

Single-Taper Window Design on the Sphere

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Abstract—These are notes from scattered projects, held in one place a bit like a notebook.

Index Terms—key, word, keyword list

I. SINGLE TAPER DESIGN

Classes of band-limited tapers for a spherical cap region are developed in [1].

A. Constraints

Here we seek a single taper $w(\hat{\mathbf{x}})$ to use as a spatial-multiplicative window for a region $\mathcal{R} \subset \mathbb{S}^2$ on the sphere. The sought properties are:

- 1) band-limited to degree L , that is,

$$(w)_\ell^m = \langle w, Y_\ell^m \rangle = 0, \quad \ell > L,$$

- 2) high spatial energy concentration, no less than some threshold $\lambda \in (0, 1)$, that is,

$$\frac{\int_{\mathcal{R}} |w(\hat{\mathbf{x}})|^2 ds(\hat{\mathbf{x}})}{\int_{\mathbb{S}^2} |w(\hat{\mathbf{x}})|^2 ds(\hat{\mathbf{x}})} = \frac{\|w\|_{\mathcal{R}}^2}{\|w\|^2} \geq \lambda$$

and such functions are referred to as λ -concentrated

- 3) close to unity in the region of interest, that is, in some sense close to the characteristic function of the region \mathcal{R}

$$\chi_{\mathcal{R}}(\hat{\mathbf{x}}) = \begin{cases} 1 & \hat{\mathbf{x}} \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$$

B. Caveats

- The threshold $\lambda \in (0, 1)$ should not exceed the maximum theoretical spatial concentration.
- The characteristic function is not band-limited.

C. Use of taper

When performing analysis a signal $f(\hat{\mathbf{x}})$ localized to a region \mathcal{R} we propose to use the modified signal

$$w(\hat{\mathbf{x}}) f(\hat{\mathbf{x}})$$

which tends to concentrates the signal simultaneously in the spatial and spectral domains.

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D. Band-limited Slepian functions

The band-limited Slepian functions for region \mathcal{R} are denoted $\varphi_n(\hat{\mathbf{x}})$ with associated real positive eigenvalues λ_n , $n = 1, 2, \dots$, that is,

$$\int_{\mathcal{R}} |\varphi_n(\hat{\mathbf{x}})|^2 ds(\hat{\mathbf{x}}) = \lambda_n.$$

The Slepian functions are ordered in n such that

$$\lambda_n \geq \lambda_{n+1}, \quad \forall n.$$

Further the Slepian functions are orthonormal on \mathbb{S}^2 and orthogonal on \mathcal{R} .

E. Formulation

If $s(\hat{\mathbf{x}})$ is an L -band-limited function then by the completeness of the band-limited Slepian functions we have

$$s(\hat{\mathbf{x}}) = \sum_{n=1}^{\infty} (s)_n \varphi_n(\hat{\mathbf{x}}), \quad (1)$$

in the sense of convergence in the mean, where

$$(s)_n = \langle s, \varphi_n \rangle = \int_{\mathbb{S}^2} s(\hat{\mathbf{x}}) \overline{\varphi_n(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}) \quad (2)$$

Define $N(\lambda)$ such that

$$\lambda_n \geq \lambda \iff n \leq N(\lambda). \quad (3)$$

Then the truncation

$$s^\lambda(\hat{\mathbf{x}}) = \sum_{n=1}^{N(\lambda)} (s^\lambda)_n \varphi_n(\hat{\mathbf{x}}) \quad (4)$$

is λ -concentrated because we only use the λ -concentrated L -band-limited Slepian functions:

$$\left\{ \varphi_n(\hat{\mathbf{x}}) : \int_{\mathcal{R}} |\varphi_n(\hat{\mathbf{x}})|^2 ds(\hat{\mathbf{x}}) = \lambda_n \geq \lambda \right\}_{n=1}^{N(\lambda)}.$$

That is, by the orthogonality of the Slepian functions on \mathcal{R} ,

$$\begin{aligned} \|s^\lambda\|_{\mathcal{R}}^2 &= \int_{\mathcal{R}} |s^\lambda(\hat{\mathbf{x}})|^2 ds(\hat{\mathbf{x}}) \\ &= \sum_{n=1}^{N(\lambda)} |(s^\lambda)_n|^2 \int_{\mathcal{R}} |\varphi_n(\hat{\mathbf{x}})|^2 ds(\hat{\mathbf{x}}) \\ &= \sum_{n=1}^{N(\lambda)} \lambda_n |(s^\lambda)_n|^2 \geq \lambda \sum_{n=1}^{N(\lambda)} |(s^\lambda)_n|^2 = \lambda \|s^\lambda\|^2. \end{aligned}$$

F. Window Design

Functions satisfying expansion (4) form a finite $N(\lambda)$ -dimensional space of λ -concentrated signals with the λ -concentrated L -band-limited Slepian functions as their basis.

Now the objective is to find a suitable $s(\hat{\mathbf{x}})$ that does not “down-weight and ultimately discard” the signal in the portions of the region \mathcal{R} [1]. A function that equally weights all parts of the region is $\chi_{\mathcal{R}}(\hat{\mathbf{x}})$, the characteristic function of the region \mathcal{R} . To obtain the optimal minimum mean square error between the inadmissible $\chi_{\mathcal{R}}(\hat{\mathbf{x}})$ and the finite $N(\lambda)$ -dimensional subspace is through an orthogonal projection,

$$w(\hat{\mathbf{x}}) = \sum_{n=1}^{N(\lambda)} (w)_n \varphi_n(\hat{\mathbf{x}}) \quad (5)$$

where

$$\begin{aligned} (w)_n &= \langle \chi_{\mathcal{R}}, \varphi_n \rangle = \int_{\mathbb{S}^2} \chi_{\mathcal{R}}(\hat{\mathbf{x}}) \overline{\varphi_n(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}) \\ &= \int_{\mathcal{R}} \overline{\varphi_n(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}) \end{aligned} \quad (6)$$

REFERENCES

- [1] M. A. Wieczorek and F. J. Simons, “Localized spectral analysis on the sphere,” *Geophys. J. Int.*, vol. 131, no. 1, pp. 655–675, May 2005.