Math- 1^{st} Term Exam.

To Solve 2^{nd} degree equation:

1) Factorize:

   a. common factor.

   b. Remarkable identities:

      i) $a^2 - b^2 = (a-b)(a+b)$

      ii) $a^2 + 2ab + b^2 = (a+b)^2$

      iii) $a^2 - 2ab + b^2 = (a-b)^2$

2) $ax^2 + bx + c = 0$ / $a \neq 0$
   $b; c \in \mathbb{R}$

Case 1: if $c = 0$

$$\Downarrow$$

$ax^2 + bx = 0 \rightsquigarrow$ take "x" common.

Case 2: if $b = 0$

$$\Downarrow$$

$ax^2 + c = 0 \rightsquigarrow x^2 = \frac{-c}{a}$

Case 3: if $b = c = 0$

$$\Downarrow$$

$ax^2 = 0 \rightsquigarrow x^2 = 0$

Solve of Quadratic Equation:

$(E): ax^2 + bx + c = 0$

$$\Delta = b^2 - 4ac$$

$\Delta > 0$ → $2 \neq$ roots

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

$\Delta = 0$ → One double root.

$$x_1 = x_2 = \frac{-b}{2a}$$

$\Delta < 0$ → No real solution.

Reduced Form of Δ:

$(E): ax^2 + \boxed{b}x + c = 0$

even $\leadsto b' = \frac{b}{2}$

$\Delta' = b'^2 - ac$

$\rightarrow \Delta' > 0 \leadsto 2 \neq$ roots

$x_1 = \frac{-b' - \sqrt{\Delta'}}{a}$ $\qquad$ $x_2 = \frac{-b' + \sqrt{\Delta'}}{a}$

$\rightarrow \Delta = 0 \leadsto$ one double root

$x_1 = x_2 = -\frac{b'}{a}$

$\rightarrow \Delta < 0 \leadsto$ No Solution.

Sum and Product.

$S = x_1 + x_2$
$= -\frac{b}{a}$

$P = x_1 x_2$
$= \frac{c}{a}$

Application 1:

$\left.\begin{array}{l} x_1 \rightarrow \text{given} \\ x_2 \rightarrow ?? \end{array}\right\} \leadsto$ Using S and P

$a + b + c = 0$

$x = 1$ $\qquad$ $x_2 = \frac{c}{a}$

$a - b + c = 0$

$x_1 = -1$ $\qquad$ $x_2 = -\frac{c}{a}$

Application 2:

Evaluate of expression in terms of x_1 and x_2

$$A = x_1 ; x_2$$

$$\to S ; P$$

Application 3:

Finding of Quadratic Equation.

Given

$x_1 ; x_2$ or $S ; P$

$$\Downarrow \quad (E): x^2 - Sx + P = 0$$

Leading to a 2^{nd} degree equation:

$$(E): ax^4 + bx^2 + c = 0$$

* let $t = x^2$

$$(E'): at^2 + bt + c = 0$$

* Solve (E') $\to t'_1 = —$

$\to t'_2 = —$

* Replace $t = x^2$

Irrational Equation:

* $\sqrt{A} = B$

- Domain of Definition (D_F)

$$\begin{cases} A \geq 0 \\ B \geq 0 \end{cases}$$

- $(\sqrt{A})^2 = B^2$

- Solve
  - * $x \in D.f \rightsquigarrow$ accepted.
  - * $x \notin D.f \rightsquigarrow$ rejected.

Rational Equation:

$$\frac{A}{B} = \frac{C}{D}$$

* Domain of Definition:

$$\begin{cases} B \neq 0 \\ \text{and} \\ D \neq 0 \end{cases}$$

* $\frac{A}{B} \times \frac{C}{D}$

$\Downarrow$ $A \times D = C \times B$

* Solve.
  - $\to$ Quadratic Equation
    - $\to$ $x \in D.f \rightsquigarrow$ accepted.
    - $\to$ $x \notin D.f \rightsquigarrow$ rejected.

Table of Sign of 2^{nd} degree Equation:

$$F(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] \rightsquigarrow \text{Canonical Form}$$

* Study of Sign of F(x)

1) $\Delta < 0 \Rightarrow$ No roots

$\rightsquigarrow$ if $a > 0 \rightsquigarrow F(x) > 0$
$\rightsquigarrow$ if $a < 0 \rightsquigarrow F(x) < 0$

$\rightsquigarrow$ F(x) takes the sign of "a"

x	$-\infty$ $\qquad$ $+\infty$
$F(x)$	same sign of "a"

ii) $\Delta = 0 \Rightarrow x_1 = x_2 = \frac{-b}{2a}$

$\Downarrow$ $f(x) = a\left(x + \frac{b}{2a}\right)^2$

$\rightsquigarrow$ if $a > 0 \rightsquigarrow f(x) > 0$

$\rightsquigarrow$ if $a < 0 \rightsquigarrow f(x) < 0$

$\rightsquigarrow$ f(x) takes the sign of "a"

Factorized Form

$a(x - x_1)^2$

or $a(x - x_2)^2$

x	$-\infty$	$\frac{-b}{2a}$	$+\infty$
$f(x)$	Same sign of "a"	0	Same sign of "a"

iii) $\Delta > 0 \Rightarrow 2 \neq$ roots

$\Downarrow$ $f(x) = a(x - x_1)(x - x_2)$ $\rightsquigarrow$ Factorized Form.

x	$-\infty$	x_1		x_2	$+\infty$
$f(x)$	Same sign of "a"	0	opp. sign of "a"	0	same sign of "a"

Remarks!!

· $f(x) > 0$ for all $x \in \mathbb{R}$

$\Downarrow$ $\Delta < 0$ and $a > 0$

$f(x) < 0$ for all $x \in \mathbb{R}$

$\Downarrow$ $\Delta < 0$ and $a < 0$

One single root

$\Downarrow$ 1^{st} degree $\rightsquigarrow a = 0$

* Reciprocal $\rightsquigarrow \begin{cases} a \neq 0 \\ \Delta \geq 0 \\ P = 1 \end{cases}$

* Opp. roots $\begin{cases} a \neq 0 \\ \Delta \geq 0 \\ S = 0 \end{cases}$

* Same sign $\begin{cases} a \neq 0 \\ \Delta \geq 0 \\ P > 0 \end{cases}$

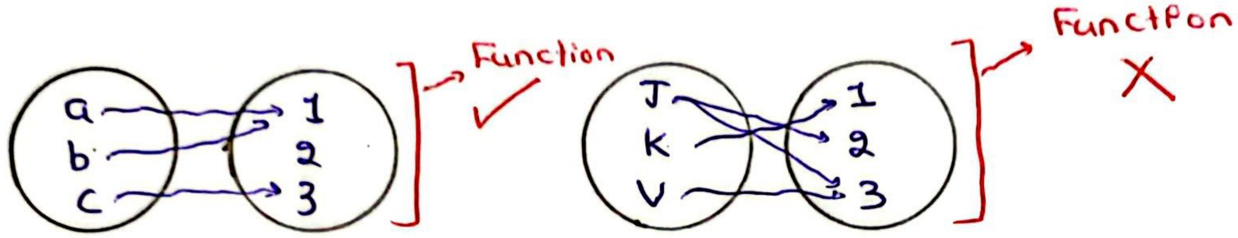
Functions:

1°> Domain of Definition.

$$y = F(x)$$

antecedent $\leftarrow X \longrightarrow 1\,Y \rightarrow$ Image

$X \downarrow$ D.F $\subseteq \mathbb{R}$ $\quad$ $Y \downarrow$ Range $\subseteq \mathbb{R}$



→ Domain of Definition for Polynomial:

$$D_F = \underbrace{\{x \in \mathbb{R} \,/\, F(x) \in \mathbb{R}\}}_{\text{defined.}}$$

$$F(x) : ax^n + bx^{n-1} + \dots + c$$

→ defined for all $x \in \mathbb{R}$

→ $D_F = R =]-\infty ; +\infty[$

→ Domain of Definition for rational Function.

$$F(x) = \frac{A}{B}$$

$\downarrow$ $D_F = R = \{x \in \mathbb{R} / B \neq 0\}$

→ Domain of Definition for irrational Function.

$$F(x) = \sqrt{A}$$

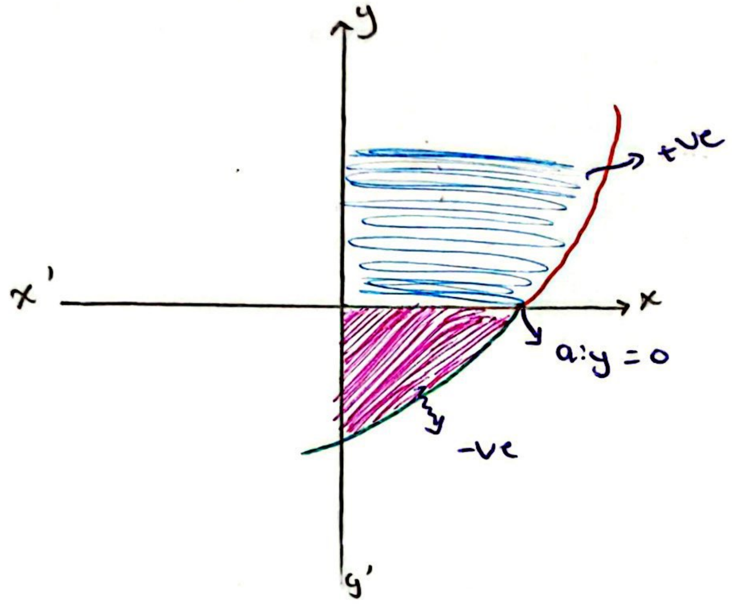
$\downarrow$ $D_F = R = \{x \in \mathbb{R} / A \geq 0\}$

Representative Curve of (C) of $y = f(x)$

Substitute "x" in $y = f(x)$

$(C): \{M(x;y) \,/\, x \in D_f \text{ and } y = f(x)\}$

Sign of Function Graphically:



* (C): above x-axis
  $f(x) > 0$

* (C): below x-axis
  $f(x) < 0$

* (C) ∩ x-axis
  $f(x) = 0$

x	$-\infty$	a	$+\infty$
$f(x)$	$-$	0	$+$

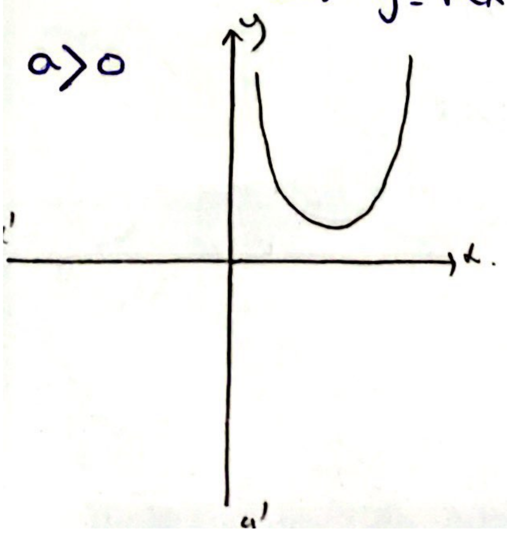
below x-axis

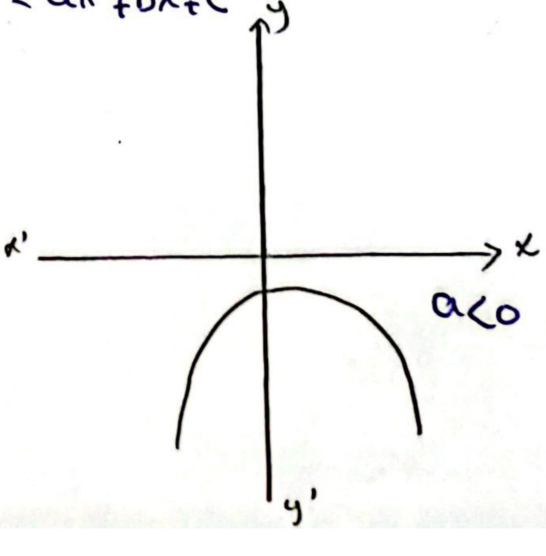
above x-axis.

Sign of Quadratic Function Graphically:

$(C): y = f(x) = ax^2 + bx + c$

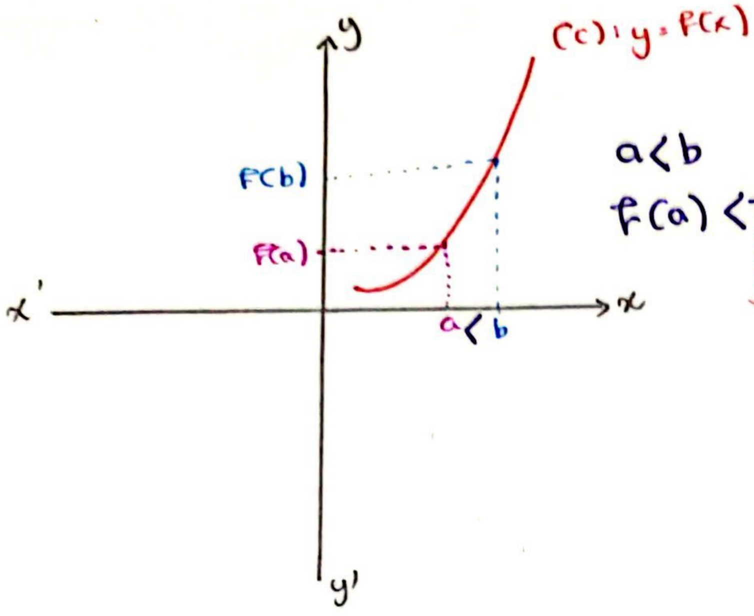
$a > 0$





$a < 0$

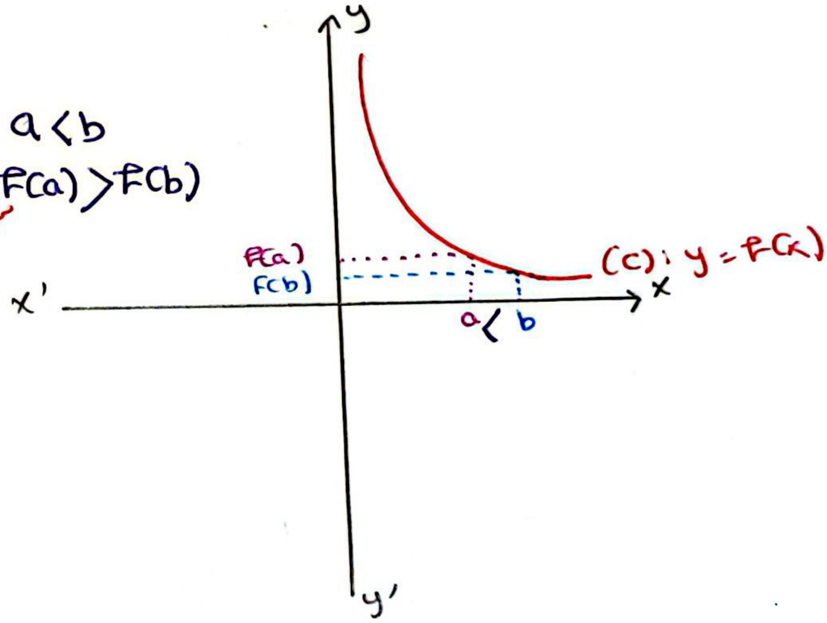
Sense of Variation of functions:



$a < b$

$f(a) < f(b)$

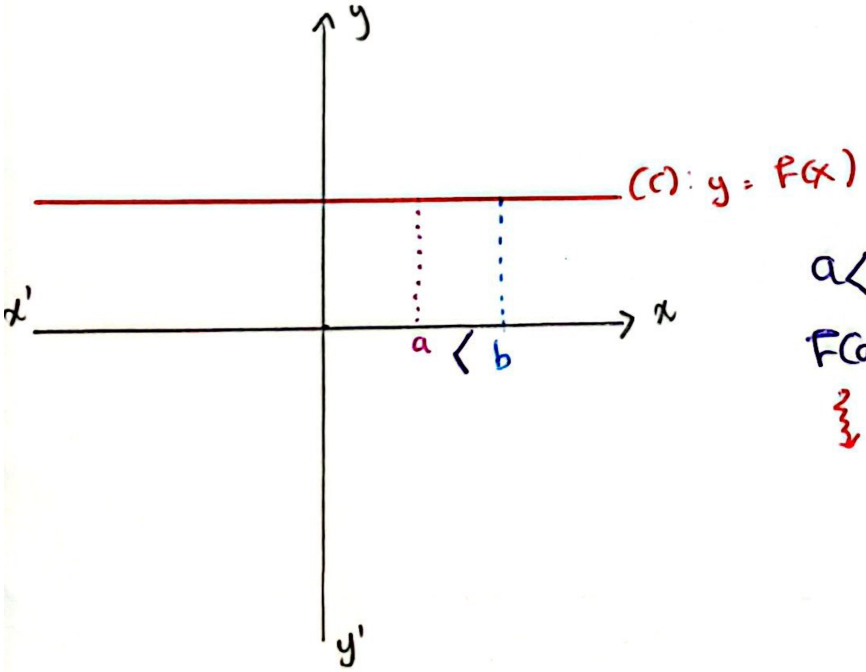
f increases (↗)



$a < b$

$f(a) > f(b)$

f decreases (↘)



$a < b$

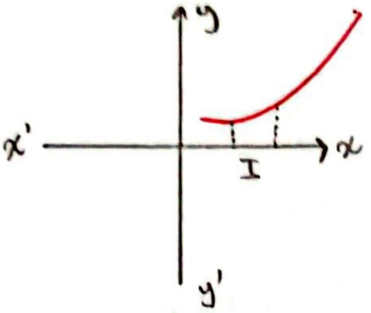
$f(a) = f(b) = K$

constant.

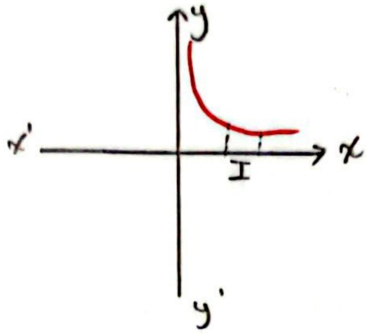
Remarks!!

* (f) is monotonic over $I \subseteq \mathbb{R}$

$f \nearrow$ over I



$f \searrow$ over I



* Limits:

Limits at infinity:

* $\lim_{x \to \pm\infty} x^{\text{even}} = +\infty$

* $\lim_{x \to +\infty} x^{\text{odd}} = +\infty$

* $\lim_{x \to -\infty} x^{\text{odd}} = -\infty$

$\lim_{x \to -\infty} x^n = (-\infty)^n$

even: $+\infty$

odd: $-\infty$

* $\lim_{x \to +\infty} \sqrt{x} = \sqrt{+\infty} = +\infty$

* $\lim_{x \to \pm\infty} \frac{K}{x} = 0$

* $\lim_{x \to +\infty} \frac{1}{x} = 0$ → $\underbrace{y = 0}_{H.A}$

* $\lim_{x \to \pm\infty} ax^n = a(\pm\infty)^n$

$= \pm\infty$ n: even → $+\infty$; n: odd → $\pm\infty$,

* $\frac{\pm\infty}{K} = \pm\infty$

* $\frac{nb}{\pm\infty} = 0$

* $\lim_{x \to 0^+} = +\infty$

* $\lim_{x \to 0^-} = -\infty$

→ $\frac{nb}{0^{\pm}} = \pm\infty$

Rule 1:

$P(x): \underbrace{ax^n}_{\text{highest degree}} + bx^{n-1} + \dots c \ / n \in \mathbb{N}$

$\to \lim\limits_{x \to \pm\infty} ax^n = \pm\infty$ — n: even → $+\infty$; n: odd → $\pm\infty$

→ If and only if: Polynomial + $x \to \pm\infty$

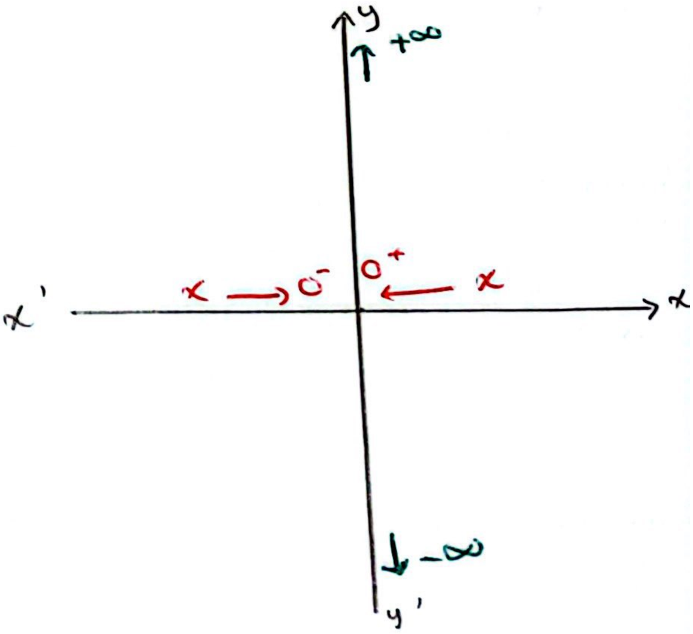
Rule 2:

$\lim\limits_{x \to \pm\infty} \dfrac{A(x)}{B(x)}$ — $A(x)$ → highest degree ; $B(x)$ → highest degree

→ Simplify → Substitute.

Remark:

$D_F = \underbrace{]-\infty; 0[}_{\lim\limits_{x\to 0^-} f(x)} \cup \underbrace{]0; +\infty[}_{\lim\limits_{x\to 0^+} f(x)}$



Graphically:

* $\lim\limits_{x \to 0^+} \dfrac{1}{x} = +\infty$
* $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$

→ $x = 0$ → V.A

Remark:

* D.F = [a, ...

→ f(a)

. D.F =]a; ...

→ $\lim\limits_{x \to a^+} f(a)$

Indeterminate Forms:

⟶ Polynomials:

$+\infty - \infty$

* Common Factor.

$$A \pm B = A\left(1 \pm \frac{B}{A}\right)$$

or $$A \pm B = B\left(\frac{A}{B} \pm 1\right)$$

⟶ Irrationals:

[illegible] $+\infty - \infty$ [illegible]

* Common Factor
or
* rationalize

} or both.

Indeterminate Form:

$\infty \times 0$

⟶ Irrationals:

* rationalize
or
* $\frac{\cancel{x}\sqrt{x^2}}{\cancel{x}}$

} or both.

Indeterminate Form:

1°) $\frac{0}{0}$

⟶ $\frac{\text{Polynomial}}{\text{Polynomial}}$

⇓

Substitute ⟶ Ind. Form.

⇓

i) Factorize.

ii) Simplify.

iii) Substitute.

$\rightsquigarrow \dfrac{\text{Irrational}}{\text{Irrational}} \rightsquigarrow$ Substitute $\rightsquigarrow$ Ind. Form

i) Rationalize $\begin{cases} \sqrt{A} \times \sqrt{A} \\ \text{or } (\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) \end{cases}$

ii) Simplify.

iii) Substitute.

g) $\dfrac{\infty}{\infty}$

$\rightsquigarrow \dfrac{\text{Polynomial}}{\text{Polynomial}} \rightsquigarrow$ Substitute $\rightsquigarrow$ Ind. Form

i) $\dfrac{\text{highest degree}}{\text{highest degree}}$

ii) Simplify.

iii) Substitute.

$\rightsquigarrow$ Irrationals $\rightsquigarrow$ Substitute $\rightsquigarrow$ Ind. Form

rationalize $\rightsquigarrow$ substitute,

or $\dfrac{\cancel{x} \cdot \sqrt{x^2}}{\cancel{x}}$

Remark:

* $(a^3 + b^3) = (a + b)(a^2 + b^2 - ab)$

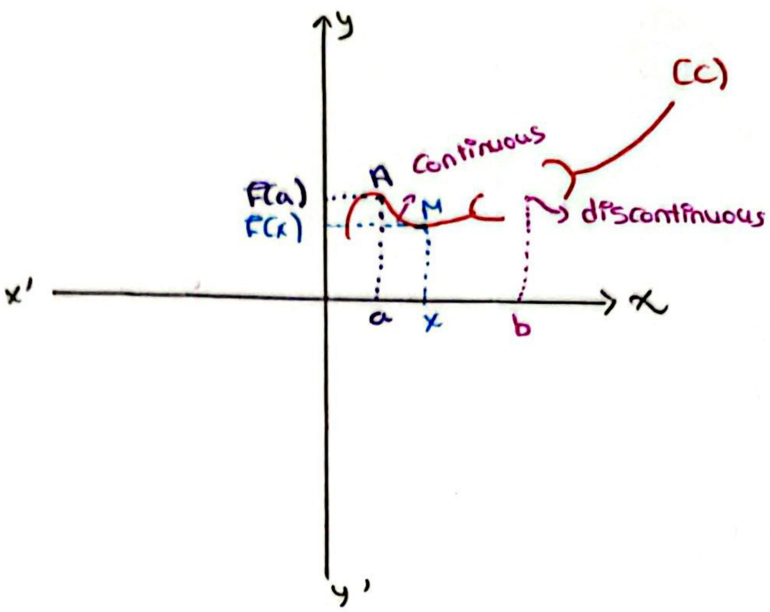
* $(a^3 - b^3) = (a - b)(a^2 + b^2 + ab)$

Continuity :

1> Continuity at a point:

* (c): $y = f(x)$; $D_f \subseteq \mathbb{R}$

* "a" : fixed value.



2> (f) continuous at "a"

$$\begin{cases} * \text{ "a"} \in D_f \; ; \; f(a). \\ * \lim_{x \to a} f(x) \\ * \lim_{x \to a} f(x) = f(a) \end{cases}$$

Derivatives :

Derivative at a point "a"

$(C) : y = f(x)$: D_f

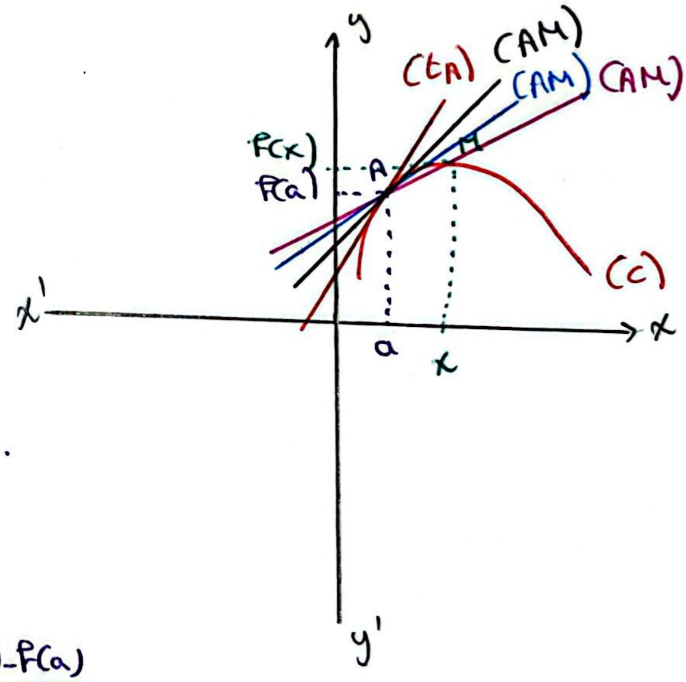
* $a \in D_f$; $f(a)$ / $A(a; f(a)) \in (C)$

* $x \in D_f$; $M(x; f(x))$

(AM) intersects (C) at 2 points A & M.

where, slope (AM) = $\dfrac{y_M - y_A}{x_M - x_A}$

Slope(AM) = $\dfrac{f(x) - f(a)}{x - a}$

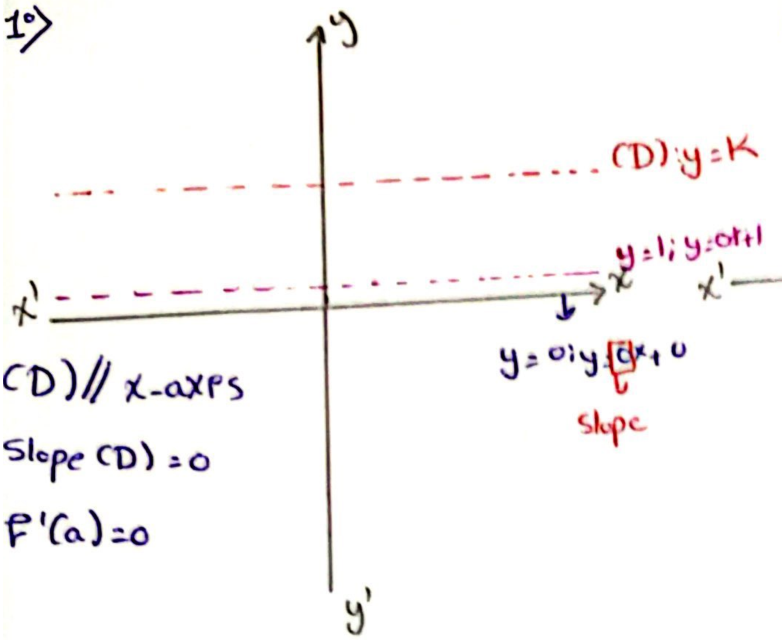


$f'(a)$ = Slope (t_A) = $\lim_{x \to a} \dfrac{f(x) - f(x)}{x - a}$

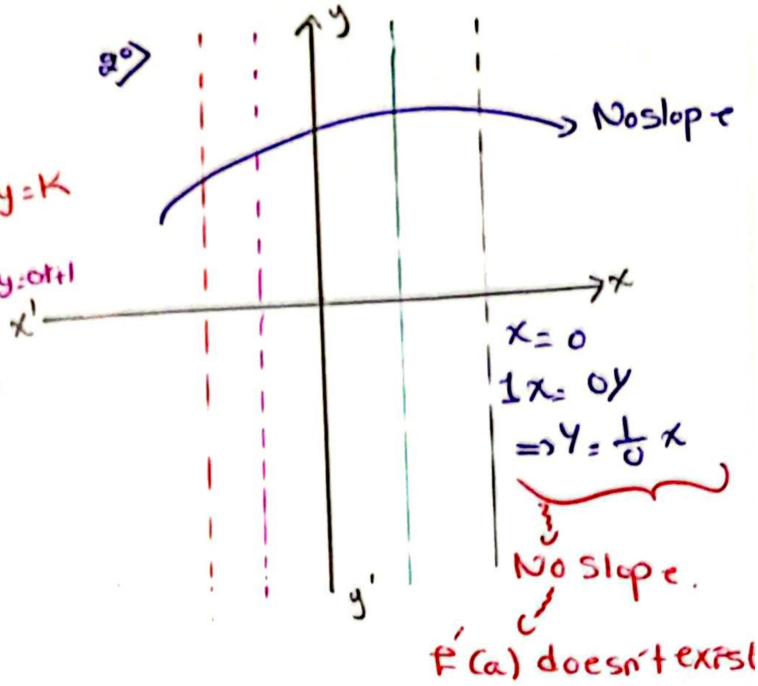
derivative of "f" at "a"

Remarks:

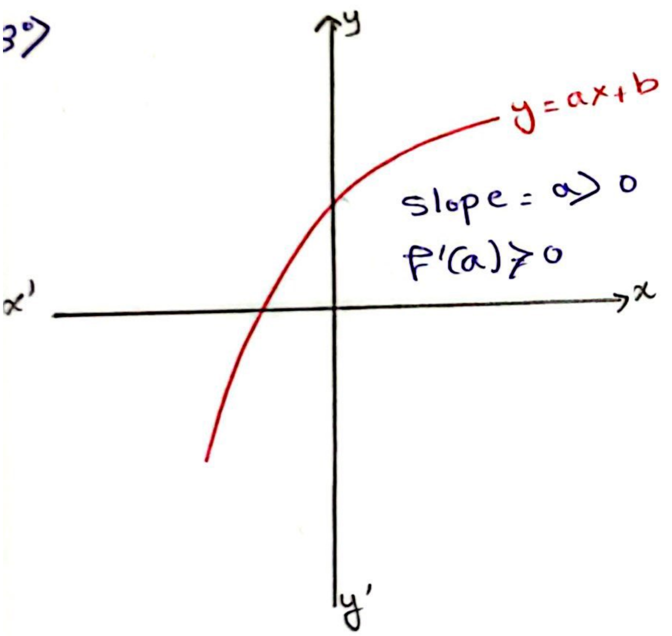
1°)

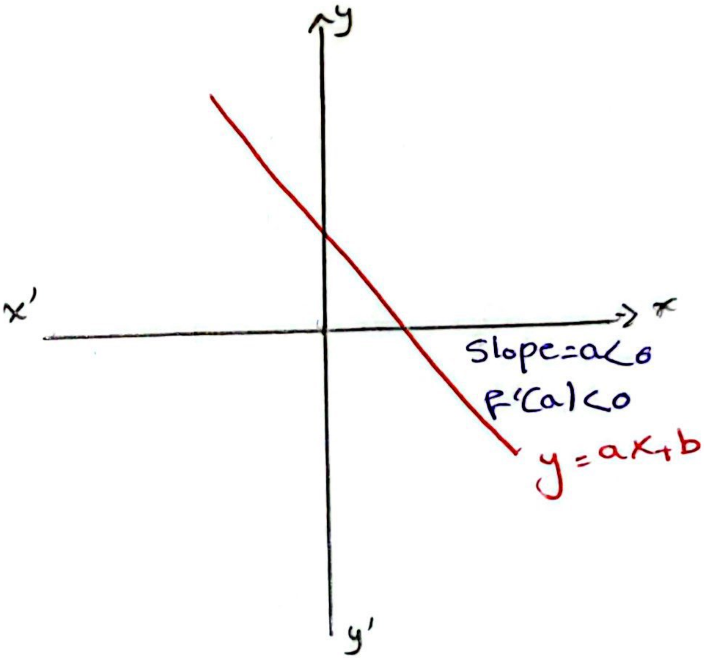


2°)



3°)





Overall Remarks:

* Form 2nd degree equation:

  i) Calculate S and P

  ii) $x^2 - Sx + P = 0$

* Verify the ∓ 1 is a root and find the other.

  i) Substitute "± 1" in the equation.

  ii) $x_2 = \frac{-c}{a}$ or $x_2 = \frac{c}{a}$

* Find evident root $\rightsquigarrow$ $a-b+c=0$

or $a+b+c=0$

* Calculate the roots (given S and P)

i> $S = x_1 + x_2$ $P = x_1 x_2$

ii> $(E): x^2 - Sx + P = 0$

iii> Solve.

: Prove : $\exists \neq$ roots without calculation.

a and c : opp. signs $\rightsquigarrow$ $\Delta > 0$

. Domain of definition $\rightsquigarrow$ value of "x".

: $f(x) = \dfrac{k}{\sqrt{A}}$ $\rightsquigarrow$ $A > 0$