Spring 2011

信號與系統
Signals and Systems

Chapter SS-9
The Laplace Transform

صفحه
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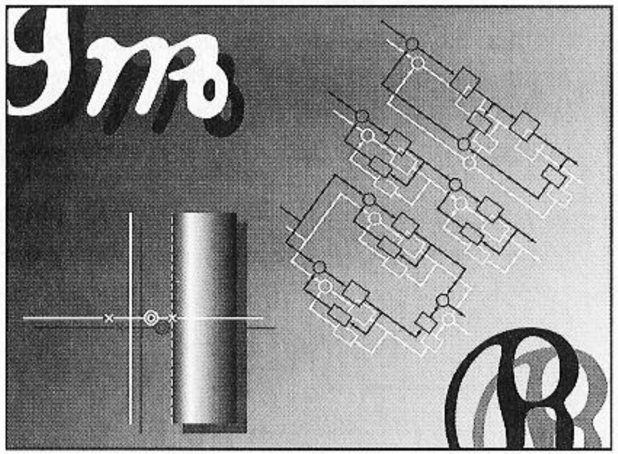
Feng-Li Lian

NTU-EE

Feb11 – Jun11



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Fourier Series, Fourier Transform, Laplace Transform, z-Transform

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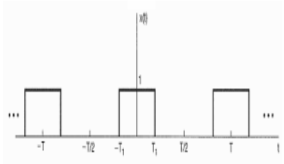
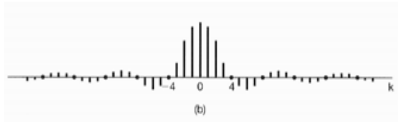
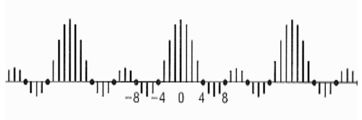
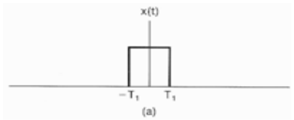
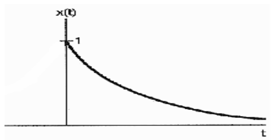
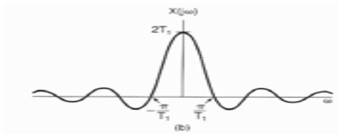
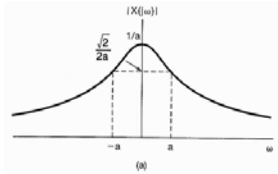
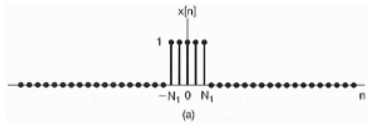
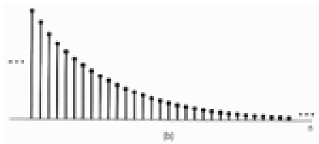
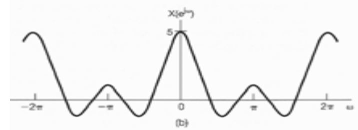
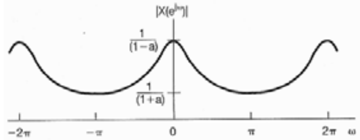
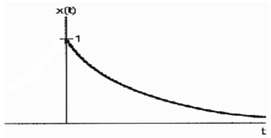
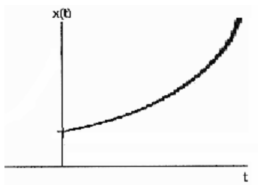
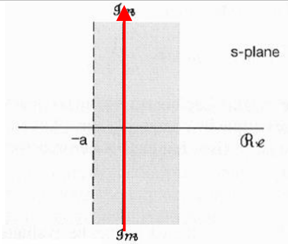
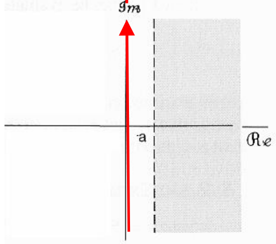
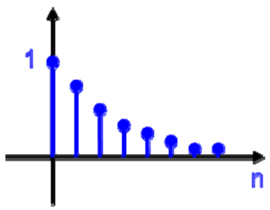
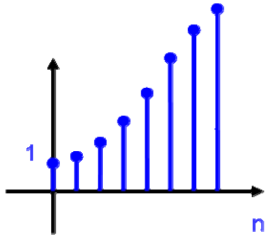
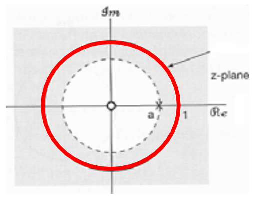
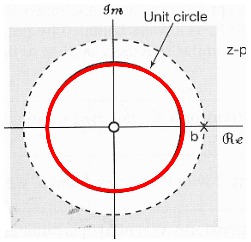
FS				
FT	 	 	 	 
LT/zT	 	 	 	 

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

Introduction

Let $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \quad = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

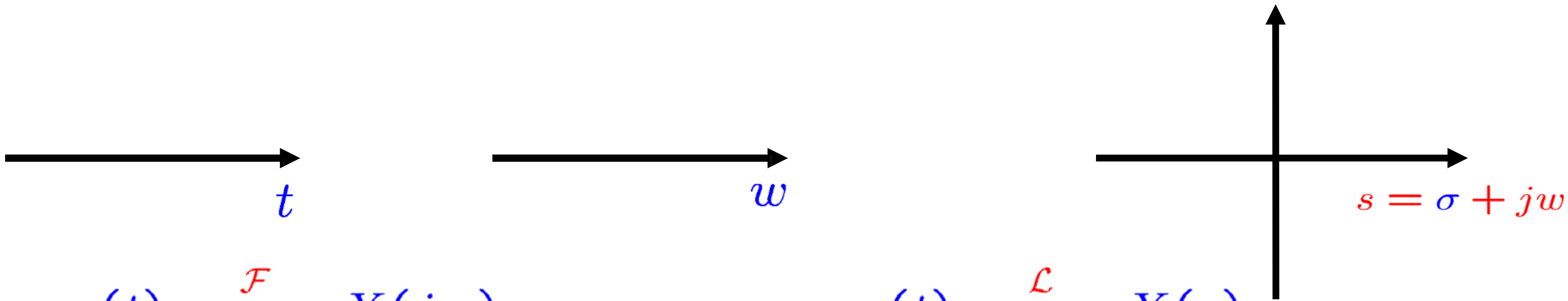
$$\Rightarrow y(t) = H(s)x(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

▪ The Laplace transform of a general signal x(t):

$s = \sigma + jw$

$$X(jw) \triangleq \int_{-\infty}^{\infty} x(t) e^{-jwt} dt \qquad X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



$$x(t) \xleftrightarrow{\mathcal{F}} X(jw) \qquad x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

$$X(jw) = \mathcal{F}\left\{x(t)\right\} \qquad X(s) = \mathcal{L}\left\{x(t)\right\}$$

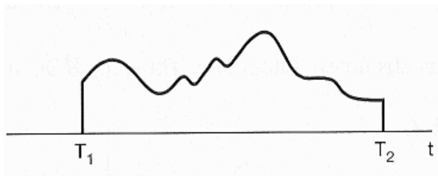
$$x(t) = \mathcal{F}^{-1}\left\{X(jw)\right\} \qquad x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$$

$$X(s)\Big|_{s=jw} = \mathcal{L}\left\{x(t)\right\}\Big|_{s=jw} = \mathcal{F}\left\{x(t)\right\} = X(jw)$$

Laplace Transform & Fourier Transform:

$$X(s)\Big|_{s=jw} = \mathcal{L}\{x(t)\}\Big|_{s=jw} = \mathcal{F}\{x(t)\} = X(jw)$$

$$\mathcal{L}\left\{x(t)\right\} = X(s) \qquad s = \sigma + jw$$



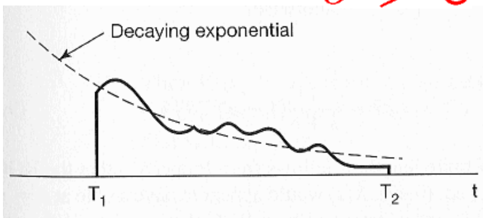
$$= X(\sigma + jw)$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + jw)t} dt$$

$$= \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t} \right] e^{-jwt} dt$$

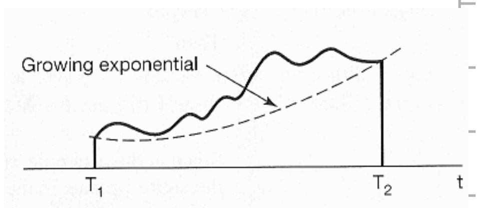
$$= \mathcal{F}\left\{ x(t) e^{-\sigma t} \right\}$$

$\sigma > 0$



(a)

$\sigma < 0$

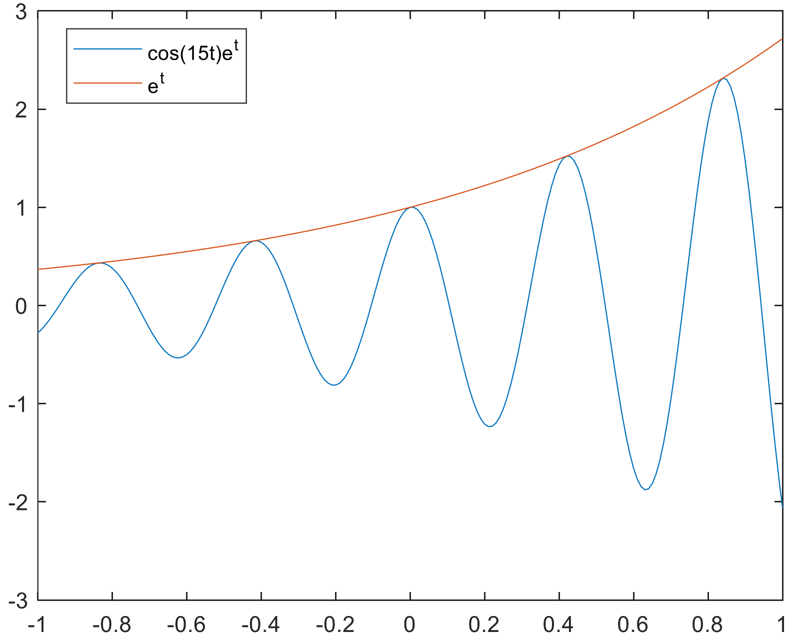


(b)

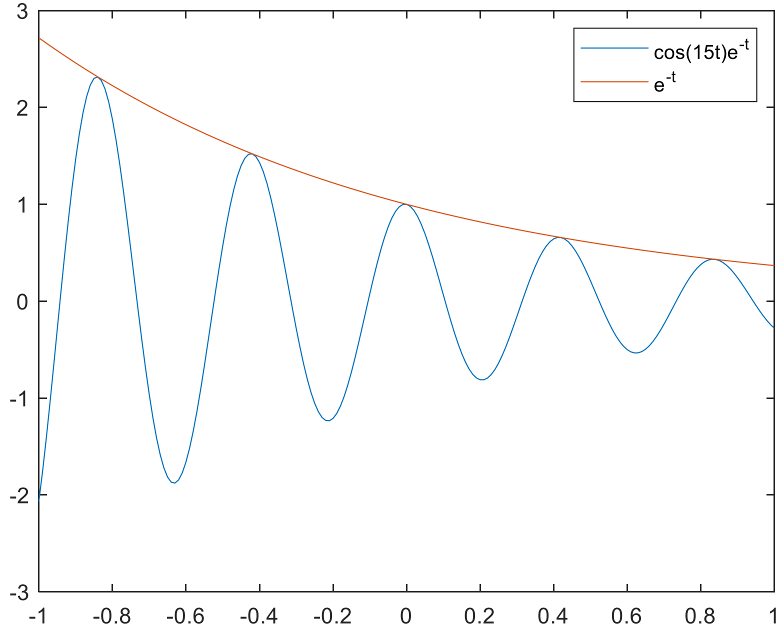
The Fourier Transform is not a special case of Laplace Transform!

$$\mathcal{L}\{\cos(t)\} = \mathcal{F}\{\cos(t)e^{-\sigma t}\}$$

$$\sigma > 0$$



$$\sigma < 0$$



Absolutely Integrability is a sufficient condition for Fourier Transform
But it is a necessary and sufficient condition for Laplace Transform!

$$\mathcal{L}\{\cos(t)\} = \mathcal{F}\{\cos(t)e^{-\sigma t}\}$$

So, even for $\sigma = 0$ the Laplace transform does not exist!

- Example 9.1:

$$X(jw) \triangleq \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = e^{-at}u(t)$$

$$X(jw) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-jwt}dt \qquad = \left. \frac{e^{-(a+jw)t}}{-(a+jw)} \right|_0^{\infty}$$

$$= \frac{1}{jw+a}, \quad a > 0$$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \qquad = \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$X(\sigma + jw) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-jwt}dt \qquad = \frac{1}{(\sigma+a)+jw}, \quad \sigma + a > 0$$

$$= \frac{1}{s+a}, \qquad \mathcal{R}e\{s\} > -a$$

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Example 9.2:

$$x(t) = -e^{-at}u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}dt$$

$$= \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

Region of Convergence (ROC)

Region of Convergence (ROC):

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

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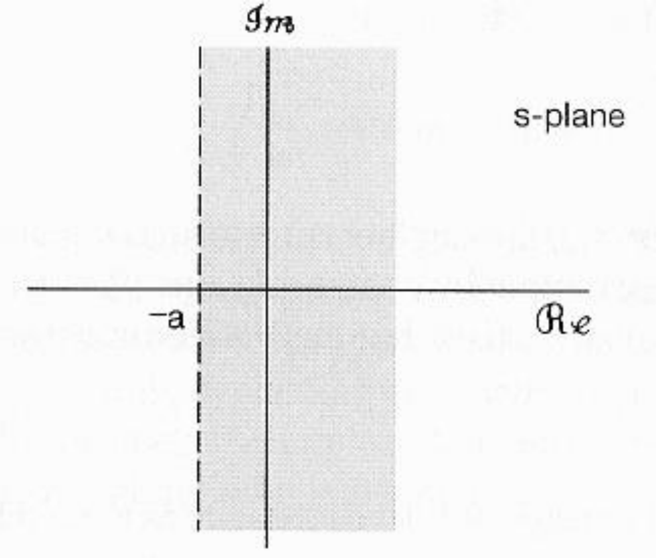
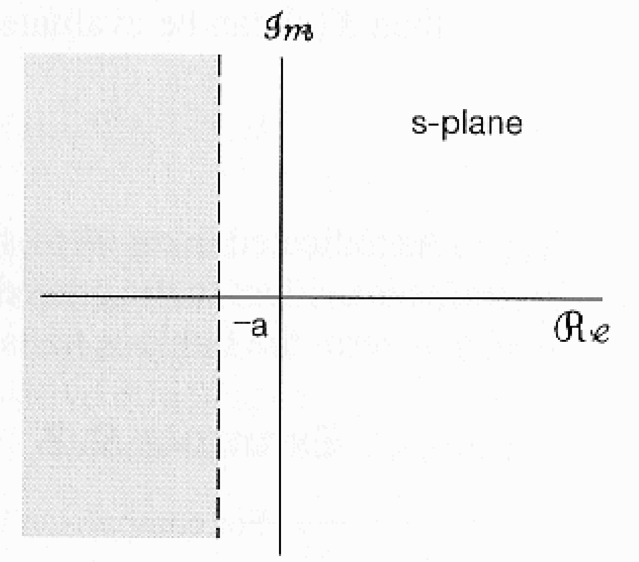
where Fourier transform of $x(t)e^{-\sigma t}$ converges



(a)



(b)

- **Example 9.3:**

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} \left[3e^{-2t}u(t) - 2e^{-t}u(t)\right] e^{-st} dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt$$

$$= 3\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+1}\right)$$

$$e^{-t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2$$

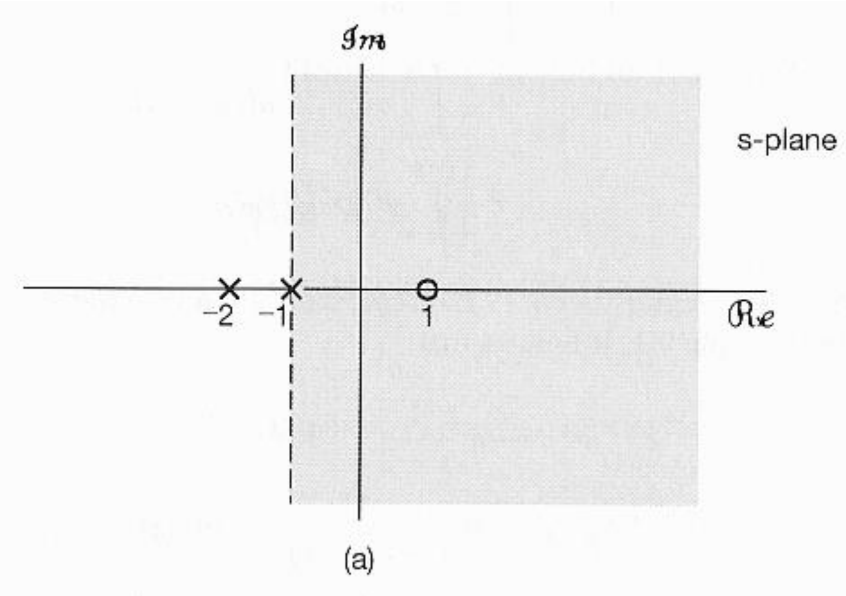
$$3e^{-2t}u(t) - 2e^{-t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{3}{s+2} - \frac{2}{s+1}, \qquad \mathcal{R}e\{s\} > -1$$

- Example 9.3:

$\mathcal{R}e\{s\} > -2 \quad \mathcal{R}e\{s\} > -1$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{3}{s+2} - \frac{2}{s+1}, \qquad \mathcal{R}e\{s\} > -1$$

$$\overset{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{(s+2)(s+1)}, \qquad \mathcal{R}e\{s\} > -1$$



- The jw-axis is included in the ROC!
- Fourier transform!
  - s = jw

Stable

- **Example 9.4:**

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -\mathcal{R}e\{a\}$$

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \qquad \mathcal{R}e\{s\} > -2 \qquad \mathcal{R}e\{s\} > -1$$

$$= \left[e^{-2t} + \frac{1}{2}e^{-t}\left(e^{j3t} + e^{-j3t}\right)\right]u(t)$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1-3j)}, \qquad \mathcal{R}e\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1+3j)}, \qquad \mathcal{R}e\{s\} > -1$$

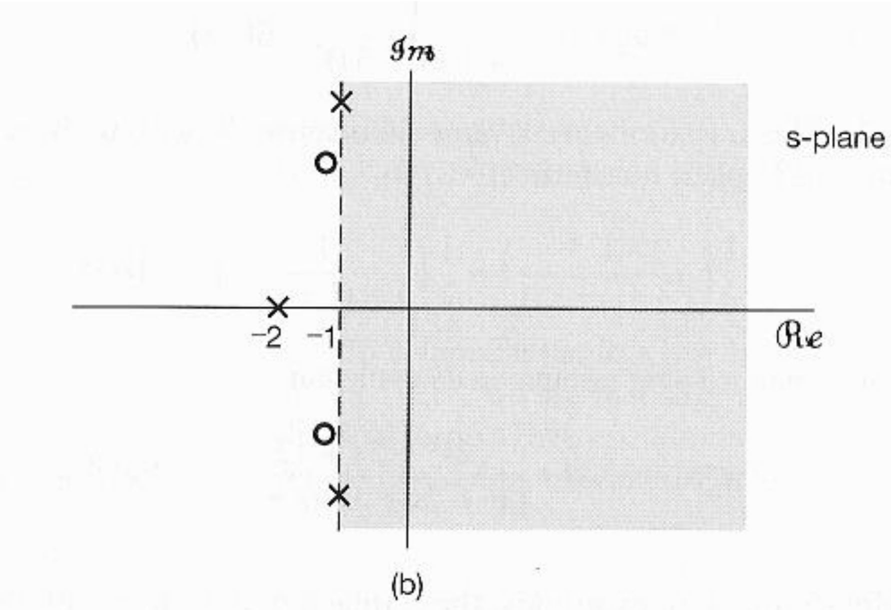
$$X(s) = \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s+(1-3j)} + \frac{1}{s+(1+3j)}\right] = \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$$

- Example 9.4:

$\mathcal{R}e\{s\} > -2$ $\mathcal{R}e\{s\} > -1$

$$e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$



- The jw-axis is included in the ROC!
- Fourier transform!
  - s = jw

Absolutely Integrability is a sufficient condition for Fourier Transform
But it is a necessary and sufficient condition for Laplace Transform!

It seems that when jw axis is not in ROC, the signal does not have the Fourier Transform.

However, there are some examples that contradict this observation!

$$x(t) = e^{jw_0 t}u(t)$$

$$X(s) = \frac{1}{s - jw}, \qquad ROC \quad \mathcal{R}\{s\} > 0,$$

jw axis is not included in ROC!

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jw axis is not included in ROC!

$$X(jw) = \frac{1}{j(w - w_0)} + \pi\delta(w - w_0)$$

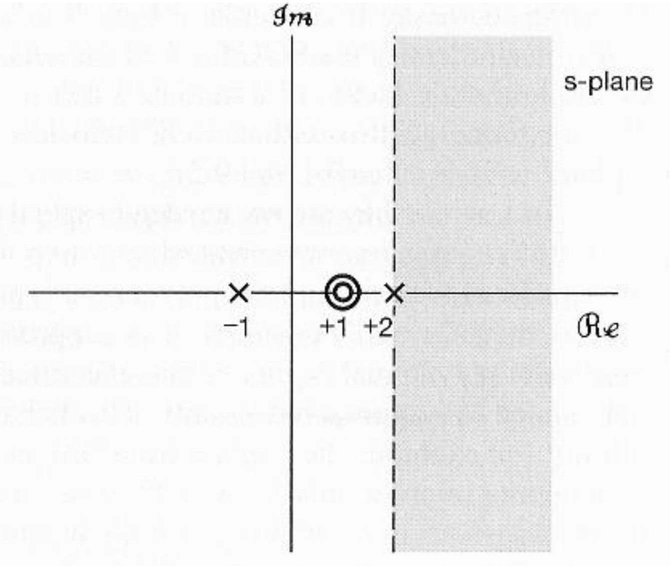
- Example 9.5:

$$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

$$x(t) = \delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t)$$

$$X(s) = 1 - \frac{4}{3} \frac{1}{s+1} + \frac{1}{3} \frac{1}{s-2}, \quad \mathcal{R}e\{s\} > 2$$

$$\delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathcal{R}e\{s\} > 2$$



- The jw-axis is not included in the ROC!
- Fourier transform?
- Why?

The Pole-Zero Plot of X(s)

In the preceding examples, the Laplace transform of the signals, were rational!

$$X(s) = \frac{N(s)}{D(s)}$$

X(s) can be determined uniquely, by the roots of its nominators and Denominators, and a constant gain.

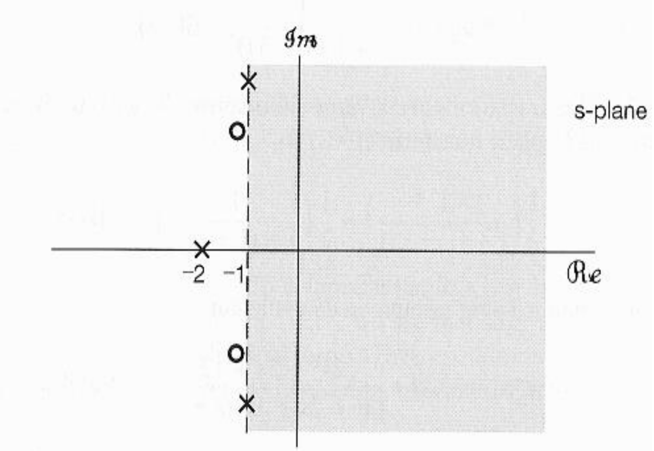
$$X(s) = A\frac{\Pi(s - z_i)}{\Pi(s - p_i)}$$

The Pole-Zero Plot of X(s)

The roots of the nominator, $\mathbf{z}_i$ are called the zeros of X(s)
The roots of the denominator, $\mathbf{p}_i$ are called the poles of X(s)

$$X(s) = \frac{N(s)}{D(s)} = A\frac{\Pi(s - z_i)}{\Pi(s - p_i)}$$

At s equals to $\mathbf{z}_i$, X(s) become zero, $X(z_i) = 0$
When s is equal to $\mathbf{p}_i$, X(s) became unbounded, $\lim_{s \to p_i} X(s) \to \infty$



(b)

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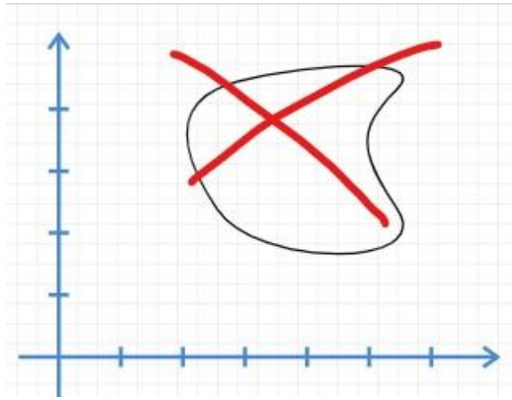
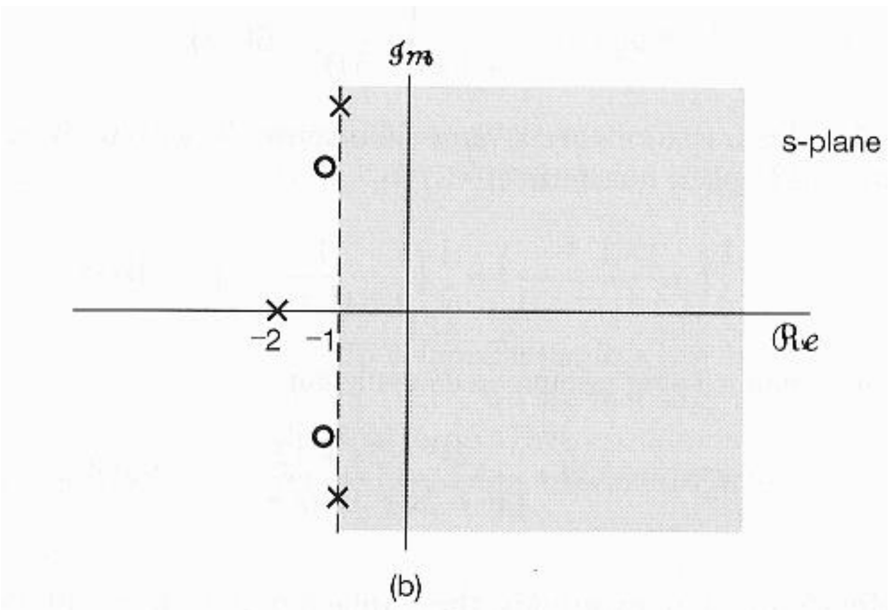
- Properties of ROC:

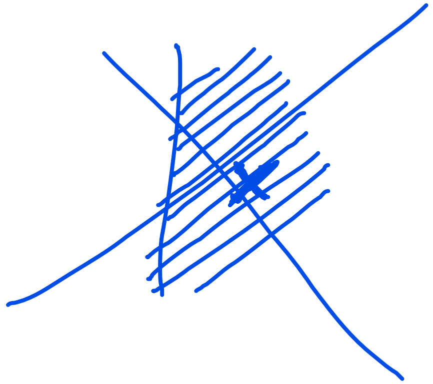
1. The ROC of X(s) consists of strips parallel to the jw-axis in the s-plane

$$\mathcal{L}\left\{x(t)\right\} = X(s) = \mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$$



2. For rational Laplace transforms, the ROC does not contain any poles



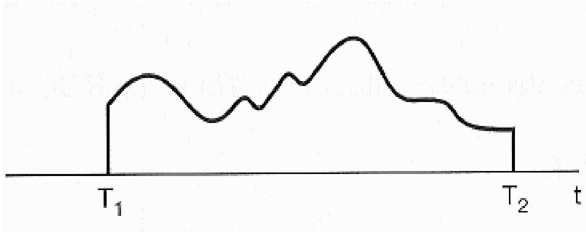


$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$

- **Types of Signals:**

1. Finite Duration

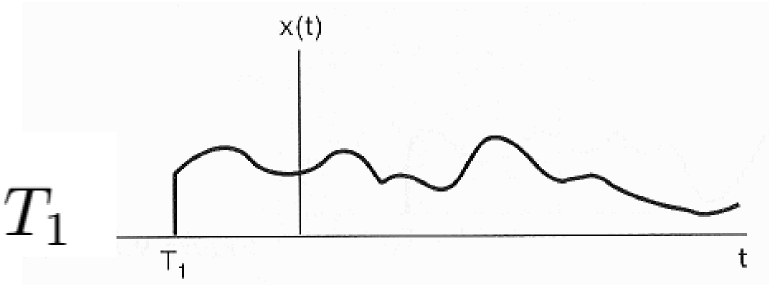
$$x(t) = 0, \quad \forall t < T_1, t > T_2$$



2. Infinite Duration

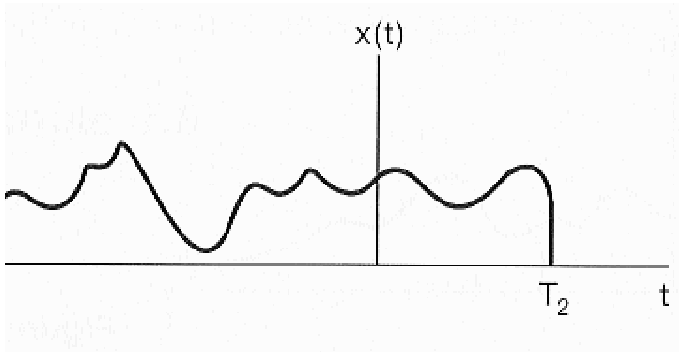
2.1. Right sided signal

$$\exists \quad T_1 < \infty, \qquad x(t) = 0, \quad \forall t < T_1$$



2.2 Left sided signal

$$\exists \quad T_2 > -\infty, \qquad x(t) = 0, \quad \forall t > T_2$$

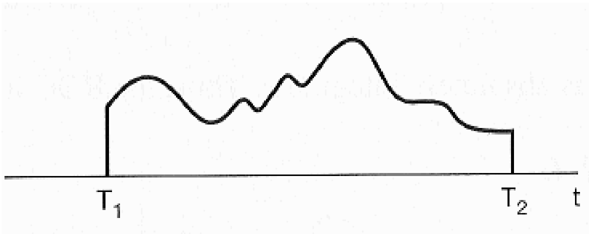


2.3 Two sided signal

Properties of ROC:

3. If x(t) is of finite duration & is absolutely integrable, then the ROC is the entire s-plane

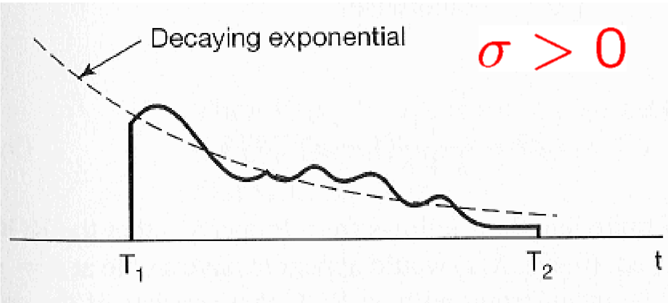
$$x(t) = 0, \quad \forall t < T_1, t > T_2 \qquad \int_{T_1}^{T_2} |x(t)| dt < \infty$$



$$s = \sigma + jw$$

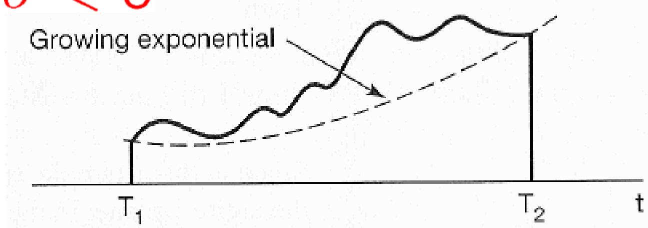
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{T_1}^{T_2} x(t) e^{-st} dt \; < \; e^{-\sigma(T_1 \text{ or } T_2)} \int_{T_1}^{T_2} |x(t)| dt$$

$\sigma > 0$



(a)

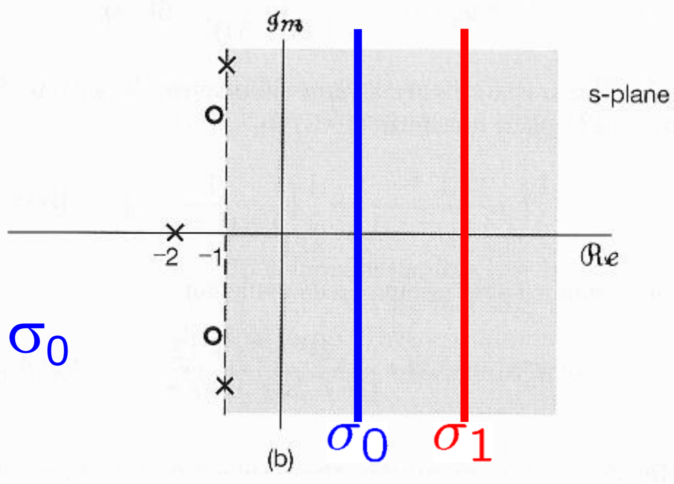
$\sigma < 0$

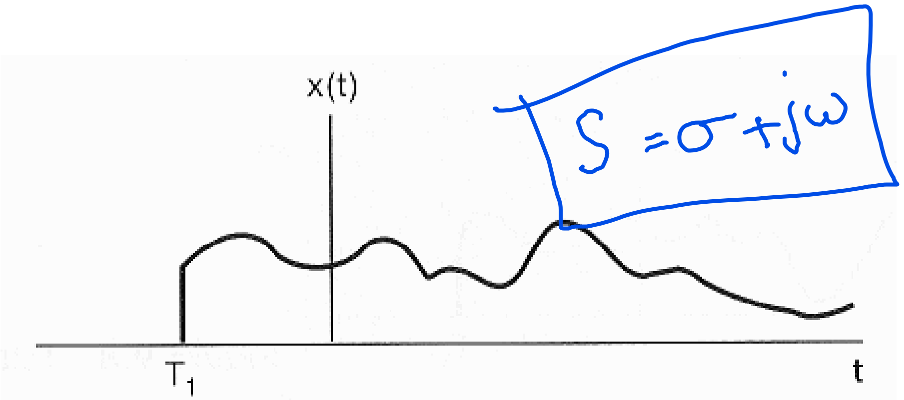


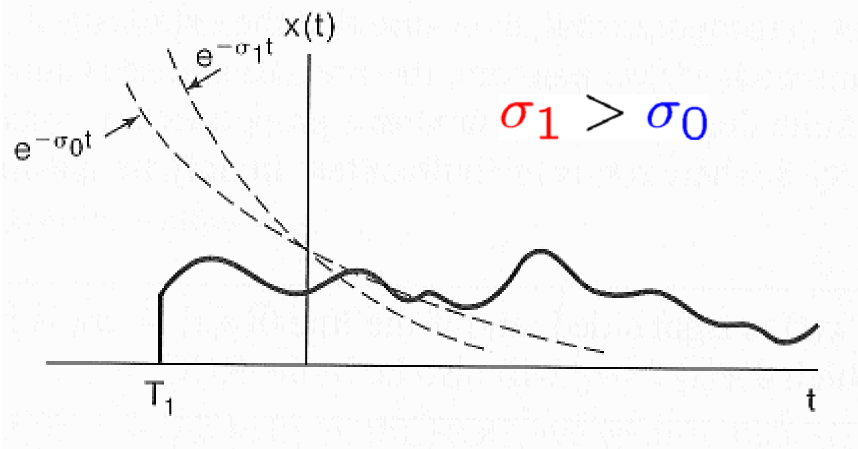
(b)

Properties of ROC:

4. If $x(t)$ is right-sided, and
   if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC,
   then all values of s for which $\mathcal{Re}\{s\} > \sigma_0$
   will also be in the ROC





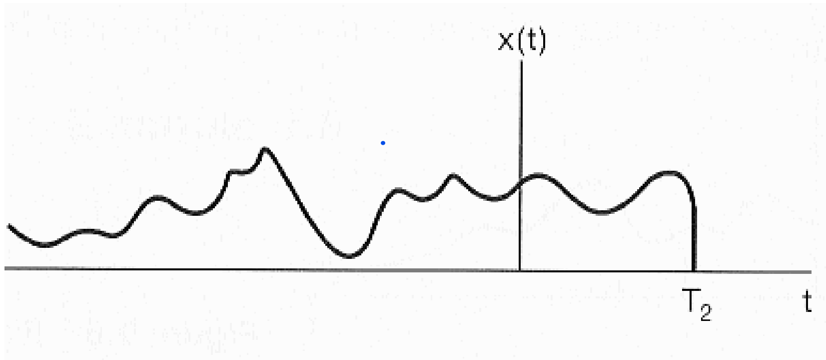


$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

$$\Rightarrow \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_1 t} dt \ \leq \ e^{-(\sigma_1 - \sigma_0) T_1} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt$$

Properties of ROC:

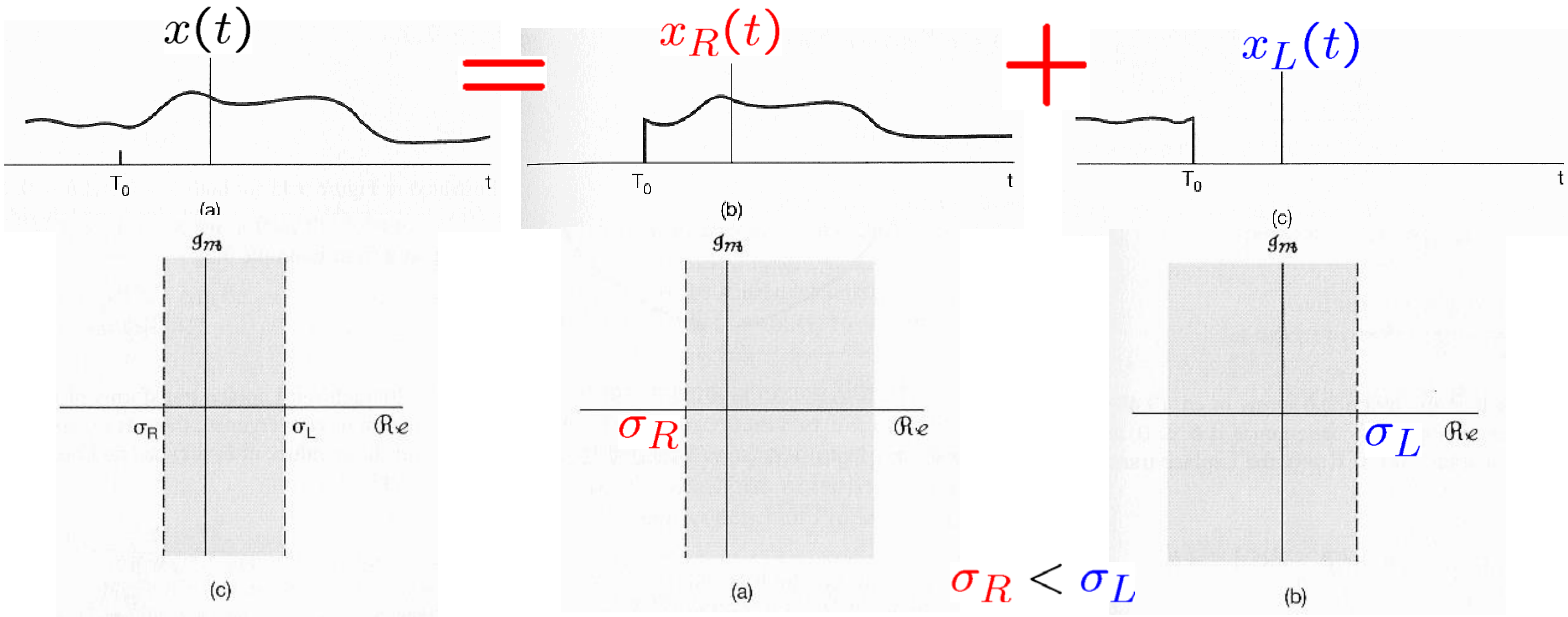
5. If $x(t)$ is left-sided, and
if the line $Re\{s\} = \sigma_0$ is in the ROC,
then all values of s for which $Re\{s\} < \sigma_0$
will also be in the ROC



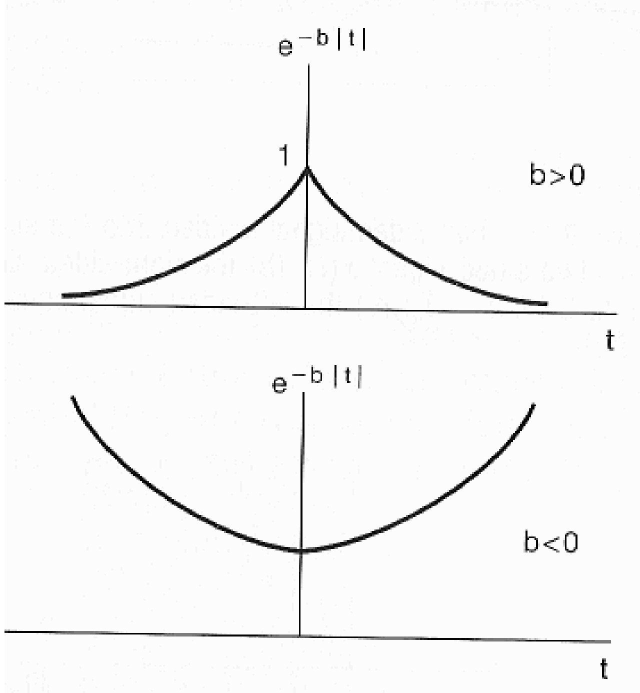
The argument is the similar to that for Property 4.

Properties of ROC:

6. If x(t) is two-sided, and
   if the line Re{s} = σ_0 is in the ROC,
   then the ROC will consist of a strip in the s-plane
   that includes the line Re{s} = σ_0



- Example 9.7:



$$x(t) = e^{-b|t|} \qquad = e^{-bt}u(t) + e^{+bt}u(-t)$$

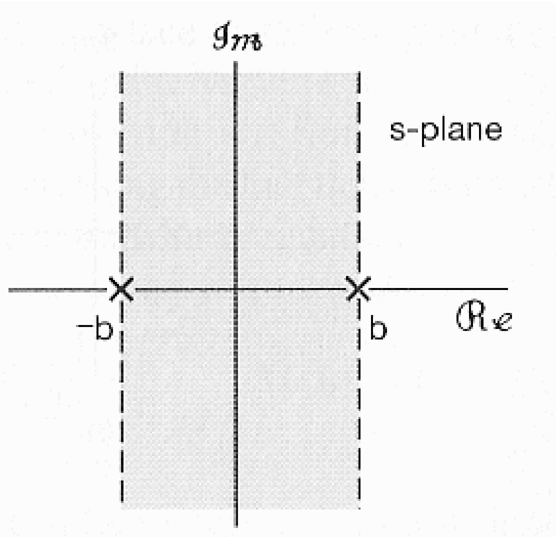
$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b}, \quad \mathcal{R}e\{s\} > -b$$

$$e^{+bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b}, \quad \mathcal{R}e\{s\} < +b$$

- $b > 0$:

$$e^{-b|t|} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} + \frac{-1}{s-b}, \quad -b < \mathcal{R}e\{s\} < +b$$

$$= \frac{-2b}{(s+b)(s-b)}$$

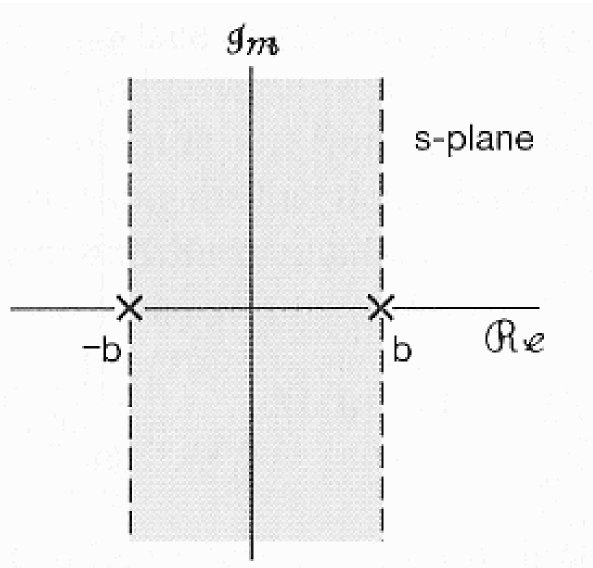


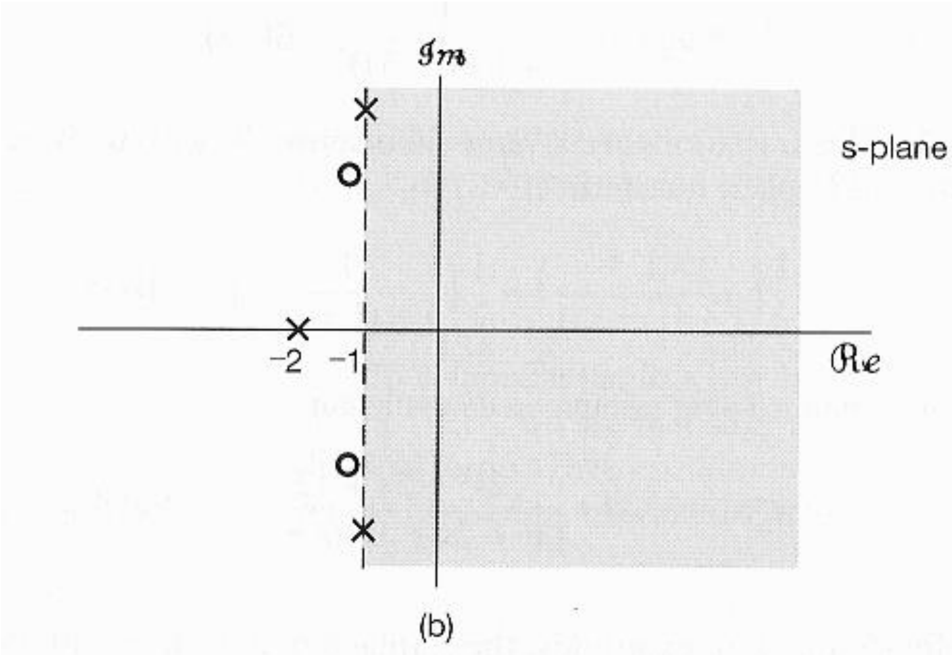
- $b \leq 0$: no common ROC

$x(t)$ has no Laplace transform

- Properties of ROC:

  7. If the Laplace transform X(s) of x(t) is rational, then its ROC is bounded by poles or extends to ∞.
  In addition, no poles of X(s) are contained in ROC

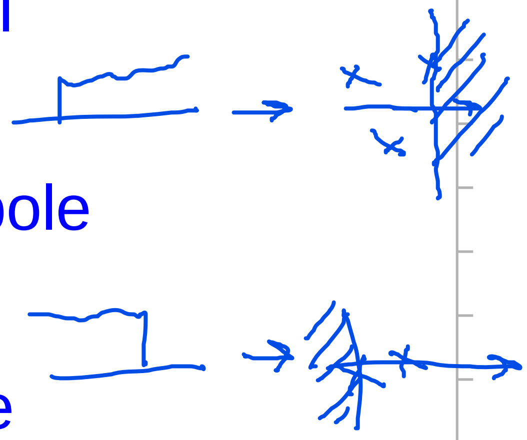




$$X(s) = \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \cdots + \frac{B_1}{s+b} + \cdots$$

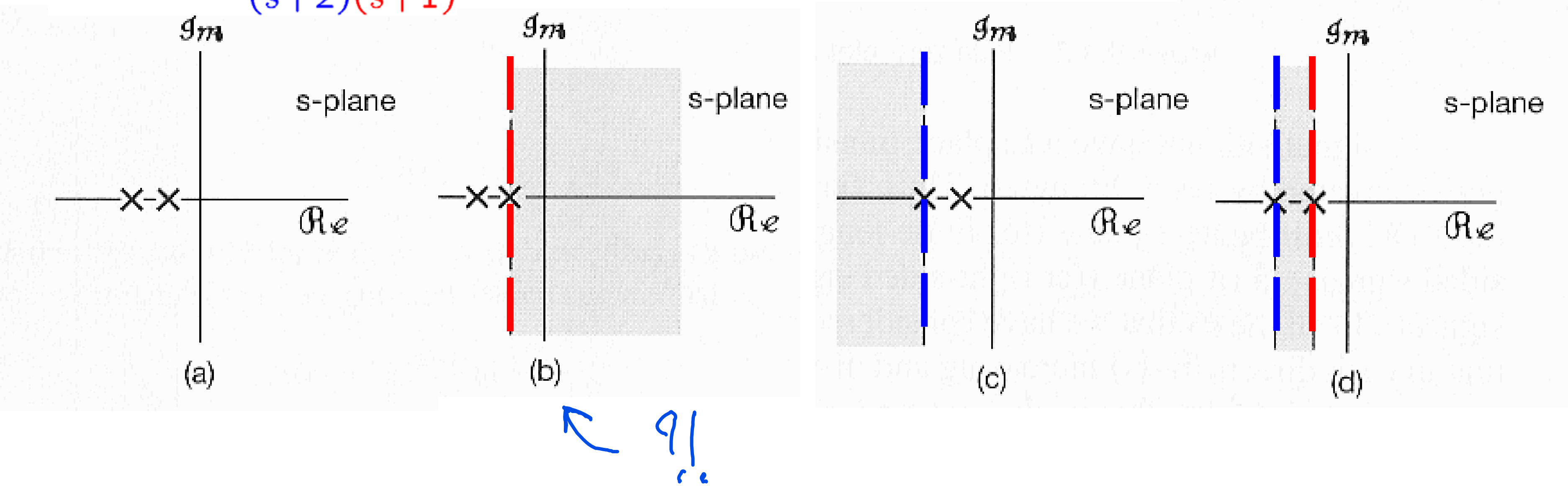
- Properties of ROC:

8. If the Laplace transform X(s) of x(t) is rational
   - If x(t) is right-sided, the ROC is the region in the s-plane to the right of the rightmost pole
   - If x(t) is left-sided, the ROC is the region in the s-plane to the left of the leftmost pole



$$X(s) = \frac{1}{(s+2)(s+1)}$$

- Which one has Fourier transform?



Outline

- The Laplace Transform
- The Region of Convergence for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

The Inverse Laplace Transform:

- By the use of contour integration

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(\sigma + jw) = \mathcal{F}\left\{ x(t) e^{-\sigma t} \right\} = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-jwt} dt$$

$\forall s = \sigma + jw$ in the ROC

$$x(t) e^{-\sigma t} = \mathcal{F}^{-1}\left\{ X(\sigma + jw) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + jw) e^{jwt} dw$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + jw) e^{(\sigma + jw)t} dw$$

$s = \sigma + jw$

$ds = j dw$

$$\Rightarrow x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

The Inverse Laplace Transform:

- By the technique of partial fraction expansion

$$X(s) = \frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \cdots + \frac{A_m}{s+a_m}$$

$$x(t) = A_1 e^{-a_1 t} u(t) - A_2 e^{-a_2 t} u(-t) + \cdots + x_m(t)$$

(if R.S.) (if L.S.)

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

- Example 9.9:

→ right_sided

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

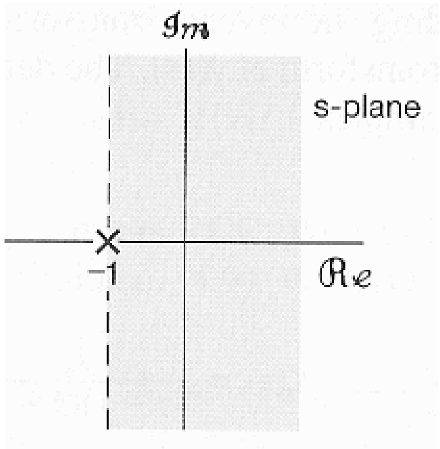
$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$= \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

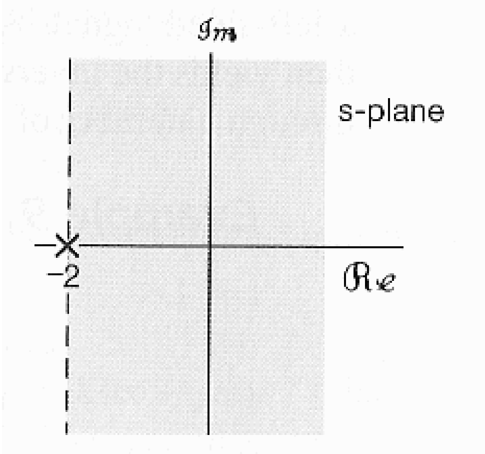
$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{R}e\{s\} > -2$$

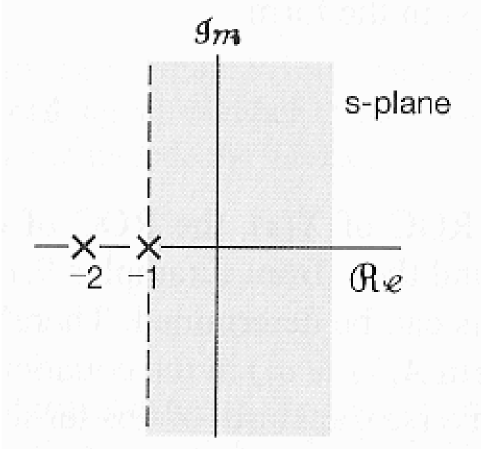
$$\left[e^{-t} - e^{-2t}\right] u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$



(b)



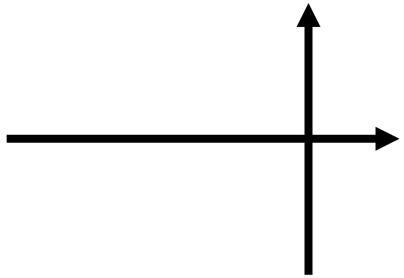
(c)



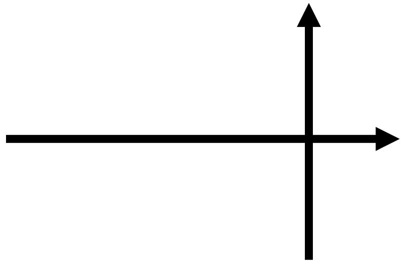
(a)

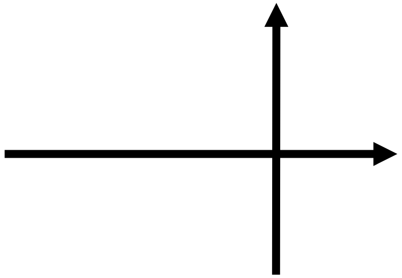
- Examples 9.9, 9.10, 9.11:

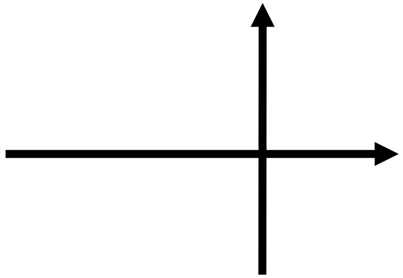
	$\mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)}$	$-\ e^{-t}u(-t)$	$e^{-t}u(t)$

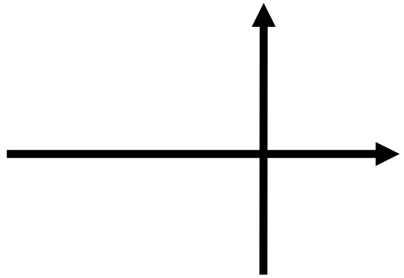


	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\}$
$\frac{1}{(s+2)}$	$-\ e^{-2t}u(-t)$	$e^{-2t}u(t)$









	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)} + \frac{1}{(s+2)}$	$-\ e^{-t}u(-t)$ $+$ $-\ e^{-2t}u(-t)$	$-\ e^{-t}u(-t)$ $+$ $e^{-2t}u(t)$	$e^{-t}u(t)$ $+$ $e^{-2t}u(t)$

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Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

- Linearity of the Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x_1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad ROC = R_1$$

$$x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad ROC = R_2$$

$$a\, x_1(t) + b\, x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} a\, X_1(s) + b\, X_2(s),$$

with ROC containing $R_1 \cap R_2$

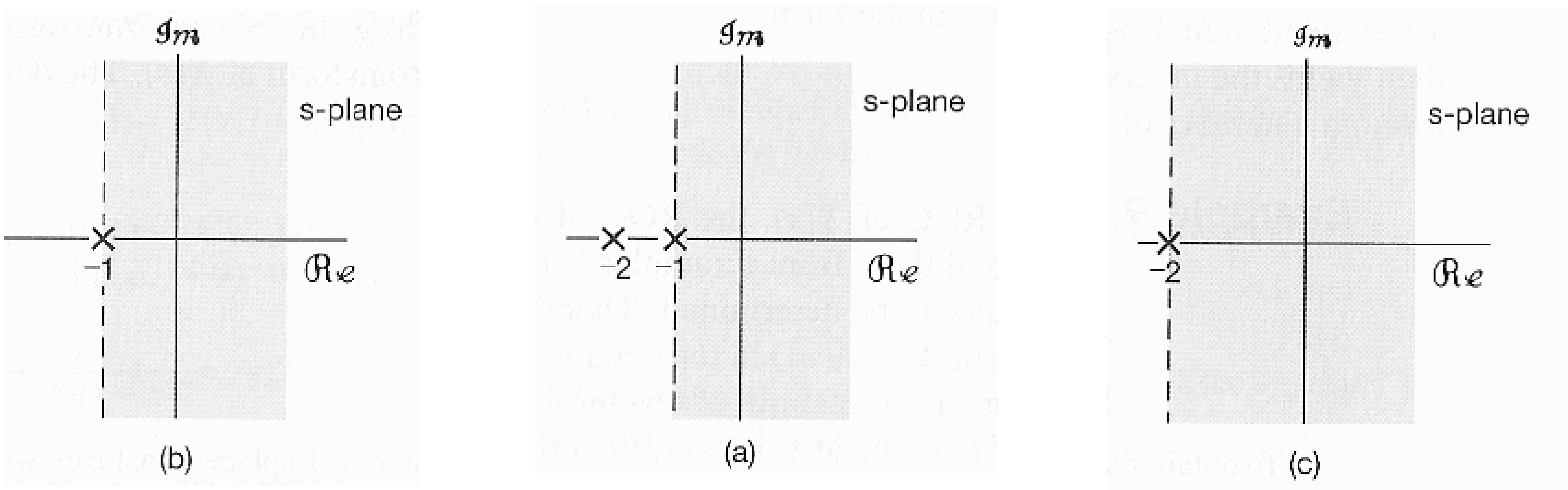
- Example 9.13:

$$x(t) = x_1(t) - x_2(t)$$

$$X_1(s) = \frac{1}{(s+1)}, \quad \mathcal{R}e\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$X(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} = \frac{1}{(s+2)}$$



- **Time Shifting:**

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$x(t - t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s), \quad ROC = R$$

$$X_0(s) = \int_{-\infty}^{\infty} x(t - t_0) e^{-st} dt$$

- **Shifting in the s-Domain:**

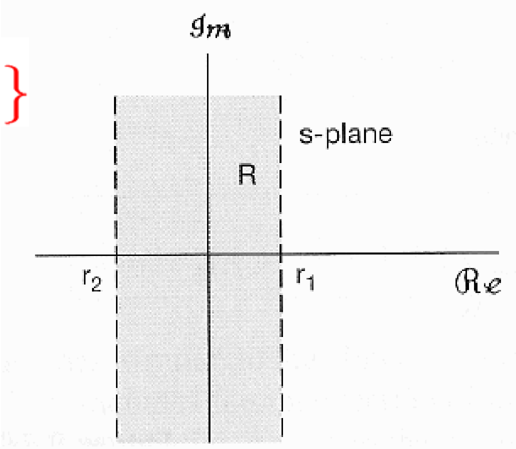
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

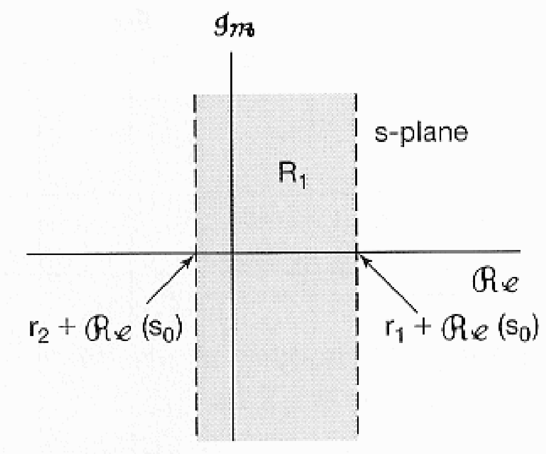
$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \quad ROC = R + \mathcal{R}e\{s_0\}$$

$$X(s - s_0) = \int_{-\infty}^{\infty} x(t) e^{-(s - s_0)t} dt$$





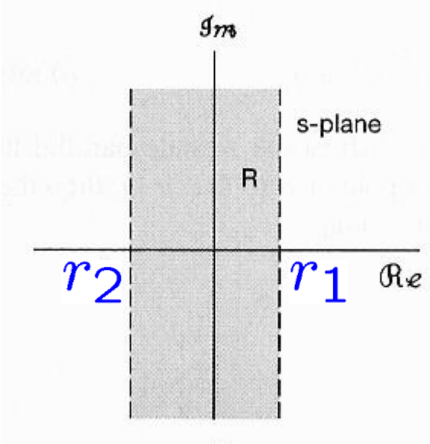
(b)

- Time Scaling:

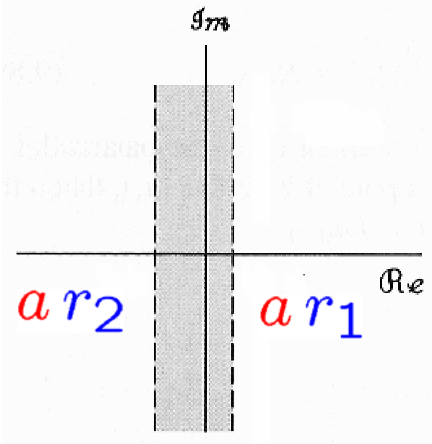
$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X(\frac{s}{a}), \quad ROC = aR$$

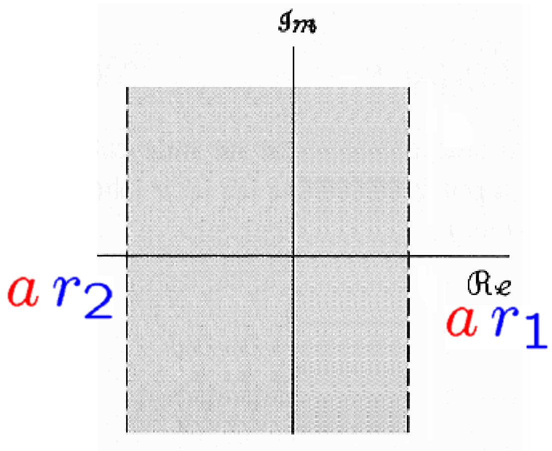
$$X_a(s) = \int_{-\infty}^{\infty} x(a\,t) e^{-st} dt$$



(a)

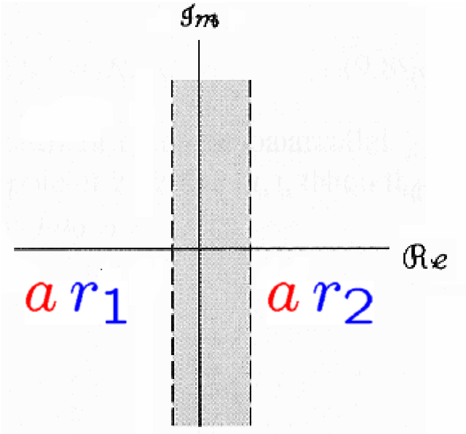


$\boxed{0 < a < 1}$

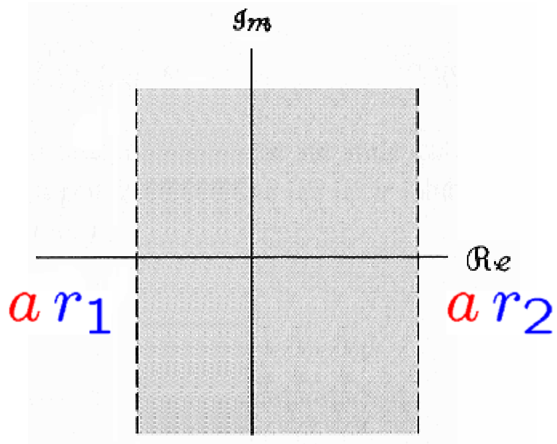


$\boxed{1 < a}$

$$s \longrightarrow \frac{s}{a}$$



$\boxed{-1 < a < 0}$



$\boxed{a < -1}$

$$x(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(-s), \quad ROC = -R$$

- Conjugation:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*), \quad ROC = R$$

$$X(s) = \int_{-\infty}^{\infty} x(t) \; e^{-st} \; dt$$

$$x(t) = x^*(t) \longleftrightarrow X(s) = X^*(s^*)$$

$$z_i \text{ is a zero of } X(s) \longleftrightarrow z_i^* \text{ is a zero of } X(s)$$

$$p_i \text{ is a pole of } X(s) \longleftrightarrow p_i^* \text{ is a peros of } X(s)$$

▪ Convolution Property:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad ROC = R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad ROC = R_2$$

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) X_2(s), \qquad ROC \text{ containing } R_1 \cap R_2$$

$$\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) d\tau$$

Differentiation in the Time & s-Domain:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$\frac{d}{dt} x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \quad ROC \text{ containing } R$$

$$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds} X(s), \quad ROC = R$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \quad e^{st} ds$$

$$X(s) = \int_{-\infty}^{\infty} x(t) \quad e^{-st} dt$$

Integration in the Time Domain:

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s),$$ ROC containing $R \cap \{\mathcal{R}e\{s\} > 0\}$

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}, \qquad \text{ROC } \mathcal{R}\{s\} > -\mathcal{R}\{a\}$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}, \qquad \text{ROC } \mathcal{R}\{s\} > 0$$

▪ The Initial-Value Theorem:

If $x(t) = 0$ for $t < 0$ $\Rightarrow$ $x(0^+) = \lim_{s \to \infty} sX(s)$

$x(t)$ does not include any singular function (impulse, doublet, ...) at $t = 0$.

▪ The Final-Value Theorem:

If $x(t) = 0$ for $t < 0$ and
$x(t)$ has a finit limit as $t \to \infty$, $\Rightarrow$ $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$

Properties of the Laplace Transform

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$$

Outline

- The Laplace Transform
- The Region of Convergence for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

Some Laplace Transform Pairs

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$

$Re\{s\} < 0$

$-[\cos\omega_0 t]u(-t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+\omega_0^2}$,

$-[e^{-\alpha t}\cos\omega_0 t]u(-t) \leftrightarrow \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$, $Re\{s\} < -\alpha$

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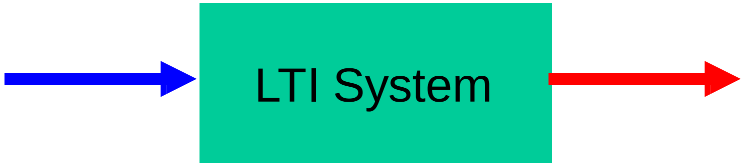
Analysis & Characterization of LTI Systems:

$h(t)$

$x(t)$

$X(s)$



$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s)X(s)$$

$$H(s) = \mathcal{L}\{h(t)\}$$

$H(s)$: system function

or transfer function

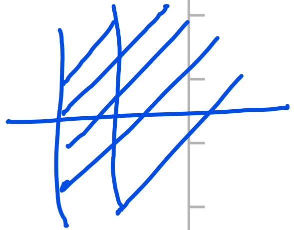
- Causality
- Stability

Causality:

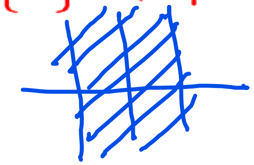
- For a causal LTI system,
  $h(t) = 0$ for $t < 0$, and thus is right sided
- The ROC associated with the system function for a causal system is a right-half plane
- For a system with a rational system function, causality of the system is equivalent to the ROC being
  the right-half plane to the right of the rightmost pole

- Examples 9.17, 9.18, 9.19:

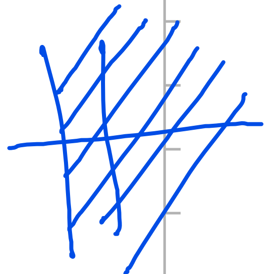
$$h(t) = e^{-t}u(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$



$$h(t) = e^{-|t|} \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{-2}{s^2-1}, \quad -1 < \mathcal{R}e\{s\} < +1$$



$$h(t) = e^{-(t+1)}u(t+1) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{e^s}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$



$$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{rational} \\ ROC: & \text{right-sided} \end{cases} \qquad \begin{cases} h(t): & \text{not causal} \\ H(s): & \text{rational} \\ ROC: & \text{not right-sided} \end{cases} \qquad \begin{cases} h(t): & \text{not causal} \\ H(s): & \text{not rational} \\ ROC: & \text{right-sided} \end{cases}$$

Anti-causality:

- For a anti-causal LTI system,
  $h(t) = 0$ for $t > 0$, and thus is left sided

- The ROC associated with the system function for a anti-causal system is a left-half plane

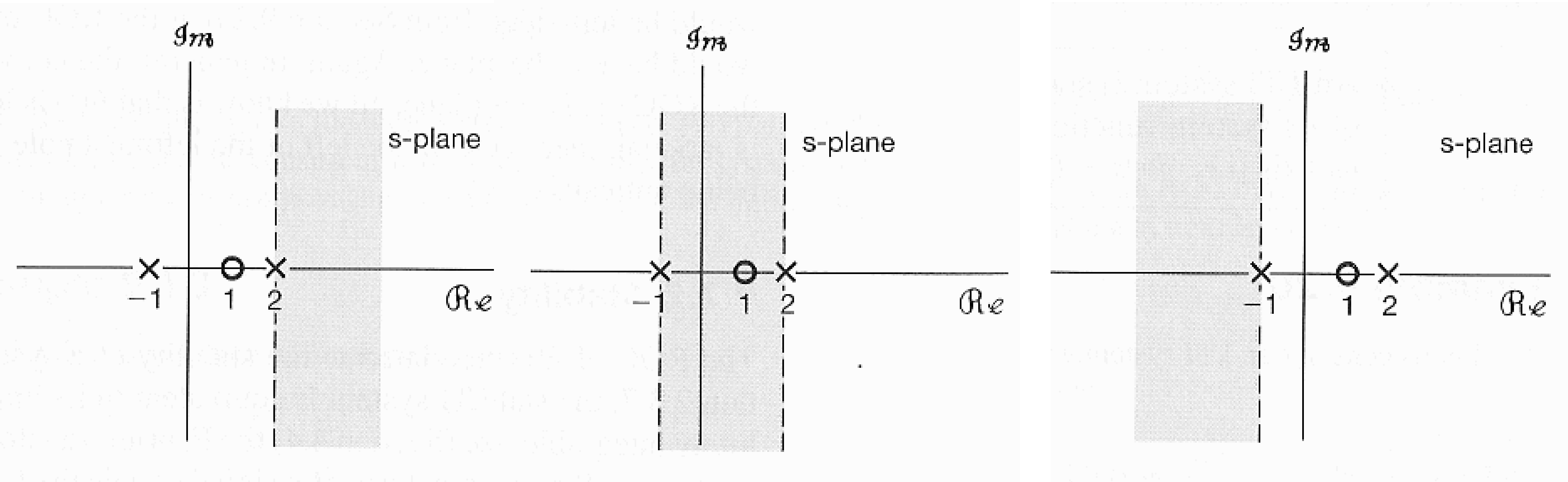
- For a system with a rational system function, anti-causality of the system is equivalent to the ROC being
  the left-half plane to the left of the leftmost pole

Stability:

- An LTI system is stable
  if and only if
  the ROC of its system function $H(s)$ includes
  the entire $j\omega$-axis [i.e., $Re\{s\} = 0$]

- Example 9.20:

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$



(a) (b) (c)

$$\boxed{h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)}$$

$$\boxed{h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)}$$

$$\boxed{h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t)}$$

causal, (unstable) stable, (not causal) unstable, anticausal

Stability:

- A causal system with rational system function $H(s)$ is stable

  if and only if

  all of the poles of $H(s)$ lie in the left-half of s-plane,

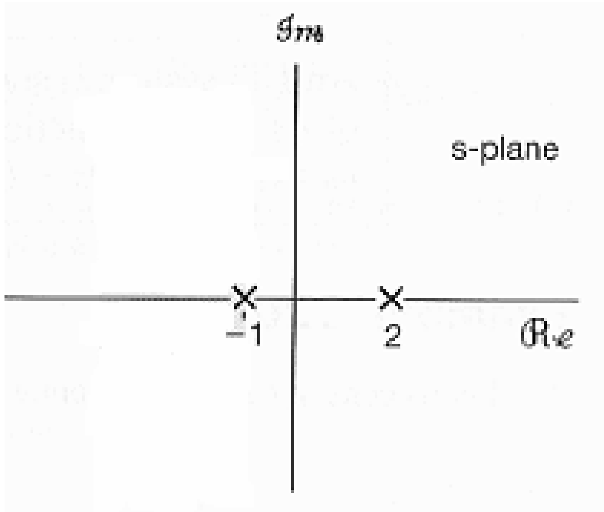
  i.e., all of the poles have negative real parts

Examples 9.17, 9.21:

$$h(t) = e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$

$$h(t) = e^{2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s-2}, \quad 2 < \mathcal{R}e\{s\}$$

$$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{stable, rational} \end{cases} \qquad \begin{cases} h(t): & \text{causal} \\ H(s): & \text{unstable, rational} \end{cases}$$



- LTI Systems by Linear Constant-Coef Differential Equations:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$Y(s) = X(s) H(s) \qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\mathcal{L}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{L}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{L}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{L}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

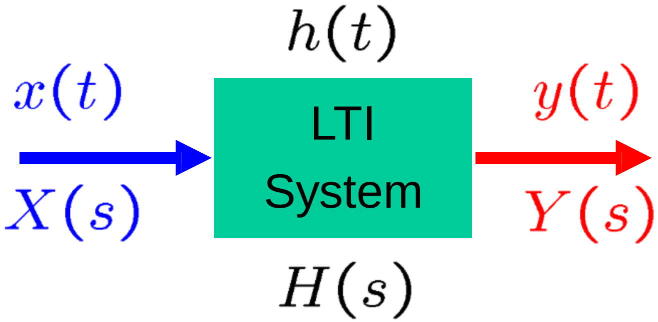
$$Y(s)\left[\sum_{k=0}^{N} a_k s^k\right] = X(s)\left[\sum_{k=0}^{M} b_k s^k\right]$$

zeros

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_M s^M + \cdots + b_1 s + b_0}{a_N s^N + \cdots + a_1 s + a_0}$$

poles

- Example 9.23:



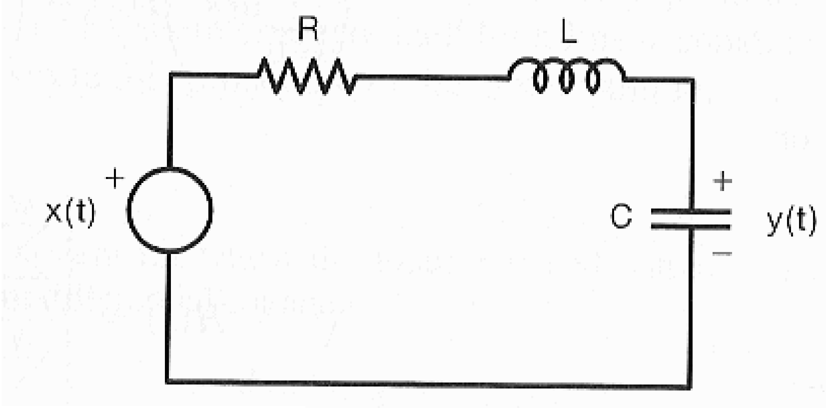
$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$\Rightarrow \; sY(s) + 3Y(s) = X(s) \qquad\qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow \; (s+3)\,Y(s) = X(s) \qquad\qquad \Rightarrow \; H(s) = \frac{1}{s+3}$$

- If causal, $\Rightarrow \; \mathcal{R}\{s\} > -3, \quad \Rightarrow \; h(t) = e^{-3t}u(t)$
- If anti-causal, $\Rightarrow \; \mathcal{R}\{s\} < -3, \quad \Rightarrow \; h(t) = -e^{-3t}u(-t)$

Example 9.24:



$$LC \frac{d^2 y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t)$$

$$\Rightarrow H(s) = \frac{\left(\frac{1}{LC}\right)}{s^2 + \left(\frac{R}{L}\right) s + \left(\frac{1}{LC}\right)} = \frac{\left(\frac{1}{LC}\right)}{(s-a)(s-b)}$$

- If $R, L, C > 0$, $\Rightarrow \mathcal{R}e\{a\}, \mathcal{R}e\{b\} < 0$

  i.e., poles with negative real parts

- Example 9.25:

?

$$x(t) = e^{-3t}u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = \left[e^{-t} - e^{-2t}\right] u(t)$$

$$X(s) = \frac{1}{s+3}, \quad -3 < \mathcal{R}e\{s\} \qquad\qquad Y(s) = \frac{1}{(s+1)(s+2)}, \quad -1 < \mathcal{R}e\{s\}$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$$

$$\text{ROC}: \ -1 < \mathcal{R}e\{s\}$$

$$\Rightarrow \quad \text{casual, stable}$$

$$\Rightarrow \frac{d^2y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + 2\,y(t) = \frac{dx(t)}{dt} + 3\,x(t)$$

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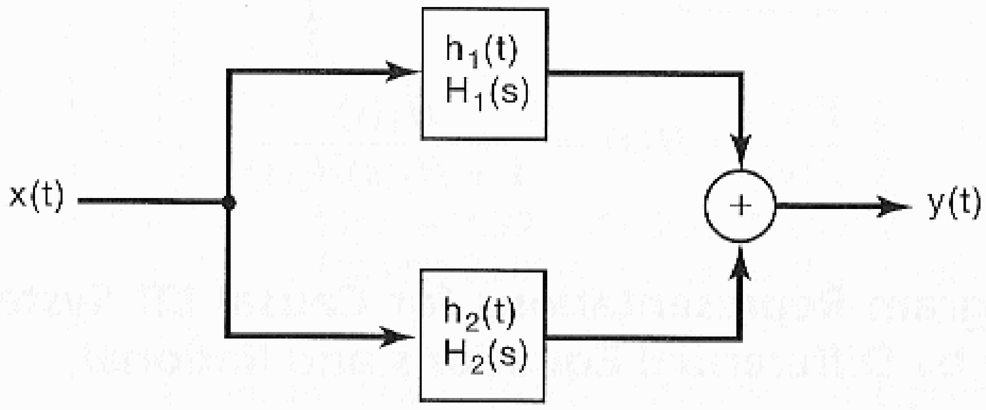
System Function Blocks:

- parallel interconnection

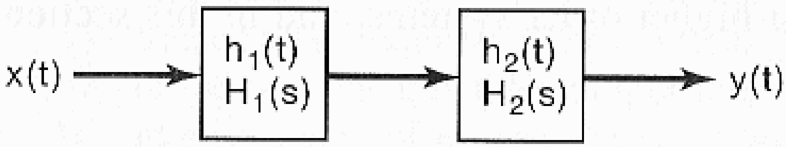
$$h(t) = h_1(t) + h_2(t)$$
$$H(s) = H_1(s) + H_2(s)$$



(a)

- series interconnection

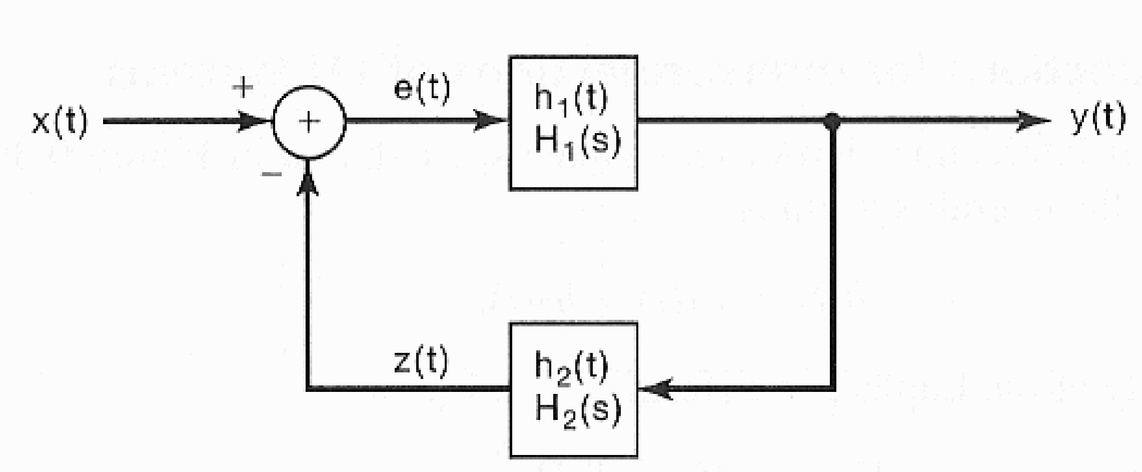
$$h(t) = h_1(t) * h_2(t)$$
$$H(s) = H_1(s)\, H_2(s)$$



(b)

System Function Blocks:

- feedback interconnection



$$H(s) = \frac{H_1(s)}{1+H_1(s)H_2(s)}$$

$$Y = H_1\, E$$

$$Z = H_2\, Y$$

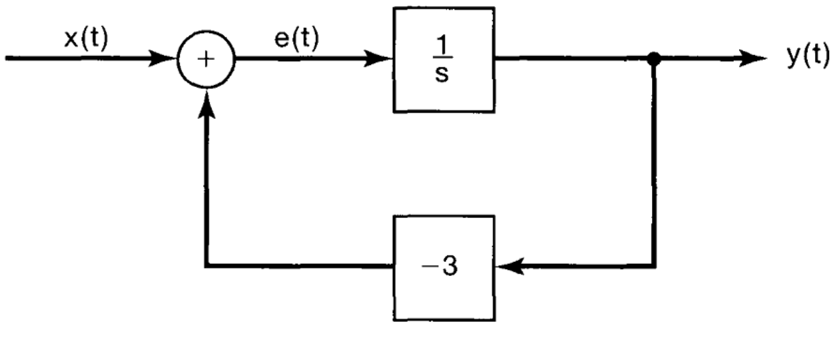
$$E = X - Z$$

- Example 9.28:

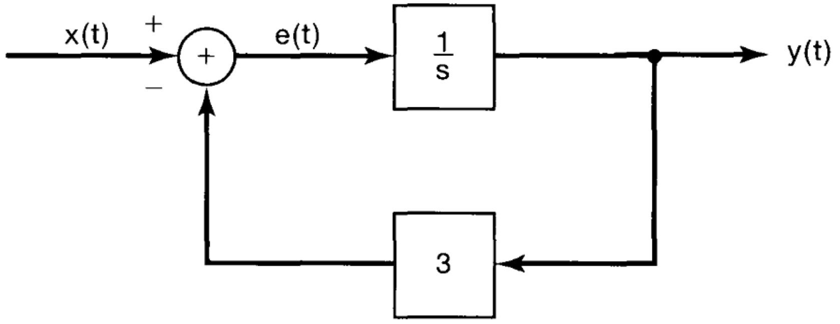
  - Consider a causal LTI system with system function

$$H(s) = \frac{1}{s+3} \quad \Rightarrow \quad Y(s) = \frac{1}{s+3}X(s) \quad \Rightarrow \quad \frac{d}{dt}y(t) + 3y(t) = x(t)$$

$$\Rightarrow \quad \frac{d}{dt}y(t) = x(t) - 3y(t)$$



(a)

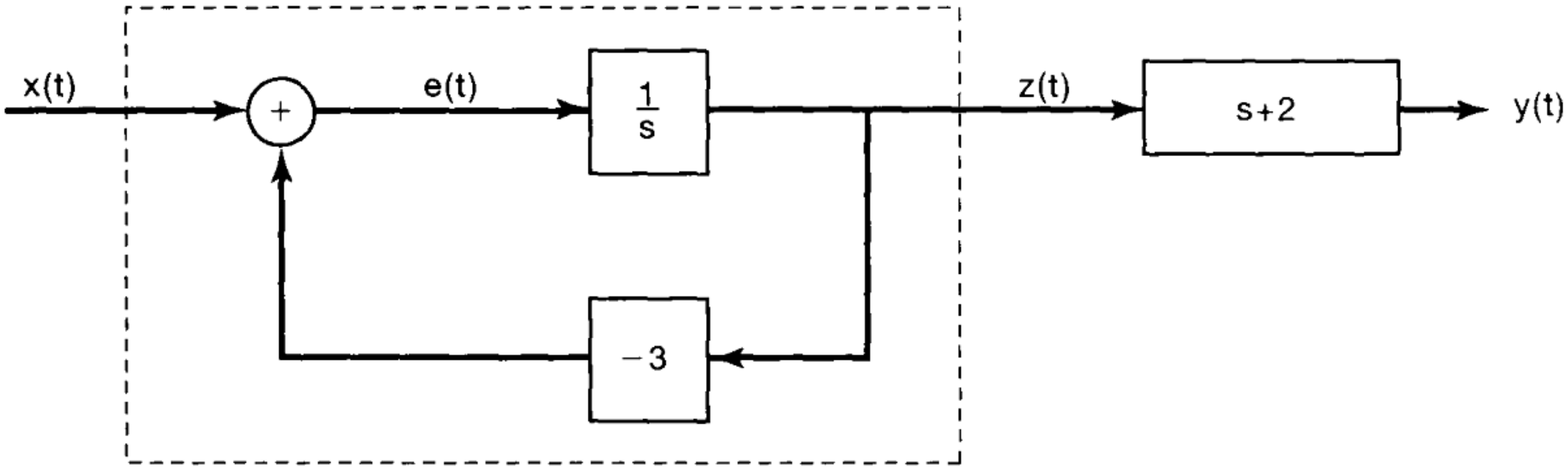


- Example 9.29:

  - Consider a causal LTI system with system function

$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3}\right)(s+2)$$

$$\Rightarrow \quad Z(s) \triangleq \frac{1}{s+3} X(s) \quad \& \quad Y(s) = (s+2) Z(s)$$

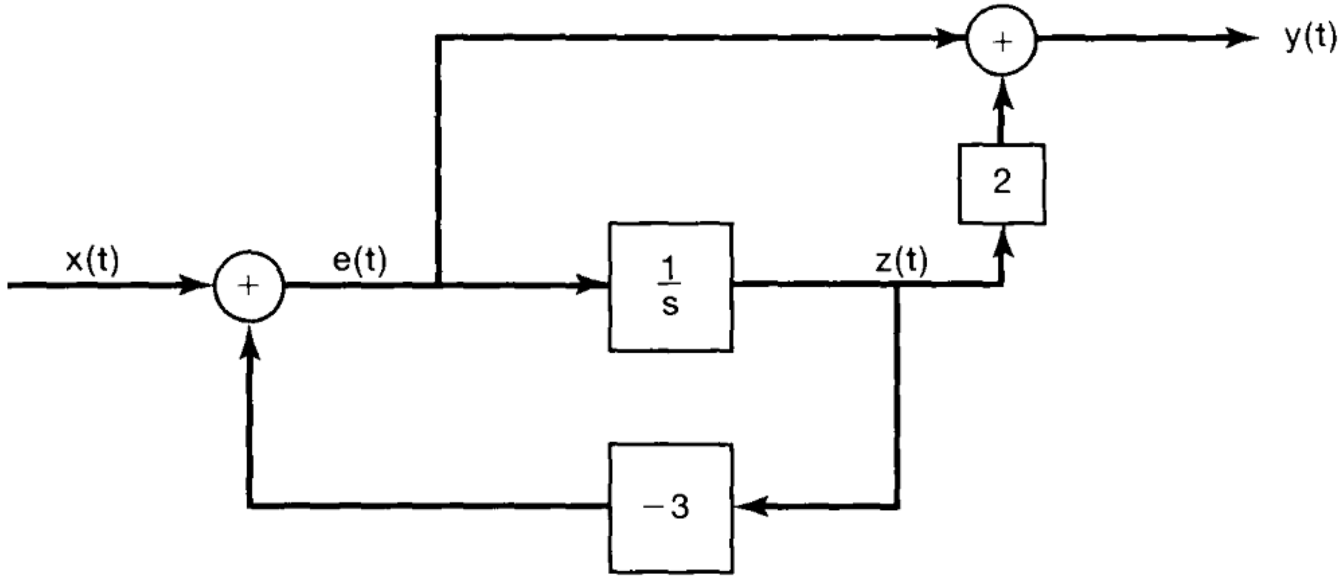


(a)

- Example 9.29:
  - Consider a causal LTI system with system function

$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3}\right)(s+2)$$

$$\Rightarrow \quad Z(s) \triangleq \frac{1}{s+3}X(s) \quad \& \quad Y(s) = (s+2)Z(s)$$



Example 9.6

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

s=-a is the root of the denominator.

x(t) is finite duration and absolutely integrable.

Example 9.6

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

Using the L'hospital's rule:

$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left[\frac{\frac{d}{ds}\left(1 - e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)}\right] = \lim_{s\to -a} Te^{-aT}e^{-sT}$$

Example 9.14

$$x(t) = te^{-at}u(t).$$

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

$$te^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \quad \mathrm{Re}\{s\} > -a$$

$$\frac{t^2}{2}e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^3}, \quad \mathrm{Re}\{s\} > -a$$

$$\boxed{\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}, \quad \mathrm{Re}\{s\} > -a}$$

Example 9.26

Suppose that we are given the following information about an LTI system:

1. The system is causal.
2. The system function is rational and has only two poles, at $s = -2$ and $s = 4$.
3. If $x(t) = 1$, then $y(t) = 0$.
4. The value of the impulse response at $t = 0^+$ is 4.

determine the system function of the system.

Example 9.26

Suppose that we are given the following information about an LTI system:

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3. If $x(t) = 1$, then $y(t) = 0$.
4. The value of the impulse response at $t = 0^+$ is 4.

determine the system function of the system.

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8},$$

the response $y(t)$ to the input $x(t) = 1 = e^{0\cdot t}$

$$H(0)\cdot e^{0\cdot t} = H(0). \quad \Rightarrow \quad p(0) = 0$$

$$\Rightarrow \quad p(s) = sq(s),$$

Example 9.26

Suppose that we are given the following information about an LTI system:

1. The system is causal.
2. The system function is rational and has only two poles, at $s = -2$ and $s = 4$.
3. If $x(t) = 1$, then $y(t) = 0$.
4. The value of the impulse response at $t = 0^+$ is 4.

determine the system function of the system.

$$\lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{s^2 q(s)}{s^2 - 2s - 8} = 4.$$

$$\lim_{s\to\infty} \frac{Ks^2}{s^2 - 2s - 8} = \lim_{s\to\infty} \frac{Ks^2}{s^2} = K.$$

$$H(s) = \frac{4s}{(s+2)(s-4)}.$$