UNIVERSITY OF ALBERTA

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ECE 760 PROJECT REPORT

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Robotics: Modelling, Planning and Control
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1 Case Study: 4DOF Revolute-Joint Planar Robot (A Redundant Robot)

In this project, a four degrees of freedom (4DOF) planar revolute-joint robot is considered as the case study. The schematic depiction of the robot is presented in Fig. 1.1.

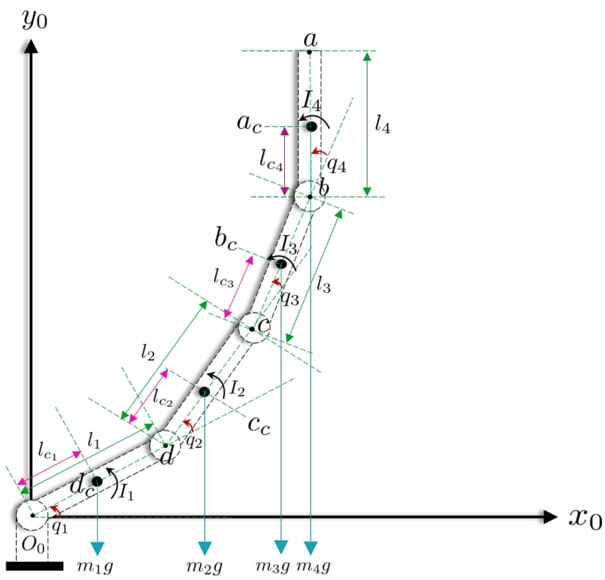

Figure 1.1: Schematic of the 4DOF planar revolute-joint robot.

1.1 Physical Parameters

The physical parameters of the robots are denoted by m_i for the mass of each link, l_i for the length of each link, I_i denotes the moment of inertia of link i about an axis coming out of the page, passing through the center of mass of link i, a is the position of end-effector, b is the position of the fourth joint, c is the position of the third joint, d is the position of the second joint, O_0 is the position of the first joint which is attached to the global coordinate frame $O_0x_0y_0z_0$. Also, a_c, b_c, c_c and d_c are the positions of the centers of mass (CoM) of the corresponding links according to the Fig. 1.1. Also, l_{c_i} is the distance between joint $i-1$ and the CoM of the link i. Furthermore, the joint position of the joint i is denoted by q_i.

1.2 Forward Kinematics

In the forward Kinematics, the aim is to find the position of any point of the robot based upon the joint positions. Let $\mathbf{X}\in\mathbb{R}^{2\times1}$ represent the Cartesian position of a given point of the robot. Therefore, the relation between the point and the joints can be according to the formula $\mathbf{X}=h(\mathbf{q})$ where $h(\mathbf{q})$:$\mathbb{R}^{4\times1}\rightarrow\mathbb{R}^{2\times1}$ describes the nonlinear mapping between the joint-space positions and task-space positions. For the case of conciseness, let have the notations as follows.

$$c_{ijkz}\triangleq\cos\left(q_i+q_j+q_k+q_z\right) \qquad s_{ijkz}\triangleq\sin\left(q_i+q_j+q_k+q_z\right) \tag{1}$$

using which the Cartesian position of the end-effector and joints can be obtained as a function of joint positions according to

$$\begin{array}{ll} \begin{cases} x_a=l_1c_1+l_2c_{12}+l_3c_{123}+l_4c_{1234} \\ y_a=l_1s_1+l_2s_{12}+l_3s_{123}+l_4s_{1234} \end{cases} & \begin{cases} x_b=l_1c_1+l_2c_{12}+l_3c_{123} \\ y_b=l_1s_1+l_2s_{12}+l_3s_{123} \end{cases} \\ \begin{cases} x_c=l_1c_1+l_2c_{12} \\ y_c=l_1s_1+l_2s_{12} \end{cases} & \begin{cases} x_a=l_1c_1 \\ y_a=l_1s_1 \end{cases} \end{array} \tag{2}$$

1.2.1 Jacobian Matrix of the 4DOF Revolute-Joint Robot

The relation between the robot's end-effector velocity and the joint velocities are $\dot{\mathbf{X}}=\mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$ where $\mathbf{J}(\mathbf{q})\in\mathbb{R}^{2\times4}$ is the Jacobian matrix defined as $\mathbf{J}(\mathbf{q})=\frac{\partial h(\mathbf{q})}{\partial \mathbf{q}}$. The relation can be shown equivalently as

$$\begin{bmatrix} x_{\text{end-effector}} \\ y_{\text{end-effector}} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} \tag{3}$$

where if we define $q_{12}\triangleq q_1+q_2$, $q_{123}\triangleq q_1+q_2+q_3$ and $q_{1234}\triangleq q_1+q_2+q_3+q_4$, using the forward kinematics, the Jacobian matrix is obtained as

$$\begin{aligned}
&J_{11}=-l_1 sin(q_1)-l_2 sin(q_{12})-l_3 sin(q_{123})-l_4 sin(q_{1234}) \\
&J_{21}=l_1 cos(q1)+l_2 cos(q_{12})+l_3 cos(q_{123})+l_4 cos(q_{1234}) \\
&J_{12}=-l_2 sin(q_{12})-l_3 sin(q_{123})-l_4 sin(q_{1234}) \\
&J_{22}=l_2 cos(q_{12})+l_3 cos(q_{123})+l_4 cos(q_{1234}) \\
&J_{13}=-l_3 sin(q_{123})-l_4 sin(q_{1234}) \qquad J_{23}=l_3 cos(q_{123})+l_4 cos(q_{1234}) \\
&J_{14}=-l_4 sin(q_{1234}) \qquad J_{24}=l_4 cos(q_{1234})
\end{aligned} \tag{4}$$

1.3 Robot Dynamics (Motion Equation)

As we know, the motion equation of the robot in Fig. 1.1 can be characterized by the Euler-Lagrange Equation as

$$\mathbf{D}(\mathbf{q})+\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}+\mathbf{G}(\mathbf{q})=\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})\boldsymbol{\Theta}=\boldsymbol{\tau} \tag{5}$$

where $\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}\in\mathbb{R}^{4\times1}$ are respectively the vectors of the joint positions, velocities and accelerations. $\mathbf{D}(\mathbf{q})\in\mathbb{R}^{4\times4}$, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\in\mathbb{R}^{4\times4}$ and $\mathbf{G}(\mathbf{q})\in\mathbb{R}^{4\times1}$ are the inertia matrix, the

Coriolis/centrifugal matrix and the gravitational vector, respectively. Moreover, $\boldsymbol{\tau} \in \mathbb{R}^{4\times1}$ shows the exerted torque on the robot. Furthermore, $\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})\boldsymbol{\Theta}$ is a linear parameterization of the first three terms of (5) such that $\boldsymbol{\Theta}$ is the vector of actual parameters of the robot and $\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})$ is the regressor matrix.

1.4 Finding the Inertia Matrix

We know the Jacobian matrix can be obtained by the investigation of the relation between the linear velocity of points on robot and the linear/angular velocity of the joints. Since, one can use the Jacobian matrices of points associated with the center of mass of each link to find the kinetic energy of the robot, which results finally in the inertia matrix, therefore, I adopt this approach as follows.

First we know that the Jacobian matrix of any point on the robot and joint positions can be obtained according to the following formula, in which the Jacobian matrix includes the sections related to the angular and linear velocity.

$$
\begin{aligned}
\mathbf{J}_{i_{point}} &= \begin{bmatrix} Z_{i-1}\times(O_{point}-O_{i-1}) \\ Z_{i-1} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{v_{i_{point}}} \\ \mathbf{J}_{\omega_{i_{point}}} \end{bmatrix} \\
\mathbf{J}_{point} &= \begin{bmatrix} \mathbf{J}_{1_{point}} & \mathbf{J}_{2_{point}} & \cdots & \mathbf{J}_{4_{point}} \end{bmatrix} \\
\mathbf{J}_{point_i} &= \begin{bmatrix} \mathbf{J}_{1_{point_i}} & \mathbf{J}_{2_{point_i}} & \cdots & \mathbf{J}_{4_{point_i}} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{J}_{v_i} \\ \mathbf{J}_{\omega_i} \end{bmatrix}
\end{aligned} \tag{6}
$$

1.4.1 Kinetic Energy of the 4DOF Robot

We know that the linear and angular velocities of any point on any link can be expressed in terms of the Jacobian matrix and the derivative of the joint variables. Since the joint variables are indeed the generalized coordinates, we have

$$
\mathbf{v}_i = \mathbf{J}_{v_i}\dot{\mathbf{q}}, \qquad \omega_i = \mathbf{J}_{\omega_i}\dot{\mathbf{q}} \tag{7}
$$

If the mass of link i is m_i and the inertia matrix of link i around a coordinate frame parallel to frame i whose origin is at the center of mass, is $\mathbf{I}_i$, then he overall kinetic energy of the manipulator equals

$$K = \frac{1}{2}\dot{\mathbf{q}}^T \left(\sum_{i=1}^{n} \left[m_i \mathbf{J}_{v_i}^T \mathbf{J}_{v_i} + \mathbf{J}_{\omega_i}^T \mathbf{R}_i \mathbf{I}_i \mathbf{R}_i^T \mathbf{J}_{\omega_i} \right] \right) \dot{\mathbf{q}} \tag{8}$$

where $\mathbf{R}$ is the orientation transformation between the body attached frame and the inertia frame. Therefore, we can conclude that the kinetic energy of the manipulator is of the form

$$K = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{D}(\mathbf{q}) \dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T [\mathbf{D}_{transnational}(\mathbf{q}) + \mathbf{D}_{rotational}(\mathbf{q})] \dot{\mathbf{q}} \tag{9}$$

1.4.2 Transnational Component of the Inertia Matrix

Based on what explained in (6), the Jacobian matrices $\mathbf{J}_{v_1}$, $\mathbf{J}_{v_2}$, $\mathbf{J}_{v_3}$ and $\mathbf{J}_{v_4}$ are found in the following:

Jacobian Matrix Related to the First Link:

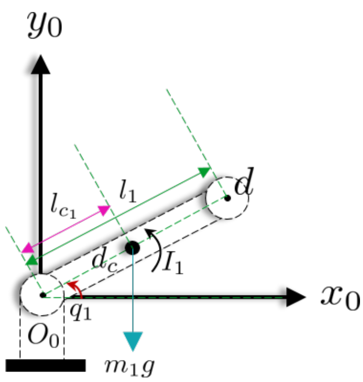

Figure 1.2: Schematic of the 1DOF planar revolute-joint robot.

Given Fig. 1.2, we have $\mathbf{J}_{dc} = \begin{bmatrix} \mathbf{J}1_{dc} & \mathbf{J}2_{dc} & \mathbf{J}3_{dc} & \mathbf{J}4_{dc} \end{bmatrix}$ where

$$\mathbf{J}1_{dc} = \begin{bmatrix} Z_O \times (O_{dc} - O_0) \\ Z_O \end{bmatrix} = \begin{bmatrix} [001]^T \times [x_{dc} y_{dc} 0]^T \\ [001]^T \end{bmatrix} = \begin{bmatrix} [-y_{dc} x_{dc} 0]^T \\ [001]^T \end{bmatrix} \tag{10}$$

and $\mathbf{J}_{2_{dc}}=\mathbf{J}_{3_{dc}}=\mathbf{J}_{4_{dc}}=\mathbf{0}$. Therefore, it is possible to see that

$$\mathbf{J}_{v_1}=\begin{bmatrix} -l_{c_1}s_1 & 0 & 0 & 0 \\ l_{c_1}c1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \hat{\mathbf{J}}_{v_1}\triangleq\mathbf{J}_{v_1}^T\mathbf{J}_{v_1}=\begin{bmatrix} \hat{\mathbf{J}}_{v_1}^{11} & \hat{\mathbf{J}}_{v_1}^{12} & \hat{\mathbf{J}}_{v_1}^{13} & \hat{\mathbf{J}}_{v_1}^{14} \\ \hat{\mathbf{J}}_{v_1}^{21} & \hat{\mathbf{J}}_{v_1}^{22} & \hat{\mathbf{J}}_{v_1}^{23} & \hat{\mathbf{J}}_{v_1}^{24} \\ \hat{\mathbf{J}}_{v_1}^{31} & \hat{\mathbf{J}}_{v_1}^{32} & \hat{\mathbf{J}}_{v_1}^{33} & \hat{\mathbf{J}}_{v_1}^{34} \\ \hat{\mathbf{J}}_{v_1}^{41} & \hat{\mathbf{J}}_{v_1}^{42} & \hat{\mathbf{J}}_{v_1}^{43} & \hat{\mathbf{J}}_{v_1}^{4s4} \end{bmatrix} \tag{11}$$

such that $\hat{\mathbf{J}}_{v_1}^{11}=l_{c_1}^2$, $\hat{\mathbf{J}}_{v_1}^{12}=\hat{\mathbf{J}}_{v_1}^{21}=0$, $\hat{\mathbf{J}}_{v_1}^{13}=\hat{\mathbf{J}}_{v_1}^{31}=0$, $\hat{\mathbf{J}}_{v_1}^{14}=\hat{\mathbf{J}}_{v_1}^{41}=0$, $\hat{\mathbf{J}}_{v_1}^{22}=0$, $\hat{\mathbf{J}}_{v_1}^{23}=\hat{\mathbf{J}}_{v_1}^{32}=0$, $\hat{\mathbf{J}}_{v_1}^{24}=\hat{\mathbf{J}}_{v_1}^{42}=0$, $\hat{\mathbf{J}}_{v_1}^{33}=0$, $\hat{\mathbf{J}}_{v_1}^{34}=\hat{\mathbf{J}}_{v_1}^{43}=0$ and $\hat{\mathbf{J}}_{v_1}^{44}=0$.

Jacobian Matrix Related to the Second Link:

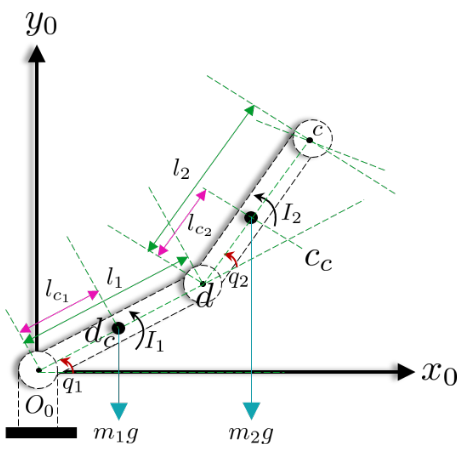

Figure 1.3: Schematic of the 2DOF planar revolute-joint robot.

Given Fig. 1.3, we have $\mathbf{J}_{cc}=\begin{bmatrix} \mathbf{J}_{1_{cc}} & \mathbf{J}_{2_{cc}} & \mathbf{J}_{3_{cc}} & \mathbf{J}_{4_{cc}} \end{bmatrix}$ where

$$\begin{aligned} \mathbf{J}_{1_{cc}}&=\begin{bmatrix} Z_O\times(O_{cc}-O_0) \\ Z_O \end{bmatrix}=\begin{bmatrix} [001]^T\times[x_{cc}y_{cc}0]^T \\ [001]^T \end{bmatrix}=\begin{bmatrix} [-y_{cc}x_{cc}0]^T \\ [001]^T \end{bmatrix} \\ \mathbf{J}_{2_{cc}}&=\begin{bmatrix} Z_d\times(O_{cc}-O_d) \\ Z_d \end{bmatrix}=\begin{bmatrix} [001]^T\times[x_{cc}-x_dy_{cc}-y_d0]^T \\ [001]^T \end{bmatrix}=\begin{bmatrix} [y_d-y_{cc}x_{cc}-x_d0]^T \\ [001]^T \end{bmatrix} \end{aligned} \tag{12}$$

and $\mathbf{J}_{3_{cc}}=\mathbf{J}_{4_{cc}}=\mathbf{0}$. Thus, we get

$$\mathbf{J}_{v_2}=\begin{bmatrix} -l_1s_1-l_{c_2}s_{12} & -l_{c_2}s_{12} & 0 & 0 \\ l_1c_1+l_{c_2}c_{12} & l_{c_2}c_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \hat{\mathbf{J}}_{v_2}\triangleq\mathbf{J}_{v_2}^T\mathbf{J}_{v_2}=\begin{bmatrix} \hat{\mathbf{J}}_{v_2}^{11} & \hat{\mathbf{J}}_{v_2}^{12} & \hat{\mathbf{J}}_{v_2}^{13} & \hat{\mathbf{J}}_{v_2}^{14} \\ \hat{\mathbf{J}}_{v_2}^{21} & \hat{\mathbf{J}}_{v_2}^{22} & \hat{\mathbf{J}}_{v_2}^{23} & \hat{\mathbf{J}}_{v_2}^{24} \\ \hat{\mathbf{J}}_{v_2}^{31} & \hat{\mathbf{J}}_{v_2}^{32} & \hat{\mathbf{J}}_{v_2}^{33} & \hat{\mathbf{J}}_{v_2}^{34} \\ \hat{\mathbf{J}}_{v_2}^{41} & \hat{\mathbf{J}}_{v_2}^{42} & \hat{\mathbf{J}}_{v_2}^{43} & \hat{\mathbf{J}}_{v_2}^{4s4} \end{bmatrix} \tag{13}$$

such that $\hat{\mathbf{J}}_{v_2}^{11}=l_{12}^2+2l_1l_{c_2}c_2$, $\hat{\mathbf{J}}_{v_2}^{12}=\hat{\mathbf{J}}_{v_2}^{21}=l_{c_2}^2+l_1l_{c_2}c_2$, $\hat{\mathbf{J}}_{v_2}^{13}=\hat{\mathbf{J}}_{v_2}^{31}=0$, $\hat{\mathbf{J}}_{v_2}^{14}=\hat{\mathbf{J}}_{v_2}^{41}=0$, $\hat{\mathbf{J}}_{v_2}^{22}=l_{c_2}^2$, $\hat{\mathbf{J}}_{v_2}^{23}=\hat{\mathbf{J}}_{v_2}^{32}=0$, $\hat{\mathbf{J}}_{v_2}^{24}=\hat{\mathbf{J}}_{v_2}^{42}=0$, $\hat{\mathbf{J}}_{v_2}^{33}=0$, $\hat{\mathbf{J}}_{v_2}^{34}=\hat{\mathbf{J}}_{v_2}^{43}=0$ and $\hat{\mathbf{J}}_{v_2}^{44}=0$.

Jacobian Matrix Related to the Third Link:

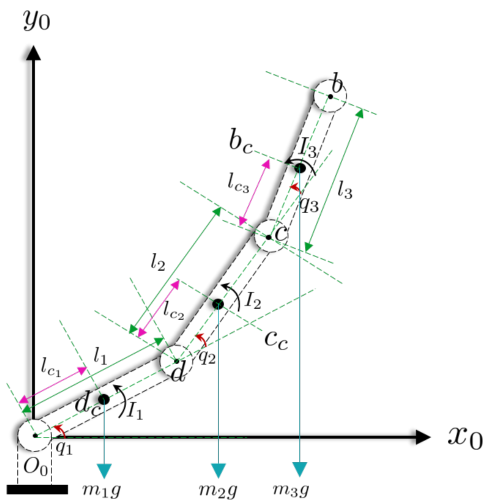

Figure 1.4: Schematic of the 3DOF planar revolute-joint robot.

Given Fig. 1.4, we have $\mathbf{J}_{bc}=\begin{bmatrix}\mathbf{J}_{1_{bc}} & \mathbf{J}_{2_{bc}} & \mathbf{J}_{3_{bc}} & \mathbf{J}_{4_{bc}}\end{bmatrix}$ where

$$\begin{aligned}
\mathbf{J}_{1_{bc}}&=\begin{bmatrix} Z_O\times(O_{bc}-O_0) \\ Z_O \end{bmatrix}=\begin{bmatrix} [001]^T\times[x_{bc}y_{bc}0]^T \\ [001]^T \end{bmatrix}=\begin{bmatrix} [-y_{bc}x_{bc}0]^T \\ [001]^T \end{bmatrix} \\
\mathbf{J}_{2_{bc}}&=\begin{bmatrix} Z_d\times(O_{bc}-O_d) \\ Z_d \end{bmatrix}=\begin{bmatrix} [001]^T\times[x_{bc}-x_dy_{bc}-y_d0]^T \\ [001]^T \end{bmatrix}=\begin{bmatrix} [y_d-y_{bc}x_{bc}-x_d0]^T \\ [001]^T \end{bmatrix} \\
\mathbf{J}_{3_{bc}}&=\begin{bmatrix} Z_c\times(O_{bc}-O_c) \\ Z_c \end{bmatrix}=\begin{bmatrix} [001]^T\times[x_{bc}-x_cy_{bc}-y_c0]^T \\ [001]^T \end{bmatrix}=\begin{bmatrix} [y_c-y_{bc}x_{bc}-x_c0]^T \\ [001]^T \end{bmatrix}
\end{aligned} \tag{14}$$

and $\mathbf{J}_{4_{bc}}=\mathbf{0}$. Accordingly, we get

$$\mathbf{J}_{v_3}=\begin{bmatrix} -l_1s_1-l_2s_{12}-l_{c_3}s_{123} & -l_2s_{12}-l_{c_3}s_{123} & -l_{c_3}s_{123} & 0 \\ l_1c_1+l_2c_{12}+l_{c_3}c_{123} & l_2c_{12}+l_{c_3}c_{123} & l_{c_3}c_{123} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\hat{\mathbf{J}}_{v_3}\triangleq\mathbf{J}_{v_3}^T\mathbf{J}_{v_3}=\begin{bmatrix} \hat{\mathbf{J}}_{v_3}^{11} & \hat{\mathbf{J}}_{v_3}^{12} & \hat{\mathbf{J}}_{v_3}^{13} & \hat{\mathbf{J}}_{v_3}^{14} \\ \hat{\mathbf{J}}_{v_3}^{21} & \hat{\mathbf{J}}_{v_3}^{22} & \hat{\mathbf{J}}_{v_3}^{23} & \hat{\mathbf{J}}_{v_3}^{24} \\ \hat{\mathbf{J}}_{v_3}^{31} & \hat{\mathbf{J}}_{v_3}^{32} & \hat{\mathbf{J}}_{v_3}^{33} & \hat{\mathbf{J}}_{v_3}^{34} \\ \hat{\mathbf{J}}_{v_3}^{41} & \hat{\mathbf{J}}_{v_3}^{42} & \hat{\mathbf{J}}_{v_3}^{43} & \hat{\mathbf{J}}_{v_3}^{4s4} \end{bmatrix} \tag{15}$$

where $\hat{\mathbf{J}}_{v_3}^{11}=l_{123}^2+2l_1(l_2c_2+l_{c_3}c_{23})+2l_2l_{c_3}c_3$, $\hat{\mathbf{J}}_{v_3}^{12}=\hat{\mathbf{J}}_{v_3}^{21}=l_{23}^2+l_1(l_2c_2+l_{c_3}c_{23})+2l_2l_{c_3}c_3$, $\hat{\mathbf{J}}_{v_3}^{13}=\hat{\mathbf{J}}_{v_3}^{31}=l_{c_3}^2+l_1l_{c_3}c_{23}+l_2l_{c_3}c_3$, $\hat{\mathbf{J}}_{v_3}^{14}=\hat{\mathbf{J}}_{v_3}^{41}=0$, $\hat{\mathbf{J}}_{v_3}^{22}=l_{23}^2+2l_2l_{c_3}c_3$, $\hat{\mathbf{J}}_{v_3}^{23}=\hat{\mathbf{J}}_{v_3}^{32}=l_{c_3}^2+l_2l_{c_3}c_3$, $\hat{\mathbf{J}}_{v_3}^{24}=\hat{\mathbf{J}}_{v_3}^{42}=0$, $\hat{\mathbf{J}}_{v_3}^{33}=l_{c_3}^2$, $\hat{\mathbf{J}}_{v_3}^{34}=\hat{\mathbf{J}}_{v_3}^{43}=0$ and $\hat{\mathbf{J}}_{v_3}^{44}=0$.

Jacobian Matrix Related to the Fourth Link:

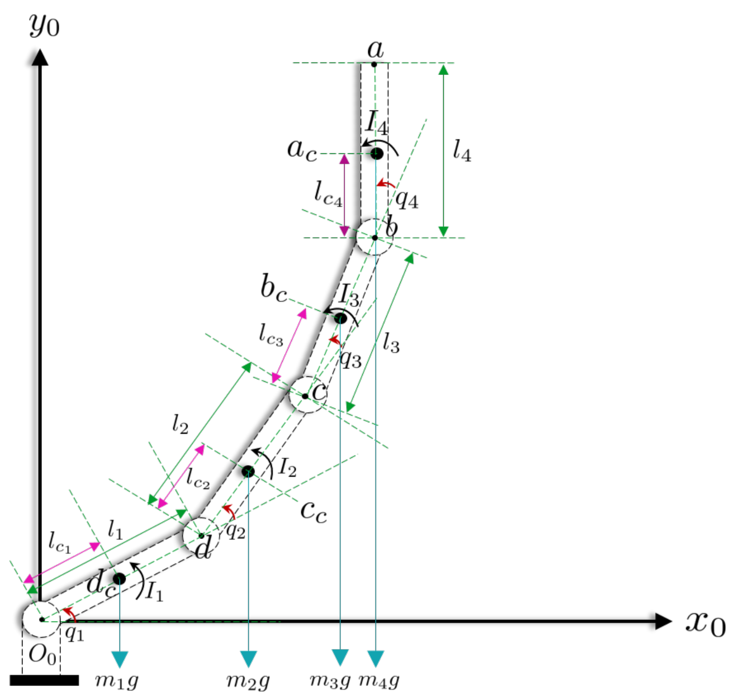

Figure 1.5: Schematic of the 4DOF planar revolute-joint robot.

Given Fig. 1.1, we have $\mathbf{J}_{ac}=\begin{bmatrix}\mathbf{J}1_{ac} & \mathbf{J}2_{ac} & \mathbf{J}3_{ac} & \mathbf{J}4_{ac}\end{bmatrix}$ where

$$\begin{aligned}
\mathbf{J}1_{ac}&=\begin{bmatrix}Z_O\times(O_{ac}-O_0)\\ Z_O\end{bmatrix}=\begin{bmatrix}[001]^T\times[x_{ac}y_{ac}0]^T\\ [001]^T\end{bmatrix}=\begin{bmatrix}[-y_{ac}x_{ac}0]^T\\ [001]^T\end{bmatrix}\\
\mathbf{J}2_{ac}&=\begin{bmatrix}Z_d\times(O_{ac}-O_d)\\ Z_d\end{bmatrix}=\begin{bmatrix}[001]^T\times[x_{ac}-x_d y_{ac}-y_d 0]^T\\ [001]^T\end{bmatrix}=\begin{bmatrix}[y_d-y_{ac}x_{ac}-x_d 0]^T\\ [001]^T\end{bmatrix}\\
\mathbf{J}3_{ac}&=\begin{bmatrix}Z_c\times(O_{ac}-O_c)\\ Z_c\end{bmatrix}=\begin{bmatrix}[001]^T\times[x_{ac}-x_c y_{ac}-y_c 0]^T\\ [001]^T\end{bmatrix}=\begin{bmatrix}[y_c-y_{ac}x_{ac}-x_c 0]^T\\ [001]^T\end{bmatrix}\\
\mathbf{J}4_{ac}&=\begin{bmatrix}Z_b\times(O_{ac}-O_b)\\ Z_b\end{bmatrix}=\begin{bmatrix}[001]^T\times[x_{ac}-x_b y_{ac}-y_b 0]^T\\ [001]^T\end{bmatrix}=\begin{bmatrix}[y_b-y_{ac}x_{ac}-x_b 0]^T\\ [001]^T\end{bmatrix}
\end{aligned}\tag{16}$$

Therefore,

$$\mathbf{J}_{v_4}=\begin{bmatrix}-l_1s_1-l_2s_{12}-l_3s_{123}-l_{c_4}s_{1234} & -l_2s_{12}-l_3s_{123}-l_{c_4}s_{1234} & -l_3s_{123}-l_{c_4}s_{1234} & -l_{c_4}s_{1234}\\ l_1c_1+l_2c_{12}+l_3c_{123}+l_{c_4}c_{1234} & l_2c_{12}+l_3c_{123}+l_{c_4}c_{1234} & l_3c_{123}+l_{c_4}c_{1234} & l_{c_4}c_{1234}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\hat{\mathbf{J}}_{v_4}\triangleq\mathbf{J}_{v_4}^T\mathbf{J}_{v_4}=\begin{bmatrix}\hat{\mathbf{J}}_{v_4}^{11} & \hat{\mathbf{J}}_{v_4}^{12} & \hat{\mathbf{J}}_{v_4}^{13} & \hat{\mathbf{J}}_{v_4}^{14}\\ \hat{\mathbf{J}}_{v_4}^{21} & \hat{\mathbf{J}}_{v_4}^{22} & \hat{\mathbf{J}}_{v_4}^{23} & \hat{\mathbf{J}}_{v_4}^{24}\\ \hat{\mathbf{J}}_{v_4}^{31} & \hat{\mathbf{J}}_{v_4}^{32} & \hat{\mathbf{J}}_{v_4}^{33} & \hat{\mathbf{J}}_{v_4}^{34}\\ \hat{\mathbf{J}}_{v_4}^{41} & \hat{\mathbf{J}}_{v_4}^{42} & \hat{\mathbf{J}}_{v_4}^{43} & \hat{\mathbf{J}}_{v_4}^{4s4}\end{bmatrix}\tag{17}$$

such that

$$\begin{aligned}
&\hat{\mathbf{J}}_{v_4}^{11}=l_{1234}^2+2l_1(l_2c_2+l_3c_{23}+l_{c_4}c_{234})+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4 \quad l_{1234}^2\triangleq l_1^2+l_2^2+l_3^2+l_{c_4}^2\\
&\hat{\mathbf{J}}_{v_4}^{12}=\hat{\mathbf{J}}_{v_4}^{21}=l_{234}^2+l_1(l_2c_2+l_3c_{23}+l_{c_4}c_{234})+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4 \quad l_{234}^2\triangleq l_2^2+l_3^2+l_{c_4}^2\\
&\hat{\mathbf{J}}_{v_4}^{13}=\hat{\mathbf{J}}_{v_4}^{31}=l_{34}^2+l_1(l_3c_{23}+l_{c_4}c_{234})+l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4 \quad l_{34}^2\triangleq l_3^2+l_{c_4}^2
\end{aligned}$$

$$\hat{\mathbf{J}}_{v_4}^{14}=\hat{\mathbf{J}}_{v_4}^{41}=l_{c_4}^2+l_1l_{c_4}c_{234}+l_2l_{c_4}c_{34}+l_3l_{c_4}c_4 \qquad \hat{\mathbf{J}}_{v_4}^{22}=l_{234}^2+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4$$

$$\hat{\mathbf{J}}_{v_4}^{23}=\hat{\mathbf{J}}_{v_4}^{32}=l_{34}^2+l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4 \qquad \hat{\mathbf{J}}_{v_4}^{24}=\hat{\mathbf{J}}_{v_4}^{42}=l_{c_4}^2+l_2l_{c_4}c_{34}+l_3l_{c_4}c_4$$

$$\hat{\mathbf{J}}_{v_4}^{33}=l_{34}^2+2l_3l_{c_4}c_4\,`cm\hat{\mathbf{J}}_{v_4}^{34}=\hat{\mathbf{J}}_{v_4}^{43}=l_{c_4}^2+l_3l_{c_4}c_4 \qquad \hat{\mathbf{J}}_{v_4}^{44}=l_{c_4}^2$$

Now, we can calculate the transnational pat of the inertia matrix.

$$D_{\text{transnational}}=m_1\hat{J}_{v_1}+m_2\hat{J}_{v_2}+m_3\hat{J}_{v_3}+m_4\hat{J}_{v_4}\triangleq\begin{bmatrix} D_t^{11} & D_t^{12} & D_t^{13} & D_t^{14} \\ D_t^{21} & D_t^{22} & D_t^{23} & D_t^{24} \\ D_t^{31} & D_t^{32} & D_t^{33} & D_t^{34} \\ D_t^{41} & D_t^{42} & D_t^{43} & D_t^{44} \end{bmatrix} \tag{18}$$

where

$$\begin{aligned}
D_t^{11}=&m_1\left[l_{c_1}^2\right]+m_2\left[l_{12}^2+2l_1l_{c_2}c_2\right]+m_3\left[l_{123}^2+2l_1(l_2c_2+l_{c_3}c_{23})+2l_2l_{c_3}c_3\right]\\
&+m_4\left[l_{1234}^2+2l_1(l_2c_2+l_3c_{23}+l_{c_4}c_{234})+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]\\
D_t^{12}=D_t^{21}=&m_2\left[l_{c_2}^2+l_1l_{c_2}c_2\right]+m_3\left[l_{23}^2+l_1(l_2c_2+l_{c_3}c_{23})+2l_2l_{c_3}c_3\right]\\
&+m_4\left[l_{234}^2+l_1(l_2c_2+l_3c_{23}+l_{c_4}c_{234})+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]\\
D_t^{13}=D_t^{31}=&m_3\left[l_{c_3}^2+l_1l_{c_3}c_{23}+l_2l_{c_3}c_3\right]\\
&+m_4\left[l_{34}^2+l_1(l_3c_{23}+l_{c_4}c_{234})+l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]\\
D_t^{14}=D_t^{41}=&m_4\left[l_{c_4}^2+l_1l_{c_4}c_{234}+l_2l_{c_4}c_{34}+l_3l_{c_4}c_4\right]\\
D_t^{22}=&m_2\left[l_{c_2}^2\right]+m_3\left[l_{23}^2+2l_2l_{c_3}c_3\right]+m_4\left[l_{234}^2+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]\\
D_t^{23}=D_t^{32}=&m_3\left[l_{c_3}^2+l_2l_{c_3}c_3\right]+m_4\left[l_{34}^2+l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]
\end{aligned}$$

$$D_t^{24}=m_4\left[l_{c_4}^2+l_2l_{c_4}c_{34}+l_3l_{c_4}c_4\right] \qquad D_t^{33}=m_3\left[l_{c_3}^2\right]+m_4\left[l_{34}^2+2l_3l_{c_4}c_4\right]$$

$$D_t^{34}=D_t^{43}=m_4\left[l_{c_4}^2+l_3l_{c_4}c_4\right] \qquad D_t^{44}=m_4\left[l_{c_4}^2\right]$$

1.4.3 Rotational Component of the Inertia Matrix

Let $\hat{\mathbf{J}}_{\omega_i} \triangleq \mathbf{J}_{\omega_i}^T \mathbf{J}_{\omega_i}$ and given previous section's calculations we can conclude that

$$\mathbf{J}_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \mathbf{J}_{\omega_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \mathbf{J}_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \mathbf{J}_{\omega_4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\hat{\mathbf{J}}_{\omega_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{\mathbf{J}}_{\omega_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{\mathbf{J}}_{\omega_3} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{\mathbf{J}}_{\omega_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (19)$$

and is obvious that $\omega_1 = \dot{q}_1 k$, $\omega_2 = (\dot{q}_1 + \dot{q}_2)k$, $\omega_3 = (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)k$ and $w_4 = (\dot{q}_1 + \dot{q}_2 + \dot{q}_3 + \dot{q}_4)k$. Since ω_i is aligned with k, the product $\mathbf{R}_i \mathbf{I}_i \mathbf{R}_i^T$ reduces simply to I_i. Therefore,

$$D_{\text{rotational}} = I_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + I_3 \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + I_4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} I_1+I_2+I_3+I_4 & I_2+I_3+I_4 & I_3+I_4 & I_4 \\ I_2+I_3+I_4 & I_2+I_3+I_4 & I_3+I_4 & I_4 \\ I_3+I_4 & I_3+I_4 & I_3+I_4 & I_4 \\ I_4 & I_4 & I_4 & I_4 \end{bmatrix} \quad (20)$$

and finally we can obtain the inertia matrix of the robot as follows

$$D_{\text{robot}} = D_{\text{transnational}} + D_{\text{rotational}} \triangleq \begin{bmatrix} D_r^{11} & D_r^{12} & D_r^{13} & D_r^{14} \\ D_r^{21} & D_r^{22} & D_r^{23} & D_r^{24} \\ D_r^{31} & D_r^{32} & D_r^{33} & D_r^{34} \\ D_r^{41} & D_r^{42} & D_r^{43} & D_r^{44} \end{bmatrix} \quad (21)$$

such that

$$
\begin{aligned}
D_r^{11}&=m_1\left[l_{c_1}^2\right]+m_2\left[l_{12}^2+2l_1l_{c_2}c_2\right]+m_3\left[l_{123}^2+2l_1(l_2c_2+l_{c_3}c_{23})+2l_2l_{c_3}c_3\right]\\
&\quad+m_4\left[l_{1234}^2+2l_1(l_2c_2+l_3c_{23}+l_{c_4}c_{234})+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]+I_{1234}\\
I_{1234}&\triangleq I_1+I_2+I_3+I_4\\
D_r^{12}&=D_r^{21}=m_2\left[l_{c_2}^2+l_1l_{c_2}c_2\right]+m_3\left[l_{23}^2+l_1(l_2c_2+l_{c_3}c_{23})+2l_2l_{c_3}c_3\right]\\
&\qquad+m_4\left[l_{234}^2+l_1(l_2c_2+l_3c_{23}+l_{c_4}c_{234})+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]+I_{234}\\
D_r^{13}&=D_r^{31}=m_3\left[l_{c_3}^2+l_1l_{c_3}c_{23}+l_2l_{c_3}c_3\right]\\
&\qquad+m_4\left[l_{34}^2+l_1(l_3c_{23}+l_{c_4}c_{234})+l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]+I_{34}\\
D_r^{14}&=D_r^{41}=m_4\left[l_{c_4}^2+l_1l_{c_4}c_{234}+l_2l_{c_4}c_{34}+l_3l_{c_4}c_4\right]+I_4\\
D_r^{22}&=m_2\left[l_{c_2}^2\right]+m_3\left[l_{23}^2+2l_2l_{c_3}c_3\right]+m_4\left[l_{234}^2+2l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]+I_{234}\\
D_r^{23}&=D_r^{32}=m_3\left[l_{c_3}^2+l_2l_{c_3}c_3\right]+m_4\left[l_{34}^2+l_2(l_3c_3+l_{c_4}c_{34})+2l_3l_{c_4}c_4\right]+I_{34}\\
D_r^{24}&=D_r^{42}=m_4\left[l_{c_4}^2+l_2l_{c_4}c_{34}+l_3l_{c_4}c_4\right]+I_4\\
D_r^{33}&=m_3\left[l_{c_3}^2\right]+m_4\left[l_{34}^2+2l_3l_{c_4}c_4\right]+I_{34}\\
D_r^{34}&=D_r^{43}=m_4\left[l_{c_4}^2+l_3l_{c_4}c_4\right]+I_4\\
D_r^{44}&=m_4\left[l_{c_4}^2\right]+I_4
\end{aligned}
\tag{22}
$$

1.5 Finding the Centrifugal/Coriolis Matrix

After finding the inertia matrix, Using the following formula we can obtain the elements of the Centrifugal/Coriolis Matrix.

$$
C_{ij}=\sum_{k=1}^{4}\frac{1}{2}\left\{\frac{\partial D_r^{ij}}{\partial q_k}+\frac{\partial D_r^{ik}}{\partial q_j}+\frac{\partial D_r^{kj}}{\partial q_i}\right\}
\tag{23}
$$

For more details, please check Appendix.

1.6 Finding the Gravitational Vector

To this end, we need to find the total potential energy of the manipulator. The potential energy of the manipulator is summation of the potential energies related to each link. Therefore, the potential energy for each link is listed below

$$\begin{aligned}
&P_1 = m_1 g l_{c_1} s_1 \\
&P_2 = m_2 g (l_1 s_1 + l_{c_2} s_{12}) \\
&P_3 = m_3 g (l_1 s_1 + l_2 s_{12} + l_{c_3} s_{123}) \\
&P_4 = m_4 g (l_1 s_1 + l_2 s_{12} + l_3 s_{123} + l_{c_4} s_{1234})
\end{aligned} \tag{24}$$

using which the total potential energy would be $P_{total} = P_1 + P_2 + P_3 + P_4$. Now we can find the gravitational vectors elements' as follows

$$\begin{aligned}
&g_{11} = \frac{\partial P}{\partial q_1} = g c_1 (m_1 l_{c_1} + m_{234} l_1) + g c_{12} (m_2 l_{c_2} + m_{34} l_2) + g c_{123} (m_3 l_{c_3} + m_4 l_{c_4}) + g c_{1234} m_4 l_{c_4} \\
&g_{21} = \frac{\partial P}{\partial q_1} = g c_{12} (m_2 l_{c_2} + m_{34} l_2) + g c_{123} (m_3 l_{c_3} + m_4 l_{c_4}) + g c_{1234} m_4 l_{c_4} \\
&g_{31} = \frac{\partial P}{\partial q_1} = g c_{123} (m_3 l_{c_3} + m_4 l_{c_4}) + g c_{1234} m_4 l_{c_4} \\
&g_{41} = \frac{\partial P}{\partial q_1} = g c_{1234} m_4 l_{c_4}
\end{aligned} \tag{25}$$

1.7 Finding the Regressor Matrix

Based one the formulation obtained for the inertia matrix, the Centrifugal/Coriolis matrix and the gravitational vector, the $\mathbf{\Theta} \in \mathbb{R}^{14 \times 1}$ was found and the regressor matrix $\mathbf{Y} \in \mathbb{R}^{4 \times 14}$ determined as

$$\begin{bmatrix}
Y_{1,1} & Y_{1,2} & Y_{1,3} & \dots & Y_{1,14} \\
Y_{2,1} & Y_{2,2} & Y_{2,3} & \dots & Y_{2,14} \\
Y_{3,1} & Y_{3,2} & Y_{3,3} & \dots & Y_{3,14} \\
Y_{3,1} & Y_{3,2} & Y_{3,3} & \dots & Y_{4,14}
\end{bmatrix} \tag{26}$$

where

$$Y_{1,1}=(2\ddot{q}_1+\ddot{q}_2)c_2-\left(2\dot{q}_1\dot{q}_2+\dot{q}_2^2\right)s_2$$

$$Y_{1,2}=(2\ddot{q}_1+\ddot{q}_2+\ddot{q}_3)c_{23}+\left(\dot{q}_3^2-2\dot{q}_1\dot{q}_2-\dot{q}_2^2\right)s_{23}$$

$$Y_{1,3}=(2\ddot{q}_1+\ddot{q}_2+\ddot{q}_3+\ddot{q}_4)c_{234}-\left(\dot{q}_2^2+\dot{q}_3^2+\dot{q}_4^2+2\dot{q}_1\dot{q}_2+2\dot{q}_1\dot{q}_3+2\dot{q}_1\dot{q}_4+2\dot{q}_2\dot{q}_3+2\dot{q}_2\dot{q}_4+2\dot{q}_3\dot{q}_4\right)s_{234}$$

$$Y_{1,4}=(2\ddot{q}_1+2\ddot{q}_2+\ddot{q}_3)c_3-\left(2\dot{q}_1\dot{q}_3+2\dot{q}_2\dot{q}_3+\dot{q}_3^2\right)s_3$$

$$Y_{1,5}=(2\ddot{q}_1+2\ddot{q}_2+\ddot{q}_3+\ddot{q}_4)c_{34}-\left(\dot{q}_3^2+\dot{q}_4^2+2\dot{q}_1\dot{q}_3+2\dot{q}_1\dot{q}_4+2\dot{q}_2\dot{q}_3+2\dot{q}_2\dot{q}_4+2\dot{q}_3\dot{q}_4\right)s_{34}$$

$$Y_{1,6}=(2\ddot{q}_1+2\ddot{q}_2+2\ddot{q}_3+\ddot{q}_4)c_4-\left(2\dot{q}_1\dot{q}_4+2\dot{q}_2\dot{q}_4+2\dot{q}_3\dot{q}_4+\dot{q}_4^2\right)s_4$$

$$Y_{1,7}=\ddot{q}_1 \qquad Y_{1,8}=\ddot{q}_2 \qquad Y_{1,9}=\ddot{q}_3 \qquad Y_{1,10}=\ddot{q}_4 \qquad Y_{1,11}=gc_1$$

$$Y_{1,12}=gc_{12} \qquad Y_{1,13}=gc_{123} \qquad Y_{1,14}=gc_{1234}$$

$$Y_{2,1}=\ddot{q}_1c_2+\dot{q}_1^2s_2$$

$$Y_{2,2}=\ddot{q}_1c_{23}+\dot{q}_1^2s_{23}$$

$$Y_{2,3}=\ddot{q}_1c_{234}+\dot{q}_1^2s_{234}$$

$$Y_{2,4}=(2\ddot{q}_1+2\ddot{q}_2+\ddot{q}_3)c_3-\left(2\dot{q}_1\dot{q}_3+2\dot{q}_2\dot{q}_3+\dot{q}_3^2\right)s_3$$

$$Y_{2,5}=(2\ddot{q}_1+2\ddot{q}_2+\ddot{q}_3+\ddot{q}_4)c_{34}-\left(\dot{q}_3^2+\dot{q}_4^2+2\dot{q}_1\dot{q}_3+2\dot{q}_1\dot{q}_4+2\dot{q}_2\dot{q}_3+2\dot{q}_2\dot{q}_4+2\dot{q}_3\dot{q}_4\right)s_{34}$$

$$Y_{2,6}=(2\ddot{q}_1+2\ddot{q}_2+2\ddot{q}_3+\ddot{q}_4)c_4-\left(2\dot{q}_1\dot{q}_4+2\dot{q}_2\dot{q}_4+2\dot{q}_3\dot{q}_4+\dot{q}_4^2\right)s_4$$

$$Y_{2,7}=0 \qquad Y_{2,8}=\ddot{q}_1+\ddot{q}_2 \qquad Y_{2,9}=\ddot{q}_3 \qquad Y_{2,10}=\ddot{q}_4$$

$$Y_{2,11}=0 \qquad Y_{2,12}=gc_{12} \qquad Y_{2,13}=gc_{123} \qquad Y_{2,14}=gc_{1234}$$

$$
\begin{aligned}
&Y_{3,1}=0\\
&Y_{3,2}=\ddot{q}_1c_{23}-\dot{q}_1^2s_{23}\\
&Y_{3,3}=\ddot{q}_1c_{234}-\dot{q}_1^2s_{234}\\
&Y_{3,4}=(\ddot{q}_1+\ddot{q}_2)c_3+\left(2\dot{q}_1\dot{q}_2-\dot{q}_1^2+\dot{q}_2^2\right)s_3\\
&Y_{3,5}=(\ddot{q}_1+\ddot{q}_2)c_{34}+\left(2\dot{q}_1\dot{q}_2-\dot{q}_1^2+\dot{q}_2^2\right)s_{34}\\
&Y_{3,6}=(2\ddot{q}_1+2\ddot{q}_2+2\ddot{q}_3+\ddot{q}_4)c_4-\left(2\dot{q}_1\dot{q}_4+2\dot{q}_2\dot{q}_4+2\dot{q}_3\dot{q}_4+\dot{q}_4^2\right)s_4\\
&Y_{3,7}=0\\
&Y_{3,8}=0\\
&Y_{3,9}=\ddot{q}_1+\ddot{q}_2+\ddot{q}_3\\
&Y_{3,10}=\ddot{q}_4\\
&Y_{3,11}=0\\
&Y_{3,12}=0\\
&Y_{3,13}=gc_{123}\\
&Y_{3,14}=gc_{1234}\\
&Y_{4,1}=0\\
&Y_{4,2}=0\\
&Y_{4,3}=\ddot{q}_1c_{234}+\dot{q}_1^2s_{234}\\
&Y_{4,4}=0\\
&Y_{4,5}=(\ddot{q}_1+\ddot{q}_2)c_{34}+\left(\dot{q}_1^2+\dot{q}_2^2+2\dot{q}_1\dot{q}_2\right)s_{34}\\
&Y_{4,6}=(\ddot{q}_1+\ddot{q}_2+\ddot{q}_3)c_4+\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2+2q_1\dot{q}_2+2\dot{q}_1\dot{q}_3+2\dot{q}_2\dot{q}_3\right)s_4\\
&Y_{4,7}=0\\
&Y_{4,8}=0\\
&Y_{4,9}=0\\
&Y_{4,10}=\ddot{q}_1+\ddot{q}_2+\ddot{q}_3+\ddot{q}_4\\
&Y_{4,11}=0\qquad Y_{4,12}=0\qquad Y_{4,13}=0\qquad Y_{4,14}=gc_{1234}
\end{aligned}
\tag{27}
$$

1.8 Finding the Dynamics Coefficients

According to the formulation obtained for the inertia matrix, the Centrifugal/Coriolis matrix and the gravitational vector, the $\Theta \in \mathbb{R}^{14\times 1}$ was found to be

$$
\begin{aligned}
&\theta_1 \triangleq m_2 l_1 l_{c_2} + m_3 l_1 l_2 + m_4 l_1 l_2 \\
&\theta_2 \triangleq m_4 l_1 l_3 + m_3 l_1 l_{c_3} \\
&\theta_3 \triangleq m_4 l_1 l_{c_4} \\
&\theta_4 \triangleq m_3 l_2 l_{c_3} + m_4 l_2 l_3 \\
&\theta_5 \triangleq m_4 l_2 l_{c_4} \\
&\theta_6 \triangleq m_4 l_3 l_{c_4} \\
&\theta_7 \triangleq m_1 l_{c_1}^2 + m_2 l_{12}^2 + m_3 l_{123}^2 + m_4 l_{1234}^2 + I_{1234} \\
&\theta_8 \triangleq m_2 l_{c_2}^2 + m_3 l_{23}^2 + m_4 l_{234}^2 + I_{234} \\
&\theta_9 \triangleq m_3 l_{c_3}^2 + m_4 l_{34}^2 + I_{34} \\
&\theta_{10} \triangleq m_4 l_{c_4}^2 + I_4 \\
&\theta_{11} \triangleq m_1 l_{c_1} + (m_2 + m_3 + m_4) l_1 \\
&\theta_{12} \triangleq m_2 l_{c_2} + (m_3 + m_4) l_2 \\
&\theta_{13} \triangleq m_3 l_{c_3} + m_4 l_3 \\
&\theta_{14} \triangleq m_4 l_{c_4}
\end{aligned}
$$

Therefore, having the full dynamics of a redundant 4DOF planar robot, I am able to implement most of the control methods that were covered in the class, including important methods such as adaptive control and redundancy resolution for implementing sub-task control.

2 Implementing Control Methods

In this chapter, the simulation results are derived for different controllers and as you will see, in all of them full tracking is achieved. Also, some videos accompany this report to explicitly illustrate the perfect modeling, implementation and performance of the robot and utilized controllers.

2.1 Adaptive Control

In this part, the adaptive control will be investigated for several scenarios including joint-space and task-space tacking, and redundancy resolution.

2.1.1 Tracking in Task-Space with Redundancy Resolution

Here, the adaptive control for the task-space tracking is combined with redundancy resolution. The physical parameters of the robot is shown in Table 2.1.

Table 2.1: Physical parameters.

	i=1	i=2	i=3	i=4
link length	1.05 m	0.96 m	0.94 m	0.92 m
link mass	3 kg	1.75 kg	1.02 kg	0.98 kg
link moment of inertia	0.12 kg.m^2	0.07 kg.m^2	0.04 kg.m^2	0.03 kg.m^2
initial joint position	$\pi/8$ *rad*	$\pi/10$ *rad*	$\pi/11$ *rad*	$\pi/8$ *rad*

For tracking in task-space, I have considered two desired trajectories for the end-effector of the robot as follows

$$Type1: \begin{cases} x=1+0.3cos(\pi t) \\ y=3+0.3sin(\pi t) \end{cases} \qquad Type2: \begin{cases} x=1+cos(0.2\pi t) \\ y=3.1+0.2sin(\pi t) \end{cases} \tag{1}$$

The utilized controller:

Defining error signal as $\mathbf{e}=\mathbf{X}_{desired}-\mathbf{X}_{end-effector}$, the following controller is utilized.

$$\begin{aligned} &\boldsymbol{\tau}=-\hat{\mathbf{M}}\mathbf{a}-\hat{\mathbf{C}}\mathbf{v}-\hat{\mathbf{G}}+\mathbf{K}\mathbf{s}+\mathbf{J}^T\bar{\boldsymbol{\tau}}=-\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{a},\mathbf{v})\hat{\boldsymbol{\Theta}}+\mathbf{K}\mathbf{s}+\mathbf{J}^T\bar{\boldsymbol{\tau}} \\ &\mathbf{s}\triangleq-\mathbf{J}^+\boldsymbol{\lambda}\mathbf{e}+\dot{\mathbf{q}}-\left(\mathbb{I}_4-\mathbf{J}^+\mathbf{J}\right)\boldsymbol{\Psi} \\ &\mathbf{v}\triangleq\mathbf{J}^+\boldsymbol{\lambda}\mathbf{e}+\left(\mathbb{I}_4-\mathbf{J}^+\mathbf{J}\right)\boldsymbol{\Psi} \\ &\mathbf{a}\triangleq\dot{\mathbf{J}}^+\boldsymbol{\lambda}\mathbf{e}+\mathbf{J}\boldsymbol{\lambda}\dot{\mathbf{e}}+\frac{d}{dt}\left[\left(\mathbb{I}_4-\mathbf{J}^+\mathbf{J}\right)\boldsymbol{\Psi}\right] \\ &\bar{\boldsymbol{\tau}}\triangleq\mathbf{K}_r\mathbf{r}-\mathbf{K}_J\dot{\mathbf{e}} \\ &\mathbf{r}\triangleq-\boldsymbol{\lambda}\mathbf{e}+\mathbf{J}\dot{\mathbf{q}} \end{aligned} \tag{2}$$

and the time-varying estimates of the uncertain parameters evolve as

$$\dot{\hat{\boldsymbol{\Theta}}}=-\boldsymbol{\Gamma}\mathbf{Y}^T\mathbf{s} \tag{3}$$

such that the initial estimate is 90% of the real values. All the constant matrices $\mathbf{K}$, $\boldsymbol{\lambda}$, $\mathbf{K}_r$ and $\mathbf{K}_J$ are positive. The auxiliary vector for redundancy resolution is slected as

$$\boldsymbol{\Psi}\triangleq\begin{bmatrix}1 & -2 & -0.9 & 1.4\end{bmatrix}^T(-0.1(q_1-2q_2-0.9q_3+1.4q_4)) \tag{4}$$

Results for Task-Space Trajectory Type 1:

Fig. 2.1 shows the desired trajectory and the trajectory of the robot's end-effector. As we see, we have achieved a perfect trajectory tracking. Please note that in this controller the redundancy resolution is also incorporated, and the attached videos clearly shows how redundancy resolution affects the robot configuration. Also, Fig. 2.2 shows the estimated values of the coefficients.

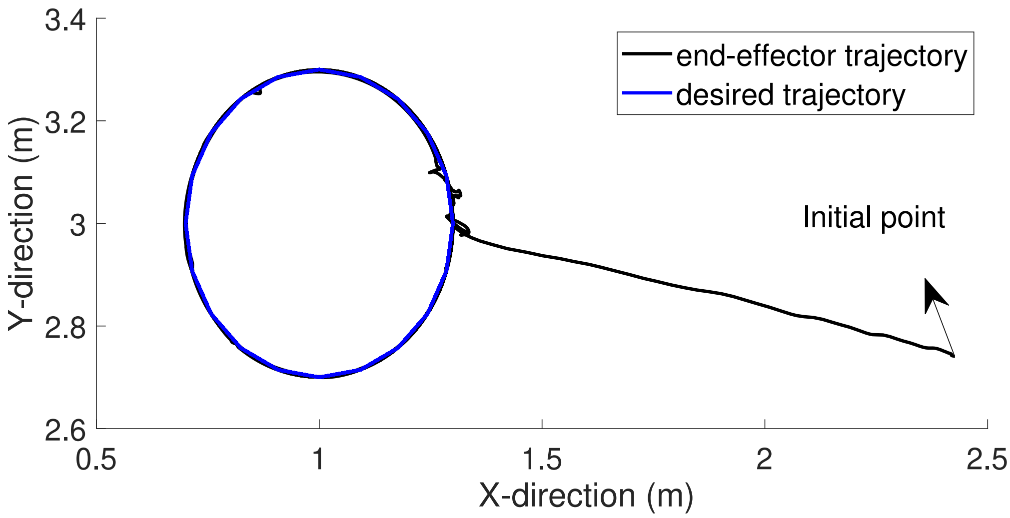

Figure 2.1: Tracking for considered trajectory type1 adaptive control

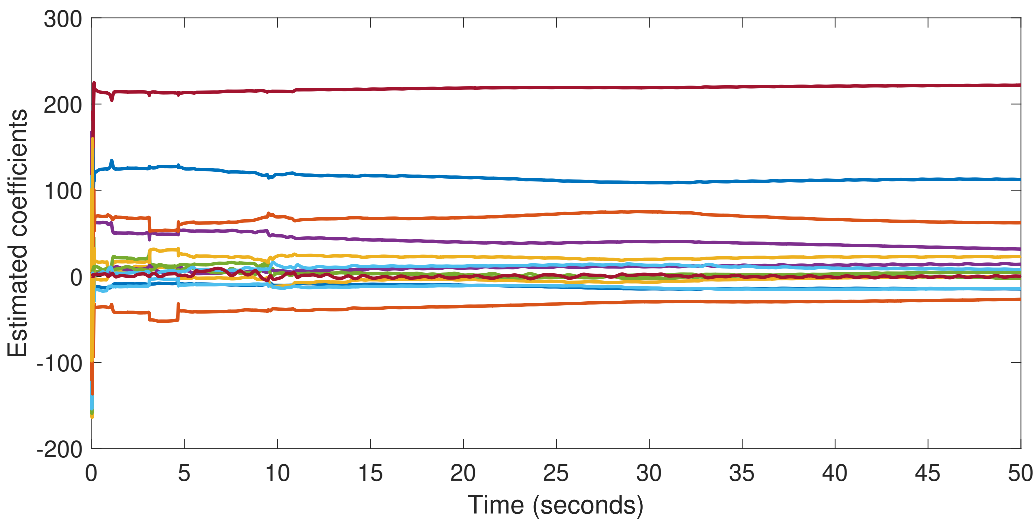

Figure 2.2: Estimated physical coefficients for type1

Results for Task-Space Trajectory Type 2:

Fig. 2.3 shows the desired trajectory type2 and end-effector tracking. As we again see, there is a perfect task-space tracking. Also, Fig. 2.4 shows the estimated values.

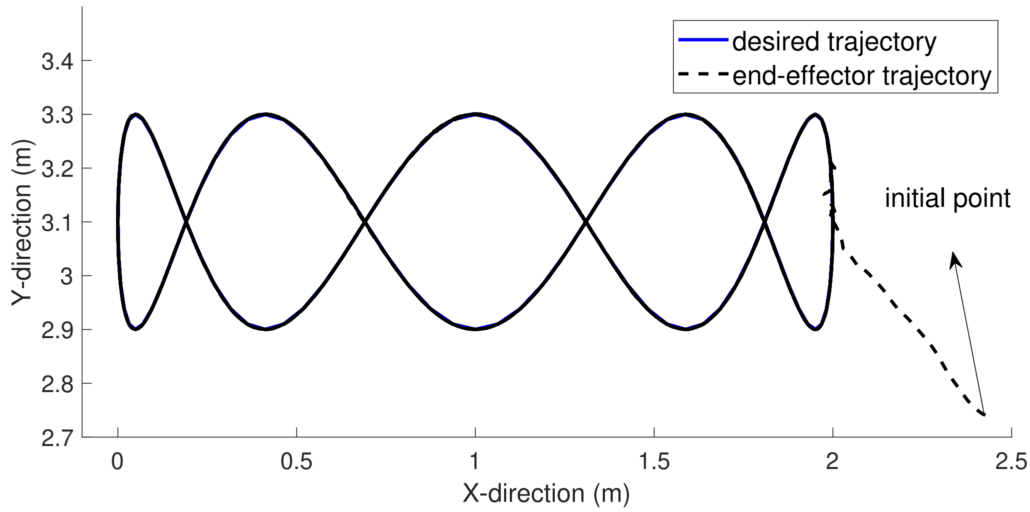

Figure 2.3: Tracking for considered trajectory type2 adaptive control

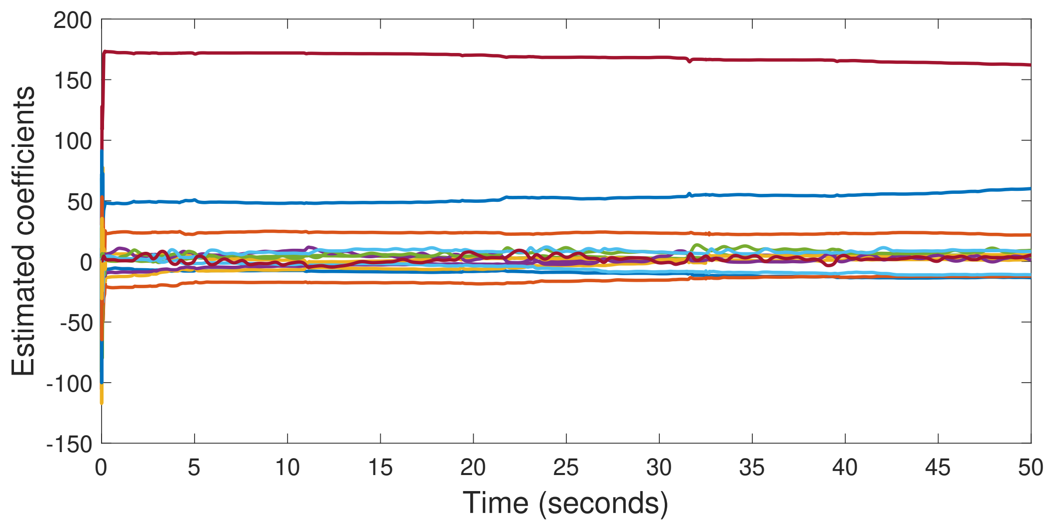

Figure 2.4: Estimated physical coefficients for type2

2.1.2 Manipulability and Singularity Avoidance

For singularity avoidance, we should aim to increase the manipulability of the robot, which in literature is characterized by the following formulation

$$\text{manipulability index} = \sqrt{det(\mathbf{J}\mathbf{J}^T)} \tag{5}$$

Therefore, the differentiable function for singularity avoidance can be defined as $f(\mathbf{q}) = \sqrt{det(\mathbf{J}\mathbf{J}^T)}$, using which the auxillary vector $\boldsymbol{\Psi}$ utilized in the controller can be defined as

$$\mathbf{\Psi} \triangleq -\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} = -\frac{\partial \sqrt{det(\mathbf{J}\mathbf{J}^T)}}{\partial \mathbf{q}} \tag{6}$$

Using this vector function, the robot would be able to regulate its configuration in order to increase the manipulability while tracking the desired trajectory. Fig. 2.5 shows the manipulability index for the case we have singularity avoidance as our sub-task control, i.e., $\mathbf{\Psi} \neq 0$, and for the case we do not incorporate the sub-task control in the controller development, i.e., $\mathbf{\Psi} = \mathbf{0}$. As you can see, by plugging the singularity avoidance task into the controller development, the manipulability of the robot has largely improved.

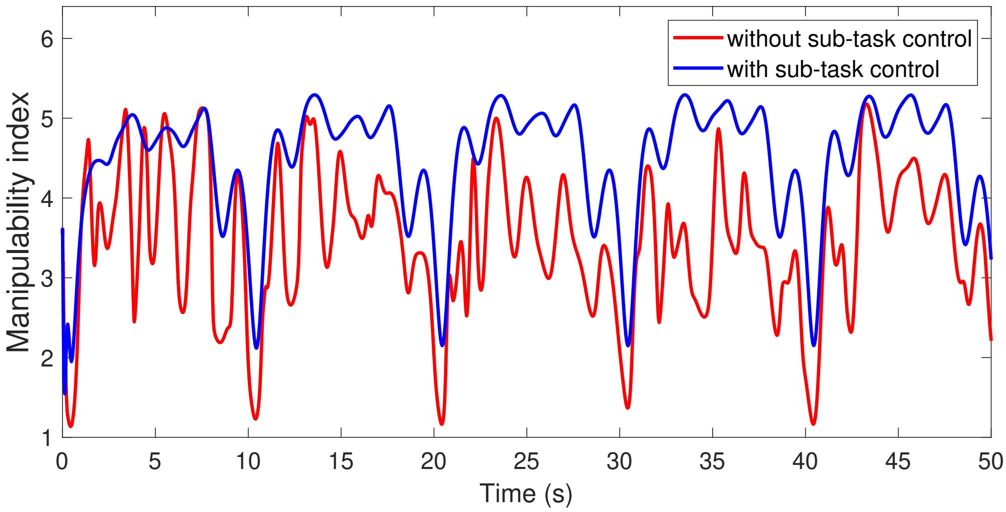

Figure 2.5: Manipulability index. Sub-task control: Singularity avoidance.

2.1.3 Tracking in Joint-Space

In this part, the tracking problem in joint-space is investigated. The physical parameters of the robot is shown in Table 2.2. For tracking in joint-space, I have considered four different trajectories for the joints of the robot as follows.

$$\text{Joints } 1-2: \begin{cases} q1_{desired} = 0.55 + 0.1cos(0.05\pi t) \\ q2_{desired} = 0.85 + 0.1cos(0.1\pi t) \end{cases} \quad \text{Joints } 3-4: \begin{cases} q3_{desired} = 0.75 + 0.1cos(0.15\pi t) \\ q4_{desired} = 1 + 0.01cos(0.5\pi t) \end{cases} \tag{7}$$

Table 2.2: Physical parameters.

	i=1	i=2	i=3	i=4
link length	0.95 m	1.1 m	0.49 m	0.88 m
link mass	1.1 kg	0.87 kg	0.56 kg	0.77 kg
link moment of inertia	0.12 kg.m^2	0.17 kg.m^2	0.15 kg.m^2	0.19 kg.m^2
initial joint position	$\pi/6\ rad$	$\pi/6\ rad$	$\pi/6\ rad$	$\pi/6\ rad$

The utilized controller:

For joint-space tracking, I have implemented the following controller.

$$\begin{aligned}
&\boldsymbol{\tau}=\hat{\mathbf{M}}\ddot{\mathbf{q}}_r+\hat{\mathbf{C}}\dot{q}_r+\hat{\mathbf{G}}+\mathbf{K}_P\mathbf{e}+\mathbf{K}_D\dot{\mathbf{e}}=\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r)\hat{\boldsymbol{\Theta}}+\mathbf{K}_P\mathbf{e}+\mathbf{K}_D\dot{\mathbf{e}}\\
&\mathbf{e}\triangleq\mathbf{q}_{desired}-\mathbf{q}\\
&\dot{\mathbf{q}}_r\triangleq\dot{\mathbf{q}}+\boldsymbol{\Lambda}\mathbf{e}\\
&\boldsymbol{\Lambda}\triangleq\mathbf{K}_D^{-1}\mathbf{K}_P
\end{aligned} \tag{8}$$

Defining $\mathbf{s}\triangleq\dot{\mathbf{q}}_\mathbf{r}-\dot{\mathbf{q}}$, the time-varying estimates of the uncertain parameters evolve as

$$\dot{\hat{\boldsymbol{\Theta}}}=\boldsymbol{\Gamma}\mathbf{Y}^T(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r)\mathbf{s} \tag{9}$$

Results for joint-space trajectory:

Fig. 2.6 shows the different desired joints positions and the robots joints' trajectories. As you can see, using the controller, the tracking has been achieved properly. Also, Fig. 2.7 depicts the time derivative of coefficients' estimations, and as it was expected, it converges to zero. In line with Fig. 2.7, Fig. 2.8 illustrates that the estimation of the dynamics coefficients have converged to constant values.

2.2 Impedance Control: Task-Space Regulation

In this section, the impedance controller is implemented to achieved regulation. A constant point is considered to be the desired position for the end-effector of the robot,

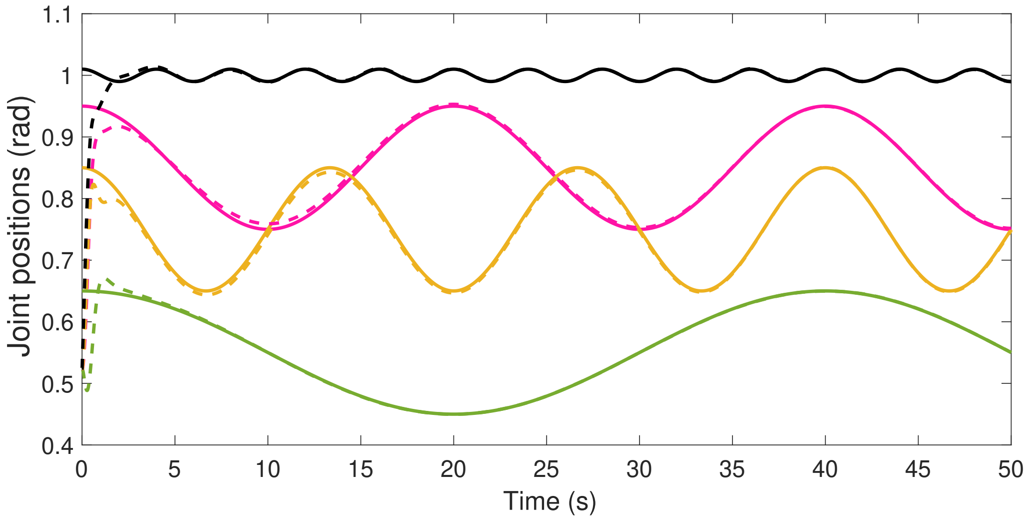

Figure 2.6: Joint positions tracking. Solid lines are desired ones and dashed ones are the real positions.

and in free motion, the performance of the impedance controller is investigated. The desired reference impedance model for the robot end-effector in Cartesian coordinates is considered to be

$$m\ddot{\tilde{\mathbf{X}}}_m + c\dot{\tilde{\mathbf{x}}}_m + k\tilde{\mathbf{X}}_m = f_{ext} \tag{10}$$

where $\tilde{X}_m = \mathbf{X}_m - \mathbf{X}_{eq}$, $\mathbf{X}_m$ is the reference model position in Cartesian coordinates, and $\mathbf{X_{eq}}$ is the desired equilibrium trajectory to be tracked by the robot in free motion. For this scenario, the robot physical parameters is considered to be as shown in Table 2.3.

Table 2.3: Physical parameters.

	i=1	i=2	i=3	i=4
link length	1.08 m	0.98 m	0.94 m	0.92 m
link mass	3.12 kg	1.85 kg	1.02 kg	0.98 kg
link moment of inertia	0.12 kg.m^2	0.07 kg.m^2	0.04 kg.m^2	0.03 kg.m^2
initial joint position	$\pi/8\ rad$	$\pi/10\ rad$	$\pi/11\ rad$	$\pi/8\ rad$

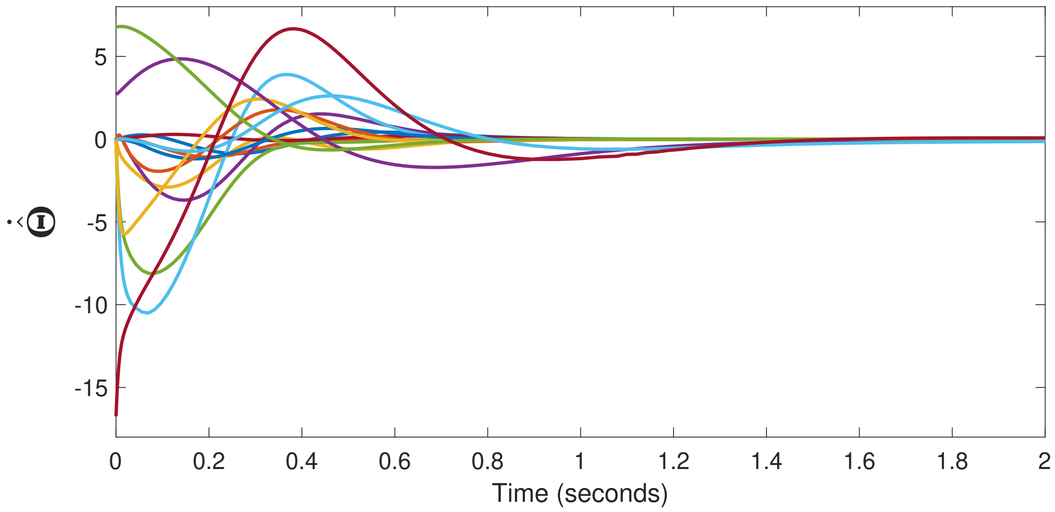

Figure 2.7: Derivation of the coefficient Estimates.

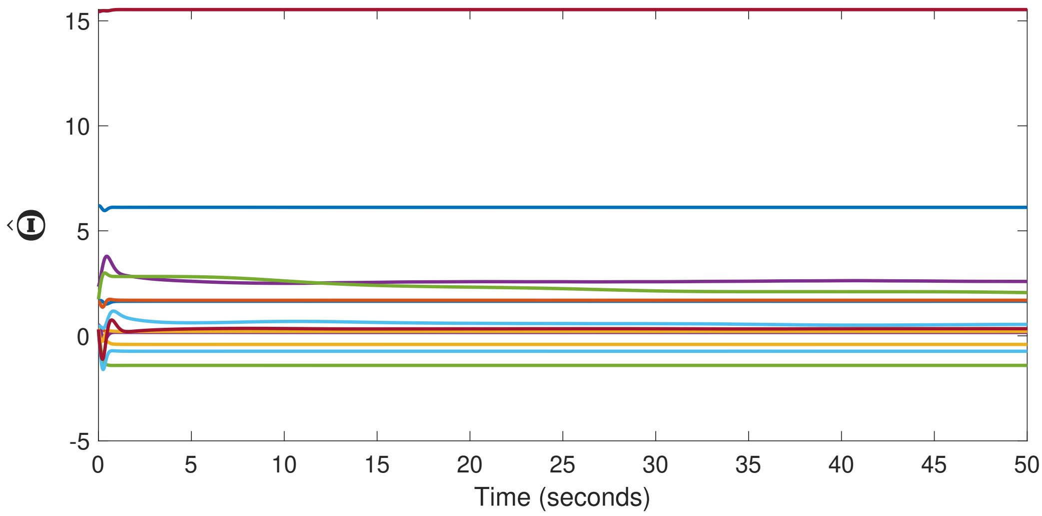

Figure 2.8: Derivation of the coefficient Estimates.

The utilized controller:

$$
\begin{aligned}
&\boldsymbol{\tau} = \mathbf{Ma} + \mathbf{Cv} + \mathbf{G} \\
&\mathbf{a} \triangleq \mathbf{J}^{-1}\left(\ddot{\mathbf{X}}_{eq} - \frac{c}{m}(\dot{\mathbf{X}} - \dot{\mathbf{X}}_{eq}) - \frac{k}{m}(\mathbf{X} - \mathbf{X}_{eq}) + \lambda_2^2 \tilde{\mathbf{X}} - \dot{\mathbf{J}}\mathbf{J}^{-1}\dot{\mathbf{X}}_r\right) \\
&\mathbf{v} \triangleq \mathbf{J}^{-1}\dot{\mathbf{X}}_r \\
&\dot{\mathbf{X}}_r \triangleq \dot{\mathbf{X}}_m - \lambda_1 \tilde{\mathbf{X}}
\end{aligned}
\tag{11}
$$

where $\lambda_1, \lambda_2 \in \mathbf{R}$ are positive values.

Regulation results for impedance control:

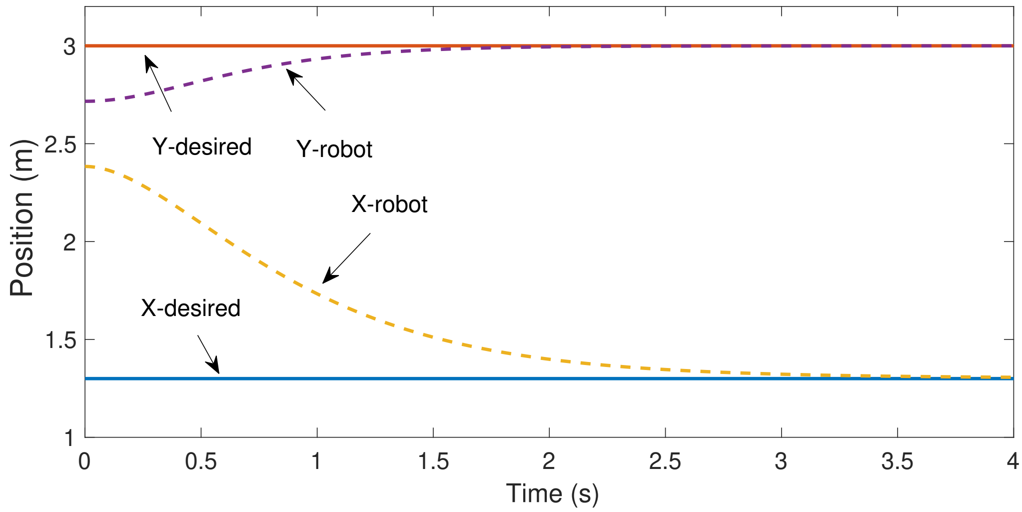

Figure 2.9: Convergence of the robot trajectory to the desired reference model's position.

Fig. 2.9 shows the position of the reference model as the desired trajectory point, and the trajectory of the end-effector of the robot.

2.3 Tracking with Feedback Linearization Control + PD

For this controller, the physical parameters of the robot is considered to be according to the following Table 2.4.

Table 2.4: Physical parameters.

	i=1	i=2	i=3	i=4
link length	0.95 m	1.1 m	0.49 m	0.88 m
link mass	1.1 kg	0.87 kg	0.56 kg	0.77 kg
link moment of inertia	0.12 kg.m^2	0.17 kg.m^2	0.14 kg.m^2	0.19 kg.m^2
initial joint position	$\pi/8$ rad	$\pi/10$ rad	$\pi/11$ rad	$\pi/8$ rad

Tracking in joint-space

In this part, the feedback linearization control + PD is implemented. The role of feedback linearization part is to cancel the nonlinearities of the dynamics and the role of

PD control is to regulate the performance of error signal. For tracking in joint-space, I have considered four different trajectories for the joints of the robot as follows

$$\text{Joints } 1-2: \begin{cases} q1_{desired} = 0.55 + 0.1cos(0.05\pi t) \\ q2_{desired} = 0.85 + 0.1cos(0.1\pi t) \end{cases} \quad \text{Joints } 3-4: \begin{cases} q3_{desired} = 0.75 + 0.1cos(0.15\pi t) \\ q4_{desired} = 1 + 0.01cos(0.5\pi t) \end{cases} \tag{12}$$

The utilized controller:

$$\boldsymbol{\tau} = \mathbf{M}(\ddot{\mathbf{q}}_d + \mathbf{K}_P(\mathbf{q}_d - \mathbf{q}) + \mathbf{K}_D(\dot{\mathbf{q}}_d - \dot{\mathbf{q}})) + \mathbf{C}\dot{\mathbf{q}} + \mathbf{G} \tag{13}$$

where the control gains $\mathbf{K}_D$ and $\mathbf{K}_P$ are positive matrices.

Results:

Fig. 2.10 shows the desired joint positions, and the trajectories of the root's joints. As you can see, the joints trajectories have converged to the desired trajectories. Also, Fig. 2.11 shows the joint velocities which all are bounded.

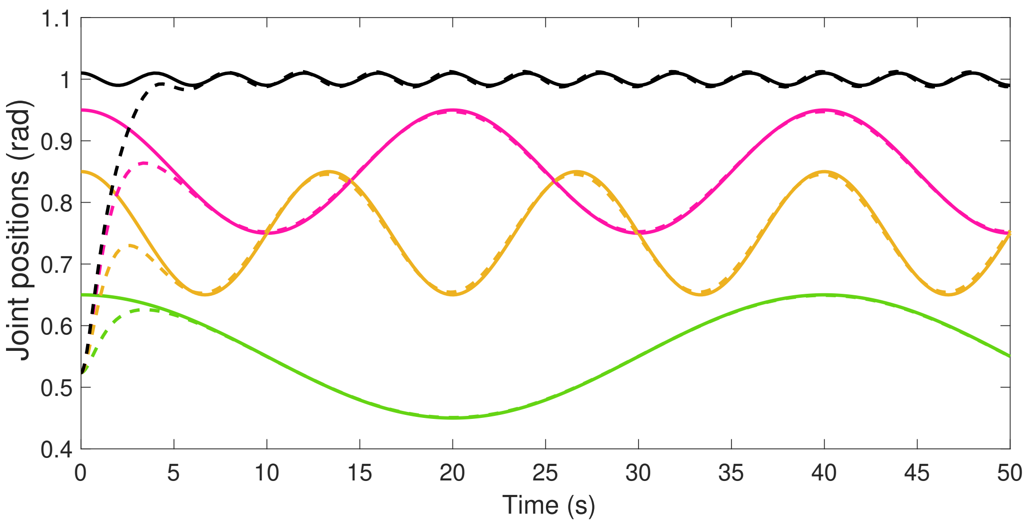

Figure 2.10: Joint-space tracking results for the robot's joints.

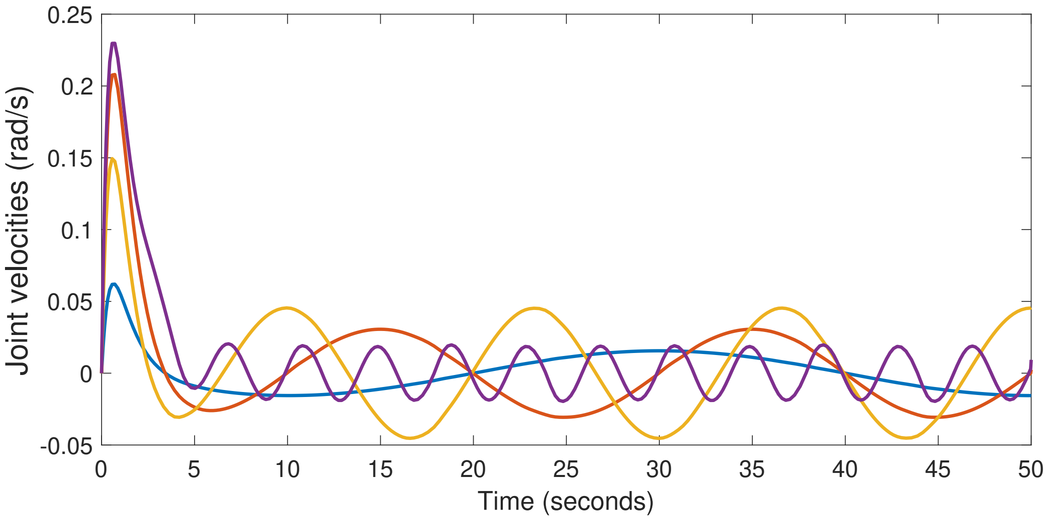

Figure 2.11: Robot joint velocities.

A1 | Appendix

A1.1 Detailed Calculation of Matrix C

Finding Matrix C

ص 1

$$\underline{\underline{C_{11}}} = \sum_{i=1}^{4} \frac{1}{2} \left\{ \frac{\partial D_r^{11}}{\partial q_i} + \frac{\partial D_r^{1i}}{\partial q_1} - \frac{\partial D_r^{i1}}{\partial q_1} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{11}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{11}}{\partial q_2} = -2m_2 l_1 l_{c2} S_2 - 2m_3 l_1 l_2 S_2 - 2m_3 l_1 l_{c3} S_{23} - 2m_4 l_1 l_3 S_{23} - 2m_4 l_1 l_{c4} S_{234} - 2m_4 l_1 l_2 S_2$$

$$= \cancel{\left(-(2m_2 l_1 l_{c2} + 2m_3 l_1 l_2\right) S_2}$$

$$= \cancel{\left(-2 l_1 (m_2+m_3+m_4)(2 l_2 + l_{c2}\right) S_2}$$

$$= \left[-2m_2 l_1 l_{c2} - 2m_3 l_1 l_2 - 2m_4 l_1 l_2\right] S_2 + \left[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\right] S_{23} - 2m_4 l_1 l_{c4} S_{234}$$

$$\frac{\partial D_r^{11}}{\partial q_3} = -2m_3 l_1 l_{c3} S_{23} - 2m_3 l_2 l_{c3} S_3 - 2m_4 l_1 l_3 S_{23} - 2m_4 l_1 l_{c4} S_{234} - 2m_4 l_2 l_3 S_3 - 2m_4 l_2 l_{c4} S_{34}$$

$$= \left[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3\right] S_3 + \left[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\right] S_{23} + \left[-2m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34}$$

$$\frac{\partial D_r^{11}}{\partial q_4} = -2m_4 l_1 l_{c4} S_{234} - 2m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{12}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{13}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{14}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{21}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{31}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{41}}{\partial q_1} = 0$$

$$\underline{\underline{C_{11}}} = \frac{1}{2}[0]\dot{q}_1 + \frac{1}{2}\Big[\left[-2m_2 l_1 l_{c2} - 2m_3 l_1 l_2 - 2m_4 l_1 l_2\right] S_2 + \left[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\right] S_{23} + \left[-2m_4 l_1 l_{c4}\right] S_{234}\Big]\dot{q}_2$$

$$+ \frac{1}{2}\Big[\left[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3\right] S_3 + \left[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\right] S_{23} + \left[-2m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34}\Big]\dot{q}_3$$

$$+ \frac{1}{2}\Big[\left[-2m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34} + \left[-2m_4 l_3 l_{c4}\right] S_4\Big]\dot{q}_4$$

ص۲

$$\underline{\underline{C_{12}}} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{12}}{\partial q_i} + \frac{\partial D_r^{1i}}{\partial q_2} - \frac{\partial D_r^{i2}}{\partial q_1} \right\} \dot{q}_i$$

ص۱

$$\frac{\partial D_r^{12}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{12}}{\partial q_2} = -m_2 l_1 l_{c2} S_2 - m_3 l_1 l_2 S_2 - m_3 l_1 l_{c3} S_{23} - m_4 l_1 l_2 S_2 - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234}$$

$$= \left[-m_2 l_1 l_{c2} - m_3 l_1 l_2 - m_4 l_1 l_2\right] S_2 + \left[-m_3 l_1 l_{c3}\right] S_{23} + \left[-m_4 l_1 l_3\right] S_{23}$$

$$\frac{\partial D_r^{12}}{\partial q_3} = -m_3 l_1 l_{c3} S_{23} - 2m_3 l_2 l_{c3} S_3 - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234} - 2m_4 l_2 l_3 S_3 - 2m_4 l_2 l_{c4} S_{34}$$

$$= \left[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3\right] S_3 + \left[-m_3 l_1 l_{c3} - m_4 l_1 l_3\right] S_{23} + \left[-m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34}$$

$$\frac{\partial D_r^{12}}{\partial q_4} = -m_4 l_1 l_{c4} S_{234} - 2m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{11}}{\partial q_2} = -2m_2 l_1 l_{c2} S_2 - 2m_3 l_1 l_2 S_2 - 2m_3 l_1 l_{c3} S_{23} - 2m_4 l_1 l_2 S_2 - 2m_4 l_1 l_3 S_{23} - 2m_4 l_1 l_{c4} S_{234}$$

$$= \left[-2m_2 l_1 l_{c2} - 2m_3 l_1 l_2 - 2m_4 l_1 l_2\right] S_2 + \left[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\right] S_{23} + \left[-2m_4 l_1 l_{c4}\right] S_{234}$$

$$\frac{\partial D_r^{12}}{\partial q_2} = -m_2 l_1 l_{c2} S_2 - m_3 l_1 l_2 S_2 - m_3 l_1 l_{c3} S_{23} - m_4 l_1 l_2 S_2 - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234}$$

$$= \left[-m_2 l_1 l_{c2} - m_3 l_1 l_2 - m_4 l_1 l_2\right] S_2 + \left[-m_3 l_1 l_{c3} - m_4 l_1 l_3\right] S_{23} + \left[-m_4 l_1 l_{c4}\right] S_{234}$$

$$\frac{\partial D_r^{13}}{\partial q_2} = -m_3 l_1 l_{c3} S_{23} - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234}$$

$$\frac{\partial D_r^{14}}{\partial q_2} = -m_4 l_1 l_{c4} S_{234}$$

$$\frac{\partial D_r^{12}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{32}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{22}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{42}}{\partial q_1} = 0$$

C_{12}

ص2

ص3

$$C_{12} = \frac{1}{2}\Big[\big[-2m_2 l_1 l_{c2} - 2m_3 l_1 l_2 - 2m_4 l_1 l_2\big]S_2 + \big[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\big]S_{23}$$

$$+ \big[-2m_4 l_1 l_{c4}\big]S_{234} - 0\Big]\dot{q}_1$$

$$+ \frac{1}{2}\Big[\overset{*2}{\big[-m_2 l_1 l_{c2} - m_3 l_1 l_2 - m_4 l_1 l_2\big]}S_2 + \overset{*2}{\big[-m_3 l_1 l_{c3} - m_4 l_1 l_3\big]}S_{23}$$

$$+ \overset{*2}{\big[-m_4 l_1 l_{c4}\big]}S_{234}\Big]\dot{q}_2$$

$$+ \frac{1}{2}\Big[\big[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3\big]S_3 + \big[-m_3 l_1 l_{c3} - m_4 l_1 l_3\big]S_{23}$$

$$+ \big[-m_4 l_1 l_{c4}\big]S_{234} + \big[-2m_4 l_2 l_{c4}\big]S_{34}$$

$$+ \big[-m_3 l_1 l_{c3} - m_4 l_1 l_3\big]S_{23} + \big[-m_4 l_1 l_{c4}\big]S_{234}\Big]\dot{q}_3$$

$$+ \frac{1}{2}\Big[\big[-m_4 l_1 l_{c4}\big]S_{234} + \big[-2m_4 l_2 l_{c4}\big]S_{34} + \big[-2m_4 l_3 l_{c4}\big]S_4$$

$$+ \big[-m_4 l_1 l_{c4}\big]S_{234}\Big]\dot{q}_4$$

4ص

$$C_{13} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{13}}{\partial q_i} + \frac{\partial D_r^{1i}}{\partial q_3} - \frac{\partial D_r^{i3}}{\partial q_1} \right\} \dot{q}_i$$

١٥

$$\frac{\partial D_r^{13}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{13}}{\partial q_1} = \frac{\partial D_r^{23}}{\partial q_1} = \frac{\partial D_r^{33}}{\partial q_1} = \frac{\partial D_r^{43}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{13}}{\partial q_2} = -m_3 l_1 l_{c3} S_{23} - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234}$$
$$= [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234}$$

$$\frac{\partial D_r^{13}}{\partial q_3} = -m_3 l_1 l_{c3} S_{23} - m_3 l_2 l_{c3} S_3 - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_3 S_3$$
$$- m_4 l_2 l_{c4} S_{34}$$
$$= [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23}$$
$$+ [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{13}}{\partial q_4} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$
$$= [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{11}}{\partial q_3} = -2m_3 l_1 l_{c3} S_{23} - 2m_3 l_2 l_{c3} S_3 - 2m_4 l_1 l_3 S_{23} - 2m_4 l_1 l_{c4} S_{234}$$
$$-2m_4 l_2 l_3 S_3 - 2m_4 l_2 l_{c4} S_{34}$$
$$= [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3] S_{23}$$
$$+ [-2m_4 l_1 l_{c4} - 2m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{12}}{\partial q_3} = -m_3 l_1 l_{c3} S_{23} - 2m_3 l_2 l_{c3} S_3 - m_4 l_3 l_1 S_{23} - m_4 l_1 l_{c4} S_{234}$$
$$-2m_4 l_2 l_3 S_3 - 2m_4 l_2 l_{c4} S_{34}$$
$$= [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_3 l_1] S_{23}$$
$$+ [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{13}}{\partial q_3} = -m_3 l_1 l_{c3} S_{23} - m_3 l_2 l_{c3} S_3 - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234}$$
$$- m_4 l_2 l_3 S_3 - m_4 l_2 l_{c4} S_{34}$$
$$= [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23}$$
$$+ [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{14}}{\partial q_3} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34} = [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34}$$

C_{13}

ص 2

ص 5

$$C_{13} = \frac{1}{2}\Big[0 + [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3] S_{23}$$
$$+ [-2m_4 l_1 l_{c4} - 2m_4 l_2 l_{c4}] S_{34} + 0 \Big] \dot{q}_1$$

$$+ \frac{1}{2}\Big[[-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234}$$
$$+ [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_3 l_1] S_{23}$$
$$+ [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] \Big] \dot{q}_2$$

$$+ \frac{1}{2}\Big[{}^{*2}[-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + {}^{*2}[-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23}$$
$$+ {}^{*2}[-m_4 l_1 l_{c4}] S_{234} + {}^{*2}[-m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_3$$

$$+ \frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$
$$+ [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_4$$

ص ٥

$$\underline{\underline{C_{14}}} = \sum_{i=1}^{4} \frac{1}{2} \left\{ \frac{\partial D_r^{14}}{\partial q_i} + \frac{\partial D_r^{1i}}{\partial q_4} - \frac{\partial D_r^{i4}}{\partial q_1} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{14}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{14}}{\partial q_2} = -m_4 l_1 l_{c4} S_{234} - \cancel{m_4 l_2 l_{c4} S_{34}}$$

$$\frac{\partial D_r^{14}}{\partial q_3} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{14}}{\partial q_4} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34} - m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{11}}{\partial q_4} = -2m_4 l_1 l_{c4} S_{234} - 2m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{12}}{\partial q_4} = -m_4 l_1 l_{c4} S_{234} - 2m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{13}}{\partial q_4} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{14}}{\partial q_4} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34} - m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{14}}{\partial q_1} = \frac{\partial D_r^{24}}{\partial q_1} = \frac{\partial D_r^{34}}{\partial q_1} = \frac{\partial D_r^{44}}{\partial q_1} = 0$$

$$\underline{\underline{C_{14}}} = \frac{1}{2} \Big[[-2m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4 \Big] \dot{q}_1$$

$$+ \frac{1}{2} \Big[[-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4 \Big] \dot{q}_2$$

$$+ \frac{1}{2} \Big[[-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4 \Big] \dot{q}_3$$

$$+ \frac{1}{2} \Big[\overset{*2}{[-m_4 l_1 l_{c4}]} S_{234} + \overset{*2}{[-m_4 l_2 l_{c4}]} S_{34} + \overset{*2}{[-m_4 l_3 l_{c4}]} S_4 \Big] \dot{q}_4$$

ص۲

$$\underline{C_{21}} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{21}}{\partial q_i} + \frac{\partial D_r^{2i}}{\partial q_1} - \frac{\partial D_r^{i1}}{\partial q_2} \right\}\dot q_i \quad -\left[-m_4 l_1 l_{c4}\right] S_{234}\Big]\dot q_4$$

$$+\frac{1}{2}\Big[\left[-m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34} + \left[-2m_4 l_3 l_{c4}\right] S_4$$

$$\frac{\partial D_r^{21}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{21}}{\partial q_2} = \left[-m_2 l_1 l_{c2} - m_3 l_1 l_2 - m_4 l_1 l_2\right] S_2 + \left[-m_3 l_1 l_{c3}\right] S_{23} + \left[-m_4 l_1 l_3\right] S_{23}$$

$$\frac{\partial D_r^{21}}{\partial q_3} = \left[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3\right] S_3 + \left[-m_3 l_1 l_{c3} - m_4 l_1 l_3\right] S_{23} + \left[-m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34}$$

$$\frac{\partial D_r^{21}}{\partial q_4} = \left[-m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34} + \left[-2m_4 l_3 l_{c4}\right] S_4$$

$$\frac{\partial D_r^{21}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{22}}{\partial q_1} = \frac{\partial D_r^{23}}{\partial q_1} = \frac{\partial D_r^{24}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{11}}{\partial q_2} = -2m_2 l_1 l_{c2} S_2 - 2m_3 l_1 l_2 S_2 - 2m_3 l_1 l_{c3} S_{23} - 2m_4 l_1 l_{c4} S_{234} - 2m_4 l_1 l_2 S_2 - 2m_4 l_1 l_3 S_{23}$$

$$= \left[-2m_2 l_1 l_{c2} - 2m_3 l_1 l_2 - 2m_4 l_1 l_2\right] S_2 + \left[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\right] S_{23} + \left[-2m_4 l_1 l_{c4}\right] S_{234}$$

$$\frac{\partial D_r^{21}}{\partial q_2} = \checkmark$$

$$\frac{\partial D_r^{31}}{\partial q_2} = -m_3 l_1 l_{c3} S_{23} - m_4 l_1 l_3 S_{23} - m_4 l_1 l_{c4} S_{234}$$

$$\frac{\partial D_r^{41}}{\partial q_2} = -m_4 l_1 l_{c4} S_{234}$$

$$\underline{C_{21}} = \frac{1}{2}\Big[-\left[-2m_2 l_1 l_{c2} - 2m_3 l_1 l_2 - 2m_4 l_1 l_2\right] S_2 - \left[-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3\right] S_{23} - \left[-2m_4 l_1 l_{c4}\right] S_{234}\Big]\dot q_1$$

$$+\frac{1}{2}\Big[\left[-m_2 l_1 l_{c2} - m_3 l_1 l_2 - m_4 l_1 l_2\right] S_2 + \left[-m_3 l_1 l_{c3}\right] S_{23} + \left[-m_4 l_1 l_3\right] S_{23} - \left[\ ''\ \right] S_2 - \left[\ ''\ \right] S_{23} - \left[\ ''\ \right] S_{23}\Big]\dot q_2$$

$$+\frac{1}{2}\Big[\left[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3\right] S_3 + \left[-m_3 l_1 l_{c3} - m_4 l_1 l_3\right] S_{23} + \left[-m_4 l_1 l_{c4}\right] S_{234} + \left[-2m_4 l_2 l_{c4}\right] S_{34} - \left[-m_3 l_1 l_{c3}\right] S_{23} - \left[-m_4 l_1 l_3\right] S_{23} + \left[-m_4 l_1 l_{c4}\right] S_{234}\Big]\dot q_3$$

ادامه بالا

ص 8

$$C_{22} = \sum_{i=1}^{4} \frac{1}{2} \left\{ \frac{\partial D_r^{22}}{\partial q_i} + \frac{\partial D_r^{2i}}{\partial q_2} - \frac{\partial D_r^{i2}}{\partial q_2} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{22}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{22}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{22}}{\partial q_3} = -2m_3 l_2 l_{c3} S_3 - 2m_4 l_2 l_3 S_3 - 2m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{22}}{\partial q_4} = -2m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$C_{22} = \frac{1}{2}[0]\dot{q}_1 + \frac{1}{2}[0]\dot{q}_2$$
$$+ \frac{1}{2}\Big[[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-2m_4 l_2 l_{c4}] S_{34}\Big] \dot{q}_3$$
$$+ \frac{1}{2}\Big[[-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_4$$

مرو

$$C_{23} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{23}}{\partial q_i} + \frac{\partial D_r^{2i}}{\partial q_3} - \frac{\partial D_r^{i3}}{\partial q_2} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{23}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{23}}{\partial q_2} = 0$$

$$+\frac{1}{2}\Big[[-m_4 l_2 l_{c4}] S_{34} + [-2 m_4 l_3 l_{c4}] S_4 + [-m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_4$$

$$\frac{\partial D_r^{23}}{\partial q_3} = -m_3 l_2 l_{c3} S_3 - m_4 l_2 l_3 S_3 - m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{23}}{\partial q_4} = -m_4 l_2 l_{c4} S_{34} - 2 m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{21}}{\partial q_3} = [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{22}}{\partial q_3} = [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-2m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{23}}{\partial q_3} = -m_3 l_2 l_{c3} S_3 - m_4 l_2 l_3 S_3 - m_4 l_2 l_{c4} S_{34} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 - m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{24}}{\partial q_3} = -m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{13}}{\partial q_2} = [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234}$$

$$\frac{\partial D_r^{23}}{\partial q_2} = 0 \qquad \frac{\partial D_r^{33}}{\partial q_2} = 0 \qquad \frac{\partial D_r^{43}}{\partial q_2} = 0$$

$$D_{23} = \frac{1}{2}\Big[[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} - [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} - [-m_4 l_1 l_{c4}] S_{234} \Big] \dot{q}_1$$

$$+\frac{1}{2}\Big[[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-2m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_2$$

$$+\frac{1}{2}\Big[[\overset{*2}{-}m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + \overset{*2}{[}-m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_3$$

ادامه دارد

$$C_{24} = \sum_{i=1}^{4} \frac{1}{2} \left\{ \frac{\partial D_r^{24}}{\partial q_i} + \frac{\partial D_r^{2i}}{\partial q_4} - \frac{\partial D_r^{i4}}{\partial q_2} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{24}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{24}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{24}}{\partial q_3} = -m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{24}}{\partial q_4} = -m_4 l_2 l_{c4} S_{34} - m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{21}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{22}}{\partial q_4} = [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{23}}{\partial q_4} = -m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{24}}{\partial q_4} = -m_4 l_2 l_{c4} S_{34} - m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{14}}{\partial q_2} = -m_4 l_1 l_{c4} S_{234}$$

$$\frac{\partial D_r^{24}}{\partial q_2} = 0 \; ; \; \frac{\partial D_r^{34}}{\partial q_2} = 0 \; ; \; \frac{\partial D_r^{44}}{\partial q_2} = 0$$

$$C_{24} = \frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4 - [-m_4 l_1 l_{c4}] S_{234}\Big] \dot{q}_1$$

$$+ \frac{1}{2}\Big[[-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_2$$

$$+ \frac{1}{2}\Big[[-m_4 l_2 l_{c4}] S_{34} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_3$$

$$+ \frac{1}{2}\Big[\overset{*2}{[-m_4 l_2 l_{c4}]} S_{34} + \overset{*2}{[-m_4 l_3 l_{c4}]} S_4\Big] \dot{q}_4$$

$$C_{31} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{31}}{\partial q_i} + \frac{\partial D_r^{3i}}{\partial q_1} - \frac{\partial D_r^{i1}}{\partial q_3} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{31}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{31}}{\partial q_2} = [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234}$$

$$\frac{\partial D_r^{31}}{\partial q_3} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{31}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2 m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{31}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{32}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{33}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{34}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{11}}{\partial q_3} = [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3] S_{23} + [-2m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{21}}{\partial q_3} = [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{31}}{\partial q_3} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{41}}{\partial q_3} = [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34}$$

$C_{31} =$

$$\frac{1}{2}\Big[[-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 + [-2m_3 l_1 l_{c3} - 2m_4 l_1 l_3] S_{23} + [-2m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_1$$

$$+ \frac{1}{2}\Big[\cancel{\partial D_r^{3}} [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234} - [-2m_3 l_2 l_{c3} - 2m_4 l_2 l_3] S_3 - [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} - [-m_4 l_1 l_{c4}] S_{234} - [-2m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_2$$

$$+ \frac{1}{2}[0] \dot{q}_3 + \frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4 - [-m_4 l_1 l_{c4}] S_{234} - [-m_4 l_2 l_{c4}] S_{34} \Big] \dot{q}_4$$

ص 12

$$\underline{C_{32}} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{32}}{\partial q_i} + \frac{\partial D_r^{3i}}{\partial q_2} - \frac{\partial D_r^{i2}}{\partial q_3} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{32}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{32}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{32}}{\partial q_3} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] s_3 + [-m_4 l_2 l_{c4}] s_{34}$$

$$\frac{\partial D_r^{32}}{\partial q_4} = [-m_4 l_2 l_{c4}] s_{34} + [-2 m_4 l_3 l_{c4}] s_4$$

$$\frac{\partial D_r^{31}}{\partial q_2} = [-m_3 l_1 l_{c3} - m_4 l_1 l_3] s_{23} + [-m_4 l_1 l_{c4}] s_{234}$$

$$\frac{\partial D_r^{32}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{33}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{34}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{12}}{\partial q_3} = [-2 m_3 l_2 l_{c3} - 2 m_4 l_2 l_3] s_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] s_{23} + [-m_4 l_1 l_{c4}] s_{234} + [-2 m_4 l_2 l_{c4}] s_{34}$$

$$\frac{\partial D_r^{22}}{\partial q_3} = [-2 m_3 l_2 l_{c3} - 2 m_4 l_2 l_3] s_3 + [-2 m_4 l_2 l_{c4}] s_{34}$$

$$\frac{\partial D_r^{32}}{\partial q_3} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] s_3 + [-m_4 l_2 l_{c4}] s_{34}$$

$$\frac{\partial D_r^{42}}{\partial q_3} = -m_4 l_2 l_{c4} s_{34}$$

$$\begin{aligned} \underline{\underline{C_{32}}} = \frac{1}{2}\Big[& [-m_3 l_1 l_{c3} - m_4 l_1 l_3] s_{23} + [-m_4 l_1 l_{c4}] s_{234} \\ & - [-2 m_3 l_2 l_{c3} - 2 m_4 l_2 l_3] s_3 - [-m_3 l_1 l_{c3} - m_4 l_1 l_3] s_{23} \\ & - [-m_4 l_1 l_{c4}] s_{234} - [-2 m_4 l_2 l_{c4}] s_{34} \Big] \dot{q}_1 \\ & + \frac{1}{2}\Big[- [-2 m_3 l_2 l_{c3} - 2 m_4 l_2 l_3] s_3 - [-2 m_4 l_2 l_{c4}] s_{34} \Big] \dot{q}_2 \\ & + \frac{1}{2}[0] \dot{q}_3 + \frac{1}{2}\Big[[-m_4 l_2 l_{c4}] s_{34} + [-2 m_4 l_3 l_{c4}] s_4 - [-m_4 l_2 l_{c4}] s_{34} \Big] \dot{q}_4 \end{aligned}$$

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$$C_{33} = \sum_{i=1}^{4} \frac{1}{2} \left\{ \frac{\partial D_r^{33}}{\partial q_i} + \frac{\partial D_r^{3i}}{\partial q_3} - \frac{\partial D_r^{i3}}{\partial q_3} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{33}}{\partial q_1} = 0 \qquad \frac{\partial D_r^{33}}{\partial q_2} = 0 \qquad \frac{\partial D_r^{33}}{\partial q_3} = 0$$

$$\frac{\partial D_r^{33}}{\partial q_4} = -2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{31}}{\partial q_3} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + [-m_3 l_1 l_{c3} - m_4 l_1 l_3] S_{23} + [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{32}}{\partial q_3} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{33}}{\partial q_3} = 0 \qquad \frac{\partial D_r^{34}}{\partial q_3} = 0$$

$$\frac{\partial D_r^{13}}{\partial q_3} = \checkmark$$

$$\frac{\partial D_r^{23}}{\partial q_3} = [-m_3 l_2 l_{c3} - m_4 l_2 l_3] S_3 + [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{33}}{\partial q_3} = 0 \qquad \frac{\partial D_r^{43}}{\partial q_3} = 0$$

$$C_{33} = \frac{1}{2}[0]\dot{q}_1 + \frac{1}{2}[0]\dot{q}_2 + \frac{1}{2}[0]\dot{q}_3 + \frac{1}{2}\left[[-2m_4 l_3 l_{c4}] S_4\right]\dot{q}_4$$

$$C_{34} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{34}}{\partial q_i} + \frac{\partial D_r^{3i}}{\partial q_4} - \frac{\partial D_r^{i4}}{\partial q_3} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{34}}{\partial q_1} = \frac{\partial D_r^{34}}{\partial q_2} = \frac{\partial D_r^{34}}{\partial q_3} = 0$$

$$\frac{\partial D_r^{34}}{\partial q_4} = -m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{31}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{32}}{\partial q_4} = [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{33}}{\partial q_4} = -2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{34}}{\partial q_4} = -m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{14}}{\partial q_3} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{24}}{\partial q_3} = -m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{34}}{\partial q_3} = 0 \qquad \frac{\partial D_r^{44}}{\partial q_3} = 0$$

$$C_{34} = \frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4 - [-m_4 l_1 l_{c4}] S_{234} - [-m_4 l_2 l_{c4}] S_{34}\Big] \dot{q}_1$$

$$+ \frac{1}{2}\Big[[-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4 - [-m_4 l_2 l_{c4}] S_{34}\Big] \dot{q}_2$$

$$+ \frac{1}{2}\Big[[-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_3$$

$$+ \frac{1}{2}\Big[[-m_4 l_3 l_{c4}] S_4 + [-m_4 l_3 l_{c4}] S_4\Big] \dot{q}_4$$

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$$\underline{\underline{C_{41}}} = \sum_{i=1}^{4} \frac{1}{2} \left\{ \frac{\partial D_r^{4i}}{\partial q_i} + \frac{\partial D_r^{4i}}{\partial q_1} - \frac{\partial D_r^{i1}}{\partial q_4} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{41}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{41}}{\partial q_2} = -m_4 l_1 l_{c4} S_{234}$$

$$\frac{\partial D_r^{41}}{\partial q_3} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{41}}{\partial q_4} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34} - m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{41}}{\partial q_1} = 0 \;/\; \frac{\partial D_r^{42}}{\partial q_1} = \frac{\partial D_r^{43}}{\partial q_1} = \frac{\partial D_r^{44}}{\partial q_1} = 0$$

$$\frac{\partial D_r^{11}}{\partial q_4} = [-2m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{21}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} + [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{31}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{41}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-m_4 l_3 l_{c4}] S_4$$

$$\underline{\underline{C_{41}}} = \frac{1}{2}\Big[-[-2m_4 l_1 l_{c4}] S_{234} - [-2m_4 l_2 l_{c4}] S_{34} - [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_1$$

$$+ \frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} - [-m_4 l_1 l_{c4}] S_{234} - [-2m_4 l_2 l_{c4}] S_{34} - [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_2$$

$$+ \frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} - [-m_4 l_1 l_{c4}] S_{234} - [-m_4 l_2 l_{c4}] S_{34} - [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_3$$

$$+ \underbrace{\frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-m_4 l_3 l_{c4}] S_4 - [\;''\;] - [\;''\;] - [\;''\;]\Big] \dot{q}_4}_{\frac{1}{2}[0]\dot{q}_4}$$

$$C_{42} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{42}}{\partial q_i} + \frac{\partial D_r^{4i}}{\partial q_2} - \frac{\partial D_r^{i2}}{\partial q_4} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{42}}{\partial q_1} = \frac{\partial D_r^{42}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{42}}{\partial q_3} = [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{42}}{\partial q_4} = [-m_4 l_2 l_{c4}] S_{34} + [-m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{41}}{\partial q_2} = -m_4 l_1 l_{c4} S_{234}$$

$$\frac{\partial D_r^{42}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{43}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{44}}{\partial q_2} = 0$$

$$\frac{\partial D_r^{12}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} - 2m_4 l_2 l_{c4} S_{34} - 2m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{22}}{\partial q_4} = [-2m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{32}}{\partial q_4} = [-m_4 l_2 l_{c4}] S_{34} + [-2m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{42}}{\partial q_4} = [-m_4 l_2 l_{c4}] S_{34} + [-m_4 l_3 l_{c4}] S_4$$

$$C_{42} = \frac{1}{2}\Big[[-m_4 l_1 l_{c4}] S_{234} - [-m_4 l_1 l_{c4}] S_{234} - [-2m_4 l_2 l_{c4}] S_{34} - [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_1$$

$$+ \frac{1}{2}\Big[-[-2m_4 l_2 l_{c4}] S_{34} - [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_2$$

$$+ \frac{1}{2}\Big[[-m_4 l_2 l_{c4}] S_{34} - [-m_4 l_2 l_{c4}] S_{34} - [-2m_4 l_3 l_{c4}] S_4\Big] \dot{q}_3$$

$$+ \frac{1}{2}[0] \dot{q}_4$$

$$C_{43} = \sum_{i=1}^{4} \frac{1}{2}\left\{ \frac{\partial D_r^{43}}{\partial q_i} + \frac{\partial D_r^{4i}}{\partial q_3} - \frac{\partial D_r^{i3}}{\partial q_4} \right\} \dot q_i$$

$$\frac{\partial D_r^{43}}{\partial q_1} = \frac{\partial D_r^{43}}{\partial q_2} = \frac{\partial D_r^{43}}{\partial q_3} = 0$$

$$\frac{\partial D_r^{43}}{\partial q_4} = -m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{41}}{\partial q_3} = -m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34}$$

$$\frac{\partial D_r^{42}}{\partial q_3} = [-m_4 l_2 l_{c4}] S_{34}$$

$$\frac{\partial D_r^{43}}{\partial q_3} = 0$$

$$\frac{\partial D_r^{44}}{\partial q_3} = 0$$

$$\frac{\partial D_r^{13}}{\partial q_4} = [-m_4 l_1 l_{c4}] S_{234} + [-m_4 l_2 l_{c4}] S_{34} + [-2 m_4 l_3 l_{c4}] S_4$$

$$\frac{\partial D_r^{23}}{\partial q_4} = -m_4 l_2 l_{c4} S_{34} - 2 m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{33}}{\partial q_4} = -2 m_4 l_3 l_{c4} S_4$$

$$\frac{\partial D_r^{43}}{\partial q_4} = -m_4 l_3 l_{c4} S_4$$

$$C_{43} = \frac{1}{2}\Big[-m_4 l_1 l_{c4} S_{234} - m_4 l_2 l_{c4} S_{34} - [-m_4 l_1 l_{c4}] S_{234} - [-m_4 l_2 l_{c4}] S_{34} - [-2 m_4 l_3 l_{c4}] S_4\Big] \dot q_1$$

$$+ \frac{1}{2}\Big[[-m_4 l_2 l_{c4}] S_{34} - [-m_4 l_2 l_{c4}] S_{34} - [-2 m_4 l_3 l_{c4}] S_4\Big] \dot q_2$$

$$+ \frac{1}{2}\Big[\cancel{+[-m_4 l_3 l_{c4}] S_4] \dot q_3} - [-2 m_4 l_3 l_{c4}] S_4\Big] \dot q_3$$

$$+ \frac{1}{2}\Big[[-m_4 l_3 l_{c4}] S_4 - [-m_4 l_3 l_{c4}] S_4\Big] \dot q_4$$

$$C_{44} = \sum_{i=1}^{4} \frac{1}{2} \left\{ \frac{\partial D_r^{44}}{\partial q_i} + \frac{\partial D_r^{4i}}{\partial q_4} - \frac{\partial D_r^{i4}}{\partial q_4} \right\} \dot{q}_i$$

$$\frac{\partial D_r^{44}}{\partial q_1} = \frac{\partial D_r^{44}}{\partial q_2} = \frac{\partial D_r^{44}}{\partial q_3} = \frac{\partial D_r^{44}}{\partial q_4} = 0$$

$$C_{44} = 0$$

A1.2 Videos

Video1

https://youtu.be/G0veBQx6jiU

Video2

https://youtu.be/dK7HXMc58tI