Spring 2011

信號與系統
Signals and Systems

Chapter SS-5
The Discrete-Time Fourier Transform

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NTU-EE

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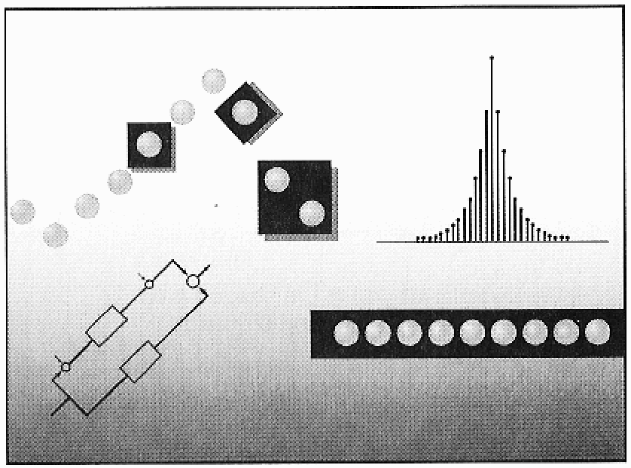
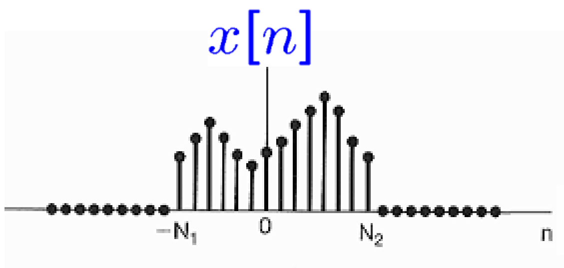


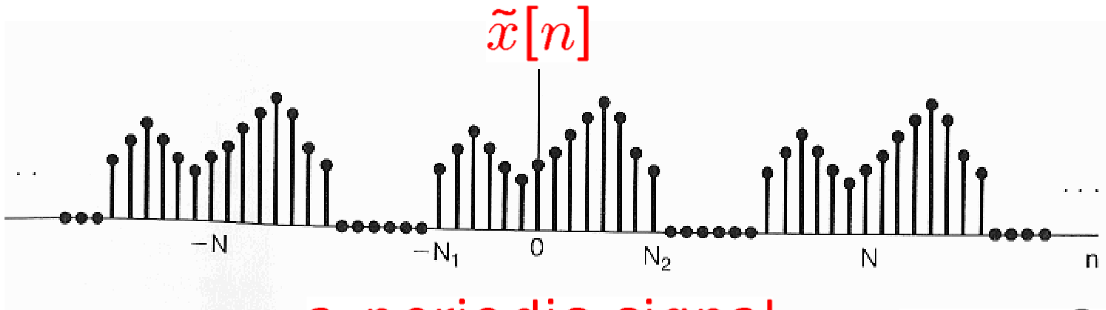
Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

▪ DT Fourier Transform of an Aperiodic Signal:



an aperiodic signal



a periodic signal

$$w_0 = \frac{2\pi}{N}$$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} = \sum_{k=<N>} a_k e^{jk(w_0)n}$$

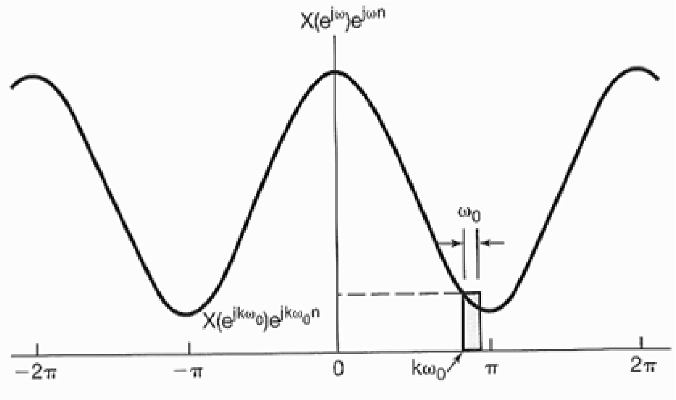
$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(w_0)n}$$

$$\Rightarrow \quad a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(w_0)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(w_0)n}$$

DT Fourier Transform of an Aperiodic Signal:

- Define $X(e^{jw})$:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$



- Then,

$$a_k = \frac{1}{N}X(e^{jkw_0})$$

$$w = kw_0$$

- Hence,

$$\boxed{w_0 = \frac{2\pi}{N}}$$

$$\boxed{\frac{1}{N} = \frac{1}{2\pi}w_0}$$

$$\boxed{w_0 N = 2\pi}$$

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N}X(e^{jkw_0})e^{jkw_0 n}$$

$$= \frac{1}{2\pi}\sum_{k=<N>} X(e^{jkw_0})e^{jkw_0 n}w_0$$

DT Fourier Transform of an Aperiodic Signal:

- As $N \to \infty$, $\tilde{x}[n] \to x[n]$

$$\boxed{w_0 N = 2\pi}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

- inverse Fourier transform eqn
- synthesis eqn

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

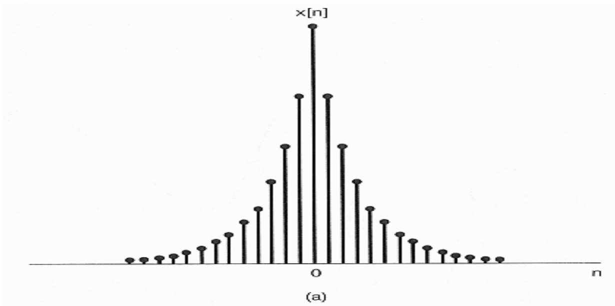
- $X(e^{jw})$: Fourier transform of $x[n]$
  spectrum
- analysis eqn

$$a_k = \frac{1}{N} X(e^{jw}) \Big|_{w = k w_0}$$

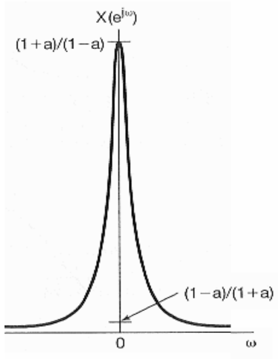
$$\boxed{w_0 = \frac{2\pi}{N}}$$

- <u>Periodicity of DT Fourier Transform:</u>

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$



$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$



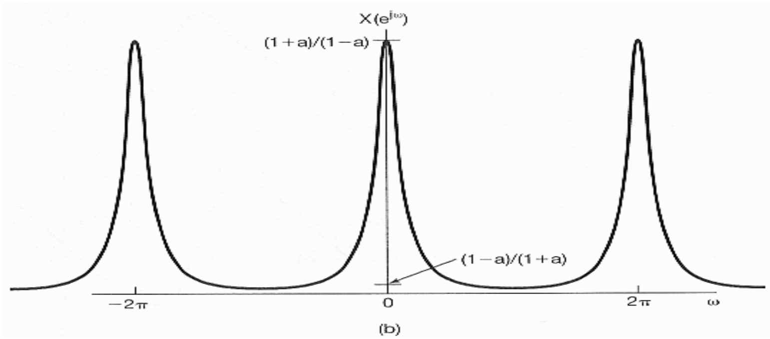
$$X(e^{j(w+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w)n}e^{-j(2\pi)n}$$

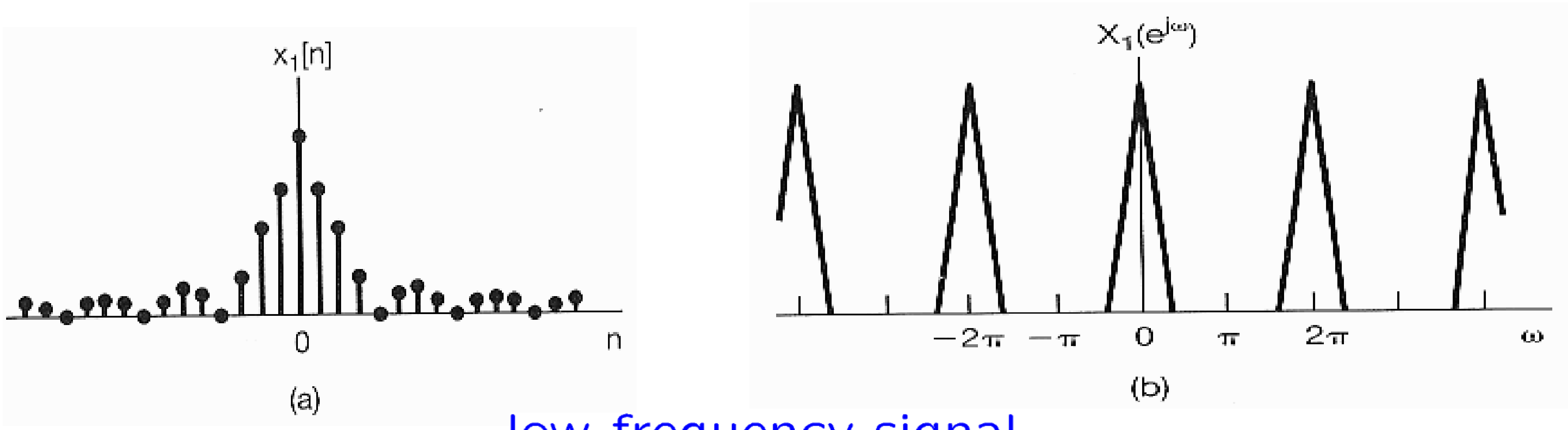
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w)n}$$

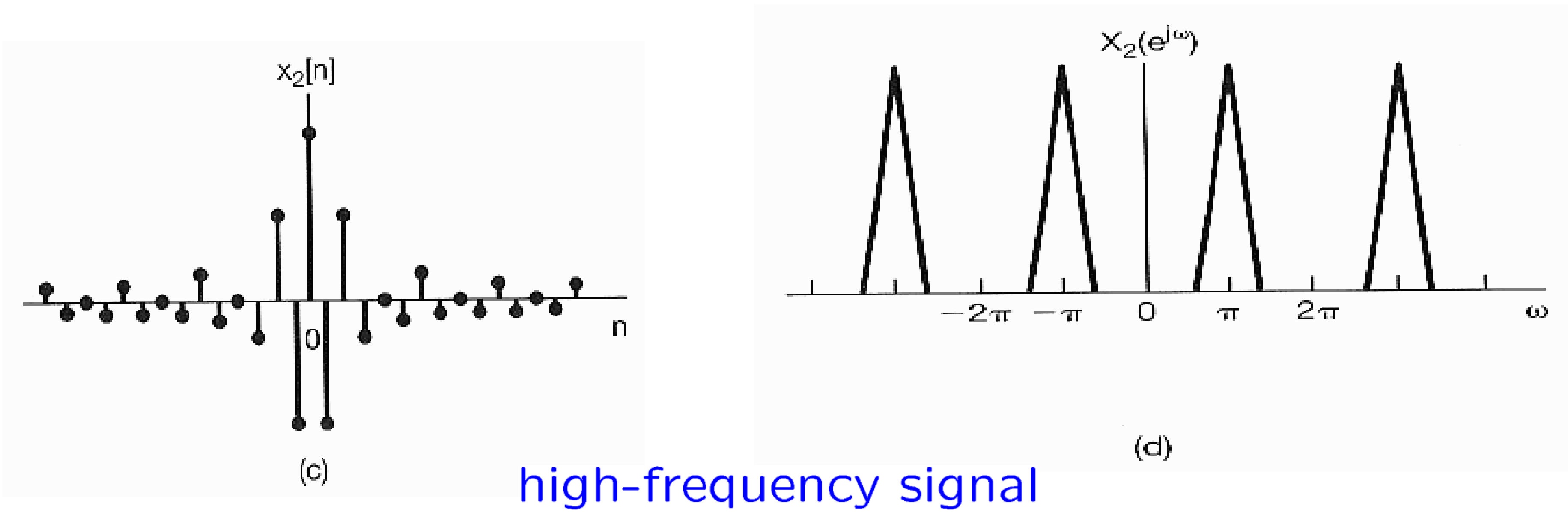
$$= X(e^{jw})$$



- High-Frequency & Low-Frequency Signals:



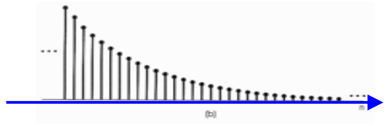
low-frequency signal



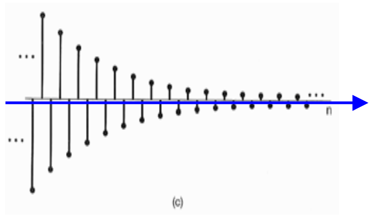
high-frequency signal

- Example 5.1:

$0 < a < 1$



$-1 < a < 0$



$$x[n] = a^n \, u[n], \quad |a| < 1$$

$$\Rightarrow \; X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, e^{-jwn} \quad = \sum_{n=0}^{\infty} (ae^{-jw})^n \quad = \frac{1}{1 - ae^{-jw}}$$

$$|X\left(e^{j\omega}\right)| = \frac{1}{\sqrt{(1 - \alpha\cos\omega)^2 + (\alpha\sin\omega)^2}} = \frac{1}{\sqrt{1 - 2\alpha\cos(\omega) + \alpha^2}}$$

$$\angle X(e^{jw}) = \angle \text{ nominator } - \angle \text{ denominator } = 0 - \tan^{-1}\frac{\alpha\sin\omega}{1 - \alpha\cos\omega}$$

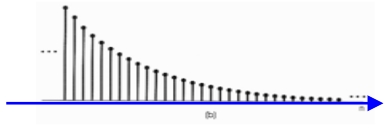
$$\frac{1}{\sqrt{1 - 2\alpha\cos(\omega) + \alpha^2}}$$

$$\frac{1}{1-\alpha} \quad \omega = 0$$

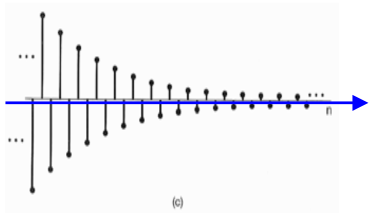
$$\frac{1}{1+\alpha} \quad \omega = \pi$$

- Example 5.1:

$0 < a < 1$



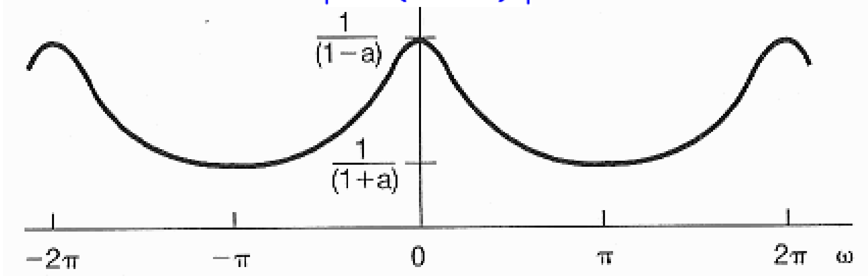
$-1 < a < 0$



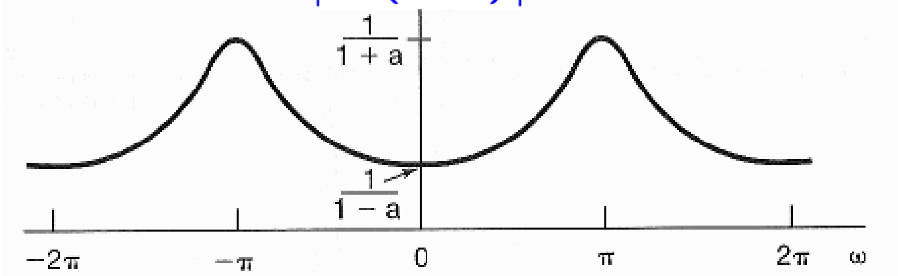
$$x[n] = a^n\, u[n], \quad |a| < 1$$

$$\Rightarrow \; X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^n\, u[n]\, e^{-jwn} = \sum_{n=0}^{\infty} (ae^{-jw})^n = \frac{1}{1 - ae^{-jw}}$$

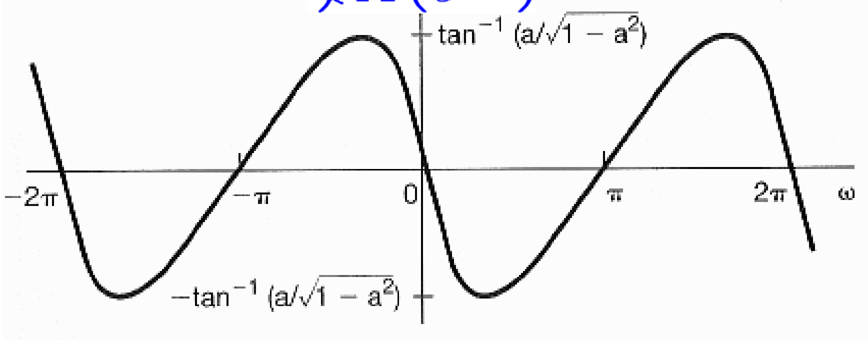
$0 < a < 1$ $\quad |X(e^{jw})|$



$-1 < a < 0$ $\quad |X(e^{jw})|$

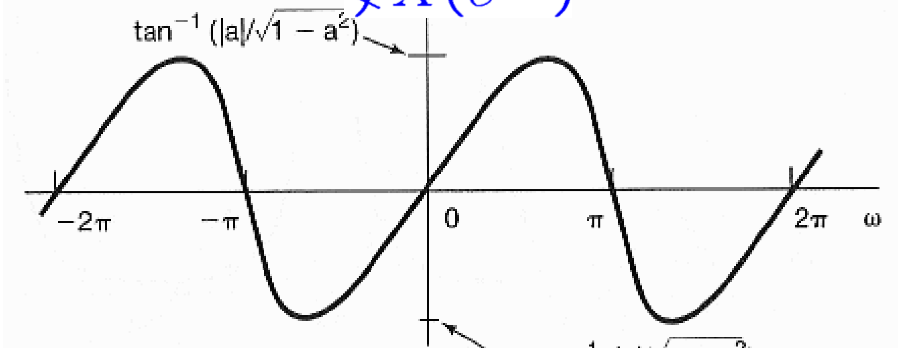


$\sphericalangle X(e^{jw})$



(a)

$\sphericalangle X(e^{jw})$



(b)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

- **Example 5.2:** $x[n] = a^{|n|}, \quad 0 < a < 1$

  $-1 < a < 0$

$$\Rightarrow X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^{|n|} \, e^{-jwn}$$

$$= \sum_{n=0}^{+\infty} a^{n} \, e^{-jwn} + \sum_{n=-\infty}^{-1} a^{-n} \, e^{-jwn}$$

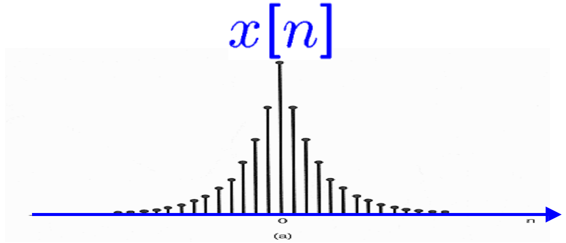
$$= \sum_{n=0}^{+\infty} (ae^{-jw})^{n} + \sum_{m=1}^{\infty} (ae^{jw})^{m}$$

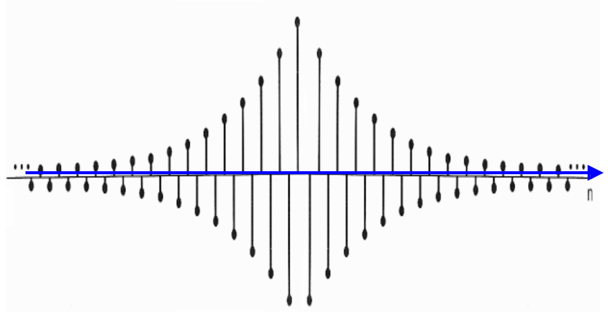
$$= \frac{1}{1 - ae^{-jw}} + \frac{ae^{jw}}{1 - ae^{jw}}$$

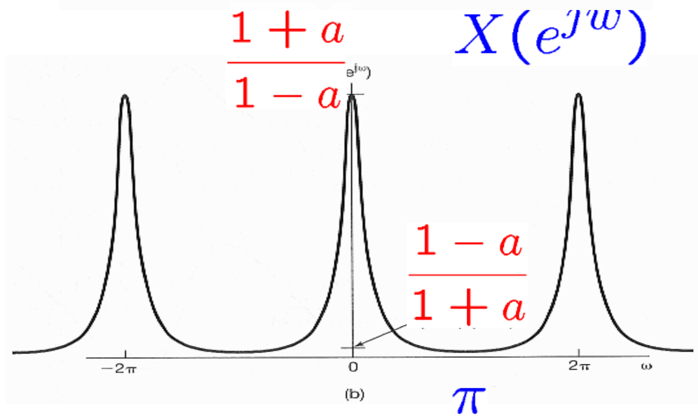
$$= \frac{1 - a^2}{1 - 2a\cos w + a^2}$$

$$= \frac{1 - a^2}{(1-a)^2}$$

$$= \frac{1 - a^2}{(1+a)^2}$$



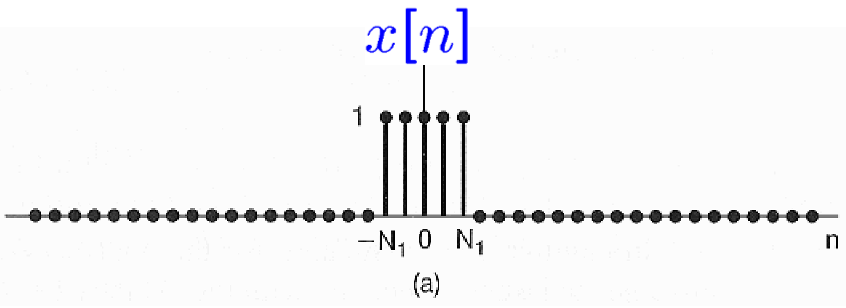




$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- Example 5.3:

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$



$$\Rightarrow X(e^{jw}) = \sum_{n=-N_1}^{N_1} e^{-jwn}$$

$$= e^{-jw(-N_1)} + \cdots + e^{-jw(N_1)} \quad = e^{-jw(-N_1)} \left(\frac{1-(e^{-jw})^{2N_1+1}}{1-(e^{-jw})} \right)$$

$$= e^{jw(N_1)} \left(\frac{(e^{-jw})^{N_1+1/2}\left((e^{jw})^{N_1+1/2}-(e^{-jw})^{N_1+1/2}\right)}{(e^{-jw/2})\left((e^{jw/2})-(e^{-jw/2})\right)} \right)$$

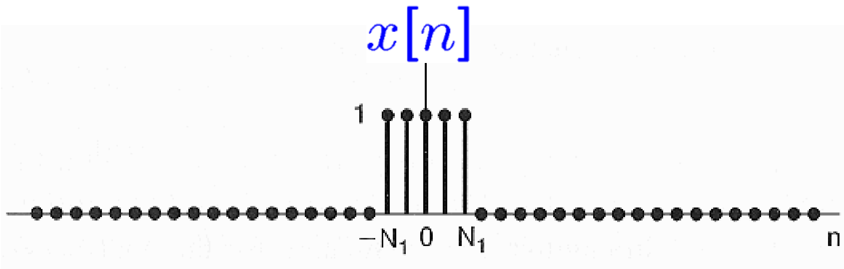
$$= \frac{\sin\left(w(N_1+\frac{1}{2})\right)}{\sin(w/2)}$$

$$\boxed{\begin{aligned} &1 - e^{-j\theta} \\ &= e^{-j\theta/2}e^{j\theta/2} - e^{-j\theta/2}e^{-j\theta/2} \\ &= e^{-j\theta/2}\left(e^{j\theta/2} - e^{-j\theta/2}\right) \end{aligned}}$$

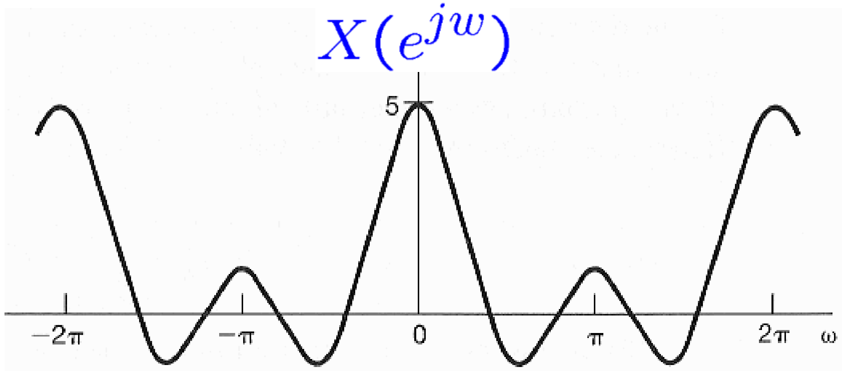
- Example 5.3:

$$\Rightarrow \quad X(e^{jw}) = \sum_{n=-N_1}^{N_1} e^{-jwn}$$

$$= \frac{\sin\left(w(N_1 + \frac{1}{2})\right)}{\sin(w/2)}$$



(a)



(b)

- Convergence of DT Fourier Transform:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$X(e^{jw}) \overset{\text{syn}}{\rightarrow} x[n] \overset{\text{analysis}}{\rightarrow} \hat{X}(e^{jw})$$

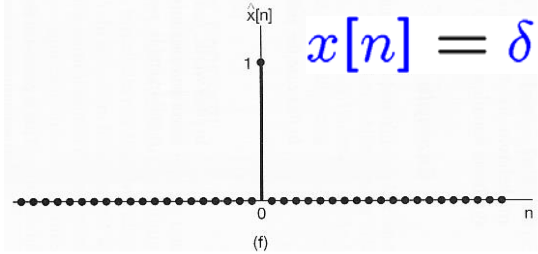
$$\hat{X}(e^{jw}) \overset{?}{\rightarrow} X(e^{jw})$$

- The analysis equation will converge:

$$\hat{X}(e^{jw}) = \frac{X(e^{jw})^- + X(e^{jw})^+}{2} \qquad \sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

$$\int_{2\pi} |X(e^{jw}) - \hat{X}(e^{jw})|^2 dw = 0 \qquad \sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

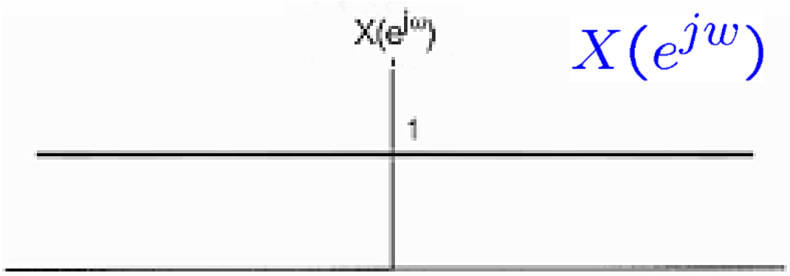
- **Example 5.4:**



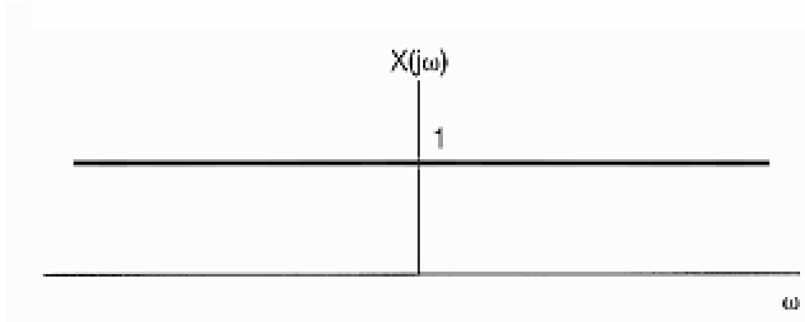
$x[n] = \delta[n]$

$$x[n] = \delta[n], \quad \text{i.e., unit impulse}$$

$$\Rightarrow \; X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} = 1$$



$X(e^{jw})$



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

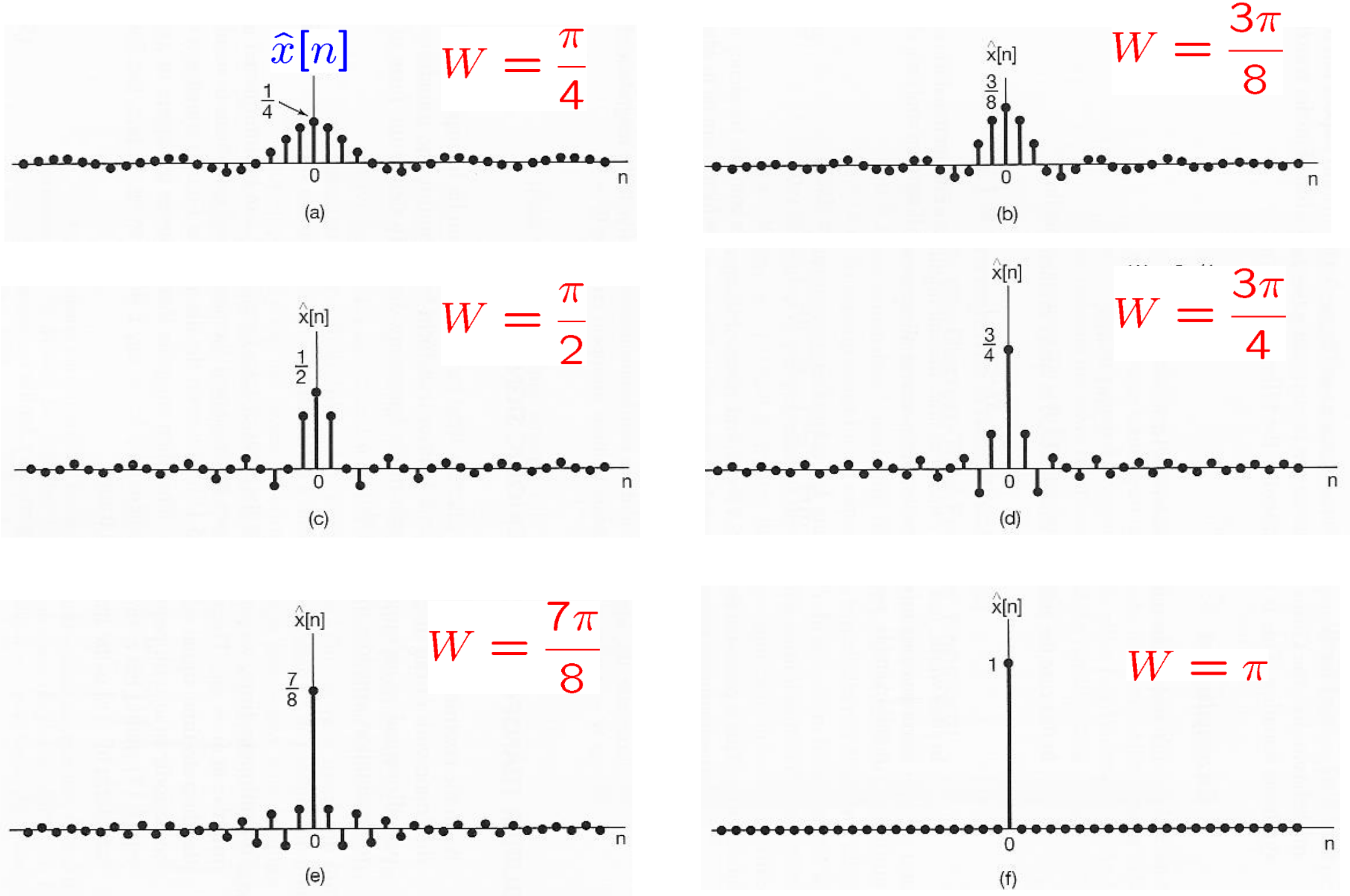
$$= \frac{1}{2\pi} \int_{2\pi} e^{jwn} dw$$

- **Approximation**

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-W}^{+W} X(e^{jw}) e^{jwn} dw = \frac{\sin Wn}{\pi n}$$

- **Approximation of an Aperiodic Signal:**

$$\hat{x}[n] = \frac{\sin W n}{\pi n}$$



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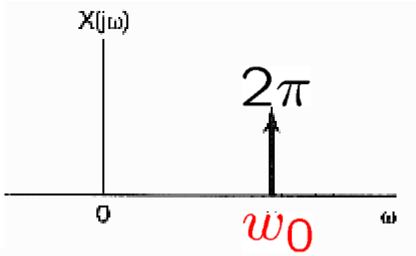
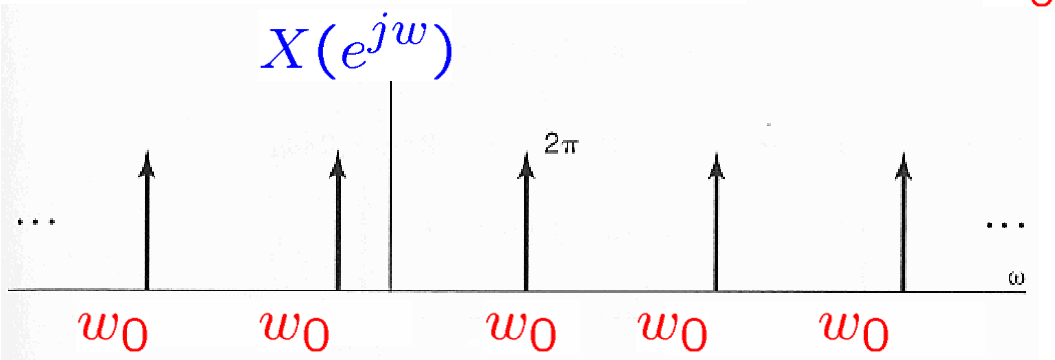
- Fourier Transform from Fourier Series:

$$x(t) = e^{jw_0 t} \xleftrightarrow{\mathcal{CTFT}} X(jw) = 2\pi\delta(w - w_0)$$



$$x[n] = e^{jw_0 n} \xleftrightarrow{\mathcal{DTFT}}$$



$$X(e^{jw}) = \cdots + 2\pi\delta(w - w_0 + 2\pi) + 2\pi\delta(w - w_0) + 2\pi\delta(w - w_0 - 2\pi) + \cdots$$

$$= \sum_{l=-\infty}^{+\infty} 2\pi\delta(w - w_0 - 2\pi l)$$

$$\boxed{w_0 = \frac{2\pi}{N}}$$

$$\frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{+\infty} 2\pi\delta(w - w_0 - 2\pi l)e^{jwn} \; dw$$

$$= e^{j(w_0 + 2\pi r)n} = e^{jw_0 n}$$

Fourier Transform from Fourier Series:

$$\boxed{w_0 = \frac{2\pi}{N}}$$

- more generally,

$$x[n] = \sum_{k=<N>} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n} \qquad = \sum_{k=<N>} a_k \, e^{jk(w_0)n}$$

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - k\frac{2\pi}{N}\right) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - kw_0\right)$$

- If $k = 0, 1, ..., N-1$

$$x[n] = a_0 + a_1 \, e^{j\,1\,\left(\frac{2\pi}{N}\right)n} + a_2 \, e^{j\,2\,\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1} \, e^{j\,(N-1)\,\left(\frac{2\pi}{N}\right)n}$$

$$= x_0 + x_1 + x_2 + \cdots + x_{N-1}$$

a linear combination of signals

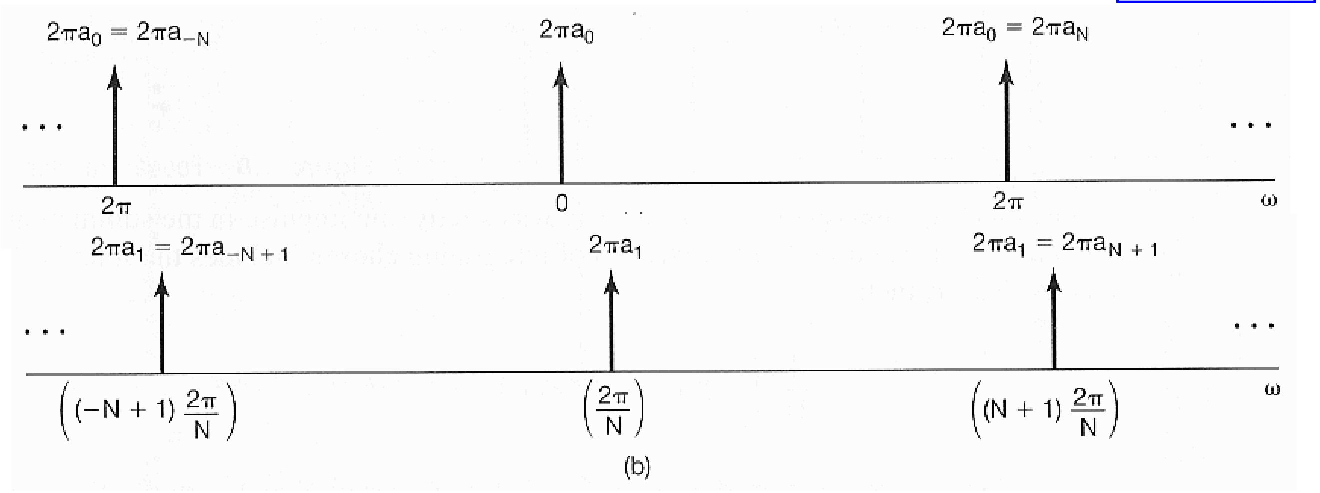
with $w_0 = 0, \frac{2\pi}{N}, \frac{2\cdot 2\pi}{N}, \cdots, \frac{(N-1)\cdot 2\pi}{N}$

- Fourier Transform from Fourier Series:

$$\boxed{w_0 = \frac{2\pi}{N}}$$

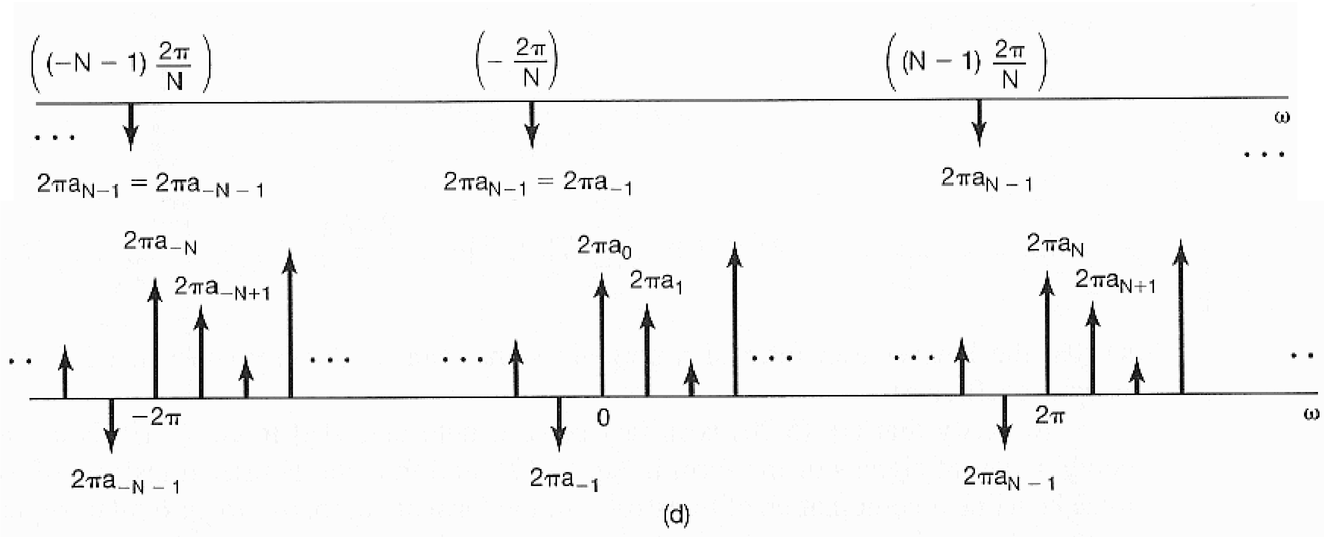
$x_0 \overset{\mathcal{F}}{\longleftrightarrow}$

$x_1 \overset{\mathcal{F}}{\longleftrightarrow}$



$x_{N-1} \overset{\mathcal{F}}{\longleftrightarrow}$

$X(e^{jw}) \quad =$

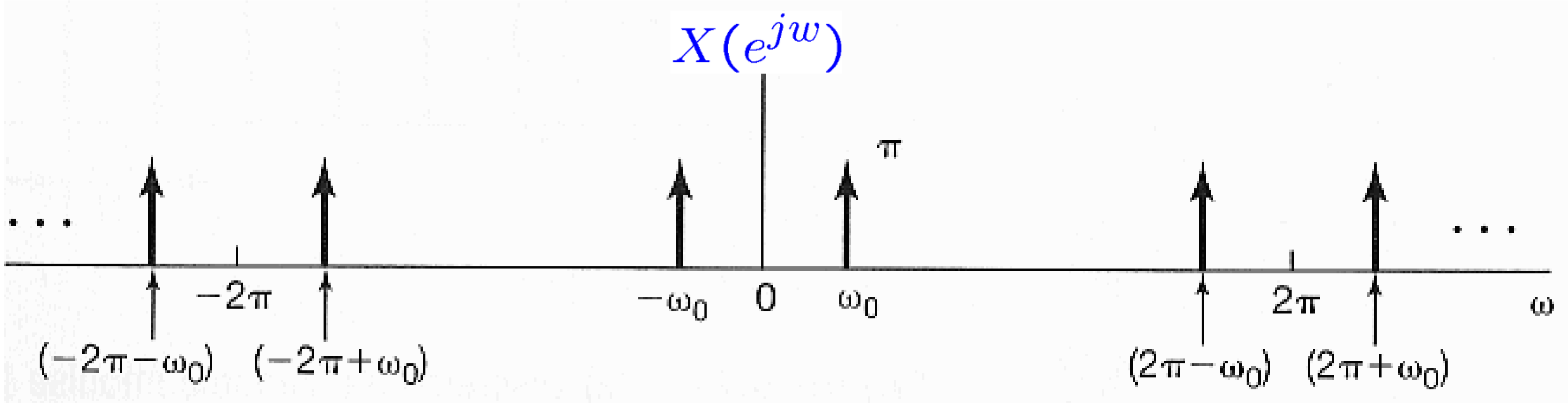


- Example 5.5:

$$x[n] = \cos(w_0 n) = \frac{e^{jw_0 n} + e^{-jw_0 n}}{2} \qquad \text{with } w_0 = \frac{2\pi}{5}$$

$$X(e^{jw}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w + \frac{2\pi}{5} - 2\pi l\right)$$

$$= \pi\,\delta\left(w - \frac{2\pi}{5}\right) + \pi\,\delta\left(w + \frac{2\pi}{5}\right), \quad -\pi \le w < \pi$$



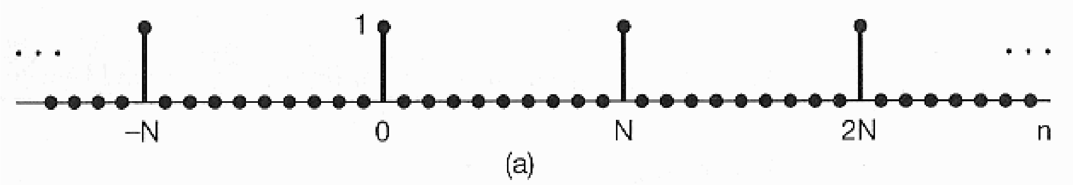
- Example 5.6:

$$x[n] = \sum_{k=<N>} a_k \, e^{jk(2\pi/N)n}$$

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - \frac{2\pi k}{N}\right)$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

$x[n]$



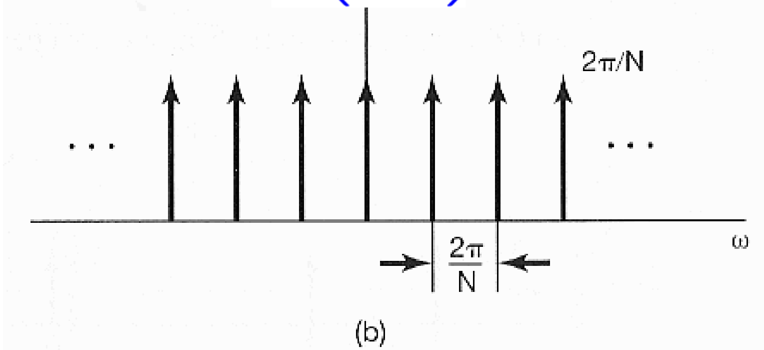
(a)

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

choose $0 \leq n \leq N-1$

$$\Rightarrow \; a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

$$\Rightarrow \; X(e^{jw}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(w - k\frac{2\pi}{N}\right)$$



(b)

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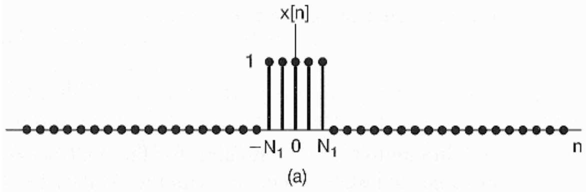
Outline

Section	Property
5.3.2	Linearity
5.3.3	Time Shifting
5.3.3	Frequency Shifting
5.3.4	Conjugation
5.3.6	Time Reversal
5.3.7	Time Expansion
5.4	Convolution
5.5	Multiplication
5.3.5	Differencing in Time
5.3.5	Accumulation
5.3.8	Differentiation in Frequency
5.3.4	Conjugate Symmetry for Real Signals
5.3.4	Symmetry for Real and Even Signals
5.3.4	Symmetry for Real and Odd Signals
5.3.4	Even-Odd Decomposition for Real Signals
5.3.9	Parseval's Relation for Aperiodic Signals

Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

Fourier Transform Pair:

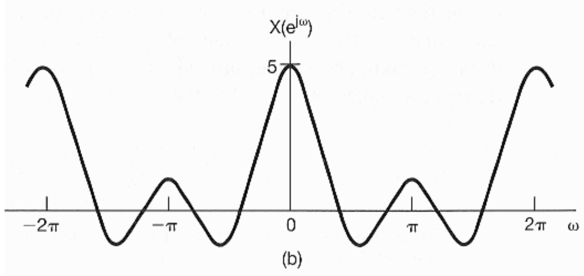


- Synthesis equation:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

- Analysis equation:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$



- Notations:

$$|a| < 1$$

$$X(e^{jw}) = \mathcal{F}\{x[n]\} \qquad \frac{1}{1-ae^{jw}} = \mathcal{F}\{a^n u[n]\}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{jw})\} \qquad a^n u[n] = \mathcal{F}^{-1}\{\frac{1}{1-ae^{jw}}\}$$

$$x[n] \xleftrightarrow{\mathcal{DTFT}} X(e^{jw}) \qquad a^n u[n] \xleftrightarrow{\mathcal{DTFT}} \frac{1}{1-ae^{jw}}$$

- **Periodicity of DT Fourier Transform:**

$$X(e^{j(w+2\pi)}) = X(e^{jw})$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- **Linearity:**

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{jw})$$

$$\Rightarrow a\, x[n] + b\, y[n] \overset{\mathcal{F}}{\longleftrightarrow} a\, X(e^{jw}) + b\, Y(e^{jw})$$

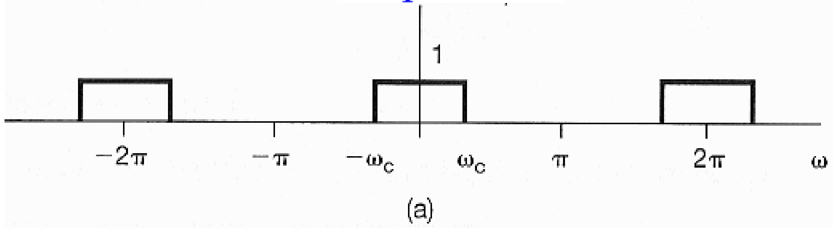
- **Time & Frequency Shifting:**

$$\Rightarrow \quad x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwn_0}X(e^{jw})$$

$$\Rightarrow \quad e^{jw_0n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(w-w_0)})$$

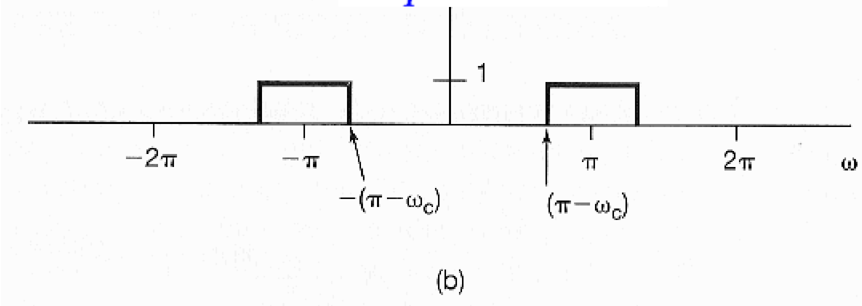
- Example 5.7:

$H_{lp}(e^{jw})$



(a)

$H_{lp}(e^{j(w-\pi)})$



(b)

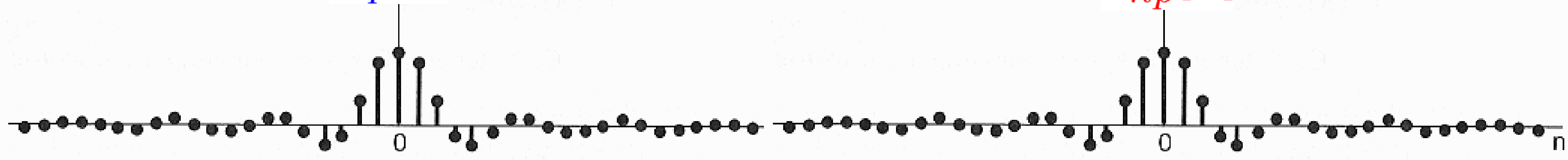
$$H_{hp}(e^{jw}) = H_{lp}(e^{j(w-\pi)})$$

$$\Rightarrow \; h_{hp}[n] = e^{j\pi n} \; h_{lp}[n]$$

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$

$$= (-1)^n h_{lp}[n]$$

$h_{lp}[n]$ $\qquad$ $h_{hp}[n]$



Conjugation & Conjugate Symmetry:

$$\boxed{X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}}$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw}) \qquad x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-jw})$$

- $x[n] = x^*[n] \Rightarrow X(e^{-jw}) = X^*(e^{jw})$

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw}$$

$x[n]$ is real $\Rightarrow$ $X(e^{jw})$ is conjugate symmetric

- $x[n] = x^*[n]$ & $x[-n] = x[n]$

$$\Rightarrow X(e^{-jw}) = X^*(e^{jw}) \;\;\&\;\; X(e^{-jw}) = X(e^{jw})$$

$$\Rightarrow X(e^{jw}) = X^*(e^{jw})$$

$x[n]$ is real & even $\Rightarrow$ $X(e^{jw})$ are real & even

- $x[n]$ is real & odd $\Rightarrow$ $X(e^{jw})$ are purely imaginary & odd

- Conjugation & Conjugate Symmetry:

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})$$

$$\mathcal{E}v\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(e^{jw})\}$$

$$\mathcal{O}d\{x[n]\} \xleftrightarrow{\mathcal{F}} j\,\mathcal{I}m\{X(e^{jw})\}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- Differencing & Accumulation:

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$x[n] - x[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} \left(1 - e^{-jw}\right)X(e^{jw})$$

$$X(e^{jw}) \qquad e^{-jw}X(e^{jw})$$

$$\sum_{m=-\infty}^{n} x[m] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-jw}}X(e^{jw}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(w - 2\pi k)$$

dc or average value

$$y[n] = \sum_{m=-\infty}^{n} x[m] \qquad \Rightarrow \; y[n] - y[n-1] = x[n]$$

$$y[n-1] = \sum_{m=-\infty}^{n-1} x[m] \qquad \Rightarrow \; \left(1 - e^{-jw}\right)Y(e^{jw}) = X(e^{jw})$$

- **Differentiation in Frequency:**

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})$$

$$\frac{1}{j} n x[n] \xleftrightarrow{\mathcal{F}} \frac{d}{dw} X(e^{jw})$$

$$n x[n] \xleftrightarrow{\mathcal{F}} j \frac{d}{dw} X(e^{jw})$$

$$\frac{d}{dw} X(e^{jw}) = \frac{d}{dw} \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} (-jn) x[n] e^{-jwn}$$

$$= (-j) \sum_{n=-\infty}^{+\infty} \left[n x[n] \right] e^{-jwn}$$

- **Time Reversal:**

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})$$

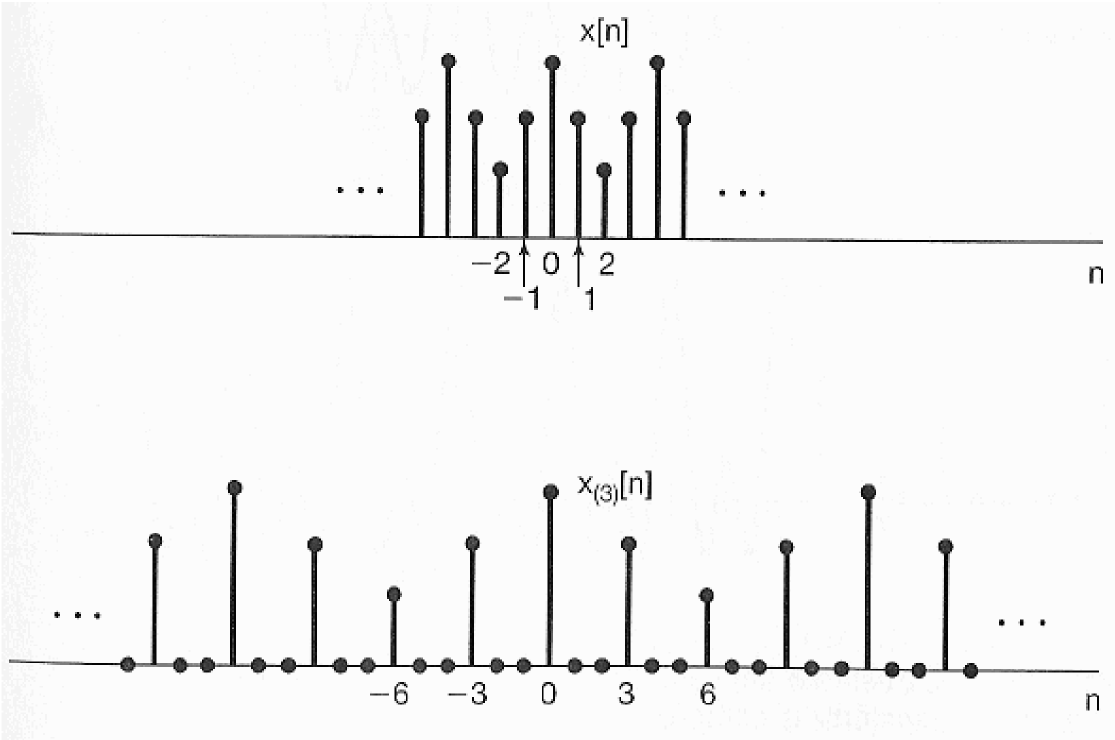
$$x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-jw})$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$X(e^{j(-w)})) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(-w)n}$$

Time Expansion:

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$



$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- **Time Expansion:**

$$\Rightarrow \; X_{(k)}(e^{jw}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-jwn}$$

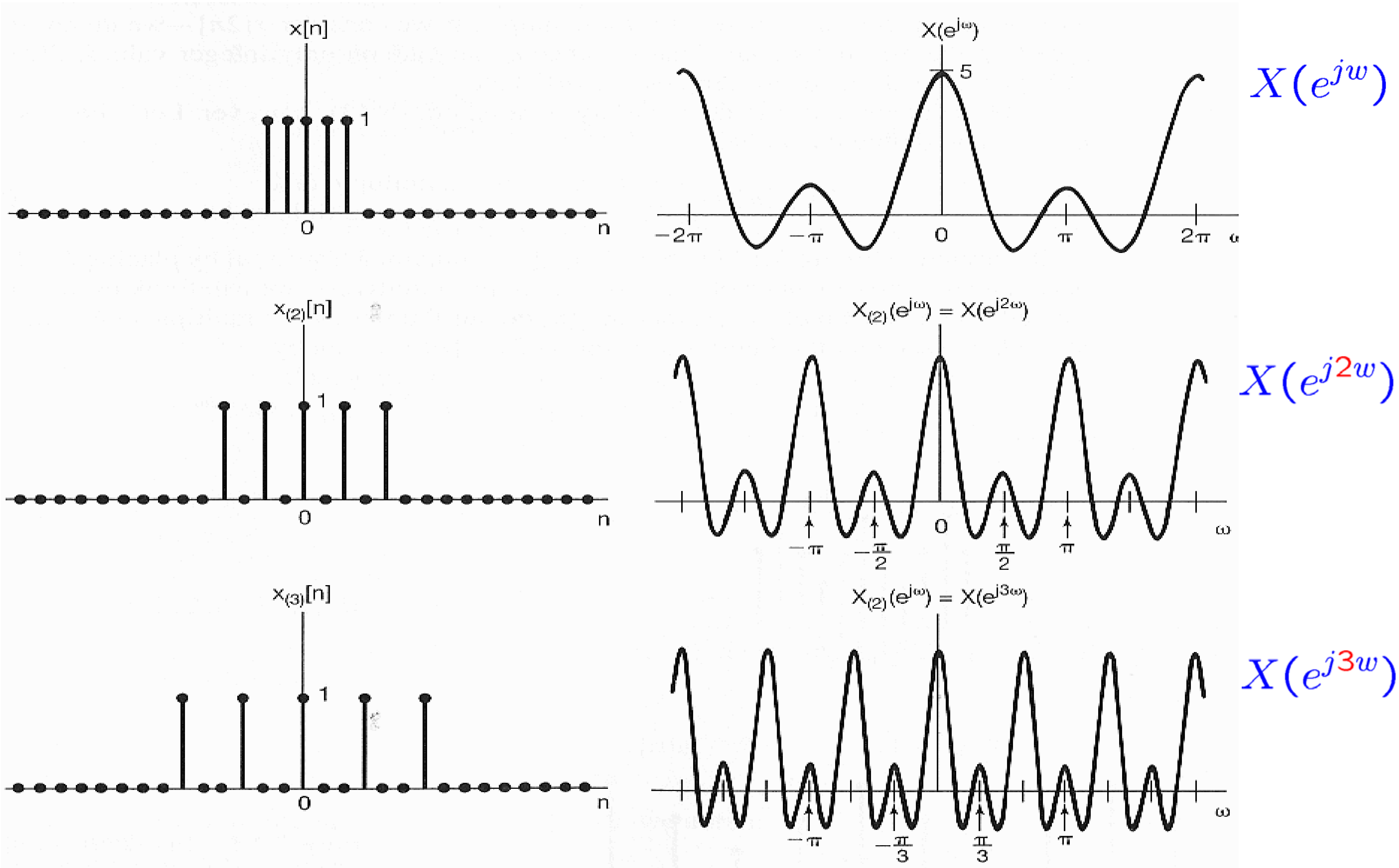
$$= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-jwrk}$$

$$= \sum_{r=-\infty}^{+\infty} x[r]e^{-j(kw)r} \qquad x_{(k)}[rk] = x[r]$$

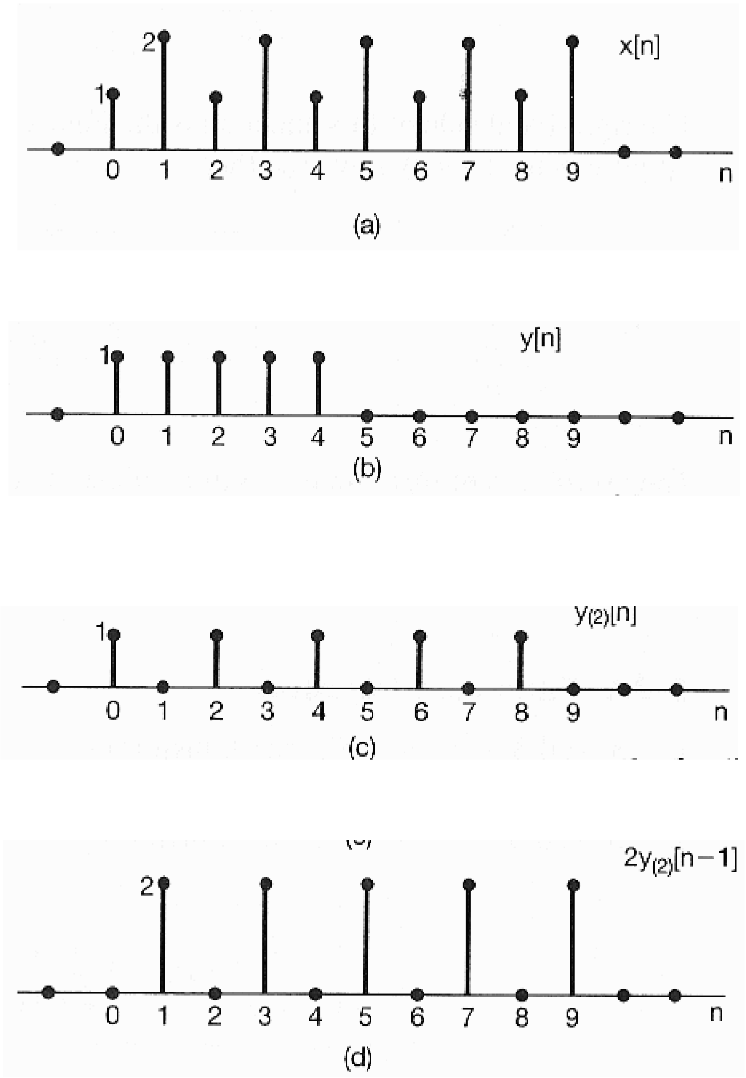
$$= X(e^{jkw})$$

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jkw})$$

Time Expansion:



- Example 5.9:



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

$$Y(e^{jw}) = e^{-j2w}\frac{\sin(5w/2)}{\sin(w/2)}$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$y_{(2)}[n] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j4w}\frac{\sin(5w)}{\sin(w)}$$

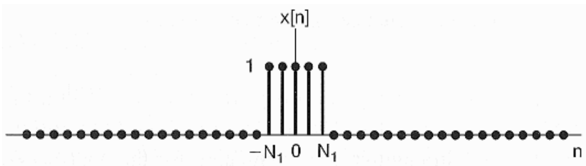
$$2y_{(2)}[n-1] \overset{\mathcal{F}}{\longleftrightarrow} 2e^{-jw}e^{-j4w}\frac{\sin(5w)}{\sin(w)}$$

$$X(e^{jw}) = (1 + 2e^{-jw}) \cdot e^{-j4w} \cdot \frac{\sin(5w)}{\sin(w)}$$

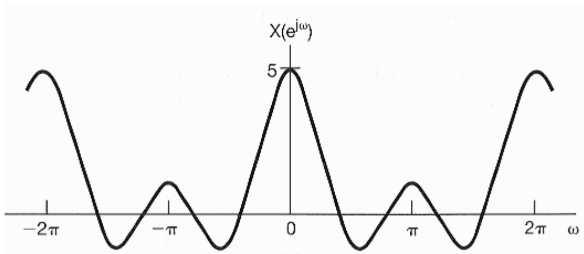
Parseval's relation:

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{jw})|^2 dw$$

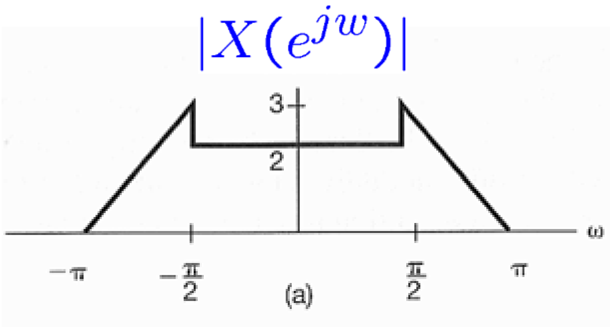


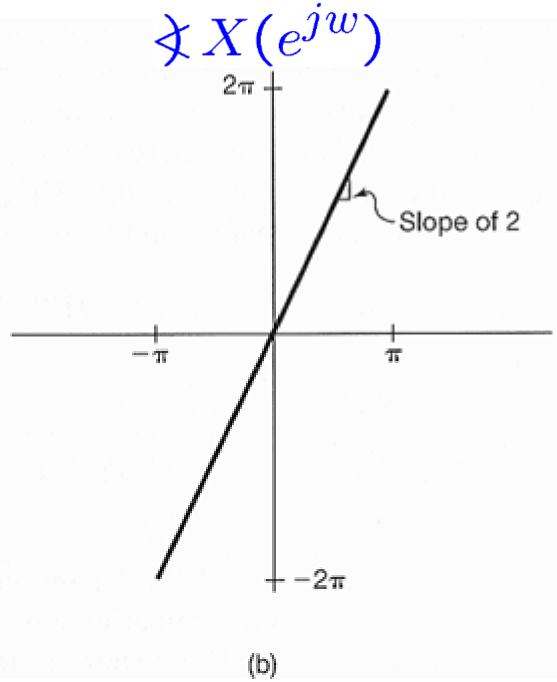
(a)



(b)

Example 5.10:





- $x[n]$ is periodic, real, even, and/or of finite energy?

$\rightarrow$ $X(e^{jw}) \neq 0$ $\Rightarrow$ $x[n]$ is NOT periodic

$\rightarrow$ even magnitude, odd phase $\Rightarrow$ $x[n]$ is real

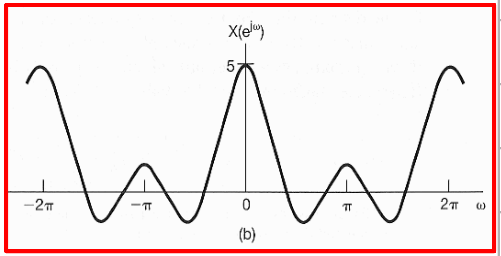
$\rightarrow$ $X(e^{jw})$ is NOT real $\Rightarrow$ $x[n]$ is NOT even

$\rightarrow$ $X(e^{jw})$ is finite $\Rightarrow$ $x[n]$ is finite

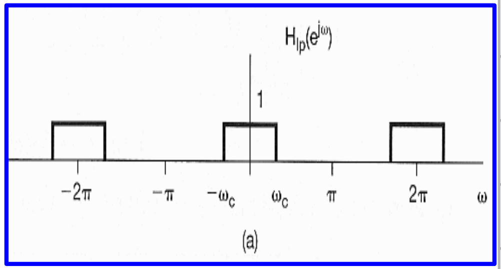
- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

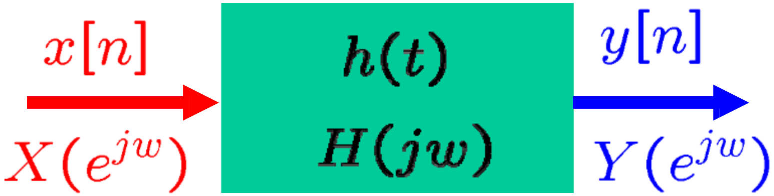
- Convolution Property:

$$y[n] = x[n] * h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

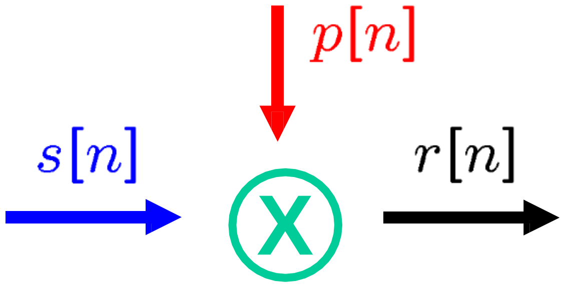


$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$





- Multiplication Property:



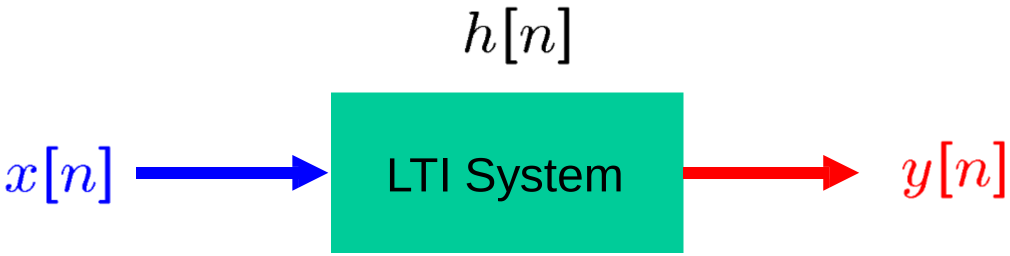
$$r[n] = s[n]p[n] \stackrel{\mathcal{F}}{\longleftrightarrow} R(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(w-\theta)})d\theta$$

Convolution Property

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- **Example 5.11:**



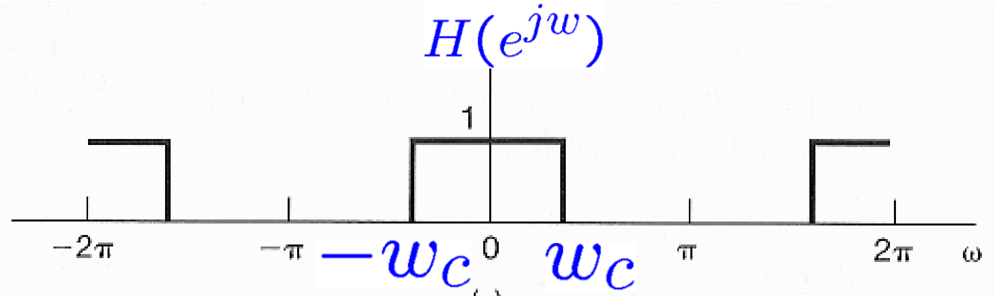
$$h[n] = \delta[n - n_0]$$

$$\Rightarrow \; H(e^{jw}) \; = \; \sum_{n=-\infty}^{+\infty} \delta[n - n_0]e^{-jwn} \; = \; e^{-jwn_0}$$

$$\Rightarrow \; Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

$$= e^{-jwn_0}\, X(e^{jw}) \qquad \Rightarrow \; y[n] = x[n - n_0]$$

- Example 5.12:

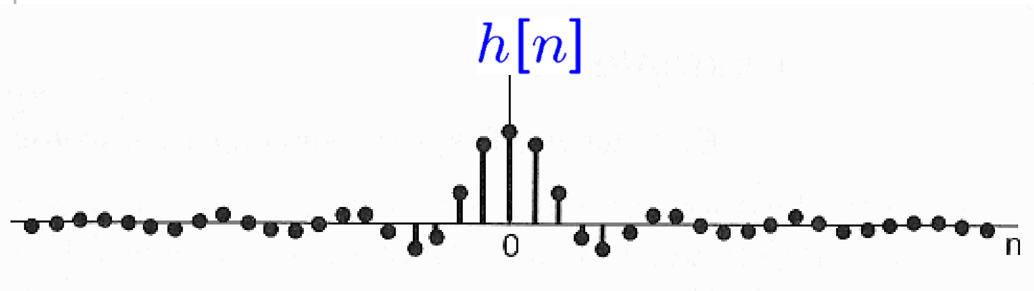


(a)

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})e^{jwn}dw$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jwn}dw$$

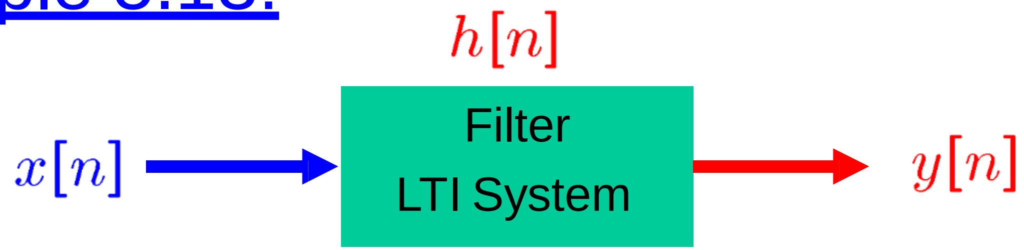
$$= \frac{\sin w_c n}{\pi n}$$



(b)

- not causal
- oscillatory

- Example 5.13:



$$h[n] = a^n u[n], \quad |a| < 1 \qquad \Rightarrow \; H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$x[n] = b^n u[n], \quad |b| < 1 \qquad \Rightarrow \; X(e^{jw}) = \frac{1}{1 - be^{-jw}}$$

$$\Rightarrow \; Y(e^{jw}) = H(e^{jw}) X(e^{jw})$$

$$= \frac{1}{1 - ae^{-jw}} \frac{1}{1 - be^{-jw}}$$

- Example 5.13:

if $a \neq b$ $\quad Y(e^{jw}) = \left[\left(\frac{a}{a-b}\right) \frac{1}{1 - ae^{-jw}} + \left(\frac{-b}{a-b}\right) \frac{1}{1 - be^{-jw}} \right]$

$$\Rightarrow \quad y[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]$$

if $a = b$ $\quad Y(jw) = \left(\frac{1}{1 - ae^{-jw}}\right)^2 = \frac{j}{a} e^{jw} \frac{d}{dw} \left(\frac{1}{1 - ae^{-jw}}\right)$

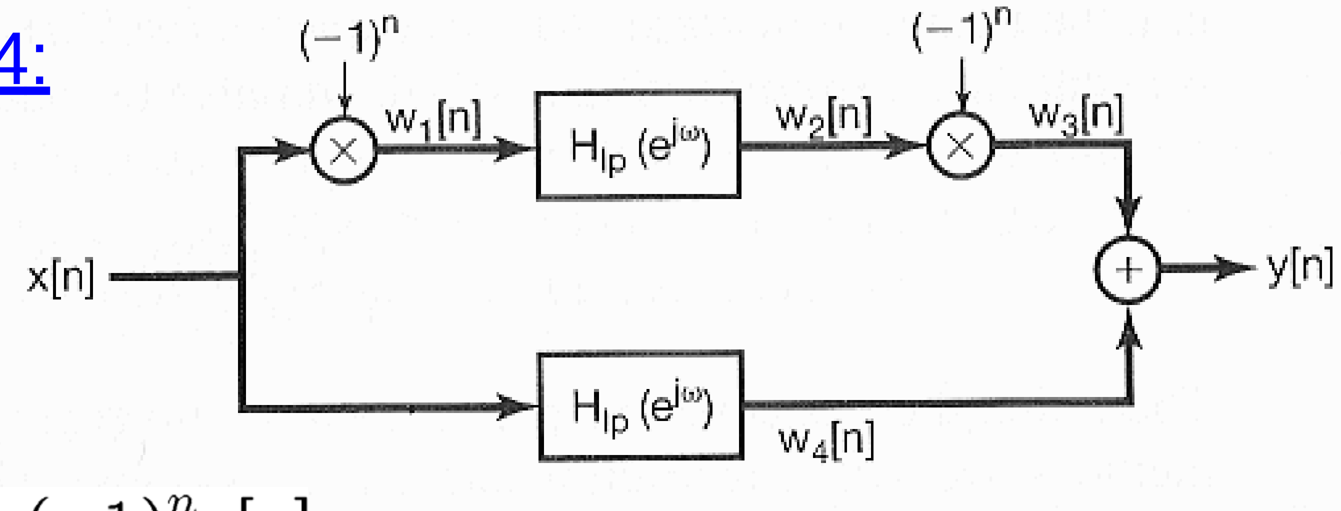
since $\quad a^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - ae^{-jw}}$

and $\quad n\, a^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j \frac{d}{dw} \left[\frac{1}{1 - ae^{-jw}}\right]$

and $\quad (n+1)\, a^{n+1} u[n+1] \stackrel{\mathcal{F}}{\longleftrightarrow} j\, e^{jw} \frac{d}{dw} \left[\frac{1}{1 - ae^{-jw}}\right]$

$$\Rightarrow \quad y[n] = (n+1) a^n u[n+1]$$

- Example 5.14:



(a)

$$w_1[n] = e^{j\pi n} x[n] = (-1)^n x[n]$$

$$\Rightarrow \quad W_1(e^{jw}) = X(e^{j(w-\pi)})$$

$$W_4(e^{jw}) = H_{lp}(e^{jw})\ X(e^{jw})$$

$$W_2(e^{jw}) = H_{lp}(e^{jw})\ X(e^{j(w-\pi)})$$

$$w_3[n] = e^{j\pi n} w_2[n] = (-1)^n w_2[n]$$

$$\Rightarrow \quad W_3(e^{jw}) = W_2(e^{j(w-\pi)}) = H_{lp}(e^{j(w-\pi)})\ X(e^{j(w-2\pi)})$$

$$= H_{lp}(e^{j(w-\pi)})\ X(e^{jw})$$

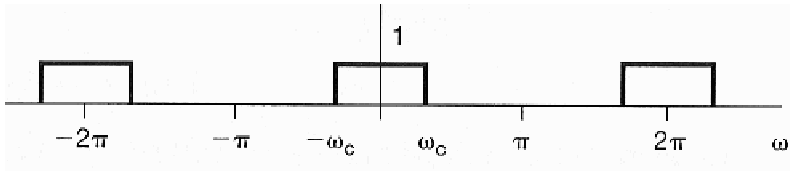
- Example 5.14:

$$Y(e^{jw}) = W_3(e^{jw}) + W_4(e^{jw})$$

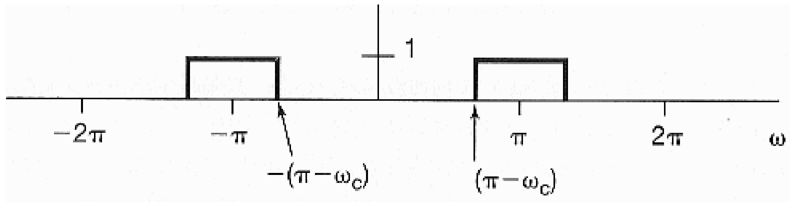
$$= H_{lp}(e^{j(w-\pi)})\ X(e^{jw}) + H_{lp}(e^{jw})\ X(e^{jw})$$

$$= \left[H_{lp}(e^{j(w-\pi)})\ + H_{lp}(e^{jw})\right]\ X(e^{jw})$$

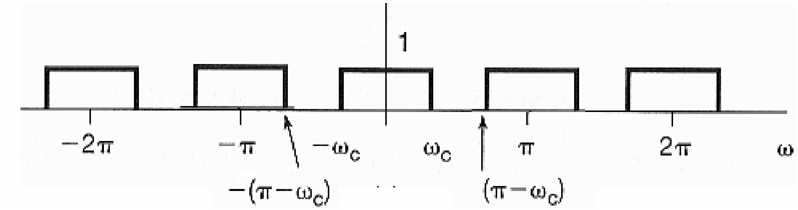
$$H(e^{jw}) = H_{lp}(e^{j(w-\pi)})\ + H_{lp}(e^{jw})$$



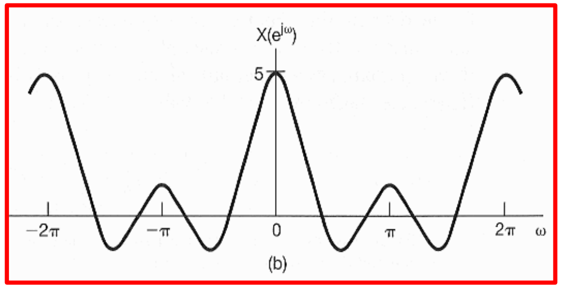
highpass + lowpass



$\rightarrow$ bandstop

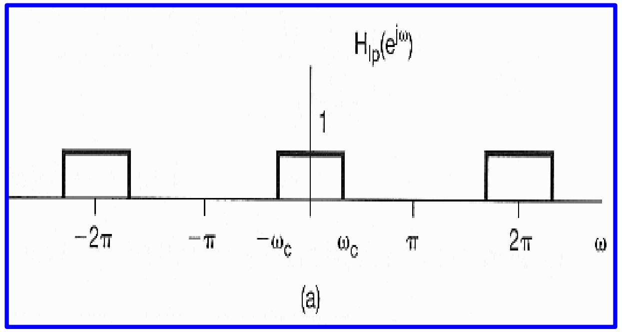


- **Convolution Property:**



$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



- **Multiplication Property:**

$$r[n] = s[n]p[n] \xleftrightarrow{\mathcal{F}} R(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(w-\theta)})d\theta$$

Multiplication Property

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$r[n] = s[n]p[n]$$

$$\Rightarrow \quad R(e^{jw}) = \sum_{n=-\infty}^{+\infty} r[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} s[n]p[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} s[n]\left\{\frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-jwn}$$

$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})\left[\sum_{n=-\infty}^{+\infty} s[n]e^{-j(w-\theta)n}\right]d\theta$$

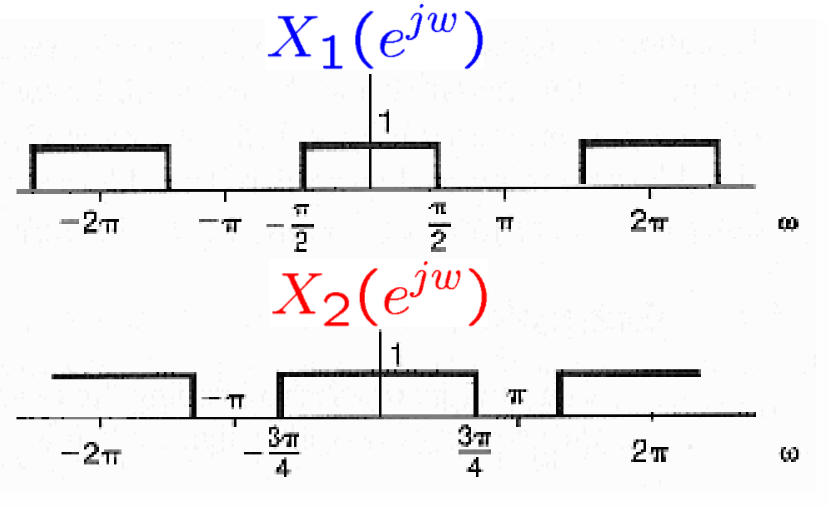
$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})S(e^{j(w-\theta)})d\theta \quad = \frac{1}{2\pi}\int_{2\pi} P(e^{j(w-\theta)})S(e^{j\theta})d\theta$$

- Example 5.15:

$$x[n] = x_1[n]x_2[n]$$

$$x_1[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$$

$$x_2[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$$



$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$

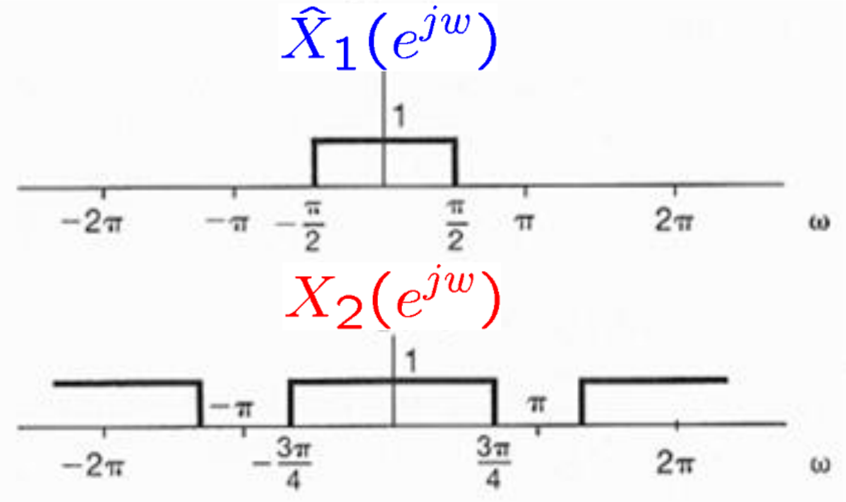
$$\hat{X}_1(e^{jw}) = \begin{cases} X_1(e^{jw}), & \text{for } -\pi < w \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

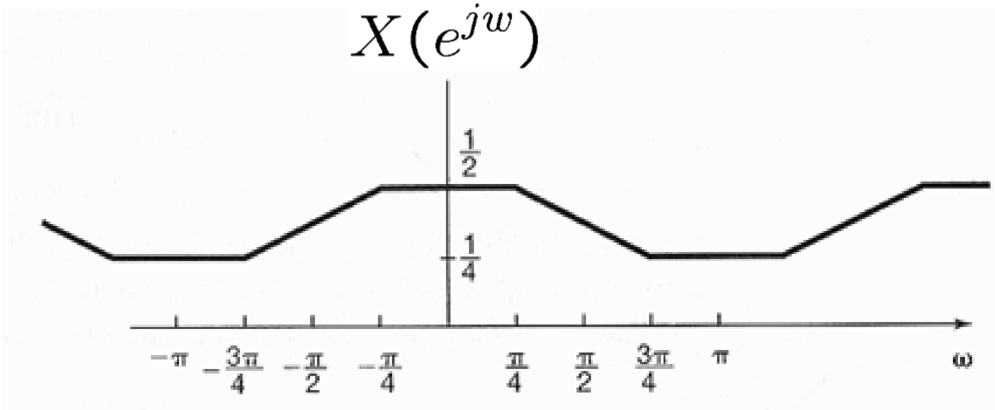
$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$

- Example 5.15:

$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta}) X_2(e^{j(w-\theta)}) d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(w-\theta)}) d\theta$$





- Example 5.15:

If $X_1(w)$ and $X_2(w)$ are periodic , then their periodic convolution can be calculated using:

$$\hat{X}_1(w) = \begin{cases} X_1(w) & |w| \leq \pi, \\ 0 & o.w. \end{cases}$$

$$\hat{X}_2(w) = \begin{cases} X_2(w) & |w| \leq \pi, \\ 0 & o.w. \end{cases}$$

$$\hat{Y}(w) = \hat{X}_1(w) * \hat{X}_2(w)$$

$$Y(w) = \sum_{k=-\infty}^{\infty} \hat{Y}(w - 2k\pi)$$

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$

5.3.9 Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k

$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n]\begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Outline

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

- DT Fourier Series Pair of Periodic Signals:

- $x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

IF $f[k] = \frac{1}{N} \sum_{n=<N>} g[n] e^{-jk(2\pi/N)n}$ $\quad g[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} f[k]$

$$f[n] = \sum_{k=<N>} \frac{1}{N} g[-k] e^{jk(2\pi/N)n} \qquad f[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} \frac{1}{N} g[-k]$$

LET $k = n, n = -k$ $\qquad a_k :=: \frac{1}{N} x[-n]$

- Duality in DT Fourier Series:

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$x[n-n_0] \xleftrightarrow{\mathcal{FS}} a_k \, e^{-jk(2\pi/N)n_0}$$

$$e^{+jm(2\pi/N)n} \, x[n] \xleftrightarrow{\mathcal{FS}} a_{k-m}$$

$$\sum_{r=<N>} x[r] \, y[n-r] \xleftrightarrow{\mathcal{FS}} N a_k \, b_k$$

$$x[n] \, y[n] \xleftrightarrow{\mathcal{FS}} \sum_{l=<N>} a_l \, b_{k-l}$$

- Duality between DT-FT & CT-FS:

$$x[n] \xleftrightarrow{\mathcal{DTFT}} X(e^{jw}) \qquad x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad a_k = \frac{1}{T}\int_{T} x(t)e^{-jkw_0t}dt$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn} \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

$$f[n] \overset{\mathcal{DFT}}{\to} g(w)$$

$$g(t) \overset{\mathcal{CFS}}{\to} f[-k]$$

Duality between DT-FS & DT-FS:

Assume that f[n] is periodic with fundamental period N and its Fourier series coefficient are a_k = g[k]

Then a_k = g[k] is periodic with fundamental period N and its Fourier series coefficient are b_k = f[-k]/N

$$f[n] \xrightarrow{\mathcal{FS}} g[k]$$

$$g[n] \xrightarrow{\mathcal{FS}} \frac{1}{N} f[-k]$$

Duality

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

Outline

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
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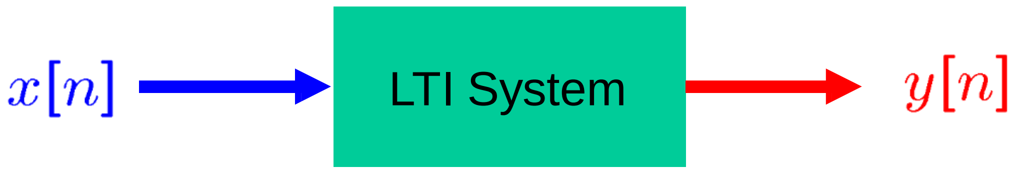
- A useful class of DT LTI systems:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$



$$Y(e^{jw}) = X(e^{jw}) H(e^{jw}) \qquad H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$\boxed{x[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-jwn_0}X(e^{jw})}$$

$$\sum_{k=0}^{N} a_k y[n-k] \quad = \quad \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k \quad y[n-k] \quad = \quad \sum_{k=0}^{M} b_k \quad x[n-k]$$

$$\sum_{k=0}^{N} a_k \; e^{-jkw} \; Y(e^{jw}) \quad = \quad \sum_{k=0}^{M} b_k \; e^{-jkw} \; X(e^{jw})$$

$$\Rightarrow \quad H(e^{jw}) \quad = \quad \frac{Y(e^{jw})}{X(e^{jw})} \quad = \quad \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}}$$

$$= \frac{b_0 + b_1 e^{-jw} + \cdots + b_M e^{-jMw}}{a_0 + a_1 e^{-jw} + \cdots + a_N e^{-jNw}}$$

- Examples 5.18 & 5.19:

$|a| < 1$

$$y[n] - ay[n-1] = x[n] \quad \Rightarrow \quad H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$Y(\cdot) \qquad e^{-jw}Y(\cdot) \qquad \Rightarrow \quad h[n] = a^n u[n]$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

$$\Rightarrow \quad H(e^{jw}) = \frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}} = \frac{2}{(1 - \frac{1}{2}e^{-jw})(1 - \frac{1}{4}e^{-jw})}$$

$$= \frac{4}{(1 - \frac{1}{2}e^{-jw})} - \frac{2}{(1 - \frac{1}{4}e^{-jw})}$$

$$\Rightarrow \quad h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

- Example 5.20:

$$h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} H(e^{jw}) = \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = ???$$

$$= x[n] * h[n]$$

$$\Rightarrow Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

$$= \left[\frac{1}{1-\frac{1}{4}e^{-jw}}\right]\left[\frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}\right]$$

$$= \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})^2}$$

$$= \frac{8}{(1-\frac{1}{2}e^{-jw})} - \frac{4}{(1-\frac{1}{4}e^{-jw})} - \frac{2}{(1-\frac{1}{4}e^{-jw})^2}$$

$$\Rightarrow y[n] = \left\{8\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n\right\}u[n]$$