = 2, \dots, n \end{array}$$

$\boldsymbol{J}_{si}$ is determined in the same way as the joint screw axis $\boldsymbol{\mathcal{S}}_i$, except that $\boldsymbol{J}_{si}$ is determined for an arbitrary $\boldsymbol{\theta}$ rather than $\boldsymbol{\theta} = \mathbf{0}$. It shows contribution to the EE twist with $\dot{\theta}_i = 1$.

Screw axis describing the ith joint axis (expressed in the fixed space frame $\{s\}$) when the robot in its zero/home configuration $\boldsymbol{\theta} = \mathbf{0}$.

Note: The space Jacobian $\boldsymbol{J}_s$ is independent of the choice of the end-effector frame $\{b\}$.

Example: Space Jacobian of a Spatial RRRP Robot

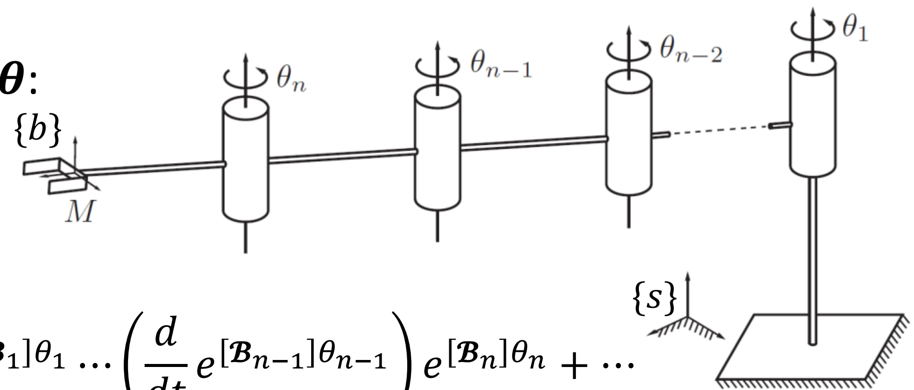


Body Jacobian

Consider an n-link open chain as configuration $\boldsymbol{\theta}$:

$$\boldsymbol{T}(\boldsymbol{\theta}) = \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}e^{[\boldsymbol{\mathcal{B}}_2]\theta_2}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}$$

(forward kinematics)



$$[\boldsymbol{\mathcal{V}}_b] = \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} \qquad (\boldsymbol{T} = \boldsymbol{T}_{sb})$$

$$\dot{\boldsymbol{T}} = \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}\left(\frac{d}{dt}e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\right) + \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots\left(\frac{d}{dt}e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}\right)e^{[\boldsymbol{\mathcal{B}}_n]\theta_n} + \cdots$$

$$= \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}[\boldsymbol{\mathcal{B}}_n]\dot{\theta}_n + \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}[\boldsymbol{\mathcal{B}}_{n-1}]e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_{n-1} + \cdots$$
$$+\boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}[\boldsymbol{\mathcal{B}}_1]e^{[\mathcal{B}_2]\theta_2}\dots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1$$

$$\boldsymbol{T}^{-1} = e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\dots e^{-[\boldsymbol{\mathcal{B}}_1]\theta_1}\boldsymbol{M}^{-1}$$

$$[\boldsymbol{\mathcal{V}}_b] = [\boldsymbol{\mathcal{B}}_n]\dot{\theta}_n + e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}[\boldsymbol{\mathcal{B}}_{n-1}]e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_{n-1} + \cdots + e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\dots e^{-[\boldsymbol{\mathcal{B}}_2]\theta_2}[\boldsymbol{\mathcal{B}}_1]e^{[\boldsymbol{\mathcal{B}}_2]\theta_2}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_1$$

$$\boldsymbol{A}^{-1}[\boldsymbol{\mathcal{B}}_i]\boldsymbol{A} = [[\mathrm{Ad}_{\boldsymbol{A}^{-1}}]\boldsymbol{\mathcal{B}}_i] \qquad \boldsymbol{A} \in SE(3)$$

$$\boldsymbol{\mathcal{V}}_b = \underbrace{\boldsymbol{\mathcal{B}}_n}_{\boldsymbol{J}_{bn}}\dot{\theta}_n + \underbrace{\left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}}\right]\boldsymbol{\mathcal{B}}_{n-1}}_{\boldsymbol{J}_{b,n-1}}\dot{\theta}_{n-1} + \cdots + \underbrace{\left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\cdots e^{-[\boldsymbol{\mathcal{B}}_2]\theta_2}}\right]\boldsymbol{\mathcal{B}}_1}_{\boldsymbol{J}_{b1}}\dot{\theta}_1$$

$$\boldsymbol{\mathcal{V}}_b = \boldsymbol{J}_{b1}(\theta_2,\dots,\theta_n)\dot{\theta}_1 + \cdots + \boldsymbol{J}_{bn-1}(\theta_n)\dot{\theta}_{n-1} + \boldsymbol{J}_{bn}\dot{\theta}_n$$

Body Jacobian (cont.)

$$\mathcal{V}_b = [J_{b_1}(\theta_2, \dots, \theta_n) \quad \cdots \quad J_{b_{n-1}}(\theta_n) \quad J_{b_n}] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

The **body Jacobian** $\boldsymbol{J}_b(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ to the end-effector (or body) twist $\mathcal{V}_b$. The ith column of $\boldsymbol{J}_b(\boldsymbol{\theta})$ is

$$\boldsymbol{J}_{bi}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{J}_{\omega_b,i}(\boldsymbol{\theta}) \\ \boldsymbol{J}_{v_b,i}(\boldsymbol{\theta}) \end{bmatrix} = \left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n} \dots e^{-[\boldsymbol{\mathcal{B}}_{i+1}]\theta_{i+1}}}\right] \boldsymbol{\mathcal{B}}_i \qquad \begin{matrix} \boldsymbol{J}_{bn} = \boldsymbol{\mathcal{B}}_n \\ i = n-1, \dots, 1 \end{matrix}$$

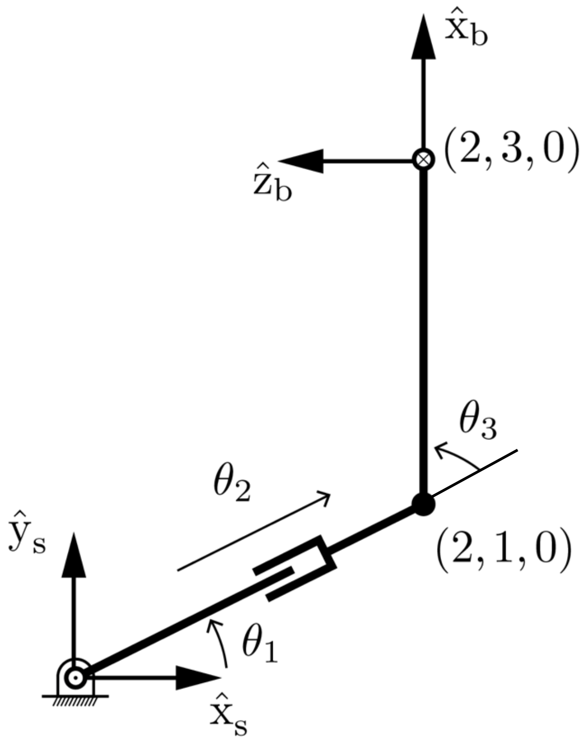
$\boldsymbol{J}_{bi}$ is determined in the same way as the joint screw axis $\boldsymbol{\mathcal{B}}_i$, except that $\boldsymbol{J}_{bi}$ is determined for an arbitrary $\boldsymbol{\theta}$ rather than $\boldsymbol{\theta} = \mathbf{0}$. It shows contribution to the EE twist with $\dot{\theta}_i = 1$.

Screw axis describing the ith joint axis (expressed in the end-effector frame $\{b\}$) when the robot in its zero/home configuration $\boldsymbol{\theta} = \mathbf{0}$.

Note: The body Jacobian $\boldsymbol{J}_b$ is independent of the choice of the space frame $\{s\}$.

Example

Find the space and body Jacobians in the given configuration.



Relationship between Space and Body Jacobian

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}$$
$$\mathcal{V}_b = J_b(\theta)\dot{\theta}$$

$$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$
$$\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$$
$$[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}]\mathcal{V} = [\mathrm{Ad}_{T_1 T_2}]\mathcal{V}$$

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} \quad \Rightarrow \quad [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = J_s(\theta)\dot{\theta} \quad \Rightarrow \quad [\mathrm{Ad}_{T_{bs}}][\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta}$$

$$\Rightarrow \quad \mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta} \quad \Rightarrow \quad J_b(\theta)\dot{\theta} = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta} \quad \overset{\forall \dot{\theta} \neq 0}{\Rightarrow}$$

$$\boxed{J_b(\theta) = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)}$$

Similarly, $\boxed{J_s(\theta) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta)}$

Note: The space and body Jacobians, and the space and body twists, are similarly related by the adjoint map because each column of the space or body Jacobian corresponds to a twist.

Another Form of Geometric Jacobian

Another form of geometric Jacobian is defined as $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

where $\boldsymbol{\omega}_s$ is the angular velocity of EE frame $\{b\}$ expressed in fixed frame $\{s\}$ ans $\dot{\boldsymbol{p}}$ is linear velocity of the origin of EE frame $\{b\}$ expressed in fixed frame $\{s\}$.

Example

Prove that the relationship between the body Jacobian $\boldsymbol{J}_b$ where $\begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$, space Jacobian $\boldsymbol{J}_s$ where $\begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$, and geometric Jacobian $\boldsymbol{J}_g$ where $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ are as follows.

$$\boldsymbol{J}_g(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{R}_{sb} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_{sb} \end{bmatrix} \boldsymbol{J}_b(\boldsymbol{\theta})$$

$$\boldsymbol{J}_g(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0} \\ -[\boldsymbol{p}] & \boldsymbol{I}_3 \end{bmatrix} \boldsymbol{J}_s(\boldsymbol{\theta})$$

Analytic Jacobian

Analytic Jacobian

There exist alternative notions of the Jacobian that are based on the representation of the end-effector configuration using a minimum set of coordinates $\boldsymbol{x}_e$ (e.g., Euler angles $\boldsymbol{\phi}$, unit quaternions $\boldsymbol{q}$, or exponential coordinates $\boldsymbol{r}$ for orientation). In these cases, the end-effector velocity is represented by the <u>time derivative of the coordinates</u> $\dot{\boldsymbol{x}}_e$, and the Jacobian is called the **Analytic Jacobian** $\boldsymbol{J}_a$, which is derived by time differentiation of the forward kinematics function with respect to the joint variables: $\boldsymbol{x}_e = \boldsymbol{f}(\boldsymbol{\theta}) \;\rightarrow\; \dot{\boldsymbol{x}}_e = \boldsymbol{J}_a(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$.

- $\mathcal{V}_s = \begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\mathcal{V}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Geometric Jacobian

- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{\phi}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$
- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,q}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\boldsymbol{q} = (q_0, q_1, q_2, q_3)$ $(\|\boldsymbol{q}\| = 1)$
- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\boldsymbol{r} = \widehat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ $(\|\widehat{\boldsymbol{\omega}}\| = 1, \theta \in [0, \pi])$

Analytic Jacobian

Note: $\boldsymbol{r}$ and $\boldsymbol{q}$ represent the orientation of EE frame $\{b\}$ expressed in fixed frame $\{s\}$ and $\boldsymbol{p}$ represents the position of the origin of $\{b\}$ expressed in $\{s\}$.

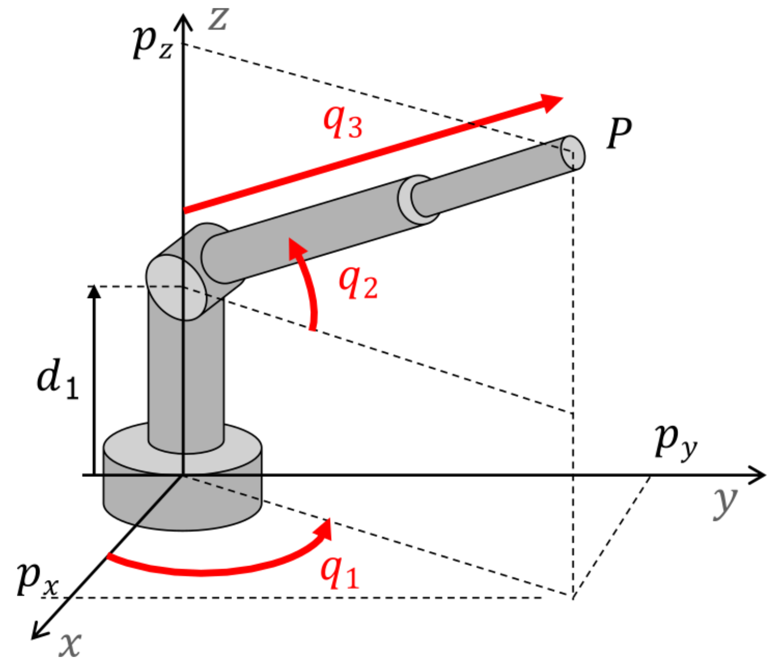
Geometric Jacobian vs Analytic Jacobian

- From a physical viewpoint, the meaning of $\boldsymbol{\omega}$ is more intuitive than that of $\dot{\boldsymbol{\phi}}$, $\dot{\boldsymbol{q}}$, and $\dot{\boldsymbol{r}}$. However, while the integral of $\dot{\boldsymbol{\phi}}$, $\dot{\boldsymbol{q}}$, and $\dot{\boldsymbol{r}}$ over time gives $\boldsymbol{\phi}$, $\boldsymbol{q}$, and $\boldsymbol{r}$, the integral of $\boldsymbol{\omega}$ does not admit a clear physical interpretation.

- The geometric Jacobian is used whenever it is necessary to refer to quantities of clear **physical meaning**, while the analytical Jacobian is used whenever it is necessary to refer to differential quantities of variables defined in the task space.

Example

Find analytic Jacobian of polar (RRP) robot.



Example

Prove that the relationship between the body Jacobian $\boldsymbol{J}_b$ where $\boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and an analytic Jacobian $\boldsymbol{J}_{a,r}$ where $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is as follows.

$$\boldsymbol{J}_{a,r}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{A}^{-1}(\boldsymbol{r}) & \boldsymbol{0} \\ \boldsymbol{0} & e^{[\boldsymbol{r}]} \end{bmatrix} \boldsymbol{J}_b(\boldsymbol{\theta})$$

Note: $\boldsymbol{\omega}_b = \boldsymbol{A}(\boldsymbol{r})\dot{\boldsymbol{r}}$ where $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{I}_3 - \dfrac{1-\cos\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^2}[\boldsymbol{r}] + \dfrac{\|\boldsymbol{r}\| - \sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^3}[\boldsymbol{r}]^2$

and we assume that the matrix $\boldsymbol{A}(\boldsymbol{r})$ is invertible.

Example

Prove that the relationship between the analytic Jacobian $\boldsymbol{J}_{a,\phi}$, where $\begin{bmatrix}\dot{\boldsymbol{\phi}}\\ \dot{\boldsymbol{p}}\end{bmatrix} = \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and $\boldsymbol{\phi} = (\varphi, \vartheta, \psi)$ is the Euler angles ZYZ in current frame, and geometric Jacobian $\boldsymbol{J}_g$ where $\begin{bmatrix}\boldsymbol{\omega}_s\\ \dot{\boldsymbol{p}}\end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is as follows.

$$\boldsymbol{J}_g(\boldsymbol{\theta}) = \begin{bmatrix}\boldsymbol{A}(\boldsymbol{\phi}) & \boldsymbol{0}_3\\ \boldsymbol{0}_3 & \boldsymbol{I}_3\end{bmatrix}\boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\ , \qquad \boldsymbol{A}(\boldsymbol{\phi}) = \begin{bmatrix}0 & -s_\varphi & c_\varphi s_\vartheta\\ 0 & c_\varphi & s_\varphi s_\vartheta\\ 1 & 0 & c_\vartheta\end{bmatrix}$$

General Form of Velocity Kinematics

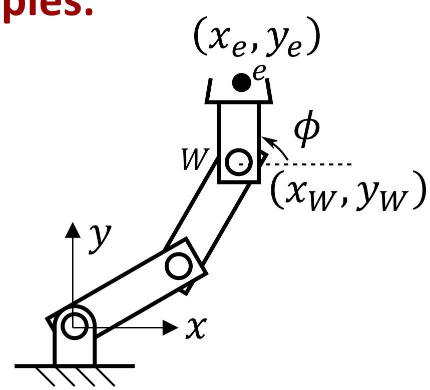
In general, depending on the dimension of task space r ($r \leq 6$), the differential kinematics equation can be represented as

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

where <u>now</u> $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$ (e.g., $\boldsymbol{\mathcal{V}}_b$, $\boldsymbol{\mathcal{V}}_s$, $(\boldsymbol{\omega}_s, \dot{\boldsymbol{p}})$, or $\dot{\boldsymbol{x}}_e$) is end-effector velocity for the specific task, $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$, and $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ is the corresponding Jacobian matrix that can be extracted from a $6 \times n$ geometric or analytic Jacobian (by removing null and irrelevant rows).

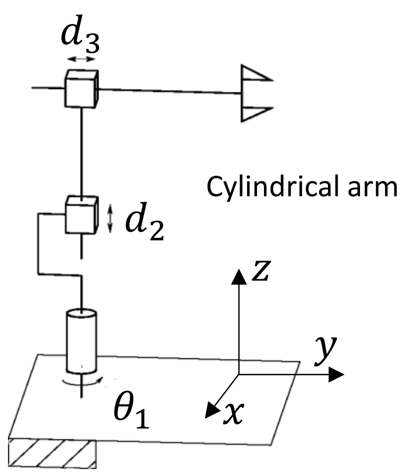
Examples:

(1)



- For the 3R planar robot, if $\boldsymbol{x}_e = (x_W, y_W, \phi)$, then $\boldsymbol{J} \in \mathbb{R}^{3\times 3}$ and if $\boldsymbol{x}_e = (x_e, y_e)$, then $\boldsymbol{J} \in \mathbb{R}^{2\times 3}$.

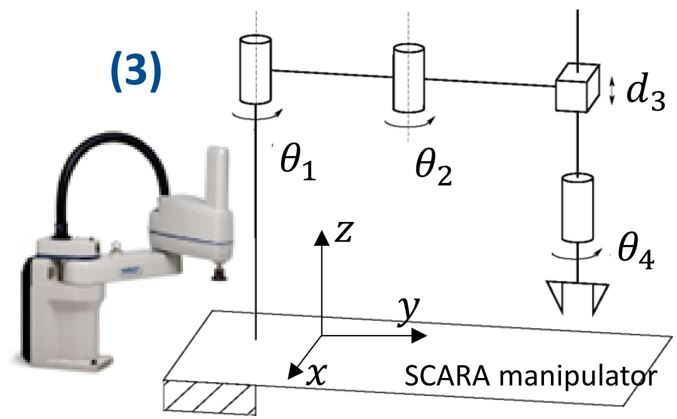
(2)



$\boldsymbol{J} \in \mathbb{R}^{3\times 3}$

$\boldsymbol{x}_e = (x_W, y_W, z_W)$

(3)



$\boldsymbol{J} \in \mathbb{R}^{4\times 4}$

$\boldsymbol{x}_e = (x_W, y_W, z_W, \phi)$

- (2) and (3) are inherently impossible to rotate about axes x and y.

Singularity Analysis

Kinematic Singularity

The configurations at which the robot's end-effector loses the ability to move instantaneously in one or more directions in the task-space is called a **Kinematic Singularity**. In these directions, the robot can resist arbitrary wrenches.

- In singular configuration $\boldsymbol{\theta}^*$, $\boldsymbol{J}(\boldsymbol{\theta}^*) \in \mathbb{R}^{r\times n}$ is rank-deficient, i.e., $\mathrm{rank}\big(\boldsymbol{J}(\boldsymbol{\theta}^*)\big) < \min(r, n)$.

  To check rank-deficiency, use the Jacobian that maps $\dot{\boldsymbol{\theta}}$ to the non-zero and independent velocities of the EE frame $\{b\}$ (i.e., after removing null and irrelevant rows of $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ where $\boldsymbol{J}$ can be $\boldsymbol{J}_a$, $\boldsymbol{J}_g$, or $\boldsymbol{J}_b$ as we are interested in the velocity of $\{b\}$ rather than $\{s\}$).

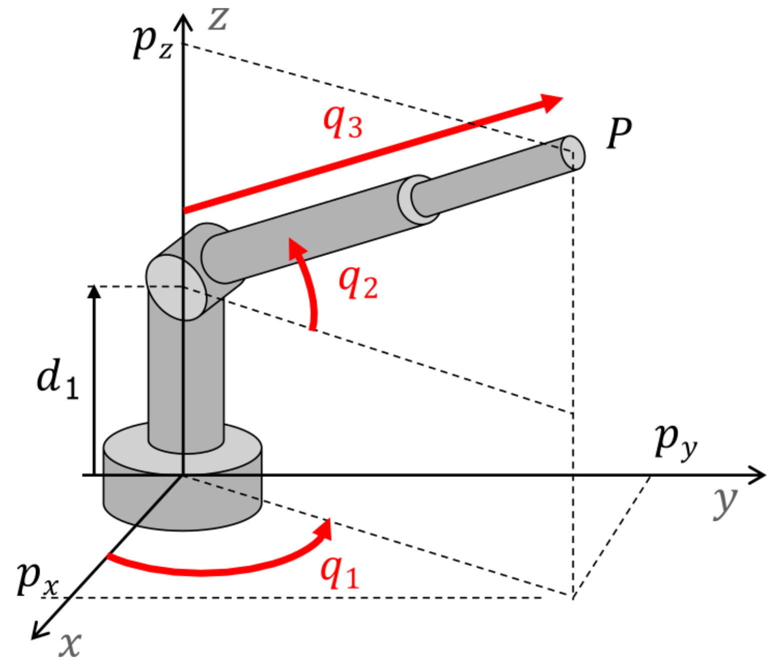
$$\begin{cases} \text{when } r = n\text{: find } \boldsymbol{\theta} \text{ such that } \det \boldsymbol{J}(\boldsymbol{\theta}) = 0 \\ \text{when } r < n\text{: find } \boldsymbol{\theta} \text{ such that } \det\big(\boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})\big) = 0. \end{cases}$$

- The kinematic singularities are typical of a mechanical structure and independent of the choice of the frames (e.g., fixed frame $\{s\}$ and end-effector frame $\{b\}$).

- In the neighborhood of a singularity, small velocities $\boldsymbol{\mathcal{V}}$ in the task space may cause large velocities $\dot{\boldsymbol{\theta}}$ in the joint space.

Example

Find singularities of polar (RRP) robot.



Kinematic Singularity

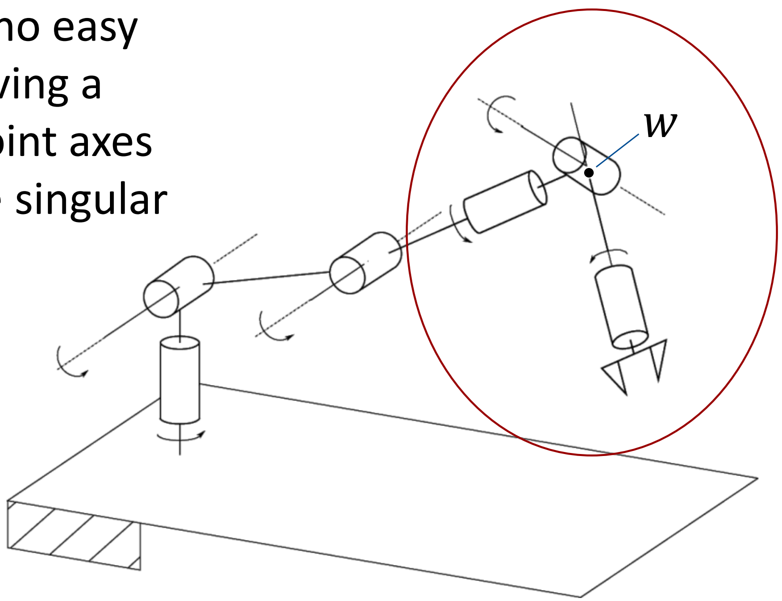
Singularities can be classified into:

- **Boundary Singularities**: They occur when the manipulator is either outstretched or retracted (it is easy to avoid).
- **Internal Singularities**: They occur anywhere inside the reachable workspace (it is hard to avoid).

Singularity Decoupling: Computation of internal singularities via the Jacobian determinant may be tedious and of no easy solution for complex structures. For manipulators having a spherical wrist (i.e., last three consecutive revolute joint axes intersect at a common point w), we can compute the singular configurations in two steps:

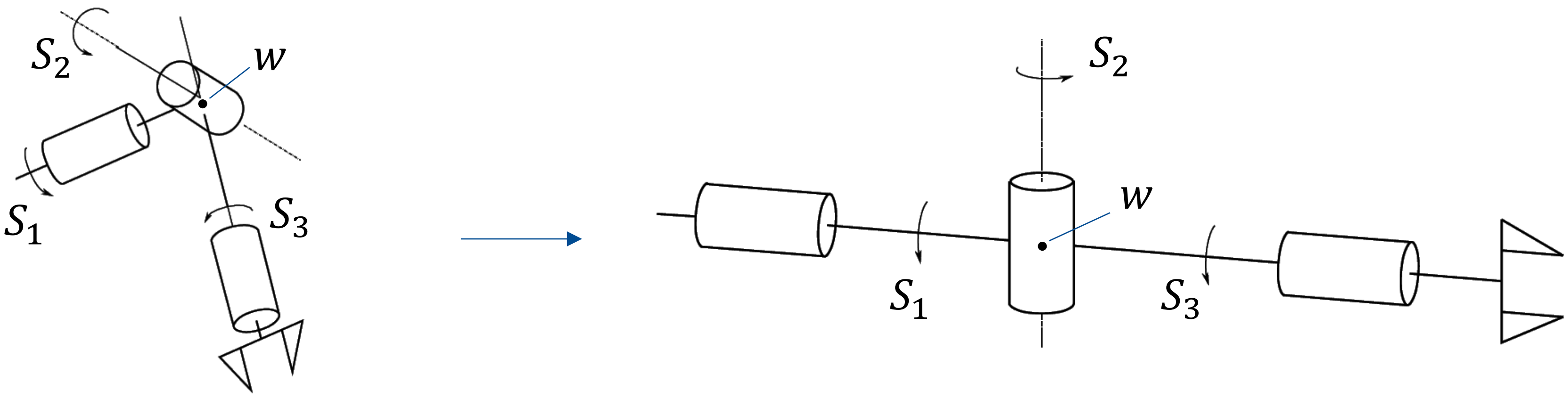
- Computation of **wrist singularities** resulting from the motion of the spherical wrist.
- Computation of **arm singularities** resulting from the motion of the first 3 or more links.

(spherical wrist)



An Example of Wrist Singularity

The **Wrist Singularity** occurs when S_1 and S_3 are aligned (see Euler angles singularities!).

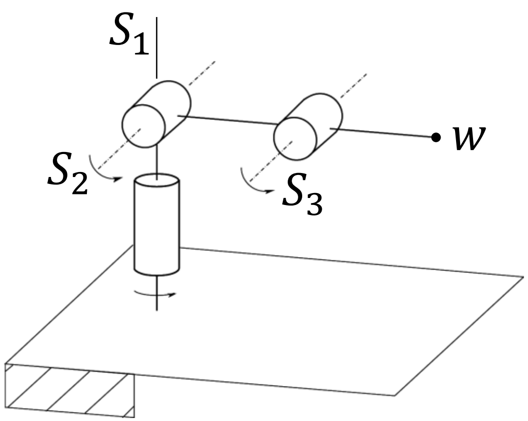


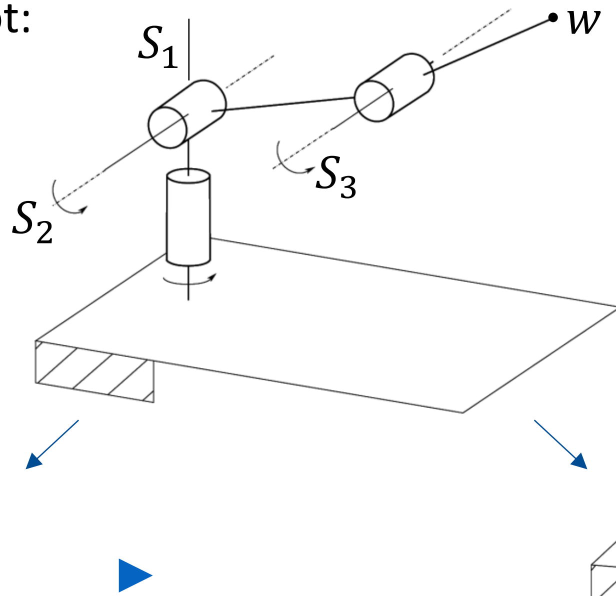
- In this configuration, the wrist cannot rotate about the axis orthogonal to S_1 and S_2.
- Rotations of equal magnitude about opposite directions on S_1 and S_3 do not produce any end-effector rotation.

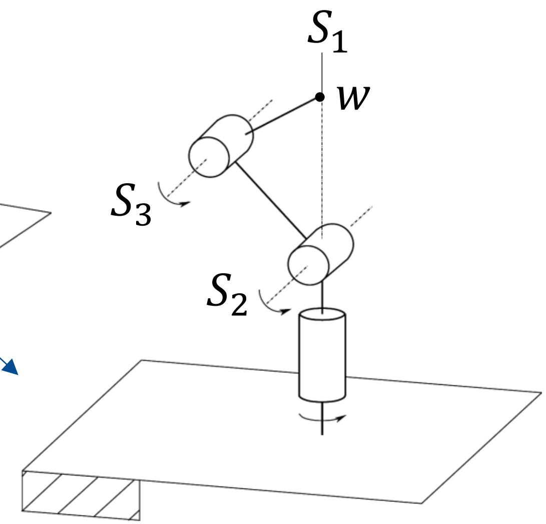
Note: Wrist singularity is naturally described in the joint space and can be encountered anywhere inside the manipulator reachable workspace.

Examples of Arm Singularities

A 3R anthropomorphic spatial robot:







Elbow Singularity: when the elbow is outstretched or retracted.

Shoulder Singularity: when the wrist point (w) lies on axis S_1 (the whole axis S_1 describes a continuum of singular configurations).

Note: Arm singularity is well identified in the task space, and thus, they can be suitably avoided in the end-effector trajectory planning stage.

Examples of Common Singular Configurations ($n \geq 3$)

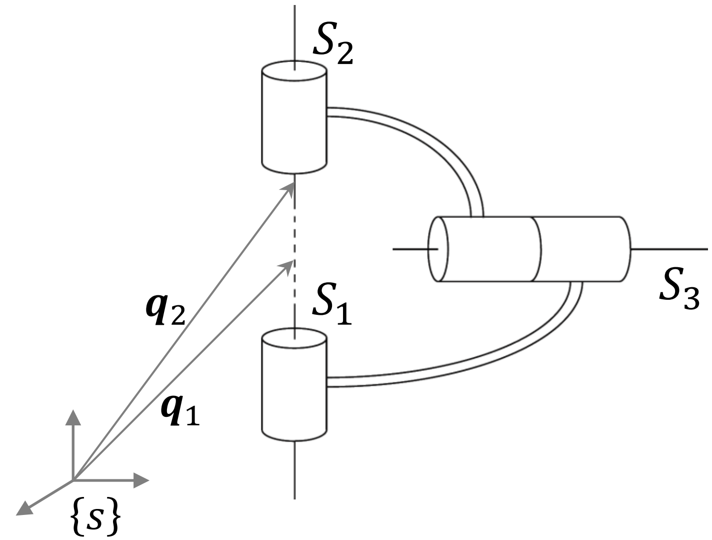
Case I: Two Collinear Revolute Joint Axes

$$J_{s1}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\omega}_{s1} \\ -\boldsymbol{\omega}_{s1} \times \boldsymbol{q}_1 \end{bmatrix} \quad J_{s2}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\omega}_{s2} \\ -\boldsymbol{\omega}_{s2} \times \boldsymbol{q}_2 \end{bmatrix}$$

$$\left.\begin{aligned} \boldsymbol{\omega}_{s1} &= \boldsymbol{\omega}_{s2} \\ \boldsymbol{\omega}_{s1} \times \boldsymbol{q}_1 &= \boldsymbol{\omega}_{s2} \times \boldsymbol{q}_2 \end{aligned}\right\} \; J_{s1} = J_{s2}$$

The set $\{J_{s1}, J_{s2}, \ldots\}$ cannot be linearly independent.

Example: Wrist singularity.

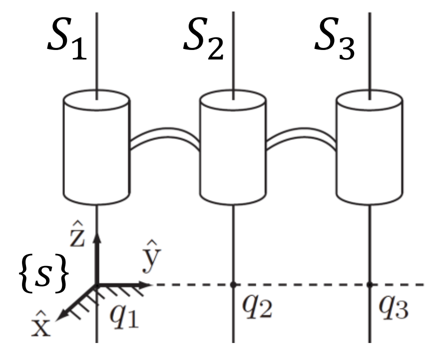


Case II: Three Coplanar and Parallel Revolute Joint Axes

$$J_s(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\omega}_{s1} & \boldsymbol{\omega}_{s1} & \boldsymbol{\omega}_{s1} & \cdots \\ \mathbf{0} & \underbrace{-\boldsymbol{\omega}_{s1} \times \boldsymbol{q}_2}_{\boldsymbol{u}} & \underbrace{-\boldsymbol{\omega}_{s1} \times \boldsymbol{q}_3}_{\alpha \boldsymbol{u}} & \cdots \end{bmatrix}$$

The set $\{J_{s1}, J_{s2}, J_{s3}, \ldots\}$ cannot be linearly independent.

Example: Singularity of a 3R planar robot.



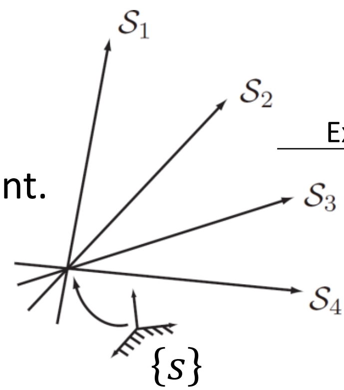
Examples of Common Singular Configurations ($n \geq 4$)

Case III: Four Revolute Joint Axes Intersecting at a Common Point

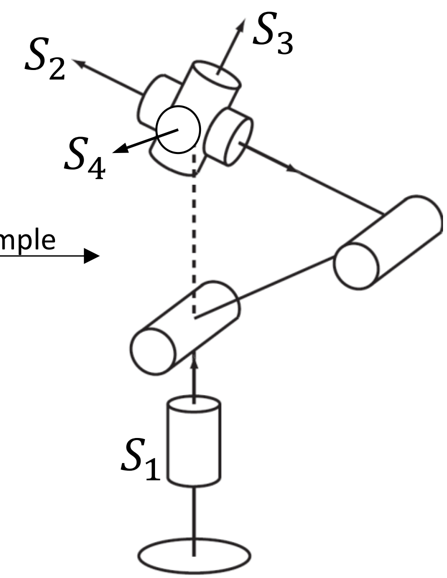
$$J_s(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\omega}_{s1} & \boldsymbol{\omega}_{s2} & \boldsymbol{\omega}_{s3} & \boldsymbol{\omega}_{s4} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \end{bmatrix}$$

The set $\{J_{s1}, J_{s2}, J_{s3}, J_{s4}, \ldots\}$ cannot be linearly independent.

Example: Shoulder singularity of a 3R anthropomorphic robot.



Example



Case IV: Four Coplanar Revolute Joints

$$\boldsymbol{\omega}_{si} = \begin{bmatrix} \omega_{six} \\ \omega_{siy} \\ 0 \end{bmatrix} \qquad \boldsymbol{q}_i = \begin{bmatrix} q_{ix} \\ q_{iy} \\ 0 \end{bmatrix} \qquad -\boldsymbol{\omega}_{si} \times \boldsymbol{q}_i = \begin{bmatrix} 0 \\ 0 \\ \omega_{siy} q_{ix} - \omega_{six} q_{iy} \end{bmatrix}$$

$$J_s(\boldsymbol{\theta}) = \begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_{s1y} q_{1x} - \omega_{s1x} q_{1y} & \omega_{s2y} q_{2x} - \omega_{s2x} q_{2y} & \omega_{s3y} q_{3x} - \omega_{s3x} q_{3y} & \omega_{s4y} q_{4x} - \omega_{s4x} q_{4y} \end{bmatrix}$$

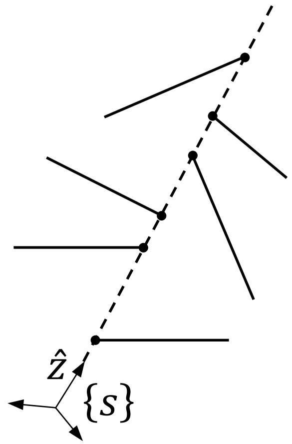
The set $\{J_{s1}, J_{s2}, J_{s3}, J_{s4}, \ldots\}$ cannot be linearly independent.

Examples of Common Singular Configurations ($n \geq 6$)

Case V: Six Revolute Joints Intersecting a Common Line

$$-\boldsymbol{\omega}_{si} \times \boldsymbol{q}_i = (\omega_{siy}q_{iz}, -\omega_{six}q_{iz}, 0)$$

$$\boldsymbol{J}_s(\boldsymbol{\theta}) = \begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} & \omega_{s5x} & \omega_{s6x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} & \omega_{s5y} & \omega_{s6y} \\ \omega_{s1z} & \omega_{s2z} & \omega_{s3z} & \omega_{s4z} & \omega_{s5z} & \omega_{s6z} \\ \omega_{s1y}q_{1z} & \omega_{s2y}q_{2z} & \omega_{s3y}q_{3z} & \omega_{s4y}q_{4z} & \omega_{s5y}q_{5z} & \omega_{s6y}q_{6z} \\ -\omega_{s1x}q_{1z} & -\omega_{s2x}q_{2z} & -\omega_{s3x}q_{3z} & -\omega_{s4x}q_{4z} & -\omega_{s5x}q_{5z} & -\omega_{s6x}q_{6z} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



The set $\{\boldsymbol{J}_{s1}, \boldsymbol{J}_{s2}, \boldsymbol{J}_{s3}, \boldsymbol{J}_{s4}, \boldsymbol{J}_{s5}, \boldsymbol{J}_{s6}, \ldots\}$ cannot be linearly independent.

Note: The closeness of a configuration $\boldsymbol{\theta}$ to a singular configuration can be numerically tested by evaluating how close the minimum singular value of $\boldsymbol{J}(\boldsymbol{\theta})$, i.e., $\sigma_{\min}\{\boldsymbol{J}(\boldsymbol{\theta})\}$, (for $n > r$) or the minimum eigenvalue value of $\boldsymbol{J}(\boldsymbol{\theta})$, i.e., $\lambda_{\min}\{\boldsymbol{J}(\boldsymbol{\theta})\}$, or $\det\big(\boldsymbol{J}(\boldsymbol{\theta})\big)$ (for $n = r$) is to **zero**.

Example

For the KUKA LBR iiwa 7R robot shown in its zero/home configuration, for a general task of manipulating a rigid body, what is the dimension of the Jacobian matrix? What is the rank of the Jacobian at the shown configuration?