Robust Slepian Functions on the Sphere

Rodney A. Kennedy, *Fellow, IEEE*, and Collaborator, *Senior Member, IEEE*

***Abstract*—Slepian functions on the sphere maximally concentrate energy inside a region for a given bandlimit.**

***Index Terms*—key, word, keyword list**

I. Introduction

A. Background

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

B. Contributions

- We use a weighting function, $h(\widehat{\boldsymbol{x}})$ to generalise the standard Slepian formulation, section II-B.
- The Slepian probem is known to yield dual-orthogonality; when it is generalised through the weighting function then this property has a generalization and further there is a three-fold orthogonality, section II-D. Further, the derivation for orthogonality is succinct and general using the notion of isomorphism.
- Regularization for the case of measurement error or noise yields robustness to noise enhancement that would occur for eigenfunctions with small eigenvalues, section III-B.

C. Paper Organization

Aliquam lectus. Vivamus leo. Quisque ornare tellus ullamcorper nulla. Mauris porttitor pharetra tortor. Sed fringilla justo sed mauris. Mauris tellus. Sed non leo. Nullam elementum, magna in cursus sodales, augue est scelerisque sapien, venenatis congue nulla arcu et pede. Ut suscipit enim vel sapien. Donec congue. Maecenas urna mi, suscipit in, placerat ut, vestibulum ut, massa. Fusce ultrices nulla et nisl.

II. Problem Formulation

A. Notation

The natural complex Hilbert space on the sphere is denoted $L^2(\mathbb{S}^2)$ with inner product $\langle f, g\rangle \triangleq \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\,\overline{g(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}})$. The spherical harmonic transform (SHT) is given by

$$(f)_\ell^m = \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\,\overline{Y_\ell^m(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}})$$

for degree $\ell \in \{0, 1, \ldots\}$ and order m where $|m| \le \ell$.

The subspace of band-limited functions on the sphere of maximum degree L is denoted $\mathcal{H}_L \in L^2(\mathbb{S}^2)$ and is N-dimensional where

$$N \triangleq (L+1)^2$$

If a signal $f(\widehat{\boldsymbol{x}})$ is band-limited to L then $\langle f, Y_\ell^m\rangle = 0$ for $\ell > L$ and it has the spectral (spherical harmonic) representation given by the vector

$$\mathbf{f} = \Big[(f)_0^0, (f)_1^{-1}, (f)_1^0, (f)_1^1, \ldots, (f)_L^L\Big] \in \mathbb{C}^N. \qquad (1)$$

This vector can be indexed with $n = 0, 1, 2, \ldots, N-1$, where $n = \ell(\ell+1)+m$. Generally when we say f is band-limited then the maximum degree L is understand.

The spatial and spectral representations are related through isomorphism [1]

$$\langle f, g\rangle = \langle \mathbf{f}, \mathbf{g}\rangle_{\mathbb{C}^N}, \quad \forall f, g \qquad (2)$$

where the spectral-domain inner product is $\langle \mathbf{f}, \mathbf{g}\rangle_{\mathbb{C}^N} = \mathbf{g}^H\mathbf{f}$. This isomorphism greatly simplifies our demonstrations of different types of orthogonality.

B. Weighted spatial-domain concentration problem

Let $h(\widehat{\boldsymbol{x}})$ be a real, non-negative weighting function bounded by unity on the unit sphere $\mathbb{S}^2$:

$$0 \le h(\widehat{\boldsymbol{x}}) \le 1, \quad \forall \widehat{\boldsymbol{x}} \in \mathbb{S}^2. \qquad (3)$$

Then we seek the band-limited signal $f(\widehat{\boldsymbol{x}}) \in \mathcal{H}_L$ that maximizes the following weighted spatial-domain concentration

$$\lambda_0 = \max_{f\in\mathcal{H}_L}\left\{\frac{\int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\,|f(\widehat{\boldsymbol{x}})|^2\,ds(\widehat{\boldsymbol{x}})}{\int_{\mathbb{S}^2}|f(\widehat{\boldsymbol{x}})|^2\,ds(\widehat{\boldsymbol{x}})}\right\}. \qquad (4)$$

1) Concentration interpretation: By specifying the signal to be band-limited then we are limiting the spread of energy in the spectral domain. By judicious choose of the weighting function, $h(\widehat{\boldsymbol{x}})$ in (4), we attempt to limit the spatial-domain distribution of energy to some localized portion of the sphere. The uncertainty principle says we cannot simultaneously concentrate energy in the spatial and spectral domains. A concentration problem is when we attempt to concentrate as much as possible in both domains.

C. Spectral-domain reformulation of concentration problem

The weighted concentration problem (4) can be written in the spectral domain as the Rayleigh quotient

$$\lambda_0 = \max_{\mathbf{f}\in\mathbb{C}^H} \frac{\mathbf{f}^H\mathbf{H}\mathbf{f}}{\mathbf{f}^H\mathbf{f}}, \qquad (5)$$

Rodney A. Kennedy and Collaborator are with the Research School of Engineering, College of Engineering and Computer Science, The Australian National University (ANU), Canberra, ACT 2601, Australia (email: {rodney.kennedy, collaborator}@anu.edu.au).

This work was supported under the Australian Research Council's Discovery Projects funding scheme (Project No. DP1094350).

where the spectral-domain Hermitian matrix $\mathbf{H}$ has elements

$$H_{\ell,p}^{m,q} \triangleq \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\, Y_p^q(\widehat{\boldsymbol{x}})\, \overline{Y_\ell^m(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}), \tag{6}$$

and $\mathbf{f}$ is the spectral representation of $f(\widehat{\boldsymbol{x}})$. Note that the rows and columns of matrix $\mathbf{H}$ are indexed consistent with $\mathbf{f}$ in (1).

The Rayleigh quotient (5) is solved by finding the eigenvector, $\mathbf{v}_0$, corresponding to the largest eigenvalue, λ_0, of $\mathbf{H}$. All eigenvalues of $\mathbf{H}$ are real and non-negative, $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, and the corresponding eigenvectors, $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$, can be chosen as orthonormal, that is,

$$\mathbf{H}\mathbf{v}_n = \lambda_n \mathbf{v}_n, \quad \text{with} \quad \langle \mathbf{v}_n, \mathbf{v}_m \rangle_{\mathbb{C}^N} = \delta_{m,n}.$$

If the components of the dominant spectral-domain eigenvector, $\mathbf{v}_0$, are $(v_0)_\ell^m$ then the spatial-domain eigenfunction is

$$\begin{aligned} v_0(\widehat{\boldsymbol{x}}) &= \sum_{\ell,m} (v_0)_\ell^m \, Y_\ell^m(\widehat{\boldsymbol{x}}) \\ &= \arg \max_{f \in \mathcal{H}_L} \left\{ \frac{\int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\, |f(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})}{\int_{\mathbb{S}^2} |f(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})} \right\}. \end{aligned} \tag{7}$$

and this is the optimal spatial function that attains (4).

D. *Three-fold spatial-domain orthogonality of eigenfunctions*

Firstly, the N eigenvectors $\mathbf{v}_n$, $n = 0, 1, 2, \ldots, N-1$, of $\mathbf{H}$ are orthonormal in $\mathbb{C}^N$. Then by isomorphism, (2), we have *orthonormality* of the N eigenfunctions $v_n(\widehat{\boldsymbol{x}})$ in $\mathcal{H}_L$:

$$\langle v_n, v_m \rangle = \langle \mathbf{v}_n, \mathbf{v}_m \rangle_{\mathbb{C}^N} = \delta_{m,n}.$$

Secondly, since we have eigenvectors then

$$\begin{aligned} \langle \mathbf{H}\mathbf{v}_n, \mathbf{v}_m \rangle_{\mathbb{C}^N} = \mathbf{v}_m^H \mathbf{H} \mathbf{v}_n &= \mathbf{v}_m^H \lambda_n \mathbf{v}_n \\ &= \lambda_n \langle \mathbf{v}_n, \mathbf{v}_m \rangle_{\mathbb{C}^N} = \lambda_n \, \delta_{n,m}. \end{aligned}$$

So, by isomorphism, spatially this is the same as

$$\int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\, v_n(\widehat{\boldsymbol{x}})\, \overline{v_m(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}) = \lambda_n \, \delta_{n,m}.$$

This is a spatial-domain *orthogonality* of the $v_n(\widehat{\boldsymbol{x}})$ with respect to a weighted spatial-domain inner product

$$\langle f, g \rangle_h \triangleq \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\, f(\widehat{\boldsymbol{x}})\, \overline{g(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}).$$

Under this weighted inner product $\|v_n\|_h^2 = \lambda_n < 1$. However, it is clear that, whenever $\lambda_n > 0$, $\{v_n(\widehat{\boldsymbol{x}})/\sqrt{\lambda_n}\}$ are *orthonormal* in the weighted inner product space.

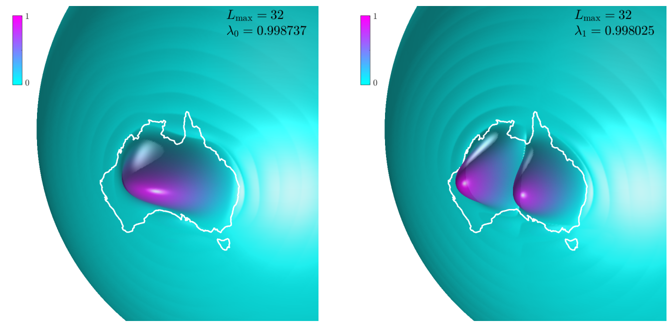
Finally, there is a third sense in which the $v_n(\widehat{\boldsymbol{x}})$ are orthogonal. Implicitly define a third inner product through

$$\langle f, g \rangle = \langle f, g \rangle_h + \langle f, g \rangle_{1-h}.$$

Then, given (3), we have $0 \leq 1 - h(\widehat{\boldsymbol{x}}) \leq 1$ and $v_n(\widehat{\boldsymbol{x}})$ are *orthogonal* with the complementary weighted inner product

$$\langle f, g \rangle_{1-h} \triangleq \int_{\mathbb{S}^2} (1 - h(\widehat{\boldsymbol{x}}))\, f(\widehat{\boldsymbol{x}})\, \overline{g(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}),$$

and can be normalized as in $\{v_n(\widehat{\boldsymbol{x}})/\sqrt{1-\lambda_n}\}$ ($\lambda_n < 1$). Further, in this case, the Hermitian matrix is $\mathbf{H}^c \triangleq \mathbf{I} - \mathbf{H}$.



(a) $v_0(\widehat{\boldsymbol{x}})$ where $\lambda_0 = 0.998737$ (b) $v_1(\widehat{\boldsymbol{x}})$ where $\lambda_1 = 0.998025$

Fig. 1: Magnitudes of the two dominant eigenfunctions for Australia including Tasmania region for band-limit $L = 32$.

In summary, the eigenfunctions satisfy the three-fold *spatial-domain orthogonality* (with spectral counterparts):

$$\begin{aligned} \langle v_n, v_m \rangle &= \delta_{m,n} & &(= \langle \mathbf{v}_n, \mathbf{v}_m \rangle_{\mathbb{C}^N}) \\ \langle v_n, v_m \rangle_h &= \lambda_n \, \delta_{m,n} & &(= \langle \mathbf{H}\mathbf{v}_n, \mathbf{v}_m \rangle_{\mathbb{C}^N}) \\ \langle v_n, v_m \rangle_{1-h} &= (1 - \lambda_n)\, \delta_{m,n} & &(= \langle \mathbf{H}^c\mathbf{v}_n, \mathbf{v}_m \rangle_{\mathbb{C}^N}) \end{aligned}$$

which implies the energy concentrations are $\|v_n\|^2 = 1$, $\|v_n\|_h^2 = \lambda_n$, and $\|v_n\|_{1-h}^2 = 1 - \lambda_n$.

E. *Slepian spatial-domain concentration*

Defining a region $R \in \mathbb{S}^2$, then selecting the real, non-negative weighting function, as $h(\widehat{\boldsymbol{x}}) = \chi_R(\widehat{\boldsymbol{x}})$, where

$$\chi_R(\widehat{\boldsymbol{x}}) \triangleq \begin{cases} 1 & \widehat{\boldsymbol{x}} \in R \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

in (4), and $\langle f, g \rangle_R = \int_R f(\widehat{\boldsymbol{x}})\, \overline{g(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}})$, is the inner product, we recover the Slepian concentration problem on the sphere

$$\lambda_0 = \max_{f \in \mathcal{H}_L} \frac{\int_R |f(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})}{\int_{\mathbb{S}^2} |f(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})}, \tag{9}$$

whose spectral-domain Hermitian matrix $\mathbf{H}$ has elements

$$H_{\ell,p}^{m,q} \triangleq \int_R Y_p^q(\widehat{\boldsymbol{x}})\, \overline{Y_\ell^m(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}). \tag{10}$$

We have the three-fold spatial-domain orthogonality of Slepian eigenfunctions: on the whole sphere, within region R and within region $\mathbb{S}^2 \setminus R$.

For illustration, on the Earth, normalized with unit radius, let the region $R \in \mathbb{S}^2$ be the Australian continent including Tasmania, and let band-limit $L = 32$. Therefore $N = 1089$ and the resulting spectral-domain Hermitian matrix is $\mathbf{H}$ is 1089×1089. The two dominant eigenfunctions $v_0(\widehat{\boldsymbol{x}})$ and $v_1(\widehat{\boldsymbol{x}})$ are shown in Fig. 1.

III. Towards Robust Modelling in a Region

A. *Expansion modelling without noise*

Using the orthonormal sequence $\{v_n(\widehat{\boldsymbol{x}})\}$ in $\mathcal{H}_L$, any band-limited function $f(\widehat{\boldsymbol{x}})$ has expansion, valid for $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$,

$$f(\widehat{\boldsymbol{x}}) = \sum_{n=0}^{N-1} (f)_n v_n(\widehat{\boldsymbol{x}}) = \sum_{n=0}^{N-1} \underbrace{\sqrt{\lambda_n}\,(f)_n}_{\triangleq\, (f)_{h;n}} \frac{v_n(\widehat{\boldsymbol{x}})}{\sqrt{\lambda_n}}, \tag{11}$$

where

$$(f)_n \triangleq \langle f, v_n \rangle = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{y}})\, \overline{v_n(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}),$$

and

$$(f)_{h;n} \triangleq \Big\langle f, \frac{v_n}{\sqrt{\lambda_n}} \Big\rangle_h = \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{y}})\, f(\widehat{\boldsymbol{y}}) \frac{\overline{v_n(\widehat{\boldsymbol{y}})}}{\sqrt{\lambda_n}}\, ds(\widehat{\boldsymbol{y}}).$$

Therefore, if band-limited $f(\widehat{\boldsymbol{x}})$ is determined from local information implicit in weighting $h(\widehat{\boldsymbol{y}})$ we can determine the coefficients of interest using

$$(f)_n = \frac{1}{\lambda_n} \langle f, v_n \rangle_h = \frac{1}{\lambda_n} \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{y}})\, f(\widehat{\boldsymbol{y}})\, \overline{v_n(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}). \quad (12)$$

For example with $h(\widehat{\boldsymbol{x}}) = \chi_R(\widehat{\boldsymbol{x}})$ then the local information is just the information within region R.

B. Expansion modelling with uncertainty in measurement

The energy associated with the nth eigenfunction with respect to the weighted localized inner product is $|(g)_{h;n}|^2$. However, this implies the energy on the sphere is $|(g)_{h;n}|^2/\lambda_n$. Therefore, we may see a significant growth in the energy on the sphere or significant noise enhancement whenever λ_n is small. We can follow the approach taken in [2] to ameliorate such noise enhancement that can occur in (12).

Suppose f is band-limited, $f \in \mathcal{H}_L$, but can only be observed in some localized portion of the sphere through a weighting function h and is also subject to noise

$$g(\widehat{\boldsymbol{x}}) = h(\widehat{\boldsymbol{x}})\, f(\widehat{\boldsymbol{x}}) + z(\widehat{\boldsymbol{x}}),$$

The regularization objective is

$$\min_{\tilde{f} \in \mathcal{H}_L} \|\tilde{f}\|^2 \quad \text{subject to} \quad \|\tilde{f} - g\|_h^2 \leq \epsilon^2$$

Write

$$\tilde{f}(\widehat{\boldsymbol{x}}) = \sum_{n=0}^{N-1} a_n v_n(\widehat{\boldsymbol{x}}) \quad \text{and} \quad g(\widehat{\boldsymbol{x}}) = \sum_{n=0}^{N-1} b_n v_n(\widehat{\boldsymbol{x}})$$

Then

$$\|\tilde{f}\|^2 = \sum_{n=0}^{N-1} |a_n|^2$$

and

$$\|\tilde{f} - g\|_h^2 = \sum_{n=0}^{N-1} \lambda_n |a_n - b_n|^2$$

IV. Ideas

- **Random matrix theory:** The square root of the Hermitian is the weighted local projection. Then we can use random matrix theory to characterize the eigenvalues.
- **Operator Reformulation:** Integral equation and kernel version of theory.
- **Regularized Hermitian:** Can the Hermitian be regularized directly to deliver more sensible eigenvectors? (The original idea)
- **Spatial-limited problem:** Do the converse Slepian problem.
- **Uncertain L:** Need spectral weighting to deal with uncertain L; simultaneous with spatial weighting? Make robust to imprecise band-limited value of L.
- **Franks framework reformulation:** Cast everything in Franks framework.
- **Errors-in-variables:** Uncertainty in spatial measurement (domain) and values (range).
- **RKHS:** Real positive definite weighting is 75% there. For weighted inner product is it amenable to kernel trick?
- **Beampattern Deconvolution:** Weighting is a single shot convolution. Does this show how to deconvolve imperfect measurement beamshapes?
- **Sampled space:** Pseudo-DH samples and working with information preserving spatial samples.

V. Conclusions

Donec molestie, magna ut luctus ultrices, tellus arcu nonummy velit, sit amet pulvinar elit justo et mauris. In pede. Maecenas euismod elit eu erat. Aliquam augue wisi, facilisis congue, suscipit in, adipiscing et, ante. In justo. Cras lobortis neque ac ipsum. Nunc fermentum massa at ante. Donec orci tortor, egestas sit amet, ultrices eget, venenatis eget, mi. Maecenas vehicula leo semper est. Mauris vel metus. Aliquam erat volutpat. In rhoncus sapien ac tellus. Pellentesque ligula.

References

[1] R. A. Kennedy and P. Sadeghi, *Hilbert Space Methods in Signal Processing*. Cambridge, UK: Cambridge University Press, Mar. 2013.

[2] Y. Alem, Z. Khalid, and R. A. Kennedy, "Band-limited extrapolation on the sphere for signal reconstruction in the presence of noise," in *Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing, ICASSP'2014*, Florence, Italy, May 2014, pp. 4141–4145.

APPENDIX A
OBSCURE THING

Fusce suscipit cursus sem. Vivamus risus mi, egestas ac, imperdiet varius, faucibus quis, leo. Aenean tincidunt. Donec suscipit. Cras id justo quis nibh scelerisque dignissim. Aliquam sagittis elementum dolor. Aenean consectetuer justo in pede. Curabitur ullamcorper ligula nec orci. Aliquam purus turpis, aliquam id, ornare vitae, porttitor non, wisi. Maecenas luctus porta lorem. Donec vitae ligula eu ante pretium varius. Proin tortor metus, convallis et, hendrerit non, scelerisque in, urna. Cras quis libero eu ligula bibendum tempor. Vivamus tellus quam, malesuada eu, tempus sed, tempor sed, velit. Donec lacinia auctor libero.

APPENDIX B
UNCLEAR THING

Curabitur ac lorem. Vivamus non justo in dui mattis posuere. Etiam accumsan ligula id pede. Maecenas tincidunt diam nec velit. Praesent convallis sapien ac est. Aliquam ullamcorper euismod nulla. Integer mollis enim vel tortor. Nulla sodales placerat nunc. Sed tempus rutrum wisi. Duis accumsan gravida purus. Nunc nunc. Etiam facilisis dui eu sem. Vestibulum semper. Praesent eu eros. Vestibulum tellus nisl, dapibus id, vestibulum sit amet, placerat ac, mauris. Maecenas et elit ut erat placerat dictum. Nam feugiat, turpis et sodales volutpat, wisi quam rhoncus neque, vitae aliquam ipsum sapien vel enim. Maecenas suscipit cursus mi.