NORM PRINCIPLE FOR SEMISIMPLE GROUPS OVER FUNCTION FIELDS OF CURVES

Abstract. In this paper, we show that for a semisimple, simply connected linear algebraic group G, if kernel of the Rost invariant for G is trivial, then the norm principle holds for G. As an application, we prove the norm principle for groups of type D_n (with D_4 non-trialitarian) over p-adic function fields, and for quasi-split groups of types E_6, E_7, and ${}^{3,6}D_4$ over perfect fields of characteristic $\neq 2, 3$. We further provide a local-global criterion ensuring the norm principle for semisimple groups over function fields of smooth, projective, geometrically integral curves over number fields.

Let L/K be a finite separable field extension, G be a semisimple linear algebraic group defined over K, and Z be the center of G. Consider the following diagram

$$\begin{array}{ccc} H^1(L,Z) & \xrightarrow{\alpha_L} & H^1(L,G) \\ {\scriptstyle cor_{L/K}}\downarrow & & \\ H^1(K,Z) & \xrightarrow{\alpha_K} & H^1(K,G), \end{array}$$

where $cor_{L/K}$ denotes the corestriction map in Galois cohomology, and the map α is induced by the inclusion $Z \hookrightarrow G$.

We say that the norm principle holds for G over K, if for every finite separable field extension L/K, we have $cor_{L/K}(\ker\ \alpha_L) \subseteq \ker \alpha_K$. It is an open question whether the norm principle holds in general for every semisimple linear algebraic group G over any field K.

Let $R_K : H^1(K,G) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ be the Rost invariant of G over K (see [14, Section 31] for details). In this paper, we prove:

Theorem: (2.2) If the kernel of R_K is trivial, then the norm principle holds for G over K.

The norm principle is open for groups of type D_n, E_6, and E_7. As an application of Theorem 2.2, we prove the norm principle for some exceptional groups (Theorem 2.3), as well as groups of type D_n over p-adic function fields (Theorem 2.4).

A basic theme in studying the arithmetic of linear algebraic groups is the passage from a global field F to the function field $K = F(C)$ of a smooth projective geometrically integral curve C over F. The norm principle is known for all semisimple groups over global fields. In Section 3, we prove a local-global principle for the norm principle (see Theorem 3.1). Using this local-global approach, we prove the norm principle for certain groups of classical type D_n over global function fields of characteristic zero.

Theorem: (3.4) Let k be a number field, and C be a smooth, projective, geometrically integral curve over k with function field $K = k(C)$. Let G be a semisimple group of type D_n (with D_4 non-trialitarian) defined over k. Then, the norm principle holds for G over K.

The paper is organized as follows. Section 1 is preliminary in nature where we recall the Norm Principle and gather known results.

Section 2 is devoted to the study of the relation between the triviality of the kernel of the Rost invariant and the norm principle (Theorem 2.2). Theorems 2.3 and 2.4 are derived from the main result (Theorem 2.2) in this section.

In Section 3, we establish a criterion under which the norm principle holds for simply connected groups defined over global function fields, and we prove Theorem 3.8. We finally recall Serre's injectivity question and then state the relation between the results of this paper and Serre's question.

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1. Introduction

Let L/K be a finite separable field extension and $N_{L/K} : L^* \to K^*$ be the norm map. Let q be a regular quadratic form defined over K. Scharlau's norm principle, a classical theorem from the algebraic theory of quadratic forms, states that if an element $\lambda \in L^*$ is a similarity factor of q over L, then the element $N_{L/K}(\lambda) \in K^*$ is a similarity factor of q over K (see [15, Chapter 7, Section 4, Page 205]).

Another norm principle due to Knebusch states that if $\lambda \in L^*$ is a spinor norm of q over L, then $N_{L/K}(\lambda) \in K^*$ is a spinor norm of q over K (see [15, Chapter 7, Section 5, Page 206]).

A norm principle was formulated in [1] for reductive linear algebraic groups, which generalizes the above classical theorems (we will shortly explain how Scharlau's and Knebusch's theorems can be viewed in terms of norm principle for linear algebraic groups). Following the notation of [20], we refer to the norm principle formulated in [1] as the H^0-variant of the norm principle (with a slight modification in the definition, though it is essentially equivalent to the one in [1]).

Before stating the definition of the norm principle, we note that for a commutative linear algebraic group T defined over K, and for any finite separable field extension L/K, the norm map $N_{L/K}$ is defined as

$$\begin{array}{rcl} N_{L/K} : T(L) & \longrightarrow & T(K) \\ t & \longmapsto & \Pi_\gamma \ \gamma(t), \end{array}$$

where γ runs over all cosets of $Gal(K^{sep}/L)$ in $Gal(K^{sep}/K)$.

We recall the H^0-variant of the norm principle below.

Definition. ***(H^0-variant of the Norm Principle, due to Merkurjev*** [1]***)*** *Let L/K be a finite separable field extension. Suppose G is a reductive linear algebraic group defined over K, and $T := G/[G, G]$ is the abelianization of G, which is a torus defined over K. Let $f : G \longrightarrow T$ be the canonical homomorphism defined over K. Consider the following diagram:*

$$\begin{array}{ccc} G(L) & \xrightarrow{f_L} & T(L) \\ & & \downarrow{\scriptstyle N_{L/K}} \\ G(K) & \xrightarrow{f_K} & T(K). \end{array}$$

The H^0-variant of the norm principle holds for G over L/K, if $\mathrm{Im}(N_{L/K} \circ f_L) \subseteq \mathrm{Im}(f_K)$. *Also, the H^0-variant of the norm principle holds for G over K, if the H^0-variant of the norm principle holds for G over every finite separable field extension L/K.*

Note that if G is commutative, then the H^0-variant of the norm principle holds for G, since $G = T$ and f will be the identity map. The validity of the H^0-variant of the norm principle for reductive groups is an open question.

Scharlau's (resp. Knebusch's) norm principle for a quadratic form q can be reformulated as the H^0-variant of the norm principle for the group of proper similitudes of q, i.e. $GO^+(q)$ (resp. the special Clifford group of q, i.e. $\Gamma^+(q)$) (see [3, Section 8.2]).

A cohomological variant of the norm principle for semisimple linear algebraic groups was introduced by Gille in [10]. Following the notation of [20], we refer to this property as the H^1-variant of the norm principle for a semisimple linear algebraic group S. We recall this definition below:

Definition. ***(H^1-variant of the Norm Principle, due to Gille*** [10]***)*** *Let L/K be a finite separable field extension, S a semisimple linear algebraic group over K, and $Z \subseteq S$ its center. Consider the following diagram*

$$\begin{array}{ccc} H^1(L,Z) & \xrightarrow{\alpha_L} & H^1(L,S) \\ {\scriptstyle cor_{L/K}}\downarrow & & \\ H^1(K,Z) & \xrightarrow{\alpha_K} & H^1(K,S). \end{array}$$

In the above diagram, $cor_{L/K}$ denotes the corestriction map in Galois cohomology, and the map α is induced by the inclusion $Z \hookrightarrow S$.

The H^1-variant of the norm principle holds for S over L/K, if $cor_{L/K}(\ker\ \alpha_L) \subseteq \ker \alpha_K$. *Also, the H^1-variant of the norm principle holds for S over K, if the H^1-variant of the norm principle holds for S over L/K, for every finite separable field extension L/K.*

Note that if S is commutative, then the H^1-variant of the norm principle holds for S, because the corestriction map of S would make the above diagram a commutative square and that would imply $cor_{L/K}(\ker\ \alpha_L) \subseteq \ker \alpha_K$.

The validity of the H^1-variant of the norm principle for semisimple groups is open in general.

For any given field K, the H^0-variant of the norm principle for all reductive groups over K, and the H^1-variant of the norm principle for all semisimple groups over K are equivalent (see [20, Theorem 2.1]). So in this paper, by the norm principle for a semisimple group S, we shall mean the H^1-variant of the norm principle for S.

We recall some known facts about the norm principle below:

- The norm principle for the simply connected cover of a semisimple group implies the norm principle for the semisimple group itself (see [20, Lemma 2.3]).
- The norm principle for a semisimple group can be obtained from the norm principle for all simple components of it (see [1, Proposition 5.2]).
- The norm principle holds for all semisimple groups with no components of type $D_n, n \geq 4$, E_6, or E_7, over arbitrary fields (see [1, Theorem 1.1]).
- The norm principle holds for all semisimple groups over number fields (see [11]).
- The norm principle holds for any semisimple, simply connected (and hence any semisimple) group over a p-adic field; because the pointed set $H^1(K, S)$ is trivial for any semisimple, simply connected group S over a p-adic field K (see [13]).
- The norm principle holds for any semisimple, simply connected (and hence any semisimple) group over a global field of positive characteristic; because the pointed set $H^1(K, S)$ is trivial for any semisimple, simply connected group S over a global field of positive characteristic K (see [12]).

Based on the reductions stated above, to settle the norm principle for all semisimple groups, one needs to prove the norm principle for simple, simply connected groups of types D_n, E_6, and E_7.

2. Triviality of the kernel of the Rost invariant and the norm principle

Let K be a field and G be a semisimple, simply connected linear algebraic group defined over K. There is an invariant due to Rost,

$$R : H^1(K, G) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

which is functorial in K (see [14, Section 31] for details). The first step of our strategy in this paper to study the norm principle, is to establish a relation between kernel of the Rost invariant for G and the norm principle for G. Towards this end, we consider the restriction of the Rost invariant to the center of G.

Let Z be the center of G, and $\alpha : H^1(-, Z) \to H^1(-, G)$ the Galois cohomology map induced by the map $Z \to G$. Consider the following maps:

$$\begin{tikzcd} H^1(K, Z) \arrow[r, "\alpha_K"] \arrow[rr, bend right, "\iota_K"'] & H^1(K, G) \arrow[r, "R_K"] & H^3(K, \mathbb{Q}/\mathbb{Z}(2)). \end{tikzcd}$$

Let L be a finite separable field extension of K. Consider the following diagram, where the vertical maps are the corestriction maps in Galois cohomology:

$$
\begin{tikzcd}
H^1(L,Z) \arrow[r, "\alpha_L"] \arrow[d, "cor_{L/K}"] \arrow[rr, bend left, "\iota_L"] & H^1(L,G) \arrow[r, "R_L"] & H^3(L,\mathbb{Q}/\mathbb{Z}(2)) \arrow[d, "cor_{L/K}"] \\
H^1(K,Z) \arrow[r, "\alpha_K"] \arrow[rr, bend right, "\iota_K"'] & H^1(K,G) \arrow[r, "R_K"] & H^3(K,\mathbb{Q}/\mathbb{Z}(2)).
\end{tikzcd} \tag{2.1}
$$

An explicit formula for the map ι_F, for any field F containing K, is given in [9]. The following theorem follows from this explicit description.

Theorem 2.1. *The above diagram (2.1) is commutative.*

Proof. Let Z° be the dual multiplicative group scheme of Z. There is a natural cup product $H^1(K,Z)\otimes H^2(K,Z^\circ)\to H^3(K,\mathbb{Q}/\mathbb{Z}(2))$ (see [9, Theorem 1.2]). Let t_G be the Tits class of G, which is an element in $H^2(K,Z)$ (see [14, Page 426] for the definition of the Tits class). There is a natural map $\hat{\rho}^* : H^2(K,Z)\to H^2(K,Z^\circ)$ (see [9, Section 5.C] for the details of the map $\hat{\rho}^*$). Let $s_G=\hat{\rho}^*(t_G)^{-1}$. In [9, Theorem 1.2], it is proved that the map ι_K is given by the formula

$$\iota_K(x) = x\cup s_G,\ \textit{for every}\ x\in H^1(K,Z).$$

The Rost invariant is preserved under scalar extension. Therefore, we have the formula

$$\iota_L(x) = x\cup s_G,\ \textit{for every}\ x\in H^1(L,Z),$$

where $res_{L/K}$ denotes the restriction map.

Let $u\in H^1(L,Z)$. We have

$$
\begin{aligned}
cor_{L/K}(\iota_L(u)) &= cor_{L/K}(u\cup res_{L/K}(s_G))\\
&= cor_{L/K}(u)\cup s_G \quad (\textit{by projection formula in Galois cohomology})\\
&= \iota_K(cor_{L/K}(u)).
\end{aligned}
$$

Note that s_G is actually defined over K, which allows us to use the projection formula. ☐

Theorem 2.2. *Let K be a field, and G be a semisimple, simply connected linear algebraic group defined over K, with Rost invariant R. Assume that the kernel of R_K is trivial. Then the norm principle holds for G over K.*

Proof. Let L/K be a finite separable field extension. Recall the maps α and ι from Diagram 2.1. Let $u\in\ker\ \alpha_L$. This implies that $u\in\ker\iota_L$, hence $cor_{L/K}(\iota_L(u))$ is trivial in $H^3(K,\mathbb{Q}/\mathbb{Z}(2))$. By Theorem 2.1, we have $cor_{L/K}(u)\in\ker\iota_K$. By assumption, $ker\ R_K$ is trivial. Therefore, $cor_{L/K}(u)\in\ker\alpha_K$. ☐

Recall that the norm principle is open in general for semisimple groups of type D_n, E_6, and E_7.

Theorem 2.3. *Let K be a perfect field of char $\neq 2, 3$, and G be a simple, simply connected quasi-split group of type E_6, E_7, or ${}^{3,6}D_4$ defined over K. Let Z be the center of G. Then the induced map of pointed sets*

$$\alpha_K : H^1(K, Z) \to H^1(K, G)$$

is the zero map. In particular, the norm principle holds for G over K.

Proof. Consider the following maps

$$\begin{tikzcd} H^1(K,Z) \arrow[r, "\alpha_K"] \arrow[rr, bend right, "\iota_K"'] & H^1(K,G) \arrow[r, "R_K"] & H^3(K,\mathbb{Q}/\mathbb{Z}(2)), \end{tikzcd}$$

where R_K is the Rost invariant of G over K.

Let $\overline{G}$ be the adjoint group G/Z. By [14, Proposition 31.6], there is only one element $\nu_G \in H^1(K, \overline{G})$ such that the twisted group $\overline{G_{\nu_G}}$ is quasi-split. Since (by assumption) G itself is quasi-split, the element ν_G has to be the distinguished element of $H^1(K, \overline{G})$. Consider the coboundary map $\delta : H^1(K, \overline{G}) \to H^2(K, Z)$. Then the Tits class $t_G = -\delta(\nu_G)$ is trivial in $H^2(K, Z)$. Hence the element s_G (introduced in the proof of Theorem 2.1) is trivial in $H^2(K, Z^\circ)$, where Z° denotes the dual multiplicative group scheme of Z. Then by [9, Theorem 1.2] the map ι_K is the zero map. The kernel of R_K is trivial by [5, Theorem 6.1 and Theorem 6.12]. Therefore the map α_K is the zero map, which clearly implies that the norm principle holds for G over K. □

In the next theorem, we prove the norm principle for groups of classical type D_n over p-adic function fields.

We recall that the u-invariant of a field K, denoted by $u(K)$, is the largest dimension of an anisotropic quadratic form over K and is ∞ if such an integer does not exist for K.

Theorem 2.4. *Let K be a field of characteristic $\neq 2$ with $cd_2(K) \leq 3$. Suppose further that $u(K) \leq 10$. Let G be a semisimple linear algebraic group of classical type D_n defined over K. Then the norm principle holds for G over K. In particular, the norm principle holds for G defined over the function field of any smooth projective geometrically integral curve over a p-adic field.*

Proof. Recall the exceptional isomorphism $D_1 \cong A_1$. The norm principle is known for all groups of type A_1 by the main theorem in [1], and hence the norm principle holds for G, if G is of type D_1.

Suppose $n \geq 2$. By the reductions mentioned in the introduction, it suffices to prove the norm principle for any simple simply connected linear algebraic group G of classical type D_n defined over K. Any such group G is the spinor group $Spin(A, \sigma)$ for a central simple K-algebra A of degree $2n$ with orthogonal involution σ. By [18, Theorem 1.8], the Rost invariant of $Spin(A, \sigma)$ has trivial kernel over K. Then the norm principle for G follows from Theorem 2.2.

If K is the function field of a smooth projective geometrically integral curve defined over a p-adic field, then K has cohomological dimension 3. Furthermore, $u(K) = 8$ (see [17]). Hence the norm principle holds for G over K. $\square$

3. Applications and open questions

Let K be a field with a collection of overfields $\{K_\nu\}_{\nu\in\Omega}$. Let G be a semisimple linear algebraic group defined over K. The main result in this section (see Theorem 3.1) asserts that if the norm principle holds for G over a class of overfields $\{K_\nu\}_{\nu\in\Omega}$ of K, and a local-global principle is valid, then the norm principle holds for G over K. As an application, we prove the norm principle for certain groups of type D_n over function fields of curves over number fields (see Theorem 3.4). Note that the validity of the norm principle for semisimple groups over function fields of curves over number fields is an open problem in general.

The following theorem relates the Norm principle to a local-global principle.

Theorem 3.1. *Let K be a field, with a collection of overfields $\{K_\nu\}_{\nu\in\Omega}$. Let G be a semisimple linear algebraic group defined over K. Assume that the kernel of the global to local map*

$$\theta_K : H^1(K,G) \longrightarrow \prod_{\nu\in\Omega} H^1(K_\nu, G)$$

is trivial. Furthermore, assume that the norm principle holds for G over K_ν, for each $\nu \in \Omega$. Then the norm principle holds for G over K.

Proof. Let Z be the center of G, and let $\alpha : H^1(-,Z) \to H^1(-,G)$ be the Galois cohomology map induced by the embedding $Z \to G$. Let L/K be a finite separable field extension. Consider the following commutative diagram:

$$\begin{tikzcd}
H^1(L,Z) \arrow[rr, "\alpha_L"] \arrow[dr, "\gamma_L"] \arrow[dd, "cor_{L/K}"'] & & H^1(L,G) \arrow[dr, "\theta_L"] & \\
& \prod_{\nu\in\Omega} H^1(K_\nu\otimes_K L, Z) \arrow[rr, "\beta_L"] \arrow[dd, "cor_{K_\nu\otimes_K L/K_\nu}" near start] & & \prod_{\nu\in\Omega} H^1(K_\nu\otimes_K L, G) \\
H^1(K,Z) \arrow[rr, "\alpha_K" near end] \arrow[dr, "\gamma_K"] & & H^1(K,G) \arrow[dr, "\theta_K"] & \\
& \prod_{\nu\in\Omega} H^1(K_\nu, Z) \arrow[rr, "\beta_K"] & & \prod_{\nu\in\Omega} H^1(K_\nu, G).
\end{tikzcd}$$

Let $u \in \operatorname{Ker}\alpha_L$, and $(w_\nu)_{\nu\in\Omega} \in \prod_{\nu\in\Omega} H^1(K_\nu\otimes_K L, Z)$ be the image of u under γ_L. Hence $(w_\nu)_{\nu\in\Omega} \in \operatorname{Ker}\beta_L$. By assumption, the norm principle holds for G over each extension $K_\nu \otimes_K L/K_\nu$, therefore

$$cor_{K_\nu\otimes_K L/K_\nu}(w_\nu)_{\nu\in\Omega} \in \operatorname{Ker}\beta_K.$$

Since

$$\theta_K \circ \alpha_K \circ\ cor_{L/K}(u) = \beta_K(cor_{K_\nu\otimes_K L/K_\nu}(w_\nu)_{\nu\in\Omega})$$

and $\operatorname{Ker}\theta_K$ is by assumption trivial, we have $cor_{L/K}(u) \in \operatorname{Ker}\alpha_K$. $\square$

Note that in [21, Theorem 1.3.2], a result similar to Theorem 3.1 is proved; however, in that setting, the field K is taken to be the function field of a curve over a number field, and the overfields K_ν are specifically the completions of K with respect to the valuations associated to the points on the curve. In Theorem 3.1, we do not assume K and K_ν are necessarily of this form.

Let k be a number field, and C be a smooth, projective, geometrically integral curve over k, with function field $K = k(C)$. We introduce two strategies for studying the norm principle for semisimple groups G over K. Both of these approaches utilize Theorem 3.1. But the overfields K_ν are different: In the first approach, the overfields K_ν are the fields obtained by completing the underlying number field k with respect to discrete valuations on k. In the second approach, the overfields K_ν are obtained by completing the field K with respect to divisorial valuations on K itself.

The first approach is as follows. Let k be a number field, and C be a smooth, projective, geometrically integral curve over k. Let $K = k(C)$ be the function field of C. We denote by Ω_k the set of places of k, and for $\nu \in \Omega_k$, let $K_\nu = k_v(C)$ denote the function field of the curve C_{k_v}, which is an example of a semiglobal field.

In [7], Colliot-Thélène conjectures the following:

Conjecture 3.2. [7] *Let k, K, and Ω_k be as above. Let G be a semisimple simply connected linear algebraic group defined over K. The following map of pointed sets has trivial kernel:*

$$H^1(K, G) \to \Pi_{\nu \in \Omega_k} H^1(K_\nu, G).$$

This conjecture has been studied in various cases. For instance, Parimala and Preeti have proved the conjecture in some cases where the group G is in fact defined over k (see [16, Theorem 6.5]).

Theorem 3.3. *Let k, K, and Ω_k be as above. Let G be a semisimple simply connected linear algebraic group defined over K. Assume that the norm principle holds for G over every K_ν, for every $\nu \in \Omega_k$, and also assume that Conjecture 3.2 holds for G over K. Then the norm principle holds for G over K.*

Proof. This is an immediate consequence of Theorem 3.1. □

Theorem 3.4. *Let k and K be as above. Let G be a semisimple linear algebraic group of type D_n defined over k. Then, the norm principle holds for G over K.*

Proof. Similar to Theorem 2.4, it suffices to prove the norm principle for simple, simply connected groups of type D_n; so without loss of generality we assume $G = Spin(A, \sigma)$, where A is a central simple algebra over K with orthogonal involution σ. Any such group $Spin(A, \sigma)$ can be realized as $Spin(h)$, where h is a hermitian form defined over (D, σ'), where (D, σ') is a central simple algebra over K, Brauer equivalent to A, with orthogonal involution. If $Spin(A, \sigma)$ is of type D_n, then $dim(h) = n$.

If $n \geq 2$, then by [16, Theorem 6.5], conjecture 3.2 holds for $Spin(h)$. Also the norm principle holds for $Spin(h)$ over each overfield K_ν by Theorem 2.4. Then the result follows from Theorem 3.3.

Finally, as mentioned in the proof of Theorem 2.4, the norm principle holds for groups of type D_1 over arbitrary fields, and thus the proof is complete. □

We finally introduce another framework for studying the norm principle for semisimple groups defined over function fields of algebraic curves over number fields, which differs from the one suggested by Theorem 3.3. However, it still relies on a local-global approach and makes essential use of Theorem 3.1.

We propose the following question, which can be regarded as a Hensel's lemma–type statement for the norm principle.

Question 3.5. *Let M be a complete discretely valued field with residue field $\overline{M}$. Assume that the norm principle holds for all simply connected linear algebraic groups defined over all finite separable extensions of $\overline{M}$. Does the norm principle hold for all simply connected linear algebraic groups defined over M?*

Recall that, since the norm principle for a semisimple group follows from that of its simply connected cover, one may replace "simply connected" with "semisimple" in the formulation of Question 3.5 without loss of generality.

An affirmative answer to Question 3.5 implies that the norm principle holds for semisimple groups over a large class of complete discretely valued fields, including all complete discretely valued fields whose residue fields are number fields, p-adic fields, or global fields of positive characteristic (over all these fields the norm principle is known for all semisimple groups, as mentioned in the introduction). One natural example of such complete discretely valued fields is the completion of the function field of an algebraic curve with respect to valuations associated to closed points of the curve, when the curve is defined over a number field or a p-adic field.

Now we propose another question, similar to the question above, which concerns only groups of type D_n. Recall that any simple, simply connected linear algebraic group of type D_n over a field of characteristic $\neq 2$, is the spinor group $Spin(D, \sigma)$ for a central simple algebra with orthogonal involution. We also recall that any central simple algebra which admits an orthogonal involution has to be of index 2^r, for some nonnegative integer r.

Question 3.6. *Let M be a complete discretely valued field with residue field $\overline{M}$ with char $(\overline{M}) \neq 2$, and let r be a nonnegative integer. Assume that for all central simple algebras $\mathfrak{D}$ of index $\leq 2^r$ with orthogonal involution $\boldsymbol{\sigma}$, defined over all finite separable field extensions F of $\overline{M}$, the norm principle holds for $Spin(\mathfrak{D}, \boldsymbol{\sigma})$ over F. Does the norm principle hold for $Spin(D, \sigma)$ over M, for all central simple algebras with orthogonal involution (D, σ) of index $\leq 2^r$ defined over M?*

For $r = 0$, i.e. when D (resp. $\mathfrak{D}$) splits, the main result proved by Bhaskhar, Chernousov, and Merkurjev in [4, Theorem 5.1] gives an affirmative answer to Question 3.6. Note that the main result in [4] is stated in terms of the H^0-variant of the norm principle for the extended Clifford group of a quadratic form q, i.e. $\Omega(q)$. See [4, Lemma 2.2] for the equivalent statement in terms of the H^1-variant of the norm principle for $Spin(q)$.

For $r = 1$, i.e. when D (resp. $\mathfrak{D}$) is a quaternion algebra, then Question 3.6 has affirmative answer by [20, Theorem 7.1].

To the best of our knowledge, there are no results in the literature concerning Question 3.6 for $r \geq 2$. Moreover, no analogous results appear to be available for groups of type E_6 and E_7.

Now we discuss the second strategy for studying the norm principle for semisimple groups over function fields of smooth projective geometrically integral curves over number fields. Recall that it is sufficient to settle the norm principle for simple, simply connected groups.

We recall the following well known open question (For the details about divisorial valuations over function field of a curve, see for instance [6]):

Question 3.7. *([4, Section 7]) Let k be a number field, K be the function field of a smooth projective, geometrically integral curve over k, and G be a simply connected linear algebraic group over K. Let Ω be a set of divisorial valuations of K and for every $\nu \in \Omega$, let K_ν be the completion of K with respect to ν. Then, is the kernel of the natural map*

$$\rho_{G,\Omega} : H^1(K, G) \longrightarrow \prod_{\nu \in \Omega} H^1(K_\nu, G)$$

trivial?

Theorem 3.8. *Let k, K, G, and Ω be as in Question 3.7. Assume that*

- *(1) For every $\nu \in \Omega$, Question 3.5 has affirmative answer for K_ν, and*
- *(2) Question 3.7 has affirmative answer for G over K, i.e., the kernel of the map $\rho_{G,\Omega}$ is trivial.*

Then, the norm principle holds for G over K.

Proof. The residue field $\overline{K_\nu}$ of each K_ν is either a finite extension of k (if the valuation ν is trivial on k), or a global field of positive characteristic (if ν is not trivial on k). In both cases, the norm principle holds for all semisimple groups over $\overline{K_\nu}$ (see the introduction). So if Question 3.5 has affirmative answer for K_ν, then the norm principle holds for all simply connected, and hence all semisimple, groups over K_ν (for all $\nu \in \Omega$). In particular, the norm principle holds for G over each K_ν. Then the result follows from Theorem 3.1. □

The norm principle is related to Serre's injectivity question. In [19, Page 233], Serre asked:

Question 3.9. *([19]) Let K be a field, and let G be a smooth connected reductive K-group. Consider a set of finite field extensions L_i/K, and assume that the greatest common divisor of the degrees $[L_i : K]$ is 1. Consider the natural map*

$$H^1(K, G) \longrightarrow \prod_i H^1(L_i, G)$$

induced by restriction. Does this map have trivial kernel?

In [2, Theorem 1.2], Bhaskhar proved that if *char* $K \neq 2$, and if the Dynkin diagram of G contains connected components only of type A_n, B_n or C_n, then Serre's question has a positive answer for G. Using our results in this paper on the norm principle for groups of type D_n, we extend Bhaskhar's result to the case where the components can also be of type D_n (D_4 non-trialitarian) over some function fields.

Theorem 3.10. *Let K, G, and L_i's be as above. Suppose the Dynkin diagram of G has components of types A_n, B_n, C_n, and D_n (D_4 non-trialitarian). Suppose one the following holds:*

- *K is function field of a smooth projective geometrically integral curve over a p-adic field, or*

- *K is function field of a smooth projective geometrically integral curve over a number field k, and G is defined over k.*

Then, Serre's question has an affirmative answer for G.

Proof. By [2, Lemma 3.4], it suffices to prove the norm principle for simple components of semisimple part of G. If the simple component has one of the types A_n, B_n, or C_n, then we know the norm principle holds for that component by the main theorem in [1]. For components of type D_n (D_4 non-trialitarian), the result follows from Theorems 2.4 and 3.4. $\square$

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