Robust Slepian Functions on the Sphere

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***Abstract*—Slepian functions on the sphere maximally concentrate energy inside a region for a given bandlimit.**

***Index Terms*—key, word, keyword list**

I. Introduction

A. Background

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The most profound work is found in [1].

B. Contributions

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II. Problem Formulation

A. Notation

The natural Hilbert space on the sphere is denoted $L^2(\mathbb{S}^2)$ with inner product $\langle f, g\rangle \triangleq \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\,\overline{g(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}})$. The spherical harmonic transform (SHT) is given by

$$(f)_\ell^m = \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\,\overline{Y_\ell^m(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}})$$

for degree $\ell \in \{0,1,\ldots\}$ and order m where $|m| \le \ell$.

The subspace of band-limited functions on the sphere of maximum degree L is denoted $\mathcal{H}_L \in L^2(\mathbb{S}^2)$ and is $N \triangleq (L+1)^2$-dimensional. If a signal $f(\widehat{\boldsymbol{x}})$ is band-limited to L then $\langle f, Y_\ell^m\rangle = 0$ for $\ell > L$ and it has the spectral (spherical harmonic) representation given by the vector

$$\mathbf{f} = \Big((f)_0^0, (f)_1^{-1}, (f)_1^0, (f)_1^1, \ldots, (f)_L^L\Big) \in \mathbb{C}^N. \tag{1}$$

This vector can be indexed with $n = 0,1,2,\ldots,N-1$, where $n = \ell(\ell+1)+m$. Generally when we say f is band-limited then the maximum degree L is understand.

The spatial and spectral representations are related through isomorphism [1]

$$\langle f, g\rangle = \langle \mathbf{f}, \mathbf{g}\rangle_{\mathbb{C}^N}, \tag{2}$$

where the spectral inner product is $\langle \mathbf{f}, \mathbf{g}\rangle = \mathbf{g}^H\mathbf{f}$. This isomorphism greatly simplifies our demonstrations of different types of orthogonality.

B. Weighted spatial concentration problem

Let $h(\widehat{\boldsymbol{x}})$ be a real, non-negative weighting function bounded by unity on the unit sphere $\mathbb{S}^2$. Then we seek the band-limited signal $f(\widehat{\boldsymbol{x}}) \in \mathcal{H}_L$ that maximizes the following weighted spatial concentration

$$\lambda_0 = \max_{f\in\mathcal{H}_L}\left\{\frac{\int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\,|f(\widehat{\boldsymbol{x}})|^2\,ds(\widehat{\boldsymbol{x}})}{\int_{\mathbb{S}^2}|f(\widehat{\boldsymbol{x}})|^2\,ds(\widehat{\boldsymbol{x}})}\right\}. \tag{3}$$

The denominator in (3) can be written $\|f\|^2$ and is usually taken to be unity.

The weighted concentration problem (3) can be written in the spectral domain as the Rayleigh quotient

$$\lambda_0 = \max_{\mathbf{f}\in\mathbb{C}^H}\frac{\mathbf{f}^H\mathbf{H}\mathbf{f}}{\mathbf{f}^H\mathbf{f}}, \tag{4}$$

where the spectral Hermitian matrix $\mathbf{H}$ has elements

$$H_{\ell,p}^{m,q} \triangleq \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\,Y_p^q(\widehat{\boldsymbol{x}})\,\overline{Y_\ell^m(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}), \tag{5}$$

and $\mathbf{f}$ is the spectral representation of $f(\widehat{\boldsymbol{x}})$. Note that the rows and columns of matrix $\mathbf{H}$ are indexed consistent with $\mathbf{f}$ in (1).

Problem (4) is solved by finding the eigenvector corresponding to the largest eigenvalue of $\mathbf{H}$. All eigenvalues of $\mathbf{H}$ are real and non-negative, $\lambda_0 \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge 0$, and the corresponding eigenvectors, $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$, can be chosen as orthonormal. If the components of the dominant spectral eigenvector, $\mathbf{v}_0$, are $(v_0)_\ell^m$ then the spatial eigen-function is

$$\begin{aligned} v_0(\widehat{\boldsymbol{x}}) &= \sum_{\ell,m}(v_0)_\ell^m\,Y_\ell^m(\widehat{\boldsymbol{x}}) \\ &= \arg\max_{f\in\mathcal{H}_L}\left\{\frac{\int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\,|f(\widehat{\boldsymbol{x}})|^2\,ds(\widehat{\boldsymbol{x}})}{\int_{\mathbb{S}^2}|f(\widehat{\boldsymbol{x}})|^2\,ds(\widehat{\boldsymbol{x}})}\right\}. \end{aligned} \tag{6}$$

In summary, to find the optimal spatial function that attains (3), being $v_0(\widehat{\boldsymbol{x}})$ in (6), we compute the spectral Hermitian matrix, using (5); then determine the dominant spectral eigenvector $\mathbf{v}_0$; and synthesize $v_0(\widehat{\boldsymbol{x}})$ using the inverse SHT in (6).

C. Three-fold spatial orthogonality of eigen-functions

Firstly, with the N eigenvectors $\mathbf{v}_n$, $n = 0,1,2,\ldots,N-1$, of $\mathbf{H}$ are orthonormal in $\mathbb{C}^N$. Then by isomorphism, (2), we have *orthonormality* of the N eigen-functions $v_n(\widehat{\boldsymbol{x}})$ in $\mathcal{H}_L$. That is, from (2),

$$\langle v_n(\widehat{\boldsymbol{x}}), v_m(\widehat{\boldsymbol{x}})\rangle = \langle \mathbf{v}_n, \mathbf{v}_m\rangle_{\mathbb{C}^N} = \delta_{m,n}.$$

Secondly, spectrally (because we have eigenvectors) then

$$\begin{aligned} \langle \mathbf{H}\mathbf{v}_n, \mathbf{v}_m\rangle_{\mathbb{C}^N} = \mathbf{v}_m^H\mathbf{H}\mathbf{v}_n &= \mathbf{v}_m^H\lambda_n\mathbf{v}_n \\ &= \lambda_n\langle \mathbf{v}_n, \mathbf{v}_m\rangle_{\mathbb{C}^N} = \lambda_n\,\delta_{n,m}. \end{aligned}$$

So, by isomorphism, spatially this is the same as

$$\int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\,v_n(\widehat{\boldsymbol{x}})\,\overline{v_m(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}) = \lambda_n\,\delta_{n,m}.$$

This is spatial *orthogonality* of the $v_n(\widehat{\boldsymbol{x}})$ with respect to a weighted spatial inner product on the sphere

$$\langle f, g\rangle_h \triangleq \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}})\,f(\widehat{\boldsymbol{x}})\,\overline{g(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}).$$

Under this weighted inner product $\|v_n(\widehat{\boldsymbol{x}})\|_h^2 = \lambda_n$, which is less than unity in general. However, it is clear that, whenever $\lambda_n > 0$,

$$\Big\{\frac{1}{\sqrt{\lambda_n}}v_n(\widehat{\boldsymbol{x}})\Big\}$$

are *orthonormal* in the weighted inner product space.

Finally, there is a third sense in which the $v_n(\widehat{\boldsymbol{x}})$ are orthogonal. Implicitly define a third inner product through

$$\langle f, g\rangle = \langle f, g\rangle_h + \langle f, g\rangle_{1-h}.$$

Then given $0 \le h(\widehat{\boldsymbol{x}}) \le 1$ we have $0 \le 1 - h(\widehat{\boldsymbol{x}}) \le 1$ and $v_n(\widehat{\boldsymbol{x}})$ are also *orthogonal* in the complementary weighted inner product space

$$\langle f, g\rangle_{1-h} \triangleq \int_{\mathbb{S}^2}\Big(1 - h(\widehat{\boldsymbol{x}})\Big)\,f(\widehat{\boldsymbol{x}})\,\overline{g(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}),$$

and can be normalized in an expected way ($\lambda_n < 1$)

$$\Big\{\frac{1}{\sqrt{1-\lambda_n}}v_n(\widehat{\boldsymbol{x}})\Big\}.$$

Further, in this case, the Hermitian matrix is $\mathbf{H}^c \triangleq \mathbf{I} - \mathbf{H}$.

In summary, the eigenfunctions satisfy the three-fold *spatial orthogonality* (with spectral counterparts):

$$\begin{aligned} \langle v_n(\widehat{\boldsymbol{x}}), v_m(\widehat{\boldsymbol{x}})\rangle &= \delta_{m,n} && (= \langle \mathbf{v}_n, \mathbf{v}_m\rangle_{\mathbb{C}^N}) \\ \langle v_n(\widehat{\boldsymbol{x}}), v_m(\widehat{\boldsymbol{x}})\rangle_h &= \lambda_n\,\delta_{m,n} && (= \langle \mathbf{H}\mathbf{v}_n, \mathbf{v}_m\rangle_{\mathbb{C}^N}) \\ \langle v_n(\widehat{\boldsymbol{x}}), v_m(\widehat{\boldsymbol{x}})\rangle_{1-h} &= (1-\lambda_n)\,\delta_{m,n} && (= \langle \mathbf{H}^c\mathbf{v}_n, \mathbf{v}_m\rangle_{\mathbb{C}^N}) \end{aligned}$$

which implies the energy concentrations are $\|v_n(\widehat{\boldsymbol{x}})\|^2 = 1$, $\|v_n(\widehat{\boldsymbol{x}})\|_h^2 = \lambda_n$, and $\|v_n(\widehat{\boldsymbol{x}})\|_{1-h}^2 = 1 - \lambda_n$.

D. Slepian spatial concentration

Defining a region $R \in \mathbb{S}^2$, then selecting the real, non-negative weighting function, $h(\widehat{\boldsymbol{x}})$, as

$$\chi_R(\widehat{\boldsymbol{x}}) \triangleq \begin{cases} 1 & \widehat{\boldsymbol{x}} \in R \\ 0 & \text{otherwise} \end{cases}, \tag{7}$$

then we recover the standard Slepian concentration problem on the sphere [1]

$$\lambda_0 = \max_{f \in \mathcal{H}_L} \frac{\int_R |f(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})}{\int_{\mathbb{S}^2} |f(\widehat{\boldsymbol{x}})|^2 \, ds(\widehat{\boldsymbol{x}})}, \tag{8}$$

whose spectral Hermitian matrix $\mathbf{H}$ has elements

$$H_{\ell,p}^{m,q} \triangleq \int_R Y_p^q(\widehat{\boldsymbol{x}}) \, \overline{Y_\ell^m(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}). \tag{9}$$

Then $\mathbf{H}\mathbf{v}_0 = \lambda_0 \mathbf{v}_0$, and $\mathbf{v}_0$ is the spectral representation of the most concentrated signal, as synthesized in (6).

We have the three-fold spatial orthogonality of Slepian eigen-functions. They are orthonormal on the whole sphere, orthogonal within region R and orthogonal within region $\mathbb{S}^2 \setminus R$. Within the regions (subregions of the sphere) the effective inner product weighting is unity, that is,

$$\langle f, g \rangle_h = \int_R f(\widehat{\boldsymbol{x}}) \, \overline{g(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}),$$

and

$$\langle f, g \rangle_{1-h} = \int_{\mathbb{S}^2 \setminus R} f(\widehat{\boldsymbol{x}}) \, \overline{g(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}).$$

For illustration, on the Earth, normalized with unit radius, let the region $R \in \mathbb{S}^2$ be the Australian continent including Tasmania, and let band-limit $L = 20$. The resulting spectral Hermitian matrix $\mathbf{H}$ is 441×441. The two dominant eigen-functions $v_0(\widehat{\boldsymbol{x}})$ and $v_1(\widehat{\boldsymbol{x}})$ are shown in Fig. 1.

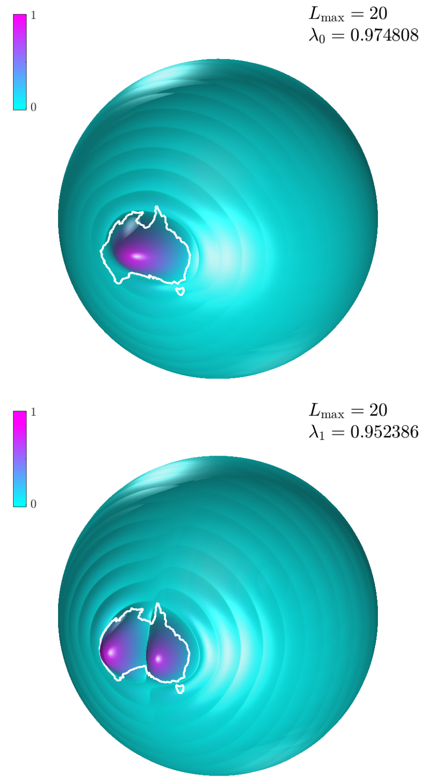

Fig. 1. The two dominant eigen-functions $v_0(\widehat{\boldsymbol{x}})$ and $v_1(\widehat{\boldsymbol{x}})$ for Australia including Tasmania for band-limit $L = 20$.

III. Dummy

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Equation example

$$\iota_{n;q}^m = 4\sqrt{1-\pi^2} \int_0^{k_u} \frac{1}{g_n(k, r_1)} \times \int_{\mathbb{S}^2} H(r_1, \widehat{\boldsymbol{x}}; k) \overline{Y_n^m(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}) \, \overline{\varphi_q(k)} \, dk. \tag{10}$$

TABLE I
MODEL PARAMETERS IN THE GENERAL FORM

Model	**Expansion**	**Coefficients** $\iota_{n;q}^{m}$	**Weighting** $g_n(k,r)$
1	Source Distribution	$\gamma_{n;q}^{m}$	$ik\,j_n(ks)h_n^{(1)}(kr)$
2	Radiating Solution	$\zeta_{n;q}^{m}$	$h_n^{(1)}(kr)$
3	Nominal Radius	$\rho_{n;q}^{m[r_0]}$	$h_n^{(1)}(kr)/h_n^{(1)}(kr_0)$

or using the `breqn` package

$$\iota_{n;q}^{m} = 4\pi \int_0^{k_u} \frac{1}{g_n(k,r_1)} \times \int_{\mathbb{S}^2} H(r_1,\widehat{\boldsymbol{x}};k)\overline{Y_n^m(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}})\,\overline{\varphi_q(k)}\,dk \tag{11}$$

IV. CONCLUSIONS

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REFERENCES

[1] R. A. Kennedy and P. Sadeghi, *Hilbert Space Methods in Signal Processing*. Cambridge, UK: Cambridge University Press, Mar. 2013.

APPENDIX A
OBSCURE THING

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APPENDIX B
UNCLEAR THING

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