Single-Taper Window Design on the Sphere

Rodney A. Kennedy, *Fellow, IEEE*, and Collaborator, *Senior Member, IEEE*

***Abstract*—These are notes from scattered projects, held in one place a bit like a notebook.**

***Index Terms*—key, word, keyword list**

I. Single Taper Design

Classes of band-limited tapers for a spherical cap region are developed in [1].

A. Constraints

Here we seek a single taper $w(\widehat{\boldsymbol{x}})$ to use as a spatial-multiplicative window for a region $\mathscr{R} \subset \mathbb{S}^2$ on the sphere. The sought properties are:

1) band-limited to degree L, that is,
$$(w)_\ell^m = \langle w, Y_\ell^m \rangle = 0, \quad \ell > L,$$
2) high spatial energy concentration, no less than some threshold $\lambda \in (0,1)$, that is,
$$\frac{\displaystyle\int_{\mathscr{R}} |w(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})}{\displaystyle\int_{\mathbb{S}^2} |w(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})} = \frac{\|w\|_{\mathscr{R}}^2}{\|w\|^2} \geq \lambda$$
and such functions are referred to as λ-concentrated
3) close to unity in the region of interest, that is, in some sense close to the characteristic function of the region $\mathscr{R}$
$$\chi_{\mathscr{R}}(\widehat{\boldsymbol{x}}) = \begin{cases} 1 & \widehat{\boldsymbol{x}} \in \mathscr{R} \\ 0 & \text{otherwise} \end{cases}$$

B. Caveats

- The threshold $\lambda \in (0,1)$ should not exceed the maximum theoretical spatial concentration.
- The characteristic function is not band-limited.

C. Use of taper

When performing analysis a signal $f(\widehat{\boldsymbol{x}})$ localized to a region $\mathscr{R}$ we propose to use the modified signal
$$w(\widehat{\boldsymbol{x}})\, f(\widehat{\boldsymbol{x}})$$
which tends to concentrates the signal simultaneously in the spatial and spectral domains.

Rodney A. Kennedy and Collaborator are with the Research School of Engineering, College of Engineering and Computer Science, The Australian National University (ANU), Canberra, ACT 2601, Australia (email: {rodney.kennedy, collaborator}@anu.edu.au).

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D. Band-limited Slepian functions

The band-limited Slepian functions for region $\mathscr{R}$ are denoted $\varphi_n(\widehat{\boldsymbol{x}})$ with associated real positive eigenvalues λ_n, $n = 1, 2, \ldots$, that is,
$$\int_{\mathscr{R}} |\varphi_n(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}}) = \lambda_n.$$
The Slepian functions are ordered in n such that
$$\lambda_n \geq \lambda_{n+1}, \quad \forall n.$$
Further the Slepian functions are orthonormal on $\mathbb{S}^2$ and orthogonal on $\mathscr{R}$.

E. Formulation

If $s(\widehat{\boldsymbol{x}})$ is an L-band-limited function then by the completeness of the band-limited Slepian functions we have
$$s(\widehat{\boldsymbol{x}}) = \sum_{n=1}^{\infty} (s)_n \, \varphi_n(\widehat{\boldsymbol{x}}), \tag{1}$$
in the sense of convergence in the mean, where
$$(s)_n = \langle s, \varphi_n \rangle = \int_{\mathbb{S}^2} s(\widehat{\boldsymbol{x}}) \, \overline{\varphi_n(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}) \tag{2}$$
Define $N(\lambda)$ such that
$$\lambda_n \geq \lambda \iff n \leq N(\lambda). \tag{3}$$
Then the truncation
$$s^\lambda(\widehat{\boldsymbol{x}}) = \sum_{n=1}^{N(\lambda)} (s^\lambda)_n \, \varphi_n(\widehat{\boldsymbol{x}}) \tag{4}$$
is λ-concentrated because we only use the λ-concentrated L-band-limited Slepian functions:
$$\Big\{ \varphi_n(\widehat{\boldsymbol{x}}) \colon \int_{\mathscr{R}} |\varphi_n(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}}) = \lambda_n \geq \lambda \Big\}_{n=1}^{N(\lambda)}.$$
That is, by the orthogonality of the Slepian functions on $\mathscr{R}$,
$$\begin{aligned} \|s^\lambda\|_{\mathscr{R}}^2 &= \int_{\mathscr{R}} |s^\lambda(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}}) \\ &= \sum_{n=1}^{N(\lambda)} |(s^\lambda)_n|^2 \int_{\mathscr{R}} |\varphi_n(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}}) \\ &= \sum_{n=1}^{N(\lambda)} \lambda_n \, |(s^\lambda)_n|^2 \geq \lambda \sum_{n=1}^{N(\lambda)} |(s^\lambda)_n|^2 = \lambda \, \|s^\lambda\|^2. \end{aligned}$$

F. Window Design

Functions satisfying expansion (4) form a finite $N(\lambda)$-dimensional space of λ-concentrated signals with the λ-concentrated L-band-limite Slepian functions as their basis.

Now the objective is to find a suitable $s(\widehat{\boldsymbol{x}})$ that does not "down-weight and ultimately discard" the signal in the portions of the region $\mathscr{R}$ [1]. A function that equally weights all parts of the region is $\chi_{\mathscr{R}}(\widehat{\boldsymbol{x}})$, the characteristic function of the region $\mathscr{R}$. To obtain the optimal minimum mean square error between the inadmissible $\chi_{\mathscr{R}}(\widehat{\boldsymbol{x}})$ and the finite $N(\lambda)$-dimensional subspace is through an orthogonal projection,

$$w(\widehat{\boldsymbol{x}}) = \sum_{n=1}^{N(\lambda)} (w)_n \, \varphi_n(\widehat{\boldsymbol{x}}) \tag{5}$$

where

$$\begin{aligned} (w)_n = \langle \chi_{\mathscr{R}}, \varphi_n \rangle &= \int_{\mathbb{S}^2} \chi_{\mathscr{R}}(\widehat{\boldsymbol{x}}) \, \overline{\varphi_n(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}) \\ &= \int_{\mathscr{R}} \overline{\varphi_n(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}) \end{aligned} \tag{6}$$

References

[1] M. A. Wieczorek and F. J. Simons, "Localized spectral analysis on the sphere," *Geophys. J. Int.*, vol. 131, no. 1, pp. 655–675, May 2005.