Finally, the weighted completeness relation on the 2-sphere is, as we had given earlier in the book (2.80), p. 95,

$$\boxed{\sin\theta \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)\overline{Y_\ell^m(\vartheta,\varphi)} = \delta(\theta-\vartheta)\delta(\phi-\varphi).} \tag{7.31}$$

Problems

7.4. Prove the identity

$$N_\ell^{-m}\frac{(\ell-m)!}{(\ell+m)!} = N_\ell^m, \tag{7.32}$$

where the normalization factor is given in (7.20).

7.5. Using (7.24) prove the identity

$$N_\ell^{-m}P_\ell^{-m}(\cos\theta) = (-1)^m N_\ell^m P_\ell^m(\cos\theta), \tag{7.33}$$

where the normalization factor is given in (7.20).

7.6. By starting with $Y_\ell^{-m}(\theta,\phi)$ and using (7.24) or otherwise prove the conjugation property (7.26) or equivalently:

$$Y_\ell^{-m}(\theta,\phi) = (-1)^m\overline{Y_\ell^m(\theta,\phi)}. \tag{7.34}$$

7.7. From the product $(x^2-1)^\ell = (x+1)^\ell(x-1)^\ell$, show that (7.23) can be written in the alternative form

$$P_\ell^m(x) \triangleq \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\sum_{s=0}^{\ell+m}\frac{(\ell+m)!}{s!(\ell+m-s)!}\frac{d^s}{dx^s}(x+1)^\ell\frac{d^{\ell+m-s}}{dx^{\ell+m-s}}(x-1)^\ell,$$

which is valid for $m \in \{-\ell,-\ell+1,\dots,\ell\}$.

7.8. Prove the identity (7.24) repeated here

$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x),$$

by comparing the expression for $P_\ell^m(x)$ and $P_\ell^{-m}(x)$ in (7.23) using the expression in Problem 7.7 or otherwise.

7.9. Show that (7.28) is valid for all m, that is, $m \in \{-\ell,-\ell+1,\dots,\ell\}$.

7.10. Show that

$$Y_\ell^{-m}(\theta,\phi) = e^{im(\pi-2\phi)}Y_\ell^m(\theta,\phi), \quad m \in \{-\ell,-\ell+1,\dots,\ell\}.$$

7.3.5 Spherical harmonic coefficients

By completeness of the spherical harmonic functions (Colton and Kress, 1998), any signal $f \in L^2(\mathbb{S}^2)$ can be expanded as

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(f)_\ell^m Y_\ell^m(\theta,\phi), \tag{7.35}$$

where

$$
\begin{aligned}
(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle &= \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{u}}) \overline{Y_\ell^m(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}) \\
&= \int_0^{2\pi}\!\!\int_0^{\pi} f(\theta,\phi)\overline{Y_\ell^m(\theta,\phi)}\, \sin\theta\, d\theta\, d\phi
\end{aligned}
\tag{7.36}
$$

are the spherical harmonic Fourier coefficients or spherical harmonic coefficients for short. The equality is understood in terms of convergence in the mean

$$
\lim_{L\to\infty} \Big\| f - \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \langle f, Y_\ell^m \rangle Y_\ell^m(\theta,\phi) \Big\|^2 = 0, \quad \forall f \in L^2(\mathbb{S}^2). \tag{7.37}
$$

7.3.6 Shorthand notation

In the following, we will use the shorthand notation introduced back in the first part of the book in (2.71), p. 88, repeated here

$$
\sum_{\ell,m} \triangleq \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}
$$

and its truncated form

$$
\sum_{\ell,m}^{L} \triangleq \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \tag{7.38}
$$

for brevity. After a short reflection, one can see there are precisely $(L+1)^2$ terms in (7.38). The other esoteric detail here is that truncating in a different way is not forbidden, but not useful. There is a precedence of degree (ℓ) over order (m). The truncated form is a truncation in the degree ℓ.

7.3.7 Enumeration

As was also described in the first part of the book, we can recast double-indexed spherical harmonics Y_ℓ^m into a single-indexed version such as Y_n for notational brevity and also compatibility with the standard convention of orthonormal sequences. This single indexing will be also useful in representing all non-zero spherical harmonic coefficients of a bandlimited function on the 2-sphere in vector form, and representing operator matrices in the spherical harmonics basis.

The bijection given in (2.73), p. 88, and shown in Figure 2.11, p. 89, does the trick. That is, given degree ℓ and order m we generate the single index n:

$$
n = \ell(\ell+1) + m. \tag{7.39}
$$

Inversely, given a single index n we generate the degree ℓ and order m,

$$
\begin{aligned}
\ell &= \lfloor \sqrt{n} \rfloor, \\
m &= n - \lfloor \sqrt{n} \rfloor \big(\lfloor \sqrt{n} \rfloor + 1 \big),
\end{aligned}
\tag{7.40}
$$

where $\lfloor \cdot \rfloor$ is the floor function.

With re-enumerated spherical harmonics, $\{Y_n\}_{n=0}^{\infty}$, the corresponding spherical harmonic coefficient can be written $(f)_n = \langle f, Y_n \rangle$.

Example 7.10. Consider $n = 12$ in the single index notation. Using (7.40), it yields $\ell = \lfloor\sqrt{12}\rfloor = 3$ and $m = 12 - 3 \times 4 = 0$, that is,

$$Y_{12}(\theta, \phi) = Y_3^0(\theta, \phi) = \frac{1}{4}\sqrt{\frac{7}{\pi}}\big(5\cos^3\theta - 3\cos\theta\big),$$

where we used the table in Example 7.9. □

7.3.8 Spherical harmonic Parseval relation

From (7.35) and the orthonormality of spherical harmonics we can verify Parseval relation on 2-sphere

$$\boxed{\int_{\mathbb{S}^2} \big|f(\widehat{\boldsymbol{u}})\big|^2 ds(\widehat{\boldsymbol{u}}) = \sum_{\ell,m} \big|(f)_\ell^m\big|^2.} \tag{7.41}$$

Remark 7.3. Parseval relation follows in a direct way from the properties of complete orthonormal sequences, Definition 2.22, p. 70,

$$\langle f, f\rangle = \sum_{n=0}^{\infty}\langle f, Y_n\rangle\langle Y_n, f\rangle = \sum_{n=0}^{\infty}\big|(f)_n\big|^2 \equiv \sum_{\ell,m}\big|(f)_\ell^m\big|^2.$$

That is, we identify the re-enumerated spherical harmonic, Y_n, with the generic φ_n used in Part I and Part II in the book. □

Problem

7.11 (Generalized Parseval relation). Using the observation in Remark 7.3, show that the generalized Parseval relation for the spherical harmonics is given by

$$\boxed{\int_{\mathbb{S}^2} f(\widehat{\boldsymbol{u}})\overline{g(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}) = \sum_{\ell,m} (f)_\ell^m \overline{(g)_\ell^m}.} \tag{7.42}$$

7.3.9 Dirac delta function on 2-sphere

For two points

$$\widehat{\boldsymbol{u}} = \big(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\big)',$$
$$\widehat{\boldsymbol{v}} = \big(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta\big)',$$

defined on the 2-sphere, the 2-sphere Dirac delta function is denoted by $\delta(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}})$. The spatial representation is

$$\boxed{\delta(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}) = (\sin\theta)^{-1}\delta(\theta - \vartheta)\delta(\phi - \varphi).}$$