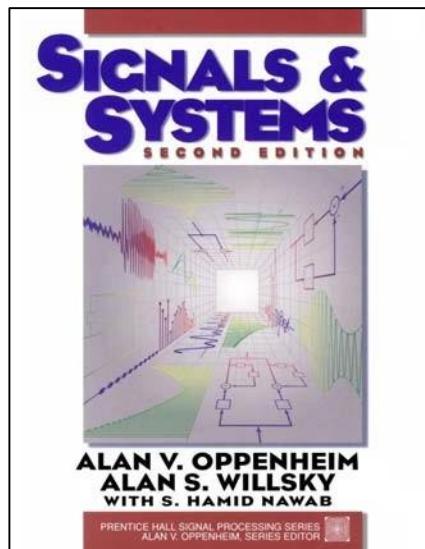


Spring 2011

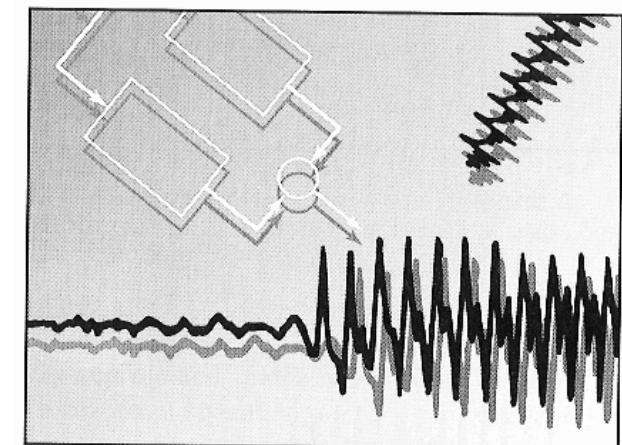
信號與系統 Signals and Systems

Chapter SS-1 Signals and Systems



Feng-Li Lian
NTU-EE
Feb11 – Jun11

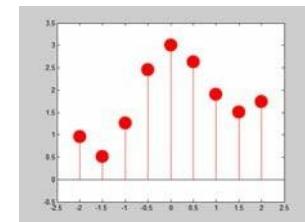
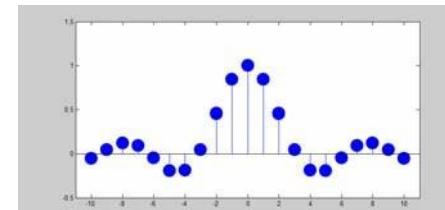
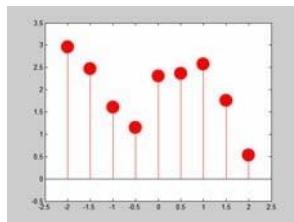
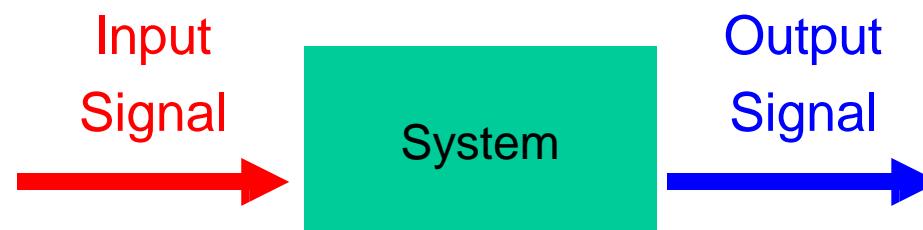
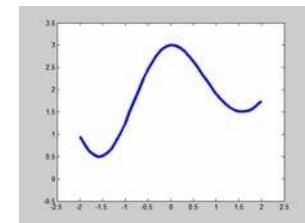
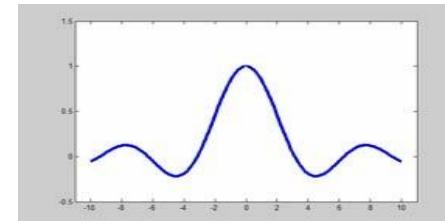
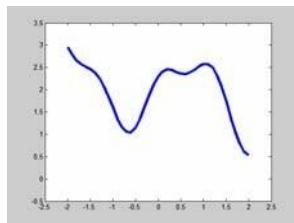
Figures and images used in these lecture notes are adopted from
“Signals & Systems” by Alan V. Oppenheim and Alan S. Willsky, 1997



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

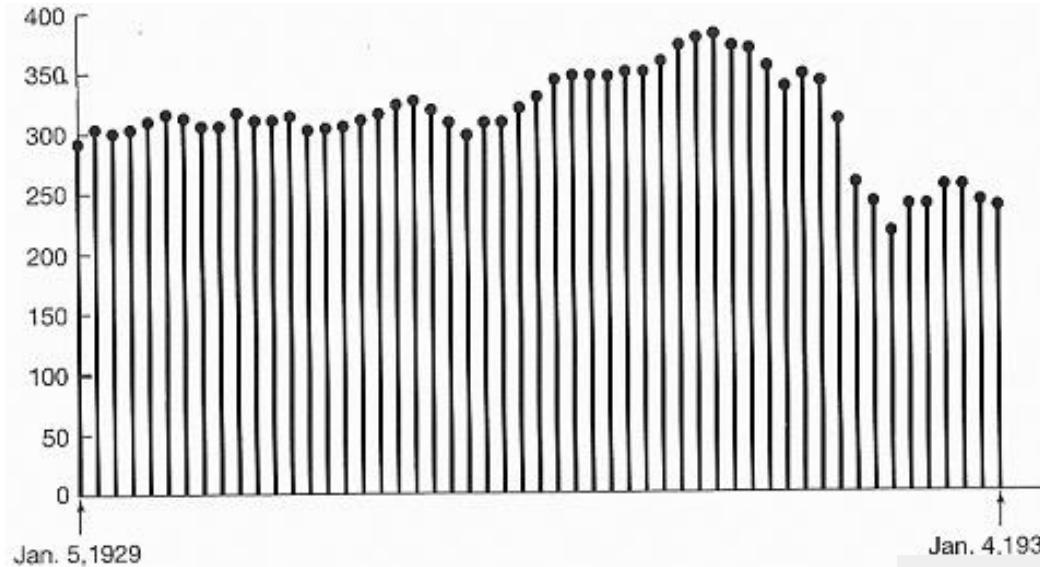
▪ Signals & Systems:

- Is about using mathematical techniques to help describe and analyze **systems** which process **signals**
 - Signals are variables that carry information
 - Systems process input signals to produce output signals



■ Discrete-Time Signals:

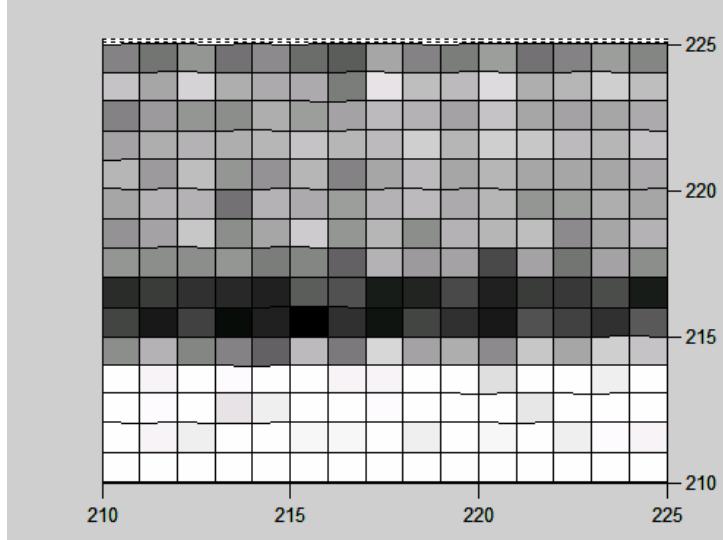
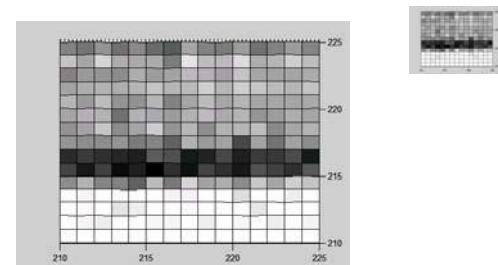
- The weekly Dow-Jones stock market index



<http://big5.jrj.com.cn/>



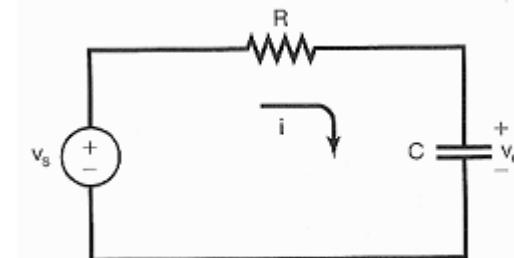
- Discrete-Time Signals:
 - A monochromatic picture



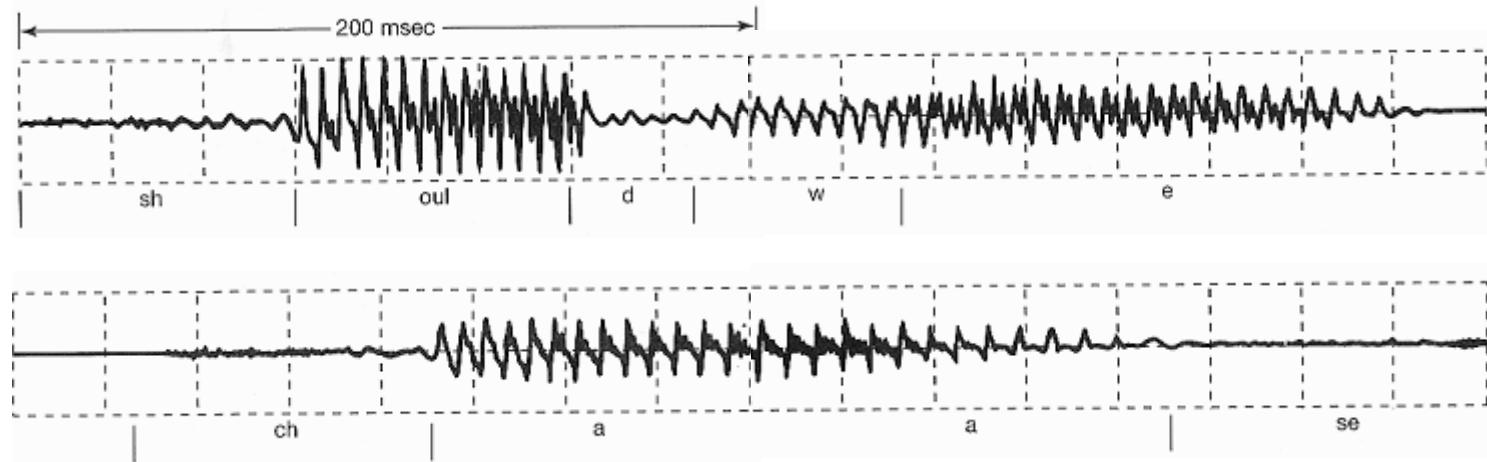
117	151	112	136	110	95	166	131	125	159	114	130	159	133
164	209	173	171	168	126	225	189	184	219	172	180	206	188
154	151	143	174	156	160	186	179	163	193	167	161	167	168
173	179	172	182	195	181	187	205	183	205	196	186	180	193
155	188	149	145	181	131	166	185	166	181	164	164	166	171
176	176	113	178	169	159	177	184	171	182	148	159	174	164
162	198	141	164	202	149	181	142	176	182	189	136	165	176
143	143	148	127	132	97	177	152	160	74	163	119	162	143
62	51	40	32	95	82	28	39	75	35	60	58	77	28
26	67	14	34	3	49	22	69	48	26	81	67	49	91
176	134	129	97	185	120	212	160	173	139	197	166	207	192
243	255	250	255	254	242	242	255	254	222	255	253	239	252
248	255	226	238	255	255	249	255	255	230	252	255	255	255
243	238	255	253	245	247	255	238	255	244	255	237	249	241

▪ Continuous-Time Signals:

- Source voltage & capacity voltage in a simple RC circuit

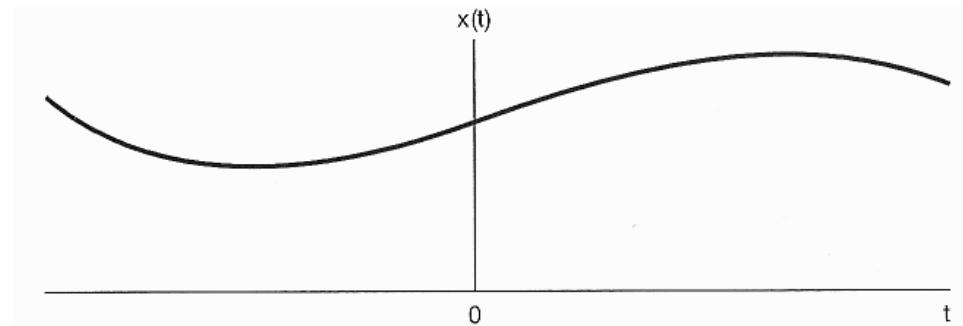


- Recording of a speech signal

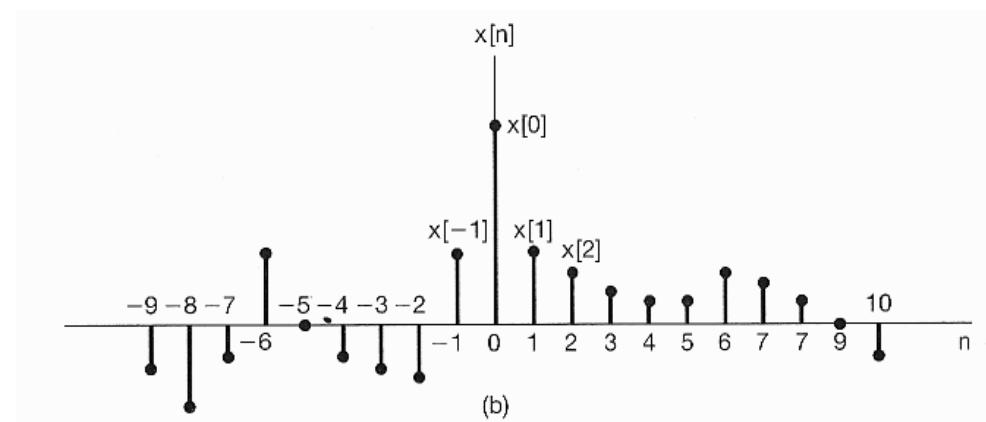


▪ Graphical Representations of Signals:

- Continuous-time signals $x(t)$ or $x_c(t)$



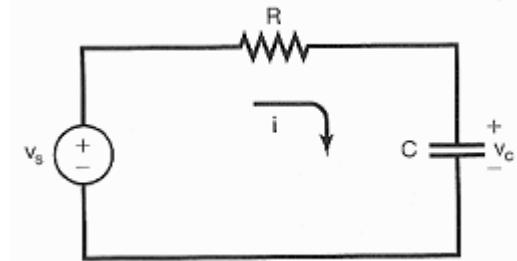
- Discrete-time signals $x[n]$ or $x_d[n]$



- Energy & Power of a resistor:

- Instantaneous power

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$



- Total energy over a finite time interval

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- Average power over a finite time interval

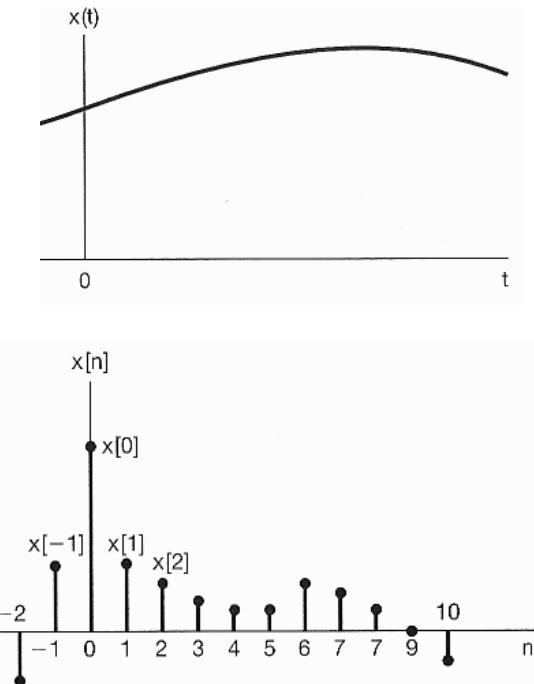
$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

■ Signal Energy & Power:

- Total energy over a finite time interval

$$E \triangleq \int_{t_1}^{t_2} |x(t)|^2 dt \quad \text{continuous-time}$$

$$E \triangleq \sum_{n=n_1}^{n_2} |x[n]|^2 \quad \text{discrete-time}$$



- Time-averaged power over a finite time interval

$$P \triangleq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \quad \text{continuous-time}$$

$$P \triangleq \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 \quad \text{discrete-time}$$

■ Signal Energy & Power:

- Total energy over an infinite time interval

$$E_{\infty} \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$E_{\infty} \triangleq \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

- Time-averaged power over an infinite time interval

$$P_{\infty} \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$P_{\infty} \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

- Three Classes of Signals:

$$E_{\infty} = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$$
$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

- Finite total energy & zero average power

$$0 \leq E_{\infty} < \infty \quad \Rightarrow \quad P_{\infty} = \lim_{T \rightarrow \infty} \frac{E_{\infty}}{2T} = 0$$

- Finite average power & infinite total energy

$$0 \leq P_{\infty} < \infty \quad \Rightarrow \quad E_{\infty} = \infty \quad (\text{if } P_{\infty} > 0)$$

- Infinite average power & infinite total energy

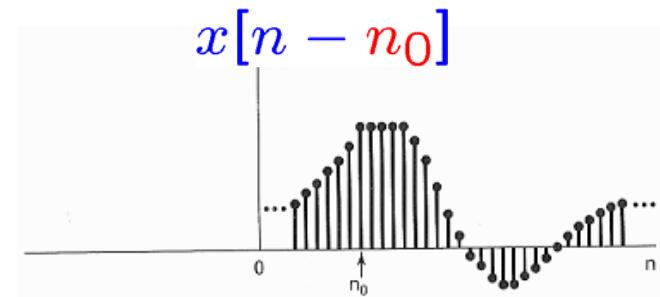
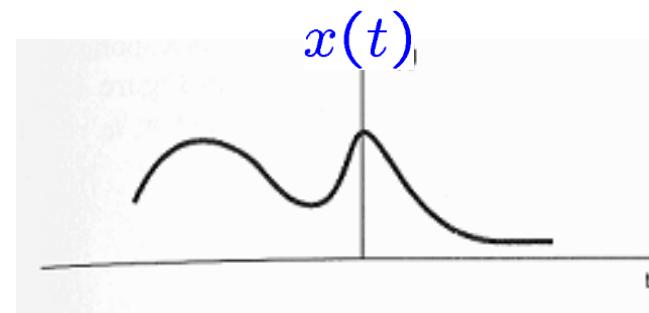
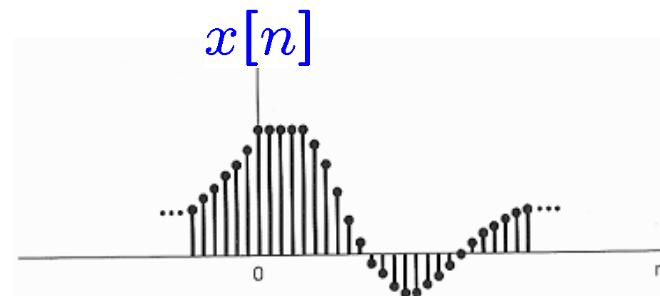
$$P_{\infty} = \infty \quad \& \quad E_{\infty} = \infty$$



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
 - Time Shift
 - Time Reversal
 - Time Scaling
 - Periodic Signals
 - Even & Odd Signals
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

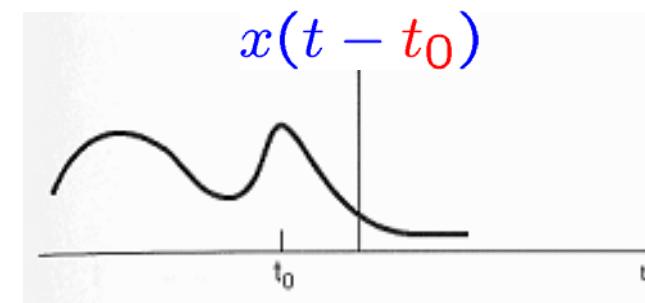
■ Time Shift:

$$\begin{cases} n_0, t_0 > 0 : \text{ delay} \\ n_0, t_0 < 0 : \text{ advance} \end{cases}$$



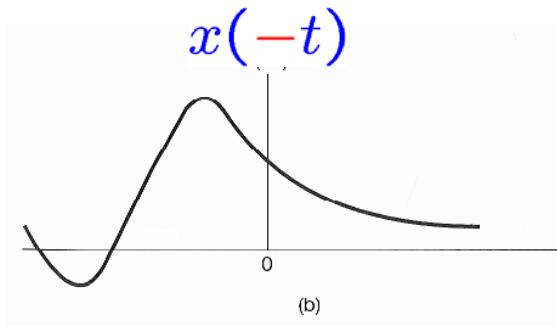
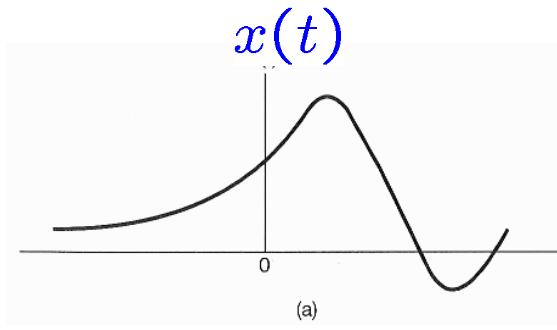
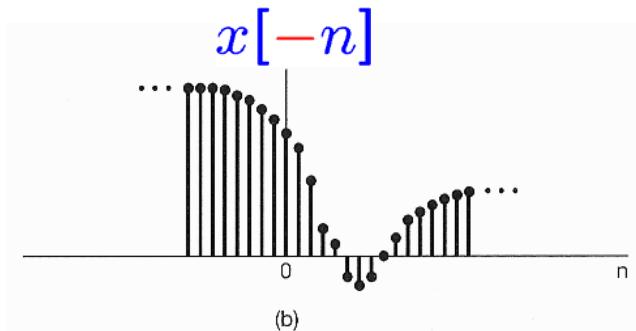
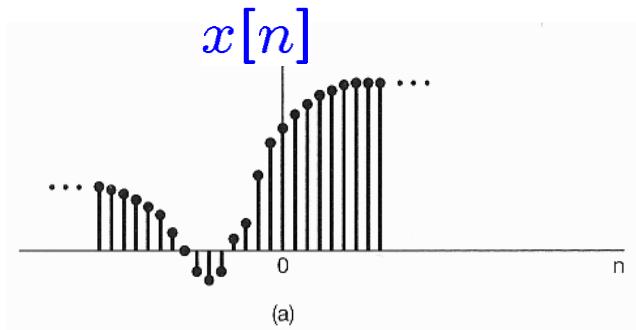
$$n_0 > 0$$

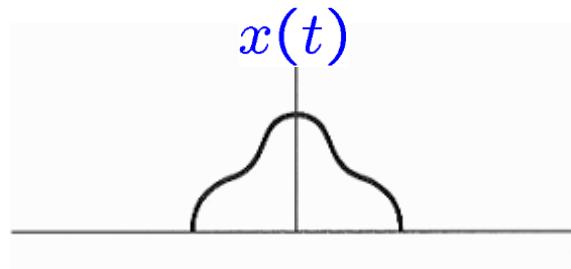
$$x[n - 8]$$



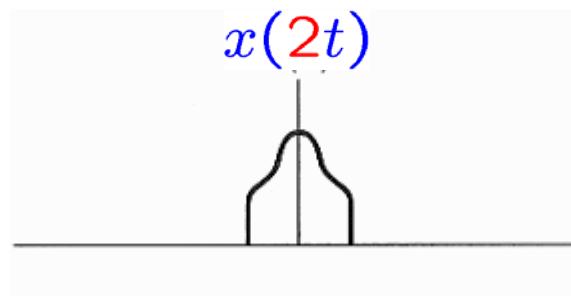
$$t_0 < 0$$

$$x(t + 5)$$

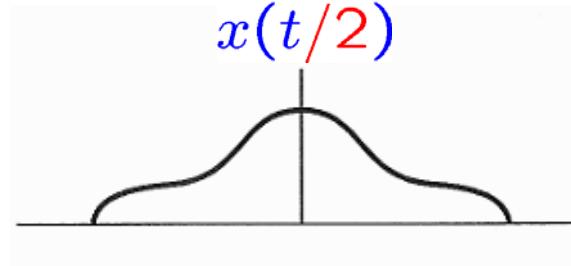
■ Time Reversal:

■ Time Scaling:

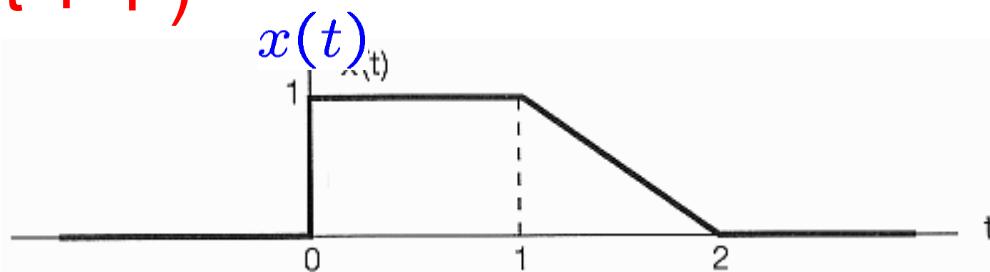
$$t \rightarrow 2t$$



$$t \rightarrow t/2$$

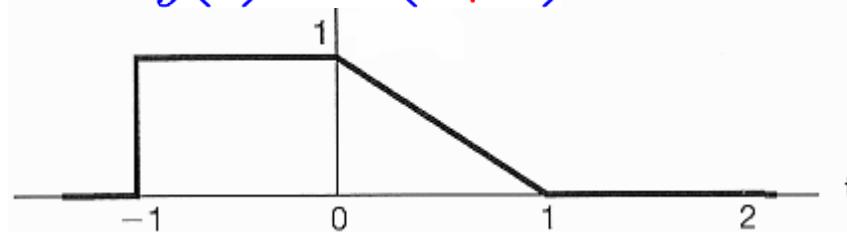


■ $x(t) \rightarrow x(-t+1)$



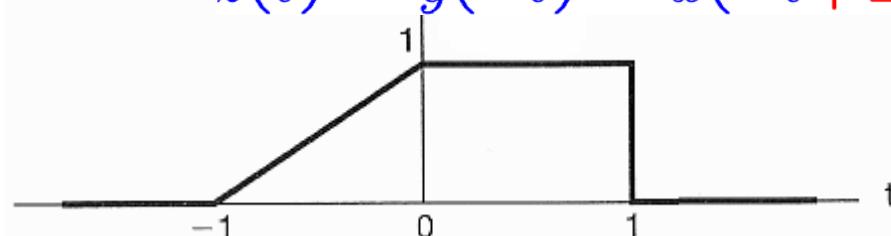
$t \rightarrow t+1$

$$y(t) = x(t+1)$$

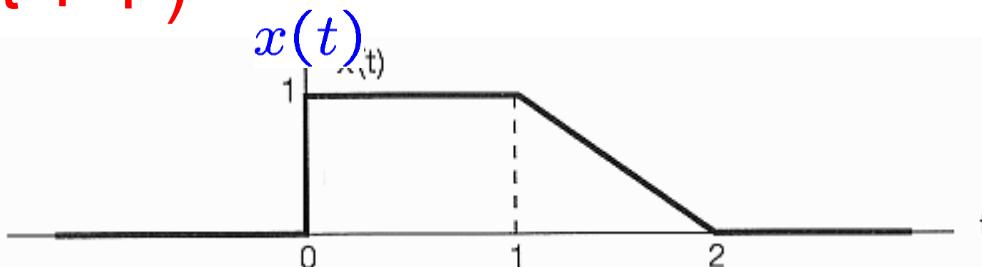


$t \rightarrow -t$

$$z(t) = y(-t) = x(-t+1)$$



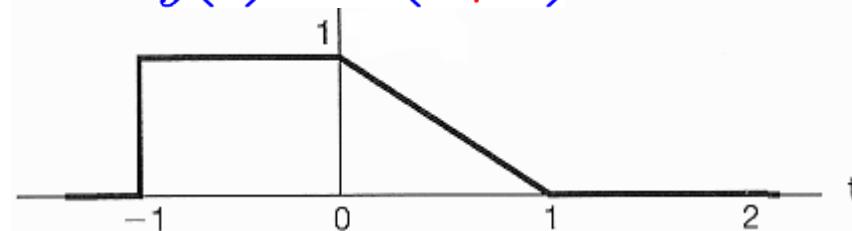
$$\blacksquare x(t) \rightarrow x(-t+1)$$



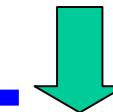
$$t \rightarrow t+1$$



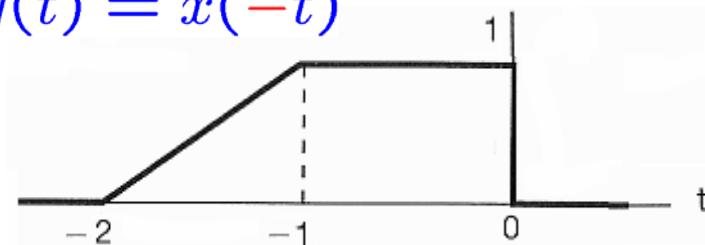
$$y(t) = x(t+1)$$



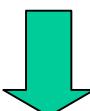
$$t \rightarrow -t$$



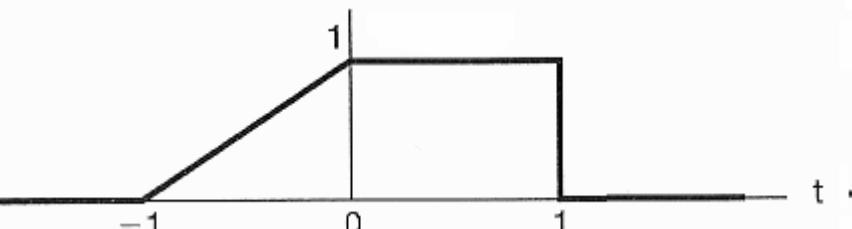
$$y(t) = x(-t)$$



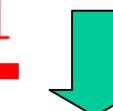
$$t \rightarrow -t$$



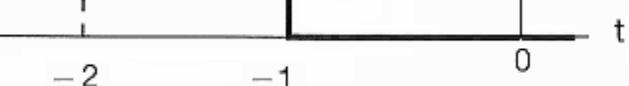
$$z(t) = y(-t) = x(-t+1)$$



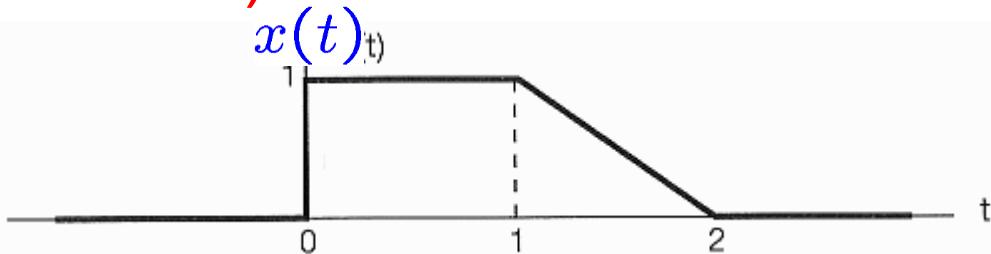
$$t \rightarrow t+1$$



$$\begin{aligned} z(t) &= y(t+1) \\ &= x(-(t+1)) = x(-t-1) \end{aligned}$$

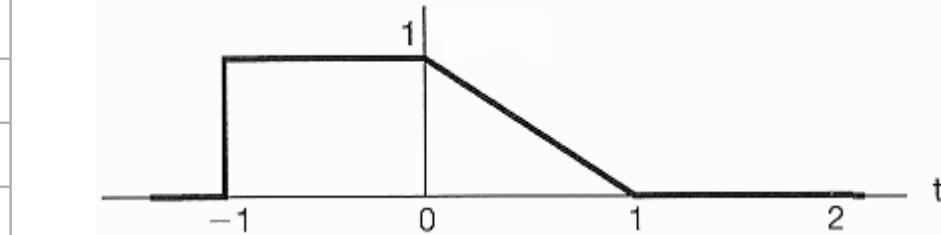


■ $x(t) \rightarrow x(3/2 t + 1)$



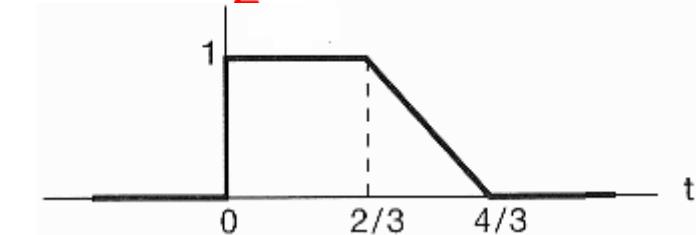
$t \rightarrow t+1$

$$y(t) = x(t+1)$$



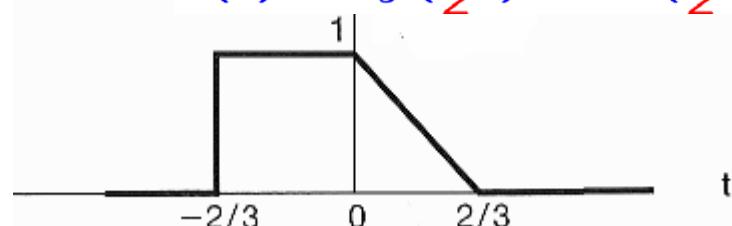
$t \rightarrow \frac{3}{2}t$

$$y(t) = x\left(\frac{3}{2}t\right)$$



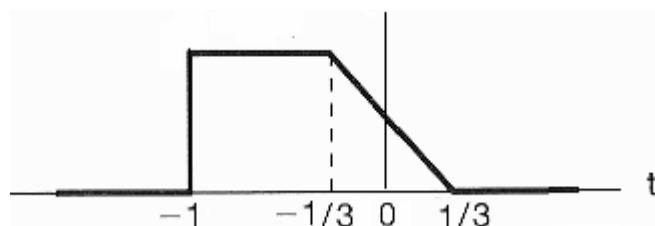
$t \rightarrow \frac{3}{2}t$

$$z(t) = y\left(\frac{3}{2}t\right) = x\left(\frac{3}{2}t+1\right)$$



$t \rightarrow t+1$

$$z(t) = y(t+1) = x\left(\frac{3}{2}t+\frac{3}{2}\right)$$

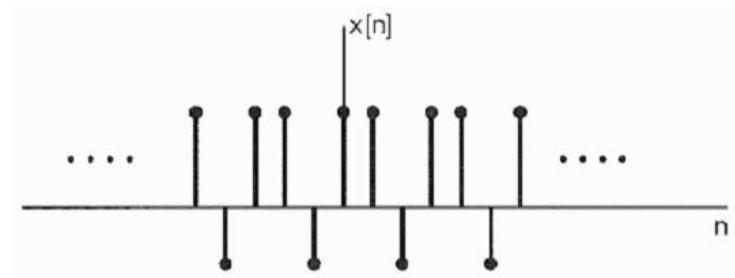
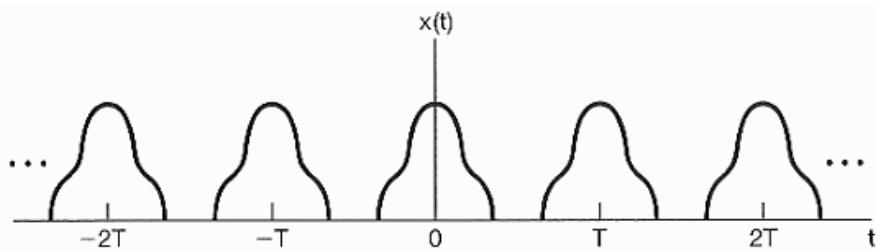


$$\blacksquare x(t) \rightarrow x(at - b)$$

- $|a| < 1$: linearly stretched
- $|a| > 1$: linearly compressed
- $a < 0$: time reversal
- $b > 0$: delayed time shift
- $b < 0$: advanced time shift

- \blacksquare Problems:
 - P1.21 for CT
 - P1.22 for DT

■ CT & DT Periodic Signals:



$$N = 3$$

$$x(t) = x(t + T) \quad \text{for } T > 0 \text{ and all values of } t$$

$$x[n] = x[n + N] \quad \text{for } N > 0 \text{ and all values of } n$$

■ Periodic Signals:

$$x(t) = x(t + T) \quad \text{for } T > 0 \text{ and all values of } t$$

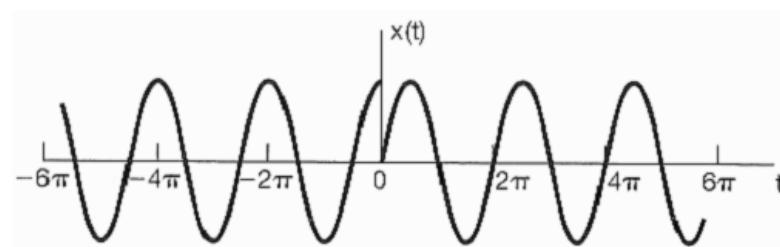
$$x[n] = x[n + N] \quad \text{for } N > 0 \text{ and all values of } n$$

- A periodic signal is **unchanged** by a **time shift** of T or N
- They are also **periodic** with period
 - $2T, 3T, 4T, \dots$
 - $2N, 3N, 4N, \dots$
- The least positive T or N is called the **fundamental period** denoted as T_0 or N_0

■ Periodic signal ?

$$x(t) = x(t + T) \quad \forall t, \quad T > 0$$

$$x(t) = \begin{cases} \cos(t), & \text{if } t < 0 \\ \sin(t), & \text{if } t \geq 0 \end{cases}$$

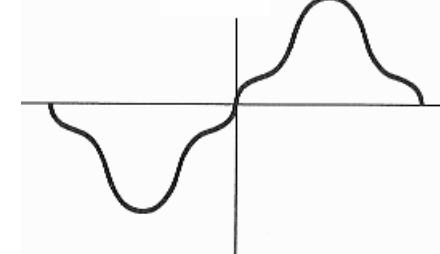
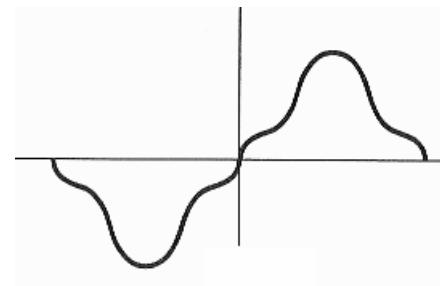
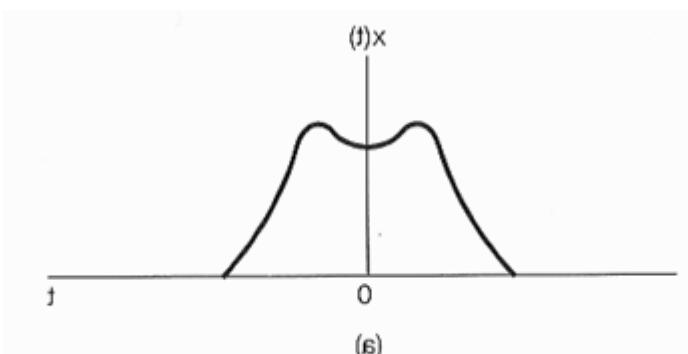
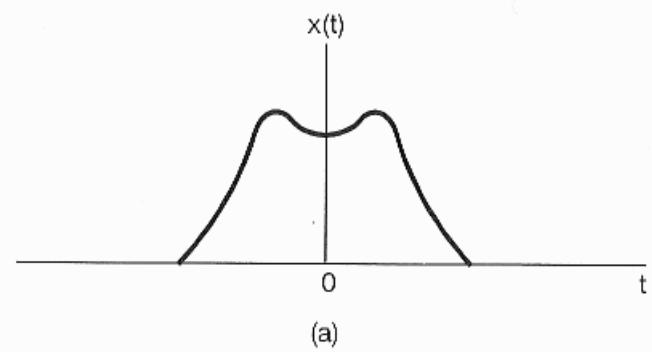


- **Problems:**
 - P1.25 for CT
 - P1.26 for DT

- Even & odd signals:

A signal is **even** if $x(-t) = x(t)$ or $x[-n] = x[n]$

A signal is **odd** if $x(-t) = -x(t)$ or $x[-n] = -x[n]$



▪ Even-odd decomposition of a signal:

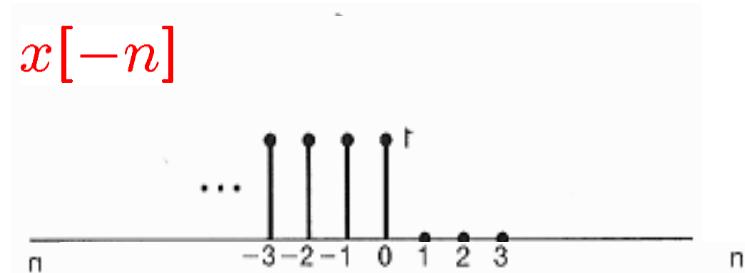
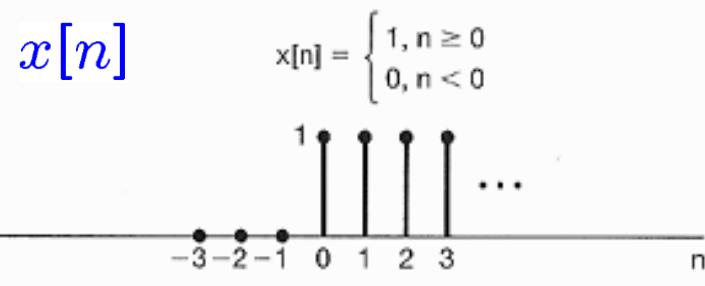
- Any signal can be broken into a **sum** of one **even signal** and one **odd signal**

$$\mathcal{Ev}\{x(t)\} = \frac{1}{2} [x(t) + x(-t)] = \frac{1}{2} [x(-t) + x(t)]$$

$$\mathcal{Od}\{x(t)\} = \frac{1}{2} [x(t) - x(-t)] = -\frac{1}{2} [x(-t) - x(t)]$$

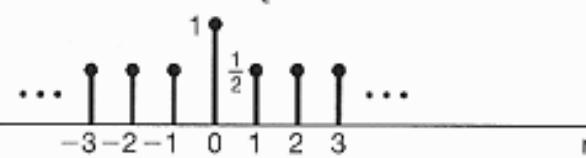
$$\Rightarrow x(t) = \mathcal{Ev}\{x(t)\} + \mathcal{Od}\{x(t)\}$$

■ Even-odd decomposition of a DT signal:



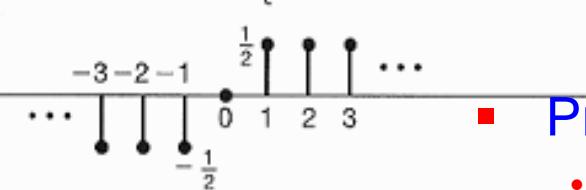
$$\mathcal{Ev}\{x[n]\} = \frac{1}{2} [x[n] + x[-n]]$$

$$\mathcal{Ev}\{x[n]\} = \begin{cases} \frac{1}{2}, n < 0 \\ 1, n = 0 \\ \frac{1}{2}, n > 0 \end{cases}$$



$$\mathcal{Od}\{x[n]\} = \frac{1}{2} [x[n] - x[-n]]$$

$$\mathcal{Od}\{x[n]\} = \begin{cases} -\frac{1}{2}, n < 0 \\ 0, n = 0 \\ \frac{1}{2}, n > 0 \end{cases}$$



■ **Problems:**

- P1.23 for CT
- P1.24 for DT

■ Uniqueness of even-odd decomposition:

Assume that $x(t) = \mathcal{E}v_1(t) + \mathcal{O}d_1(t)$

and $x(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$

So, $\mathcal{E}v_1(t) + \mathcal{O}d_1(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$

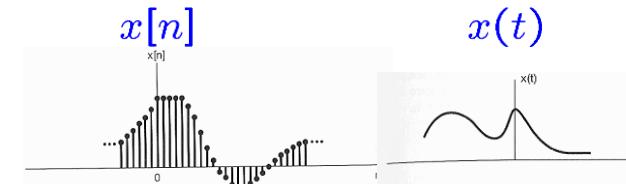
and $\mathcal{E}v_1(-t) + \mathcal{O}d_1(-t) = \mathcal{E}v_2(-t) + \mathcal{O}d_2(-t)$

Because $\begin{cases} \mathcal{E}v_1(-t) = \mathcal{E}v_1(t) \\ \mathcal{E}v_2(-t) = \mathcal{E}v_2(t) \end{cases}$ and $\begin{cases} \mathcal{O}d_1(-t) = -\mathcal{O}d_1(t) \\ \mathcal{O}d_2(-t) = -\mathcal{O}d_2(t) \end{cases}$

Then, $\mathcal{E}v_1(t) - \mathcal{O}d_1(t) = \mathcal{E}v_2(t) - \mathcal{O}d_2(t)$

$\Rightarrow 2\mathcal{E}v_1(t) = 2\mathcal{E}v_2(t)$ or, $\mathcal{E}v_1(t) = \mathcal{E}v_2(t)$

$\Rightarrow 2\mathcal{O}d_1(t) = 2\mathcal{O}d_2(t)$ or, $\mathcal{O}d_1(t) = \mathcal{O}d_2(t)$

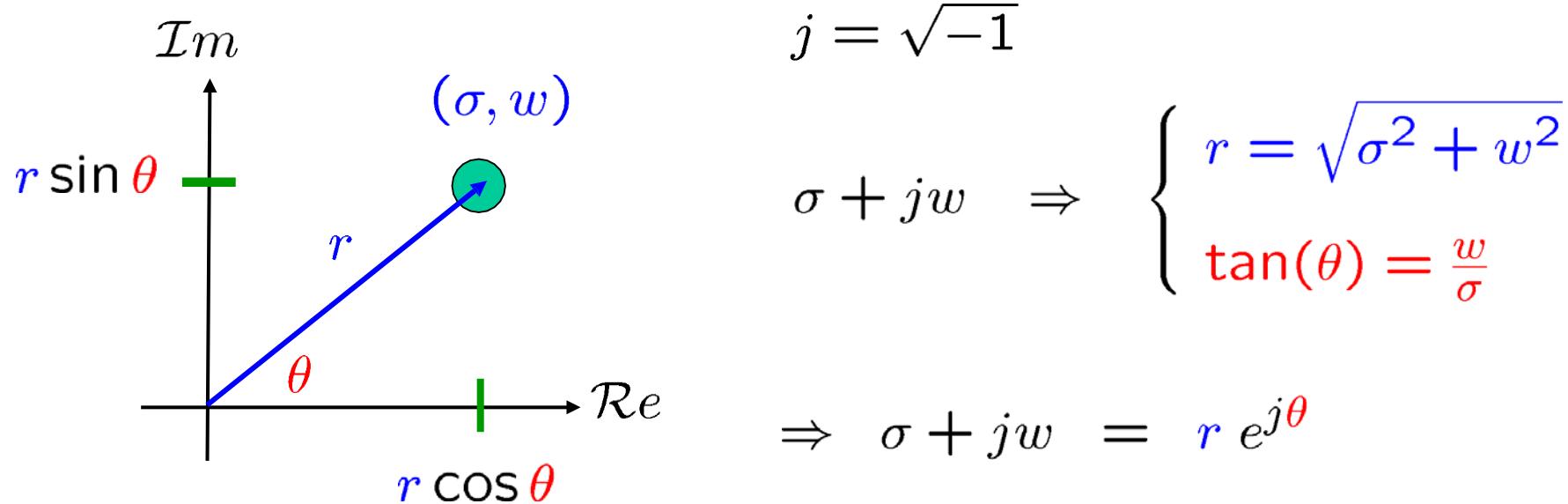


- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable

– Time Shift	$x[n - n_0]$	$x(t - t_0)$	$x(-t) = x(t), x[-n] = x[n]$
– Time Reversal	$x[-n]$	$x(-t)$	$x(-t) = -x(t), x[-n] = -x[n]$
– Time Scaling	$x[an]$	$x(at)$	$\mathcal{E}_v \{x[n]\} = \frac{1}{2} [x[n] + x[-n]]$
– Periodic Signals	$x(t) = x(t + T)$		$\mathcal{O}_d \{x[n]\} = \frac{1}{2} [x[n] - x[-n]]$
– Even & Odd Signals	$x[n] = x[n + N]$		

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

- Magnitude & Phase Representation:



- Euler's relation:

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$\Rightarrow \sigma + jw = r (\cos \theta + j \sin \theta)$$

$$= (r \cos \theta) + j(r \sin \theta)$$

■ CT Complex Exponential Signals:

$$x(t) = \textcolor{red}{C} e^{\textcolor{blue}{a}t}$$

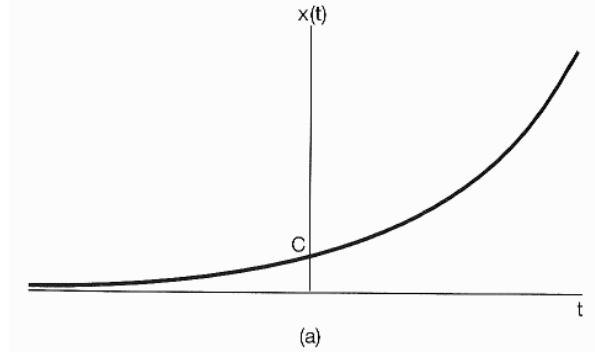
- where $\textcolor{red}{C}$ & $\textcolor{blue}{a}$ are, in general, complex numbers

$$\textcolor{blue}{a} = \sigma + jw$$

$$\textcolor{red}{C} = |C| e^{j\theta}$$

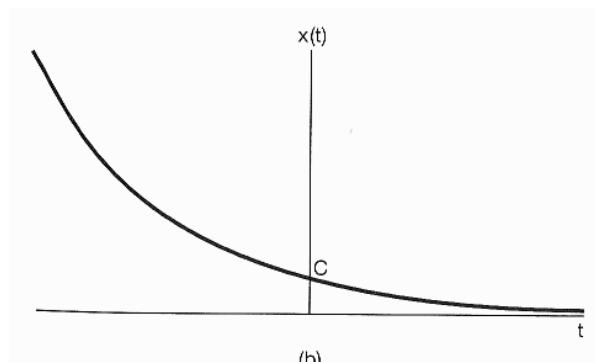
▪ Real exponential signals:

- If C & a are real



$$x(t) = Ce^{at}$$

$$a > 0$$



$$a < 0$$

■ Periodic complex exponential signals: $e^{j\theta} = \cos \theta + j \sin \theta$

- If a is purely imaginary

$$a = \sigma + jw$$

$$x(t) = e^{jw_0 t}$$

- It is periodic

— Because let

$$T_0 = \frac{2\pi}{|w_0|}$$

— Then

$$e^{jw_0 T_0} = e^{jw_0 \frac{2\pi}{|w_0|}} = \cos(\pm 2\pi) + j \sin(\pm 2\pi) = 1$$

— Hence

$$e^{jw_0(t+T_0)} = e^{jw_0 t} e^{jw_0 T_0} = e^{jw_0 t}$$

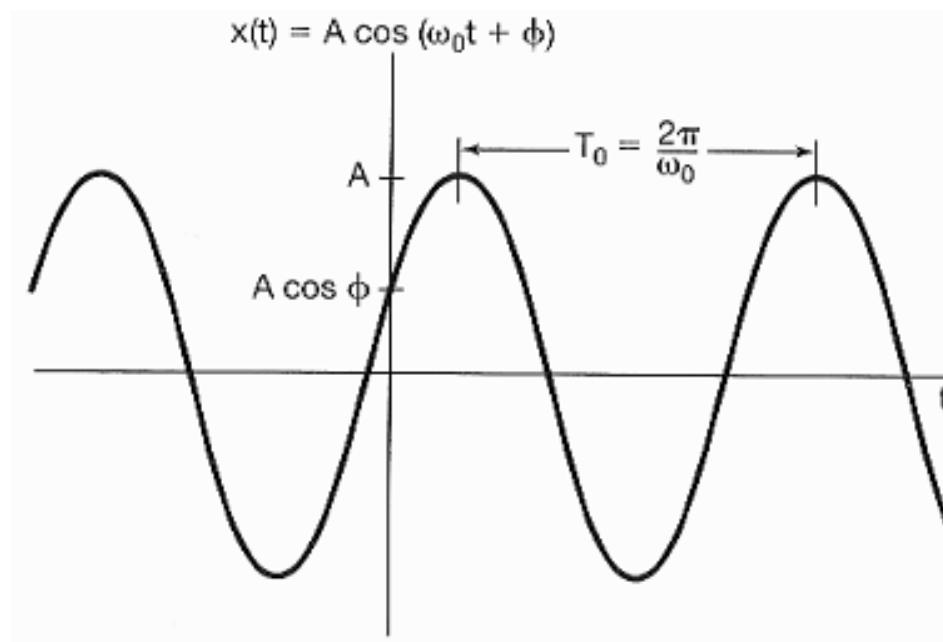
$$x(t + T_0) = x(t)$$

▪ Periodic sinusoidal signals:

$$x(t) = A \cos(\omega_0 t + \phi)$$

$$T_0 = \frac{2\pi}{\omega_0}$$
$$\omega_0 = 2\pi f_0$$

$$T_0 = \frac{1}{f_0}$$



T_0 : (sec)

ω_0 : (rad/sec)

f_0 : (1/sec = Hz)

■ Period & Frequency:

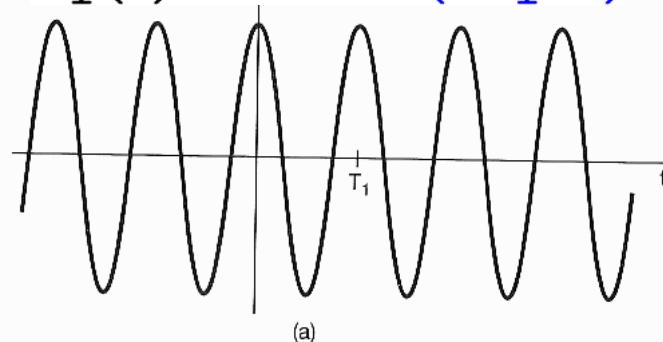
$$T_0 = \frac{2\pi}{w_0}$$

$$w_0 = 2\pi f_0$$

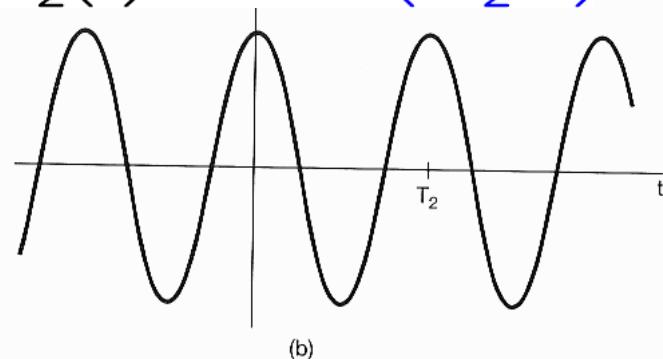
$$T_0 = \frac{1}{f_0}$$

$$\begin{array}{ccc} w_1 & w_2 & w_3 \\ T_1 & T_2 & T_3 \end{array}$$

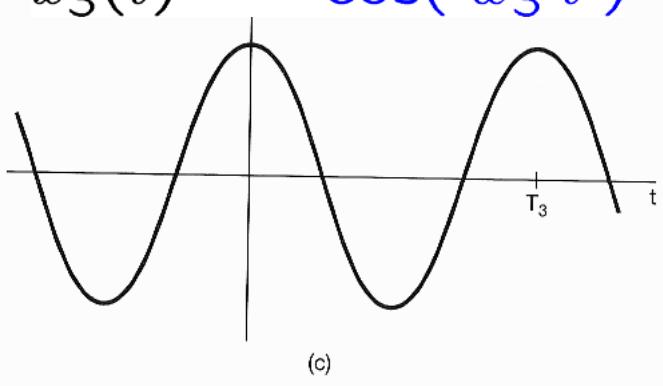
$$x_1(t) = \cos(w_1 t)$$



$$x_2(t) = \cos(w_2 t)$$



$$x_3(t) = \cos(w_3 t)$$



■ Euler's relation:

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$\cos(\theta) = \Re \{ e^{(j\theta)} \}$$

$$\sin(\theta) = \Im \{ e^{(j\theta)} \}$$

$$e^{j(-\theta)} = \cos(-\theta) + j \sin(-\theta) \Rightarrow \cos(\theta) = \frac{e^{(j\theta)} + e^{-(j\theta)}}{2}$$

$$= \cos(\theta) - j \sin(\theta) \Rightarrow \sin(\theta) = \frac{e^{(j\theta)} - e^{-(j\theta)}}{2j}$$

$$\Rightarrow A \cos(w_0 t + \phi) = \frac{A}{2} e^{j(\phi + w_0 t)} + \frac{A}{2} e^{-j(\phi + w_0 t)}$$

$$= \frac{A}{2} e^{j\phi} e^{jw_0 t} + \frac{A}{2} e^{-j\phi} e^{-jw_0 t}$$

■ Total energy & average power:

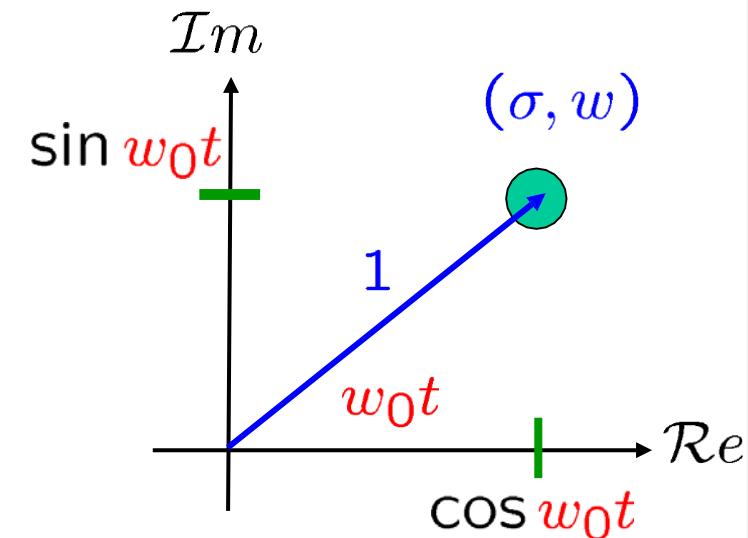
$$E_{\text{period}} = \int_0^{T_0} |e^{jw_0 t}|^2 dt$$

$$= \int_0^{T_0} 1 \cdot dt = T_0$$

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1$$

$$E_{\infty} = \infty$$

$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |e^{jw_0 t}|^2 dt = 1$$



- Problem:
- P1.3

■ General complex exponential signals:

$$C e^{at} = (|C| e^{j\theta})(e^{(r+jw_0)t})$$

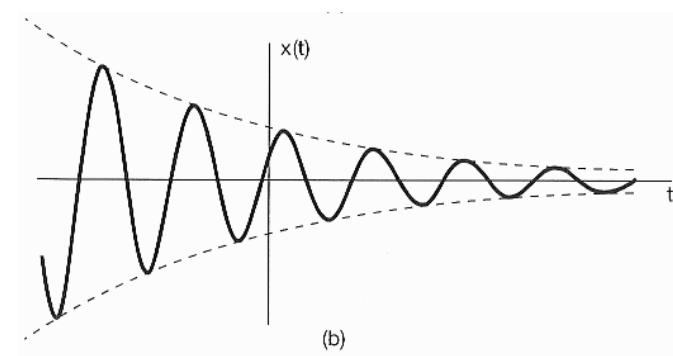
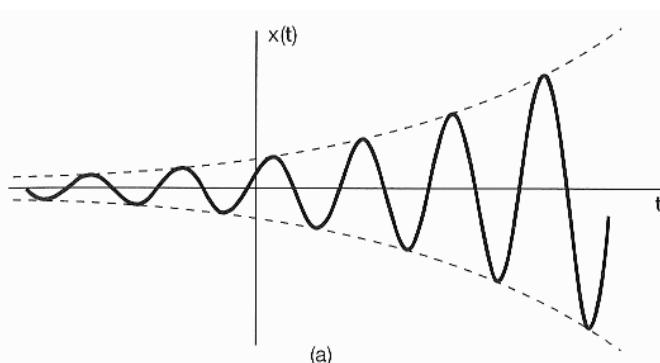
$$\sigma + jw = r e^{j\theta}$$

$$= (|C| e^{j\theta}) (e^{rt} e^{jw_0 t})$$

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$= |C| e^{rt} e^{j(w_0 t + \theta)}$$

$$= |C| e^{rt} \cos(w_0 t + \theta) + j |C| e^{rt} \sin(w_0 t + \theta)$$



$$|C| e^{rt} \cos(w_0 t + \theta)$$

▪ Harmonically related periodic exponentials

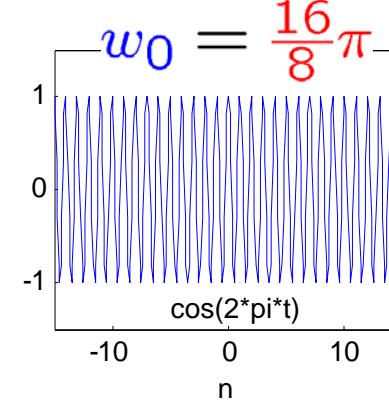
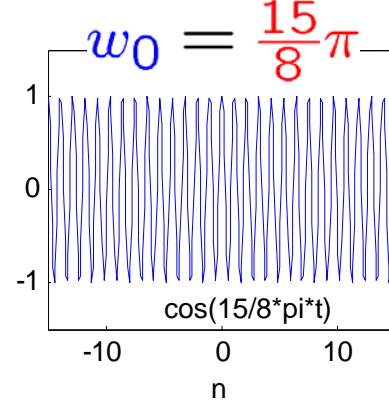
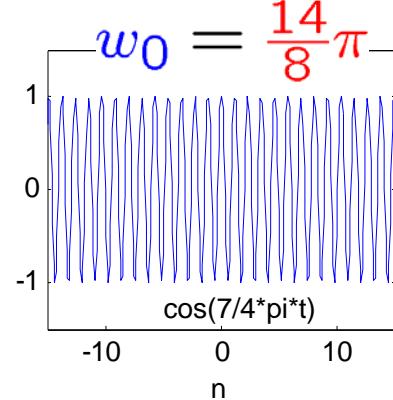
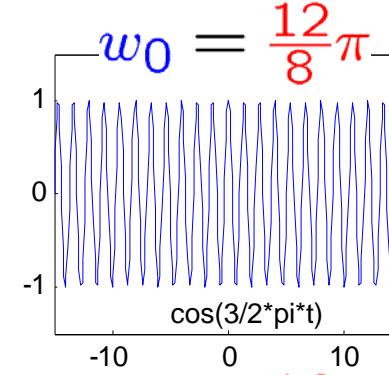
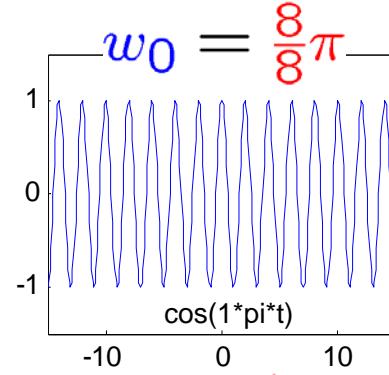
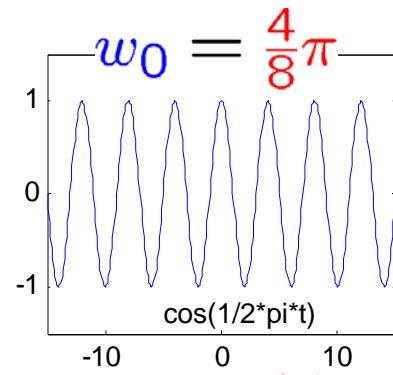
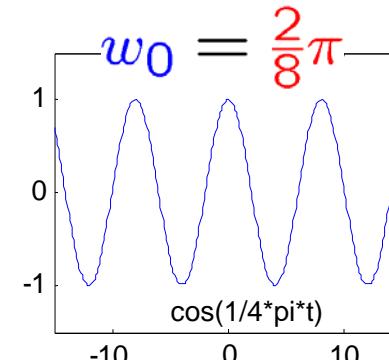
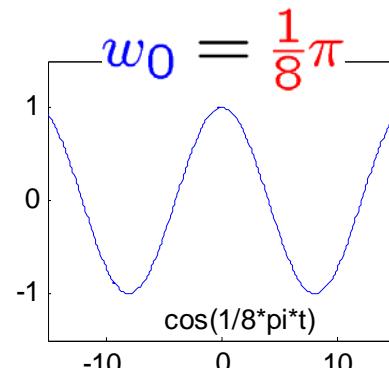
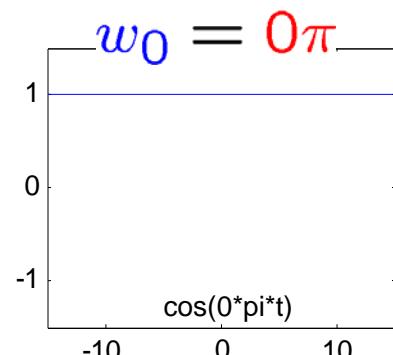
$$e^{j0w_0t}, \quad e^{j1w_0t}, \quad e^{j2w_0t}, \quad e^{j3w_0t}, \quad \dots,$$

$$e^{j(-1)w_0t}, \quad e^{j(-2)w_0t}, \quad e^{j(-3)w_0t}, \quad \dots$$

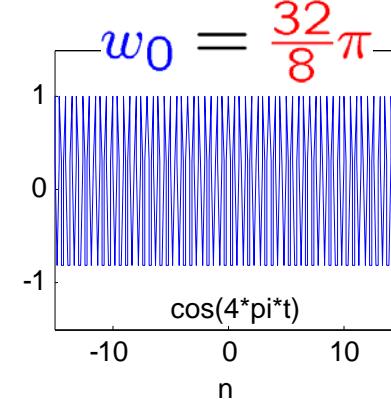
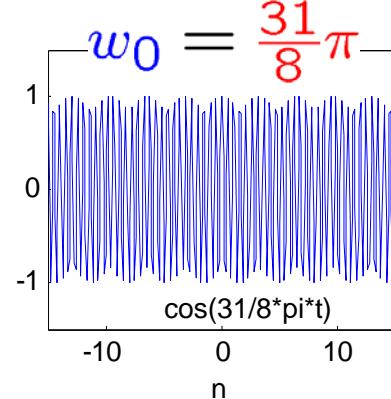
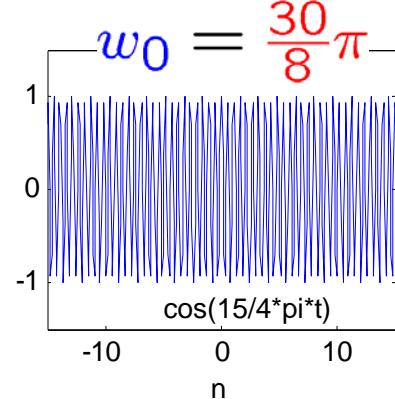
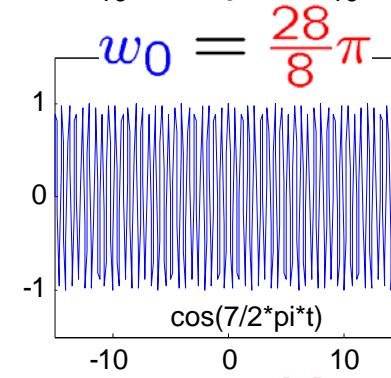
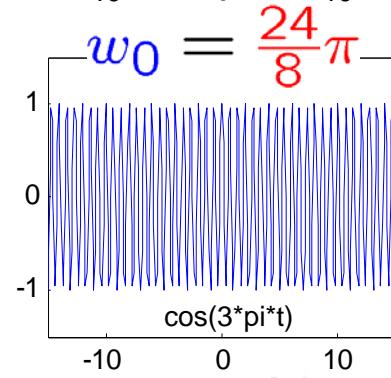
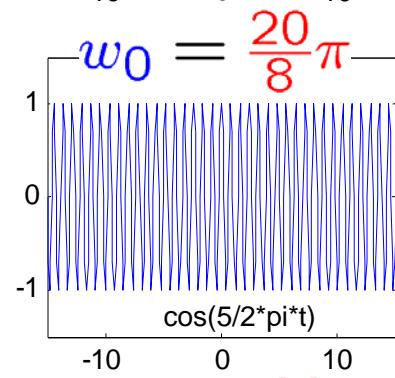
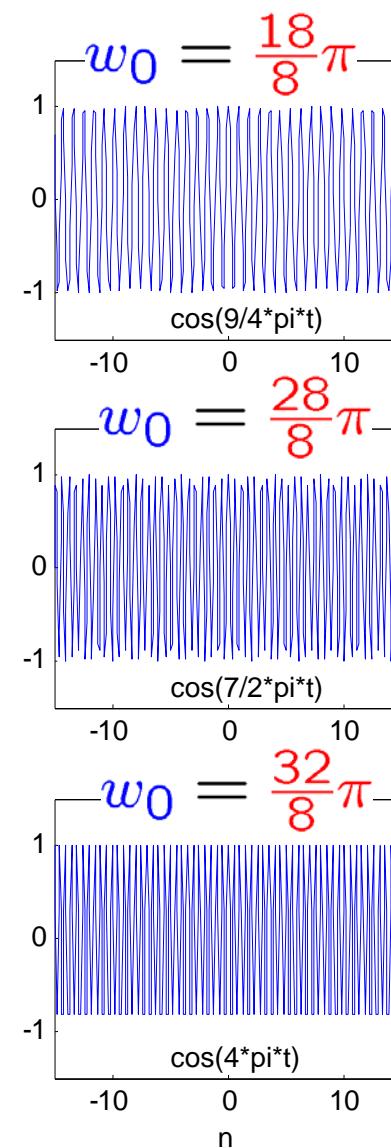
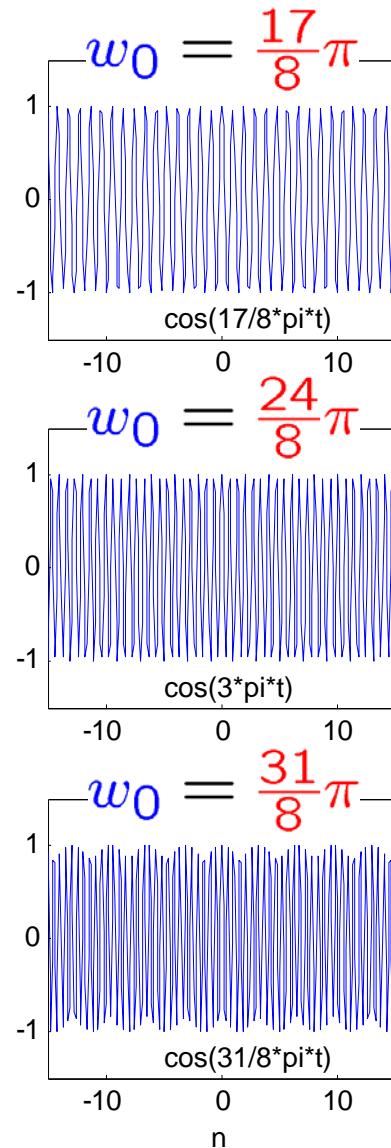
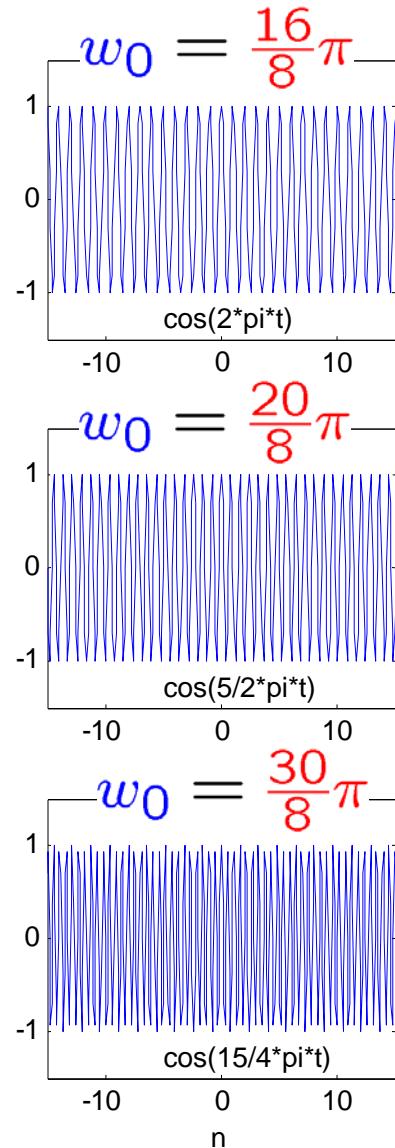
$$\phi_k(t) = e^{jk w_0 t}, \quad k = 0, \pm 1, \pm 2, \dots$$

- For $k = 0$, $\phi_k(t)$ is constant
- For $k \neq 0$, $\phi_k(t)$ is periodic with
 - fundamental frequency $|k|w_0$ and
 - fundamental period $\frac{T_0}{|k|}$

■ CT exponential signals



■ CT exponential signals



- DT complex exponential signal or sequence:

$$x(t) = \textcolor{red}{C} e^{\textcolor{blue}{a}t}$$

$$x[n] = \textcolor{blue}{C} e^{bn}$$

$$= \textcolor{blue}{C}(e^b)^n \quad \text{with } a = e^b$$

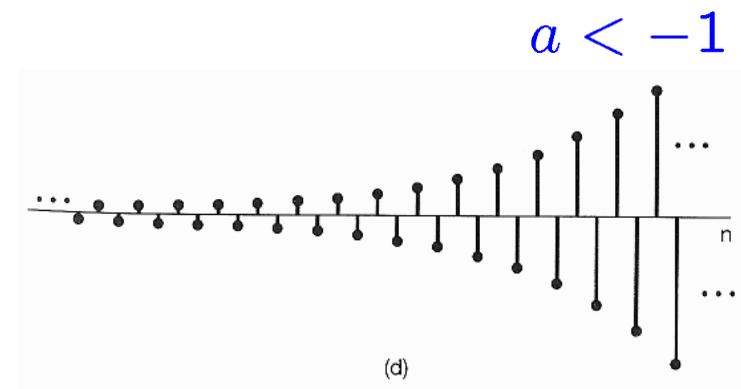
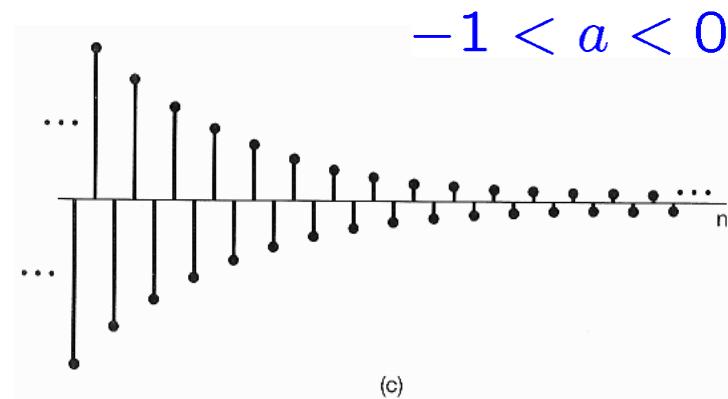
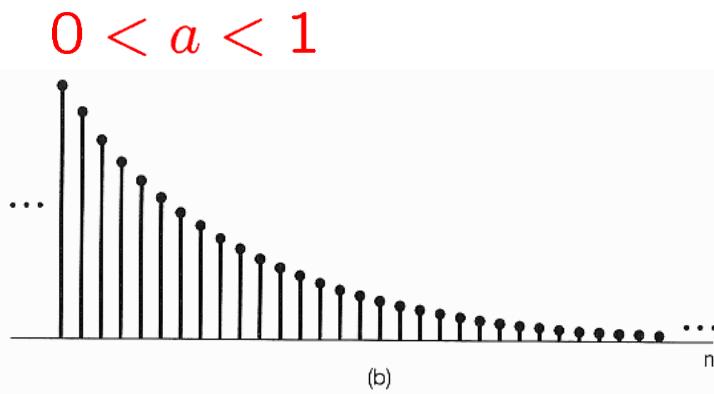
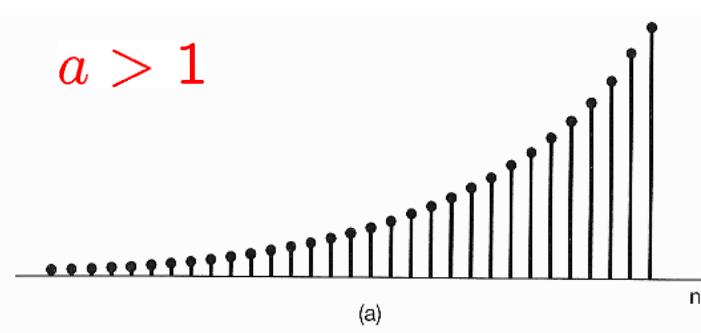
$$x[n] = \textcolor{blue}{C} a^n$$

- where **C** & **a** are, in general, complex numbers

▪ Real exponential signals:

- If C & a are real

$$x[n] = Ca^n$$



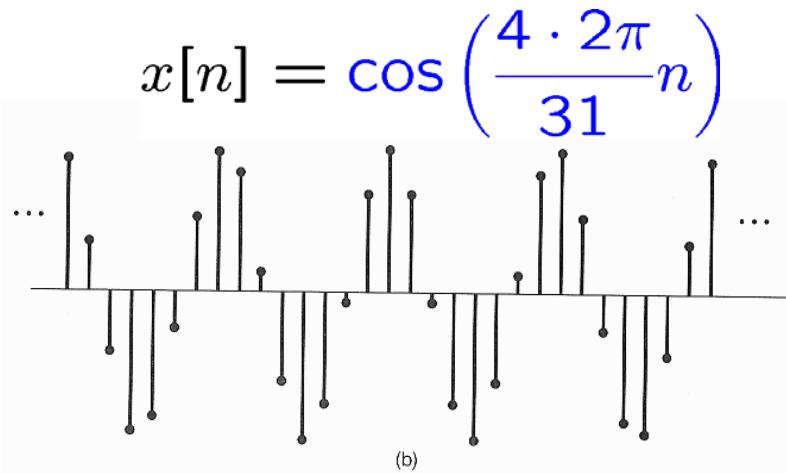
■ DT Complex Exponential & Sinusoidal Signals

- If b is purely imaginary (or $|a| = 1$)

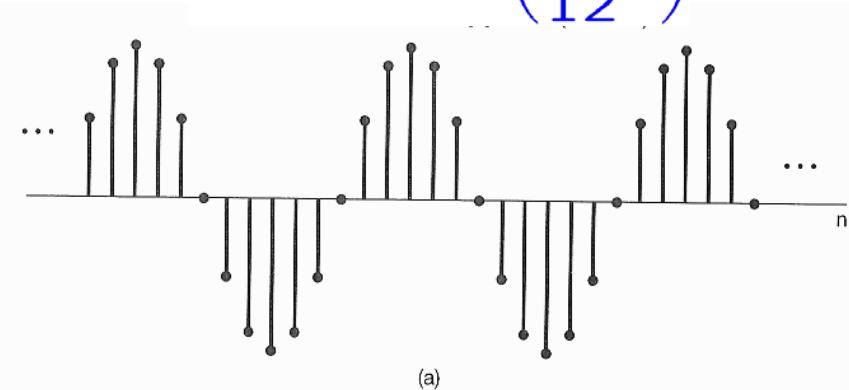
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\begin{aligned} x[n] &= e^{jw_0 n} \\ &= \cos(w_0 n) + j\sin(w_0 n) \end{aligned}$$

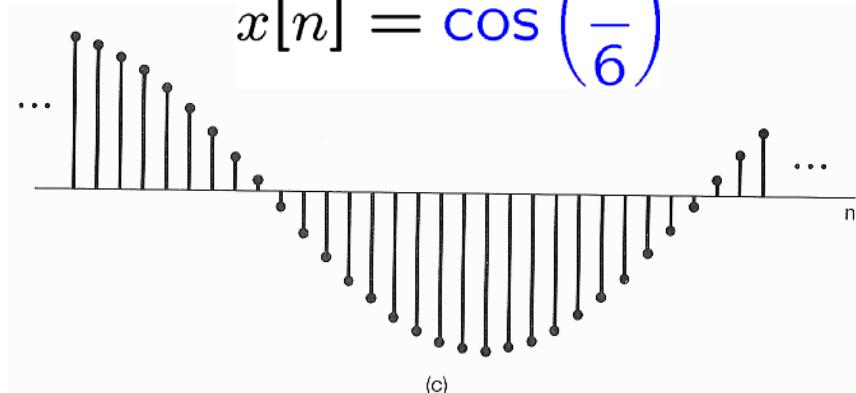
$$x[n] = \cos(w_0 n)$$



$$x[n] = \cos\left(\frac{2\pi}{12}n\right)$$



$$x[n] = \cos\left(\frac{n}{6}\right)$$



- Euler's relation:

$$e^{jw_0n} = \cos w_0 n + j \sin w_0 n$$

- And,

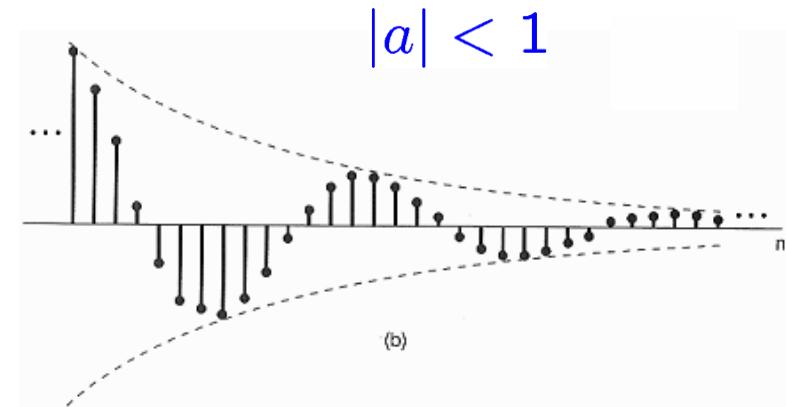
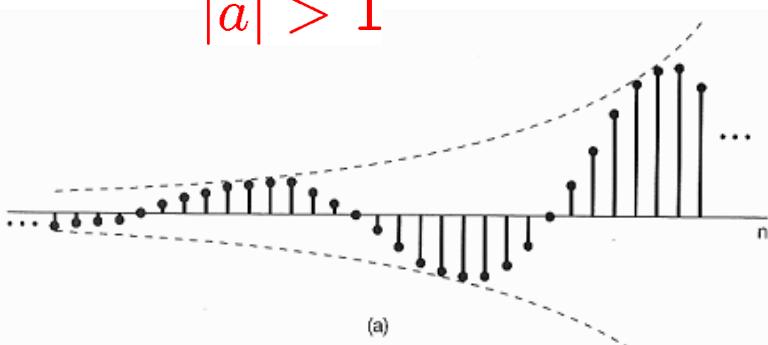
$$A \cos(w_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{jw_0n} + \frac{A}{2} e^{-j\phi} e^{-jw_0n}$$

■ General complex exponential signals:

$$Ca^n = (|C|e^{j\theta})((|a|e^{jw_0})^n)$$

$$= |C||a|^n \cos(w_0 n + \theta) + j|C||a|^n \sin(w_0 n + \theta)$$

$$|a| > 1$$



$$|C||a|^n \cos(w_0 n + \theta)$$

■ Periodicity properties of DT exponential signals

- Periodicity $e^{jw_0 n}$ for $N > 0$

- Problem:
 - P1.35

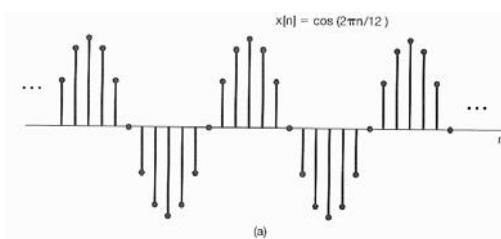
$$e^{jw_0(n+N)} = e^{jw_0n} \cdot e^{jw_0N} = e^{jw_0n} \quad \text{or} \quad e^{jw_0N} = 1$$

- That is, $w_0 N = 2\pi m$ or $\frac{w_0}{2\pi} = \frac{m}{N}$

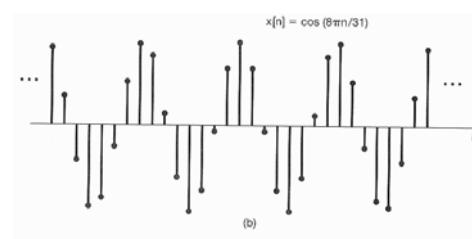
- Hence, $e^{jw_0 n}$ is periodic

if $\frac{w_0}{2\pi}$ is a rational number

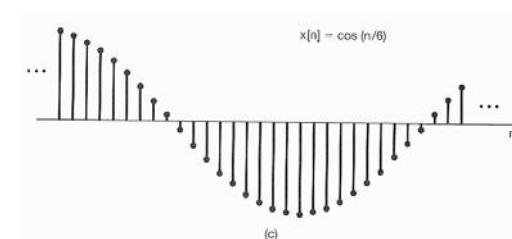
$$x[n] = \cos\left(\frac{2\pi}{12}n\right)$$



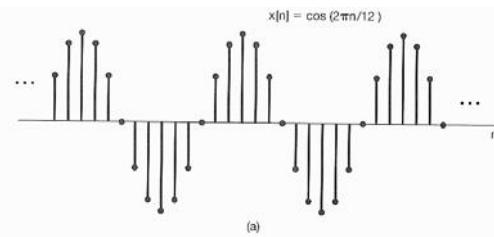
$$x[n] = \cos\left(\frac{4 \cdot 2\pi}{31}n\right)$$



$$x[n] = \cos\left(\frac{n}{6}\right)$$

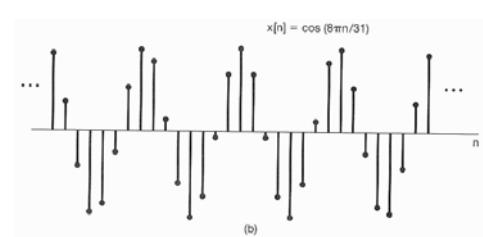


▪ Periodicity properties of DT exponential signals



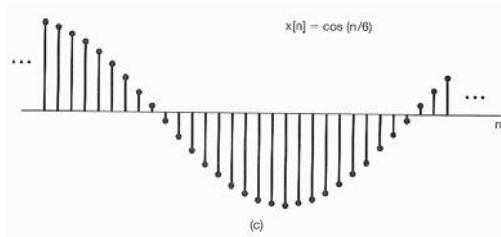
$$x(t) = \cos\left(\frac{2\pi}{12}t\right) \quad T = 12?$$

$$x[n] = \cos\left(\frac{2\pi}{12}n\right) \quad N = 12?$$



$$x(t) = \cos\left(\frac{4 \cdot 2\pi}{31}t\right) \quad T = \frac{31}{4}?$$

$$x[n] = \cos\left(\frac{4 \cdot 2\pi}{31}n\right) \quad N = \frac{31}{4}?$$



$$x(t) = \cos\left(\frac{1}{6}t\right) \quad T = 12\pi?$$

$$x[n] = \cos\left(\frac{1}{6}n\right) \quad N = 12\pi?$$

■ Periodicity properties of DT complex exponentials:

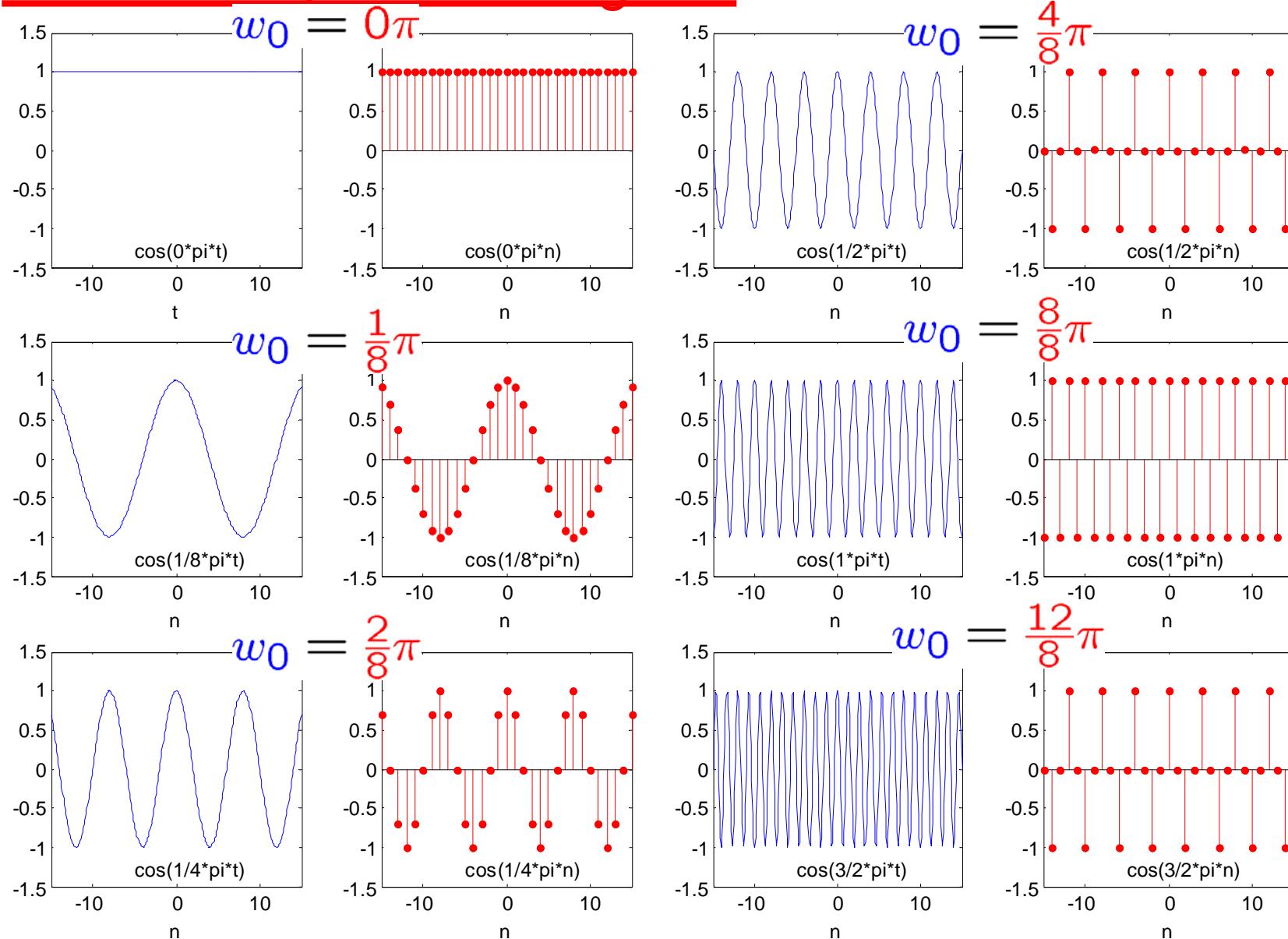
$$e^{j2\pi n} = \cos 2\pi n + j \sin 2\pi n$$

$$e^{j(w_0+2\pi)n} = e^{j2\pi n} \quad e^{jw_0n} = e^{jw_0n}$$

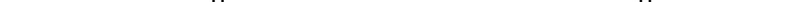
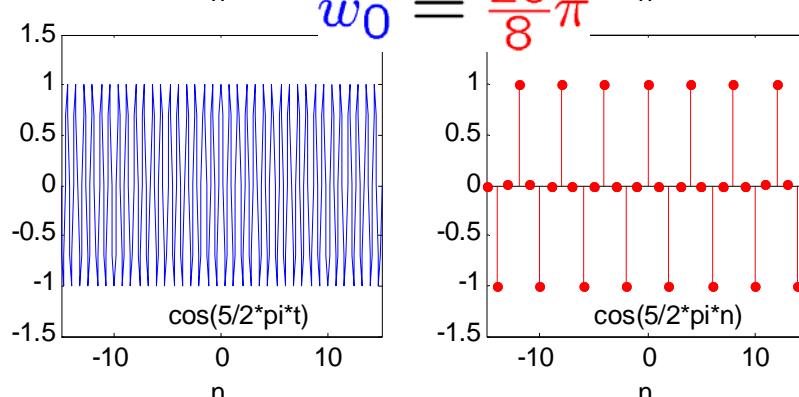
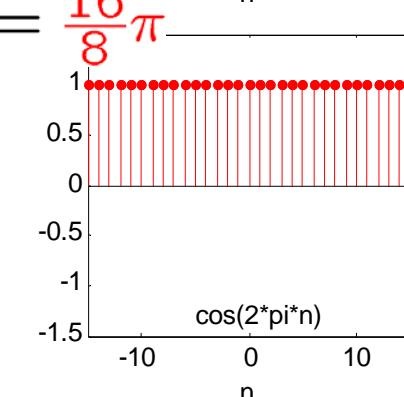
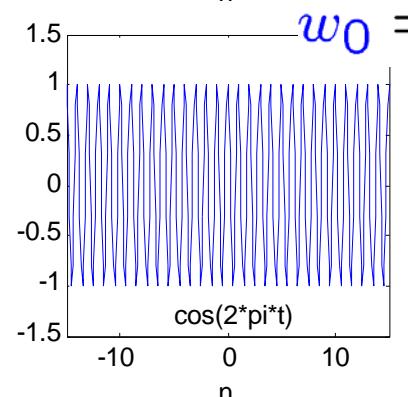
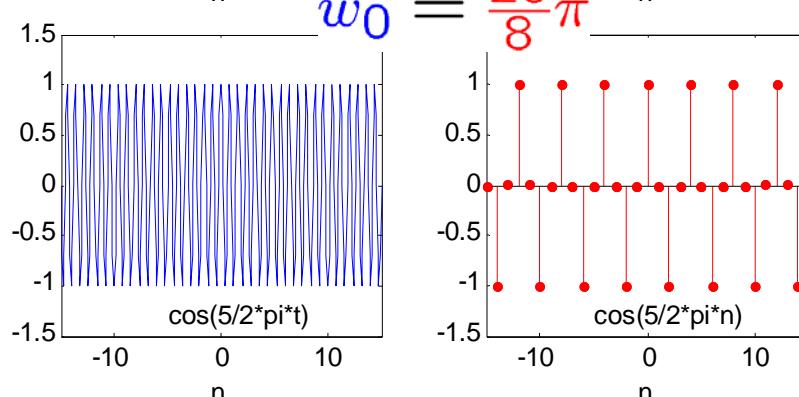
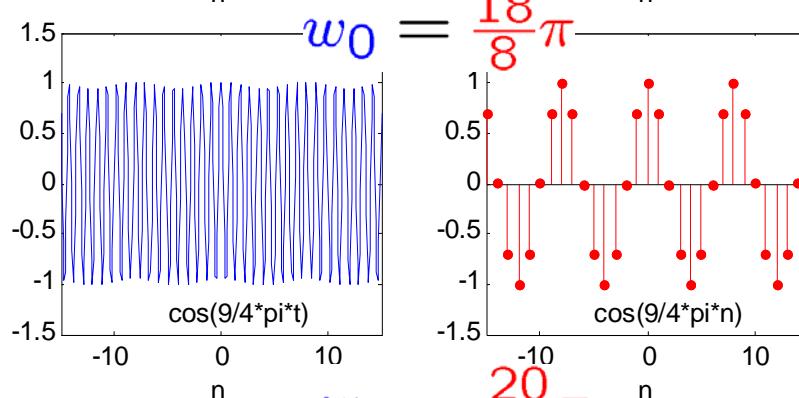
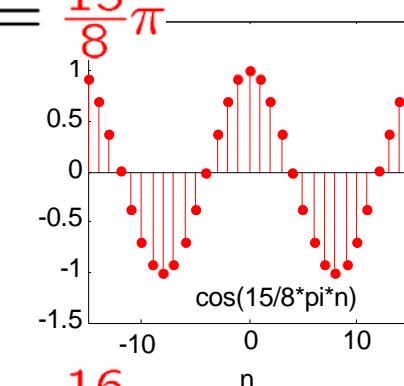
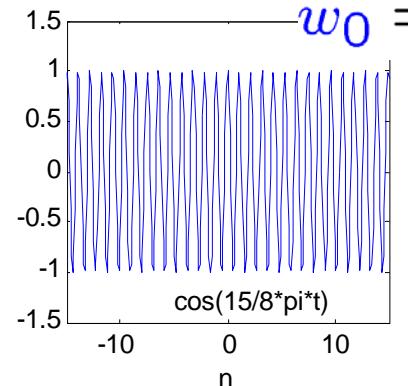
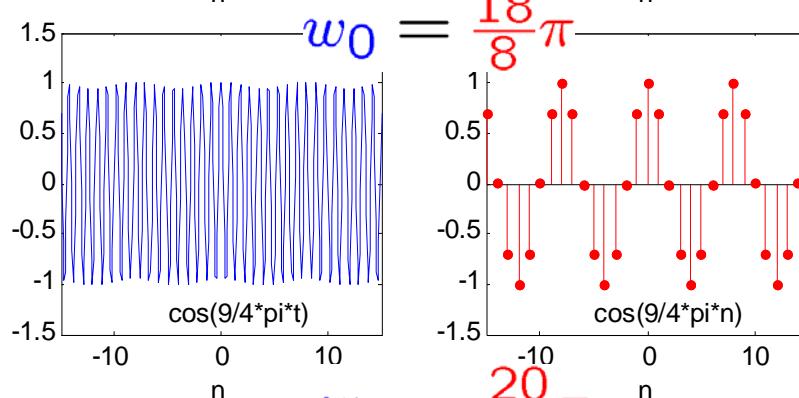
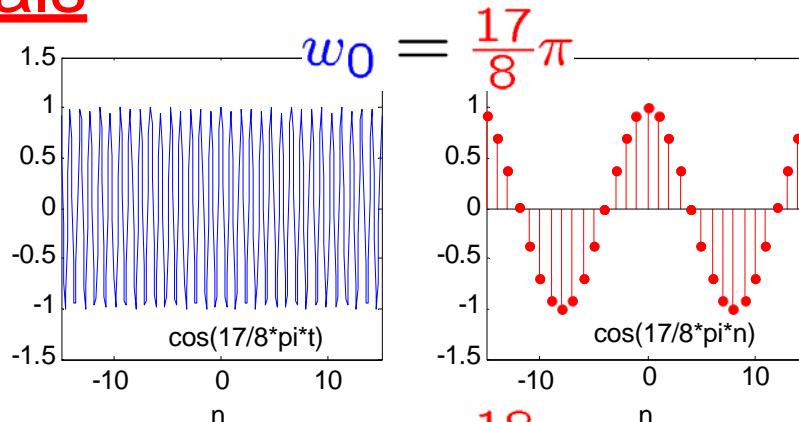
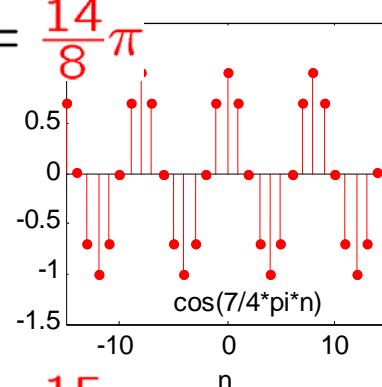
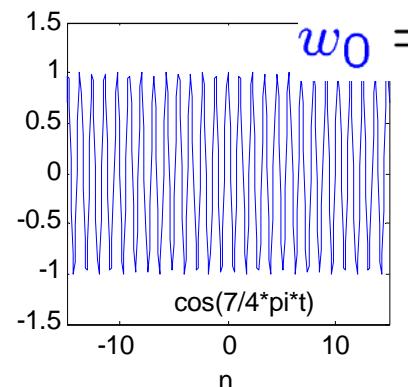
- The signal with frequency w_0 is identical to the signals with frequencies $w_0 \pm 2\pi, w_0 \pm 4\pi, w_0 \pm 6\pi, \dots$
- Only need to consider a frequency interval of length 2π
 - Usually use $0 \leq w_0 < 2\pi$ or $-\pi \leq w_0 < \pi$,
- The **low** frequencies are located at $w_0 = 0, \pm 2\pi, \dots$
The **high** frequencies are located at $w_0 = \pm \pi, \pm 3\pi, \dots$

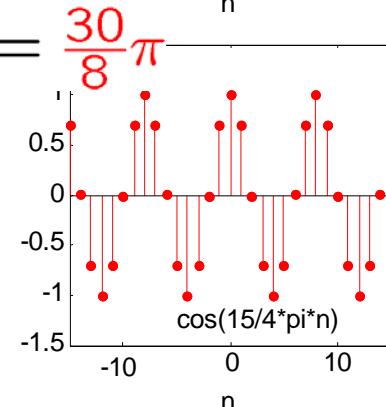
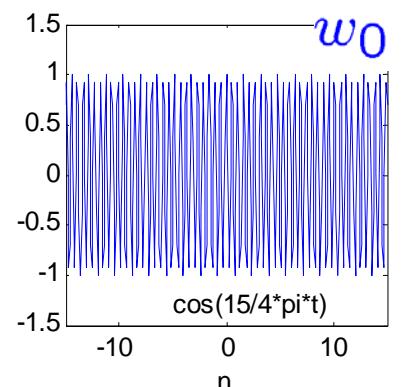
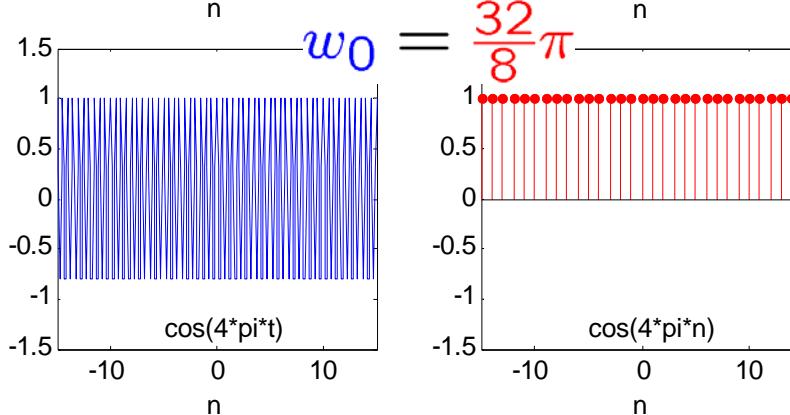
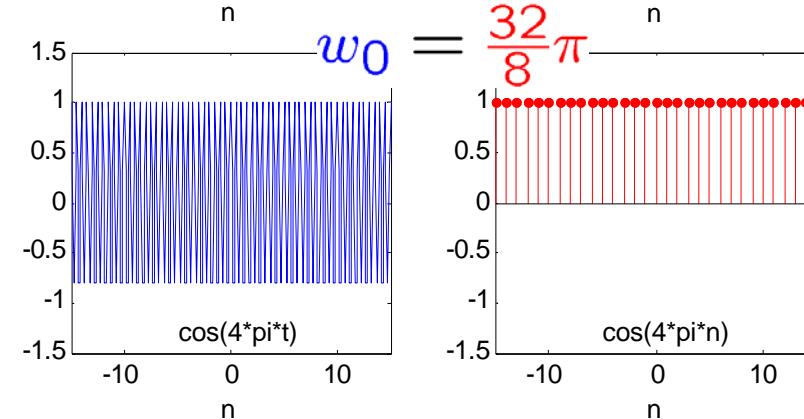
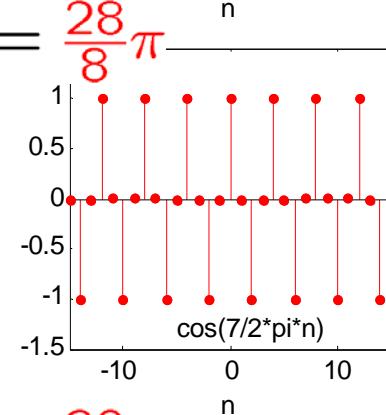
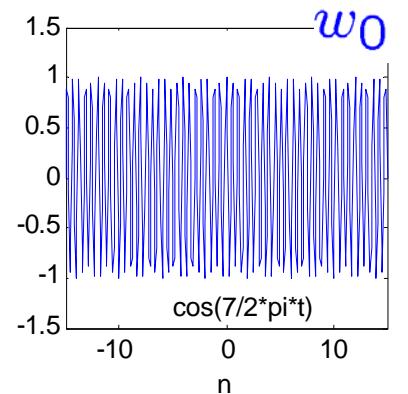
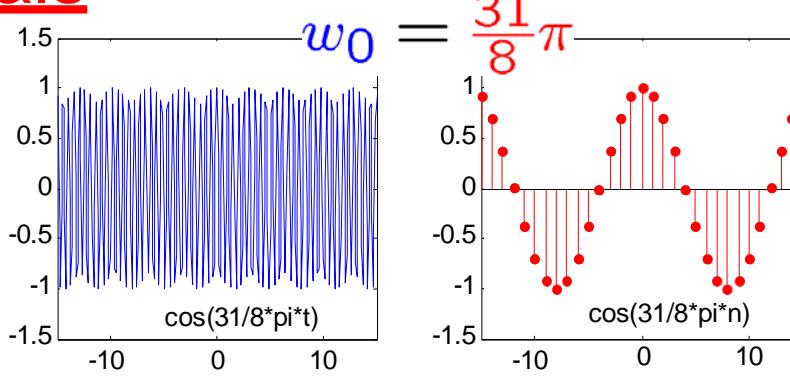
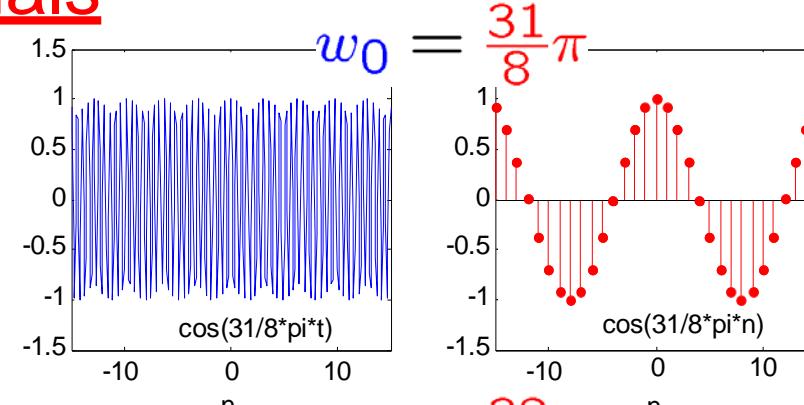
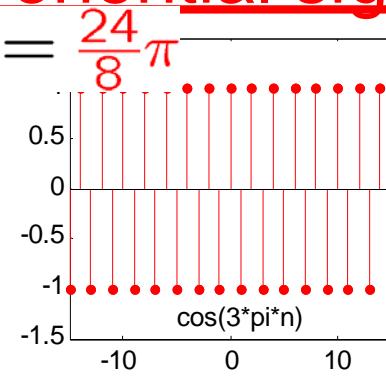
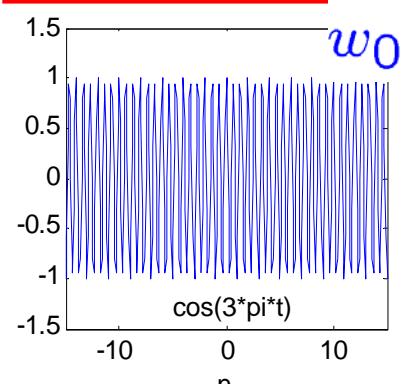
$$e^{j(0)n} = 1 \quad \text{and} \quad e^{j(\pi)n} = (e^{j(\pi)})^n = (-1)^n$$

■ CT & DT exponential signals



■ CT & DT exponential signals



■ CT & DT exponential signals

▪ Harmonically related periodic exponentials

$$\phi_k[n] = e^{j\mathbf{k}(w_0)n}, \quad = e^{j\mathbf{k}(\frac{2\pi}{N})n}, \quad k = 0, \pm 1, \pm 2, \dots$$

$$\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n}$$

$$= e^{jk(\frac{2\pi}{N})n} e^{jN(\frac{2\pi}{N})n} = \phi_k[n]$$

- Only N distinct periodic exponentials in the set

$$\phi_0[n] = 1, \quad \phi_1[n] = e^{j(\frac{2\pi}{N}n)}, \quad \phi_2[n] = e^{j(2\frac{2\pi}{N}n)},$$

$$\dots, \quad \phi_{N-1}[n] = e^{j(N-1)\frac{2\pi}{N}n}$$

$$\phi_N[n] = e^{j(N)\frac{2\pi}{N}n} = e^{j2\pi n} = 1 = \phi_0[n], ; \quad \phi_{N+1}[n] = \phi_1[n], \dots$$

■ Comparison of CT & DT signals:

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

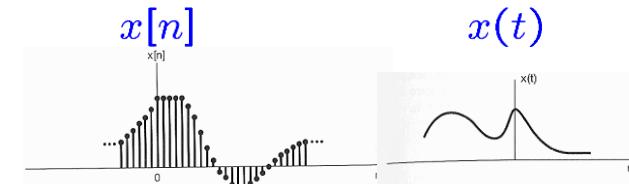
CT	$e^{j\omega_0 t}$	DT	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0		Identical signals for values of ω_0 separated by multiples of 2π	
Periodic for any choice of ω_0		Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m .	
Fundamental frequency ω_0		Fundamental frequency* ω_0/m	
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$		Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$	

*Assumes that m and N do not have any factors in common.

DT $1, e^{j(\frac{2\pi}{N}n)}, e^{j(2\frac{2\pi}{N}n)}, \dots, e^{j(N-1)\frac{2\pi}{N}n}$

CT $1, e^{j1\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}, \dots,$
 $e^{j(-1)\omega_0 t}, e^{j(-2)\omega_0 t}, e^{j(-3)\omega_0 t}, \dots$

- Problem:
- P1.36



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable

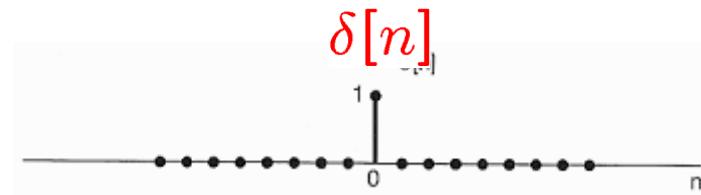
– Time Shift	$x[n - n_0]$	$x(t - t_0)$	$x(-t) = x(t), x[-n] = x[n]$
– Time Reversal	$x[-n]$	$x(-t)$	$x(-t) = -x(t), x[-n] = -x[n]$
– Time Scaling	$x[an]$	$x(at)$	$\mathcal{E}_v \{x[n]\} = \frac{1}{2} [x[n] + x[-n]]$
– Periodic Signals	$x(t) = x(t + T)$		$\mathcal{O}_d \{x[n]\} = \frac{1}{2} [x[n] - x[-n]]$
– Even & Odd Signals	$x[n] = x[n + N]$		$\phi_k(t) = e^{j k \omega_0 t}, k = 0, \pm 1, \dots$
			$\phi_k[n] = e^{j k \omega_0 n}, k = 0, \dots, N - 1$

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

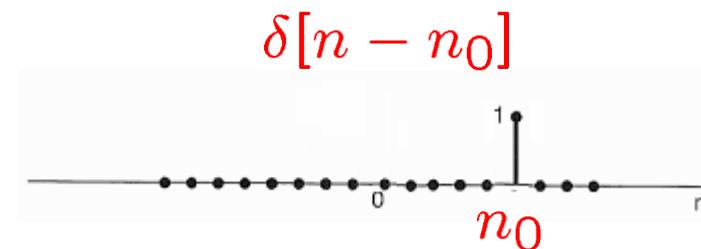
■ DT Unit Impulse & Unit Step Sequences

■ Unit impulse (or unit sample)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$



■ Unit step

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



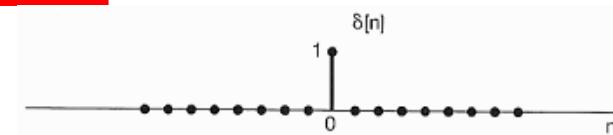
$$u[n - n_0] = \begin{cases} 0, & n < n_0 \\ 1, & n \geq n_0 \end{cases}$$



▪ Relationship Between Impulse & Step

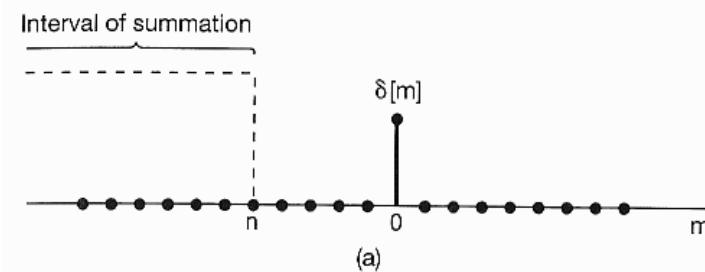
▪ First difference

$$\delta[n] = u[n] - u[n - 1]$$

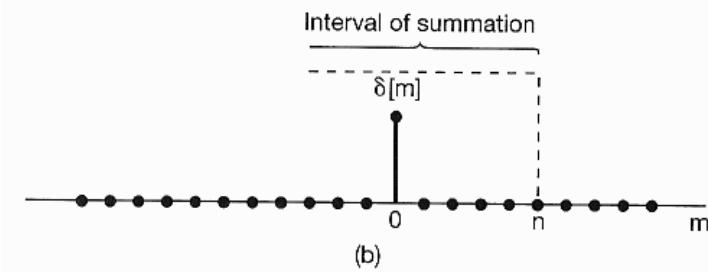


▪ Running sum

$$u[n] = \sum_{m=-\infty}^n \delta[m] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



$n < 0$



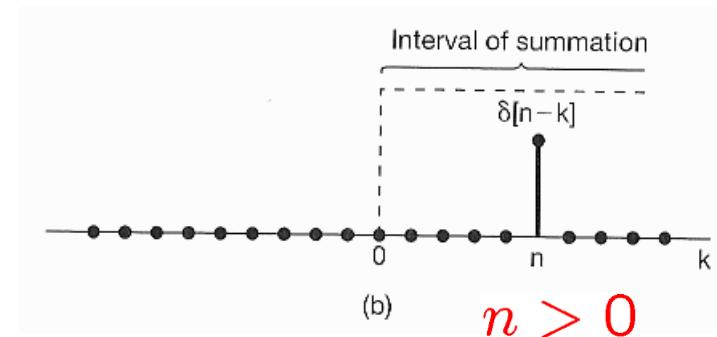
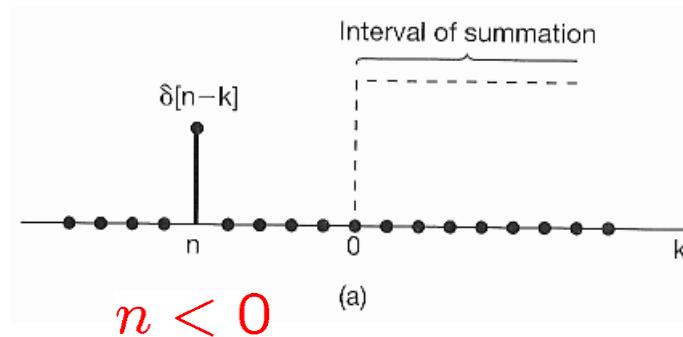
$n > 0$

▪ Relationship Between Impulse & Step

- Alternatively,

$$u[n] = \sum_{k=-\infty}^0 \delta[n-k], \quad \text{with } m = n - k$$

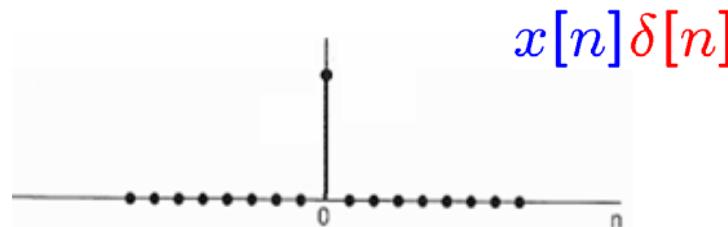
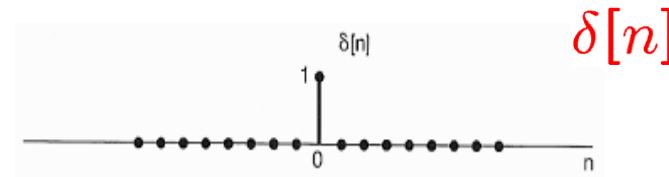
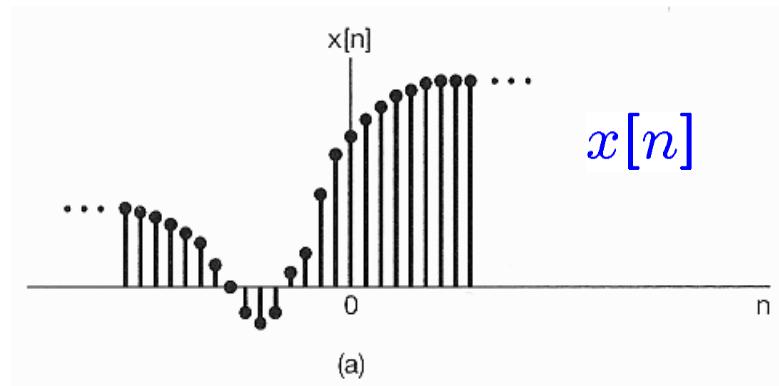
or, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$



■ Sample by Unit Impulse

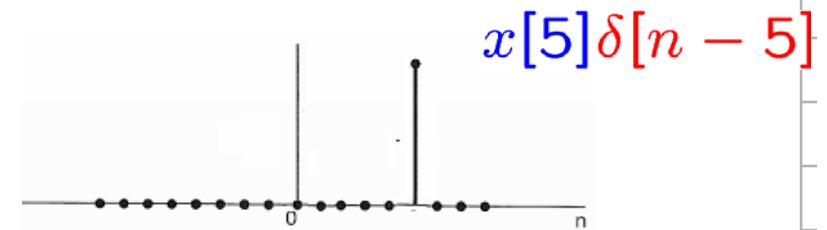
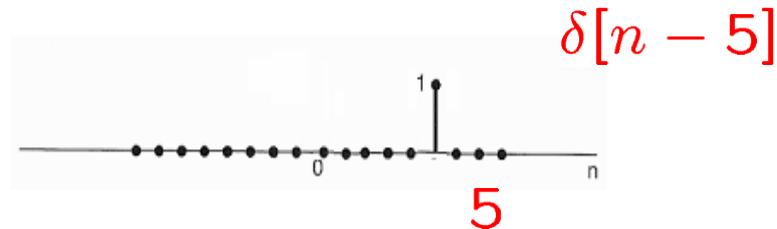
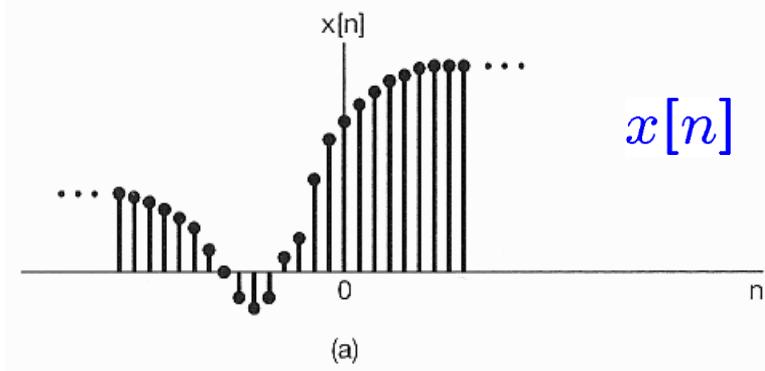
■ For $x[n]$

$$x[n]\delta[n] = x[0]\delta[n]$$



■ More generally,

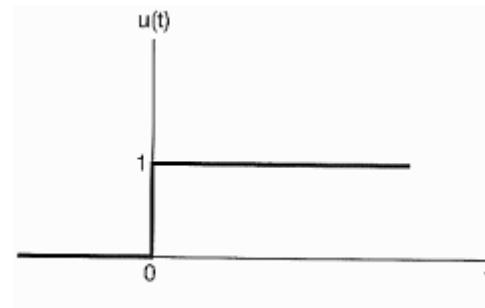
$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$



■ CT Unit Impulse & Unit Step Functions

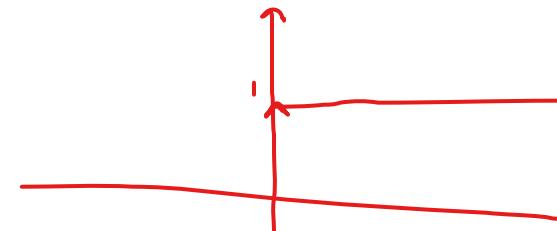
■ Unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



■ Unit impulse function

Similar to DT case, we would like to define an CT impulse function, to have



$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

▪ Relationship Between Impulse & Step

▪ Running integral

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

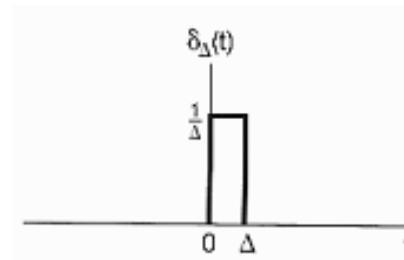
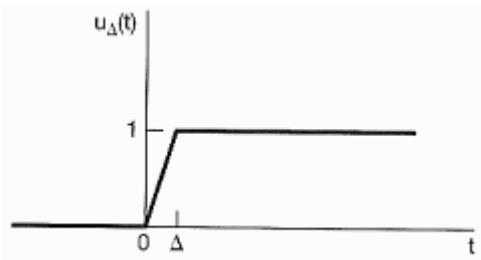
▪ First derivative

$$\delta(t) = \frac{du(t)}{dt}$$

- But, $u(t)$ is **discontinuous** at $t = 0$, hence, **not differentiable**
- Use approximation

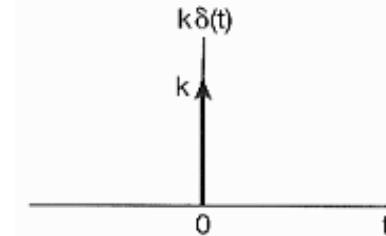
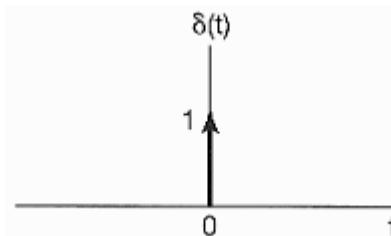
▪ Relationship Between Impulse & Step

- Use approximation



$$\delta_{\Delta}(t) = \frac{du_{\Delta}(t)}{dt}$$

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$



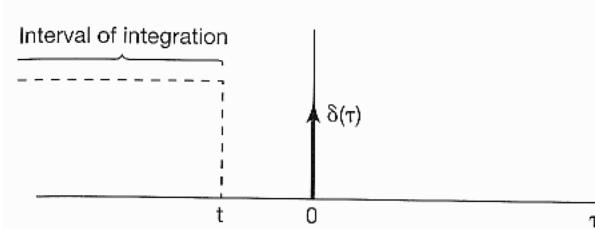
■ Relationship Between Impulse & Step

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau = \int_0^\infty \delta(t - \sigma) (-d\sigma) = \int_0^\infty \delta(t - \sigma) (d\sigma)$$

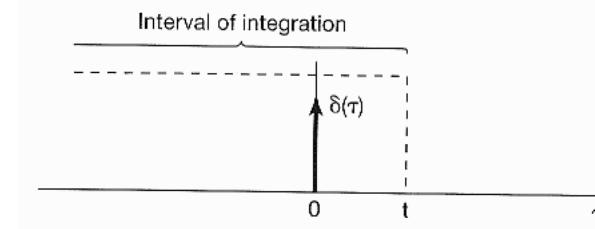
$$\tau = t - \sigma$$

$$d\tau = -d\sigma$$

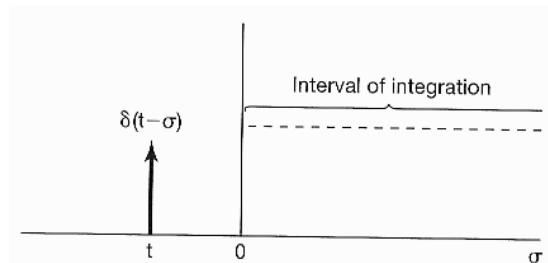
$$= \int_0^\infty \delta(t - \tau) (d\tau)$$



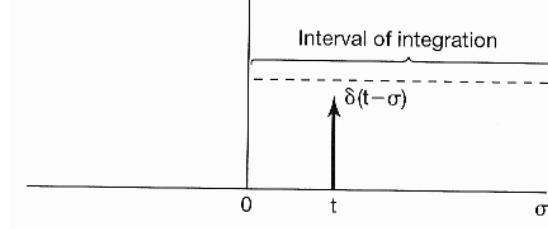
(a)



(b)



(a)



(b)

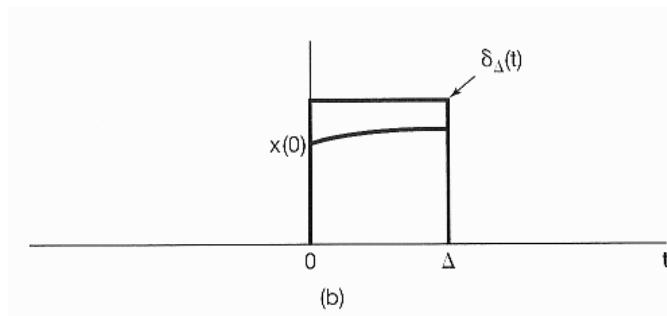
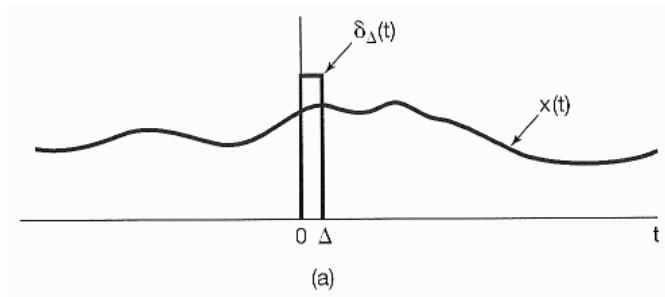
▪ Sample by Unit Impulse Function

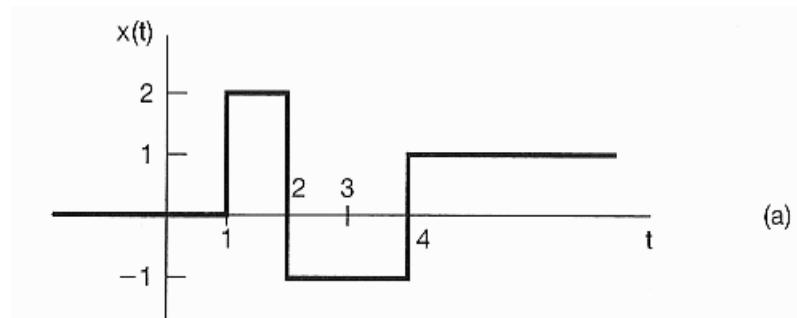
- For $x(t)$ continuous at $t=0$:

$$x(t)\delta(t) = x(0)\delta(t)$$

- More generally,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

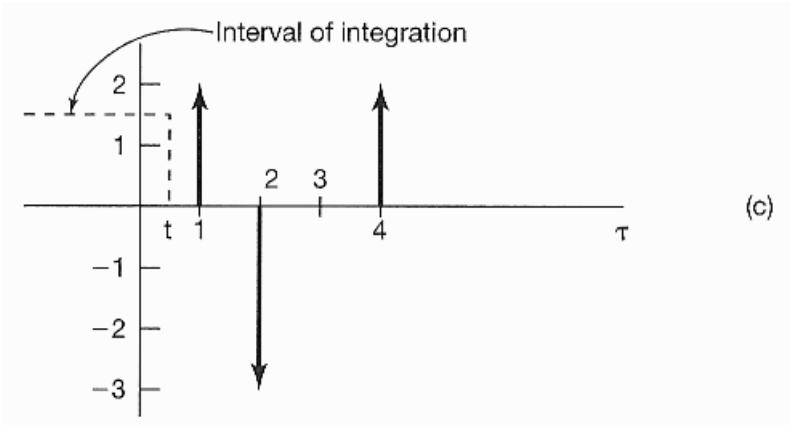
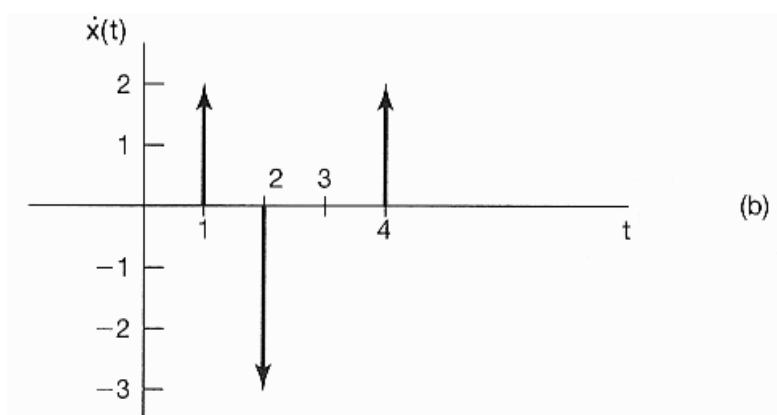


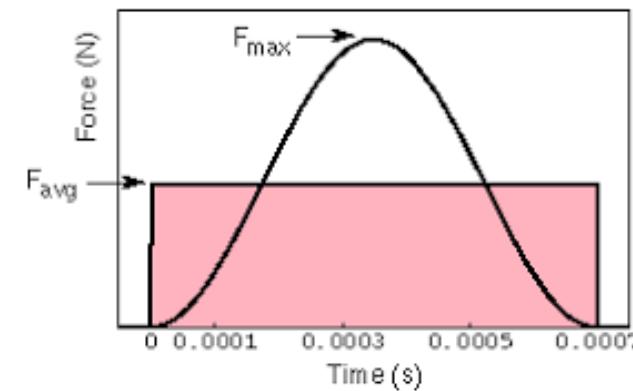
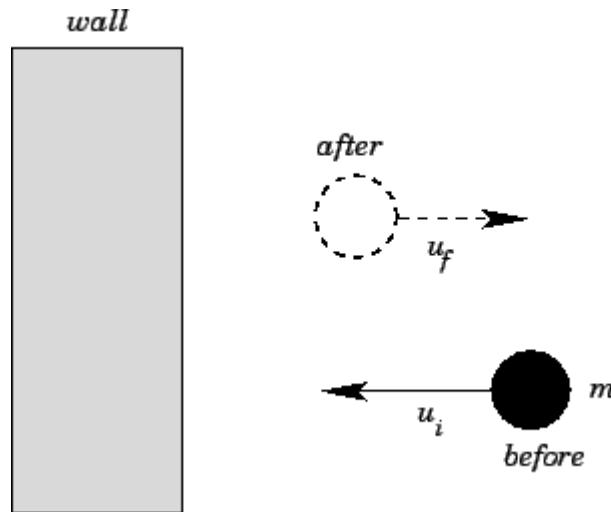
■ Example 1.7:

$$\delta(t) = \frac{du(t)}{dt}$$

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

$$x(t) = \int_0^t \dot{x}(\tau) d\tau$$





Assume that the solid ball hits the wall at t=0

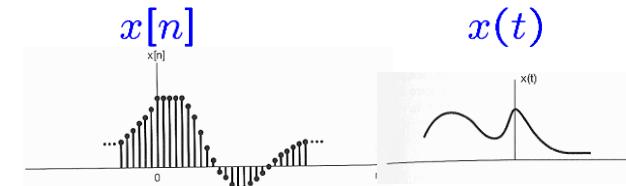
How do you describe the force imposed on the ball?

$$f(t)|_{t<0} = 0$$

$$f(t)|_{t>0} = 0$$

$$f(t) = m \frac{dV}{dt}$$

$$u_f - (-u_i) = u_f + u_i = \frac{1}{m} \int_{t_0^-}^{t_0^+} f(\tau) d\tau$$



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable

– Time Shift	$x[n - n_0]$	$x(t - t_0)$	$x(-t) =$	$x(t), x[-n] =$	$x[n]$
– Time Reversal	$x[-n]$	$x(-t)$	$x(-t) =$	$-x(t), x[-n] =$	$-x[n]$
– Time Scaling	$x[an]$	$x(at)$	$\mathcal{E}_v \{x[n]\} =$	$\frac{1}{2} [x[n] + x[-n]]$	
– Periodic Signals	$x(t) = x(t + T)$		$\mathcal{O}_d \{x[n]\} =$	$\frac{1}{2} [x[n] - x[-n]]$	
– Even & Odd Signals	$x[n] = x[n + N]$		$\phi_k(t) = e^{j k \omega_0 t}, k = 0, \pm 1, \dots$		

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

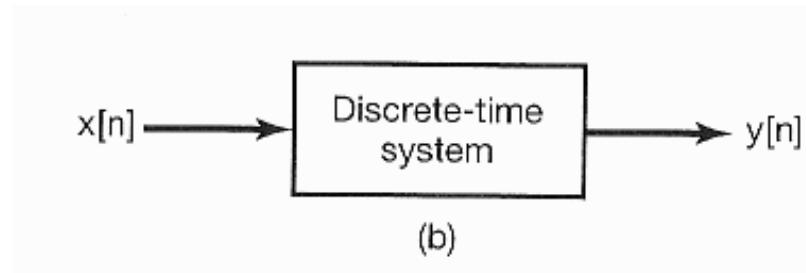
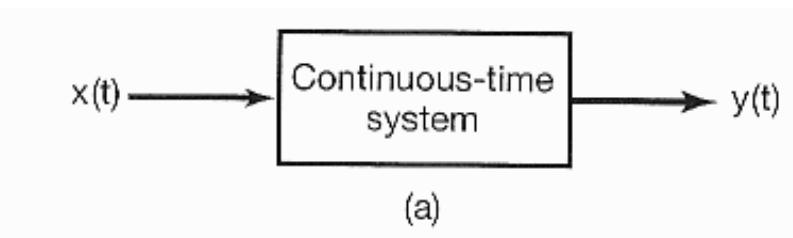
$$\phi_k[n] = e^{j k \omega_0 n}, k = 0, \dots, N - 1$$

$\delta[n], u[n]$

$\delta(t), u(t)$

▪ Physical Systems & Mathematical Descriptions

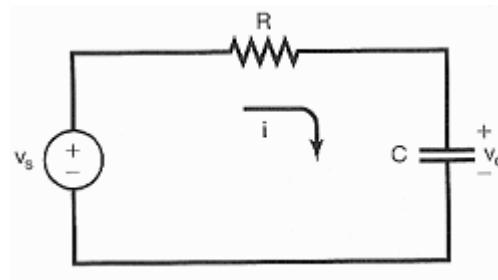
- Examples of **physical systems** are signal processing, communications, electromechanical motors, automotive vehicles, chemical-processing plants
- A **system** can be viewed as a **process** in which **input signals** are **transformed** by the system or **cause** the system to **respond** in some way, resulting in **other signals** or **outputs**



▪ Simple examples of CT systems

▪ RC circuit

Input signal: $v_s(t)$

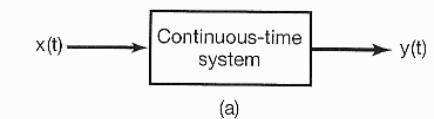


Output signal: $v_c(t)$

$$i(t) = \frac{v_s(t) - v_c(t)}{R} \quad i(t) = C \frac{dv_c(t)}{dt}$$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$



$$x(t) \rightarrow y(t)$$

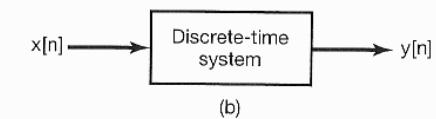
- Simple examples of DT systems

- Balance in a bank account

$$y[n] = 1.01y[n - 1] + x[n]$$

or, $y[n] - 1.01y[n - 1] = x[n]$

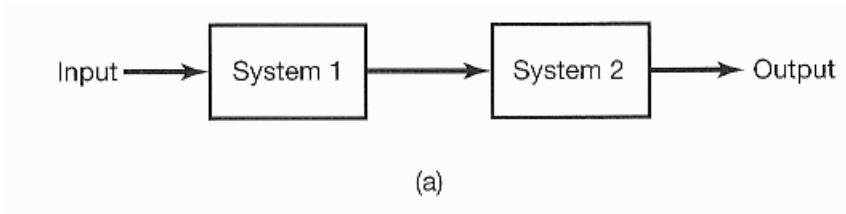
$$\Rightarrow y[n] + ay[n - 1] = bx[n]$$



$$x[n] \rightarrow y[n]$$

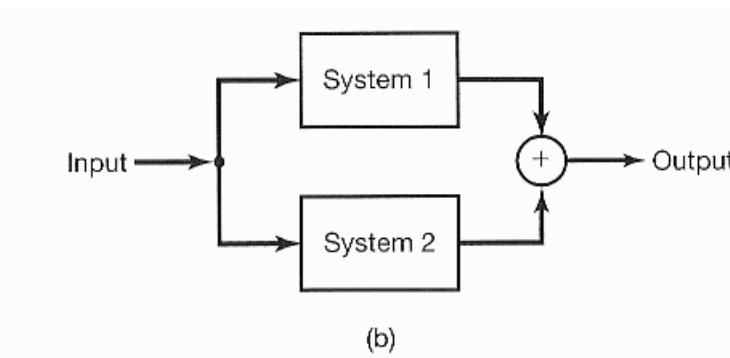
■ Interconnections of Systems

- Series or cascade interconnection of 2 systems



> e.x. radio receiver + amplifier

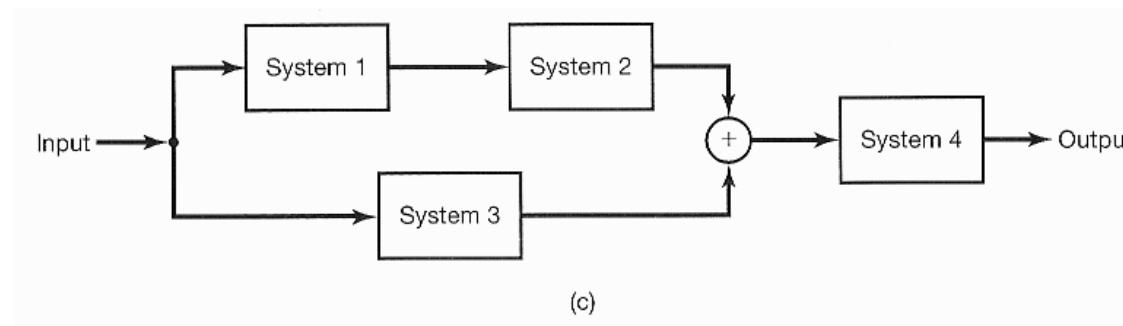
- Parallel interconnection of 2 systems



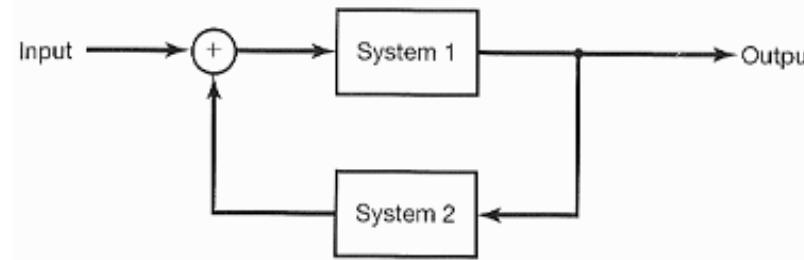
> e.x. audio system with several microphones or speakers

■ Interconnections of Systems

■ Series-parallel interconnection



■ Feedback interconnection



> e.x. cruise control, electrical circuit

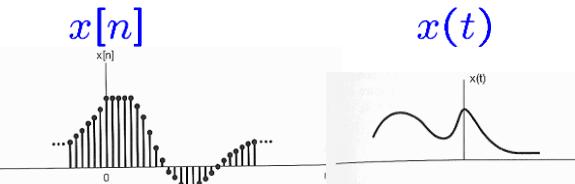
- Introduction
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- Time Shift
- Time Reversal
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- Even & Odd Signals

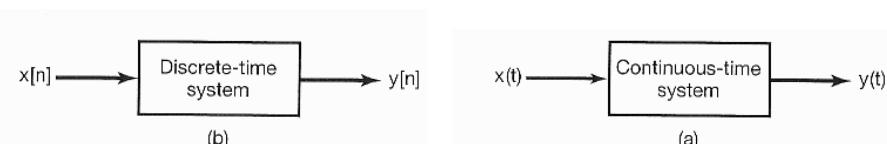
$$\begin{array}{lll} x[n - n_0] & x(t - t_0) & x(-t) = x(t), x[-n] = x[n] \\ x[-n] & x(-t) & x(-t) = -x(t), x[-n] = -x[n] \\ x[an] & x(at) & \mathcal{E}_v \{x[n]\} = \frac{1}{2} [x[n] + x[-n]] \\ x(t) = x(t + T) & & \mathcal{O}_d \{x[n]\} = \frac{1}{2} [x[n] - x[-n]] \\ x[n] = x[n + N] & & \end{array}$$

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

- Systems with or without memory
- Invertibility & Inverse Systems
- Causality
- Stability
- Time Invariance
- Linearity



$$\begin{aligned} \phi_k(t) &= e^{j k \omega_0 t}, k = 0, \pm 1, \dots \\ \phi_k[n] &= e^{j k \omega_0 n}, k = 0, \dots, N-1 \\ \delta[n], u[n] & \\ \delta(t), u(t) & \end{aligned}$$



$$x[n] \rightarrow y[n] \quad x(t) \rightarrow y(t)$$

■ Systems with or without memory

■ Memoryless systems

- Output depends only on the input **at that same time**

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t) \quad (\text{resistor})$$

■ Systems with memory

$$y[n] = \sum_{k=-\infty}^n x[k] \quad (\text{accumulator})$$

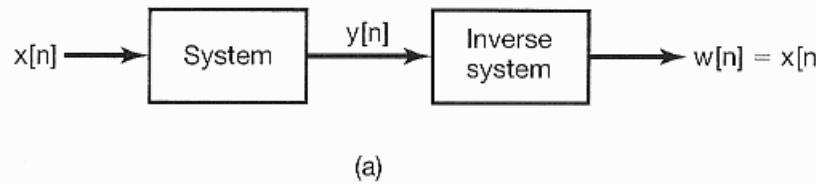
$$y(t) = \frac{1}{C} \int_{-\infty}^t x(\tau) d\tau$$

$$y[n] = x[n - 1] \quad (\text{delay})$$

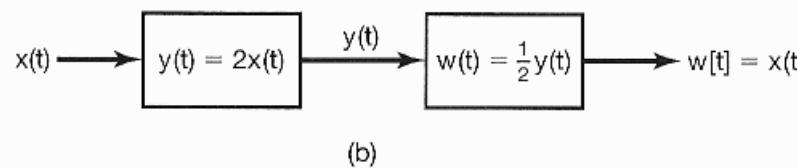
▪ Invertibility & Inverse Systems

▪ Invertible systems

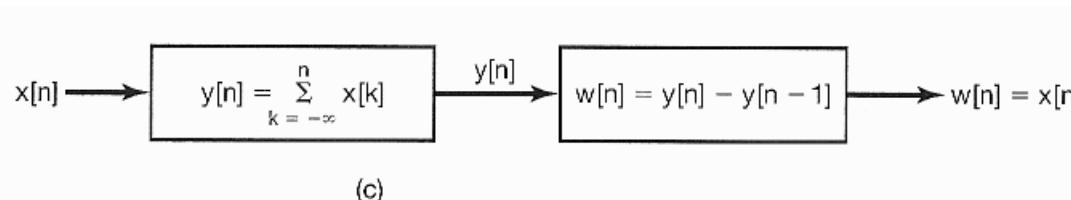
- Distinct inputs lead to distinct outputs



(a)



(b)



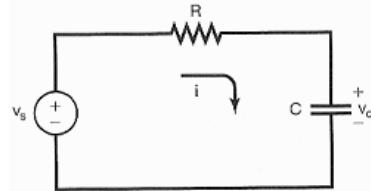
(c)

$y(t) = x(t)^2$ is not invertible

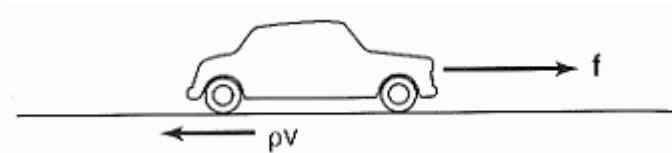
▪ Causality

▪ Causal systems

- Output depends only on input at **present** time & in the **past**



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

- Non-causal systems

$$y[n] = x[n] - x[n+1]$$

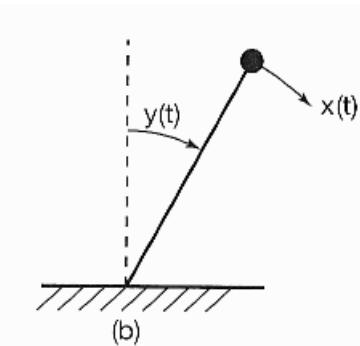
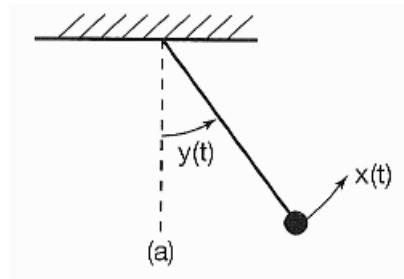
$$y(t) = x(t+1)$$

$$y(t) = x(t) \cos(t+1) \text{ ???}$$

■ Stability

■ Stable systems

- Small inputs lead to responses that **do not diverge**
- Every bounded input excites a **bounded output**
 - Bounded-input bounded-output stable (**BIBO stable**)
 - For all $|x(t)| < a$, then $|y(t)| < b$, for all t



- Balance in a bank account?

$$y[n] = 1.01y[n - 1] + x[n]$$

- Example 1.13: Stability

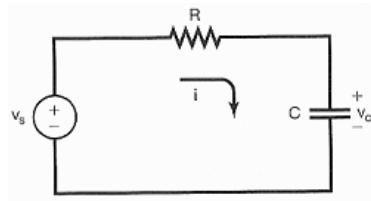
$$S_1 : \quad y(t) = t x(t)$$

$$S_2 : \quad y(t) = e^{x(t)}$$

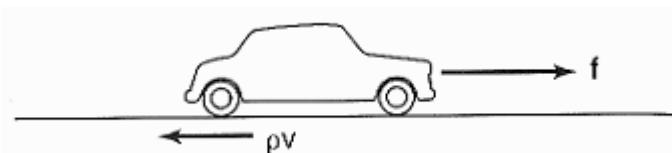
■ Time Invariance

■ Time-invariant systems

- Behavior & characteristics of system are **fixed over time**



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

- A **time shift** in the **input** signal results in an **identical time shift** in the **output** signal

$$x[n] \rightarrow y[n] \iff x[n - n_0] \rightarrow y[n - n_0]$$

■ Time Invariance

- Example of time-invariant system (Example 1.14)

$$y(t) = \sin [x(t)]$$

$$x_1(t)$$

$$y_1(t) = \sin [x_1(t)]$$

$$x_2(t) = x_1(t - t_0)$$

$$y_2(t) = \sin [x_2(t)] = \sin [x_1(t - t_0)]$$

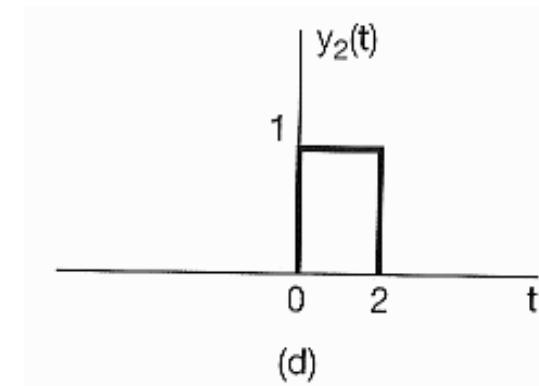
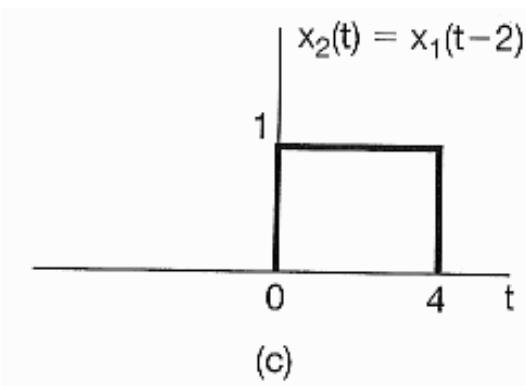
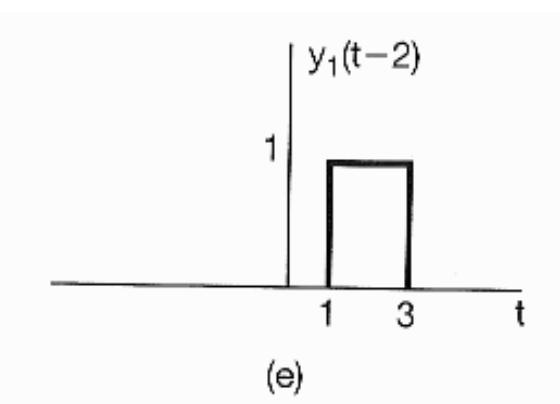
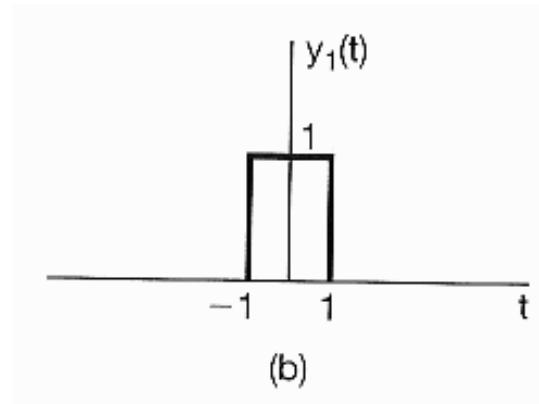
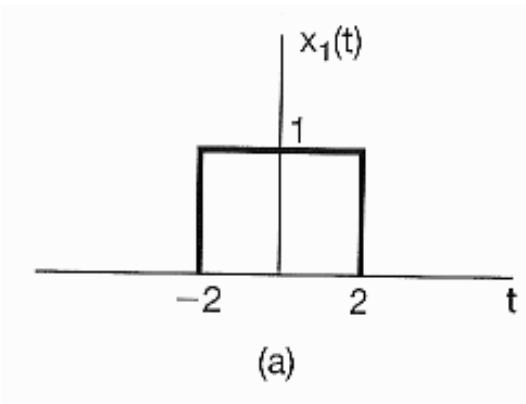
$$y_1(t - t_0) = \sin [x_1(t - t_0)]$$

$$y_2(t) = y_1(t - t_0)$$

■ Time Invariance

- Example of time-varying system (Example 1.16)

$$y(t) = x(2t)$$



■ Linearity

■ Linear systems

- If an **input** consists of the **weighted sum** of several signals, then the **output** is the **superposition** of the **responses** of the system to **each** of those signals

$$x_1[n] \rightarrow y_1[n]$$

$$x_2[n] \rightarrow y_2[n]$$

IF (1) $x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$ (additivity)

(2) $a x_1[n] \rightarrow a y_1[n]$ (scaling or homogeneity)

a : any complex constant

THEN, the system is **linear**

■ Linearity

■ Linear systems

- In general,

a, b : any complex constants

$$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n] \quad \text{for DT}$$

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t) \quad \text{for CT}$$

- OR,

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + \dots$$

$$\rightarrow y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + \dots$$

This is known as the **superposition property**

■ Linearity

■ Example 1.17:

$$S : y(t) = tx(t)$$

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$

$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

$$x_3(t) = ax_1(t) + bx_2(t)$$

$$\rightarrow y_3(t) = tx_3(t)$$

$$= t(ax_1(t) + bx_2(t)) = atx_1(t) + btx_2(t)$$

$$= a y_1(t) + b y_2(t)$$

■ Linearity

■ Example 1.18: $S : y(t) = (x(t))^2$

$$x_1(t) \rightarrow y_1(t) = (x_1(t))^2$$

$$x_2(t) \rightarrow y_2(t) = (x_2(t))^2$$

$$x_3(t) = ax_1(t) + bx_2(t)$$

$$\rightarrow y_3(t) = (x_3(t))^2 = (ax_1(t) + bx_2(t))^2$$

$$= a^2(x_1(t))^2 + b^2(x_2(t))^2 + 2abx_1(t)x_2(t)$$

$$= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t)$$

■ Linearity

■ Example 1.20: $S : y[n] = 2x[n] + 3$

$$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3$$

$$x_3[n] = ax_1[n] + bx_2[n]$$

$$\rightarrow y_3[n] = 2x_3[n] + 3$$

$$= 2(ax_1[n] + bx_2[n]) + 3$$

$$= a(2x_1[n] + 3) + b(2x_2[n] + 3) + 3 - 3a - 3b$$

$$= ay_1[n] + by_2[n] + 3(1 - a - b)$$

■ Linearity

- Example 1.20: $S : y[n] = 2x[n] + 3$

$$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3$$

- However,

$$\begin{aligned}y_1[n] - y_2[n] &= (2x_1[n] + 3) - (2x_2[n] + 3) \\&= 2[x_1[n] - x_2[n]]\end{aligned}$$

It is a **incrementally linear system**

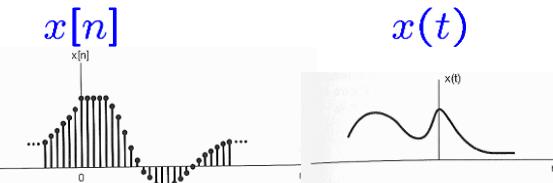
- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable

- Time Shift
- Time Reversal
- Time Scaling
- Periodic Signals
- Even & Odd Signals

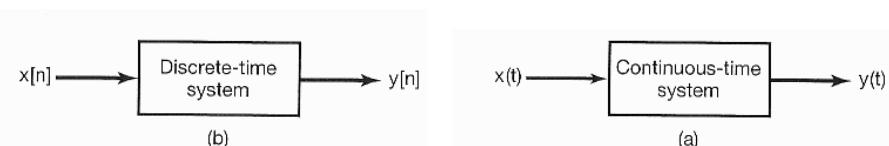
$$\begin{array}{lll} x[n - n_0] & x(t - t_0) & x(-t) = x(t), x[-n] = x[n] \\ x[-n] & x(-t) & x(-t) = -x(t), x[-n] = -x[n] \\ x[an] & x(at) & \mathcal{E}_v \{x[n]\} = \frac{1}{2} [x[n] + x[-n]] \\ x(t) = x(t + T) & & \mathcal{O}_d \{x[n]\} = \frac{1}{2} [x[n] - x[-n]] \\ x[n] = x[n + N] & & \end{array}$$

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems

- Basic System Properties
 - Systems with or without memory
 - Invertibility & Inverse Systems
 - Causality
 - Stability
 - Time Invariance
 - Linearity



$$\begin{aligned} \phi_k(t) &= e^{j k \omega_0 t}, k = 0, \pm 1, \dots \\ \phi_k[n] &= e^{j k \omega_0 n}, k = 0, \dots, N-1 \\ \delta[n], u[n] & \\ \delta(t), u(t) & \end{aligned}$$



$$x[n] \rightarrow y[n] \quad x(t) \rightarrow y(t)$$