Appendix A
Proof of Theorem 1

Proof: The proof is give in three steps.

Step I. Healthy intervals: Consider LKF $V = V_1 + V_2 + V_3$ in $t \in [t_k, t_{k+1})$ where

$$V_1 = \boldsymbol{x}^T(t)\bar{P}_1\boldsymbol{x}(t), \quad V_2 = \int_{t-h}^{t} e^{2\zeta_1(s-t)}\boldsymbol{x}^T(s)\bar{P}_2\boldsymbol{x}(s)ds,$$
$$V_3 = \int_{-h}^{0}\int_{t+s}^{t} e^{2\zeta_1(v-t)}\dot{\boldsymbol{x}}^T(v)\bar{P}_3\dot{\boldsymbol{x}}(v)dvds, \tag{1}$$

with $\bar{P}_i > 0$, $(1 \le i \le 3)$. The time derivative of V is

$$\dot{V}_1(t) = 2\dot{\boldsymbol{x}}^T(t)\bar{P}_1\boldsymbol{x}(t), \tag{2}$$
$$\dot{V}_2(t) = \boldsymbol{x}^T(t)\bar{P}_2\boldsymbol{x}(t) - e^{-2\zeta_1 h}\boldsymbol{x}^T(t-h)\bar{P}_2\boldsymbol{x}(t-h) - 2\zeta_1 V_2(t),$$
$$\dot{V}_3(t) \le h\dot{\boldsymbol{x}}^T(t)\bar{P}_3\dot{\boldsymbol{x}}(t) - 2\zeta_1 V_3(t) - e^{-2\zeta_1 h}\int_{t-h}^{t}\dot{\boldsymbol{x}}^T(v)\bar{P}_3\dot{\boldsymbol{x}}(v)dv.$$

Let $g(v) = \dot{\boldsymbol{x}}^T(v)\bar{P}_3\dot{\boldsymbol{x}}(v)$. With $[t-h, t] = [t-h, t_k] \cup [t_k, t]$, we obtain the following expression

$$\int_{t-h}^{t} g(v)dv = \int_{t-h}^{t_k} g(v)dv + \int_{t_k}^{t} g(v)dv, \tag{3}$$

Let $\boldsymbol{a}_1 = [\boldsymbol{x}^T(t), \boldsymbol{x}^T(t_k)]^T$. For free matrices $\bar{F} = [\bar{F}_1^T \;\; \bar{F}_2^T]^T$ and $\bar{G} = [\bar{G}_1^T \;\; \bar{G}_2^T]^T$ with proper dimensions it holds that

$$2\boldsymbol{a}_1^T\bar{F}\Big[\boldsymbol{x}(t_k) - \boldsymbol{x}(t-h) - \int_{t-h}^{t_k}\dot{\boldsymbol{x}}(s)ds\Big] = 0, \tag{4}$$
$$2\boldsymbol{a}_1^T\bar{G}\Big[\boldsymbol{x}(t) - \boldsymbol{x}(t_k) - \int_{t_k}^{t}\dot{\boldsymbol{x}}(s)ds\Big] = 0, \tag{5}$$

Let $\bar{Y} = \begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} \\ * & \bar{Y}_{22} \end{bmatrix}$ for free $n \times n$ matrices $\bar{Y}_{ij}$, $(i, j = 1, 2)$. It holds true that

$$\int_{t-h}^{t_k}\boldsymbol{a}_1^T\bar{Y}\boldsymbol{a}_1\, ds + \int_{t_k}^{t}\boldsymbol{a}_1^T\bar{Y}\boldsymbol{a}_1\, ds = h\,\boldsymbol{a}_1^T\bar{Y}\boldsymbol{a}_1. \tag{6}$$

From (9b), the following expression holds for $\bar{Q} \in \mathbb{R}^{n\times n} > 0$

$$-\boldsymbol{e}^T(t_k)\bar{Q}\boldsymbol{e}(t_k) + \boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k) = 0 \tag{7}$$

where $\boldsymbol{b}(t) = \exp(\int_0^t\{A - LC\}dt)\boldsymbol{e}(0)$. Based on (9a), the following null condition holds for free $E \in \mathbb{R}^{n\times n}$

$$2\Big(\boldsymbol{x}^T(t) + \dot{\boldsymbol{x}}^T(t) + \boldsymbol{x}^T(t_k)\Big)E^{-1}\Big(-\dot{\boldsymbol{x}}(t) \tag{8}$$
$$+ A\boldsymbol{x}(t) + BK(\boldsymbol{x}(t_k) + \boldsymbol{e}(t_k))\Big) = 0.$$

Let $\boldsymbol{\nu} = [\boldsymbol{x}^T(t), \boldsymbol{x}^T(t_k), \boldsymbol{x}^T(t-h), \dot{\boldsymbol{x}}^T(t), \boldsymbol{e}^T(t_k)]^T$ and $\boldsymbol{a}_2 = [\boldsymbol{a}_1^T, \;\; \dot{\boldsymbol{x}}^T(s)]^T$. Considering (9a) and all expressions given in (1) to (8), we obtain that

$$\dot{V}(t) + 2\zeta_1 V(t) \le \boldsymbol{\nu}^T\bar{W}\boldsymbol{\nu} - \int_{t-h}^{t_k}\boldsymbol{a}_2^T\bar{U}_1\boldsymbol{a}_2 ds - \int_{t_k}^{t}\boldsymbol{a}_2^T\bar{U}_2\boldsymbol{a}_2 ds$$
$$+ \boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k), \qquad t \in [t_k, t_{k+1}), \tag{9}$$

where $\bar{W} = \begin{bmatrix} \bar{W}_{11} & \bar{W}_{12} & -\bar{F}_1 & \bar{W}_{14} & \bar{W}_{15} \\ * & \bar{W}_{22} & -\bar{F}_2 & \bar{W}_{24} & \bar{W}_{25} \\ * & * & \bar{W}_{33} & 0 & 0 \\ * & * & * & \bar{W}_{44} & \bar{W}_{45} \\ * & * & * & * & \bar{W}_{55} \end{bmatrix}$, $\bar{U}_1 = \begin{bmatrix} \bar{Y} & \bar{F} \\ * & e^{-2\zeta_1 h}\bar{P}_3 \end{bmatrix}$, $\bar{U}_2 = \begin{bmatrix} \bar{Y} & \bar{G} \\ * & e^{-2\zeta_1 h}\bar{P}_3 \end{bmatrix}$, $\bar{W}_{11} = \bar{P}_2 + \bar{G}_1 + \bar{G}_1^T + E^{-1}A + (E^{-1}A)^T + h\bar{Y}_{11} + 2\zeta_1\bar{P}_1$, $\bar{W}_{12} = \bar{F}_1 - \bar{G}_1 + \bar{G}_2^T + (E^{-1}A)^T + E^{-1}BK + h\bar{Y}_{12}$, $\bar{W}_{14} = \bar{P}_1 - E^{-1} + (E^{-1}A)^T$, $\bar{W}_{22} = \bar{F}_2 + \bar{F}_2^T - \bar{G}_2 - \bar{G}_2^T + h\bar{Y}_{22} + E^{-1}BK + (E^{-1}BK)^T$ $\bar{W}_{24} = (E^{-1}BK)^T - E^{-1}$, $\bar{W}_{33} = -e^{-2\zeta_1 h}\bar{P}_2$, $\bar{W}_{44} = h\bar{P}_3 - E^{-1} - (E^{-1})^T$, $\bar{W}_{15} = \bar{W}_{25} = \bar{W}_{45} = E^{-1}BK$, $\bar{W}_{55} = -\bar{Q}$. Now, we show that if $\bar{W} < 0$, $\bar{U}_1 \ge 0$, and $\bar{U}_2 \ge 0$ system (9a) is stable. From (9), if $\bar{W} < 0$, $\bar{U}_1 \ge 0$, and $\bar{U}_2 \ge 0$ it follows that

$$V(t) < e^{-2\zeta_1 t}V(0) + (\boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k))/(2\zeta_1), \tag{10}$$

and system (9) is stable. Note that $A - LC$ is Hurwitz stable and $b(t_k)$ in (10) asymptotically approaches zero. To guarantee $\bar{W} < 0$, $\bar{U}_1 \ge 0$, and $\bar{U}_2 \ge 0$, we first construct matrix $W = \Pi_1\bar{W}\Pi_1^T$ where $\Pi_1 = \mathrm{diag}(E, E, E, E, E)$. In a similar fashion, block matrices $\bar{U}_1$ and $\bar{U}_2$ are pre- and post multiplied by Π_2 and Π_2^T, where $\Pi_2 = \mathrm{diag}(E, E, E)$. Then, we employ the following alternative variables $P_i = E\bar{P}_iE^T$, $(1 \le i \le 3)$, $F_i = E\bar{F}_iE^T$, $(i = 1, 2)$, $G_i = E\bar{G}_iE^T$, $(i = 1, 2)$, $Y_{ij} = E\bar{Y}_{ij}E^T$, $(i = 1, 2)$, $(j = 1, 2)$, $Q = E\bar{Q}E^T$, and $M = KE^T$.

Step II. FO-TSA intervals: If $t \in Z_m$, the system diverges. There exists a divergence rate $\zeta_2 > 0$ such that

$$\dot{V}(t) < 2\zeta_2 V(t), \quad t \in Z_m. \tag{11}$$

where V is given in (1). We expand (11) as

$$\dot{V} - 2\zeta_2 V = \sum_{i=1}^{3}\dot{V}_i - 2\zeta_2\sum_{i=1}^{3}V_i \le \sum_{i=1}^{3}\dot{V}_i - 2\zeta_2 V_1 < 0. \tag{12}$$

The following expressions hold for the derivatives of LKF (1) for $t \in Z_m$

$$\dot{V}_1 = 2\dot{\boldsymbol{x}}^T(t)\bar{P}_1\boldsymbol{x}(t),\ \dot{V}_2 \le \boldsymbol{x}^T(t)\bar{P}_2\boldsymbol{x}(t),\ \dot{V}_3 \le h\dot{\boldsymbol{x}}^T(t)\bar{P}_3\dot{\boldsymbol{x}}(t). \tag{13}$$

The following null equality holds based on (9b) for $t \in Z_m$

$$2\Big(\boldsymbol{x}^T(t) + \dot{\boldsymbol{x}}^T(t)\Big)E^{-1}\Big(-\dot{\boldsymbol{x}}(t) + A\boldsymbol{x}(t)\Big) = 0. \tag{14}$$

Considering (13) and (14), we revise (12) as follows

$$(\dot{V}_1 + \dot{V}_2 + \dot{V}_3) - 2\zeta_2 V_1 = \boldsymbol{a}_3^T\bar{S}\boldsymbol{a}_3 < 0, \tag{15}$$

where $\boldsymbol{a}_3 = [\boldsymbol{x}^T(t), \dot{\boldsymbol{x}}^T(t)]^T$ and $\bar{S} = \begin{bmatrix} \bar{s} & (E^{-1}A)^T - E^{-1} + \bar{P}_1 \\ * & h\bar{P}_3 - E^{-1} - (E^{-1})^T \end{bmatrix}$, with $\bar{s} = E^{-1}A + (E^{-1}A)^T - 2\zeta_2\bar{P}_1 + \bar{P}_2$. From (12) and (15), if $\bar{S} < 0$ then $\dot{V}(t) - 2\zeta_2 V(t) < 0$ is guaranteed for $t \in Z_m$. We pre- and post multiply $\bar{S}$ by Π_3 and Π_3^T, where $\Pi_3 = I_2 \otimes E$, and LMI $S < 0$ given in Theorem 1 is obtained. From $M = KE^T$, control gain K is computed reversely.

Step III. Amount of resilience to FO-TSA: Now, we merge (10) and (11) to obtain the *overall* stability condition and the maximum tolerable amount of FO-TSA. Expressions (10) and (11) are expanded as follows

$$V(t) \le e^{-2\zeta_1(t-\nu_{m-1})}V(\nu_{m-1}) + c(t_k), \quad t \in R_m, \tag{16}$$
$$V(t) \le e^{2\zeta_2(t-\xi_m)}V(\xi_m), \quad t \in Z_m. \tag{17}$$

where $c(t_k) = \frac{\boldsymbol{b}^T(t_k)\bar{Q}\boldsymbol{b}(t_k)}{2\zeta_1}$. From [1, expression (55)] and [2, Lemma 3 and 4], the following condition is obtained by combining (16) and (17)

$$V(t) < \rho^2 e^{-2\zeta t}V(0) + \eta, \quad t \ge 0, \tag{18}$$

where $\rho = e^{(\zeta_1+\zeta_2)(\alpha_0+h\beta_0)}$, $\zeta = \zeta_1 - (\zeta_1+\zeta_2)\left(\frac{1}{\alpha_1}+\frac{h}{\beta_1}\right)$, $\eta = \max_{k\in\mathbb{N}_0}\{c(t_k)\}\sum_{\substack{m\in\mathbb{N}_0\\ \xi_m\leq t}} e^{-2\zeta_1|R(\nu_{m-1},t)|}e^{2\zeta_2|Z(\xi_m,t)|}$.

Based on (18), if $\zeta > 0$ then the system remains stable under FO-TSAs. This implies that the FO-TSAs should satisfy $(\frac{1}{\alpha_1}+\frac{h}{\beta_1}) < \Omega$ with $\Omega = \zeta_1/(\zeta_1+\zeta_2)$. We call Ω the FO-TSA resilience level, as it gives an upper-bound for tolerable amount of FO-TSA. With $\zeta > 0$ and $A-LC$ being Hurwitz stable, parameter η asymptotically approaches zero. This completes the proof. ■

References

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[2] C. De Persis and P. Tesi, "Input-to-state stabilizing control under denial-of-service." *IEEE Trans. Automat. Contr.*, vol. 60, no. 11, pp. 2930–2944, 2015.