Appendix: Secure Sampled-data Observer-based Control for Wind Turbine Oscillation Under Cyber Attacks

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***Abstract*—This note includes the proofs of the proposed theorems in the article entitled "Secure Sampled-data Observer-based Control for Wind Turbine Oscillation Under Cyber Attacks".**

Appendix A
Proof of Theorem 1

Proof. Consider the LKF $V = V_1 + V_2 + V_3$ in $t \in [t_k, t_{k+1})$ for system (19), where

$$V_1 = \boldsymbol{x}^T(t)\bar{P}_1\,\boldsymbol{x}(t), \quad V_2 = \int_{t-h}^{t} e^{2\zeta_1(s-t)}\boldsymbol{x}^T(s)\,\bar{P}_2\,\boldsymbol{x}(s)ds,$$

$$V_3 = \int_{-h}^{0}\int_{t+s}^{t} e^{2\zeta_1(v-t)}\dot{\boldsymbol{x}}^T(v)\,\bar{P}_3\dot{\boldsymbol{x}}(v)dvds, \tag{a.1}$$

with positive definite matrices $\bar{P}_i$, $(1 \le i \le 3)$. We obtain the time derivative of V as follows

$$\begin{aligned}
&\dot{V}_1(t) = 2\dot{\boldsymbol{x}}^T(t)\,\bar{P}_1\boldsymbol{x}(t),\\
&\dot{V}_2(t) = \boldsymbol{x}^T(t)\,\bar{P}_2\boldsymbol{x}(t) - e^{-2\zeta_1 h}\boldsymbol{x}^T(t-h)\,\bar{P}_2\boldsymbol{x}(t-h) - 2\zeta_1 V_2(t),\\
&\dot{V}_3(t) \le -2\zeta_1 V_3(t) + h\dot{\boldsymbol{x}}^T(t)\bar{P}_3\dot{\boldsymbol{x}}(t)\\
&\qquad - e^{-2\zeta_1 h}\int_{t-h}^{t}\dot{\boldsymbol{x}}^T(v)\,\bar{P}_3\dot{\boldsymbol{x}}(v)dv,
\end{aligned} \tag{a.2}$$

Let $g(v) = \dot{\boldsymbol{x}}^T(v)\bar{P}_3\dot{\boldsymbol{x}}(v)$. With $[t-h, t] = [t-h, t_k] \cup [t_k, t]$, we obtain the following expressions

$$\int_{t-h}^{t} g(v)dv = \int_{t-h}^{t_k} g(v)dv + \int_{t_k}^{t} g(v)dv, \tag{a.3}$$

Let $\boldsymbol{a}_1 = [\boldsymbol{x}^T(t), \boldsymbol{x}^T(t_k)]^T$. For free matrices $\bar{F} = [\bar{F}_1^T\ \bar{F}_2^T]^T$ and $\bar{G} = [\bar{G}_1^T\ \bar{G}_2^T]^T$ with proper dimensions it holds that

$$2\boldsymbol{a}_1^T\bar{F}\Big[\boldsymbol{x}(t_k) - \boldsymbol{x}(t-h) - \int_{t-h}^{t_k}\dot{\boldsymbol{x}}(s)ds\Big] = 0, \tag{a.4}$$

$$2\boldsymbol{a}_1^T\bar{G}\Big[\boldsymbol{x}(t) - \boldsymbol{x}(t_k) - \int_{t_k}^{t}\dot{\boldsymbol{x}}(s)ds\Big] = 0, \tag{a.5}$$

We define $\bar{Y} = \begin{bmatrix}\bar{Y}_{11} & \bar{Y}_{12}\\ * & \bar{Y}_{22}\end{bmatrix}$ for free $n \times n$ matrices $\bar{Y}_{ij}$, $(i, j = 1, 2)$. It holds true that

$$\int_{t-h}^{t_k}\boldsymbol{a}_1^T\bar{Y}\boldsymbol{a}_1\,ds + \int_{t_k}^{t}\boldsymbol{a}_1^T\bar{Y}\boldsymbol{a}_1\,ds = h\,\boldsymbol{a}_1^T\bar{Y}\boldsymbol{a}_1. \tag{a.6}$$

From (22), the following expression holds for $\bar{Q} \in \mathbb{R}^{n\times n} > 0$

$$-\boldsymbol{e}^T(t_k)\bar{Q}\boldsymbol{e}(t_k) + \boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k) = 0 \tag{a.7}$$

where $\boldsymbol{b}(t) = \exp(\int_0^t \sum_{i=1}^{r}\mu_i\{A_i - L_iC\}dt)\boldsymbol{e}(0)$. For free matrix $E \in \mathbb{R}^{n\times n}$, the following null condition holds true from (19)

$$2\Big(\boldsymbol{x}^T(t) + \dot{\boldsymbol{x}}^T(t) + \boldsymbol{x}^T(t_k)\Big)E^{-1}\Bigg(-\dot{\boldsymbol{x}}(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big\{A_i\boldsymbol{x}(t) + B_iK_j(\boldsymbol{x}(t_k) + \boldsymbol{e}(t_k))\Big\}\Bigg) = 0. \tag{a.8}$$

Let $\boldsymbol{\nu} = [\boldsymbol{x}^T(t), \boldsymbol{x}^T(t_k), \boldsymbol{x}^T(t-h), \dot{\boldsymbol{x}}^T(t), \boldsymbol{e}^T(t_k)]^T$ and $\boldsymbol{a}_2 = [\boldsymbol{a}_1^T,\ \dot{\boldsymbol{x}}^T(s)]^T$. Considering (19) and all expressions given in (a.1) to (a.8), we obtain that

$$\dot{V}(t) + 2\zeta_1 V(t) \le \boldsymbol{\nu}^T\bar{W}\boldsymbol{\nu} - \int_{t-h}^{t_k}\boldsymbol{a}_2^T\bar{U}_1\boldsymbol{a}_2 ds - \int_{t_k}^{t}\boldsymbol{a}_2^T\bar{U}_2\boldsymbol{a}_2 ds + \boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k), \qquad t \in [t_k, t_{k+1}), \tag{a.9}$$

where

$$\bar{W} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\begin{bmatrix}\bar{W}_{i,11} & \bar{W}_{ij,12} & -\bar{F}_1 & \bar{W}_{i,14} & \bar{W}_{ij,15}\\ * & \bar{W}_{ij,22} & -\bar{F}_2 & \bar{W}_{ij,24} & \bar{W}_{ij,25}\\ * & * & -e^{-2\zeta_1 h}\bar{P}_2 & 0 & 0\\ * & * & * & \bar{W}_{44} & \bar{W}_{ij,45}\\ * & * & * & * & \bar{W}_{55}\end{bmatrix},$$

$$\bar{U}_1 = \begin{bmatrix}\bar{Y} & \bar{F}\\ * & e^{-2\zeta_1 h}\bar{P}_3\end{bmatrix}, \qquad \bar{U}_2 = \begin{bmatrix}\bar{Y} & \bar{G}\\ * & e^{-2\zeta_1 h}\bar{P}_3\end{bmatrix},$$

$$\begin{aligned}
&\bar{W}_{i,11} = \bar{P}_2 + \bar{G}_1 + \bar{G}_1^T + E^{-1}A_i + (E^{-1}A_i)^T + h\bar{Y}_{11} + 2\zeta_1\bar{P}_1,\\
&\bar{W}_{ij,12} = \bar{F}_1 - \bar{G}_1 + \bar{G}_2^T + (E^{-1}A_i)^T + E^{-1}B_iK_j + h\bar{Y}_{12},\\
&\bar{W}_{i,14} = \bar{P}_1 - E^{-1} + (E^{-1}A_i)^T,\\
&\bar{W}_{ij,22} = \bar{F}_2 + \bar{F}_2^T - \bar{G}_2 - \bar{G}_2^T + h\bar{Y}_{22} + E^{-1}B_iK_j\\
&\qquad + (E^{-1}B_iK_j)^T,\\
&\bar{W}_{ij,24} = (E^{-1}B_iK_j)^T - E^{-1}, \quad \bar{W}_{44} = h\bar{P}_3 - E^{-1} - (E^{-1})^T,\\
&\bar{W}_{ij,15} = \bar{W}_{ij,25} = \bar{W}_{ij,45} = E^{-1}B_iK_j, \qquad \bar{W}_{55} = -\bar{Q}.
\end{aligned}$$

Now, we show that if $\bar{W} < 0$, $\bar{U}_1 \ge 0$, and $\bar{U}_2 \ge 0$ system (19) is stable. From (a.9), if $\bar{W} < 0$, $\bar{U}_1 \ge 0$, and $\bar{U}_2 \ge 0$ it follows that

$$\dot{V}(t) < -2\zeta_1 V(t) + \boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k), \quad t \in [t_k, t_{k+1}), \tag{a.10}$$

or equivalently

$$V(t) < e^{-2\zeta_1 t}V(0) + \frac{\boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k)}{2\zeta_1}, \quad t \in [t_k, t_{k+1}) \tag{a.11}$$

We try to convey (a.11) in terms of $\|\boldsymbol{x}(t)\|$ and $\|\boldsymbol{x}(0)\|$. A lower-bound for the left hand side of (a.11) is $\lambda_{\min}(\bar{P}_1)\|\boldsymbol{x}(t)\|^2 \leq V_1(t) \leq V(t)$. As for $V(0)$, we have $V_1(0) \leq \lambda_{\max}(\bar{P}_1)\|\boldsymbol{x}(0)\|^2$. For $V_2(0)$, the region for the integral variable s is $s\in[-h,0]$. Since $s\in[-h,0]$, it is true that $0 < e^{2\zeta_1 s} \leq 1$. Therefore, one can obtain $V_2(0) = \int_{-h}^{0} e^{2\zeta_1 s}\boldsymbol{x}^T(s)\bar{P}_2\,\boldsymbol{x}(s)ds \leq \int_{-h}^{0}\boldsymbol{x}^T(s)\bar{P}_2\,\boldsymbol{x}(s)ds$. Note that $\boldsymbol{x}(s)=\boldsymbol{x}(0)$ for $s\in[-h,0]$. Hence, $V_2(0) \leq \int_{-h}^{0}\boldsymbol{x}^T(s)\bar{P}_2\,\boldsymbol{x}(s)ds \leq \lambda_{\max}(\bar{P}_2)\|\boldsymbol{x}(0)\|^2\int_{-h}^{0}ds = h\,\lambda_{\max}(\bar{P}_2)\|\boldsymbol{x}(0)\|^2$. As for $V_3(0)$, it holds that $\boldsymbol{x}(s)=\boldsymbol{x}(0)$ for $s\in[-h,0]$, which implies $\dot{\boldsymbol{x}}(s)=0$ and $V_3(0)=0$. In conclusion, the following bounds are obtained from (a.11)

$$\|\boldsymbol{x}(t)\|^2 \leq k_1 e^{-2\zeta_1 t}\|\boldsymbol{x}(0)\|^2 + k_2, \tag{a.12}$$

where $k_1 = \frac{\lambda_{\max}(\bar{P}_1)+h\,\lambda_{\max}(\bar{P}_2)}{\lambda_{\min}(\bar{P}_1)}$ and $k_2 = \frac{\boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k)}{2\zeta_1\lambda_{\min}(\bar{P}_1)}$ From (a.12) it is clear that system (19) is stable. Note that $A_i - L_iC$ is Hurwitz stable and thus k_2 in (a.12) asymptotically approaches zero. Recall that expression (a.12) is derived under conditions $\bar{W} < 0$, $\bar{U}_1 \geq 0$, and $\bar{U}_2 \geq 0$. To guarantee these conditions we first construct matrix $W = \Pi_1\bar{W}\Pi_1^T$ where $\Pi_1 = \mathrm{diag}(E,E,E,E,E)$. In a similar fashion, block matrices $\bar{U}_1$ and $\bar{U}_2$ are pre- and post multiplied by Π_2 and Π_2^T, where $\Pi_2 = \mathrm{diag}(E,E,E)$. Then, we employ the following alternative variables $P_i = E\bar{P}_iE^T$, $(1\leq i\leq 3)$, $F_i = E\bar{F}_iE^T$, $(i=1,2)$, $G_i = E\bar{G}_iE^T$, $(i=1,2)$, $Y_{ij} = E\bar{Y}_{ij}E^T$, $(i=1,2)$, $(j=1,2)$, $Q = E\bar{Q}E^T$, and $M = KE^T$. According to [1, Theorem 2.2], condition $W<0$ is fulfilled if $W_{ii}<0$ and $\frac{2}{r-1}W_{ii}+W_{ij}+W_{ji}<0$ which are given in (23) and (24). Control gains K_i and $i=1,\ldots,r$ are computed from (26). This completes the proof. $\square$

Appendix B
Proof of Theorem 2

Proof. In the presence of DoS, the system is either in Healthy intervals ($t\in R_m$) or in Attack intervals ($t\in Z_m$). The proof follows in two steps.

Step I. Healthy intervals ($t\in R_m$): From Theorem 1 we previously showed that if LMIs (23), (24), and (25) are guaranteed then trajectories of the system follows (a.10) which is reproduced below for ease of reference

$$\dot{V}(t) < -2\zeta_1\,V(t) + \boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k), \quad t\in R_m. \tag{a.13}$$

Step II. DoS intervals ($t\in Z_m$): If $t\in Z_m$, the states of (20) may diverge. There exists a divergence rate $\zeta_2>0$ such that

$$\dot{V}(t) < 2\zeta_2 V(t), \quad t\in Z_m. \tag{a.14}$$

where $V = V_1+V_2+V_3$ is given in (a.1). We expand (a.14) as

$$\dot{V} - 2\zeta_2 V = \sum_{i=1}^{3}\dot{V}_i - 2\zeta_2\sum_{i=1}^{3}V_i \leq \sum_{i=1}^{3}\dot{V}_i - 2\zeta_2 V_1 < 0. \tag{a.15}$$

The following expressions hold for the derivatives of LKF (a.1) in $t\in Z_m$

$$\begin{aligned}&\dot{V}_1 = 2\dot{\boldsymbol{x}}^T(t)\bar{P}_1\boldsymbol{x}(t), \qquad \dot{V}_2 \leq \boldsymbol{x}^T(t)\bar{P}_2\boldsymbol{x}(t),\\ &\dot{V}_3 \leq h\dot{\boldsymbol{x}}^T(t)\bar{P}_3\dot{\boldsymbol{x}}(t).\end{aligned} \tag{a.16}$$

The following null equality holds based on (20) for $t\in Z_m$

$$2\Big(\boldsymbol{x}^T(t)+\dot{\boldsymbol{x}}^T(t)\Big)E^{-1}\Big(-\dot{\boldsymbol{x}}(t)+\sum_{i=1}^{r}\mu_iA_i\boldsymbol{x}(t)\Big)=0. \tag{a.17}$$

Considering (a.16) and (a.17), we revise (a.15) as follows

$$\sum_{i=1}^{3}\dot{V}_i - 2\zeta_2 V_1 = \boldsymbol{a}_3^T\bar{S}_i\boldsymbol{a}_3 < 0, \tag{a.18}$$

where $\boldsymbol{a}_3 = [\boldsymbol{x}^T(t),\,\dot{\boldsymbol{x}}^T(t)]^T$ and

$$\bar{S}_i = \sum_{i=1}^{r}\mu_i\begin{bmatrix}\bar{s}_i & (E^{-1}A_i)^T - E^{-1} + \bar{P}_1\\ * & h\bar{P}_3 - E^{-1} - (E^{-1})^T\end{bmatrix}, \tag{a.19}$$

with $\bar{s}_i = E^{-1}A_i + (E^{-1}A_i)^T - 2\zeta_2\bar{P}_1 + \bar{P}_2$. Considering (a.15) and (a.18), if $\bar{S}_i<0$ then $\dot{V}(t) - 2\zeta_2 V(t) < 0$ is guaranteed for $t\in Z_m$. We pre- and post multiply $\bar{S}_i$ by Π_3 and Π_3^T, where $\Pi_3 = I_2\otimes E$. Employing the same alternative variables used at the end of proof of Theorem 1 leads to LMIs $S_i<0$, $i=1,\ldots,r$ given in (30).

Now, we merge (a.13) and (a.14) to obtain the *overall* stability condition and the maximum tolerable amount of DoS. Expressions (a.13) and (a.14) are expanded as follows

$$V(t) \leq e^{-2\zeta_1(t-\nu_{m-1})}V(\nu_{m-1}) + c(t_k), \quad t\in R_m, \tag{a.20}$$

$$V(t) \leq e^{2\zeta_2(t-\xi_m)}V(\xi_m), \quad t\in Z_m. \tag{a.21}$$

where $c(t_k) = \frac{\boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k)}{2\zeta_1}$. Assume that $t\in Z_m$. From (a.20) and (a.21) we obtain that [1]

$$\begin{aligned}V(t) &\leq e^{2\zeta_2(t-\xi_m)}V(\xi_m)\\ &\leq e^{2\zeta_2(t-\xi_m)}\Big(e^{-2\zeta_1(\xi_m-\nu_{m-1})}V(\nu_{m-1}) + c(t_k)\Big)\\ &\leq e^{2\zeta_2(t-\xi_m)}e^{-2\zeta_1(\xi_m-\nu_{m-1})}e^{2\zeta_2(\nu_{m-1}-\xi_{m-1})}V(\xi_{m-1})\\ &\quad + c(t_k)\,e^{2\zeta_2(t-\xi_m)}\\ &\leq \ldots\\ &\leq e^{-2\zeta_1|R(0,t)|}e^{2\zeta_2|Z(0,t)|}V(0)\\ &\quad + \bar{c}\sum_{\substack{m\in\mathbb{N}_0\\ \xi_m\leq t}}e^{-2\zeta_1|R(\nu_{m-1},t)|}e^{2\zeta_2|Z(\xi_m,t)|}\end{aligned} \tag{a.22}$$

where $\bar{c} = \max_{k\in\mathbb{N}_0}\{c(t_k)\}$. It is straightforward to show that (a.22) also holds if we start with $t\in R_m$. From (11), (12), (17), and (18), we derive that

$$e^{-2\zeta_1|R(0,t)|}e^{2\zeta_2|Z(0,t)|} \leq \rho^2\,e^{-2\zeta t}, \tag{a.23}$$

where

$$\rho = e^{(\zeta_1+\zeta_2)(\alpha_0+h\beta_0)}, \quad \zeta = \zeta_1 - (\zeta_1+\zeta_2)\left(\frac{1}{\alpha_1}+\frac{h}{\beta_1}\right). \tag{a.24}$$

[1]For visualization of inequalities in (a.22) refer to Fig. 4.

Condition (a.23) leads to

$$V(t) < \rho^2 e^{-2\zeta t} V(0) + \eta, \quad t \geq 0, \tag{a.25}$$

where $\eta = \bar{c} \sum_{\substack{m \in \mathbb{N}_0 \\ \xi_m \leq t}} e^{-2\zeta_1 |R(\nu_{m-1}, t)|} e^{2\zeta_2 |Z(\xi_m, t)|}$. Based on (a.25), if $\zeta > 0$ then the system remains stable under DoS attacks. This implies that the DoS attacks should satisfy $(\frac{1}{\alpha_1} + \frac{h}{\beta_1}) < \Omega$ with $\Omega = \zeta_1/(\zeta_1 + \zeta_2)$. Parameter Ω is referred to as the DoS resilience, since it represents the upper-bound for tolerable amount of DoS. Note that with $\zeta > 0$ and $A_{L_i} = A_i - L_i C$ being Hurwitz stable, parameter η approaches zero asymptotically. This completes the proof. □

Appendix C
Proof of Theorem 3

For free matrix $E \in \mathbb{R}^{n \times n}$, the following holds true from the closed-loop system under deception attacks given in (21)

$$\begin{aligned} &2\Big(\boldsymbol{x}^T(t) + \dot{\boldsymbol{x}}^T(t) + \boldsymbol{x}^T(t_k)\Big) E^{-1} \Bigg(- \dot{\boldsymbol{x}}(t) \\ &\quad + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \Big\{ A_i \boldsymbol{x}(t) + B_i K_j (\boldsymbol{x}(t_k) + \boldsymbol{e}(t_k)) \Big\} \\ &\quad + \sum_{i=1}^{r} \mu_i B_i \left(\boldsymbol{\eta}_1(t) + \boldsymbol{\eta}_2(t) + \boldsymbol{u}_s(t_k) \right) \Bigg) = 0, \end{aligned} \tag{a.26}$$

where $\boldsymbol{\eta}_1(t) = \Gamma(t) \sum_{j=1}^{r} \mu_j K_j \boldsymbol{x}(t_k)$ and $\boldsymbol{\eta}_2(t) = \Gamma(t) \sum_{j=1}^{r} \mu_j K_j \boldsymbol{e}(t_k)$. From (14) it holds that

$$\boldsymbol{\eta}_1^T(t) \boldsymbol{\eta}_1(t) \leq \bar{\gamma}^2 \boldsymbol{x}^T(t_k) \Big(\sum_{j=1}^{r} \mu_j K_j \Big)^T \Big(\sum_{j=1}^{r} \mu_j K_j \Big) \boldsymbol{x}(t_k), \tag{a.27}$$

$$\boldsymbol{\eta}_2^T(t) \boldsymbol{\eta}_2(t) \leq \bar{\gamma}^2 \boldsymbol{e}^T(t_k) \Big(\sum_{j=1}^{r} \mu_j K_j \Big)^T \Big(\sum_{j=1}^{r} \mu_j K_j \Big) \boldsymbol{e}(t_k). \tag{a.28}$$

Let $\boldsymbol{\nu} = [\boldsymbol{x}^T(t), \boldsymbol{x}^T(t_k), \boldsymbol{x}^T(t-h), \dot{\boldsymbol{x}}^T(t), \boldsymbol{e}^T(t_k), \boldsymbol{\eta}_1^T(t), \boldsymbol{\eta}_2^T(t), \boldsymbol{u}_s^T(t)]^T$ and $\boldsymbol{a}_2 = [\boldsymbol{x}^T(t), \boldsymbol{x}^T(t_k), \dot{\boldsymbol{x}}^T(s)]^T$. Similar to (a.9), we consider (a.26), (a.27), (a.28), and all expressions in (a.1)-(a.7) to obtain the following condition

$$\begin{aligned} \dot{V}(t) + 2\zeta_1 V(t) \leq & \boldsymbol{\nu}^T \begin{bmatrix} \bar{W} & \bar{N} \\ * & \bar{R} \end{bmatrix} \boldsymbol{\nu} - \int_{t-h}^{t_k} \boldsymbol{a}_2^T \bar{U}_1 \boldsymbol{a}_2 ds \\ & - \int_{t_k}^{t} \boldsymbol{a}_2^T \bar{U}_2 \boldsymbol{a}_2 ds + \boldsymbol{b}^T(t_k) \bar{Q} \boldsymbol{b}(t_k) + \bar{u}_s^2, \qquad t \in [t_k, t_{k+1}), \end{aligned} \tag{a.29}$$

Block matrix $\bar{W}$ in (a.29) has the same structure given below (a.9) except for the following two blocks

$$\begin{aligned} \bar{W}_{ij,11} =& \bar{P}_2 + \bar{G}_1 + \bar{G}_1^T + E^{-1} A_i + (E^{-1} A_i)^T + h \bar{Y}_{11} + 2\zeta_1 \bar{P}_1 \\ & + \bar{\gamma}^2 K_j^T K_j, \\ \bar{W}_{ij,55} =& - \bar{Q} + \bar{\gamma}^2 K_j^T K_j. \end{aligned}$$

Based on (a.29), if $\bar{C} = \begin{bmatrix} \bar{W} & \bar{N} \\ * & \bar{R} \end{bmatrix} < 0$, $\bar{U}_1 \geq 0$, and $\bar{U}_2 \geq 0$, then

$$\dot{V}(t) < -2\zeta_1 V(t) + \bar{u}_s^2 + \boldsymbol{b}^T(t_k) \bar{Q} \boldsymbol{b}(t_k), \quad t \in [t_k, t_{k+1}), \tag{a.30}$$

where $\bar{U}_1$, $\bar{U}_2$, and $\boldsymbol{b}(t_k)$ are defined previously and

$$\bar{N} = \sum_{i=1}^{r} \mu_i \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes E^{-1} B_i, \quad \bar{R} = -\mathrm{diag}(I_m, I_m, I_m).$$

To guarantee $\bar{C} < 0$, $\bar{U}_1 \geq 0$, and $\bar{U}_2 \geq 0$, we first construct matrix $C = \Pi_1 \bar{C} \Pi_1^T$ where $\Pi_1 = \mathrm{diag}(E, E, E, E, E, I, I, I)$. In a similar fashion, block matrices $\bar{U}_1$ and $\bar{U}_2$ are pre- and post multiplied by Π_2 and Π_2^T, where $\Pi_2 = \mathrm{diag}(E, E, E)$. The Schur complement Lemma is used for the term $\bar{\gamma}^2 K_j^T K_j$ in $\bar{W}_{ij,22}$ and $\bar{W}_{ij,55}$. Then, we employ the following alternative variables $P_i = E \bar{P}_i E^T$, $(1 \leq i \leq 3)$, $F_i = E \bar{F}_i E^T$, $(i = 1, 2)$, $G_i = E \bar{G}_i E^T$, $(i = 1, 2)$, $Y_{ij} = E \bar{Y}_{ij} E^T$, $(i = 1, 2)$, $(j = 1, 2)$, $Q = E \bar{Q} E^T$, and $M = K E^T$. According to [1, Theorem 2.2], condition $C < 0$ is fulfilled if $C_{ii} < 0$ and $\frac{2}{r-1} C_{ii} + C_{ij} + C_{ji} < 0$ which are given in (34) and (35). Control gains K_i and $i = 1, \ldots, r$ are computed from (38). This completes the proof.

References

[1] H. D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 324–332, 2001.