17

Asset Classes (I)

Everything has its beauty but not everyone sees it.

Confucius

A hybrid derivative is a multi-asset derivative whose underlyings do not belong to the same asset class. The structures introduced in previous chapters are focused around equities, although much of the analysis regarding structuring and risk analysis can be extended to other asset classes. In the case of hybrids we again have to think about the effects of volatility and correlation. On the side of volatility, different asset classes have different volatility structures and can have very different implied volatility skews compared to equities. Also, the different asset classes have different futures and forward curves. The issues of liquidity and transaction costs also arise and must be understood for each asset class.

In this and the next chapter on hybrid derivatives we present these other asset classes and explain the intricacies in each of them. By describing the markets, the forward curves, the vanilla derivatives and then some exotic derivatives that exist in each, we can then translate this understanding to combine these asset classes with each other in a meaningful manner in Chapter 19. Understanding these asset classes will also allow us to extend all the analysis done so far on various structures to all the asset classes, individually, and ultimately combined. The chapters on hybrid derivatives also provide us with tools for tackling dynamic strategies involving multiple asset classes in Chapter 21.

The most obvious motivation for the use of hybrids is that products structured on multiple asset classes can provide a valid source of diversification. While we have already seen volatility as a valid diversifier for an equity portfolio owing to its negative correlation with equity, we can also find low correlations, if not negative correlations, among the other asset classes. Correlation structures are discussed in Chapter 19 where we describe some examples of macro-economic views that can be structured into hybrid derivatives and used for purposes of diversification or yield enhancement.

Other than equities, we discuss the following major asset classes:

1. Interest Rates
2. Commodities
3. Foreign Exchange (FX)
4. Inflation
5. Credit

Hybrid derivatives enable an investor to take a view on combinations of these asset classes whether for speculative purposes or even as a hedging strategy for multi-asset class exposures. This chapter covers interest rates and commodities and Chapter 18 covers FX, inflation and credit.

17.1 INTEREST RATES

We have already touched upon interest rates in Chapter 1 when we discussed basic instruments. In this section we go into more detail, in particular with respect to the aspects of interest rate derivatives that we will later use to construct multi-asset derivatives. The section will cover forward rate agreements and swaps, explaining constant maturity swaps (CMS). We discuss bonds, yield curves and interest rate swaptions. This leads us to a discussion of the SABR model that is now a market standard in interest rate derivatives. The section ends with a discussion of some popular interest rate exotics.

This section is by no means a comprehensive discussion of interest rate derivatives; it is designed to arm us with the knowledge and tools that are necessary to understand hybrid derivatives, although much more can be said. The interest rate products described here are standard, and for more elaborate discussions on interest rate markets and derivatives we refer the reader to Brigo and Mercurio (2006) and Rebonato (2002).

17.1.1 Forward Rate Agreements

A forward rate agreement (FRA) is an OTC contract that specifies an interest rate that will be paid or received as part of an obligation that starts at a future date. The relevant dates and the notional amounts will also be specified in the contract. A typical contract involves two parties exchanging a fixed rate for a floating one equal to some reference rate that underlies the contract, typically LIBOR or EURIBOR. Payments are calculated on the basis of the notional amount, and it is the difference between the fixed and floating legs that is ultimately paid. The party receiving the floating leg (who is paying the fixed rate) is said to be long the FRA, and the party paying floating is short the FRA. These are important as a swap is a combination of FRAs.

Let $R_{\text{ref}}(t)$ denote the reference rate at time t and R_{fixed} the fixed rate. On the effective date, T_{eff}, the payment made by the FRA is the netted amount given by

$$\text{FRA}_{\text{payoff}}(T_{\text{eff}}) = \text{Notional} \times \frac{(R_{\text{ref}}(T_{\text{eff}}) - R_{\text{fixed}}) \cdot d}{1 + R_{\text{ref}}(T_{\text{eff}}) \cdot d}$$

d is the day count fraction, which is given by the day count convention of the relevant currency on which the FRA is written. It defines the number of days in the year over which interest rates are calculated. As a fraction, for GBP this is typically 365, while for EUR and USD it is 360, and d is given by the number of days divided by 365 (or 360).

The fixed rate R_{fixed} is the rate at which both parties agree. Both the fixed and reference rates are those that begin to accrue on the effective date T_{eff}, and in turn are paid on the termination date of the contract. The discount factor, which is represented in the denominator, is specific to the case where – because the payments are known on the effective date – they are paid on such a date.

Consider a simplified example to illustrate the contract where party A enters into an FRA with counterparty B such that party A will receive a fixed rate of 3% for 1 year on a notional amount of $10,000,000 in 2 years' time. Party B will receive a floating rate, the 1-year LIBOR in this example, which is determined in 2 years' time on the effective date. The same notional applies to the 1-year LIBOR rate prevailing at that point in time and is used to compute the net payments needed to be made.

In 2 years' time, according to the FRA contract, and assuming that the 1-year LIBOR is 3.2%, which is higher than the agreed fixed rate of 3%, party A who is paying the fixed portion will have to make a net payment of $(3.2\% - 3\%) \times \text{Notional} = \$20,000$ to party B. Here we ignored the day count fraction and did not discount.

The reference rate to be used in computing the net payment depends on the difference between the effective date and the termination date. For example, if the FRA has an effective date in 3 years and termination date in 3.5 years, then the USD 6-month LIBOR rate would be specified in the FRA contract. Similarly, an FRA with an effective date in 4 months and termination date in 5 months would use the USD 1-month LIBOR rate.

17.1.2 Constant Maturity Swaps

An interest rate swap is an OTC instrument in which two counterparties agree to exchange a stream of interest payments for another stream of cashflows. In a typical fixed for floating interest rate swap, one party makes payments based on a reference rate (the floating leg of the swap) in exchange for the other party making payments based on a fixed rate (the fixed leg of the swap). The fixed rate in such a swap is computed so that the swap has a net present value of zero at initiation, and, as such, the fixed rate is called a swap rate.

These swap rates form the swap rate curve, also known as a swap curve or LIBOR curve. This curve must be related to the zero coupon yield curve. The swap curve gives the relationship between swap rates at different maturities. In some cases, particularly in emerging market currencies, where sovereign debt is not liquidly traded, a swap curve can sometimes be more complete, and can thus be the better indicator of the term structure of interest rates in such currency.

Consider a swap, of maturity T, between two counterparties: A will pay a fixed interest rate on a notional in currency C, and B will pay the floating leg on the same notional in the same currency C, indexed to reference rate. Party A pays fixed and receives floating and is thus said to be long the interest rate swap. For example, the swap can involve exchanges of cashflows on a quarterly basis where the floating leg is given by the prevailing 3-month USD LIBOR plus a spread of 25 bp.

A constant maturity swap (CMS) is a swap in which the buyer is able to fix the duration of the cashflows he will receive in the swap. In the swap described above, the floating leg is reset on a quarterly basis to the LIBOR rate prevailing at the time, whereas in a CMS the floating leg is fixed against a point on the swap curve on a periodic basis. That is, the floating leg is reset with reference to a market swap rate rather than a LIBOR rate. The second leg of the swap is typically a LIBOR rate but can be a fixed rate or even another constant maturity rate. Again the structure of these can be based on a single currency or as a cross-currency swap. The value of the CMS depends on the volatilities of different forward rates and also the correlations between them. As such, pricing a CMS requires an interest rate model or at least what is known as a convexity adjustment (Brigo and Mercurio, 2006).

As an example of a CMS, consider an investor who believes the 3-month USD LIBOR rate will fall with respect to the 5-year swap rate. To play this view the investor can go long the constant maturity swap that pays the 3-month USD LIBOR and receives the 5-year swap rate. The 5-year swap rate here can be specified to be a point even further down the curve, thus allowing one to take a view on the longer section of the yield curve. The CMS thus also serves as a tool for hedging exposures to the long end of the yield curve, and in the case above where the investor receives the 5-year swap rate, a CMS as such will hedge against a sharp increase in this 5-year swap rate.

17.1.3 Bonds

Continuing the Chapter 1 introduction to bonds, we discuss here various types of bonds and then bond price sensitivities. A bond is a debt instrument in which the issuer will need to pay the holder interest, usually in the form of a coupon, and repay the notional on the maturity date of the bond.

Government Bonds

When a national government issues a bond it is known as a government or sovereign bond, and these are denominated in the currency of the relevant country. Interest from some government bonds is generally considered to be risk free, although there are cases where governments have defaulted on their sovereign bonds. In the US, Treasury securities, denominated in USD, are the least risky USD investments.

Such bonds may carry a low default risk, but do carry other risks. For example, if an investor buys a 5-year Treasury bond in USD and receives interest plus her money back in 5 years, the notional amount received back may be worth less owing to depreciation in the USD w.r.t. other currencies and also the risk of inflation. When we discuss inflation in the next chapter we see government issued bonds that are inflation-indexed and as such protect investors from exposure to inflation.

Bond Futures

Bond futures, which are traded on a futures exchange market, are contracts in which the holder is obliged to buy or sell a bond at a specifically agreed point of time in the future and for a specific price. Being traded on an exchange, these contracts are standardized and their trading regulated. As with trading any future, we are exposed to price fluctuations between the initial trade date and the exercise date of the futures contract. That is to say, the price of a bond can fluctuate like anything else, and although a bond may be purchased and its interest payments may be safe, its price between the initial agreement of a futures contract and exercise date can vary by a significant amount.

Bond Market Indices

A bond market index is a weighted index of bonds or other interest rate instruments. Like a stock market index, a bond index is used as a method to measure the composite value of its constituents. As an index it can also serve as a benchmark for comparison with other bond portfolios. However, bond indices are generally more complex than stock market indices and harder to replicate. It is still possible, however, to replicate such an index using bond futures. A Treasury bond index, for example, is a portfolio of outstanding Treasury bonds and notes.

Floating Rate Notes

A floating rate note (FRN) is a bond in which the coupon varies according to a reference rate, for example LIBOR plus a constant spread. A typical FRN has quarterly coupons and, at each payment date, the value of the reference rate is monitored and added to the agreed spread. For example, the quarterly coupon can be the 1-month USD LIBOR plus a spread of 30 bp.

Inverse Floating Rate Note

An inverse floating rate note (IFRN) again offers a variable coupon, but in this case the coupon has an inverse relationship with a specified reference rate. An instrument as such is designed to offer a higher coupon as the reference rate declines. The price of a typical bond is inversely proportional to the interest rates used in computing its value – as rates go up the value of the bond in the market goes down. The inverse floater is designed so that as short-term interest rates fall, both the bond's yield and value increase. The opposite holds for such a structure if rates were to rise and the inverse floating rate note's value would decrease accordingly. As an example of the formulation of the inverse floating rate:

$$\text{Floating Rate} = \underbrace{6\%}_{\textit{fixed rate}} - \underbrace{2}_{\textit{gearing}} \times \underbrace{\text{6m USD LIBOR}}_{\textit{reference rate}}$$

Bond Price Sensitivities

The *duration* of a financial asset refers to the sensitivity of this asset's price to a movement in interest rates. In the context of bonds, duration is the percentage change in the price of the bond with respect to interest rates, i.e. the absolute change in the price w.r.t. interest rates, divided by the current bond price. Duration is measured in years and is between 0 and T years, where T is the maturity of the bond. For small movements in interest rates, duration gives the first-order effect, i.e. linear effect, as the approximation of the drop in price of the bond with an increase of 1% per annum in interest rates. A bond with 10-year maturity and a duration of 5 would fall in price by approximately 5% if interest rates increased by 1% per annum.

The second-order effect, or bond *convexity*, measures the sensitivity of the duration of the bond to a change in interest rates. This is used similarly to the way the *Gamma* of a derivative, or book of derivatives, is used. If the duration of a book of bonds is low, but the overall convexity is still high, then a movement in interest rates will have a large impact on the duration of the book. If that book has both low duration and convexity, then it is far better hedged against large interest rate movements. The concept of Taylor series expansions discussed at the start of Chapter 5 applies, and we can write the change in the price of a bond $\Delta(P)$, where $P = P(r)$ is the bond price written as a function of (flat) interest rates, in terms of the first- and second-order terms of the series:

$$\frac{\Delta(P)}{P} = -D\Delta(r) + \frac{C}{2}(\Delta(r))^2$$

where D is the bond duration and C is the bond convexity.

17.1.4 Yield Curves

A yield curve is a plot of the interest rate yields of bonds of different maturities versus these maturities, and yield curves are widely monitored. The bonds used to form a yield curve must all be of the same credit quality. For example, the highly looked at yield curve of US Treasury securities of different maturities. The significance of this yield curve is that owing to the government's influence in this curve it is often used to infer future information about economic growth.

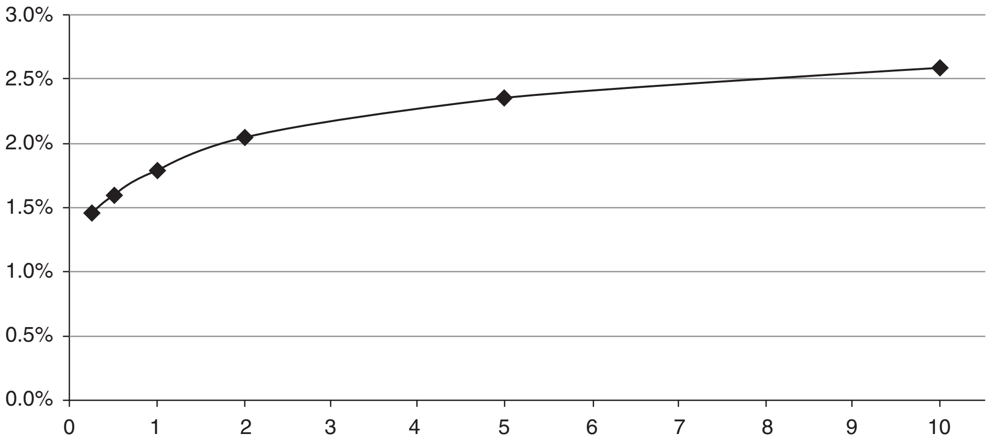

Figure 17.1 A typical yield curve. The 3-month, 6-month, 1-year, 2-year, 5-year and 10-year rates are plotted. Notice the upward-sloping nature of the curve as well as how this increase slows down as maturities get longer.

Yield curves are typically upward sloping with the longer maturities accompanied by higher yields, although this increase in yield by maturity slows down as the maturities increase (see Figure 17.1). An upward-sloping curve generally reflects that the market expects higher interest rates in the future. Because the Federal reserve controls short-term rates,[1] an upward-sloping yield curve indicates that the market believes Fed policy will be favourable to financial markets. The steeper the yield curve, the more positive this is believed to be.

The opposite case, where we see a downward-sloping yield curve, is when the short-term interest rates are higher than the longer term rates. This is also known as an inverted or negative yield curve. This type of yield curve reflects that there are market expectations for lower rates in the future. This is primarily because high short-term interest rates can imply that the government is trying to slow down the economy.

We may also see a humped shape yield curve. Assume the hump peaks at the 2-year point, then such a curve will indicate that the market expects rates to rise over the next 2 years but then decline.

Yield curves can make parallel shifts, and this is when all the points along the curve rise (or fall) by the same amount. This generally signals a change in economic conditions and expectations regarding inflation. We come across these when structuring macro-economically meaningful multi-asset options. A non-parallel shift of the yield curve is when the various points move by different amounts.

One can structure options to take a view on any of the yield curve shapes or moves, specifically the flattening or steepening of the yield curve, and movements up and down in the yield curve. One can also take a view not only on the slope of the yield curve but also on its curvature. With regards to the slope, one can have a yield curve option on the spread between two rates on the yield curve corresponding to two different maturities. Based on how these rates are chosen, the option will pay off when the yield curve either flattens or steepens.

[1] The Fed controls the federal funds rate. This is the rate that banks charge each other for overnight loans of reserves.

In the example of the CMS above, receiving the 5-year swap rate for example in a constant maturity swap allows for the view that rates will rise in the future as a result of a steepening of the yield curve. The opposite holds and one can pay the swap rate in a CMS with the view that the long-term swap rates will not end up as high as the market is currently implying through the yield curve. When an investor receives the 7-year CMS, for example, and pays a floating rate of, say, LIBOR plus a spread, then the exposure is primarily to the slope of the yield curve and not to its level. This means that such a structure is not sensitive to parallel shifts in the yield curve.

17.1.5 Zero Coupon, LIBOR and Swap Rates

In this subsection we establish the relationships between some of the interest rate concepts we have seen so far. Firstly, the relationship between a zero coupon bond $B(t, T)$ at time t and maturity T, and the instantaneous interest rate r_t.

$$B(t, T) = E\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\right]$$

Here we point out the difference between zero coupon bonds and discount factors. Discount factors are not random as we can always get the current discount factors $D(T)$ by stripping the yield curve (Hagan, 2003). $D(T) = B(0, T) =$ today's discount factor for maturity T. However, zero coupon bonds $B(t, T)$ will remain random until the present time reaches time t.

The spot LIBOR rate at time t and maturity T is

$$L(t, T) = \frac{1 - B(t, T)}{(T - t)B(t, T)}$$

in terms of bonds $B(t, T)$.

A forward LIBOR at time t, with expiry T_{i-1} and maturity T_i, is

$$F_i(t) = \frac{1}{T_i - T_{i-1}}\left(\frac{B(t, T_{i-1})}{B(t, T_i)} - 1\right)$$

This is a market rate, and is the underlyer of the forward rate agreement contracts discussed above. The Treasuries used to form a yield curve only have a finite number of maturities, and to see the yield at a maturity for which no Treasury security is available one will have to interpolate the yield curve. When doing so it should be checked that the forward rates computed as described here are all positive to ensure that the interpolation of the yield curve is arbitrage free.

A swap rate described above can also be written in terms of bonds. Let $S_{\alpha,\beta}(t)$ be the swap rate at time t with tenor $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$.

$$S_{\alpha,\beta}(t) = \frac{B(t, T_\alpha) - B(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})B(t, T_i)}$$

This is a market rate and it underlies interest rate swaps discussed above. At time t these are all known from the bond prices $B(t, T)$.

17.1.6 Interest Rate Swaptions

An interest rate swap option (called a swaption) is an option that gives the holder the right but not obligation to enter into an underlying swap. Which leg of the swap the holder of a swaption can potentially enter into is determined by the type of swaption: the owner of a payer swaption has the right but not obligation to enter into a swap where they pay the fixed leg and receive floating. A receiver swaption gives the opposite: the right but not obligation to enter into a swap where they pay the floating leg and receive fixed.

Swaptions are OTC derivatives but there exists an interbank swaption market. Typically swaptions are valued using Black's model, and from the swaption market one can obtain implied swaption volatilities. One key difference between a swaption and an option on a stock is that two swaptions of the same maturity (the option) can be on two swaps of quite different tenors. So for swaption volatilities there are not only volatilities for swaptions of different maturities but also different volatilities for underlying swaps of different tenors. The dominant factor in the swaption market is the time to expiry of the swaption compared to the tenor of the underlying swap.

Black's model (see Black, 1976) is specifically designed to have as underlying a forward contract on a swap – a forward swap rate. Instead of having a call option on a spot rate, we have a call option on a forward rate. In 1976 Black applied this model to price calls and puts on physical commodities, forwards and futures, and applying it to the case of European swaptions the underlying is a single forward swap rate:

$$\text{Call Option}_{\text{price}} = \mathrm{e}^{-rt_{\text{set}}}\,[f\mathcal{N}(d_1) - K\mathcal{N}(d_2)]$$

$$\text{Put Option}_{\text{price}} = \mathrm{e}^{-rt_{\text{set}}}\,[K\mathcal{N}(-d_2) - f\mathcal{N}(-d_1)]$$

where f is the current underlying forward rate and K is the strike price. The values of d_1 and d_2 are

$$d_{1,2} = \frac{\log f/K \pm \frac{1}{2}\sigma_B^2 t_{\exp}}{\sigma_B\sqrt{t_{\exp}}}$$

This resembles the Black–Scholes formula but has some key differences, in particular, forward prices exhibit a different form of randomness to spot prices. In the above formula the discount factor is taken between time zero and the settlement date t_{set}. The time parameter used in computing d_1 and d_2, which in turn gives the value inside the brackets, is the expiry date $t_{\exp}$.

The volatility σ_B will be chosen from the swaption implied volatilities in the market. Typically in the swaption market there are ATM swaption implied volatilities for various swaption maturities and various tenors of the underlying swaps. A swaption is quoted by the maturity of the option, the tenor of the swap, whether the swap is a receiver or payer swap, the strike and the implied volatility. For example, a 1 into 2 receiver at 5.4% for 16.8%. The 1-year represents the maturity of the option, the 2-year is the tenor on the underlying (receiver) swap, the option is struck at 5.4% and has an implied volatility of 16.8%.

Using an ATM volatility to price an OTM European swaption would require an adjustment (Figure 17.2). Black's model is popular because it is extremely fast, and recently we saw the emergence of the SABR model that is the simplest extension of Black's model that accounts for skew and has decent implied dynamics. To manage a book with potentially thousands of swaptions, speed is a key factor, and the SABR model is as instantaneous as Black's model. We discuss SABR in a section of its own below.

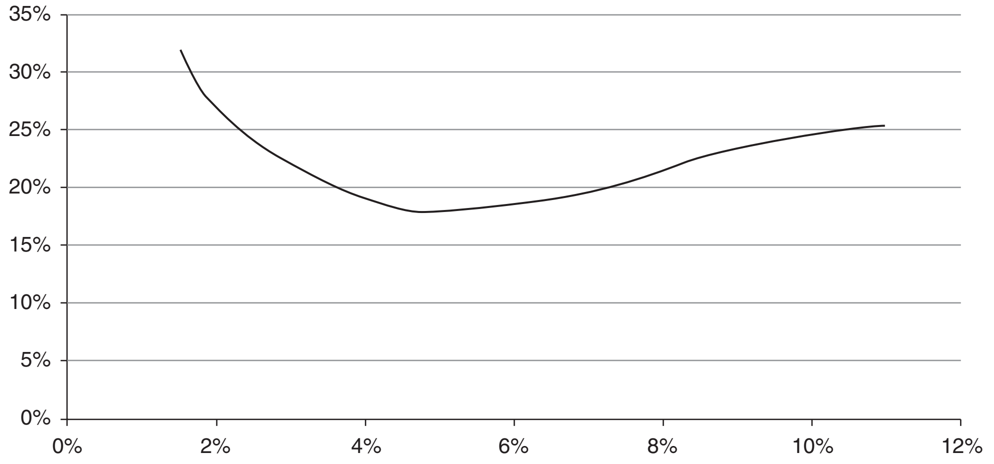

Figure 17.2 The implied volatility smile of a European swaption. Deep OTM call option implied volatilities tend to flatten out the deeper one goes OTM. The smile shape is different to the equity skews.

17.1.7 Interest Rate Caps and Floors

An interest rate cap is an OTC derivative that makes a payment to the holder when a specified short-term interest rate rises above a specified *cap* rate. Interest rate cap maturities typically range from 1 year to 7 years, and the derivative makes periodic payments where, at each such payment date, the difference between the underlying reference rate and the cap rate is paid. Consider, for example, a 2-year cap in which each quarter a payment is made in the amount by which LIBOR exceeds 1.5%. The premium for the cap is usually paid up front.

The interest rate cap is a series of interest rate *caplets*, one for each period up to the cap's maturity. A caplet is simply a European call option on the reference rate with strike rate equal to the cap. The payoff of a caplet on a rate L struck at K is

$$\text{Caplet}_{\text{payoff}}(T) = N \cdot \alpha \cdot \max(L(T) - K, 0)$$

where N is the notional value exchanged and α is the day count fraction corresponding to the period to which L applies. This is essentially a call option on the LIBOR rate observed at time T.

The interest rate cap allows its holder to limit a floating rate exposure to interest rates rising. By buying an interest rate cap on the rate to which the investor is exposed, he receives payments when the rate exceeds the cap rate (the strike). The longer the maturity of the cap the more it offers this protection, but the more expensive it is.

An interest rate floor is defined similarly as a series of *floorlets*, each of which is a put option on a reference rate, typically LIBOR. The buyer of the interest rate floor receives payments on the maturity of any of the individual floorlets; the reference rate is below the specified floor.

Both the caplets and floorlets are valued using Black's model, and again the relevant implied caplet volatility will be used.

17.1.8 The SABR Model

The SABR model was pioneered by Hagan *et al.* (2002). SABR stands for Stochastic Alpha, Beta and Rho, as these (along with a parameter ν) are the parameters that form the model. The SABR model deserves a subsection of its own; in fact it deserves a whole lot more. The key point is that although in equities, for example, there is no real market standard for managing skew risk, in interest rates and FX there is: it is this SABR model. Broker quotes for swaption volatilities of different moneyness can be quoted in terms of SABR parameters.

SABR models a single forward rate, allowing for both local volatility and stochastic volatility. As such it allows for the modelling of swaption implied volatility smiles. The motivation behind its creation was to find an extension of Black's model that allowed for smiles and skews but also offered correct smile dynamics. One key feature of this model is that the prices of European options can be computed in closed form using the SABR formula that gives the correct implied volatility to use in Black's formula. This makes the model as quick as using Black's formula, only with the added advantage that it knows about volatility smiles and also offers correct smile dynamics which, in turn, gives stable hedge ratios.

The SABR formula is based on an approximation and this can lead to troubles, these are discussed in Appendix A, section A.3.2. Mainly, it assumes a small vol of vol, which may be reasonable in interest rates or foreign exchange but not so much so in equities or commodities. The SABR formula is given by

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\log^2 f/K+\frac{(1-\beta)^4}{1920}\log^4 f/K+\cdots\right\}}.$$

$$\left(\frac{z}{x(z)}\right)\cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{\text{ex}}+\cdots\right.$$

where

$$z=\frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K$$

and $x(z)$ is defined by

$$x(z)=\log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$$

where f is the forward price. The first thing to note is that although this formula appears complicated, it is in fact closed form, and it involves nothing more than computing logarithms and powers of the various parameters. This is the original form of the SABR model, and is the volatility parameter that is plugged into the Black–Scholes formula to return the prices of European options at the relevant strikes. Note that in the above formula the dots indicate the left-out higher-order terms that are typically ignored when the formula is implemented and used. When the parameters are calibrated to an implied volatility skew across strikes, the set of SABR parameters: α, β, ρ, ν gives a parameterization of this skew.

To explain the parameters we start with β. This parameter is specified within the range $0 \le \beta \le 1$, and appears above mainly as an exponent of f. This is because, in the model from which this formula came, β in fact represents the power parameter of a specific type of local volatility. When $\beta = 0$ the underlying forward is normally distributed, and when $\beta = 1$, the underlying forward is log-normally distributed. The case where β is in between represents a dynamic for the forward that is neither Normal nor log-normal.

The β parameter is typically chosen first and set to be constant, for example, if we like to model our forward as a log-normal random variable, we set $\beta = 1$ and work with the rest of the parameters. In the case where $\beta = 1$, the model is a simple stochastic volatility extension of Black's formula in that the underlying forward is modelled as log-normal in both cases, but in SABR the volatility is also modelled as a log-normal random variable compared to Black's case where it is just a constant.

The α is a volatility-like parameter for the forward (West, 2005). Its volatility ν is thus the vol-of-vol, and ρ is the (instantaneous) correlation between the underlying and its volatility. α thus controls the height of the ATM implied volatility level. The correlation ρ controls the slope of the implied skew and ν controls its curvature.

17.1.9 Exotic Interest Rate Structures

Here we discuss some popular interest rate exotics, specifically range accruals, target redemption notes and CMS steepeners. The selection of an interest rate structure, exotic or not, can be to hedge a specific set of cashflows, or to take a speculative view on interest rates. The exotics case can allow for a more tailored hedge or a more specific view.

Callable Features

Compared to equities, we can find some long-dated structures, although longer structures in interest rates tend to have callable features. In a callable interest rate swap, for example, the payer of the fixed rate has the right to end the swap at some specified set of dates in the future (possibly only one date). In exchange for this right, the investor paying the fixed leg in such a swap would expect to have to pay an above-market rate.

These break down into European and Bermudan style callable features. For a fixed for floating swap to be European style callable, it means that the payer of the fixed leg has one date in the future at which the swap may be terminated. For example in a 7-year swap callable after 2 years the payer of the fixed leg has one and only one opportunity, at the 2-year mark to decide whether or not to terminate the swap. If not called at this point in time, the swap will remain active till its prespecified maturity, here the 7-year point. During these first 2 years the payer of the floating leg will receive an above-market fixed rate, but if rates decline then the payer of the fixed leg will most probably cancel the swap.

In a Bermudan style callable swap, the payer of the fixed leg is given the right to call the swap at a set of dates in the future. For example, a 7-year swap where the payer of the fixed leg can cancel the swap on an annual basis on or after the second year. Similar to the European case, if rates move against the payer of the fixed leg it may be cancelled at one of these dates. The Bermudan case obviously adds more flexibility, and the right to a Bermudan callable feature will cost more than a European version; however, the Bermudan feature is the more popular of the two.

Range Accruals

Range accruals are relatively popular. Similar to the range accrual we saw in section 11.5.2, the range accrual in the interest rate case pays a coupon proportional to the number of days that an underlying reference rate stays within a prespecified range. A coupon as such can be paid as an exotic coupon in a swap where the investor pays a fixed (or even a floating amount) and receives this range-dependent coupon. The range accrual coupon can also be used in a floating rate note.

There are also interest rate digitals where a digital call option makes a payment if a reference interest rate is above a specified barrier. Or even a digital that makes a payoff if the reference interest rate at maturity is within a specified range.

Target Redemption Notes

A target redemption note (TARN) pays a set of coupons that are linked to a reference rate, with the possibility of early redemption. The coupons are computed, typically using an inverse floating LIBOR rate such that once the sum of all coupons paid reaches a *target* amount, the note is redeemed at par. For example, consider a 7-year TARN with annual coupons in which the coupons of each year T_i are computed, based on the formula

$$\text{Coupon}(T_i) = 2 \times \max\left[0, 4\% - L(T_i)\right]$$

where $L(T_i)$ is the USD 12-month LIBOR at time T_i. The coupons are paid annually until either the note reaches maturity or the sum of all coupons paid has reached the target amount of 14%. In both cases the notional is returned as the note is redeemed at par. The appeal of such a note is the possibility that the investor may get his money back at par plus the target coupon in what could be a relatively short amount of time.

CMS Steepener Options

A CMS steepener option, also known as a CMS spread option, pays a coupon based on a multiple of the spread between two CMS rates. The most popular of these is the option on the 30-year to 10-year CMS rate spread. The 10-year to 2-year and 30-year to 2-year structures are also popular. The maturity of an option on such a spread does not have to be nearly as long; for example, one can have a 1-year option on the 10-year to 2-year CMS rate spread. If the yield curve was to steepen, the spread between these two CMS rates would increase. A product as such can be appealing in an environment where the yield curve is flat or even inverted, and a CMS steepener option as described provides a leveraged play on the view that the yield curve will steepen.

17.2 COMMODITIES

Moving to our next asset class, here we discuss commodities. The idea is to understand commodities as an asset class of their own. Commodities can be broken down into categories:

- Energy, which includes crude oil
- Precious metals, which includes gold
- Industrial metals, which include copper
- Agricultural products, which include wheat

There is additionally the category of Livestock and Meat.

These can be split into two categories: hard commodities and soft commodities. Hard commodities are those whose supply is limited to the finite availability of natural resources;

these include metals such as gold and energy commodities such as crude oil. Soft commodities include agricultural products and livestock that are affected by other factors such as the weather.

In this section we want to understand the properties of commodities, in particular those with significance to commodity derivatives. This entails understanding the forward curves and the volatilities of each. For more details we refer the reader to the comprehensive books by Geman (2005) and Schofield (2007).

The rise in commodity investing, which has been made even easier through commodity derivatives, comes from the broad range of possible commodities that are accessible through financial markets, in addition to the general belief that commodities markets are not strongly correlated to equity markets. Commodities are also an interesting asset class as they are believed to serve as an inflationary hedge. Commodities were in fact involved in some of the first derivatives where farmers tried to secure certain prices for their crops that were yet to be produced. Although such derivatives were originally designed for risk management purposes, today investing in commodities is done by many people with no such risks.

The Chicago Board of Trade (CBOT), the first established exchange, offers standardized commodity contracts. The CBOT along with the Chicago Mercantile Exchange (CME) are two of the largest exchanges in the world offering commodity contracts. In addition to trading in standardized commodity contracts, there are many ETFs that provide exposure to commodity prices directly, for example ETFs that track the spot price of gold. As such an investor can take a view on the price of gold without having to take hold of the physical commodity himself. Commodity indices such as the GSCI and the DJAIG can also be accessed through ETFs.

In addition to the expansion in commodity derivatives, both vanilla and exotic, there has been a rise in the appearance of commodities as an asset class in multi-asset derivatives: hybrids. An investor can add commodities to such a derivative in combination with other asset classes for diversificational benefits, as a hedge for a multi-asset class exposure that involves commodities and to even take a combined view on commodities along with another asset class.

Although commodities are generally believed to be negatively correlated to equities, in an environment such as that of the credit crunch of 2008, it is possible that owing to supply and demand and the global impact of the credit crunch, commodities such as oil also fall as equities do. The point of this chapter is to explain this asset class and then explain in the subsequent chapter how to structure and price hybrid derivatives involving commodities, not to discuss investment ideas in commodities.

17.2.1 Forward and Futures Curves, Contango and Backwardation

The forward curve is the plot of forward prices with maturity T versus this maturity, and similarly for a futures curve. A futures contract is basically a standardized forward contract, while futures are traded on an exchange, forwards are done OTC. The word *nearby* refers to how close the forward contract is to expiry. The first nearby is the closest to expiry, the second nearby is the second closest to expiry, and so on. As the first nearby expires, the second nearby contract becomes the first nearby, and so on.

An upward-sloping curve, which means that the price for the future delivery of a commodity (or asset in general) is higher than its current spot price, is referred to as *contango*. This means that the amount the market charges for the delivery of an asset in the future is more than what would be charged to receive delivery of the asset today. *Backwardation* is the opposite case where the forward curve is downward sloping, i.e. the price of future delivery of the commodity is lower than its current spot price. It is also possible to see humped forward curves.

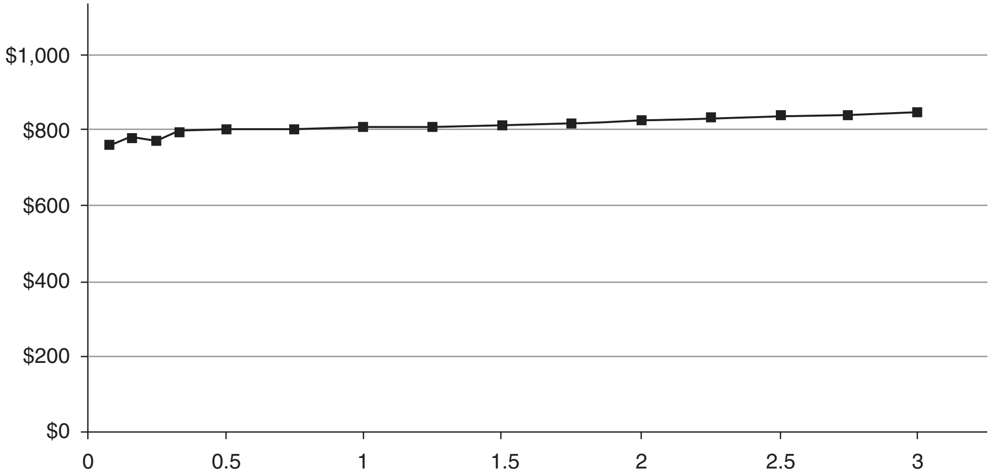

Figure 17.3 A typical futures curve for gold. Notice the curve is in contango, and also that the long end of the curve is more stable than the short end. Although the curve is bumpy at first, we still say it is in contango as the first few nearbys are cheaper than the longer term futures.

The short end of a commodities forward curve is affected directly by supply and demand in the short term, and thus this end of the forward curve often looks less stable than the longer end of the curve, which is generally smoother. When the short end of the curve is higher than the long end – that is, we are in backwardation – it generally means that the commodity in question is in short supply (compared to demand). When in contango and the short end of the forward curve is lower than the long end, it means that the commodity is generally in good supply. A hump at an intermediate point along the forward curve reflects the market's expectation of a high demand at that point in time, and multiple humps can represent a seasonal change in demand (for example, natural gas in the winter).

As an example, assume the spot price of gold is $800 per ounce. If the futures price for the delivery of gold in 3 months is $780, where payment is made on delivery, then this implies that gold is currently in short supply. The lower futures price means that the market believes that delivering gold today will be more costly than delivering it in 3 months' time, thus the backwardation. Someone who believes that in 3 months the delivery of gold will cost more than $780 can enter into the futures contract for the delivery of gold at $780 in 3 months' time, thus taking a view on the forward curve.

If the 3-month futures price for gold were $850, it reflects a healthy supply of gold today, with the view that the delivery of gold in 3 months' time is worth more than the spot price. Again a view can be taken that this futures curve will not be realized by taking the opposite side of a forward contract. In fact, the use of gold as an example is not the best as backwardation rarely occurs in the forward curve of gold. Figure 17.3 gives an example of a typical futures curve for gold.

Figure 17.4 gives an example of a futures curve for crude oil. There are several different types of crude oil: West Texas Intermediate (WTI), for example, is a type of crude oil often used as a benchmark for oil prices. WTI futures are traded on the New York Mercantile Exchange (NYMEX) where one can obtain a futures curve up to 6 years in maturity.

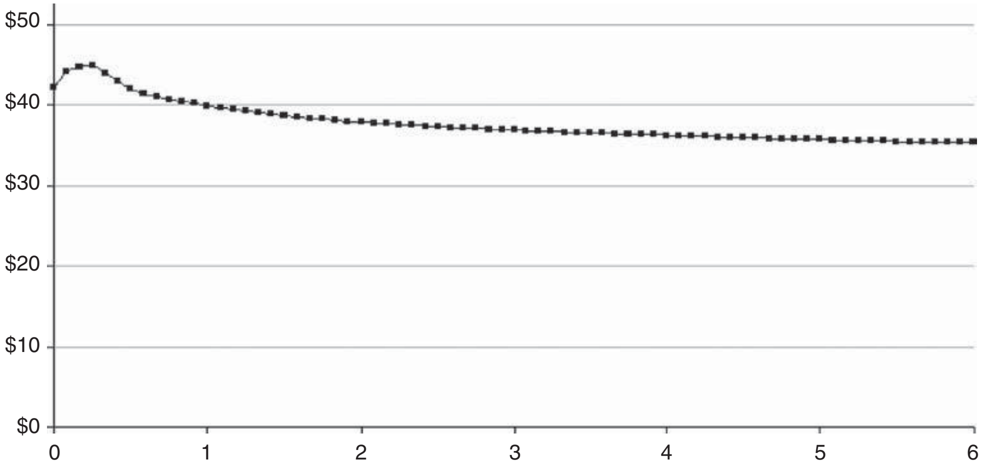

Figure 17.4 An example of a futures curve for crude oil. Notice that the curve is in contango in the start then goes into backwardation as the long-term futures are worth less than the near-term futures contracts.

To understand the case of backwardation we explain the existence of a *convenience yield* in commodities. This yield is nothing more than an adjustment to the cost of carry, which is the cost of taking on a financial position. In the case of equities there is no cost of storage but interest rates and dividends both appear in the computation of the non-arbitrage price of the forward contracts on a stock. Buying a forward contract on a stock would involve interest payments to pay for borrowed funds when buying on margin. Dividends have an opposite effect as the opportunity cost of the decision to buy a forward instead of the stock itself. The inclusion of rates and dividends reflects this in the price of forwards in the equity case. In the case of commodities, a convenience yield should be included as an adjustment to the cost of carry to account for the fact that the physical commodity itself cannot be shorted and that its delivery today might be a necessity for which a market participant is willing to pay a premium. The convenience yield is defined as the premium that a consumer is willing to pay to be able to attain the commodity now rather than at some time in the future (Schofield, 2007).

The forward price, which also serves as an approximation of the futures price, including the cost of carry, is

$$F(T) = S(0)\mathrm{e}^{(r+s-c)T}$$

where F is the forward price, T is time to delivery, S is the spot price, r is the risk-free interest rate, s represents the cost of storage, and c is the convenience yield. The value of such a convenience yield is affected by the demand for a commodity. The case where there is a shortage of a commodity with respect to the demand for it will be reflected in a high convenience yield. In the case where there is an abundance of supply for a commodity, this yield shrinks to zero.

The convenience yield helps to explain the reason behind backwardation. In the case of oil, and generally for the energy market, delivery might be a necessity, and penalties high for not

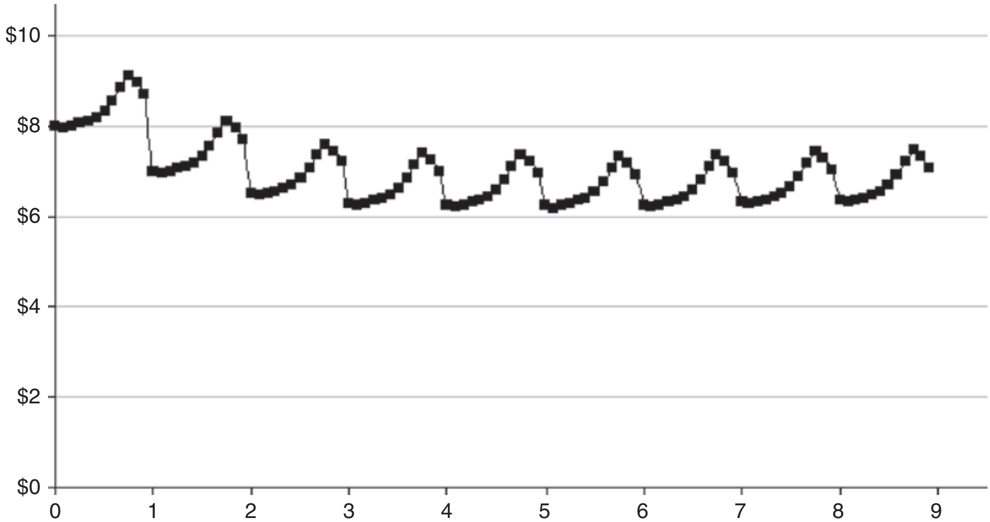

Figure 17.5 An example of a futures curve for natural gas. Notice that the curve goes between contango and backwardation during each season. In this curve the maturities start at a point during the summer season, and the curve peaks during each winter.

delivering. Obtaining oil today will carry a higher convenience yield, reflecting the premium on the price for the delivery of the spot today.

One can ask the questions: Is it possible to arbitrage the futures curve in the commodities market? Is it possible for the oil futures curve to remain in backwardation? The answers are: commodities have a non-arbitrageable forward. When the curve is in backwardation, the shorter dated contracts are worth more than the longer dated ones, making the latter seem cheap or incorrectly priced. To take advantage one would need to buy the longer dated contracts and short the front end of the curve. Firstly, the spot market can be quite illiquid and the asset hard to obtain and thus quite difficult to short. If one were to sell a contract at the front end of the curve and buy the long end, then the commodity itself would need to be obtained and delivered in order to honour the obligation of the first contract. If the commodity is not readily available in the spot market, then this forward curve is non-arbitrageable and can remain in backwardation for this reason. In the case where one does take physical delivery of the commodity, storage and other costs will be incurred. As such, and since the market for these contracts is driven by participants' fair value of future delivery, a given slope of the curve will account for these factors and not allow for an arbitrage.

17.2.2 Commodity Vanillas and Skew

Since commodity futures and forwards are more liquid than the asset's spots, these forwards and futures are used as the underlyings in commodity derivatives. From a hedging perspective, the illiquidity of commodity spots and the resulting lack of shorting ability, make hedging with spots not feasible.

A call option on a commodity, for example, will give the holder the right but not the obligation at the maturity of the option to buy a commodity future at the agreed strike price.

The underlying future will also be of an agreed maturity. In a manner similar to how we previously specified strike prices as a percentage of the current spot, the strike of an option on a commodity future can be specified as a percentage of the price of the underlying future at the time of pricing.

Pricing an option on a spot price of a seasonal commodity can also add complexity. For example, one would expect the spot price for natural gas to be higher in winter. A seasonal effect must be priced into the derivative and a Black–Scholes log-normal assumption for the spot price is unreasonable. However, since the seasonality effect is already priced into the forward curve, it makes sense to price derivatives on points of this curve compared to using the spot price. Figure 17.5 provides an example of a futures curve for natural gas.

In Black (1976) he modelled forward price instead of the spot price, which solved the aforementioned issue. We have already seen Black's model in section 17.1.6 in interest rates, his paper was written to address this issue. Black's model is also the standard in commodity markets for pricing European options on physical commodities, forwards and futures.

Commodities will also exhibit some form of skew in the implied volatilities of European options. This is the volatility that, when used in Black's model, gives the correct price of the European option, and again implied volatility is the market's consensus of future volatility. The commodity volatility surface consists of the implied volatilities of options on futures, for which we have one option maturity for each futures contract maturity. The option's maturity is typically only a few days after the date the futures contract is set to expire.

This term structure is downward sloping most of the time. Volatility in commodity markets is closely linked to issues of supply and demand. If the supply of a commodity is struggling to keep up with demand, volatility will rise. In the short term, supply and demand fluctuations have a large impact on volatility causing it to be very high in cases where there is a supply shortage, whereas the longer dated implied volatilities are governed more by long-term expectations regarding the economy and are generally lower. Long-term factors, such as the mining of limited resources and even the weather, can play a role depending on the commodity.

With respect to moneyness, the implied volatilities surface will have decreasing liquidity in European options struck away from the ATM point. Commodity implied volatilities also imply that commodity returns are positively skewed compared to equities in which returns are negatively skewed. For example, in oil, greater uncertainty is generally associated with higher oil prices. Market participants, particularly in industries, who rely on having oil in order to conduct business, will need to hedge the upside risk of price increases to which they are adverse. Those producing oil will want downside protection in order to protect profits in the event that oil prices may decline; however, upside hedging sees more demand and comes with higher implied volatilities.

18

Asset Classes (II)

No complaint, however, is more common than that of a scarcity of money.

Adam Smith

Continuing the discussion of asset classes, in this chapter we discuss foreign exchange (FX), inflation and credit, each as a separate asset class.

18.1 FOREIGN EXCHANGE

The foreign exchange market, often referred to as FX, encompasses everything to do with currency trading. Again, we focus in this section on understanding the underlying market, FX forward curves, FX vanillas, FX implied volatility smiles and some FX exotics. For a more detailed account of FX derivatives we refer the reader to Wystup (2007).

The FX market is highly liquid with many quite different parties acting as market participants, from corporations, to speculators and even governments. To define an exchange rate, one must specify two currencies: a domestic and a foreign. These are not related to geography but simply define a standard by which values will be measured. An exchange rate is defined as the amount of the domestic currency required to buy one unit of the foreign currency. A transaction in FX will involve the exchange of an amount of one currency for an amount in another currency; for example, exchanging euros for US dollars.

To be clear on notation, we use the convention of foreign–domestic where, for example, the exchange rate between the euro and the US dollar is written as EUR–USD and represents the amount of US dollars needed to buy one euro. The USD–EUR exchange rate is the inverse of this where the domestic currency is taken to be the euro. If the EUR–USD is 1.4356 then the USD–EUR is the inverse of this, given by $1/1.4356 = 0.69657$.

18.1.1 Forward and Futures Curves

Futures are standardized forward contracts that are traded through an exchange, and these are the more popular choice among currency speculators; many futures positions are closed out before they reach expiry. An FX future is a contract that allows the holder to buy or sell a currency for a specific price at a specific date in the future. Forward contracts in FX are traded OTC and are the more popular choice for parties hedging FX exposures. FX forward contracts do not need to be settled with the delivery of the foreign currency, and can be cash settled in the domestic currency. If the investor actually wanted to take hold of an amount of foreign currency, then the contract can be specified so that it is settled in the foreign currency.

The FX forward rate is affected by the interest rates in both the domestic and foreign currencies. In fact the most important factor in determining an FX forward rate is the spread between the two interest rates. If we recall the concept of the cost of carry – that is, the cost of taking on a financial position, including the opportunity costs – in the case of FX we can expect the buyer of an FX forward to be long interest rates in the domestic currency and short

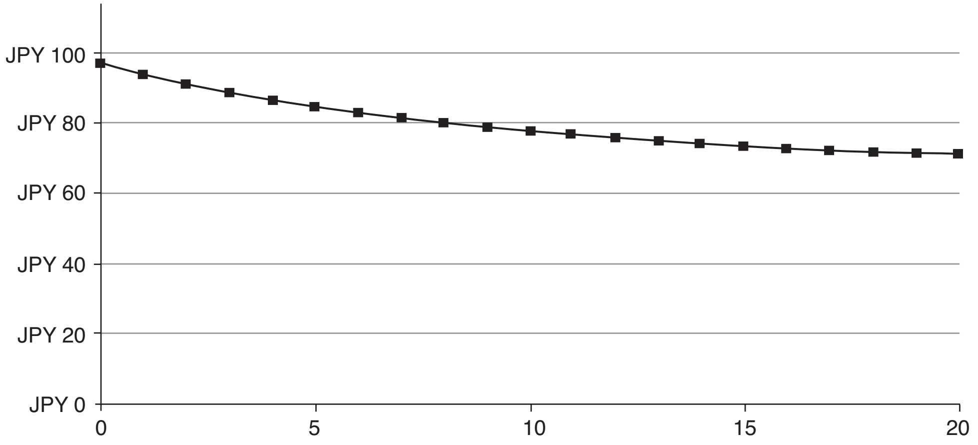

Figure 18.1 A USD–JPY forward FX curve. Notice that the curve is in backwardation, reflecting the case where interest rates in Japan are relatively lower.

interest rates in the foreign currency. The FX forward, which delivers an amount of a currency at time T is given by the formula

$$F(T) = S\,\mathrm{e}^{(r_\mathrm{d}-r_\mathrm{f})T} \tag{18.1}$$

where S is the FX spot price, r_f is the risk-free rate on the foreign currency, and r_d is the risk-free rate in the domestic currency. This formula is explained by understanding the effect of interest rates and what is known as interest rate parity.

Different countries usually have different interest rates, and this means that money in one currency can grow at the local risk-free rate, which is different to how it would grow at the risk-free rate in another currency. This must be reflected in the price of an FX forward or there would be an arbitrage opportunity, and interest rate parity relates FX forward prices to interest rates in the form of a non-arbitrage condition. Assume that an investor borrows money in one currency and converts it to another currency in which the risk-free rate on interest is made on this amount. An FX forward, that would be used to lock in an exchange rate at a future date to convert this money back to the original currency, must be priced so that the risk-free returns from such a trade match those in the original currency.

Formula (18.1) describes this parity in the case where we assume continuous compounding on interest. When both interest rates are equal, the forward price must equal the spot price. When rates are different, and depending on which is higher, the forward curve can be in contango or backwardation. Different interest rate term structure shapes can also impact the shape of the forward curve and it is not necessarily smooth. Figure 18.1 shows a USD–JPY forward curve in backwardation. The bigger the difference between the two rates, the steeper the slope of the forward curve.

Carry Trades

A carry trade is a play on the interest rate differentials between two currencies that involves the FX rate. Assume that a forward curve is quite steep, reflecting a large interest rate differential, then an investor can enter into a carry trade that involves selling the currency with the lower interest rate and buying the currency with the higher interest rate. The play on the FX rate is that the investor is paying the lower risk-free rate on the money borrowed, and receiving the higher interest rate yield in the second currency, thus netting a profit.

This is not an arbitrage strategy as it will only be profitable if the currency does not move against the investor. If the currency does not move then at the end of the investment period the investor can change money back to the original currency and pay off the loan in that currency, having netted the profit from the difference in interest rates. If the currency with the higher rate were to depreciate relative to the currency in which the investor is borrowing, then when the time comes to exchange the money back to cover the original loan, the value in the original currency will be lower and it is possible to make a loss.

The opposite can also happen where the investor makes additional gains if the currency with the higher yield appreciates. One can put on a carry trade and continuously net the interest rate differential, and close out the trade once the FX rate moves in the wrong direction; however, exchange rates can jump suddenly in the wrong direction leaving the investor with an immediate loss. Since the forward price is governed by interest rate parity, it is not possible to hedge this FX risk completely and a view must be taken on the exchange rate between the two currencies.

A typical currency to borrow in is the Japanese Yen because interest rates in Japan have been very low. An investor can borrow money in JPY paying very little interest and exchange yens for another currency with a high interest rate that can be in excess of 10% per annum. Carry trades can also involve borrowing in a low-yielding currency and investing in a high-yielding asset other than a bond, but again the same risk remains, in addition to the risks the asset in question also adds that its yield will not be paid.

18.1.2 FX Vanillas and Volatility Smiles

FX options are mostly OTC and are highly liquid – particularly those written on a forward rate. Exchange-listed FX options exist, and although these greatly reduce counterparty credit risk, the OTC market is much bigger. An FX option is a derivative in which the holder has a right but not the obligation to exchange money from one currency to another at a fixed strike rate and specified maturity date, both specified in the terms of the option.

If we take a call option in USD on one unit of a stock for a fixed strike price, then this strike price is the amount of dollars that will be paid for one unit of the stock if the option is exercised at maturity. An FX option involves the exchange of two notionals in different currencies, and the strike is given by the ratio of these notionals. As FX can be confusing, we will clarify by use of an example.

Assume that an investor buys 1,000,000 options with a GBP–EUR strike at 1.3464, where these options give the holder the right but not the obligation, at maturity, to sell GBP 1,000,000 and buy EUR 1,346,400 ($1.3464 \times 1{,}000{,}000$). The GBP notional of the trade is GBP 1,000,000, and the EUR notional of the trade is EUR 1,346,400 so that the ratio of the two is equal to the strike price. If the investor decided on a different strike for the option, then one of these notionals must change in order that the new strike is still the ratio of the two notionals. To set the option ATM, the strike would have to equal the current spot exchange rate between the two currencies, and if one notional is fixed then the second notional will have to be set so that the ratio of the two matches the spot.

In the above option, if the EUR appreciates w.r.t. the GBP, then the GBP–EUR exchange rate has decreased (the amount of EUR needed to buy one GBP has gone down). The holder of the above option has the right to sell GBP 1,000,000 and receive EUR 1,346,400 at a time when the EUR has appreciated, which means that GBP 1,000,000 is currently worth less than EUR 1,346,400. Such an option is thus a put option on GBP, and it is the market convention to quote an exchange rate strike using the terminology of a GBP–EUR strike in reference to the put option that increases in moneyness as EUR gains w.r.t. the GBP. It also acts like a call option on the EUR. The opposite option can be specified where the put is on the EUR (call on GBP) and the strike, assuming the same notionals, will be the inverse of the previous strike, i.e. $1/1.3464 = 0.7427$, and this would be referred to as a EUR–GBP strike.

The premium charged for entering into an FX option depends on how many contracts are being bought. This premium can be computed using the Garman and Kohlhagen (1983) model, which is essentially an extension of the Black–Scholes model that accounts for the two interest rates. As such, the model, like Black–Scholes, assumes that the exchange rate is log-normally distributed. The idea is to use this model to obtain the value of one contract and the premium can thus be computed by multiplying the cost of one contract by the number of contracts.

Following the same notation as before, let r_{d} denote the risk-free rate in the domestic currency, and r_{f} that of the foreign currency. The domestic currency refers to that in which the option will be denominated. Caution must be taken when specifying the spot and strike. As the strike and spot of a call option on a stock will be specified in terms of dollars per stock, the strike and spot in the FX case must also be specified in terms of the same units.

In the above example involving GBP and EUR, the call option will be denominated in GBP and the EUR–GBP strike price will be used for a call option. The strike and the spot must be specified in the same units: here we are using the number of domestic currency units per unit of foreign currency. It is the domestic currency that is exchanged for the foreign currency at maturity if the option is exercised. If we denote the spot and strike by S and K, then the values of calls and puts under the Garman and Kohlhagen model are given by

$$\text{FX Call Option}_{\text{price}} = S\,\mathrm{e}^{-r_{\mathrm{f}}T}\mathcal{N}(d_1) - K\,\mathrm{e}^{-r_{\mathrm{d}}T}\mathcal{N}(d_2)$$

The value of a put option has value

$$\text{FX Put Option}_{\text{price}} = K\,\mathrm{e}^{-r_{\mathrm{d}}T}\mathcal{N}(-d_2) - S\,\mathrm{e}^{-r_{\mathrm{f}}T}\mathcal{N}(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r_{\mathrm{d}} - r_{\mathrm{f}} + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

and the parameter σ, which should be used, is the implied volatility. Much like the previous cases we have seen, the market for vanilla options is driven by the premiums at which market makers are willing to trade, and the observed implied volatilities represent the consensus of all views. The implied distribution for FX will not be log-normal, but using the correct implied volatility obtained from the market in the above formula will yield the market price of the option.

The implied volatilities in the case of FX typically exhibit smiles, not skews. This means that ATM options will have lower implied volatilities than both ITM and OTM call options. This is the market's way of pricing a premium on FX options struck away from the money to include the risk that an exchange rate can have extreme moves in either direction. The

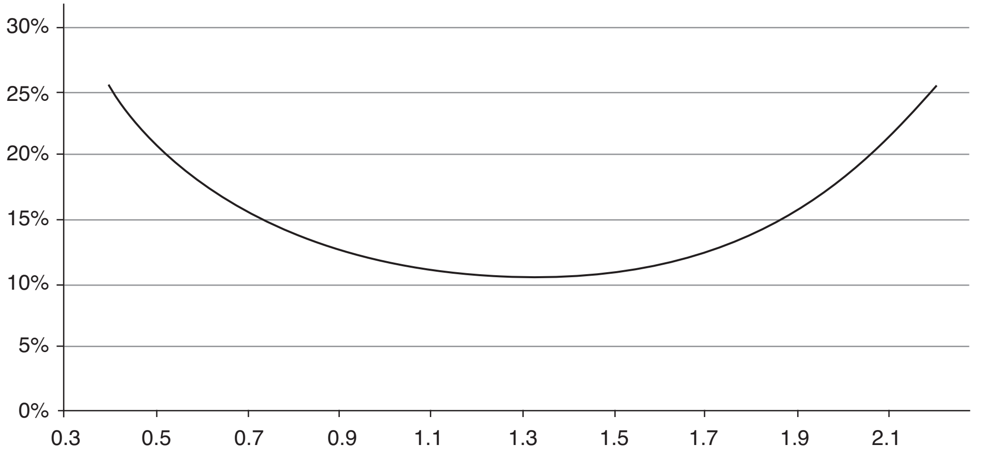

Figure 18.2 An example of an implied volatility smile for the EUR–USD. The smile is (almost) symmetric.

existence of the smile means that the market believes such are more probable than a log-normal distribution implies. An extreme move in an exchange rate between two currencies is the result of a depreciation of one currency w.r.t. the other, and the smile in FX markets prices this uncertainty in both directions (Figure 18.2). The implied volatility smile is not necessarily symmetric, and a steeper skew on one side can reflect more uncertainty in one of the currencies.

FX vanilla options must satisfy a put–call parity relationship. This is the equilibrium relationship that must exist between the prices of put and call options on the same underlying and with the same strike and expiry.

$$C(K,T) - P(K,T) = S\,\mathrm{e}^{-r_\mathrm{f}T} - K\,\mathrm{e}^{-r_\mathrm{d}T}$$

from this the parity of the Deltas of the call and put is

$$\Delta_{\text{call}} - \Delta_{\text{put}} = \mathrm{e}^{-r_\mathrm{f}T}$$

which for small maturities, and low foreign interest rates, we have $\Delta_{\text{call}} - \Delta_{\text{put}} \approx 1$.

Risk Reversals

The risk reversal is an option strategy that involves a call and a put. A long risk reversal consists of a long OTM call option and a short position in an OTM put option, both with the same maturity but different strikes. The strikes are specified in terms of the Deltas of the two options (Figure 18.3). In a typical risk reversal, the Deltas of both the call and put are chosen to be 0.25. As such it would be referred to as a 25-Delta risk reversal, and the market quotes the 25-Delta risk reversal as it gives information regarding expectations on the currencies composing the underlying exchange rate through the skewness of the implied volatility surface.

The risk reversal expresses the difference in the implied volatilities of the OTM call and the OTM put. By showing a higher implied volatility for either the call or the put, and thus implying a higher price for such an option, this tells us the direction in which the market expects the underlying to move. If the volatility of put options is higher, it means that the implied

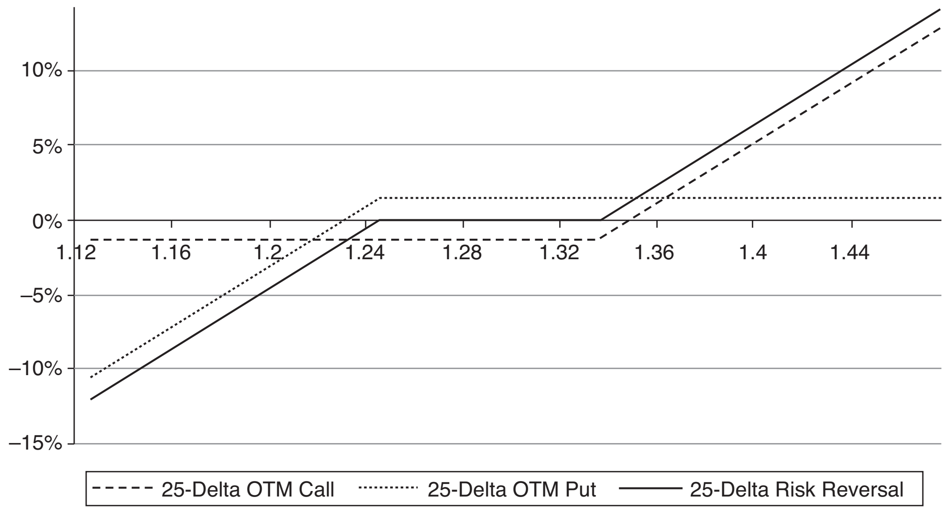

Figure 18.3 The payout of a risk reversal. It combines calls and puts in what is usually referred to as a collar.

distribution of returns is skewed in its expectation of a possible large downward movement and small but more frequent upward movements, much as for the case of the equity skew. This case is referred to as a negative risk reversal. In a positive risk reversal the opposite holds and the market places higher volatilities on call options, suggesting that the distribution is skewed in the opposite manner, and reflects expectations of large possible upward movements but with low frequency, compared to smaller downward movements of higher frequency – the opposite of the equity case. Risk reversals on a currency pair can change sign over time, and thus the FX market is said to have a stochastic (i.e. random) skew.

Risk reversals are typically quoted by giving the Delta of the call and put, a volatility spread that gives the bid–ask spread on the derivative, and an indication of which is higher between the volatilities of the calls and puts (i.e. which is favoured). Though the 25-Delta risk reversal is somewhat of a standard, one can obtain a quote for a risk reversal with any Delta, for example a 10-Delta corresponding to the risk reversal formed from Europeans with a Delta of 10%. For example, a 1-month 25-Delta risk reversal on USD–EUR can be quoted at 0.2/0.3 with EUR calls favoured over EUR puts, Here, 0.2 and 0.3 correspond to spreads over some mid-volatility, typically the ATM volatility, and these give the bid–ask spreads for the risk reversal in question. Assuming this mid-level to be 10%, then this quote means that the trader is willing to do two things: firstly, to buy the 25-Delta USD put (EUR call) for $10\% + 0.2\% = 10.2\%$ and sell the 25-Delta USD call (EUR put) for 10%, and, secondly, to sell the 25-Delta USD put (EUR call) for $10\% + 0.3\% = 10.3\%$ and buy the 25-Delta USD call (EUR put) for 10%. The favouring of the EUR call over the EUR put means that the trader is bid–asking the EUR call price at 10.2–10.3%, and leaving the EUR put volatility at the mid level of 10%.

Straddles, Strangles and Butterfly Spreads

Straddles and strangles are also option strategies that are constructed using European options, and these were discussed in Chapter 6. Both of these, like the risk reversal, are quite popular in FX and there is a liquid market for them in many currencies. In fact, these are also standard in the sense that it is quotes from these, like the risk reversal, that are used to give information regarding the market's view on the underlying currencies through their implied volatilities. Again the strikes of the European options composing these products are specified in terms of their Deltas.

A straddle position consists of two long positions, one in a call and one in a put. If we recall, a straddle can be constructed at a strike that is specified so that the Delta of the straddle is zero at initiation. In the case of FX, the ATM straddle refers to the straddle that is struck at a strike for which the Delta of the position is zero. As in the case of equities, this strike will be at or very close to the forward price, and it is this zero Delta feature that made the straddle a traditional method for trading volatility in equity derivatives. A European option comprising a straddle must, of course, have the same maturity.

Going long a strangle, one is again long two call options but with different strikes. A 25-Delta strangle is composed of a long position in a 25-Delta call option and a long position in a 25-Delta put option. The strangle, combined with knowledge of the ATM straddle implied volatility, gives us a measure of convexity. The reason is that these two combine to give a butterfly spread.

Butterfly spreads, often referred to as *Flys* in FX, are another options strategy, and also one that we have seen in section 4.2.1 of Chapter 4 under the discussion of convexity of implied volatility skews, and section 6.3.1 of Chapter 6 under option strategies. A risk reversal contains two options and it gives us information regarding which way the market is skewed, whereas the butterfly spread involves three European option strikes and gives us a measure of the convexity of the smile. In FX, the butterfly spread will involve European options struck ATM and at the two strikes corresponding to a specific Delta for each European option on either side.

Butterflys that are constructed from straddles and strangles take an inverted shape to those we saw before. Figure 18.4 shows the payoff of a short position in a butterfly

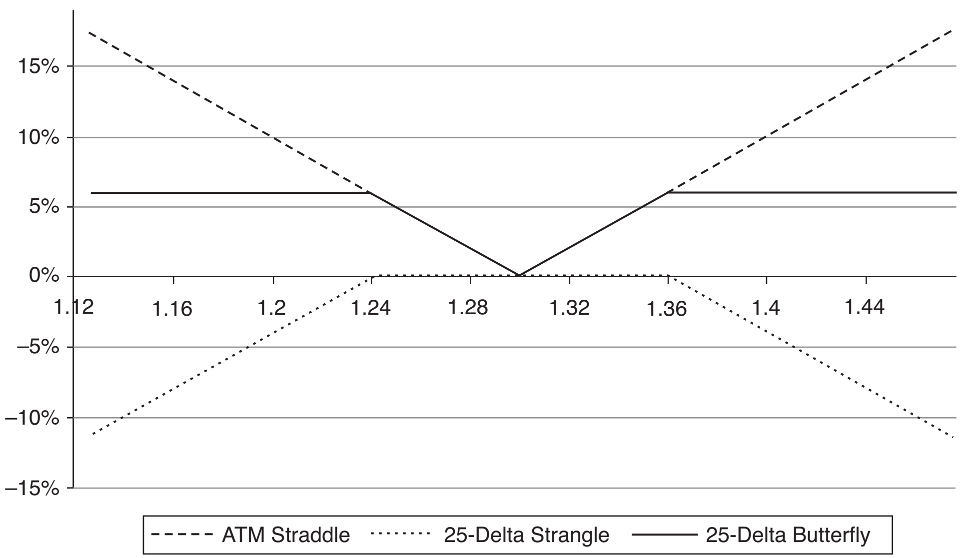

Figure 18.4 The payout of a short position in a butterfly spread. This is formed using an ATM straddle and a 25-Delta strangle.

spread as the combination of a long position in an ATM straddle and a short position in a strangle.

The method for quoting a butterfly spread, other than the maturity and Delta specification of the away-from-the-money options involved in its construction, is again a volatility spread. For example, a 1-month 25-Delta strangle quoted at 0.2/0.38 implies that the trader is willing to buy 25-Delta strangles at a volatility spread of 0.2% over the implied volatility of the ATM (zero Delta) straddle. It also means that she is willing to sell the 25-Delta strangle for a volatility of 0.38% over the same ATM straddle. The ATM straddle's implied volatility will govern whether the trader would rather buy or sell the strangle. If it is closer to the 0.2% then she favours buying the strangle over selling it; if closer to the 0.38%, then selling the strangle is favoured, and if it is right in the middle of this spread, i.e. 0.29%, then the trader is equally willing to buy or sell the strangle.

Garman–Kohlhagen, SABR and Smiles

When quoting Deltas for any of the options above, these always refer to the Black–Scholes Deltas that are computed using the implied volatilities of the quotes. To be correct, it is the Garman and Kohlhagen formula that is used. This allows for a standard method of computing the strikes of the above options and agreeing on Deltas. It is key to have a standard method to compute the Delta if this is to be exchanged as part of the transaction. So the 25-Delta call with an implied volatility of 10% is the call option whose strike is such that when this strike and implied volatility are used in the Garman and Kohlhagen formula, and the Delta is computed, it will be equal to 0.25.

To capture the smile, one has to venture beyond the Black–Scholes framework. A local volatility can be used, although the SABR model discussed in section 17.1.8 is a market standard in FX. Since most vanilla contracts in FX are written on forward rates, the SABR model, which models a forward rate, lends itself perfectly to the pricing and risk management of a portfolio of FX vanillas. Further details on the SABR model appear in section A.3.2 in, Appendix A.

The parameters of the SABR model have direct links to the discussed options. The volatility parameter, α, will control the level of the smile, and this volatility level is known through the market's quotes for ATM straddles. The correlation ρ between the underlying and the volatility controls the skew, $\partial\sigma(K)/\partial K$, which in the FX market we know through quotes of risk reversals. The vol-of vol parameter ν controls the smile's curvature, $\partial^2\sigma(K)/\partial K^2$, for which we also have the market values of strangles. Calibrating a SABR model to these points will give us the set of parameters that capture the market information of level, skew and curvature of the implied volatility smile. The β parameter can be used in the calibration, and as the local volatility parameter of the model it controls the deterministic skew; however, it is usually specified in advance based on *a priori*, and held constant. For example, the case of $\beta = 1$ refers to log-normal dynamics for the underlying and all other parameters are calibrated with this β fixed.

What makes this model significant is not only the intuitiveness of its parameters and their interpretations, but the dynamics generated by this model are consistent with what is observed in the market regarding how the skew moves as the underlying moves, $\partial\sigma_{\mathrm{ATM}}/\partial S$. As such, the model yields much more realistic hedge ratios. The benefits that were clear in the interest rate case also hold here in that this model is essentially a closed form extension to Black–Scholes, is lightning fast, and serves as an excellent tool for risk managing a portfolio of European options in FX.

18.1.3 FX Implied Correlations

The nature of exchange rates means that in certain cases one can compute the implied correlation between two exchange rates. Consider two exchange rates with the same domestic currency, for example the EUR–USD and the JPY–USD, then we know the EUR–JPY exchange rate as the quotient of these two. Call the two exchange rates of common domestic currency R_1 and R_2, and their quotient $R_3 = R_1/R_2$. Assume that we have implied volatilities for each of these three exchange rates, obtained from vanilla options on each, then applying the Black–Scholes assumption of log-normality of these rates, we know that the quotient of two log-normals is also log-normal.

$$\ln\left(\frac{R_1}{R_2}\right) = \ln R_1 - \ln R_2$$

both of $\ln R_1$ and $\ln R_2$ are normally distributed (recall a log-normal random variable is one whose logarithm is normally distributed), and their difference thus also normally distributed. On the one hand, we know the implied volatility of R_3 from the market, but we also know it must be a function of the volatilities of R_1 and R_2 according to the above equation. Working with variances, we know $\mathrm{Var}(\ln R_3) = \sigma^2(R_3)$ on the one hand, but also from the FX relationship above that

$$\begin{aligned}
\mathrm{Var}(\ln R_3) &= \mathrm{Var}(\ln R_1/R_2) \\
&= \mathrm{Var}(\ln R_1) + \mathrm{Var}(\ln R_2) - 2\mathrm{Cov}(\ln R_1, \ln R_2) \\
&= \sigma^2(R_1) + \sigma^2(R_2) - 2\,\rho_{1,2}\,\sigma(R_1)\sigma(R_2)
\end{aligned}$$

Substituting the market value of $\mathrm{Var}(R_3)$ into this equation and solving for $\rho_{1,2}$ we find that

$$\rho_{1,2} = \frac{\sigma^2(R_1) + \sigma^2(R_2) - \sigma^2(R_3)}{2\,\sigma(R_1)\sigma(R_2)}$$

where the right-hand side of this equation is all implied from the vanilla markets of each of the three exchange rates.

Wystup (2002) gives a geometric interpretation of this and describes how the same concept can be extended to multiple currencies. The key observation is that because of the nature of the FX market and in light of the above formulas, FX correlation risk can be transformed into a volatility risk, and hedging this correlation risk is thus possible by using adjusted Vegas. This can be extremely useful when handling baskets of currencies.

18.1.4 FX Exotics

FX exotics, like the other asset classes, can be used for tailoring specific market views, hedging specific FX risks, or even be structured to offer higher yields or better diversification. Digital options and barriers are especially popular in FX, and it is possible to find broker quotes for some barrier and digital options, making them more liquid in comparison to their counterparts in different asset classes. Many different types of barrier options exist in FX and below we mention a few of them.

An interesting example of a barrier option is the *Parisian* barrier. This is a knock-out option, where the option knocks out only if the underlying spends a certain amount of time above (or below, depending on the position of the barrier) the barrier. As such, this differs from the simple barrier that knocks out if the underlying crosses the barrier at any point in time. The idea behind the Parisian barrier is that if the underlying is close to the barrier, a short spike in the underlying won't cause the option to knock out (or knock in). For example, consider an up-and-out ATM call option that has a Parisian barrier at 125%, monitored daily, and of length five business days. Then the call option will only knock out if the barrier spends five or more days above this 125% barrier.

The Parisian barrier option will obviously be worth more than the simple barrier and will increase in price the longer the length of time the underlying has to spend beyond the barrier. A key feature of the Parisian barrier option is that its Greeks are far smoother at the barrier than the simple barrier option case, and as such alleviate some of the related issues that arose in Chapter 10. An interesting variation on the barrier option would be to decrease the participation in the call option above for each day the underlying spends beyond the barrier. For example, the call option can start with 100% participation and decrease by 12.5% each day the underlying spends beyond the barrier.

It is of course possible to structure a basket option where the underlying assets are all exchange rates. These will typically have a common domestic currency and the specification of different foreign currencies gives the different exchange rates in the basket. The analysis of baskets in Chapter 7 generally holds, and the addition of multiple assets helps to lower the overall volatility of the basket, compared to the volatility of a single underlying. For example, the basket can consist of differently weighted emerging market currencies' exchange rates with the EUR.

Forward skew-dependent payoffs such as cliquets are also found in the FX market. The cliquet style and related payoffs seen in Chapters 13 and 14 can be structured with FX rates as underlyings. The issues regarding forward skew and Vega convexity will still hold, but depending on the style of cliquet, the skew effect may be different, as it will, even for regular digitals, compared to equities, because of the presence of a smile.

Variance and volatility swaps are also traded in FX, and these are the same payoffs we saw in Chapter 16. Again these are essentially forward contracts on the future realized variance (or volatility) of an FX rate. The general model free replication results will still hold for the payoff of variance swaps on an FX, however the effect of skew may be slightly different to equities because of the FX smile. The formulas will also change to allow for foreign interest rates instead of dividends. The methods for trading skew using corridor and conditional variance swaps will also be different for this reason. Volatility derivatives in FX again serve the purpose of providing pure exposure to volatility, and as such allow an investor to trade the volatility of an FX rate directly.

18.2 INFLATION

This section deals with inflation as an asset class. The inflation market may not be as liquid as the previous ones, but it does have many uses, and it is a market that has witnessed considerable growth recently and does not appear to be losing momentum. In this section we discuss inflation and focus on the inflation products that exist from inflation swap and bond to inflation derivatives, with the ultimate goal of combining other asset classes with inflation in multi-asset class payoffs. In the literature, pricing inflation derivatives and related products are discussed, among others, by Hughston (1998). Inflation products and in-depth uses and motivations are covered extensively in Deacon *et al.* (2004).

18.2.1 Inflation and the Need for Inflation Products

Inflation is the increase in the price of a basket of goods and services over time. The basket in question is a weighted collection of goods and services that serves as a representative for a whole economy. Essentially, when inflation occurs, the general price of goods rises, which means that the purchasing power of a specific currency is lower, meaning that the real value of a unit of currency is lower. Deflation is the opposite phenomenon referring to the case where the same basket of goods has declined in price.

An inflation rate is a measurement of how inflation has changed over a period of time. To have an inflation rate there must be an inflation index – that is, the dynamically weighted basket as above that is modified to continuously reflect the general price of goods and services in an economy – and these are referred to as "Consumer Price Indices" (CPIs). The percentage changes of such an index are a measurement of the inflation rate, and the main CPI indices of specific countries are typically computed by government agencies – for example, the UK RPI (retail price index), the US CPI, and the Eurozone's HICP. Such indices are widely monitored because the inflation rate is a key consideration for many financial decisions, including the interest rate policies of central banks. These indices are not directly tradeable, but are accessible via futures markets on such indices (for example, futures contracts on the CPI trade on the Chicago Mercantile Exchange). The price of these futures is informative because their prices represent some form of market consensus regarding the index itself. Some inflation index futures are, however, not very long dated.

In the context of inflation, it is key to define the terms *nominal* interest rate and *real* interest rate. The nominal interest rate is a rate of interest that has not been adjusted to account for inflation. Take a government bond that pays 4% per annum, then the rate of 4% is a nominal rate of interest. Assume that an investor purchases such a bond and collects the 4% on top of the notional back after the first year, then it is not necessary for the buying power of the amount of money he currently holds (104% of the notional amount) to have gone up by 4%. If the rate of inflation that same year were 1.5%, then it means the buying power of the new notional is in fact 1.5% less. The real interest rate is (approximately) the difference between the nominal interest rate and the inflation rate for the same period. The real interest rate is thus the inflation-adjusted nominal interest rate, in this case given by $4\% - 1.5\% = 2.5\%$.

Many investors are concerned with the buying power of their money, and even in the case of, for example, a capital guaranteed note, with redemption of at least 100%, the 100% of notional is in fact worthless in real terms if the currency has witnessed (positive) inflation. When inflation occurs, the time value of such a notional is in fact decreasing, and many investors want to be protected against these inflation risks. The need for inflation products thus arises, and the client base is quite large because many investors need to have inflation-linked returns. For example, a large corporation that must give annual salary increases equal to the inflation rate, does not know in advance what the rate of inflation will be for the upcoming year(s), and thus needs a method of hedging this risk. Other examples would be investors who want to take a direct view on an inflation rate going up or down, and need a method of expressing such speculative views.

18.2.2 Inflation Swaps

In a typical inflation swap, two counterparties exchange cashflows where one party pays a fixed (or possibly floating) rate in exchange for a floating rate that is linked to a measure of inflation such as a CPI index. The inflation swap, like other swaps, is entered into at zero cost, which means that the fixed rate side of the swap reflects the market's consensus on inflationary expectations. Inflation swaps are typically computed on the basis of compounded rates, meaning that the payer of the fixed leg will pay a compounded fixed rate on the swap dates and receive the cumulative rate of change in the inflation index. For example, a 5-year swap at a fixed swap rate of 1.78% will pay the rate of 1.78% compounded, on the notional, in exchange for the compounded change in inflation, thus hedging the notional from inflation in exchange for the fixed cost. This is known as a zero coupon inflation swap, and is a fundamental building block in inflation markets.

A different inflation swap would be the year-on-year inflation swap. In this swap, the payer of the fixed rate at the end of each year will pay an amount equal to the notional times this fixed rate, and receive the annual period to period return of an inflation index given by

$$\text{Notional} \times \left(\frac{\text{CPI Index}(n)}{\text{CPI Index}(n-1)} - 1 \right)$$

where n goes from 1 up to the number of years specified in the contract of the swap. This type of swap would be better suited to an investor who needs to receive annual payments linked to the inflation rate that was realized during that year in order to meet some obligation. Additionally it is useful to a speculative investor who believes that the current market consensus on the fair swap rate is high and is willing to pay inflation and receive fixed with the view that inflation will be lower than anticipated.

The case of deflation is also included in the swap, as described above, in that if inflation decreases over one or more years, then the return on the index will be negative and the payer of the fixed leg will end up paying both legs of the swap. To avoid this, a floor can be introduced into the floating leg payments.

18.2.3 Inflation Bonds

Instead of offering a nominal rate of interest that is common to most bonds, an inflation bond's rates are based upon real interest rates. The nominal rate appearing in bonds does not account for inflation and is thus generally higher than the real interest rate yields offered on inflation-linked bonds. Inflation bonds are not new as many sovereign entities have issued inflation-linked bonds for many years. These are known as inflation-indexed bonds, where the notional is indexed to inflation. In the UK, inflation-linked gilts are linked to the UK RPI, and in the US, for example, there are Treasury inflation-protected securities that are linked to the US CPI. OTC inflation-linked bonds offered by some corporations will have a yield that is just the rate of a government issued inflation-linked bond plus a spread corresponding to the creditworthiness of the issuer.

18.2.4 Inflation Derivatives

Inflation swaps and bonds allow investors to gain exposure to inflation, or transfer inflation risk, but more tailored views or needs regarding inflation may be required. This has spurred the development of inflation derivatives that allow parties to transfer inflation risk in many different forms. It is possible to trade options on some CPI indices, although many of these are OTC and one has to rely on broker quotes for prices.

Inflation caps and floors are a good way to take a leveraged view on inflation. Payment from an inflation cap, like the interest rate cap, comes from a series of inflation caplets. The cap can be based on the year-on-year inflation rate, and each caplet has a payoff based on an inflation index I, and the payoff of the ith caplet is given by

$$\text{Caplet}_{\text{payoff}}(i) = \max\left[0, \frac{I(t_i) - I(t_{i-1})}{I(t_{i-1})} - K\right]$$

where the index i runs over all maturity dates for the caplets. Each of these payoffs is essentially a call option on the inflation rate with strike K. The payoff will be multiplied by a notional amount. Other than the first, the caplets based upon year-on-year inflation are forward starting. If the periods over which the caplet is taken are unequal, then the strike can be specified differently as $K \times (t_i - t_{i-1})$ to reflect this. Inflation floors and floorlets are the put option analogy of caps and caplets.

A caplet, as such, is related to the yields of a coupon paying a year-on-year inflation-linked bond if these rates are floored to avoid negative coupons from deflation. Quotes for inflation caps will exhibit an inflation implied volatility smile where looked at across strikes, and this will need to be captured by a modelling assumption to price more elaborate inflation-linked derivatives whose hedges rely on caps of different strikes. The smile can be explained as normal by the market's supply and demand for options of different strikes. Quotes involving implied volatilities will again correspond to prices through Black's model.

The inflation cap can be based on the zero coupon inflation swap, and in this case the caplets are just call options on the zero coupon swaps and are typically spot starting. The more popular option involving the zero coupon inflation swap is the inflation swaption, that is, the analogy of the interest rate swaption. In an inflation receiver swaption, the holder has the right to pay the floating leg of an inflation swap, and in the payer inflation swaption the right to enter into an inflation swap and pay the fixed leg.

18.3 CREDIT

Credit generally refers to borrowing power. In a transaction where one party lends money to another, credit risk arises, and this is the risk that the debtor will be unable to repay the amount of money, the interest charged on it, or both. The borrowing power of an individual or corporation is based upon how much credit risk the party lending the money will have to assume in the transaction. It is the party that owed (or will be owed) money that is subject to this risk and the greater the credit risk, the more the lender will want to be compensated for lending money. Both the ability and the willingness of the borrower to repay are factors.

Credit ratings are used to assess the creditworthiness of an entity, be it an individual, a company or even a country. Credit rating agencies use the levels of assets and liabilities, and the entity's financial history when allocating a rating. Credit ratings will in turn be used by a lender to assess the risk of lending money to such an entity, giving an idea about the borrower's ability to meet whatever obligations there are within a financial transaction.

Credit derivatives are financial contracts involving two counterparties that derive value by removing (at least part of) the credit risk from a financial instrument, and allowing it to be transferred between the two parties. The financial instrument whose credit risk underlies the credit derivative need not involve the two counterparties of the credit derivative itself. These have allowed for tailored solutions for the previously unhedgeable credit risk involved in many facets of business. The credit market witnessed huge growth in recent years, but certain aspects of the credit market were to blame for starting the crash of 2008.

The key instruments in the credit markets are

- Bonds with default risk
- Credit default swaps (CDS)
- Collateralized debt obligations (CDO).

18.3.1 Bonds and Default Risk

In a bond, the issuer must repay the principal to the holder at maturity. The coupon offered on a bond will reflect the issuer's creditworthiness because the holder of the bond has essentially loaned the notional to the issuer, and the coupon will reflect the risk the holder is taking on by buying the bond. The idea behind a bond is that it offers a return but at a lower volatility to an equity for example; however, bonds are not risk free: the buyer still runs the (credit) risk that the issuer of the bond will be unable to repay the notional at maturity (or perhaps not even the interest on the bond).

A higher credit risk of an issuer will generally be reflected in a higher yield on the bonds being issued. Treasury bonds are considered the safest of all bonds because they carry a low probability of default. Other bonds will offer a higher yield, of which the spread over the rate of government bonds will be a function of the risk involved in loaning the money to the issuer along with the length of time the money is to be loaned. The greater the maturity of the bond, the greater the risk of default on the bond and thus the higher the spread. Figure 18.5 gives an example of some yields of bonds of differently rated issuers.

A corporation, for example, may issue a bond that is not backed by collateral, but in the event that the corporation does default, a liquidation follows, and bond holders receive money

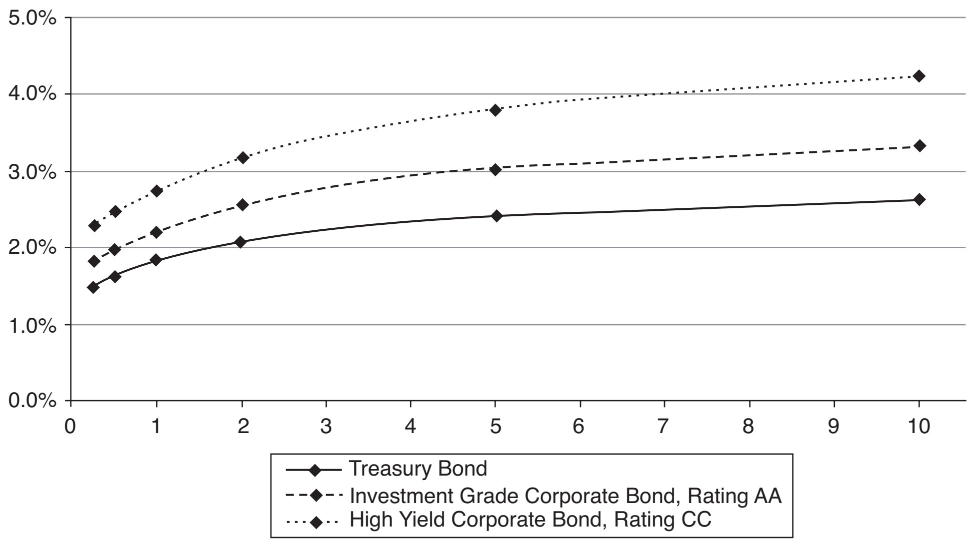

Figure 18.5 This figure depicts the yields of bonds along maturity for three differently rated bonds.

based on seniority. In some cases they may receive all their money, in others only cents on the dollar. The greater this risk, the greater the investor (holder) in the bond will expect to be compensated through higher coupons on the bond.

Junk bonds are higher yielding bonds that are rated as "below investment grade" by the relevant credit-rating agencies. For a bond to be investment grade at the time of issuance, the credit-rating agencies must rate the issuer as likely to be able to meet all related obligations. Junk bonds do not fall into this category, and are judged at the time of issuance to be more risky than investment grade bonds. The higher yield on such bonds reflects this.

18.3.2 Credit Default Swaps

A credit default swap (CDS) is an OTC credit derivative based upon the debt of a company, a sovereign entity, and possibly a credit index such as the iTraxx or the Dow Jones. The buyer of the CDS makes some form of payment to the seller (this leg is known as the premium leg), in exchange for a payoff in the event the issuer of the underlying debt defaults (known as the default leg). In the event of default, the holder of the CDS on debt in the form of a bond, has bought the right to sell the bond, at par, to the party who sold the CDS protection.

The price of a CDS that insures a notional amount of a specific debt, is given by the percentage of this notional that the seller of the CDS will receive, on an annual basis, in exchange for offering this protection. This percentage is referred to as a CDS *spread*. The annual payments typically continue until either the CDS contract expires, or a default occurs.

A CDS can thus serve as a tool for managing credit (default) risk by protecting invested notionals from the event of a default. For example, if an investor is long a corporate bond, then a CDS can provide a hedge that offsets the losses incurred if the issuing corporation defaults on the bond. A *negative basis trade* is a single-name credit trading strategy in which the investor buys both a bond and the CDS on the same name. When the CDS spread is less than the spread on the bond, the investor receives a spread while being protected from default risk.

The CDS spread on the debt of an issuing entity will generally be greater the less creditworthy is the issuer. Vice versa, as an entity's creditworthiness increases, the CDS spread on its debt will decrease. The greater the CDS spread of an entity, the greater the market perceives the possibility of the entity defaulting. For example: the cost of protecting $1,000,000 of debt issued by a corporation can be 25 bp per annum, which means that the buyer of a CDS on this notional of debt will have to pay the seller of the CDS $2,500 per year. If this corporation were to be downgraded by the rating agencies, then this CDS spread will increase, reflecting the greater risk in insuring its debt. CDSs thus allow an investor to take a speculative view on an entity's creditworthiness and the probability that he will or will not default. This is also made easier because CDS contracts can be cash settled.

CDS Indices

The popularity of CDSs has spurred the development of tradeable CDS indices that are standardized credit securities, compared to CDSs that are OTC derivatives. These indices allow for exposure to multiple credits, and can thus be used to hedge credit risk or take speculative positions in a range of credits. The standardization of these indices is key to enhancing their liquidity, allowing for investors to take positions for small bid–ask spreads. This makes trading a CDS index the more cost-efficient method to hedge or take a view on a basket of CDSs, compared to buying all the CDSs separately.

The composition of a CDS index is designed to offer specific exposure to underlyings from a sector, region or similar credit quality. Each has an index methodology, typically focused on the liquidity of the underlying CDSs. Based upon the triggering of a specific credit event on one of the names, it may be removed from the index. Futures are traded on such indices, through an exchange, and these are cash settled on expiry. Additionally, OTC derivatives are traded on the CDS indices. Two examples of such indices are the iTraxx and the CDX. For a comprehensive discussion of credit derivatives we refer the reader to O'Kane (2008).

19
Structuring Hybrid Derivatives

Where there is no strife there is decay: The mixture which is not shaken decomposes.
Heraclitus

In this chapter we combine the asset classes we have seen so far, and give examples of various hybrids. A hybrid product serves as an investment vehicle with a specific payoff and risk profile that is based upon underlyings from various asset classes, compared to a multi-asset equity derivative whose underlyings are from the same asset class. The functions of hybrid derivatives are no different to the equity derivatives seen throughout the book, with assets from multi-asset classes now replacing multiple equities, and the standard breakdown of uses for hybrid derivatives follows suit. Hybrids can serve various purposes including diversification, yield enhancement, serve as speculation vehicles for multi-asset class views and as hedging instruments for multi-asset class risk. It is possible to combine these asset classes in a suitable payoff to serve any of these purposes. In the literature, Overhaus *et al.* (2007) discuss hybrid derivatives with a focus on pricing models, and Hunter and Picot (2005/2006) discuss the functionality of hybrid derivatives.

We have at our disposal the wealth of possible structures and multi-asset payoffs addressed in the previous chapters, many, if not all, of which can be extended to work on multiple asset classes – with some caution and a dose of common sense. The possible range of investors is thus quite broad. Hybrids can be structured for retail clients of banks seeking features such as capital protection, yield enhancement or risk reduction. Institutional investors can also obtain hybrid products to express their macro-economic views in a single instrument through a multi-asset class option – however specific these views. Additionally, any corporation or portfolio manager with multi-asset class exposures can use hybrid derivatives to hedge multi-asset class risk in an efficient manner and through a single derivative.

Following suit with all the multi-asset options we have seen so far, the investor might be taking a view on a set of paths for assets of various asset classes, but in different structures the investor can be long or short the forwards of the underlyings, their volatilities and the correlations between them. In an outperformance option, the long investor is short the forward of one underlying and long the other, and also short the correlation between the underlyings of the outperformance. In a basket call option, for example, the buyer of the option is long both the volatilities of the underlyings, the correlations between them and the forwards of the underlying assets. A hybrid derivative can be constructed to express a view on any one or more of these.

We break the chapter down into sections, each based upon one of the motivations behind the creation of hybrid derivatives. Multiple examples involving possible structures are given in each section.

19.1 DIVERSIFICATION

In Chapter 7 we discussed the concept of horizontal diversification, which involves the use of multiple assets of a similar type. When combining assets from different asset classes, we now refer to this form of diversification as *vertical*. A *horizontal* diversification can reduce the risk to one single asset, but when considering multiple stocks, for example, its effectiveness is weakened when all such stocks are impacted in the same manner by economic events, and adding additional stocks to a portfolio of stocks will not reduce this risk.

The risk reduction obtained from adding multiple assets to a portfolio is not only dependent on the individual asset volatilities, but also on the correlations between these assets. Recall the approximate formula for the variance of a portfolio:

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

By selecting assets from multi-asset classes one can find assets that have relatively low historical correlations, compared to those observed among equities. Although there is no guarantee that these correlations will stay low, they do provide a more diversified portfolio than the horizontal diversification case. In recent years, globalization and other factors have contributed to the increases in the correlations between various assets from both different and identical asset classes. However, there exist economic reasons as to why we expect certain asset classes to be affected very differently by economic events, and their cycles to look quite different. As such, using multi-asset classes to diversify, should over time offer enhanced and consistent returns, while reducing the overall risk.

19.1.1 Multi-asset Class Basket Options

The simplest multi-asset class option that serves this purpose is the basket option discussed in section 7.3 of Chapter 7, where the option's payoff is contingent on the performance of a basket of assets, here from multi-asset classes. The effects of volatility and correlation on the basket option hold even if the assets are from different asset classes. The more non-perfectly correlated assets we add, the more the overall volatility of the basket is reduced and the price of a basket call or put option decreases accordingly. The volatility of the returns of an asset will affect that of the basket and thus the decision to add it as an additional asset class depends on this volatility. The choice to add an additional asset class to such a basket to enhance this diversification effect will also be based on its correlation with the remaining elements of the basket. The seller of a call or put option on a basket will be short both the volatilities of each of the components and the correlations between them. The weights of individual assets in the basket option do not have to be equal, but their sum must be unity.

Even if an investor were bullish on the US economy, buying a single option on the S&P index, or even a basket option on a number of US stocks, the risk of a market crash that would decrease all such equities still remains. This risk can only be removed by adding an additional asset class, for example gold. The basket of the S&P 500 index (equities) and the USD price of gold (commodities), in different weights even, will lower the volatility compared to both individual options. The correlation between gold and equities is not necessarily negative, but gold, which is generally considered to be a safe haven, will offer quite different returns to equities, particularly during a crash in equities.

A basket option combining the two, along with some form of averaging to capture the possible different market regimes and include these quite possibly different returns, would be a much more diversified bet. A simple capital protected structure combined with the call option on just the S&P 500 would not cost the investor any money. However, the downside of possibly not getting any return above the 100% guaranteed notional is lowered at the cost of a perhaps lower return, should the equity markets rally in a big way.

Another example can be to combine an equity index (EuroStoxx50), a commodity index (Dow Jones AIG commodity index), and a bond index (iBoxx). These three possible underlyings have very different return structures and volatilities, making them an interesting combination to have in a diversification instrument. Allowing for some averaging will allow the basket to pick up these differences more easily.

The correlations between equities and interest rates go through periods of different levels, but historically bonds exhibit a low correlation to equities. Bonds are typically used along with a portfolio of equities to lower the overall volatility and benefit from this low correlation, which allows for diversification. Additionally, bonds will still provide some income, compared to using just cash. Though the potential returns from bonds are lower than returns from the other two asset classes, the inclusion of bonds will help to generate at least some return in the event that equities perform quite poorly.

The correlation between equities and interest rates is affected by many things, including inflation and the business cycle. The addition of the commodity index serves as an inflationary hedge in the basket. Historically, commodity indices are positively correlated with inflation, but not strongly correlated with either bonds or equities, and as such the basket is quite well diversified.

Additionally, one can enter inflation directly into the payout of the basket as an additional asset to enhance diversification. Inflation has low to negative historical correlations with equities, although inflation does exhibit strong correlations to commodities such as oil. Alternatively, the deal can be structured to include some form of inflation bond, in addition to a capital guarantee feature to also protect against rising inflation.

19.1.2 Multi-asset Class Himalaya

The Himalaya structure described in section 15.2 of Chapter 15 on mountain range options, allows the investor to lock-in the performance of an asset before it is subsequently removed from the basket. This type of payoff can be quite well suited to assets that have very different return structures because it will allow the investor to pick up these different returns as the market cycles change. This can be more suited to investors who are looking for a different exposure and possible set of returns on the options than the simple basket.

An example would be a 3-year Himalaya option for the investor who believes that a bear market will persist in the short–medium term (oil to possibly outperform US equities and bonds), but is bullish long term (US equities and bonds to outperform oil), and wants to obtain a return each year above a fixed market rate. Additionally, the USD–EUR rate can be added (USD to appreciate), in relevance to its historically negative correlation with oil prices, adding diversification. The Himalaya on a basket of the four assets of US equities, US bonds, USD vs EUR (USD to appreciate), and oil, works by taking the assets with the best return each year, locking in this return, and removing this asset from the basket. Payments can be made annually to make this into an income product, or the returns paid at maturity as some weighted average. Other mountain range options from Chapter 15 can also be structured to include multi-asset classes instead of just multiple equity underlyings.

19.2 YIELD ENHANCEMENT

The next possible use for hybrids would be to benefit from the possible low correlations between some assets from different asset classes in order to enhance yields. The idea behind yield enhancement is to form a payoff of multi-asset classes that increases the leverage to offer a higher expected return but with increased risk. Investors seeking higher yields in light of the low interest rate environment currently observed in the global economy can turn to hybrids as a possibility. The possible set of enhanced yields will obviously be a function of how much additional risk is taken, but we differentiate between an investor seeking a higher yield through hybrids and an investor who is taking an aggressive and specific view on multi-asset classes.

Again we have at our disposal many yield enhancing payoffs from the multi-asset equity cases discussed in previous chapters. Other than explicit dispersion payoffs we also have the entire mountain range option that could potentially be extended to include multi-asset classes. These serve a different purpose to diversification and use multiple assets in order to increase leverage as discussed in Chapters 8 and 9. Barriers and digitals on multi-asset classes can also provide enhanced yields. We also make use of callable features that can make the prices of such options more favourable to the buyer, and thus a callable feature can allow for an increased participation for the same price as the non-callable version with a lower participation.

19.2.1 Rainbows

The rainbow, discussed in Chapter 9 under dispersion options, pays a weighted average of the performances of a basket of underlying assets where the weights are specified according to how the assets performed during the life of the option. These lie between basket options and best-of and worst-of options – depending on how the weights are distributed. For example, a 50%, 30%, 20% rainbow that pays 50% of the best performing asset, 30% of the second best, and 20% of the third lies between a best-of option (100% on the best performing) and the equally weighted basket option. This will be priced higher than the basket option but offers a better return, reflected in this higher premium, and still serves the purpose of diversification.

To leverage the rainbow payout to serve yield enhancement is to make it a more leveraged instrument than the diversifying basket option. Lower weights can be placed on the best performing in order to lower the price and increase leverage. In this case the rainbow behaves more like a worst-of option, and the resulting lower price can allow for a higher participation rate and lead to potentially enhanced returns in exchange for the increased risk.

To have a 3-coloured rainbow option, the underlying basket will need to consist of at least three different assets. For example, a retail product providing emerging market exposure along with some asset class diversity can be constructed by adding a rainbow option to a capital guaranteed note. Taking Brazil as an example of an emerging market that has performed well in recent times, especially with respect to similar economies, one can have a 50%, 30%, 20% rainbow option on a basket consisting of the Brazilian equity index the Bovespa, the Brazil Real versus the USD (BRL to appreciate w.r.t. the USD), and a commodity such as oil (WTI futures, for example). This provides a diverse exposure, and will definitely be cheaper than the sum of single options on each of the underlyings, but will have a higher expected yield than the basket option.

To use the rainbow as a yield enhancer, for example, consider a bullish view on the global economy to recover with equities to rise, real estate to rise and oil to rise. This view can be structured into a leveraged payoff by taking a 25%, 35%, 40% rainbow on the basket of these three, for example the S&P 500, the EPRA real estate index and WTI futures. The 25% weighting on the best performing, compared to the 40% weight on the worst, makes this rainbow behave more like a worst-of option. This rainbow will thus be cheaper than the call option on the basket (assuming equal weights) and participation in it could be increased to offer higher leverage for the same cost. A more extreme leveraging would be to simply take 100% weight on the worst-of, making it a worst-of option.

The correlations and volatilities in this example would be higher than the above basket, and the return structures more similar. The seller of a rainbow option, however, is not necessarily short correlation. A higher correlation adds to the overall volatility of the basket of underlying assets, however it also increases the effect of dispersion; the net effect depending on the weights of the rainbow. The discussion of Chapter 9 holds in regard to volatility and correlation in rainbow options. The skew effect must also be looked at, especially because now the assets included may have very differently behaving implied volatility skews – or smiles. This is discussed in further detail when modelling multi-asset class skews in the next chapter on the pricing of hybrid derivatives.

19.2.2 In- and Out-barriers

Adding barriers to payoffs as we saw in Chapter 10 can also reduce the price and offer a similar yield but with higher risk reflected in the decreased premium. A simple knock-out call option is cheaper than its vanilla counterpart, reflected in the higher risk that the option will knock out. An investor willing to take potentially enhanced yields in exchange for a view that the underlying will follow a specific path, can include barriers to decrease prices and increase leverage.

We refer to the case of an in-barrier for the barrier option in which the barrier is triggered by an underlying that is included in the final payout (for example, a call option on a basket with a knock-out clause on one of the underlyings), and the out-barrier for the case where the option's knock-out (or knock-in) clause is specified on an asset not included in the payout (for example, a knock-out call option on S&P 500 that knocks out if interest rates breach a certain barrier). Interesting examples of both can be constructed using multi-asset classes.

An example of a hybrid derivative with a knock-out in-barrier can be obtained by adding the knock-out feature on the first basket example of equities and gold. The barrier can be specified on the basket or on one of the individual assets; for example, a 3-year basket option that pays the weighted average performance of the basket of S&P 500 and gold, as long as the price of gold does not go above 140% of its price at the onset. The addition of the barrier decreases the price of the option reflecting the added knock-out risk, and the participation can be increased and the potential yield thus enhanced, in exchange for this additional risk.

A hybrid option with an out-barrier can, for example, be an ATM call option on the S&P 500, with a knock-out barrier on gold. The investor is bullish on US equities, but also believes that even if gold increases it won't go above 130% of its initial value during the life of the option. As long as gold never breaches this barrier on any of the observation periods for the barrier, the option is still alive. The more frequent the observation on gold, the more possible the barrier will be breached and the S&P 500 call option knocked out, and the cheaper the option. Lowering the barrier closer to the 100% level has the same effect.

19.2.3 Multi-asset Class Digitals

Digital options can be structured to involve a digital view on multi-asset classes. For example, an option that pays a fixed coupon of $X\%$ if both oil and equities are above their current levels in a year from now represents a bullish view on both asset classes. At maturity T the payoff is given by

$$\text{Double Digital}_{\text{payoff}} = X\% \times \mathbf{1}_{\{\text{S\&P 500}(T)>\text{S\&P 500}(0)\}} \times \mathbf{1}_{\{\text{Oil}(T)>\text{Oil}(0)\}}$$

In this payoff the digital strike for each asset is the same; we are comparing time T prices of the assets with time 0 prices. When this is the case, the payoff is in fact a worst-of digital, which has been discussed in Chapter 11. The strikes, however, need not be equal, and one could compare with 110% of the time 0 price for example, of one asset or both.

The higher the correlation between these two assets, the more chance the option has of ending in-the-money, and the seller of the option is thus short this correlation. The seller is short or long the individual volatilities depending on the position of the forwards. This option will be worth less than the sum of two individual ATM digital call options, so taking the combined view will allow for enhanced leverage. The oil section of the hybrid could be based upon the price of WTI futures, of a specific maturity, increasing during the life of the hybrid. The investor would as such be taking a view on the forward curve. The same analysis of digitals involving multiple equities holds here with regard to the impact of forward price movements on the digital, with attention to the possibility of different smiles and skews.

19.2.4 Multi-asset Range Accruals

By bounding one or more of the underlyings in a range and considering a range accrual on one or more of the possible asset classes, we can again enhance leverage. For example, a payoff can consist of a fixed coupon, paid at maturity, multiplied by the percentage of days that each of two assets spend within their individually specified range. By subjecting the coupon to two assets, the potential coupons, being range bound, will be relatively higher. For example, consider an investor who believes that US equities will rise but not by more than 20%, and that the USD could appreciate versus the EUR but not gain more than 15% against the EUR in the next year. Instead of buying options with barriers or outright digitals, the investor can play both ranges through a double range accrual that only accrues on the days when both these underlyings are within their respective ranges. The leverage will be quite high, and the offered coupon that is multiplied by the percentage of days that both are within the range, would look quite appealing.

Callable Dual Range Accrual: Oil and Equities

Adding a callable feature to an option, where the option is callable by the seller, will make the price more favourable for the buyer. In the case of a callable range accrual on two assets, a more appealing coupon can be offered based upon the same ranges. The idea of the callable feature is the same as the autocallable option of Chapter 12 in that the investor hopes to obtain an above-market fixed rate coupon and the structure to be called in order to get his money back early.

Take an investor who believes that a recession will persist for some time and also believes that oil and equities will stay bound within the 90–115% range for at least the next year and possibly up to 18 months. A semi-annually callable dual range accrual of 18-month maturity will capture this. The investor pays LIBOR in exchange for this potentially above-market coupon, computed semi-annually and paid on maturity, or an earlier date if called.

19.3 MULTI-ASSET CLASS VIEWS

To construct some of the hybrid derivatives seen so far, some market views on multi-asset classes were necessary. Here we describe some slightly more detailed views as examples for which we structure hybrids offering the exact required exposure. The hybrid derivative can be structured as a cheaper alternative to expressing each asset class view separately.

ICBC–CMS Steepener Hybrid

As a first example, consider an investor who is bullish on a specific basket of stocks and also believes that the yield curve will steepen, specifically the difference between the 30-year and 10-year CMS rates. A cheap structure to express the bullish view on the basket is the ICBC of section 9.2 in Chapter 9 where the returns of each stock in the basket are capped from above at a fixed cap, and the returns are then averaged to form the payout. Additionally, the view on the yield curve can be expressed through a CMS steepener option described in section 17.1.9. Combining both of these into a best-of option allows the investor to express this view in a combined manner that is cheaper than two separate views, and in one payoff that will offer a return even if one of these two parts ends out-of-the-money. A payoff for such an option with maturity T could look like

$$\max\left[0,\ \frac{1}{N}\sum_{j=1}^{N}\min\left[\mathrm{Ret}_j(T),\ \mathrm{Cap}\right],\ X\%\times\left(\mathrm{CMS}_{30\mathrm{y}}(T)-\mathrm{CMS}_{10\mathrm{y}}(T)-K\right)\right]$$

where $\mathrm{Ret}_j(T)$ is the return of the jth stock among the N stocks of the basket, Cap is the prespecified cap rate and this first part makes up the equity part of the hybrid. In the second component, the two rates are just the time T 30-year and 10-year CMS rates. X is to add increased gearing to the interest rate section of the hybrid, and K is the strike for the CMS steepener, which can be specified so that this part of the option also starts from zero. The zero at the start of the payoff acts as a global floor.

The option contains one parameter for each section that can be modified either (a) to reach a specific overall price for the hybrid, or (b) to increase the potential upside on one of the parts at the cost of lowered potential upside in the other, in the case where we want the price to remain the same. The cap of the ICBC serves as this parameter in the equity part, and the strike of the CMS steepener for the rates part. The gearing of the CMS can also be modified for this purpose. Even though this is a best-of option, and best-of can be expensive (compared to a basket option on the same underlyings), the cap on the ICBC can greatly reduce the price contribution of the equity part (compared to a capped basket call), and the strike and gearing on the CMS part can reduce the interest rate part by lowering the participation or setting it out-of-the-money.

Equity Reverse Geared Basket of Oil and CHF

An investor believes that instability in the middle east in the next year will cause oil prices to rise. Additionally the investor thinks that the CHF (Swiss franc) will appreciate with respect to the EUR, because the Swiss franc has historically been considered a safe haven during times of uncertainty. The investor also expects that this may have a negative impact on US equity markets, and is willing to include an equity view as part of a gearing on an option on the

CHF–EUR rate and oil.

$$\left(200\% - \min\left(\frac{\text{S\&P 500}(T)}{\text{S\&P 500}(0)}, 100\%\right)\right) \times \max\left[0, \frac{\text{Ret}_{\text{oil}}(T) + \text{Ret}_{\text{FX}}(T)}{2}\right]$$

For the investor, this option provides a tailored interpretation of the suggested view. The reverse gearing adds the investor's view that equities may decline, and the participation according to the above payoff is bounded between 100% and 200%, where the higher participation corresponds to a decrease in the S&P 500. The equity gearing will enhance the leverage of this view, and depending on the premium the investor is willing to spend, the basket can be made into a rainbow (whether moving towards a best-of or a worst-of) on the oil and FX rate part of the hybrid. One can set the oil and FX part of the hybrid to be written on an oil future and an FX forward so that the option is more in line with the liquid vanilla options of each of these.

Tail End of Economic Cycle: Equity – Inverse Floating Rate Hybrid

An investor believes that we are at the tail end of an economic cycle, and wants to take the view that, over the next 2 years, equities may still perform well but that equities will begin to fall at some point and this will be accompanied by a decrease in interest rates. The investor expects a weakening of the economy and interest rate cuts by the US Federal Reserve. The investor also believes that at this point rates could stay low for at least another year.

A 3-year cliquet style trigger option can be constructed where the S&P 500 is observed on a quarterly basis, and if a quarter has a negative return, the investor is then entered into an interest rate inverse floater on the USD 3-month LIBOR. The inverse floater will accrue on a quarterly basis, with a coupon that is inversely related to this reference rate. At the point where the negative equity return occurs, the equity exposure is cancelled while quarterly returns are locked in in a period-to-period cliquet style. The investor continues to accrue returns from the inverse floating rate till maturity.

The equity part involves an ATM quarterly cliquet and the structure locks in positive returns, up till the first negative one, and a cap can be included to cheapen the structure if necessary. The reverse floating part of the structure can be floored at zero and have a quarterly coupon of the form

$$\text{Floating Rate} = \max\left[0, 5\% - 2 \times \text{3-month USD LIBOR}\right]$$

where the components are as described in the inverse floating rate note of section 17.1.3. The 5% fixed rate and the gearing of 2 can be adjusted to better fit both the view of the investor and modify the price if needed. The floor in the payout turns this into a geared put option on the LIBOR rate, which is just a geared interest rate floor. The floor can also be adjusted to increase the leverage on the interest rate part.

Emerging Market Currencies, Equities and Default

An investor believes that several emerging market countries will run into serious problems in the next 2 years, primarily because their currencies and equity markets have declined severely, and that a big portion of their large amount of debt accumulated recently is in USD, against which their currencies have declined. The investor is quite bearish on their equity indices and currencies, and wants to take such a view. Additionally, the investor believes that some of these countries will be unable to repay these debts and will default, under which scenario the investor wants to receive an additional income for each default on sovereign debt that occurs among the countries of the corresponding basket of emerging market indices.

A downside put on the basket or a rainbow style modification would capture the downward view on the equities and exchange rates, and a fixed coupon can be specified additionally for each default that occurs. The basket put or rainbow can be struck OTM to lower the cost of the structure and in turn possibly serve to enhance participation.

Oil-Geared Equity Outperformance

The investor is more bullish on the Eurozone than the US in the next 4 years, and believes that the EuroStoxx50 will outperform the S&P 500 over this period and, additionally, that oil prices will rise. Making use of the outperformance payoff, we can structure an outperformance option of the EuroStoxx50 versus the S&P 500 and gear it by the price of oil. The payoff for a maturity T will look like

$$\frac{\text{Oil}(T)}{\text{Oil}(0)} \times \max\left[0, \frac{\text{S\&P 500}(T)}{\text{S\&P 500}(0)} - \frac{\text{Stoxx50}(T)}{\text{Stoxx50}(0)}\right]$$

19.4 MULTI-ASSET CLASS RISK HEDGING

Last but not least we describe how hybrid derivatives can be used as a single instrument to hedge risks from multi-asset classes. An investor looking to hedge a portfolio covering multi-asset classes would typically hedge each of these separately; however, depending on the nature of the portfolio, it may be possible to hedge with just one hybrid option. The hybrid derivative will cost less than the sum of the individual hedges and still provide the required hedge for the portfolio of multi-asset classes as a whole. This is not magic: the reality is that hedging each asset class separately is an over-hedge that does not account for correlations in the returns of the various assets.

Protective Multi-asset Class Puts

Consider an investor who holds a portfolio of commodities and equities, and wants to protect the value of the portfolio against a decline in its value. A fall in the price of either can potentially lower the value of this portfolio, and the traditional hedge would be to buy protective puts on each asset class component. The inclusion of put options reduces the overall portfolio risk because the puts will offer downside protection when the market moves against the long portfolio.

The cheaper but equally effective hedge is to buy a hybrid that serves as a put to protect the entire portfolio of these two asset classes, instead of the two separate puts. As long as correlation is less than 1, which it generally will be, the hybrid hedge will be cheaper and involves only one transaction. The strikes of the put options in the naive hedge on each component are chosen according to how much risk the investor is willing to take on the long portfolio, and the amount of put options is a function of the weights of the underlying components of the portfolio. Likewise, the weights in the hybrid hedge reflect those of the portfolio and the hybrid hedge on the portfolio will again be struck at the level beyond which the investor wants to hedge the downside of the entire portfolio.

For example, assume that the investor is long a portfolio of equities and commodities, and is not willing to lose more than 30% of the portfolio's value, then he can buy a hybrid put option on the weighted basket of the components of the portfolio with a strike of 70%. If the value of the portfolio drops by more than 30% then the investor can exercise the right from the put option to sell the portfolio at 70% even if it has dropped further. The notional of the hybrid put will need to match that of the portfolio for this to work, and the investor cannot lose more than 30% on this portfolio.

Inflation and Downside Equity Protection

Payoffs can also be constructed to protect the portfolio of an equity investor from inflation risks. If the equity portfolio performs well but inflation is high, then the effective rate of return will be less. Additionally the investor wants to protect the portfolio from downside risk, in which scenario inflation is not the primary concern. A best-of option combining a call option on an inflation index and a put option on the equity portfolio will provide a hedge for whichever of these two scenarios occurs. An example of a payoff would be

$$\max\left[0, (K - 100\%) - \text{Portfolio}_{\text{Ret}}(T),\ \text{Inflation}_{\text{Ret}}(T)\right]$$

where T is the maturity of the option and represents the duration of time over which this hedge applies. The notional of the trade should match that of the portfolio, and the gearing of both parts of the hybrid kept the same in order that the protection is in line with the portfolio's value.

The first part involves the put feature on the equity portfolio, which only ends in-the-money if the performance of the portfolio is lower than the strike. In the event where it does decline beyond this strike, the investor can sell the equity position at the strike K at maturity, and not lose more than the difference $100\% - K$. If the equity portfolio were above the strike, and ideally above the ATM point, any excess returns would be unaffected by inflation because the put feature is out-of-the-money and the positive performance (if any) of inflation is paid through the structure to hedge this risk.

20
Pricing Hybrid Derivatives

Common sense is that which judges the things given to it by other senses.

Leonardo da Vinci

In this chapter we discuss the pricing of hybrid derivatives. We have already seen various structures in the discussion of asset classes and here we discuss the various risks and modelling issues by providing modelling frameworks for each asset class. In the discussion of each asset class we came across the market standards, and Black's formula came up more than once as the standard formula for which market-implied volatilities are quoted accordingly. In this chapter we venture into the models that allow for more elaborate options in the various asset classes to be priced. The focus is kept on the models and exotic products that are of direct relevance to the pricing of hybrid derivatives that involve more than one asset class, drawing on Chapter 19 for examples of structures.

As always, when we specify a model, we are exposed to model risk. As discussed in Chapter 4 in the context of volatility models, the choice of which models to use depends on the different risks involved in the option. If the payoff exhibits convexity to the price of one of its underlying assets' prices, then we should take the underlying's price to be a random process. Again, the model inputs must be correctly specified, and will be those that are relevant to the hedging of the option. We point out that what we have learned regarding hedging throughout the book is applicable to hybrid derivatives. Additionally, the liquidity of the various assets from each asset class, and the ability to trade individual options within these asset classes is also paramount to one's ability to Delta, Vega and Gamma hedge an option.

A general problem with pricing hybrid derivatives is that correlations between the various asset classes cannot be implied from liquid instruments and is very difficult to hedge. Understanding how each of these correlations affects the price of a hybrid is important, specifically the magnitude of the correlation sensitivity of the hybrid and whether the seller of the hybrid is long or short these correlations. As always, the price of the derivative must reflect the cost of hedging it. The parameters used in the model will be chosen while bearing in mind any residual risks that cannot be hedged, and technical margins based on these risks will need to be taken. We also discuss copulas that are increasingly popular and can be necessary in hybrid derivatives. Copulas allow us to formulate multi-variate distributions using separate processes for each asset class, allowing for different types of dependence to be modelled. Under copulas, assets can be correlated in more meaningful ways, and the result is more meaningful hedge ratios.

20.1 ADDITIONAL ASSET CLASS MODELS

20.1.1 Interest Rate Modelling

There is more than one approach to interest rate modelling. The decision of which approach will depend on the structure and risks entailed in the product being priced. Possible choices of which rate to model include the short rate, LIBOR rates and forward LIBOR rates. In the market we know today's yield curve, the prices (implied volatilities) of both swaptions and interest rate caps and floors, and these can all serve as calibration instruments. If an instrument, for example an OTM swaption, is to be used in the hedging of an exotic option, then it must appear in the model's calibration in order that the model shows risk against it and includes this in the pricing. The two modelling frameworks we discuss here are short-rate models and market models.

Short-Rate Models

The price of a zero coupon bond $B(t, T)$, at time t and maturity T, is related to the instantaneous interest rate r_t by

$$B(t, T) = E\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\right] \tag{20.1}$$

The rate r_t is known as the short rate, and is the interest rate prevailing over a very small period of time. One can think of the short rate as the interest rate cost of borrowing money from time t to time $t + \mathrm{d}t$, where $\mathrm{d}t$ is a small increment. A short-rate model describes the evolution of this short rate r_t as a random variable.

The dynamics of a short-rate model are given by its drift and volatility function; these uniquely specify the first and second moments of the process. In short-rate models, the drift is often taken to be mean-reverting so that the process reverts to a long-term mean, reflecting the view that interest rates are mean-reverting. The volatility function specifies whether the process in question is normal, log-normal or something else (for example, a square root process).

In equation (20.1) the relationship between the short rate and bonds means that if we specify a process for the short rate, then the prices of all such bonds are given by the paths of this short-rate process. Bond prices are thus a function of the parameters of the model, and one step in the calibration of short-rate models is to make sure that the prices of bonds generated by the model match as closely as possible those observed in the market through the yield curve. Time dependence is often introduced to the drift of short-rate processes to allow for perfect fits to the initial yield curve.

Time dependency in the volatility structure is also necessary in a single-factor short-rate model in order that it can also be calibrated to a set of ATM swaptions or caps, in order to complete the calibration. One-factor short-rate models do suffer from the fact that with a single driving source of randomness, forward rates are perfectly correlated in the model; a contradiction to market observations. This renders such models unsuitable for pricing interest rate structures that are sensitive to the correlations between forward rates. Including additional factors by allowing the drift or the volatility to be random is necessary to calibrate such a model to an entire swaption cube (strike and maturities). The additional factor will allow for a richer volatility structure and allow us to calibrate to swaptions or caps of different strikes should this be necessary.

Despite the fact that the short rate itself is not observable in the market, these models have the distinct advantage of being quite tractable, making them handy for risk management purposes. Additionally, these models lend themselves to tree implementations that in turn allow for the pricing of more exotic structures. For example, the Bermudan swaption of section 17.1 can thus be priced using backward induction through a tree implementation of a short-rate model. Structures that involve early exercise features like American and Bermudan style options are priced as backward-looking structures.

One starts at the maturity of the option and works backwards in order to find the optimal exercise points of the option making the pricing of early exercise features possible. Trees allow for this, whereas Monte Carlo implementations, that are by nature forward looking, are not best suited for pricing options with early exercise features, although adjustments can be made.

A commonly used short-rate model is the Hull–White model (see Hull and White, 1990). Technical details of this model and some extensions of it are given in Appendix A, section A.5.

LIBOR Market Models

Market models are a quite different class of models, and in these the variable being modelled is directly observable in the market. The most successful of these is the LIBOR Market Model, abbreviated to LMM and also known as BGM after (some of) its pioneers (Brace *et al.*, 1997; Jamshidian, 1997). In an LMM the underlying variables modelled are a set of forward LIBOR rates, all of which are observable in the market, compared to the short rate that is not.

Additionally, these forward LIBOR rates are the underlyings of liquidly traded interest rate derivatives which means that their volatilities are also observable and can be implied from such options. The modelling assumption of the log-normality of each of the forward rates means that the prices of vanilla options will be given by Black's formula, and the LMM thus consistent with the market standard for the pricing of such options (caps, floors and swaptions). Short-rate models, on the other hand, do not offer such features.

A LIBOR rate $L(t, T)$, expiring at time t and paying at time T is the underlying in a caplet, a series of which forms the interest rate cap described in section 17.1.7. The LMM consistently models a whole set of n forward rates $L_i(T_i, T_{i+1})$, $i = 1, ..., n$, each corresponding to a different period. Black's formula for caps is recovered from the model for each of the forward rates being modelled, and the implied volatilities of the corresponding options are readily available to serve as calibration instruments. The key difference between the LMM and Black's model is that the LMM can consistently model an entire set of forward rates, compared to Black's model that takes a single forward rate as the underlying.

This modelling framework can also be applied to model forward swap rates, and the model in this case recovers Black's formula for European swaptions. The forward swap rate is again observable, and European swaptions are liquidly traded instruments, making calibration to interest rate swaptions instead of caps also possible.

The LMM lends itself in a more natural way than short-rate models to being calibrated to many traded instruments. The framework models a whole set of forward rates, each as a random process, and needs the correlations between these forward rates to be specified. This is also a key difference between the LMM and short rate specifications: The forward rates in a short-rate model are perfectly correlated, whereas the LMM allows for a *de-correlation* of such rates making it much more realistic, and applicable to exotics that are sensitive to the correlations between forward rates. The choice between using swaptions or caps in the calibration depends on the instruments with which we want the model to show risk and be consistent, in order to correctly price an exotic structure.

Pricing using an LMM is done using Monte Carlo simulation and the model is thus well suited for structures that are forward looking. At any point in time the simulation includes the history of each path up to that point, and thus its relevance to the exotic structure in question.

The LMM framework, in its simplest form, does not capture the interest rate skew. Stochastic volatility and local volatility extensions of the LMM do exist (see, for example, Rebonato, 2007). An interesting combination is that of the SABR model with the LMM, particularly because the SABR model is a market standard for quoting implied volatility skews in swaptions or caps. Making the LMM consistent with the SABR pricing formula will allow for a stochastic volatility extension of the model that will combine the benefits of both these models.

Interest Rate/Equity Hybrids

The interest rate modelling framework choice depends on the interest rate part of the hybrid derivative. We use the two equity/interest rate hybrids described in section 19.3 of Chapter 19: the equity/inverse floating rate hybrid, and the ICBC–CMS steepener hybrid. In both cases one will need to allow for randomness in both the interest rate and the equity, but in a quite different manner in each case.

The equity/inverse floating rate hybrid of section 19.3 is a 3-year structure which, from the onset, is a quarterly cliquet, with a local floor ATM and a local cap. The returns from the cliquet are locked in up until the point where the underlying equity index has a negative quarterly return, at which point the structure switches to accruing quarterly returns based on an inverse floating rate. If we floor the inverse floating part at zero, then the interest rate part is reduced to a set of interest rate floorlets. These are just geared put options on the relevant LIBOR rate, in this hybrid taken to be the USD 3-month LIBOR rate. The floating interest rate part for each quarter is given by

$$\begin{aligned} \text{Floating Rate}(t_i) &= \max\left[0, 5\% - 2 \times \text{ 3-month USD LIBOR}(t_{i-1}, t_i)\right] \\ &= 2 \times \max\left[0, 2.5\% - \text{3-month USD LIBOR}(t_{i-1}, t_i)\right] \end{aligned} \tag{20.2}$$

Adjusting this by the day count function is the payoff of a twice geared floorlet, each quarter.

This is a hybrid structure on equity and a USD short-term interest rate, and because it is a forward-looking structure, it can be priced using Monte Carlo simulation of a correctly calibrated model. On the equity side, we are dealing with a cliquet style payoff, which means that we immediately have forward skew risk (recall the discussion of Chapter 13), thus making the simulation of the equity path non-trivial. Some form of stochastic volatility will be needed in this case, and the calibration made consistent with a non-interest rate related quarterly cliquet. It is imperative that the forward skew risks and Vega convexities of the cliquet part of the hybrid are priced.

The interest rate part is forward looking and involves forward LIBOR rates thus making it suited to treatment with a LIBOR market model. The interest rate floor quotes are obtained from the market, with the correct strike corresponding to that of the floating rate payoff of equation (20.2), and their implied volatilities used to calibrate the LMM.

The two processes will need to be correlated and, as in the case of the autocallable swap of Chapter 12, we decide what to do about interest rate/equity correlation. The buyer of the structure is essentially playing on this correlation in that he expects declining equities to be accompanied by a decline in rates. If correlation decreases between interest rates and equities then the chance of these two moving in opposite directions and against the client increases. The seller of the structure will need to price into these correlations the fact that it is difficult to hedge interest rate/equity through liquidly traded instruments. This correlation cannot be implied from the market for the same reasons, and historical estimates are the best starting point. Risk-related margins will need to be taken.

The ICBC–CMS steepener hybrid described in section 19.3 is a best-of option between the ICBC component on a basket of underlying equities, and the (geared) spread between the

30-year and 10-year CMS rates. The main risk in the equity part of this hybrid is volatility skew, as the ICBC is quite sensitive to the equity skew. A local volatility model for the equity processes will be sufficient to capture and price this effect sufficiently. Equity correlations must also be dealt with as in the standard ICBC discussed in Chapter 9.

As for the CMS steepener part of the hybrid, spread options are sensitive to implied volatility smiles or skews, and this must be captured by the interest rate model. A constant maturity swap spread option is an example of an option that is sensitive to the correlations between the forward rates, suggesting that a LIBOR market model is better suited. The LMM used will have to be one that knows about skew, meaning either a local volatility or stochastic volatility extension thereof. In this case it may be possible to use a two-factor short-rate model on the basis that it allows for a calibration to a swaption implied volatility cube (strikes and maturities), but also allows for some de-correlation in the forward rates. From a modelling perspective, combining this with a local volatility process for each of the equities may be simpler.

The correlation between the equities and the two CMS rates is specified keeping in mind the nature of the best-of payoff, discussed in Chapter 8. On the one hand we know that the seller of this option is short the correlations between the various equity underlyings, because of the nature of the ICBC payoff, as discussed in Chapter 9. In a simple best-of option the seller is long the correlation between the two underlyings, but here, the second part of the option consists of a spread payoff. The correlations between the equity underlyings and each of the CMS rates thus affect the payoff in opposite ways. The seller of the hybrid option will be long the correlation between the first CMS rate and the equities, and short the correlations with the second CMS rate of the spread. Again, margins need to be taken on these correlation parameters.

20.1.2 Commodity Modelling

Black's model greatly simplifies commodity modelling, but is not applicable to the spot price. Additional effects such as seasonality must be priced when modelling commodity spot prices. This depends on the commodity itself, but for examples such as oil, a model of the spot price must include information regarding the convenience yield described in section 17.2.1. This yield varies considerably as supply and demand for the physical commodity change. Modelling the convenience yield as a constant will not capture the varying forms of the futures curve, and the resulting models will be unrealistic. Thus, models such as Gibson and Schwartz (1990) involve two factors: one for the oil spot price, and one for the convenience yield. The short end of the futures curve is somewhat de-correlated with the longer end of the curve, reflecting short- and long-term expectations, and multiple factors can better explain this de-correlation.

Mean reversion similar to the case of short-rate models can be used, and seasonality effects built into models to make them more realistic. Additionally, oil spot prices for example can exhibit large jumps, suggesting the need for a jump model. Jump models, like stochastic volatility models, will allow for a calibration to the volatility skew observed in the vanilla options market. It is possible to combine all these features in models, but at the cost of greater complexity.

A commodity exotic such as a digital is very sensitive to skew. A digital on the price of oil is sensitive to the volatility skew implied by vanilla options. The implied returns are positively skewed, which means that there is a skew benefit in the digital, compared to a skew cost in the equity case. The double digital of section 19.2.3 involves a digital on oil and one on equities. The skew of each has a different impact on the price of the hybrid because the markets are different in nature, and in a product like this, it is key to capture both these skews. Liquidity-based barrier shifts can be taken on each to smoothen out the Greeks near the barrier to ensure that one can hedge. The nature of digital contracts, on single or multiple underlyings, as discussed in Chapter 11, emphasizes the importance of liquidity in the underlying assets.

The correlation between oil and equities is again a quantity that cannot be implied or hedged efficiently. The historical correlation between WTI futures and the S&P 500 index may be historically low, but it changes through time and there are points where this correlation has increased far beyond its long-term average. The seller of the structure is short the correlation between oil and equities, and as usual a margin needs to be taken over the historical correlation.

20.1.3 FX Modelling

During the discussion of FX vanillas we saw the Garman and Kohlhagen formula, which is the Black–Scholes equivalent for FX, and to quote FX vanillas in terms of their Deltas, this model is used. To manage the presence of the FX smile, the SABR model can be used for a book of vanilla options. Other stochastic volatility models are also popular, for example Heston's model discussed in Appendix A (A.3.1) and Bates's model in section A.4. In both these, it is the spot FX rate that is modelled, and the variance of the spot rate modelled as a random variable. Additionally Bates's model allows for jumps in the underlying spot rate, and the addition of jumps is the main difference between Bates's and Heston's models. Once calibrated to vanillas, each of these models gives the same prices for such options, but different hedge ratios depending on the model assumptions.

FX markets exhibit what is known as a stochastic skew, as we saw, meaning that the way implied returns from the market are skewed (positively or negatively), as observed through the quotes of risk reversals, can change sign. Models such as Carr and Wu (2007) are designed to pick up this additional risk. A change in the skewness of the implied returns can have a large impact on the prices of barrier options and digitals. Pricing in this additional risk factor can thus be important. Since European options are involved in the hedging of such instruments, it is important that the model used for pricing these captures the possible changes in the skew.

When moving to exotic FX structures, things get more complicated. Particularly in callable long-dated FX structures, the options are sensitive to movements in the interest rates of both currencies of the FX. The longer the maturity of the structure, the greater this sensitivity. As we saw in Chapter 18, the FX forward is a function of the spot FX rate, and also the spread between two interest rates geared by the time to maturity. The effect of the drift involving this interest rate spread has a greater effect on the long-term evolution of the FX process than, for example, allowing its volatility to be stochastic.

As such, for long-dated callable FX structures, and all interest rate/FX hybrids, one must use a model that allows for randomness in not only the FX rate, but also the two interest rates. In particular, the interest rate part of the model will need to be calibrated to at least capture each of the yield curves correctly, and a Hull–White style model for each of these would work. Interest rate volatility also impacts and the volatilities of the interest rate part must be calibrated to interest rate volatilities, for example using interest rate swaption-implied volatilities. The FX part will need to capture the information given by the vanilla surface. The modelling choices regarding the interest rate curves and their calibration, as well as the FX process and its calibration, will have a huge impact on the hedge ratios given by the model.

Composite Options

A composite option is an option on a foreign underlying with a strike in the domestic currency of the investor. As the option is priced in the domestic currency, it pays in the domestic currency if exercised. Consider a call option on an underlying S, then a composite call option involving an exchange rate FX has the payoff

$$\text{Composite Call}_{\text{payoff}} = \max\left[0, S(T) \times \text{FX}(T) - K\right]$$

where $S(T)$ is denominated in the foreign currency, the FX rate converts this into the price of the underlying in the domestic currency, and the strike K is in the domestic currency. This is different to the quanto option described in Chapter 7 where the FX rate is fixed up front. The holder of the composite option is thus still exposed to currency movements, but benefits from having fixed the strike in his domestic currency.

For example, consider a call option on Microsoft (denominated in USD) with a strike of 15.6 GBP, designed for a GBP investor. At maturity T the payoff of the composite option is determined by the product of the USD–GBP exchange rate at time T and the price of Microsoft's stock price at time T. If Microsoft is trading at USD 24 at maturity and the exchange rate is 0.7 (that is, 0.7 GBP to buy 1 USD) then the product is 16.8 and the option expires in-the-money and pays $16.8 - 15.6 = 1.2$ GBP. If the stock price were 25 USD and the exchange rate 0.6, then the product is 15 and the option expires OTM and worthless.

The pricing of the quanto option involves using a quanto adjustment to the risk-neutral drift, but with no change in the volatility used. The composite, on the other hand, is essentially a call option on a product basket of both the exchange rate and the underlying equity. If we assume that both the equity and the FX are modelled using Black–Scholes assumptions, in particular log-normality for both, then the product of the FX rate and the equity can also be modelled by a log-normal process. A Black–Scholes formula applies where one used the risk-free rate of the foreign currency, and a volatility given by

$$\sigma_S^2 + \sigma_{\text{FX}}^2 + 2\,\rho_{S,\text{FX}} \cdot \sigma_S \cdot \sigma_{\text{FX}}$$

This volatility we recognize as the approximate volatility for a standard basket option where the sum, instead of the product of the underlyings, is taken. In this case the volatility of the product is exact for the product of two log-normals. The volatility of the product, which is used in the composite option's pricing, contains both volatilities because the composite option has risk to both the FX rate and the underlying equity.

The risk-neutral drift is taken to be the domestic rate of the investor, for the simple reason that the drift of the product of two log-normals is the sum of their drifts. These are given by $r_{\text{domestic}} - r_{\text{foreign}}$ for the exchange rate, and r_{foreign} for the risk-neutral drift of the underlying equity in its denominated currency (here this is the foreign currency). Adding these two, the r_{foreign} cancels out, and we are left with just r_{domestic}. This means that the expected growth of the product of these two is the risk-free rate of the domestic currency.

This isn't really a hybrid product, but it does combine the two asset classes in a payoff, and again we see the appearance of a two-asset-class correlation. In general this correlation is hard to imply and to hedge, unless one is able to obtain a quote for a relatively liquid stock/currency composite pair. The seller of both the composite call option and the composite put option is short the FX volatility, the volatility of the underlying equity, and the FX/equity correlation. Any more elaborate combination of FX and equities must be studied accordingly. The ability to trade the underlyings to Delta hedge is as always key, i.e. the question of liquidity. To hedge Vega or Gamma one must be able to trade options on the individual underlyings. The correlation structure is again analysed and residual risk factors built into the price.

20.2 COPULAS

A copula is defined as "something that connects or links together". In a statistical context, the word copula refers to a function that combines two or more univariate distribution functions to form multi-variate distribution functions, allowing for different types of dependence to be modelled. Applying a copula method to finance, the univariate distributions in question will be those of financial variables.

The dependencies obtained through a copula are much more realistic when interpreting the correlated behaviour of these financial assets, compared to using standard correlation coefficients. The standard correlation, as defined in Chapter 7, measures linear dependencies between two random variables.

If two variables are independent, then their correlation must be zero. The opposite is not true, however: two variables with a zero correlation are not necessarily independent. Two random variables can have a strong dependency on each other, i.e. far from being independent, but have a correlation of zero. Figure 20.1 depicts the discrete time series of two variables that have a correlation coefficient of zero, but obviously some form of dependency exists between them.

The importance of copulas in hybrid derivatives, and multi-asset derivatives in general, is that copulas provide a method of expressing joint distributions between assets, allowing for the simulation of these variables, and thus the pricing of multi-asset options. Hybrids present an interesting set of applications because of the quite different dependencies that are observed between the various asset classes. In this section we aim to explain the theory behind copulas and their importance. In the literature, some interesting discussions of the theory and uses of copulas in finance are discussed in, among others, Cherubini *et al.* (2004), Overhaus *et al.* (2007), Nelsen (1999), Schmidt (2007), and Trivedi and Zimmer (2007).

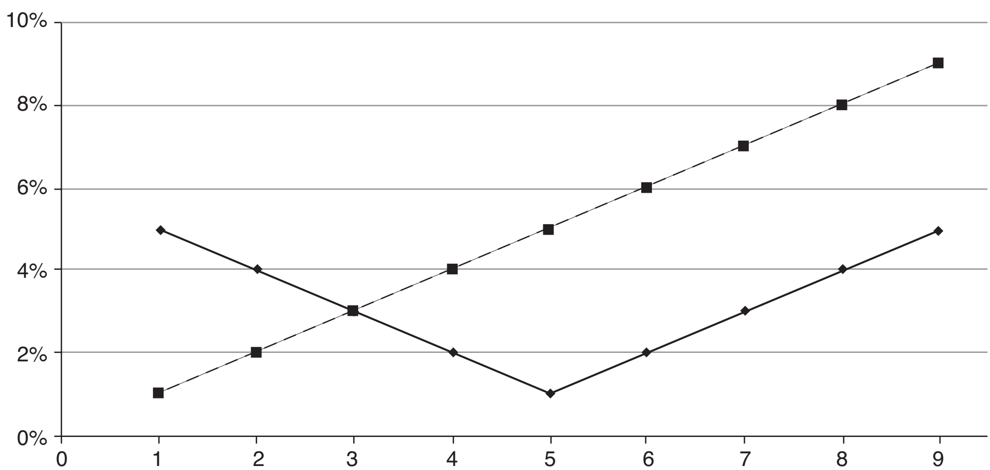

Figure 20.1 Two series with some dependency, but zero correlation. Zero correlation does not imply independence.

As we have seen, the correlation between the various asset classes can rarely be implied or hedged, and the specification of this correlation has an impact on both pricing and hedging. Although the traditional correlation does provide information regarding how different assets behave with respect to each other, it doesn't allow us to specify different behaviours for different parts of the distribution. A copula does precisely this, and modelling with copulas can thus provide more meaningful hedge ratios. For example, in the cases of out-barriers, the Greeks with respect to the variable on which the payoff is written, Delta, Vega, etc., are sensitive to movements in the asset on which the barrier knock-out clause is written. As such these Greeks will be affected by the method by which the two assets are correlated, and the more meaningful hedge ratios can result from using copulas.

20.2.1 Some Copula Theory

Let's start with the simplest case where we have only two random variables X and Y to consider. Assume that each of these has its own probability distribution, represented by two functions f_X and f_Y; these are our two univariate distributions. From these we know the cumulative distribution functions of each, denoted by $F_X(x)$ and $F_Y(y)$ for X and Y respectively. By definition, $F_X(x) = \Pr(X \leq x)$, i.e. the probability that the variable X is less than or equal to x, and in terms of the probability distribution function is given by

$$F_X(x) = \int_a^x f(s)\,\mathrm{d}s$$

where a is the lower bound over which the distribution is defined. An important property of the CDF is that, being a probability itself, means that it will take a value between 0 and 1. So how does one formulate a joint distribution $J(x, y) = \Pr(X \leq x, Y \leq y)$ out of these, where $J(x, y)$ preserves the marginal distributions of both X and Y? The answer is the copula.

Firstly, consider a uniform distribution defined on the interval [0, 1]. This has a probability distribution function equal to the constant 1 on the interval [0, 1] and 0 everywhere else. Now, regard the CDF of the random variable X above (and the same thing for Y) as a transformation of the distribution of X given by the above integral. This connects the probability distribution of X with the CDF of X. If we assume X and Y to be continuous random variables, and their CDFs both strictly increasing, then the variables $U = F_X(x)$ and $V = F_Y(y)$ are both uniformly distributed on the interval [0, 1].

So, starting with the two distributions, one for X and one for Y, we are able to transform these to uniform distributions on [0, 1]. A copula is only a joint distribution function of two random variables, defined on $[0, 1] \times [0, 1]$, such that both marginal distributions are also uniformly distributed on the interval [0, 1]. The random variables on which the copula acts are the two transformed random variables. The selection of a copula (out of many possible functions satisfying the requirements of copulas) will specify the dependency between X and Y. Note that the copula described is a 2-dimensional one, which we work with for simplicity, although the general theory is extendible to more than two random variables.

The concept of the copula was pioneered by Sklar (1959). Sklar's theorem, in simplified form, tells us that if $J(x, y)$ is a joint distribution function of two random variables X and Y, and the marginal distributions computed from $J(x, y)$ are given by $F_X(x)$ and $F_Y(y)$, then there exists a copula C such that $J(x, y) = C(F_X(x), F_Y(y))$. The converse also holds: given any two uniform marginal distributions $F_X(x)$ and $F_Y(y)$, and any copula function C, then the function $J(x, y) = C(F_X(x), F_Y(y))$ is a joint distribution function for X and Y, and it preserves their marginal distributions.

The main point from this is that the existence of the copula that gives the joint distribution is not dependent on the individual marginal distributions. This means that, in theory, whatever the distributions of X and Y, one can apply a copula to obtain a joint distribution. Also, any joint distribution function can be expressed as a copula. So through the copula, a joint distribution is formed by two separate parts: the marginal distributions of both variables, and the choice of dependency specified independently by our choice of copula.

20.2.2 Modelling Dependencies in Copulas

The standard correlation coefficient measures the strength and direction of the linear relationship between the random variables, but does not provide information about how the relationship between the two variables changes as we traverse the distribution. A copula, on the other hand, allows us to impose dependencies of different strengths between the random variables based on different sections of the distributions.

We have seen a simple example of this in equity markets. We generally see different regimes of correlation: relatively low correlation during periods of relative stability, and spikes in correlations during market crashes. The copula allows us to translate this into a model by imposing a stronger dependency between two stocks of an index on this tail in the two distributions. This is known as tail dependence, and can only be accomplished through the use of a copula. Recall that the tail of a single distribution corresponds to extreme events; and the equity market prices the higher probability of an extreme event through the skew, giving a distribution with a fatter tail than the log-normal one. Pricing an exotic structure on a single asset would generally involve capturing this through a skew model, but moving to the two (or more) asset case where we may need to capture the dependency at the part of the distribution where these fat tails occur. Given two (or more) such distributions, each representing one asset, a copula will allow us to model dependencies between extreme events in both.

An important feature of copulas is that the dependence captured by a copula is invariant under increasing and continuous transformations of the marginal distributions. This means that a copula used to join two variables X and Y can also be used on their logarithms $\ln X$ and $\ln Y$, for example. The idea is that if two variables are transformed by increasing transformations (the logarithm for example), then the transformed versions give the same information as the original variables. So specifying a dependency between X and Y, or their logarithms, is in essence the same. This means that we can work with logarithms and returns should we need to, instead of using prices for example when doing applications. The same concept does not apply to the standard coefficient: the correlation between two asset prices and the two assets' returns (or log-returns) is not necessarily the same.

In order that a copula be useful, we must specify a parametric form for it. Recall, the copula is a joint distribution function acting on $[0, 1] \times [0, 1]$ (in the case of two variables) such that both marginal distributions are uniform. As it turns out, we have at our disposal a whole range of possible parametric forms that will satisfy the criteria to be a copula. The choice of which parametric form depends on where we want to stress dependency, and below we give different copulas, each of which stresses a different part of the distribution.

We start by specifying a bivariate joint distribution J that has a useable parametric form. For example, the bivariate Normal distribution leads to what is known as the Gaussian copula described below. Our copula function C, acting on two variables u and v in [0, 1], is given by

$$C(u, v) = J_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right) \tag{20.3}$$

Let's elaborate on what this formula is saying. Firstly, X and Y are the two random variables for which we are introducing dependencies using the copula. In this formula, $F_X^{-1}(u)$ represents the inverse function of the CDF $F_X(x)$ of the random variable X. We assume that the CDFs of the random variables we are modelling are strictly increasing and continuous, which means that they have an inverse. The CDF $F_X(x)$ maps the domain where X is distributed to [0, 1], and so its inverse $F_X^{-1}(u)$ maps this back, and takes as an argument a value u in [0, 1]. Similarly, $F_Y^{-1}(v)$ is the inverse of the CDF of our second variable Y, again acting on [0, 1].

Equation (20.3) comes from the earlier point where the joint distribution was defined by the copula as $J(x, y) = C(F_X(x), F_Y(y))$, combined with the use of the inverse function: if $u = F_X(x)$ then $x = F_X^{-1}(u)$. Both inverse CDFs go from [0, 1] to the domain of their respective random variable, and the joint distribution $J_{X,Y}$ then acts on these. As such, we can build a copula by starting with the marginal distributions of both X and Y and then imposing a distribution we know to link them.

Consider the case where we are not trying to introduce any dependency at all, i.e. we want X and Y to be independent. The product copula is the only way to express this and it is simply the copula we obtain when the joint distribution of X and Y is simply the product of their CDFs: $J_{X,Y}(x, y) = F_X(x)F_Y(y)$. The following are just a few possible choices:

- The Gaussian copula allows for dependency, but does not show any dependency in the tails.
- The Gumbel copula should be used to emphasize lower tail dependencies (i.e. the downside of the distributions), it only has lower tail dependency.
- The Clayton copula has upper tail dependency.

The Gumbel and Clayton copulas are referred to as asymmetric because they are skewed; in opposite ways, however. The Student-t copula is symmetric in both up and low tail dependency. The main difference between these, and copulas in general, is specifically the section of the distribution on which they show dependency; more so than how much dependency. As such, if one decides to use copulas, the choice of the part of the distribution on which to increase dependency must have a justifiable reason behind it as the copula will stress the dependency in that part.

20.2.3 Gaussian Copula

The Gaussian copula is quite popular in finance – even though it is a symmetric copula and returns among many financial assets exhibit tail dependencies. It was originally applied to finance in the context of credit derivatives to model defaults among multiple corporations, as proposed by Li (2000). Given its symmetric nature, it is perhaps not the best copula to use for modelling defaults, but does have many other applications. Following the above theory, we have two assets, each with its own distribution. The Gaussian copula uses the Normal distribution, which in the two-asset case is the bivariate Normal distribution, and the Gaussian copula is given by

$$C(u, v) = \mathcal{N}_2\left(\mathcal{N}^{-1}(u), \mathcal{N}^{-1}(v)\right) \tag{20.4}$$

where $\mathcal{N}$ is the cumulative Normal distribution. As such, in the Gaussian copula the Normal marginal distributions are made dependent via a bivariate Normal distribution. The correlation

ρ is the only free parameter in the bivariate distribution $\mathcal{N}_2(x, y)$ above, and its value controls the dependency in this copula.

We must be clear on this important point: the Gaussian copula is not modelling two asset prices using a bivariate Normal distribution, it is modelling the dependence between their two distributions using a bivariate Normal distribution. To be precise, it actually models the dependence between the uniform distributions obtained from these. Assume that the two random variables we are correlating via this copula are X and Y, with distribution functions $F_X(x)$ and $F_Y(y)$ respectively. In formula (20.4), the variables u and v are these CDFs, which are the transformed version of X and Y whose result is in [0, 1]. We discuss the calibration and simulation of this copula in order to clarify this theory.

Calibrating the Copula

Here we discuss the calibration of copulas, focusing on the case of the Gaussian copula. Assume we have two random variables X and Y for which we have selected a joint distribution that makes financial sense. Our copula takes parametric form from this distribution, and along with marginals of X and Y, must be calibrated to market data. The use of a known parametric form for the joint distribution (the bivariate Normal in the case of the Gaussian copula) becomes important at this step because it will simplify the process of fitting the copula.

Market data will consist of a set of traded options on X, and a separate set of traded options on Y. We can fit the marginals of each to represent the individual implied distributions, and then specify the parameter of the copula, or the marginals and the parameter of the copula can be fitted at the same time.

In the latter, for example, we can compute the density function of the copula. Using the Gaussian copula described here, we have

$$\begin{aligned} c(u, v) &= \frac{\partial^2 C(u, v)}{\partial u\, \partial v} \\ &= \frac{\partial^2 \mathcal{N}_2\left(\mathcal{N}^{-1}(u), \mathcal{N}^{-1}(v)\right)}{\partial u\, \partial v} \\ &= \frac{\Phi_2(\mathcal{N}^{-1}(u), \mathcal{N}^{-1}(v))}{\Phi(\mathcal{N}^{-1}(u))\Phi(\mathcal{N}^{-1}(v))} \end{aligned}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$$

is the Normal (Gaussian) probability distribution function, and

$$\Phi_2(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[x^2 + y^2 - 2\rho x y\right]\right)$$

is the bivariate probability density function. This will be fitted to the set of data using a Maximum Likelihood Estimation method.

If we want to capture the skewness and fat tails of the individual marginals, we will need a skew model for each marginal. In this case we can apply the SABR model, specifically because, even though it is a stochastic volatility model, it has been solved to offer a simple form for the prices of European options. Not only do we know the prices of Europeans via a modified function in place of the volatility, enabling us to use Black's formula, but we also have simplified but accurate approximations of the distributions implied by the model. This is also made possible by the fact that we are not constrained in the choice of how to model our marginals, as long as the distribution function used is invertible. Thus, each of our marginals can be set to be a random process defined under the SABR model. See Appendix A, section A.3.2, for more details on the SABR model.

Simulating the Gaussian Copula

If we assume that we know the marginal distributions of two random variables X and Y, given by $F_X(x)$ and $F_Y(y)$ respectively, the following algorithm allows us to simulate the Gaussian copula that introduces the dependency to these two.

1. Simulate a vector of two independent uniform random variables $\{u_1, u_2\}$.
2. Transform these to Normal random variables $\{\epsilon_1, \epsilon_2\}$. This can be done via the Box–Muller transform:

$$\epsilon_1 = \sqrt{-2\ln u_1}\cos(2\pi u_2)\,, \qquad \epsilon_2 = \sqrt{-2\ln u_1}\sin(2\pi u_2)$$

   These are still independent, and will be standard Normal random variables.
3. Correlate these by multiplying by the decomposed correlation matrix, giving $\{n_1, n_2\}$.
4. Map these back to [0, 1] by applying the Normal CDF

$$\{u, v\} = \{\mathcal{N}(n_1), \mathcal{N}(n_2)\}.$$

5. Apply the inverse marginal distributions to u and v to obtain X and Y as $F_X^{(-1)}(u) = X$ and $F_Y^{(-1)}(v) = Y$.

In the case of standardized random variables, the correlation matrix mentioned in step 3, is the same as the covariance matrix, and is given (in the case of the standard bivariate Normal) by

$$\mathbf{M} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

This is decomposed into $\mathbf{M} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$, where $\mathbf{L}$ is a lower triangular matrix, and $\mathbf{L}^{\mathrm{T}}$ its transpose. Given this matrix $\mathbf{M}$, the matrix $\mathbf{L}$ we want is given by

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}$$

The matrix $\mathbf{L}$ is multiplied by the vector $\{\epsilon_1, \epsilon_2\}$ of independent Normal random variables to give a vector of two normally distributed random variables with correlation ρ. Up to step 3, we are essentially just generating random variables from the bivariate Normal distribution. Step 4 maps these back to the interval [0, 1] via the Normal CDF. Finally, the inverse distributions $F_X^{-1}(u)$ and $F_Y^{-1}(v)$ give us back the variables whose marginals we want to combine in the Gaussian copula in the first place. The procedure described allows us to simulate such variables, with their individual marginals intact and, combined via the Gaussian copula, controlled by the correlation ρ. The tractability and relative ease of this simulation process is an important factor in the popularity of the Gaussian copula.

The inverse distributions used in step 5 can be freely specified. They are chosen to be the inverses of the two distributions we want to have as our marginals. So, assuming that our choice of the distributions of each is invertible, we can use the inverse CDF at this step in conjunction with the Gaussian copula to get the required result. The example of the SABR model can be used here; in step 5 the inverse SABR distribution is used. As such the copula will allow for a calibration to the implied skews of the different assets, and allow this to be combined in a Gaussian copula. The copula models the dependency, and this method is particularly useful for hybrid payoffs that are skew sensitive. Caution must be taken because the SABR model works on a forward process, and the result of the simulation in step 5 is a forward.

Multi-variate Case

All the theory presented for the bivariate case extends to the multi-variate case: the existence of the copula in the multi-variate case, and its ability to model dependency among multiple variables. In the Gaussian copula, for example, the same theory holds, only the multi-variate function $\mathcal{N}_n$ replaces the bivariate cumulative Normal distribution $\mathcal{N}_2$ in order to combine the marginals of n different variables instead of two. The simulation process holds, and the calibration process too.

20.2.4 Pricing with Copulas

Other than the possibility of simulating the copula to price various payouts on multiple variables, there are some cases where the copula itself gives us the answer directly.

Bivariate Digitals in Copulas

Consider a digital that pays a coupon if both the price of oil and the S&P 500 index are greater than or equal to their values today in a year's time.

$$\text{Double Digital}_{\text{payoff}} = X\% \times \mathbf{1}_{\{\text{S\&P 500}(T) \geq \text{S\&P 500}(0)\}} \times \mathbf{1}_{\{\text{Oil}(T) \geq \text{Oil}(0)\}}$$

It is important to model the dependency here correctly, and a Gaussian copula should suffice. Referring to the two underlyings as S_1 and S_2, when writing the payoff as an expectation we need to compute the discounted value of the joint probability

$$\Pr\left[\{S_1(T) \geq S_1(0)\}\,;\{S_2(T) \geq S_2(0)\}\right]$$

More generally, consider the same payoff where the digital pays only if both are above the respective strikes K_1 and K_2, which in this example are today's value of each.

$$\Pr\left[\{S_1(T) \geq K_1\}\,;\{S_2(T) \geq K_2\}\right]$$

This is in fact closely related to the joint distributions function of the two. Assume that we chose a Gaussian copula and calibrated the copula and both the underlyings to have the marginals consistent with the market of each, then this probability is given by plugging values into the calibrated copula.

Firstly, let P_1 and P_2 be the probabilities of each individually, ending in-the-money. Let $B(T)$ be the price of the relevant risk-free asset (bond) that we know, so

$$\frac{P_1}{B(T)} = \Pr\left(S_1(T) \geq K_1\right), \quad \frac{P_2}{B(T)} = \Pr\left(S_2(T) \geq K_2\right)$$

As such, P_1 and P_2 are the prices of the individual digitals. Sklar's theorem allows us to write the bivariate joint distribution as a copula taking the marginals as arguments (see Cherubini and Luciano, 2002). Thus we can write

$$\begin{aligned}\frac{Price\,(S_1(T) \geq K_1, S_2(T) \geq K_2)}{B(T)} &= \Pr(S_1(T) \geq K_1, S_2(T) \geq K_2) \\ &= C\left(\frac{P_1}{B(T)}, \frac{P_2}{B(T)}\right)\end{aligned}$$

The copula allows the bivariate pricing to be broken into the dependency structure, and the marginal univariate distributions. The marginals we obtain from the market as the call spread proxy for each individual digital, including all relevant information regarding the skew. We refer the reader to Cherubini and Luciano (2002) for additional discussions regarding bivariate pricing with copulas.