Spring 2011

信號與系統
Signals and Systems

Chapter SS-3
The Continuous-Time Fourier Series

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NTU-EE

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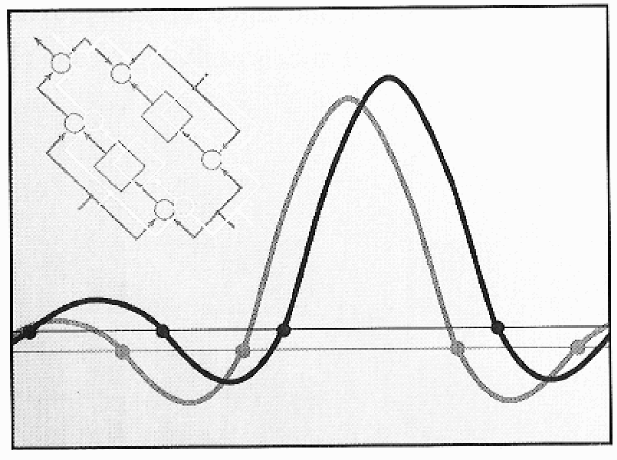


Outline

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous- Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Representation of Aperiodic CT Signals: Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the CT Fourier Transform
- Applications of Fourier Transform

- **Basic Idea:**
  - To represent signals as linear combinations of basic signals $\phi_i(t)$

- **Key Properties:**

  1. The set of basic signals can be used to construct a broad and useful class of signals $x(t)$

  2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signals constructed as linear combination of basic signals

- One of Choices:
  - The set of complex exponential signals

signals of form e^{st} in CT

- The Response of an LTI System:

input $\longrightarrow$ LTI $\longrightarrow$ output

$x(t)$ $\quad h(t) \quad$ $y(t)$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

CT: $\quad e^{st} \longrightarrow H(s)e^{st}$

eigenfunction

eigenvalue

Let $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \quad = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$\Rightarrow y(t) = H(s)x(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Eigenfunctions and Superposition Properties:

$$e^{s_k t} \rightarrow \boxed{\text{LTI}} \rightarrow H(s_k)\, e^{s_k t}$$

$$a_1 \; e^{s_1 t} \longrightarrow a_1 \; H(s_1) \; e^{s_1 t}$$

$$a_2 \; e^{s_2 t} \longrightarrow a_2 \; H(s_2) \; e^{s_2 t}$$

$$a_3 \; e^{s_3 t} \longrightarrow a_3 \; H(s_3) \; e^{s_3 t}$$

$$x(t) = a_1 \, e^{s_1 t} + a_2 \, e^{s_2 t} + a_3 \, e^{s_3 t}$$

$$y(t) = a_1 \, H(s_1) \, e^{s_1 t} + a_2 \, H(s_2) \, e^{s_2 t} + a_3 \, H(s_3) \, e^{s_3 t}$$

$$\Rightarrow \; x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

- Harmonically related complex exponentials

$$w_0 = \frac{2\pi}{T}$$

$$\phi_k(t) = e^{jkw_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, \qquad k = 0, \pm 1, \pm 2, ...$$

- The Fourier Series Representation:

$$x(t) = \cdots a_{-2}\,\phi_{-2}(t) + a_{-1}\,\phi_{-1}(t) + a_0\,\phi_0(t) + a_1\,\phi_1(t) + a_2\,\phi_2(t) + \ldots$$

$$= \sum_{k=-\infty}^{+\infty} a_k\,\phi_k(t) = \sum_{k=-\infty}^{+\infty} a_k\,e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k\,e^{jk\left(\frac{2\pi}{T}\right)t}$$

$k = +1, -1$: the first harmonic components
or, the fundamental components

$k = +2, -2$: the second harmonic components

$\cdots$ etc.

Example 3.2:

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk(2\pi)t}$$

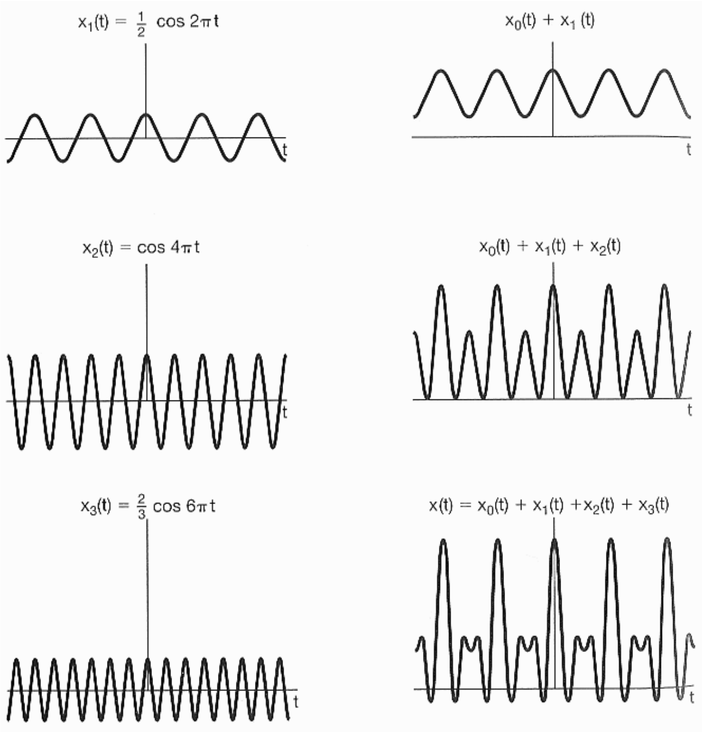
$$a_0 = 1$$
$$a_1 = a_{-1} = \frac{1}{4}$$
$$a_2 = a_{-2} = \frac{1}{2}$$
$$a_3 = a_{-3} = \frac{1}{3}$$

$$\Rightarrow x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})$$
$$+ \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

$$\Rightarrow x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$
$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$



- Procedure of Determining the Coefficients:

$$w_0 = \frac{2\pi}{T}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t)e^{-jnw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t}$$

$$\int_0^T x(t)e^{-jnw_0 t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t} dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0 t} dt \right]$$

$$\int_0^T e^{j(k-n)w_0 t} dt = \int_0^T \cos\left((k-n)w_0 t\right)\ dt + j\int_0^T \sin\left((k-n)w_0 t\right)\ dt$$

- Procedure of Determining the Coefficients:

$$\int_0^T e^{j(k-n)w_0 t}dt = \int_0^T \cos\left((k-n)w_0 t\right)\ dt + j\int_0^T \sin\left((k-n)w_0 t\right)\ dt$$

$$= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$\Rightarrow \int_0^T x(t)e^{-jnw_0 t}dt = a_n T \qquad \Rightarrow a_n = \frac{1}{T}\int_0^T x(t)e^{-jnw_0 t}dt$$

$$\Rightarrow a_k = \frac{1}{T}\int_0^T x(t)e^{-jkw_0 t}dt$$

- Furthermore,

$$\int_T e^{j(k-n)w_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \qquad \Rightarrow a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$$

In Summary:

- The synthesis equation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The analysis equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

- $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: CT Fouries series pair

- $\{a_k\}$: the Fourier series coefficients
  or the spectral coefficients of $x(t)$

- $a_0 = \frac{1}{T}\int_T x(t)dt$, the dc or constant component of $x(t)$

Fourier Series of Real Periodic Signals:

- If $x(t)$ is real, then $x^*(t) = x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$(a+b)^* = (a^*+b^*)$$
$$(a\times b)^* = (a^*\times b^*)$$

$$\Rightarrow x(t) = x(t)^* = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* e^{-jkw_0 t}$$

$$= \sum_{m=+\infty}^{-\infty} {a_{-m}}^* e^{jmw_0 t} \qquad m = -k$$

$$= \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jkw_0 t} \qquad k = m$$

$$\Rightarrow a_{-k}^* = a_k \quad \text{or,} \quad a_k^* = a_{-k}$$

$x(t) \in \mathbb{R}$ فقط برای

Alternative Forms of the Fourier Series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$\Rightarrow \; x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jkw_0 t} + a_{-k} e^{-jkw_0 t} \right]$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jkw_0 t} + a_k^* e^{-jkw_0 t} \right]$$

$$a_k e^{jkw_0 t} + a_k^* e^{-jkw_0 t} = (R+jI)(C+jS) + (R-jI)(C-jS)$$

$$= (RC-IS) + j(RS+IC) + (RC-IS) - j(RS+IC)$$

$$= 2(RC - IS)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\, \mathcal{R}e \left\{ a_k e^{jkw_0 t} \right\}$$

Alternative Forms of the Fourier Series:

- If $a_k = A_k \, e^{j\theta_k}$

$$\Rightarrow \; x(t) = a_0 + \sum_{k=1}^{\infty} 2 \, \mathcal{R}e\left\{A_k \, e^{j\theta_k} \, e^{jkw_0 t}\right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2 \, \mathcal{R}e\left\{A_k \, e^{j(kw_0 t+\theta_k)}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(kw_0 t + \theta_k)$$

- If $a_k = B_k + j \, C_k$

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

$$\boxed{(a+jb)(c+jd) = (ac-bd) + j(ad+bc)}$$

$$\Rightarrow \; x(t) = a_0 + \sum_{k=1}^{\infty} 2 \, \mathcal{R}e\left\{\left(B_k + j \, C_k\right) \, e^{jkw_0 t}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos(kw_0 t) - C_k \sin(kw_0 t)\right]$$

- Example 3.4:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$e^{j\theta} = \cos(\theta)+j\sin(\theta); \quad \cos(\theta) = \frac{1}{2}(e^{j\theta}+e^{-j\theta}); \quad \sin(\theta) = \frac{1}{2j}(e^{j\theta}-e^{-j\theta})$$

$$x(t) = 1 + \sin w_0 t + 2\cos w_0 t + \cos\left(2w_0 t + \frac{\pi}{4}\right)$$

$$\Rightarrow\ x(t) = 1+\frac{1}{2j}\left[e^{jw_0 t} - e^{-jw_0 t}\right]+\left[e^{jw_0 t} + e^{-jw_0 t}\right]$$

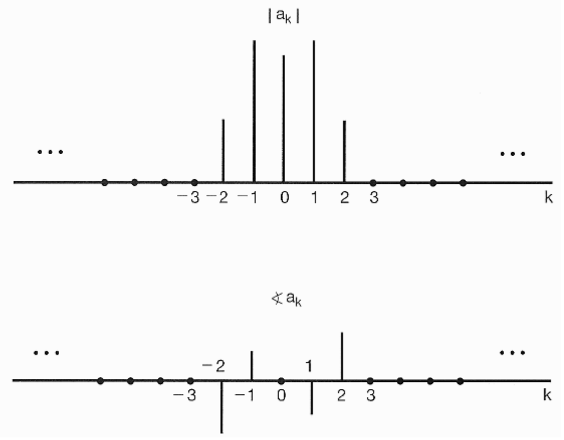
$$+\frac{1}{2}\left[e^{j(2w_0 t+\pi/4)} + e^{-j(2w_0 t+\pi/4)}\right]$$

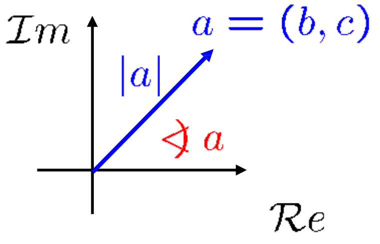
$$\Rightarrow\ x(t) = 1 + \left(1+\frac{1}{2j}\right)e^{jw_0 t} + \left(1-\frac{1}{2j}\right)e^{-jw_0 t}$$

$$+\left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2w_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2w_0 t}$$

Example 3.4:

$$\Rightarrow \begin{cases} a_0 &= 1, \\ a_1 &= \left(1+\frac{1}{2j}\right) = 1-\frac{1}{2}j, \\ a_{-1} &= \left(1-\frac{1}{2j}\right) = 1+\frac{1}{2}j, \\ a_2 &= \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1+j), \\ a_{-2} &= \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j), \\ a_k &= 0, \quad |k|>2. \end{cases}$$





$$\boxed{a = |a|e^{j\sphericalangle a}}$$

$$\boxed{a = |a|\left[\cos(\sphericalangle a) + j\sin(\sphericalangle a)\right]}$$

$$\boxed{a = b + jc = \sqrt{b^2+c^2}\left[\frac{b}{\sqrt{b^2+c^2}} + j\frac{c}{\sqrt{b^2+c^2}}\right]}$$

Convergence of the Fourier Series

- Fourier maintained that
  "any" periodic signal could be represented by a
  Fourier series
- The truth is that
  Fourier series can be used to represent
  an extremely large class of periodic signals
- The question is that
  when a periodic signal x(t) does in fact have a Fourier
  series representation?

$$x(t)$$

$$x_{FS}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk(2\pi/T)t}$$

Convergence of the Fourier Series

- One class of periodic signals:
  - Which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \qquad \Rightarrow \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt < \infty$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$

$$e_N(t) = x(t) - x_N(t) \qquad\qquad e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

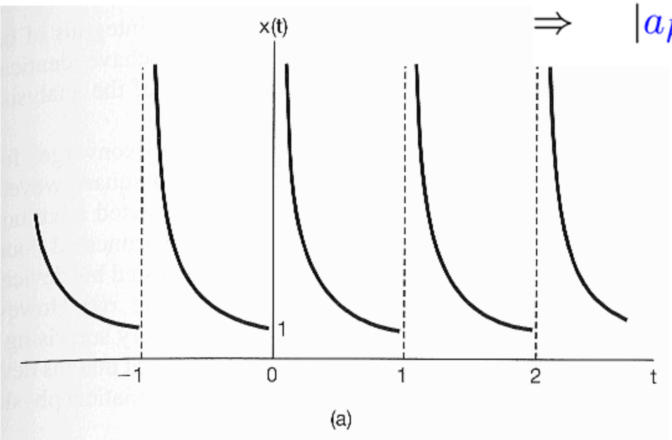
$$E_N(t) = \int_T |e_N(t)|^2 \, dt \qquad\qquad E(t) = \int_T |e(t)|^2 \, dt \;=\; 0$$

$$\rightarrow \; 0 \quad \text{as } N \rightarrow \infty \qquad\qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}, \quad \forall t \text{ ???}$$

Convergence of the Fourier Series

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 1:
    - Over any period, x(t) must be absolutely integrable, i.e.,

$$\int_T |x(t)|\, dt < \infty$$



(a)

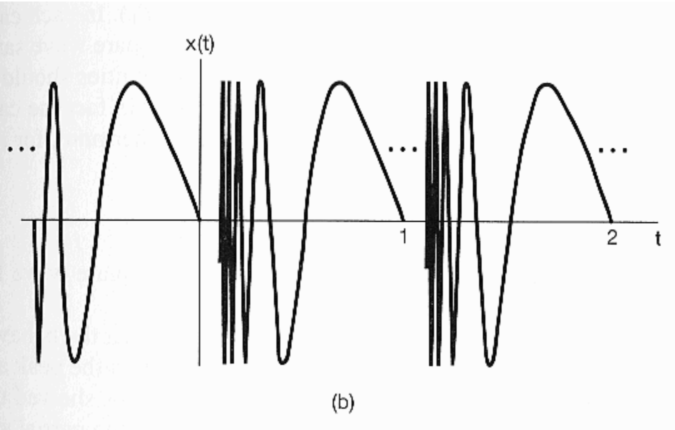
$$\Rightarrow \quad |a_k| \;\le\; \frac{1}{T}\int_T \left| x(t) e^{-jkw_0 t} \right| dt$$

$$= \frac{1}{T}\int_T \left| x(t) \right| dt \;<\; \infty$$

$$x(t) = \frac{1}{t}, \quad 0 < t \le 1$$

Convergence of the Fourier Series

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 2:
    - In any finite interval, x(t) is of bounded variation; i.e.,
    - There are no more than
      a finite number of maxima and minima
      during any single period of the signal



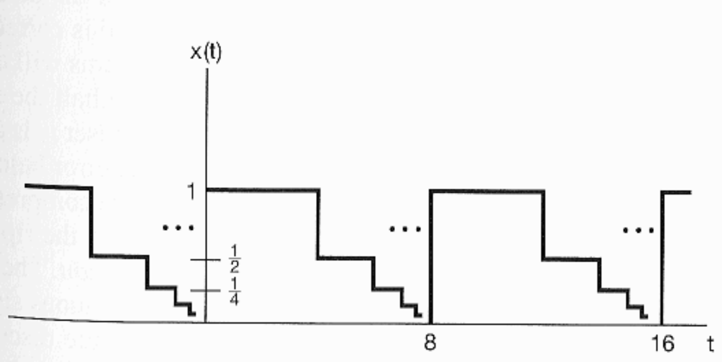
(b)

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1$$

$$\int_0^1 |x(t)|\, dt < 1$$

Convergence of the Fourier Series

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 3:
    - In any finite interval,
      x(t) has only finite number of discontinuities.
    - Furthermore, each of these discontinuities is finite



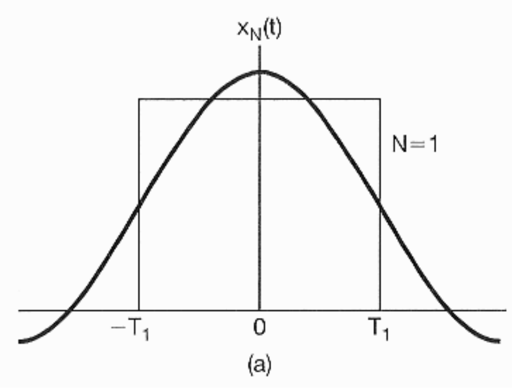
(c)

- How the Fourier series converges for a periodic signal with discontinuities

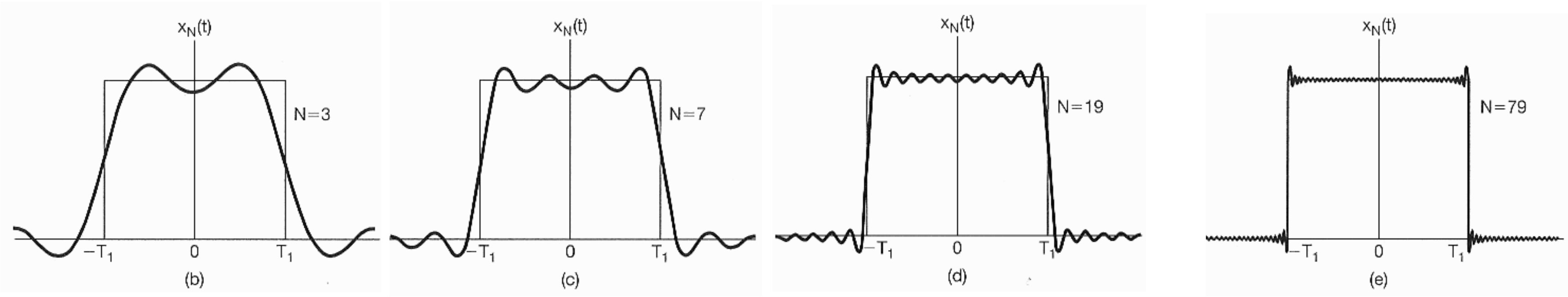
  - In 1898,
    Albert Michelson (an American physicist)
    used his harmonic analyzer
    to compute
    the truncated Fourier series approximation
    for the square wave

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$

$$x_1(t) = a_{-1} e^{-j \cdot 1 \cdot w_0 t} + a_0 + a_1 e^{j \cdot 1 \cdot w_0 t}$$

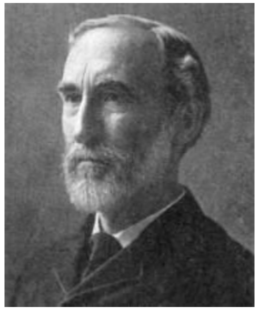


• The Gibbs phenomenon



$$x_N(T_1) = \frac{x(T_1^-) + x(T_1^+)}{2}$$

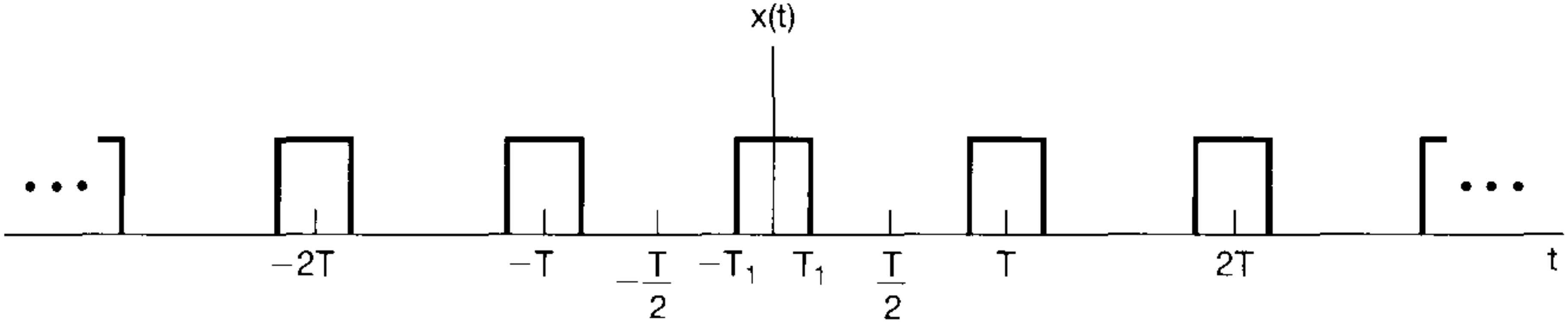
- In 1899, Gibbs showed that
  - – the partial sum near discontinuity exhibits ripples &
  - – the peak amplitude remains constant with increasing N



Josiah Willard Gibbs
1839-1903

- Example 3.5:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



$k = 0 \qquad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

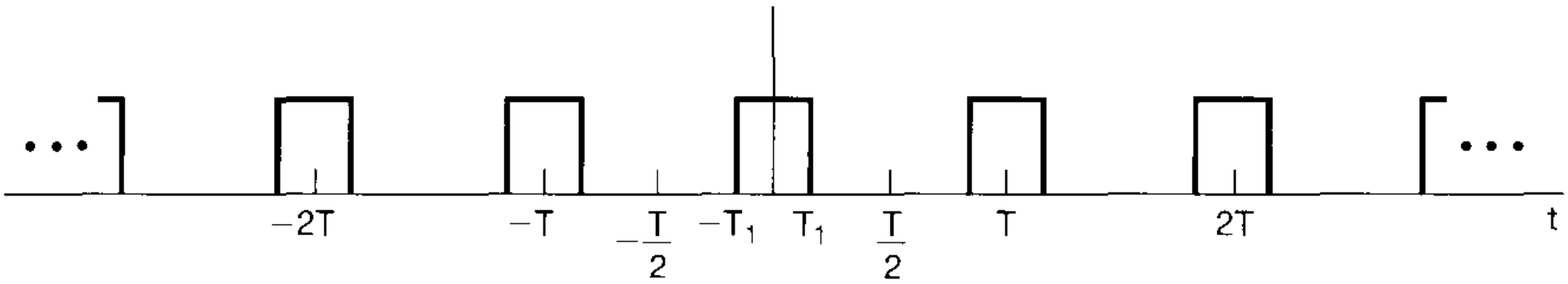
$k \neq 0 \qquad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t}dt = \frac{1}{T}\frac{1}{(-jkw_0)}e^{-jkw_0 t}\Big|_{-T_1}^{T_1} \qquad w_0 = \frac{2\pi}{T}$

$$= \frac{1}{jkw_0 T}\left[e^{jkw_0 T_1} - e^{-jkw_0 T_1}\right] /$$

$$= \frac{2\sin(kw_0 T_1)}{kw_0 T} = \frac{\sin(kw_0 T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

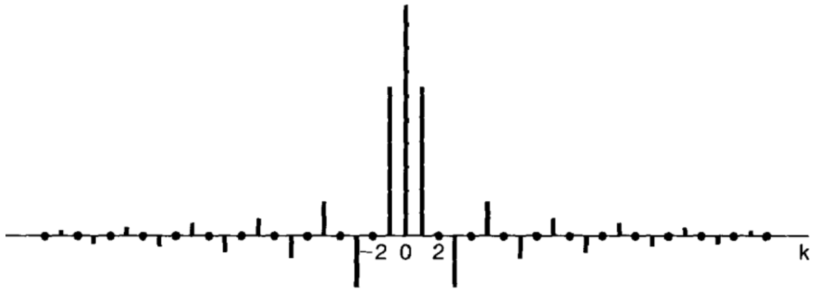
- Example 3.5: $\boxed{T = 4T_1}$

$\text{sinc}\,\theta = \frac{\sin \pi\theta}{\pi\theta}$



$$a_k = \frac{\sin(k 2\pi \frac{T_1}{T})}{k\pi} = \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

$$T\ a_k = T\ \frac{\sin(k 2\pi \frac{T_1}{T})}{k\pi} = T\ \frac{\sin(k\frac{\pi}{2})}{k\pi}$$



$$a_0 = \frac{1}{2}.$$

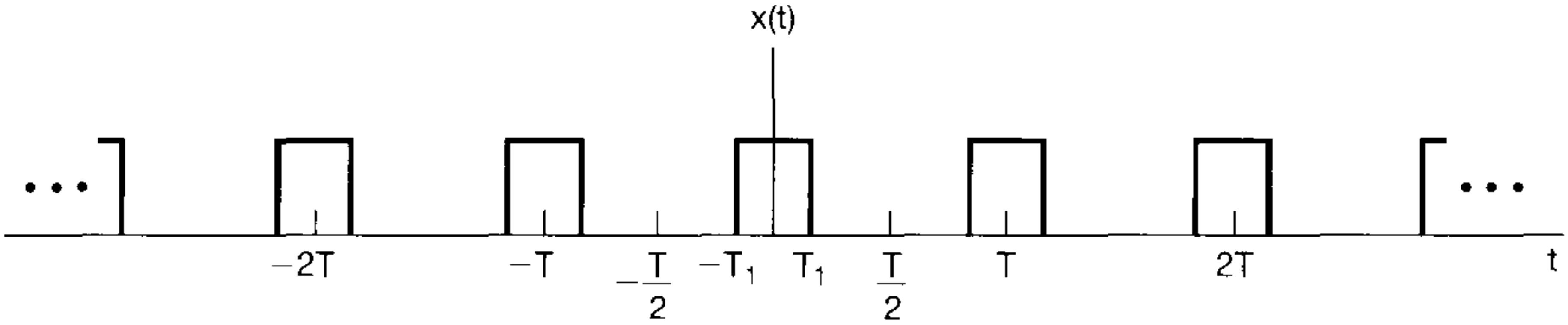
$$a_1 = a_{-1} = \frac{1}{\pi},$$

$$a_3 = a_{-3} = -\frac{1}{3\pi},$$

$$a_5 = a_{-5} = \frac{1}{5\pi},$$

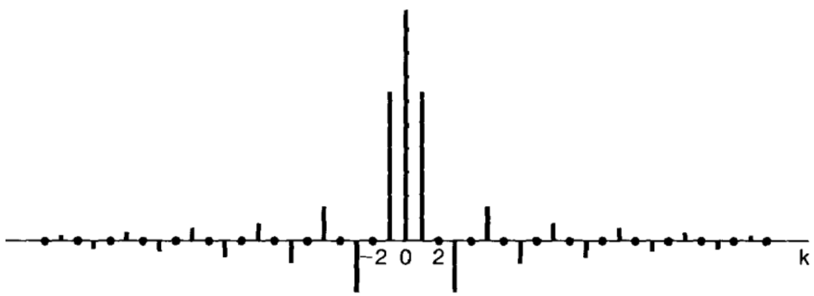
$$\vdots$$

- Example 3.5: $T = 4T_1$



$$a_k = \frac{\sin(k2\pi\frac{T_1}{T})}{k\pi} = \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

$$T\ a_k = T\ \frac{\sin(k2\pi\frac{T_1}{T})}{k\pi} = T\ \frac{\sin(k\frac{\pi}{2})}{k\pi}$$



$$a_0 = \frac{1}{2}.$$

$$a_1 = a_{-1} = \frac{1}{\pi},$$

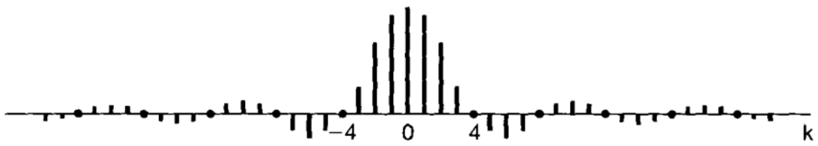
$$a_3 = a_{-3} = -\frac{1}{3\pi},$$

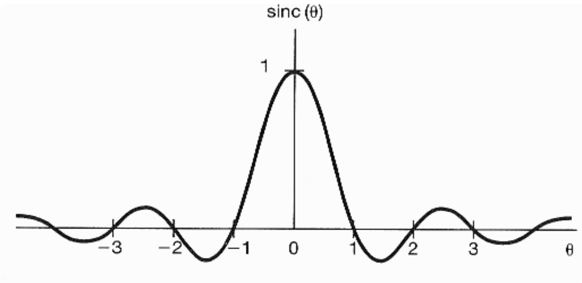
$$a_5 = a_{-5} = \frac{1}{5\pi},$$

$$\vdots$$

$T = 8T_1$

Ta_k



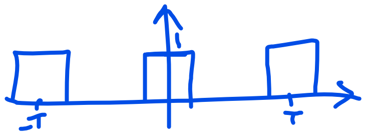


$$Ta_k = T\,\frac{\sin(k\frac{\pi}{4})}{k\pi}$$

$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$= \frac{1}{4}\;T\;\frac{\sin(\pi\frac{k}{4})}{\pi\frac{k}{4}}$$

$$= \frac{1}{4}\;T\;\mathrm{sinc}(\frac{k}{4})$$



$a_K = d\,\mathrm{sinc}(Kd)$

$d =$ duty cycle

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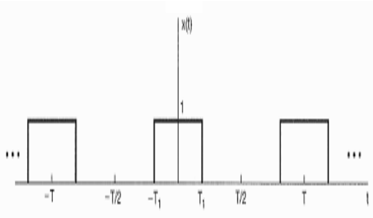
CT Fourier Series Representation:

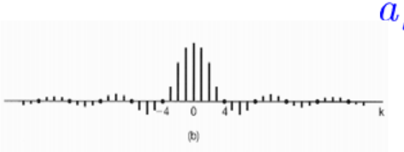


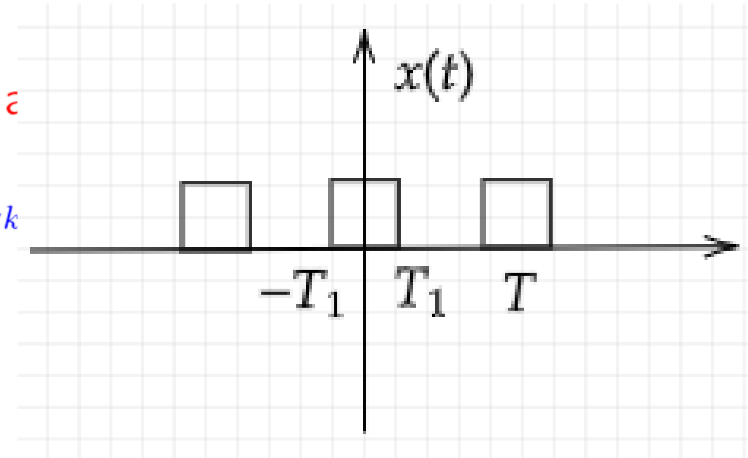
- The synthesis equation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The a

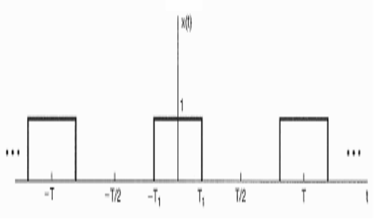
a_k ... $\pi/T)t\,dt$





- $x(t) \longleftrightarrow a_k$: Fourier series pair

CT Fourier Series Representation:

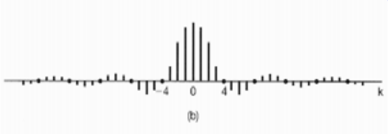


- The synthesis equation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The analysis equation:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$



- $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: Fouries series pair

Section	Property
3.5.1	Linearity
3.5.2	Time Shifting
	Frequency Shifting
3.5.6	Conjugation
3.5.3	Time Reversal
3.5.4	Time Scaling
	Periodic Convolution
3.5.5	Multiplication
	Differentiation
	Integration
3.5.6	Conjugate Symmetry for Real Signals
3.5.6	Symmetry for Real and Even Signals
3.5.6	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.5.7	Parseval's Relation for Periodic Signals

- Linearity:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0t}$$

$$\Rightarrow \quad z(t) = Ax(t)+By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k+Bb_k$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0t}$$

- Time Shifting:
  - $x(t)$: periodic signal with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x(t-t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 t_0} a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k$$

b/c $$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-jkw_0 t}dt$$

$$= \frac{1}{T}\int_T x(\tau)e^{-jkw_0(\tau+t_0)}d\tau$$

$$= e^{-jkw_0 t_0}\ \frac{1}{T}\int_T x(\tau)e^{-jkw_0\tau}d\tau$$

- Time Reversal:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$\Rightarrow x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k} \qquad x(-t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{-j\,k\left(\frac{2\pi}{T}\right)t}$$

$$= \sum_{m=-\infty}^{+\infty} a_{-m} \, e^{j\,m\left(\frac{2\pi}{T}\right)t}$$

- If $x(t)$ is even, i.e., $x(-t) = x(t)$

  $\Rightarrow a_k$ is even, i.e., $a_{-k} = a_k$

- If $x(t)$ is odd, i.e., $x(-t) = -x(t)$

  $\Rightarrow a_k$ is odd, i.e., $a_{-k} = -a_k$

- Time Scaling:

  - $x(t)$: periodic signals with period T
    and fundamental frequency w_0
  - $x(\alpha t)$: periodic signals with period $\frac{T}{\alpha}$
    and fundamental frequency αw_0

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\,w_0\,(\alpha\,t)} = \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\,\alpha\left(\frac{2\pi}{T}\right)t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\,(\alpha\,w_0)\,t} = \sum_{k=-\infty}^{+\infty} a_k \, e^{j\,k\left(\frac{2\pi}{(\frac{T}{\alpha})}\right)t}$$

- Multiplication:

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}$$

$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$\Rightarrow x(t)y(t)$: also periodic with T

$$z(t) = x(t)y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

- **Differentiation:**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

  - $x(t)$: periodic signals with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\frac{d}{dt}x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} jkw_0\, a_k$$

- **Integration:**

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{FS}}{\longleftrightarrow} \frac{1}{jkw_0}\, a_k$$

Conjugation & Conjugate Symmetry:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t)^* \overset{\mathcal{FS}}{\longleftrightarrow} {a_{-k}}^*$$

- $x(t) = x(t)^* \Rightarrow a_{-k} = a_k^*$

  $x(t)$ is real $\Rightarrow$ $\{a_k\}$ are conjugate symmetric

- $x(t) = x(t)^*$ & $x(-t) = x(t) \Rightarrow a_{-k} = a_k^*$ & $a_{-k} = a_k$

  $x(t)$ is real & even $\Rightarrow$ $\{a_k\}$ are real & even $\Rightarrow a_k = a_k^*$

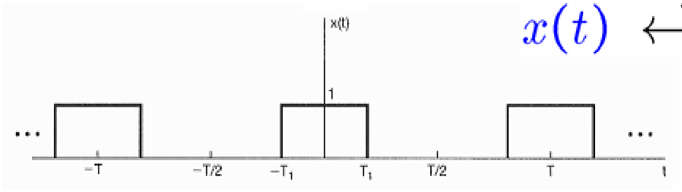
- $x(t)$ is real & odd $\Rightarrow$ $\{a_k\}$ are purely imaginary & odd

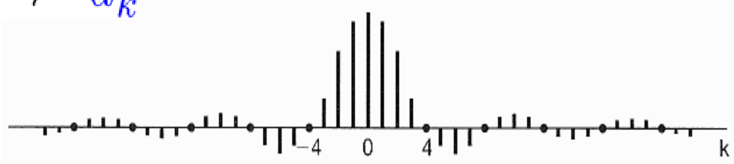
  $\Rightarrow a_k^* = -a_k$

Parseval's relation for CT periodic signals:

- As shown in Problem 3.46:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$





$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$

$$\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|a_k\right|^2$$

- Parseval's relation states that
  the total average power in a periodic signal equals
  the sum of the average powers in all of its harmonic components

Properties of CT Fourier Series

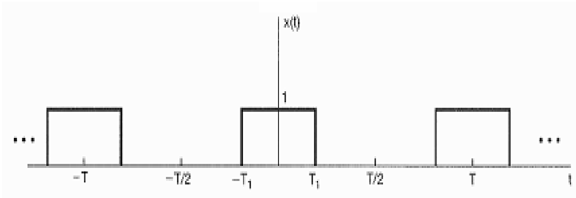
TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

- **Example 3.6:**

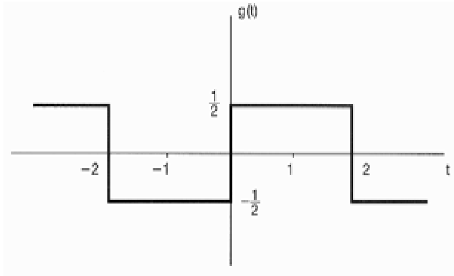


$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$a_0 = \frac{2T_1}{T}$$

$$a_k = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$



$$g(t) = x(t-1) - 1/2$$

$$x(t-1) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_k e^{-jk\pi/2}$$

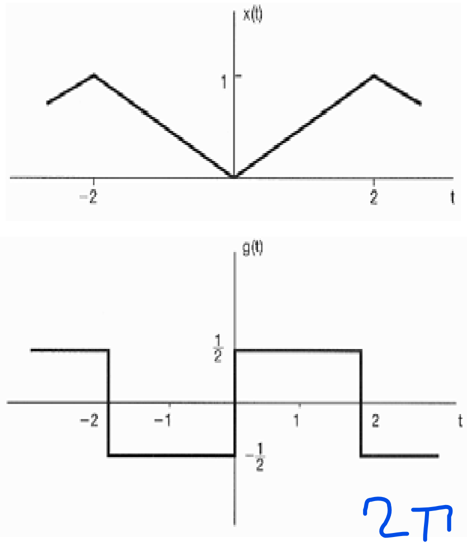
$$g(t) = x(t-1) - 1/2 \overset{\mathcal{FS}}{\longleftrightarrow} \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - 1/2, & \text{for } k = 0 \end{cases}$$

with $T = 4, T_1 = 1$

$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} \begin{cases} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

- Example 3.7:

$T = 4$



$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} e_k$$

$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} d_k$$

$$\frac{d}{dt}x(t) \overset{\mathcal{FS}}{\longleftrightarrow} jkw_0\, e_k$$

$\frac{2\pi}{4}$

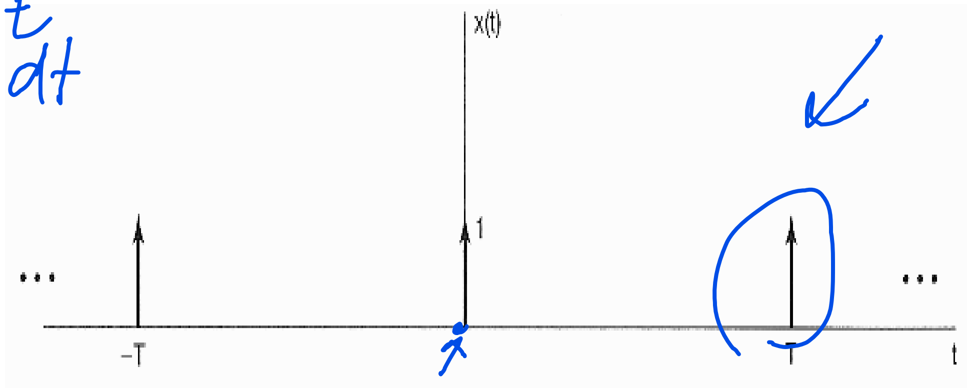
$$g(t) = \frac{d}{dt}x(t) \iff d_k = jk(\pi/2)e_k$$

$$e_k = \begin{cases} \frac{2}{jk\pi}\, d_k = \frac{2\sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

Impulse Train

$$a_k = \frac{1}{T}\int_{T} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\frac{1}{T}\int x(0)\, e^{0}\, dt$$



(a)

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT);$$

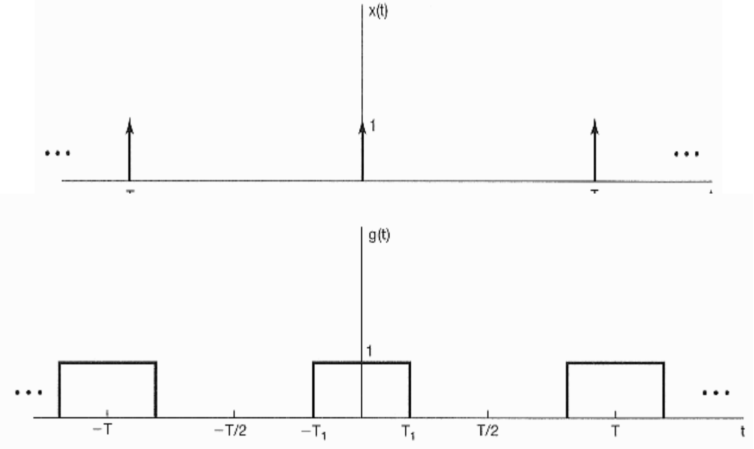
-3 3

2

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk2\pi t/T}\, dt = \frac{1}{T}.$$

$-2\delta(t-1)$

- Example 3.8:

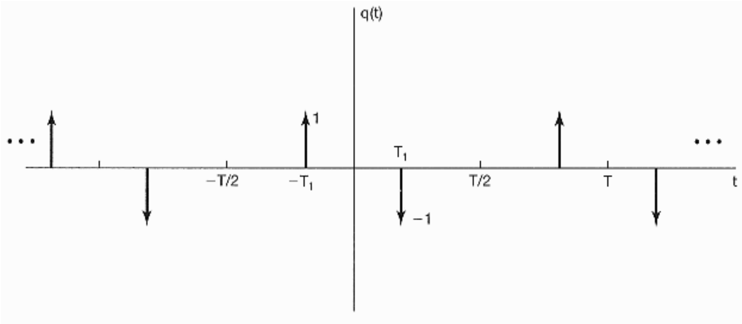


(b)

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$$ ← آنالیز

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k = \frac{1}{T}$$

$$g(t) \xleftrightarrow{\mathcal{FS}} c_k$$



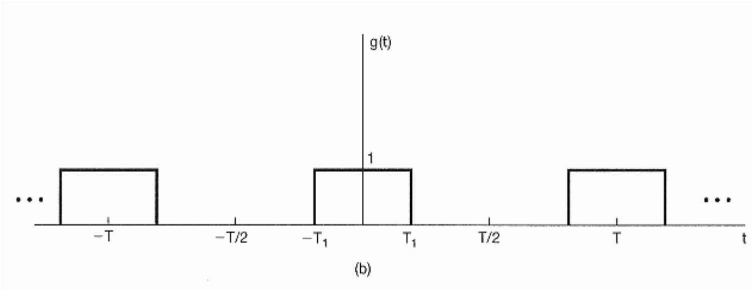
(c)

$$q(t) = \frac{d}{dt}g(t) \Longleftrightarrow b_k = jkw_0c_k$$

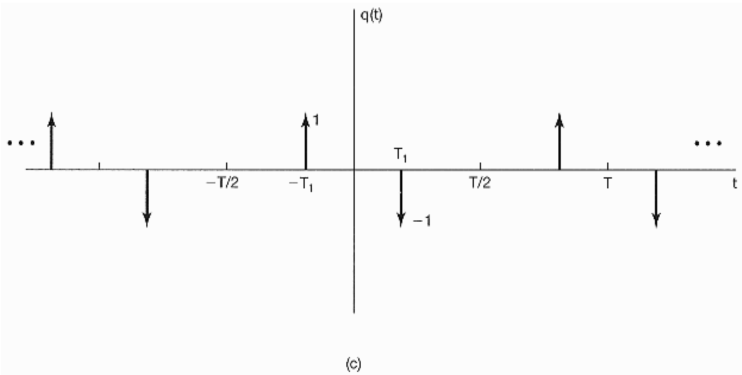
$$q(t) = x(t+T_1) - x(t-T_1) \Longleftrightarrow b_k = e^{jkw_0T_1}a_k - e^{-jkw_0T_1}a_k = \frac{1}{T}\left[e^{jkw_0T_1} - e^{-jkw_0T_1}\right]$$

$$= \frac{2j\sin(kw_0T_1)}{T}$$

- Example 3.8:



$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$



$$q(t) = \frac{d}{dt} g(t) \Longleftrightarrow b_k = jkw_0 c_k$$

$$q(t) = x(t + T_1) - x(t - T_1) \Longleftrightarrow b_k = e^{jkw_0T_1} a_k - e^{-jkw_0T_1} a_k = \frac{1}{T}\left[e^{jkw_0T_1} - e^{-jkw_0T_1}\right]$$

$$= \frac{2j\sin(kw_0T_1)}{T}$$

$$k \neq 0 \qquad c_k = \frac{b_k}{jkw_0} = \frac{2j\sin(kw_0T_1)}{jkw_0T} = \frac{\sin(kw_0T_1)}{k\pi}$$

$$k = 0 \qquad c_0 = \frac{2T_1}{T}$$

- Example 3.9:

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal. $\rightarrow x(t) = x^*(t) \rightarrow a_k = a^*_{-k} \Rightarrow \boxed{a_1 = a^*_{-1}}$
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k. $\rightarrow \omega = \pi/2$
3. $a_k = 0$ for $|k| > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd. $\rightarrow$ odd $\rightarrow x(t-1) = -x(-t+1)$
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$. $\rightarrow$ Parseval $|a_0|^2 + |a_{-1}|^2 + |a_1|^2 = 1/2$

$$-e^{-j\pi/2} a_{-1} = e^{j\pi/2} a_1 \qquad \frac{a_1}{a_{-1}} = -e^{-j\pi} = -(\cos\pi - j\sin\pi) \rightarrow \boxed{a_1 = -a_1}$$

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}.$$

$$\boxed{|a_0|^2 + 2|a_1|^2 = 1/2}$$

$\{a_{-1}\} = \mathrm{Re}\{a_{-1}\}$

$$x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}.$$

Real

$j, -j$

- Example 3.9:

$a_k = 0$ for $|k| > 1$ $\Rightarrow$ $x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}.$

$x(t)$ is a real signal. $\Rightarrow$ $x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}.$

$b_k = e^{-j\pi k/2} a_{-k}$ $\Rightarrow$

coefficients b_k correspond to the signal $x(-(t-1)) = x(-t+1)$.

, $b_0 = 0$ and $b_{-1} = -b_1$

$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = 1/2.$

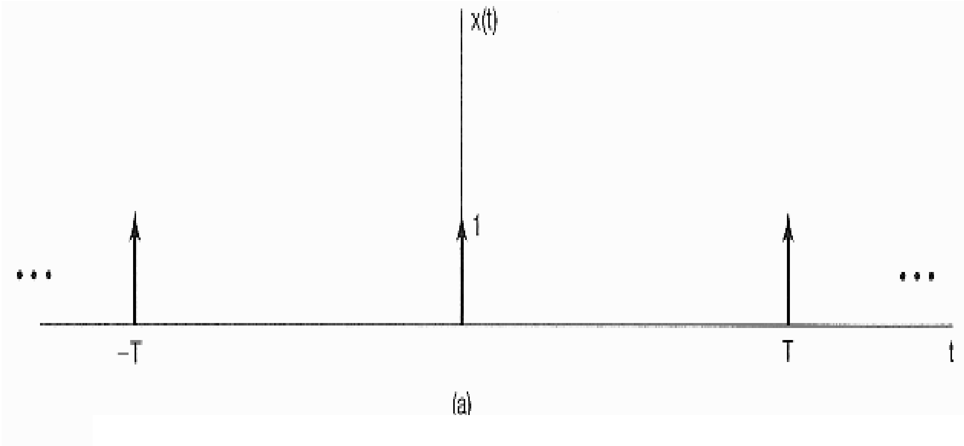
$\Rightarrow$ $|b_1|^2 + |b_{-1}|^2 = 1/2.$ $b_1 = -b_{-1}$ $\Rightarrow$ $|b_1| = 1/2$

$b_1 = j/2$ or $-j/2.$ $b_0 = 0.$ $\Rightarrow$ $a_1 = e^{-j\pi/2} b_{-1} = -jb_{-1} = jb_1$

$b_1 = j/2$ $\Rightarrow$ $x(t) = -\cos(\pi t/2)$

$b_1 = -j/2$ $\Rightarrow$ $x(t) = \cos(\pi t/2).$

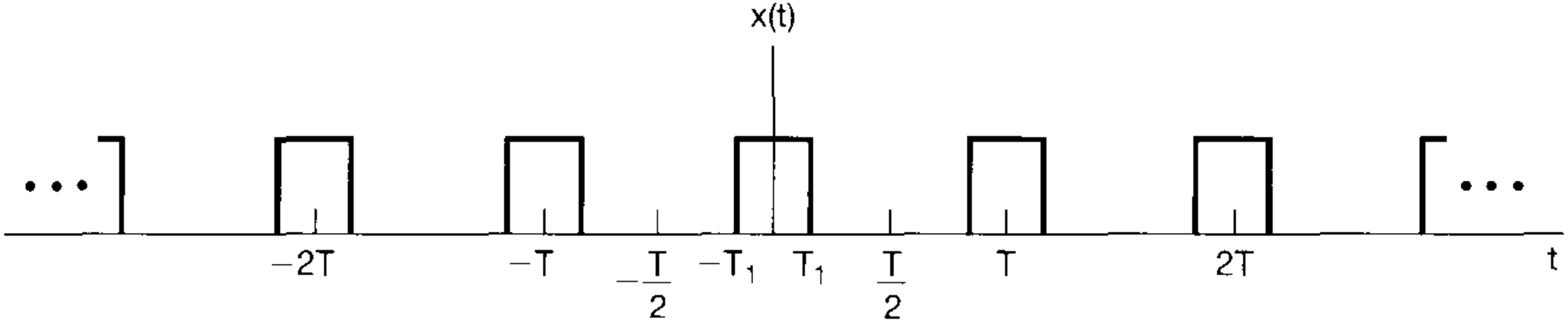
Impulse Train

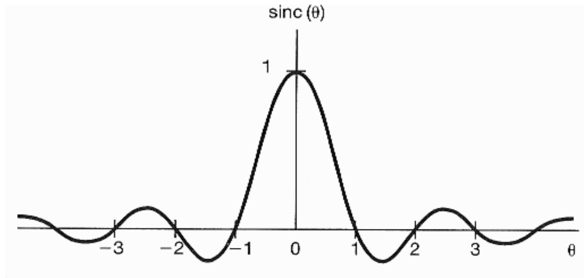


$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT);$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi t/T} dt = \frac{1}{T}.$$

- Example 3.5:





$$a_k = \frac{\sin(2k\pi T_1/T)}{k\pi} \qquad k \neq 0$$

$$d = \frac{2T_1}{T} = \frac{\text{duration of being 1}}{\text{period}}$$

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$a_0 = d$$

$$a_k = d\,sinc(kd) \qquad k \neq 0$$

- Example

$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \dfrac{\sin k\pi/4}{k\pi}, & \text{otherwise} \end{cases}$$

Determine the signal $x(t)$ that is periodic with period 4

$x(t) = x(t-4)$

$(a+bj)(c+dj) = (ac-bd) + (bc+ad)j$

$= a$

$(bc+ad) = b$

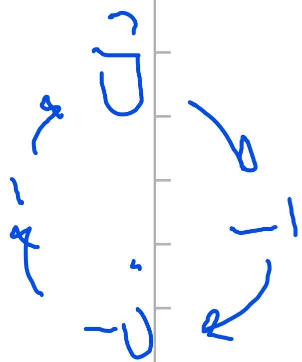
$x(t) \longleftrightarrow a_k$

$x(t-4) \longrightarrow e^{-4j\frac{\pi}{2}k} a_k = \begin{cases} 0 \\ j^k \dfrac{\sin k\pi/4}{\frac{k\pi}{4}} \times \dfrac{1}{4} \end{cases}$

$\cos 2\pi k$

$\begin{cases} \text{Real} \to 1 \\ \text{Im} \to 0 \end{cases}$

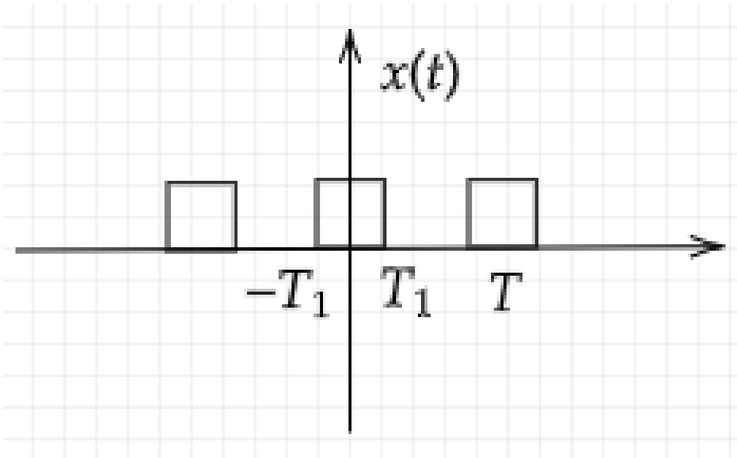
$j^k \boxed{\frac{1}{4}\operatorname{sinc}\left(\frac{k}{4}\right)} e^{-4j\frac{\pi}{2}k}$



- Example

$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \dfrac{\sin k\pi/4}{k\pi}, & \text{otherwise} \end{cases}$$

Determine the signal $x(t)$ that is periodic with period 4



$$\frac{\sin(k\pi/4)}{k\pi} = \frac{1}{4} sinc(k/4) \rightarrow T_1 = \frac{1}{2}, T = 4$$

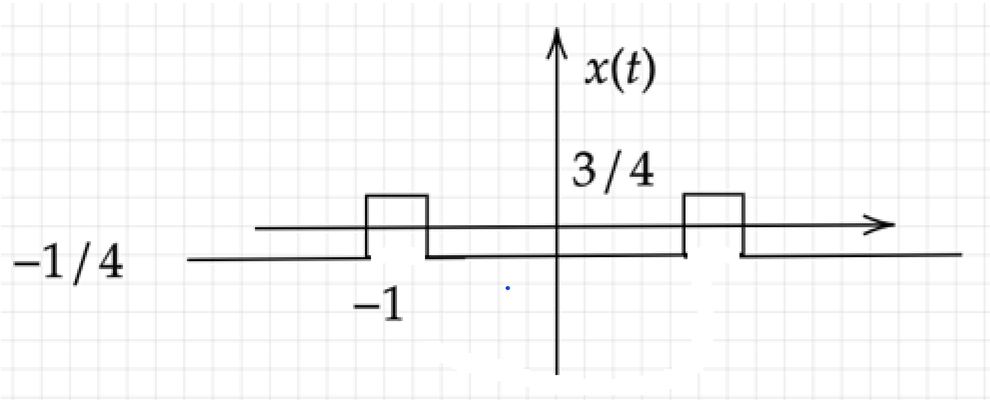
- Example 3.16

$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \dfrac{\sin k\pi/4}{k\pi}, & \text{otherwise} \end{cases}$$

Determine the signal $x(t)$ that is periodic with period 4

$$j^k = e^{kj\pi/2} \Longrightarrow \text{one step shift to the left}$$

$$\frac{\sin(k\pi/4)}{k\pi} = \frac{1}{4} sinc(k/4) \rightarrow T_1 = \frac{1}{2}, T = 4$$



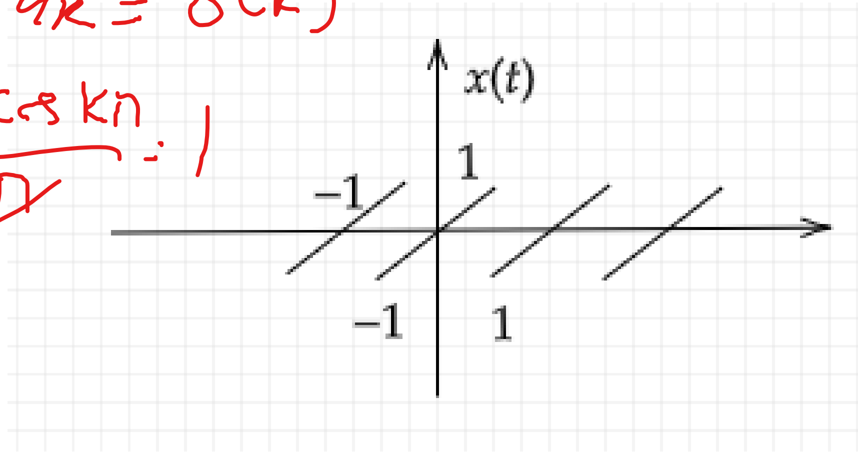
- Example

$\sin c(k)$

$a_k = \delta(k)$

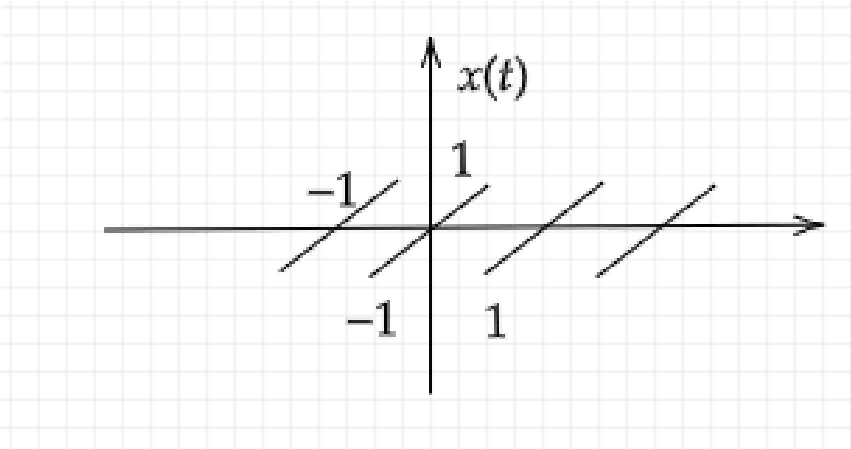
$\lim_{k \to 0} \frac{\sin k\pi}{k\pi} = \lim_{k \to 0} \frac{\pi \cos k\pi}{\pi} = 1$

$-\frac{2}{2} \times e^{-jk\pi k}$



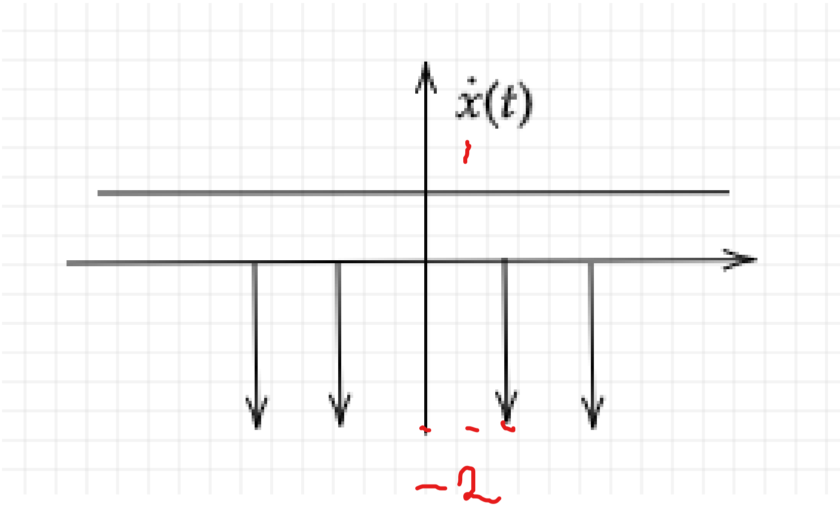
$$x(t) \leftrightarrow a_k$$

- Example



$$x(t) \leftrightarrow a_k$$

$$\dot{x}(t) \leftrightarrow b_k$$

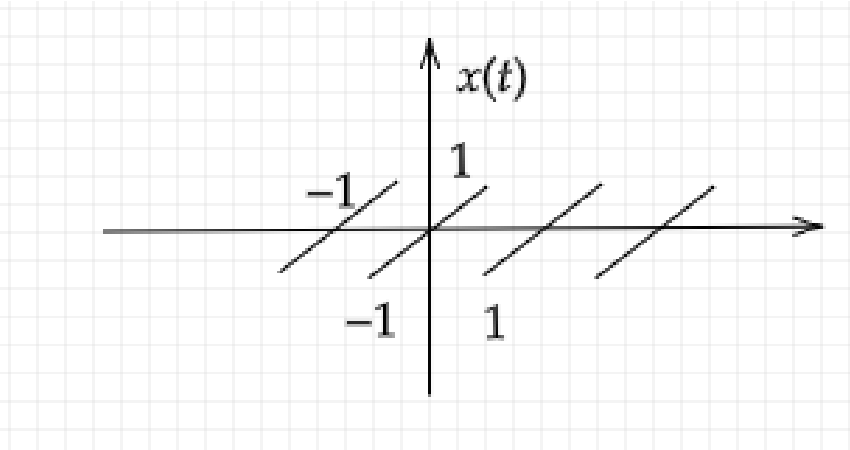


$$\dot{x}(t) = 1 - 2s(t-1)$$

$$b_k = -2e^{-jk2\pi/T}\frac{1}{T}$$

$$\frac{jk2\pi}{T}a_k = -2e^{-jk2\pi/T}\frac{1}{T}$$

- Example



$$x(t) \leftrightarrow a_k$$
$$\dot{x}(t) \leftrightarrow b_k$$

$$jk\pi a_k = -e^{-jk\pi}$$

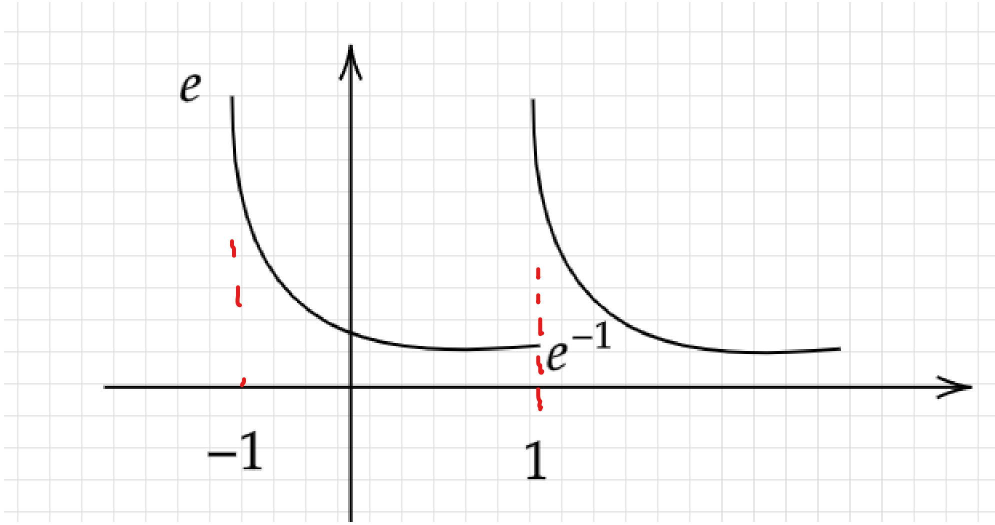
$$a_k = \frac{-e^{-jk\pi}}{jk\pi} = \frac{j(-1)^k}{k\pi} \quad k \neq 0$$

$$a_0 = 0$$

- Example

T

$x(t)$ periodic with period 2 and $x(t) = e^{-t}$ for $-1 < t < 1$

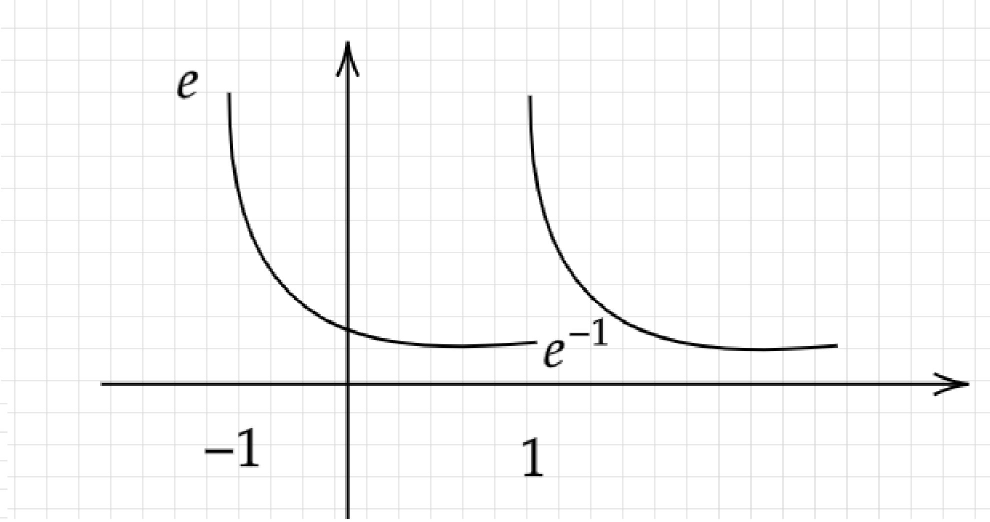


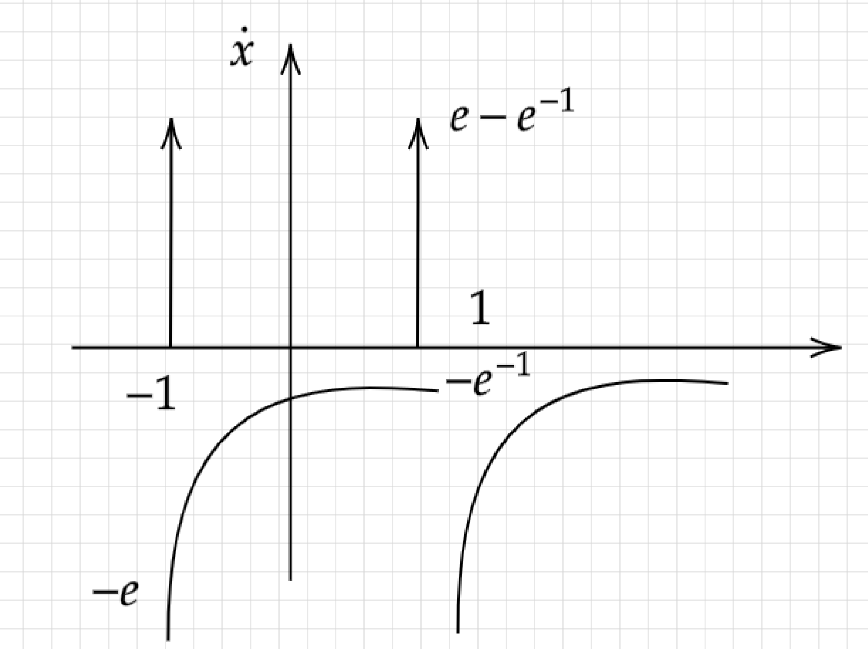
$$a_k = \frac{1}{2}\int_{-1}^{+1} e^{-t}\,dt = \frac{-1}{2}\,e^{-t}\Big]_{-1}^{1} = \frac{-1}{2}\left(e^{-1} - e^{1}\right)$$

$$= \frac{e^{+1} - e^{-1}}{2}$$

- Example

$x(t)$ periodic with period 2 and $x(t) = e^{-t}$ for $-1 < t < 1$





$$\dot{x}(t) = -x(t) + As(t+1)$$

$$A = e - e^{-1}$$

- Example

$x(t)$ periodic with period 2 and $x(t) = e^{-t}$ for $-1 < t < 1$

$$\dot{x}(t) = -x(t) + As(t+1)$$

$$jkw_0 a_k = -a_k + Ae^{jkw_0} \frac{1}{T}$$

$$jk\pi a_k = -a_k + Ae^{jk\pi} \frac{1}{2}$$

$$a_k = \frac{e - e^{-1}}{2} \frac{(-1)^k}{1 + jk\pi}$$

Outline

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous- Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series & LTI Systems
- Representation of Aperiodic CT Signals: Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the CT Fourier Transform
- Applications of Fourier Transform

The Response of an LTI System:

e^{st} $\qquad$ $H(s)e^{st}$

$in \longrightarrow$ LTI $\longrightarrow out$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

$\Rightarrow$ the impulse response

$\Rightarrow$ the system functions

- If $s = jw$

$$H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$$

$\Rightarrow$ the frequency response

$$H = |H| e^{j\sphericalangle H}$$

$$in \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ H(s//w)\end{array}} \longrightarrow out \qquad \text{CT:} \quad e^{s_i t} \longrightarrow H(s_i) e^{s_i t}$$

$$(\; s_i = j w_i$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{jkw_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k \, H(jkw_0) \, e^{jkw_0 t}$$

- <u>In Summary:</u>

$$\begin{cases} x(t) \longleftrightarrow a_k \\ y(t) \longleftrightarrow a_k H(jkw_0) \end{cases}$$

- For a periodic input signal, an LTI system can be uniquely characterized by its frequency response!

- The response of an LTI systems to a periodic signal with a_k Fourier series Coefficients is a periodic signal with $a_k H(jkw_0)$ coefficients.

- Example 3.16

LTI system with impulse response $h(t) = e^{-t}u(t)$.

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

$$\frac{1}{4}\left(e^{2\pi jt} + e^{-2\pi jt}\right) \qquad \frac{1}{2}\left(e^{4\pi jt} + e^{-4\pi jt}\right) \qquad \frac{1}{3}\left(e^{6\pi jt} + e^{-6\pi jt}\right)$$

calculate the Fourier series coefficients of the output $y(t)$

$b_k = ?$

$$a_k = \begin{cases} 1 & k=0 \\ \frac{1}{4} & k=\pm 1 \\ \frac{1}{2} & k=\pm 2 \\ \frac{1}{3} & k=\pm 3 \\ 0 & \text{o.w.} \end{cases}$$

$$H(jk\omega_0) = \int_{-\infty}^{+\infty} e^{-t} u\, e^{-jk\omega_0 t}\, dt$$

$$= \int_0^{\infty} e^{t(-1-jk\omega_0)}\, dt = \boxed{\frac{+1}{1+jk\omega_0}} = H(jk\omega_0)$$

$$\rightarrow b_k = \begin{cases} \text{--} \\ \text{--} \end{cases}$$

- Example 3.16

LTI system with impulse response $h(t) = e^{-t}u(t)$.

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

- Solution $H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$

$$H(j\omega) = \int_0^\infty e^{-\tau}e^{-j\omega\tau}\,d\tau = -\frac{1}{1+j\omega}e^{-\tau}e^{-j\omega\tau}\Big|_0^\infty$$

$$= \frac{1}{1+j\omega}.$$

- Example 3.16
  - Solution

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

$$a_0 = 1, a_1 = a_{-1} = \frac{1}{4}, a_2 = a_{-2} = \frac{1}{2}, a_3 = a_{-3} = \frac{1}{3}.$$

$$H(j\omega) = \frac{1}{1 + j\omega}.$$

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}, \quad b_k = a_k H(jk2\pi)$$

- Example 3.16

$$a_0 = 1, a_1 = a_{-1} = \frac{1}{4}, a_2 = a_{-2} = \frac{1}{2}, a_3 = a_{-3} = \frac{1}{3}.$$

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}, b_k = a_k H(jk2\pi) \quad H(j\omega) = \frac{1}{1 + j\omega}.$$

$$b_0 = 1, b_1 = \frac{1}{4}\left(\frac{1}{1 + j2\pi}\right), \qquad b_{-1} = \frac{1}{4}\left(\frac{1}{1 - j2\pi}\right),$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1 + j4\pi}\right), \qquad b_{-2} = \frac{1}{2}\left(\frac{1}{1 - j4\pi}\right),$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1 + j6\pi}\right), \qquad b_{-3} = \frac{1}{3}\left(\frac{1}{1 - j6\pi}\right).$$

- Example 3.16

$$a_k = A_k e^{j\theta_k},$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k),$$

$$y(t) = 1 + 2\sum_{k=1}^{3} D_k \cos(2\pi kt + \theta_k),$$

$$a_k = B_k + jC_k,$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t],$$

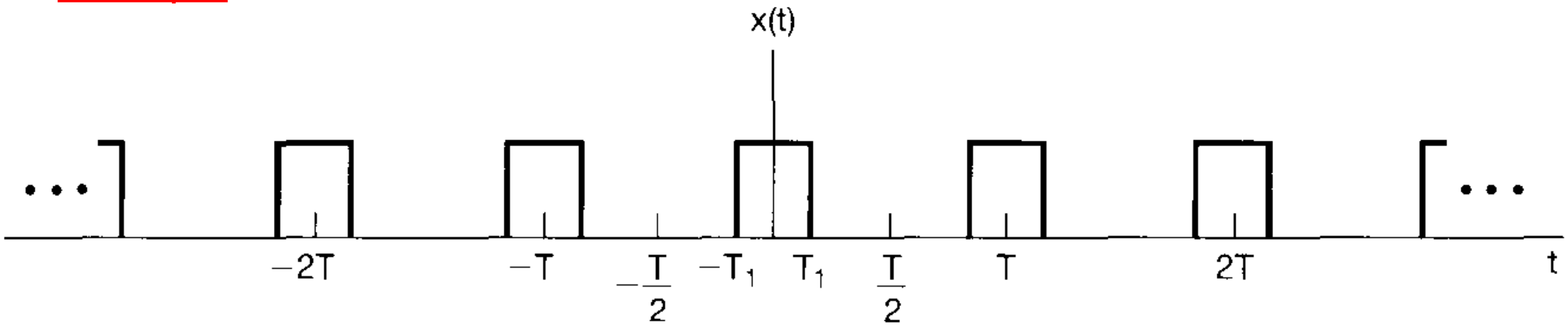
$$y(t) = 1 + 2\sum_{k=1}^{3} [E_k \cos 2\pi kt - F_k \sin 2\pi kt],$$

$$b_k = D_k e^{j\theta_k} = E_k + jF_k, \quad k = 1, 2, 3.$$

- Periodic Convolution

$$x(t) \otimes y(t) := \int_T x(\tau)y(t-\tau)d\tau \qquad c_k = T a_k b_k$$

- Example

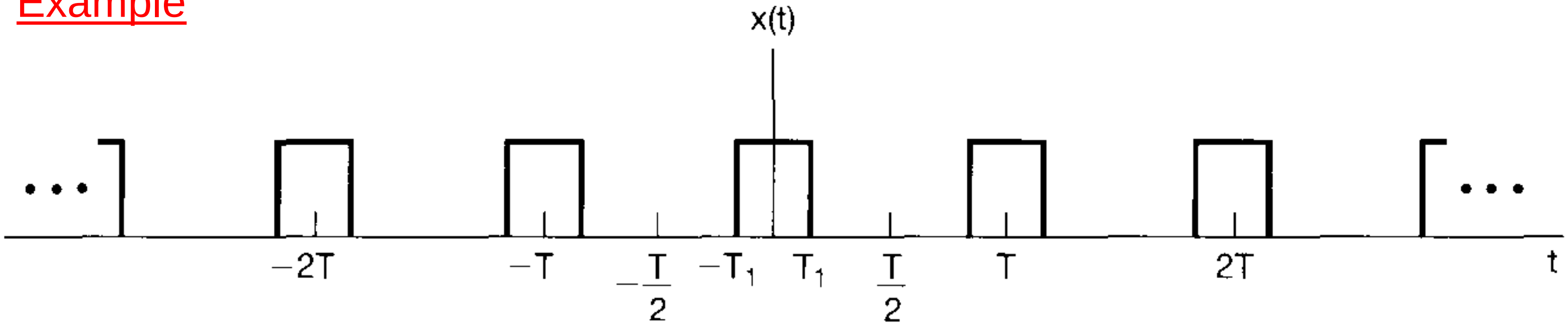


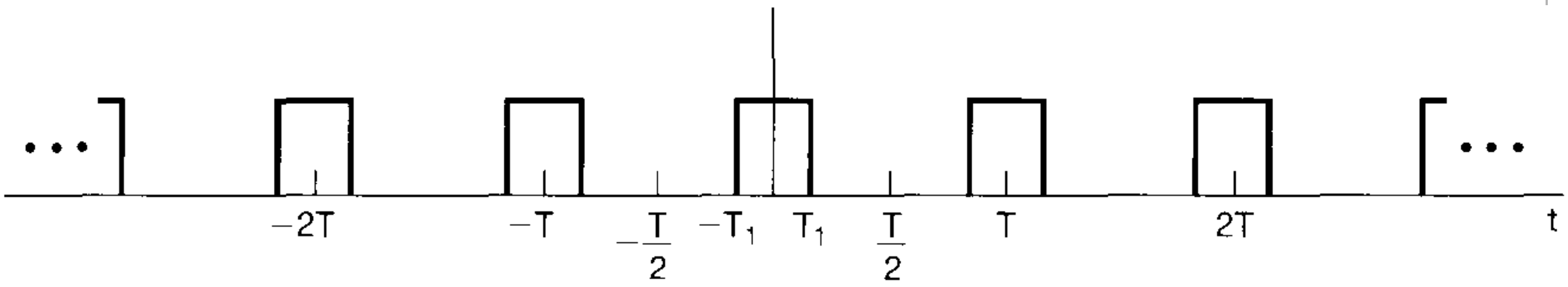
$$x(t) = y(t) \longrightarrow x(t) \otimes y(t) \quad ?$$

$$A) T_1 = T/8$$

$$B) T_1 = 3T/8$$

- Example





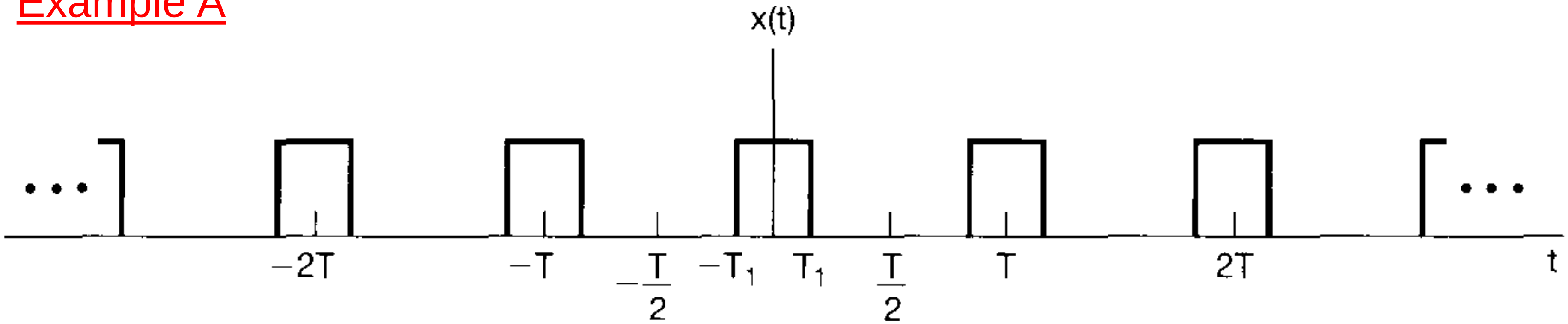
$$-T/2 < t < -T/4 \longrightarrow z(t) = 0$$

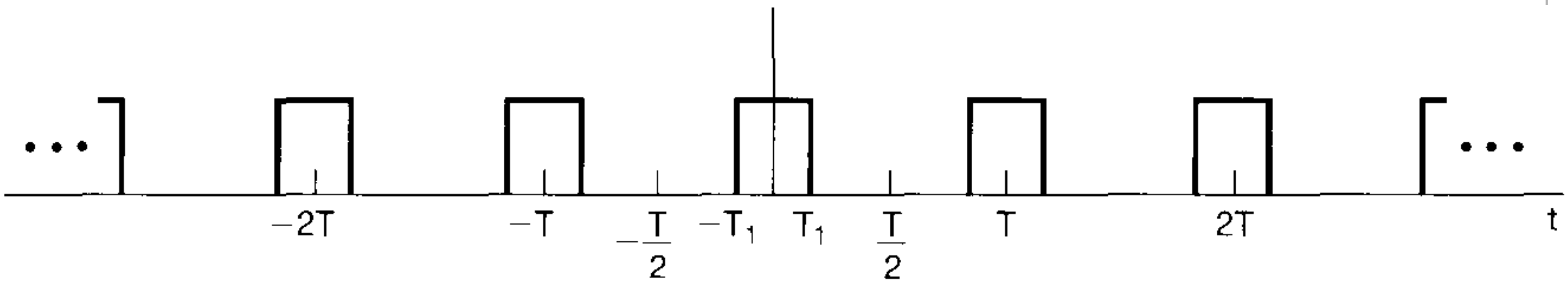
$$-T/4 < t < 0 \longrightarrow z(t) = (t + T/4)$$

$$0 < t < T/4 \longrightarrow z(t) = (-t + T/4)$$

$$T/4 < t < T/2 \longrightarrow z(t) = 0$$

- Example A





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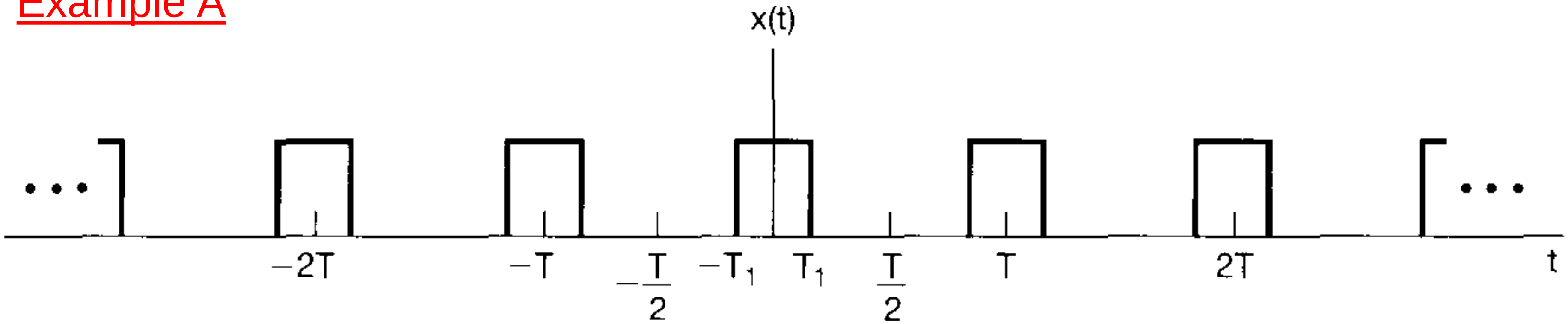

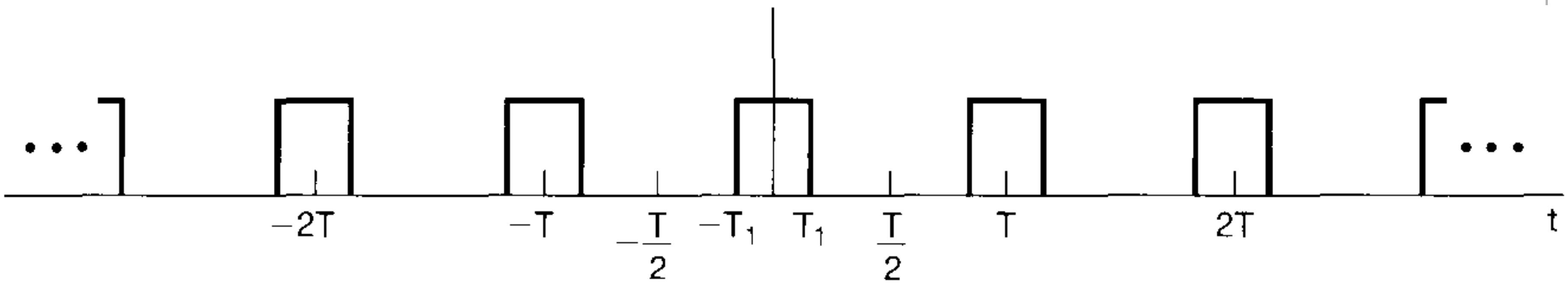

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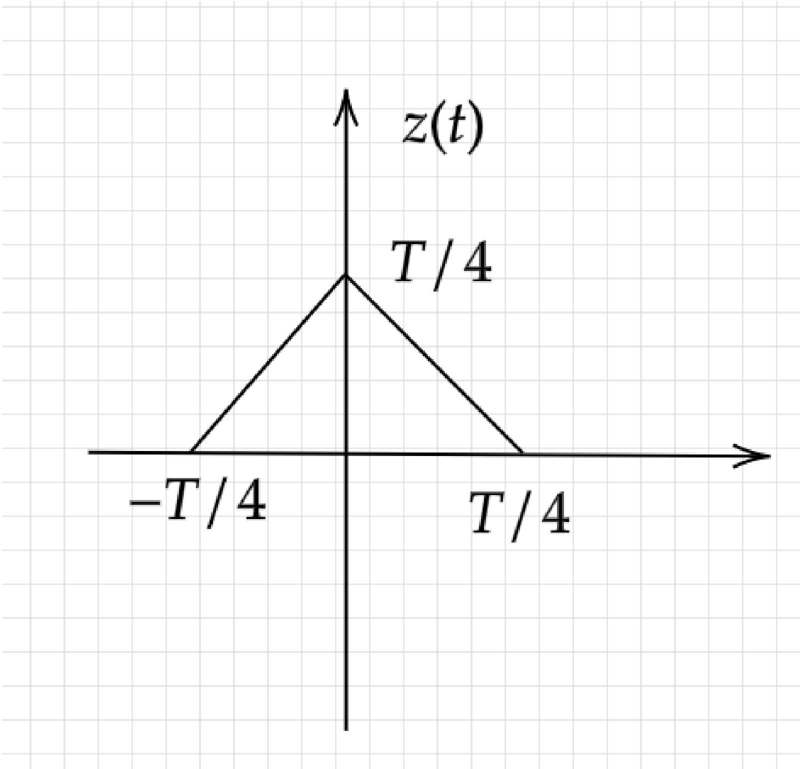
- Example A



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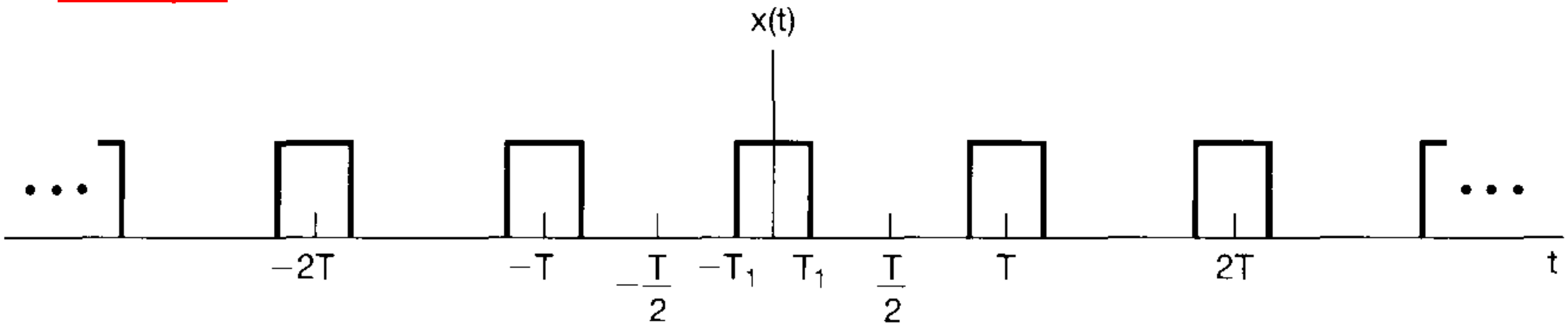
- Example

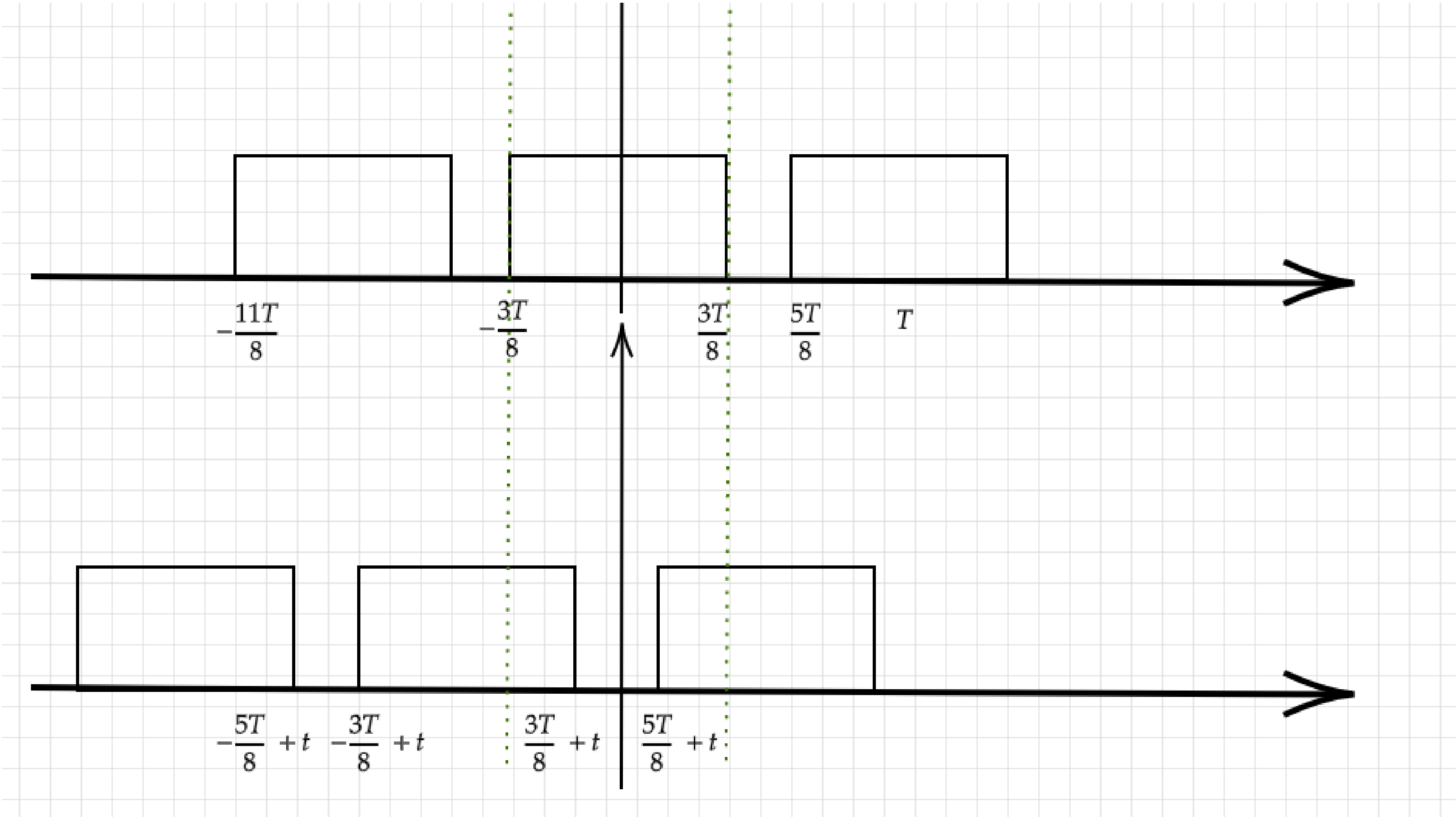


$$x(t) = y(t) \longrightarrow x(t) \otimes y(t) \quad ?$$

$A) T_1 = T/8$

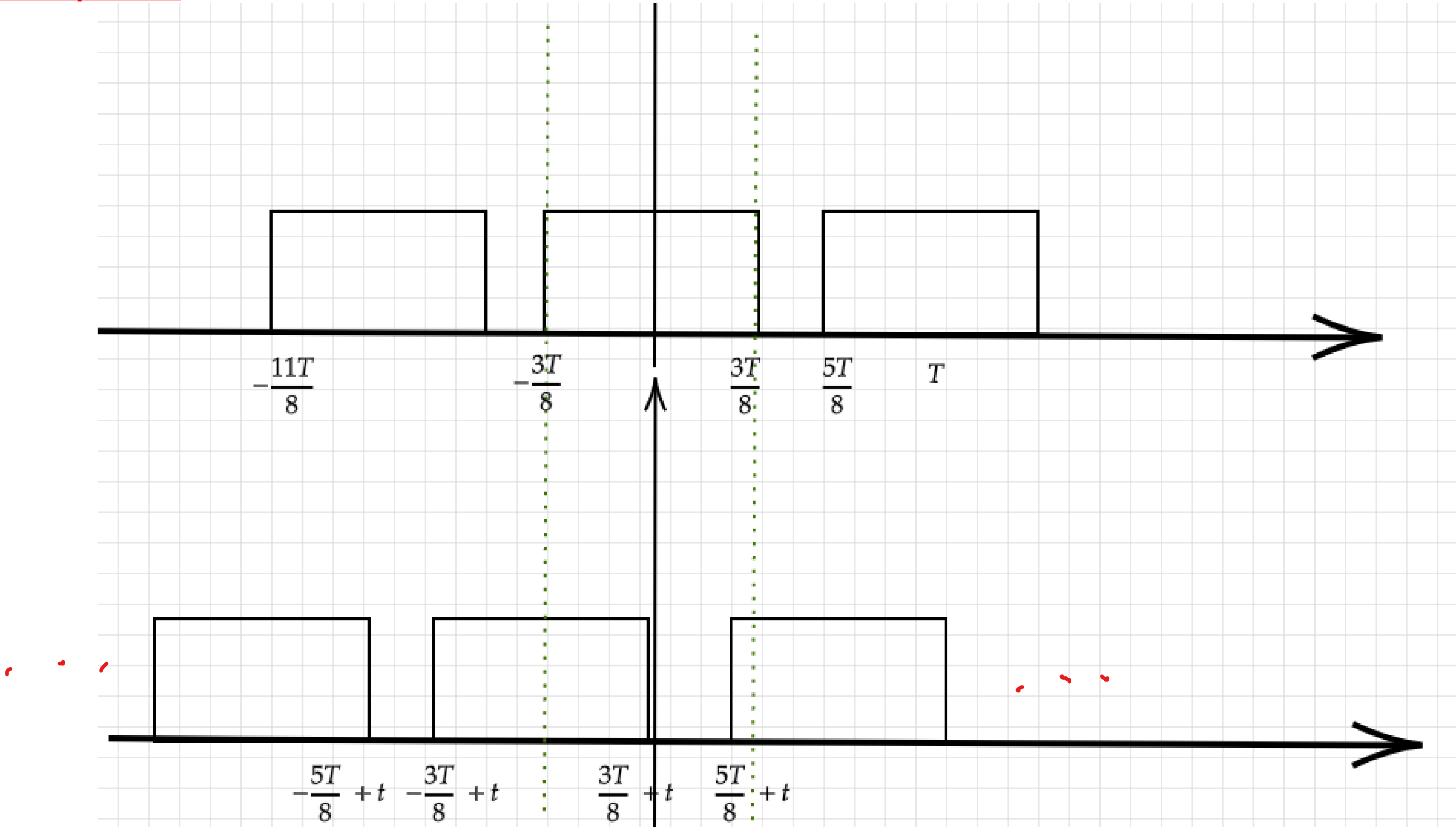
$B) T_1 = 3T/8$

- Example B



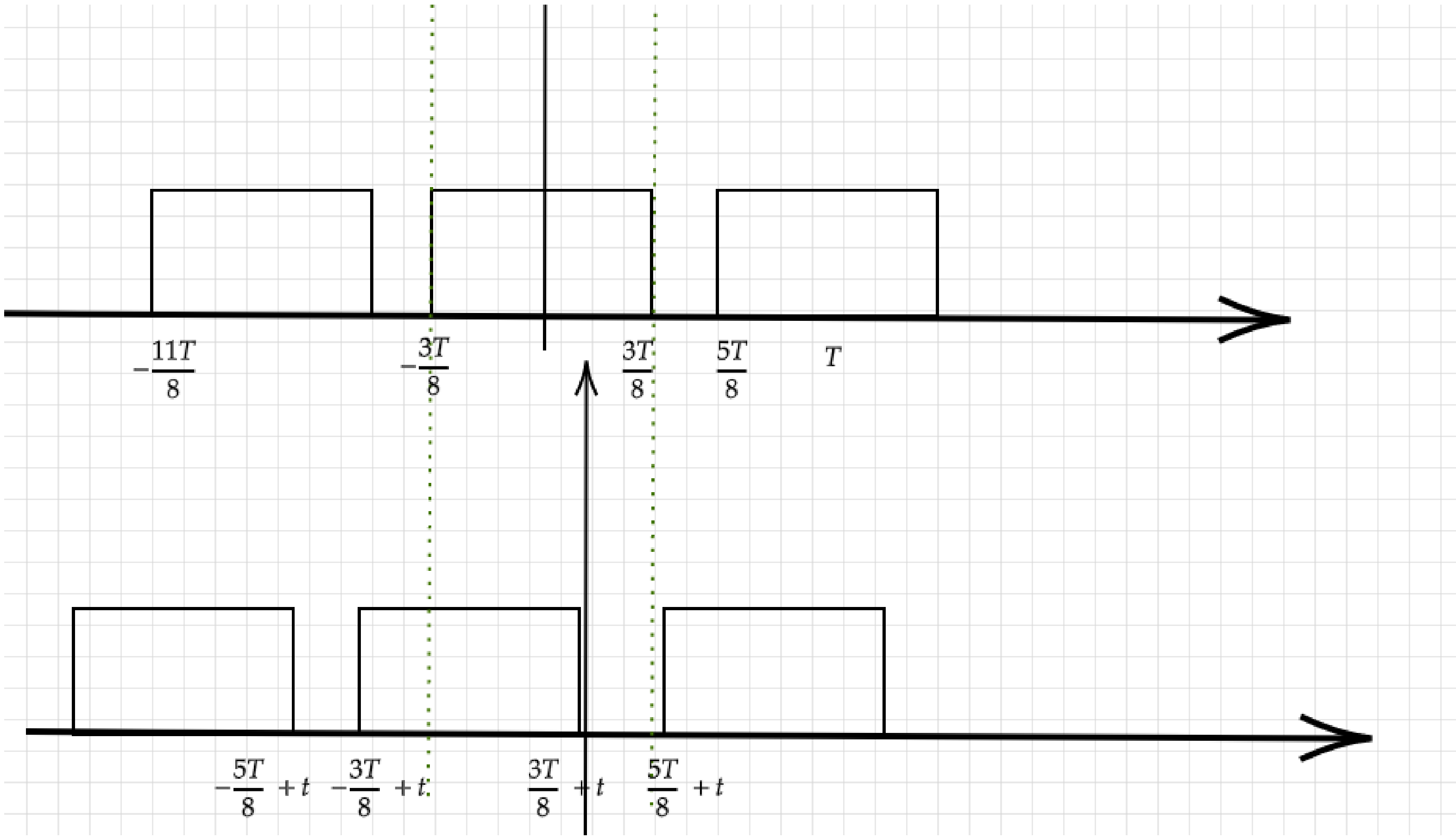
$$-\frac{T}{2} \le t \le -\frac{T}{4}, \quad z(t) = \frac{T}{2}$$

- Example B



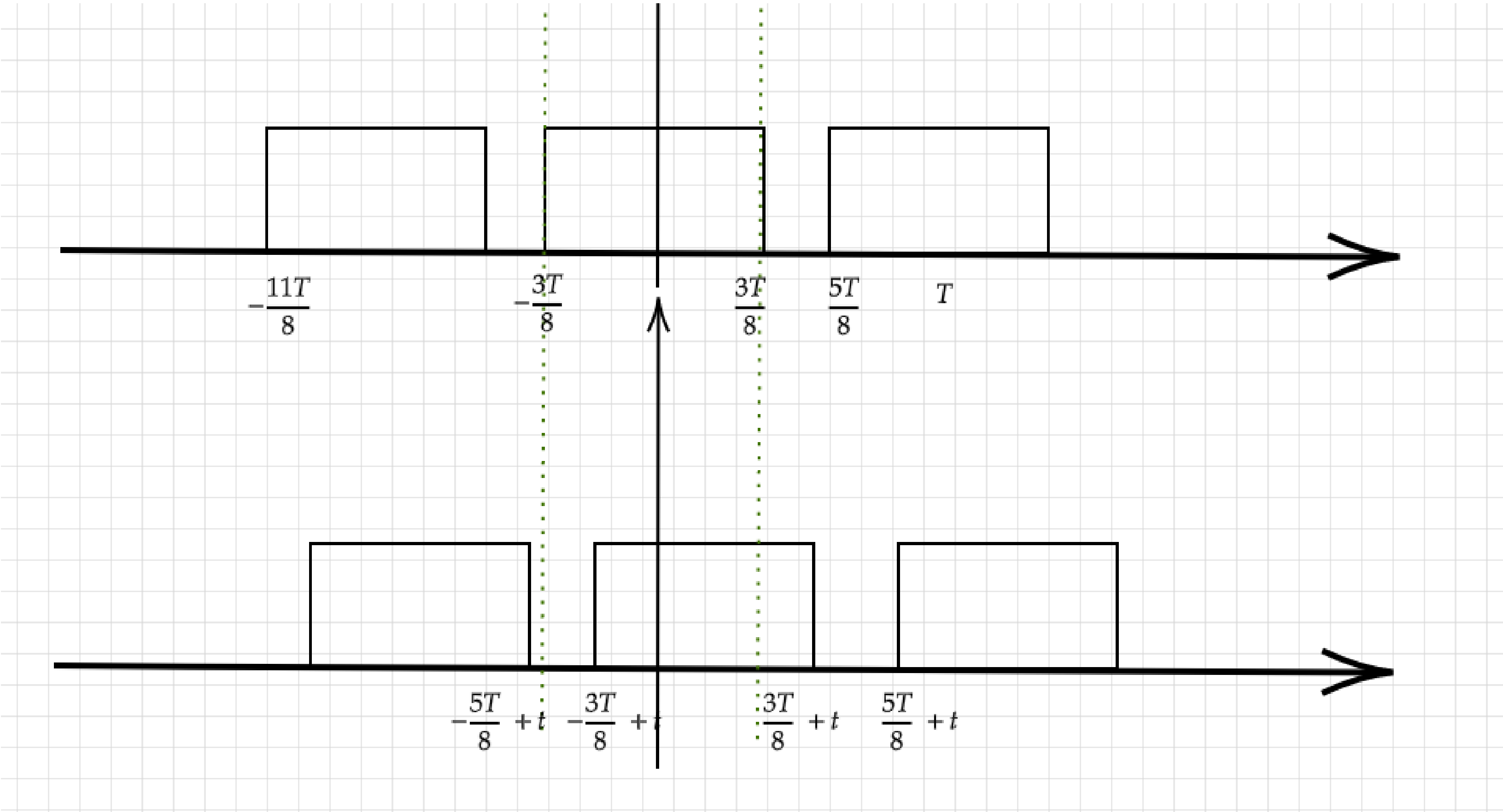
$$-\frac{T}{2} \leq t \leq -\frac{T}{4}, \quad z(t) = \frac{T}{2}$$

- Example B



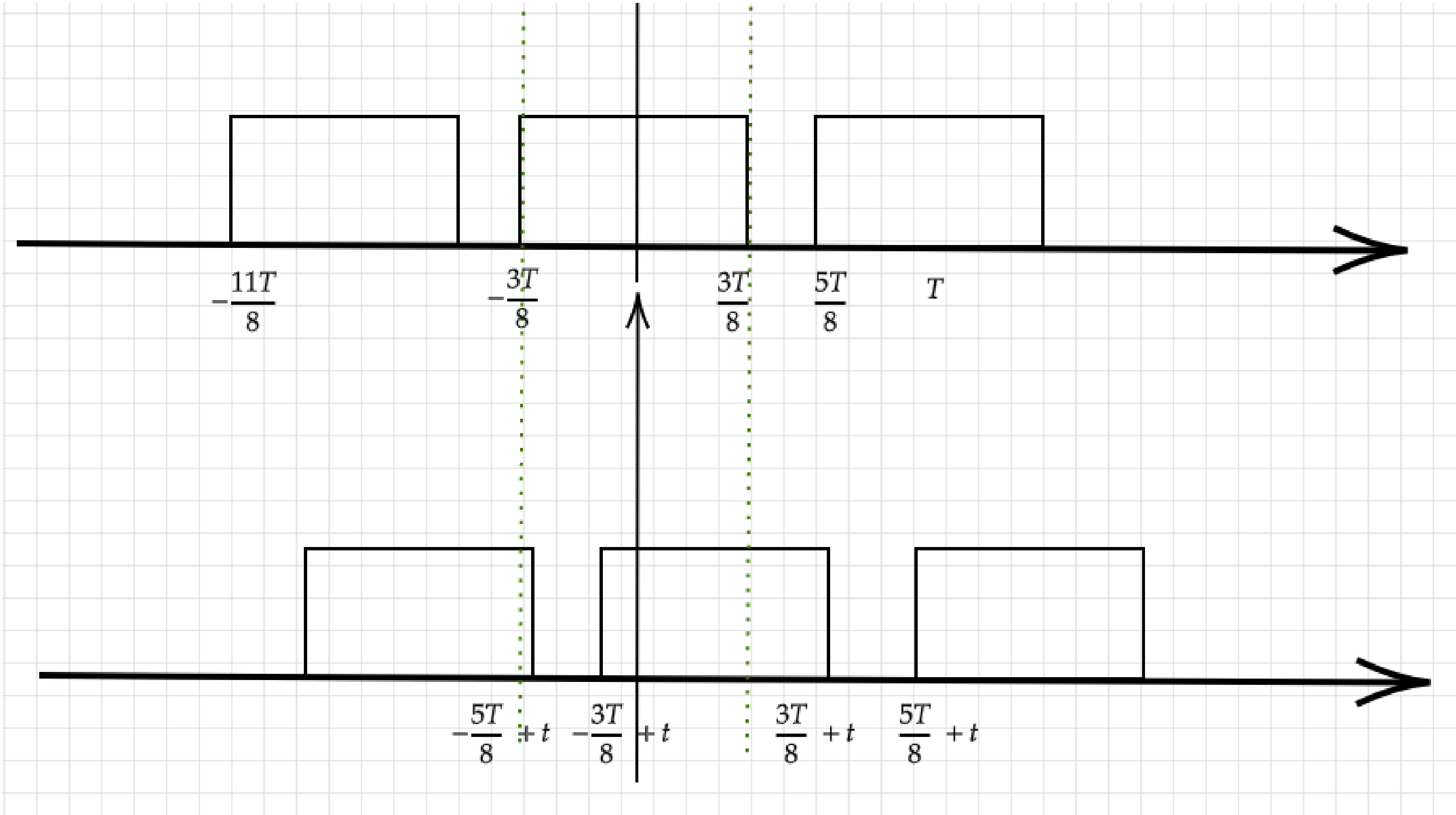
$$-\frac{T}{4} \leq t \leq 0, \quad z(t) = t + \frac{6T}{8}$$

- Example B



$$0 \leq t \leq \frac{T}{4}, \quad z(t) = -t + \frac{6T}{8}$$

- Example B



$$\frac{T}{4} \leq t \leq \frac{T}{2}, \quad z(t) = \frac{T}{2}$$

- Example B