we can achieve $\|\mathcal{B}_T f\|^2 = \alpha$, $\|f\|^2 = 1$ and more importantly, the bound

$$\|\mathcal{B}_W f\| = \sqrt{\beta} = \cos(\cos^{-1}\sqrt{\lambda_0} - \cos^{-1}\sqrt{\alpha}).$$

To see the last assertion, we follow similar steps as in (8.22)–(8.25) and some algebraic simplifications to write

$$\begin{aligned}\beta = \|\mathcal{B}_W f\|^2 &= \big\|\mathcal{B}_W(p\Psi_0 + q\mathcal{B}_T\Psi_o)\big\|^2 \\ &= p^2 + \lambda_0^2 q^2 + 2pq\lambda_0 \\ &= \big(\sqrt{(1-\alpha)(1-\lambda_0)} + \sqrt{\alpha\lambda_0}\big)^2.\end{aligned}$$

Noting that $\sqrt{\alpha} < 1$, $\sqrt{\lambda_0} < 1$ and defining $\sqrt{\alpha} = \cos\phi$ and $\sqrt{\lambda_0} = \cos\theta$, we obtain

$$\beta = (\sin\phi\sin\theta + \cos\phi\cos\theta)^2 = \big(\cos(\theta - \phi)\big)^2,$$

or simply

$$\sqrt{\beta} = \cos\big(\cos^{-1}\sqrt{\lambda_0} - \cos^{-1}\sqrt{\alpha}\,\big).$$

Case 4

The fourth case can be considered as a limiting case of the third case, except for the extra condition that $\beta = 0$ has to be excluded.

8.3 Introduction to concentration problem on 2-sphere

The objective is to find bandlimited functions belonging to $\mathcal{H}_L(\mathbb{S}^2)$, having maximum degree L, on the 2-sphere that are *optimally concentrated* within a spatial region of interest R. Optimal concentration is a bit vague here and is yet to be defined. In fact, there may be different measures of concentration. In (Simons et al., 2006), the most natural Slepian-type concentration problem is formulated and investigated in detail where the measure of concentration is the fraction of signal energy in the region of interest. We will later discuss generalized concentration measures. The dual problem is to find spacelimited functions belonging to $\mathcal{H}_R(\mathbb{S}^2)$ on 2-sphere whose spectral representation is *optimally concentrated* within the spectral degree $\ell \in \{0, 1, \ldots, L\}$.

Having reviewed the time-frequency concentration problem, we will highlight the similarities and differences of the spatio-spectral concentration problem. While there are many expected similarities, there are some important differences. For example, the arbitrary shape of the spatial concentration region R on the sphere “enriches” the problem on 2-sphere (Simons et al., 2006).

Motivation

Finding optimally concentrated signals in spatial or spectral domains has many applications in practice, which are similar in spirit and can be as diverse as applications of the Slepian concentration problem in time-frequency domain. For example, we may need to smooth a signal on the sphere to reduce the effects of noise or high spectral components by performing local *convolutional* averaging or filtering, which will be the topic of Chapter 9, p. 293. Naturally, we would like to design a filter kernel such that it is best concentrated around the point of interest and has minimal *leakage* outside the desired neighborhood. Because otherwise, other parts of the signal might significantly contribute to the filtered output. In other examples, optimally concentrated signals are applied for spectral estimation in (Dahlen and Simons, 2008) and localized spectral analysis of two windowed (masked) functions in (Wieczorek and Simons, 2005). In Section 8.13, p. 279, we will discuss an application of optimally concentrated windows that allow accurate joint spatio-spectral analysis of signals on the sphere.

8.4 Optimal spatial concentration of bandlimited signals

We first need a measure of concentration. To start with, we take the energy of a signal as our measure, which is physically relevant in many applications. The idea is very simple. The signal we are looking for is denoted by f and is bandlimited to a maximum degree L. That is $f \in \mathcal{H}_L(\mathbb{S}^2)$. We wish to maximize the ratio of signal energy in the desired region R to the total energy over the 2-sphere. Mathematically, we wish to maximize

$$\boxed{\alpha = \frac{\displaystyle\int_R \left|f(\widehat{\boldsymbol{u}})\right|^2 ds(\widehat{\boldsymbol{u}})}{\displaystyle\int_{\mathbb{S}^2} \left|f(\widehat{\boldsymbol{u}})\right|^2 ds(\widehat{\boldsymbol{u}})},} \tag{8.26}$$

where α is a measure of spatial concentration. Using the spherical harmonic representation of signal f in (7.72), p. 211, we can expand the numerator in (8.26) as

$$\int_R \left|f(\widehat{\boldsymbol{u}})\right|^2 ds(\widehat{\boldsymbol{u}}) = \sum_{\ell,m}^{L} \overline{(f)_\ell^m} \sum_{p,q}^{L} (f)_p^q D_{\ell,p}^{m,q},$$

where we have defined

$$\boxed{D_{\ell,p}^{m,q} \triangleq \int_R Y_p^q(\widehat{\boldsymbol{u}}) \overline{Y_\ell^m(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}).} \tag{8.27}$$

Note that $Y_\ell^m(\widehat{\boldsymbol{u}})$ and $Y_p^q(\widehat{\boldsymbol{u}})$ are not orthogonal over a strict subregion of $\mathbb{S}^2$ and $D_{\ell,p}^{m,q}$ should be computed numerically. We will discuss some examples later. Using the Parseval relation in (7.73), p. 213, for bandlimited signals, we can rewrite the concentration ratio in (8.26) in matrix form as

$$\boxed{\alpha = \frac{\mathbf{f}^H \mathbf{D} \mathbf{f}}{\mathbf{f}^H \mathbf{f}},} \tag{8.28}$$

where $\mathbf{f}$ is the vector containing the spectral representation of f and $(\cdot)^H$ denotes Hermitian transpose. In the above equation, matrix $\mathbf{D}$ gathers all elements such as $D_{\ell,p}^{m,q}$ and has a dimension of $(L+1)^2 \times (L+1)^2$. Since there are four indices in

$D_{\ell,p}^{m,q}$, one needs to be careful about writing them in a two-dimensional matrix format. It is important to be consistent with the way elements in the vector spectral representation $\mathbf{f}$ are arranged or interpreted. Other than this, it is somewhat arbitrary.

Here we use the single indexing convention (7.39), p. 195, which is consistent with our presentation of $\mathbf{f}$. Define

$$n \triangleq \ell(\ell+1)+m \quad \text{and} \quad r \triangleq p(p+1)+q$$

to write $D_{\ell,p}^{m,q}$ as $D_{n,r}$, which occupies $\mathbf{D}$ at row n and column r. Therefore, letting $L' \triangleq L^2+2L$, matrix $\mathbf{D}$ is

$$\mathbf{D} = \begin{pmatrix} D_{0,0} & D_{0,1} & D_{0,2} & \cdots & D_{0,L'} \\ D_{1,0} & D_{1,1} & D_{1,2} & \cdots & D_{1,L'} \\ D_{2,0} & D_{2,1} & D_{2,2} & \cdots & D_{2,L'} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{L',0} & D_{L',1} & D_{L',2} & \cdots & D_{L',L'} \end{pmatrix}$$

$$= \begin{pmatrix} D_{0,0}^{0,0} & D_{0,-1}^{0,1} & D_{0,0}^{0,1} & \cdots & D_{0,L}^{0,L} \\ D_{-1,0}^{1,0} & D_{-1,-1}^{1,1} & D_{-1,0}^{1,1} & \cdots & D_{-1,L}^{1,L} \\ D_{0,0}^{1,0} & D_{0,-1}^{1,1} & D_{0,0}^{1,1} & \cdots & D_{0,L}^{1,L} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{L,0}^{L,0} & D_{L,-1}^{L,1} & D_{L,0}^{L,1} & \cdots & D_{L,L}^{L,L} \end{pmatrix}.$$

At stationary points of the ratio (8.28), the vector $\mathbf{f}$ is a solution to the following eigenfunction problem:

$$\boxed{\mathbf{D}\mathbf{f} = \lambda\mathbf{f}.} \tag{8.29}$$

Or more explicitly for each element $(f)_\ell^m$ in $\mathbf{f}$ we have

$$\sum_{p,q}^{L} D_{\ell,p}^{m,q}(f)_p^q = \lambda(f)_\ell^m. \tag{8.30}$$

Determining matrix $\mathbf{D}$ with elements given by (8.27) is enough for (numerically) solving the eigenfunction problem (8.29).

Remark 8.1. Compared with the time concentration problem of bandlimited signals, this analogy is somewhat simpler conceptually and mathematically because the problem is finite-dimensional. □

8.4.1 Orthogonality relations

Since $\mathbf{D}$ with elements defined in (8.27) is Hermitian symmetric and positive definite, all eigenvalues are real and the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. We can sort the eigenvalues in descending order

$$1 > \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{L^2+2L} > 0. \tag{8.31}$$

We consider unit-norm orthonormal eigenvectors $\mathbf{f}_n$ and $\mathbf{f}_k$ corresponding to two distinct eigenvalues λ_n and λ_k such that

$$\langle \mathbf{f}_n, \mathbf{f}_k \rangle = \mathbf{f}_n' \overline{\mathbf{f}_k} = \delta_{n,k}, \tag{8.32}$$

where

$$\mathbf{f}_n = \Big(\big((f)_0^0\big)_n, \big((f)_1^{-1}\big)_n, \big((f)_1^0\big)_n, \dots, \big((f)_L^L\big)_n \Big)'.$$

The eigenvector $\mathbf{f}_n$ in spectral domain, corresponding to eigenvalue λ_n, gives rise to a signal on the 2-sphere $f_n(\widehat{\boldsymbol{u}})$ such that

$$f_n(\widehat{\boldsymbol{u}}) = \sum_{\ell,m}^{L} \big((f)_\ell^m\big)_n Y_\ell^m(\widehat{\boldsymbol{u}}), \quad \widehat{\boldsymbol{u}} \in \mathbb{S}^2. \tag{8.33}$$

Therefore, the corresponding eigenfunctions $f_n(\widehat{\boldsymbol{u}})$ and $f_k(\widehat{\boldsymbol{u}})$ are orthonormal on the whole 2-sphere:

$$\int_{\mathbb{S}^2} f_n(\widehat{\boldsymbol{u}}) \overline{f_k(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}) = \delta_{n,k}. \tag{8.34}$$

The concentration ratio of $f_n(\widehat{\boldsymbol{u}})$ in region R is $\lambda_n < 1$. All eigenvalues are strictly smaller than one because a bandlimited signal cannot be simultaneously spacelimited and inevitably part of its energy leaks out to the region $\bar{R} \triangleq \mathbb{S}^2 \setminus R$. $f_0(\widehat{\boldsymbol{u}})$, corresponding to the largest eigenvalue, has the largest spatial concentration in R and is our "favorite" eigenfunction, while $f_{L^2+2L}(\widehat{\boldsymbol{u}})$ is least appealing as far as concentration in R goes.

Remark 8.2. If we reverse the ordering of eigenvalues in (8.31) from smallest to biggest, we obtain the corresponding eigenfunctions that are optimally *excluded* from region R. The eigenfunction $f_{L^2+2L}(\widehat{\boldsymbol{u}})$ is the most excluded function in R and hence most concentrated in complementary region $\bar{R}$. □

In addition to satisfying the orthonormality property on the whole 2-sphere, $f_n(\widehat{\boldsymbol{u}})$ and $f_k(\widehat{\boldsymbol{u}})$ have the interesting property that they are also orthogonal (but not orthonormal) in region R. To see this, we first note that

$$\mathbf{f}_n' \overline{\mathbf{D}\mathbf{f}_k} = \lambda_k \mathbf{f}_n' \overline{\mathbf{f}_k} = \lambda_k \delta_{n,k},$$

where we have used (8.29) and $\mathbf{f}_n' \overline{\mathbf{f}_k} = \delta_{n,k}$. Now using (8.33) and the definition of $D_{\ell,p}^{m,q}$ in (8.27) we can verify that

$$\begin{aligned} \int_R f_n(\widehat{\boldsymbol{u}}) \overline{f_k(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}) &= \sum_{\ell,m}^{L} \big((f)_\ell^m\big)_n \sum_{p,q}^{L} \overline{D_{\ell,p}^{m,q} \big((f)_p^q\big)_k} \\ &= \mathbf{f}_n' \overline{\mathbf{D}\mathbf{f}_k} = \lambda_k \mathbf{f}_n' \overline{\mathbf{f}_k} = \lambda_k \delta_{n,k}. \end{aligned} \tag{8.35}$$

Below, we summarize the orthonormality and orthogonality relations

$$\boxed{\mathbf{f}_n' \overline{\mathbf{f}_k} = \delta_{n,k} \iff \int_{\mathbb{S}^2} f_n(\widehat{\boldsymbol{u}}) \overline{f_k(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}) = \delta_{n,k}}$$

and

$$\boxed{\mathbf{f}_n'\overline{\mathbf{D}\mathbf{f}_k} = \lambda_k\delta_{n,k} \iff \int_R f_n(\widehat{\boldsymbol{u}})\overline{f_k(\widehat{\boldsymbol{u}})}\,ds(\widehat{\boldsymbol{u}}) = \lambda_k\delta_{n,k}.}$$

8.4.2 Eigenfunction kernel representation

The eigenfunction problem in (8.29) has an equivalent kernel representation in $L^2(\mathbb{S}^2)$. To see this, we multiply the left-hand side of (8.30) by spherical harmonic $Y_\ell^m(\widehat{\boldsymbol{u}})$ and sum over all possible degrees and orders to obtain

$$\begin{aligned}\sum_{p,q}^{L}(f)_p^q\sum_{\ell,m}^{L}D_{\ell,p}^{m,q}Y_\ell^m(\widehat{\boldsymbol{u}}) &= \sum_{p,q}^{L}(f)_p^q\sum_{\ell,m}^{L}\int_R Y_p^q(\widehat{\boldsymbol{v}})\overline{Y_\ell^m(\widehat{\boldsymbol{v}})}\,ds(\widehat{\boldsymbol{v}})Y_\ell^m(\widehat{\boldsymbol{u}})\\ &= \int_R\sum_{p,q}^{L}(f)_p^qY_p^q(\widehat{\boldsymbol{v}})\sum_{\ell,m}^{L}Y_\ell^m(\widehat{\boldsymbol{u}})\overline{Y_\ell^m(\widehat{\boldsymbol{v}})}\,ds(\widehat{\boldsymbol{v}})\\ &= \int_R\sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell}Y_\ell^m(\widehat{\boldsymbol{u}})\overline{Y_\ell^m(\widehat{\boldsymbol{v}})}f(\widehat{\boldsymbol{v}})\,ds(\widehat{\boldsymbol{v}}).\end{aligned}$$

This equation is in the form of an integral operator $\mathcal{D}$ on $L^2(\mathbb{S}^2)$:

$$\big(\mathcal{D}f\big)(\widehat{\boldsymbol{u}}) = \int_{\mathbb{S}^2}D(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}})f(\widehat{\boldsymbol{v}})\,ds(\widehat{\boldsymbol{v}}),$$

with kernel

$$\boxed{D(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}}) = \mathbf{1}_R(\widehat{\boldsymbol{v}})\sum_{\ell,m}^{L}Y_\ell^m(\widehat{\boldsymbol{u}})\overline{Y_\ell^m(\widehat{\boldsymbol{v}})} = \mathbf{1}_R(\widehat{\boldsymbol{v}})\sum_{\ell=0}^{L}\frac{(2\ell+1)}{4\pi}P_\ell(\widehat{\boldsymbol{u}}\cdot\widehat{\boldsymbol{v}}),} \tag{8.36}$$

where $\mathbf{1}_R(\widehat{\boldsymbol{v}})$ is the indicator function for region R, and we have used the addition theorem (7.30), p. 193.

Applying the same procedure to the right-hand side of (8.30) yields

$$\lambda\sum_{\ell,m}^{L}(f)_\ell^mY_\ell^m(\widehat{\boldsymbol{u}}) = \lambda f(\widehat{\boldsymbol{u}}), \quad \widehat{\boldsymbol{u}}\in\mathbb{S}^2. \tag{8.37}$$

So the end result is

$$\boxed{\int_{\mathbb{S}^2}D(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}})f(\widehat{\boldsymbol{v}})\,ds(\widehat{\boldsymbol{v}}) = \lambda f(\widehat{\boldsymbol{u}}), \quad \widehat{\boldsymbol{u}}\in\mathbb{S}^2,} \tag{8.38}$$

where the kernel $D(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}})$ is given by (8.36).