Spring 2011

信號與系統
Signals and Systems

Chapter SS-10
The z Transform

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NTU-EE

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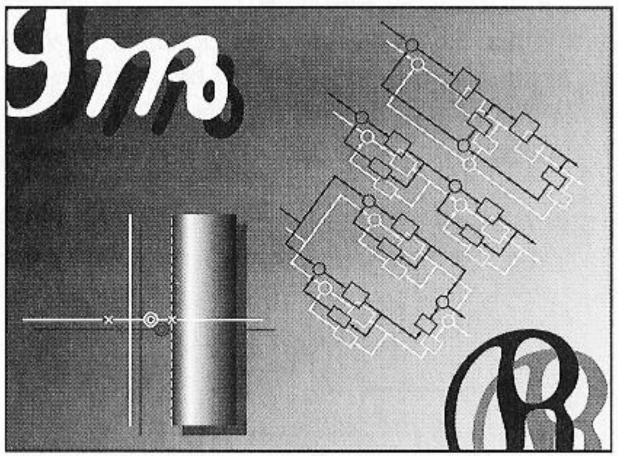


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs
- Analysis & Characterization of LTI Systems Using the z-Transforms

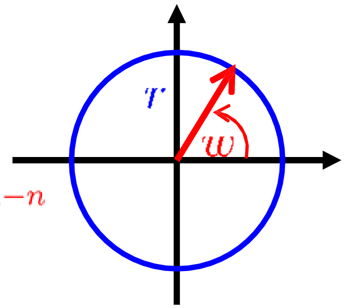
The z-Transform of a General Signal x[n]:

$$z = e^{jw}$$

$$X(e^{jw}) \triangleq \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$z = re^{jw}$$

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$



$$e^{jw} =$$

$$re^{jw} =$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw}) \qquad x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$$X(e^{jw}) = \mathcal{F}\left\{x[n]\right\} \qquad X(z) = \mathcal{Z}\left\{x[n]\right\}$$

$$x[n] = \mathcal{F}^{-1}\left\{X(e^{jw})\right\} \qquad x[n] = \mathcal{Z}^{-1}\left\{X(z)\right\}$$

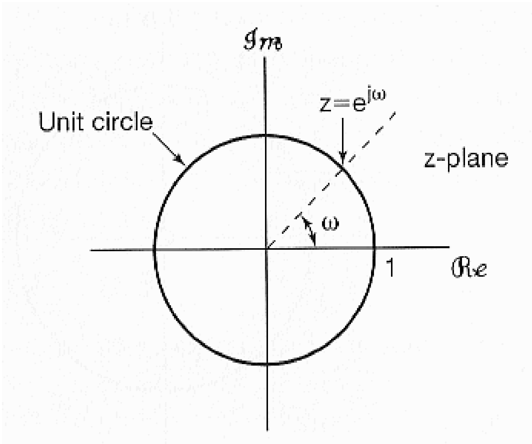
$$X(z)\Big|_{z=e^{jw}} = \mathcal{Z}\left\{x[n]\right\}\Big|_{z=e^{jw}} = \mathcal{F}\left\{x[n]\right\} = X(e^{jw})$$

- z-Transform & Fourier Transform:

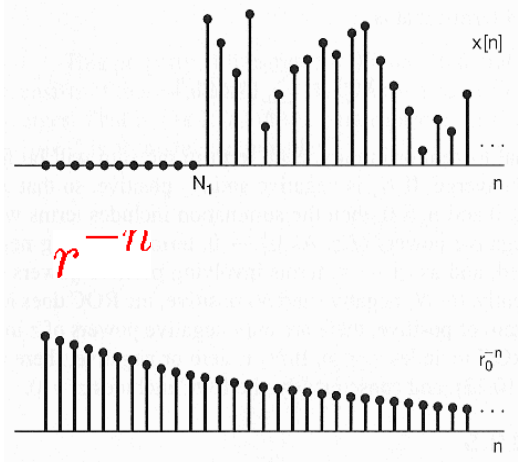
$$X(re^{jw}) = \sum_{n=-\infty}^{+\infty} x[n](re^{jw})^{-n} = \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\}e^{-jwn}$$

$$= \mathcal{F}\left\{x[n]r^{-n}\right\}$$

$$z = e^{jw}$$



$x[n]$



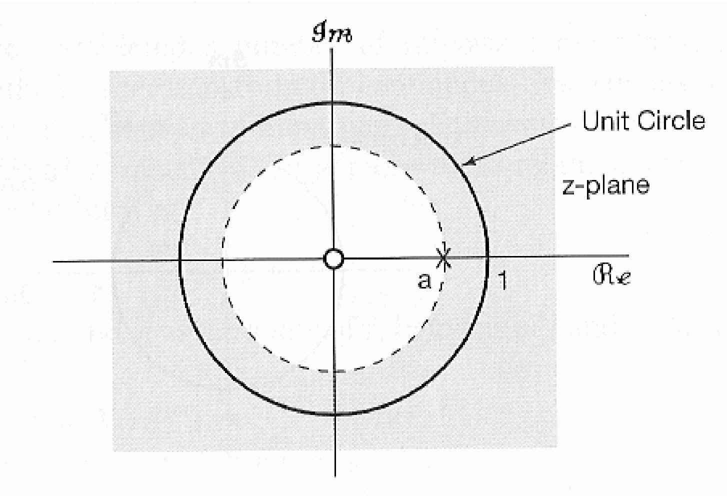
- Example 10.1:

$$x[n] = a^n u[n]$$

$$\Rightarrow X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^n\, u[n]\, e^{-jwn} \quad = \sum_{n=0}^{\infty} (ae^{-jw})^n \quad = \frac{1}{1 - ae^{-jw}}$$

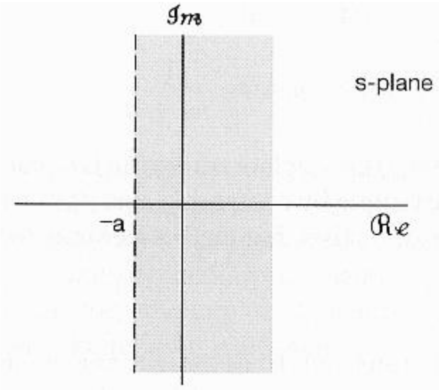
$$\Rightarrow X(z) = \sum_{n=-\infty}^{+\infty} a^n\, u[n]\, z^{-n} \quad = \sum_{n=0}^{\infty} (az^{-1})^n \qquad |az^{-1}| < 1$$

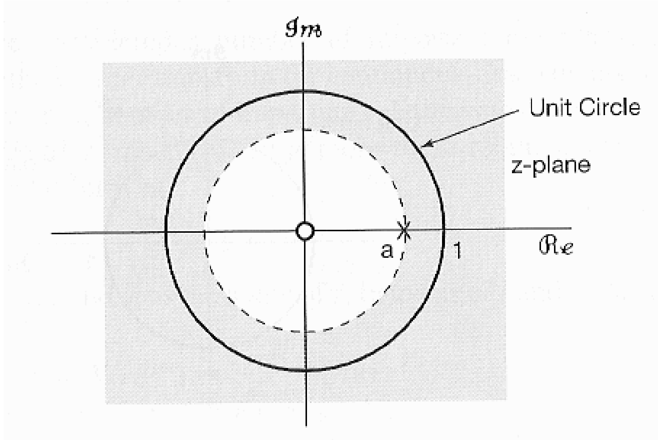
$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$



- For $|a| > 1$,

ROC does not include the unit circle, $\mathcal{F}\left\{a^n u[n]\right\}$ does not converge





$$s = \sigma + jw$$

$$z = re^{jw}$$

$e^{-at}u(t)$ $\quad e^{-\sigma t} \quad e^{-jwt}$

e^{-st}

$a^n u[n]$ $\quad r^{-n} \quad (e^{jw})^{-n}$

$(z)^{-n}$

- Example 10.2:

$$x[n] = -a^n u[-n-1]$$

$$\Rightarrow X(z) = -\sum_{n=-\infty}^{+\infty} a^n \, u[-n-1] \, z^{-n}$$

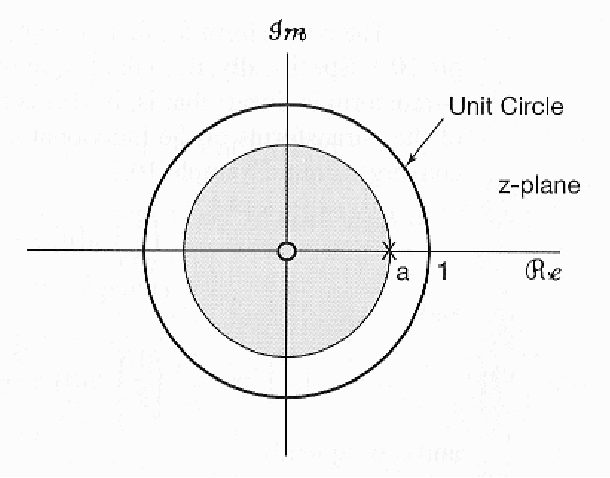
$$= -\sum_{n=-\infty}^{-1} a^n \, z^{-n} \quad = -\sum_{n=1}^{\infty} a^{-n} \, z^n \quad = 1 - \sum_{n=0}^{\infty} \left(a^{-1} \, z\right)^n$$

$|a^{-1}z| < 1$

$$= 1 - \frac{1}{1 - a^{-1}z}$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

$|z| < |a|$

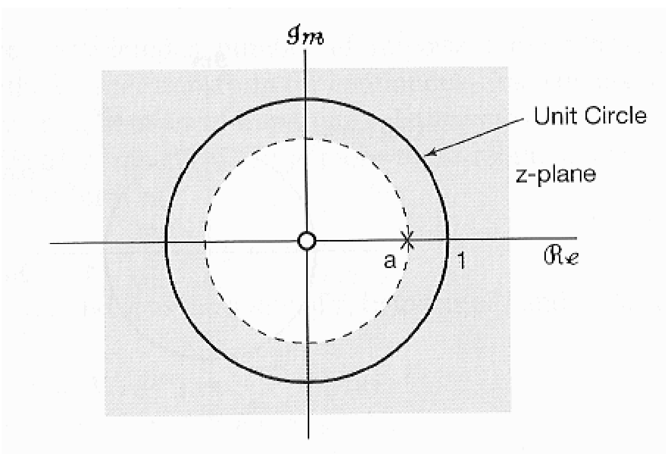


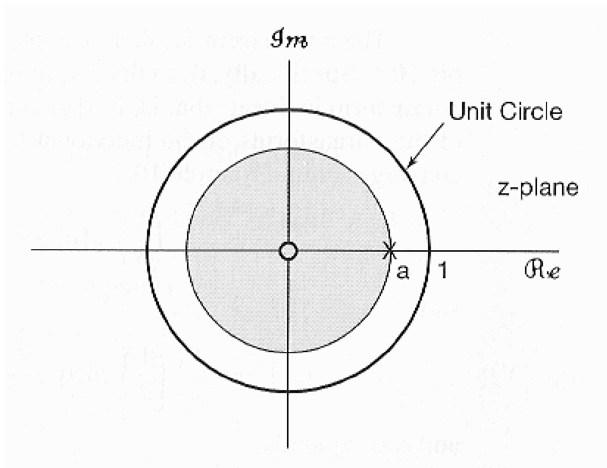
- Region of Convergence (ROC):

$$a^n u[n] \stackrel{z}{\longleftrightarrow} \frac{z}{z-a}, \quad |z| > |a|$$

$$-a^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{z}{z-a}, \quad |z| < |a|$$

where Fourier transform of $x[n]r^{-n}$ converges





- Example 10.3:

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \left\{7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]\right\} z^{-n}$$

$$= 7\sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u[n] z^{-n} - 6\sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n}$$

$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

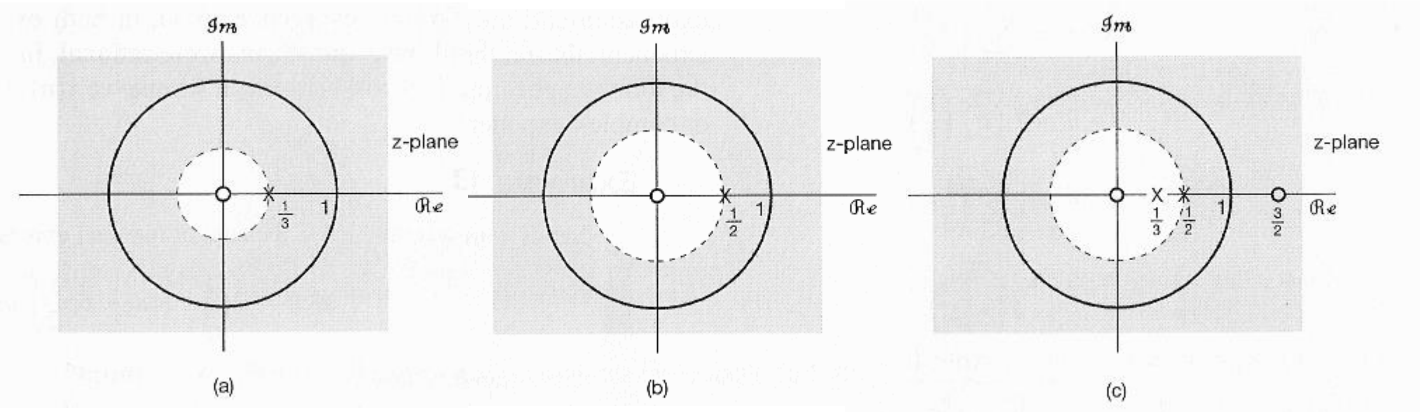
$$7\cdot\left(\frac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} 7\cdot\frac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$6\cdot\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} 6\cdot\frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

- Example 10.3:

$$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$



- Example 10.4:

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$$

$$\begin{aligned}\sin\left(\frac{\pi}{4}n\right) &= \frac{1}{2j}\left(e^{j\pi/4n} - e^{-j\pi/4n}\right)\\ &= \frac{1}{2j}\left(\left(e^{j\pi/4}\right)^n - \left(e^{-j\pi/4}\right)^n\right)\end{aligned}$$

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j}\left(\left(\frac{1}{3}\,e^{j\pi/4}\right)^n - \left(\frac{1}{3}\,e^{-j\pi/4}\right)^n\right)$$

- Example 10.4:

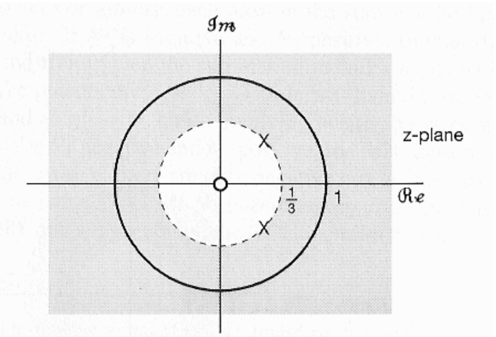
$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a}, \quad |z| > |a|$$

$$\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-\frac{1}{3}e^{j\pi/4}}, \quad |z| > \frac{1}{3}$$

$$\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-\frac{1}{3}e^{-j\pi/4}}, \quad |z| > \frac{1}{3}$$

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2j}\left(\frac{z}{z-\frac{1}{3}e^{j\pi/4}} - \frac{z}{z-\frac{1}{3}e^{-j\pi/4}}\right), \quad |z| > \frac{1}{3}$$

$$\frac{\frac{1}{3\sqrt{2}}z}{(z-\frac{1}{3}e^{j\pi/4})(z-\frac{1}{3}e^{-j\pi/4})}$$

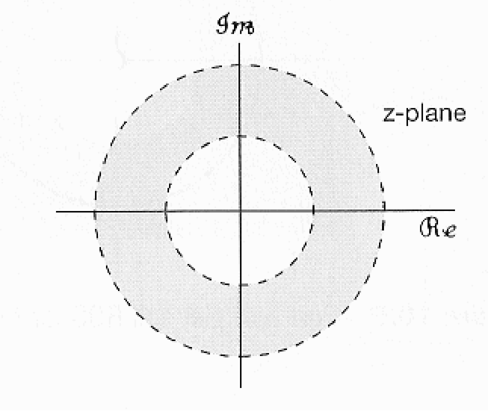


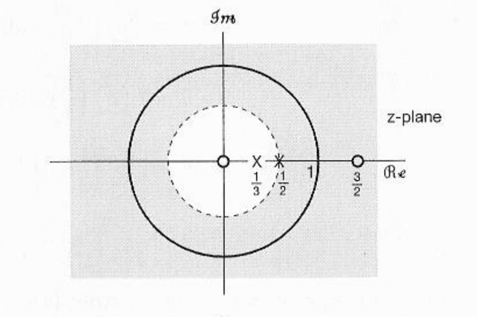
Properties of ROC:

1. The ROC of X(z) consists of a ring in the z-plane centered about the origin

2. The ROC does not contain any poles

$$\frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \qquad |z| > \frac{1}{2}$$

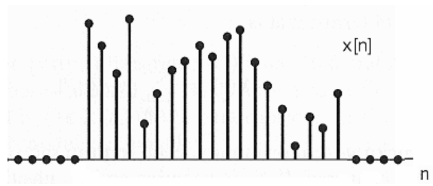




(c)

Properties of ROC:

3. If x[n] is of finite duration,
then the ROC is the entire z-plane,
except possibly $z = 0$ and/or $z = \infty$



$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=N_1}^{N_2} x[n] z^{-n} \quad \text{is bounded}$$

- However,

$$|z| \to 0 \quad \Rightarrow \quad |z|^N \to \infty \quad \text{if } N \text{ is negative}$$

$$|z| \to \infty \quad \Rightarrow \quad |z|^N \to \infty \quad \text{if } N \text{ is positive}$$

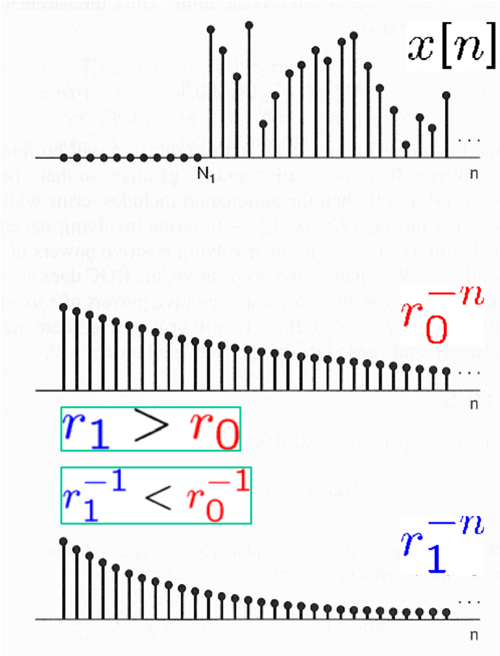
- **Properties of ROC:**

4. If x[n] is right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC

$$X(r_0 e^{jw}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

$$X(r_1 e^{jw}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_1^{-n} \right\} e^{-jwn}$$

$$< \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

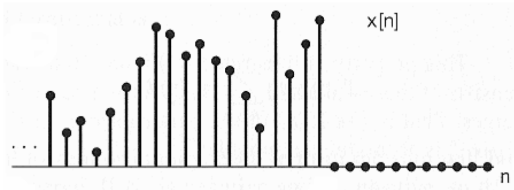


- <u>Properties of ROC:</u>

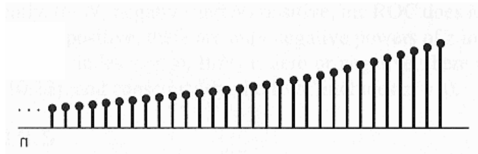
5. If $x[n]$ is left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which

$0 < |z| < r_0$

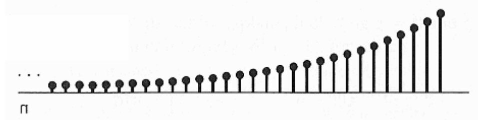
will also be in the ROC



$$X(re^{jw}) = \sum_{n=-\infty}^{N} \left\{ x[n] r^{-n} \right\} e^{-jwn}$$

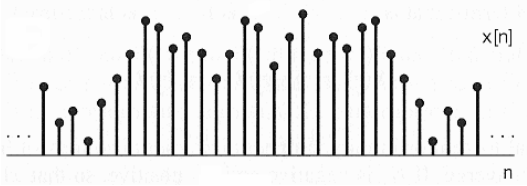


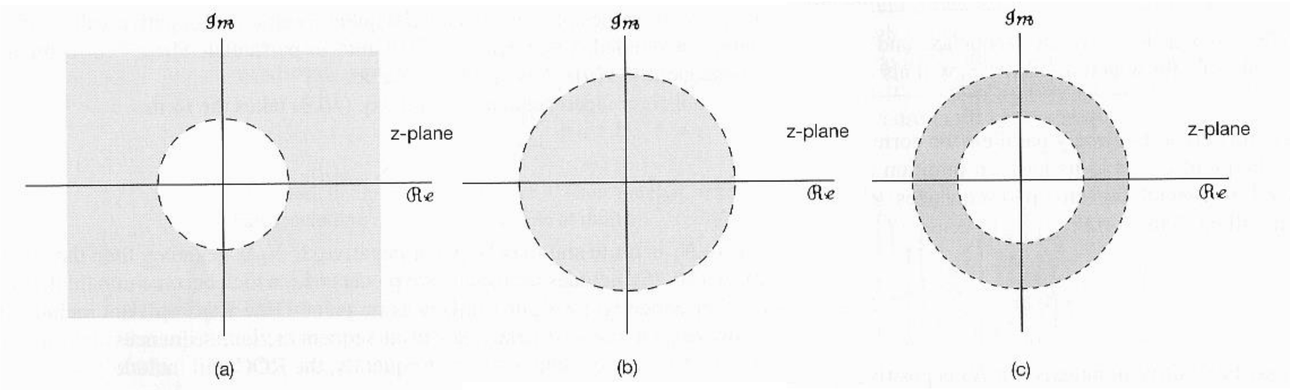
$$\boxed{0 < r_1 < r_0}$$



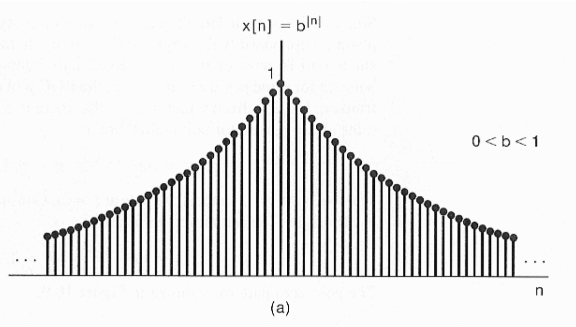
- **Properties of ROC:**

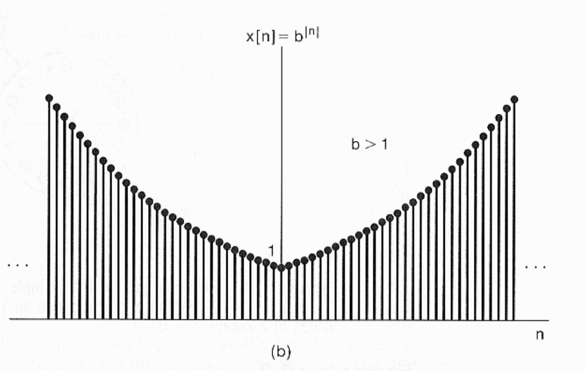
6. If $x[n]$ is two-sided, and
if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring
in the z-plane that includes the circle $|z| = r_0$





- Example 10.7:





$$x[n] = b^{|n|}, \quad b > 0$$

$$= b^{n}u[n] + b^{-n}u[-n-1]$$

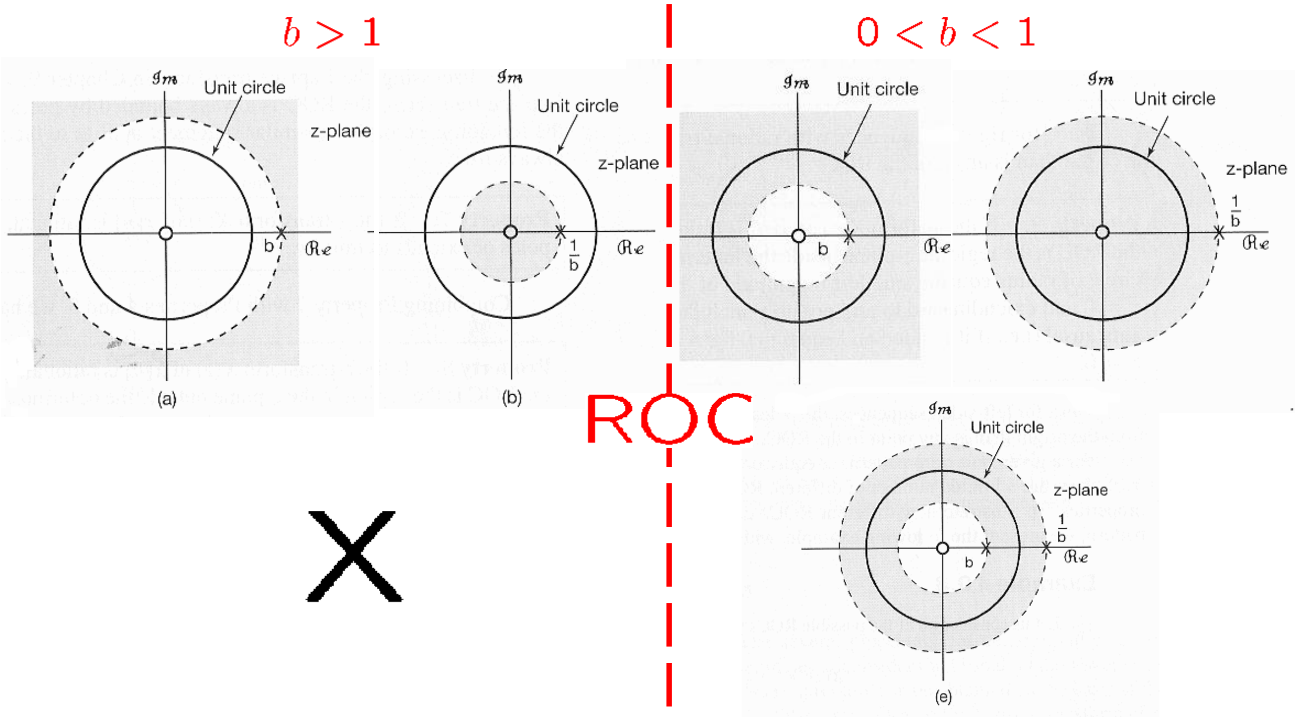
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}, \quad b < |z| < \frac{1}{b}$$

$$= \left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \quad b < |z| < \frac{1}{b}$$

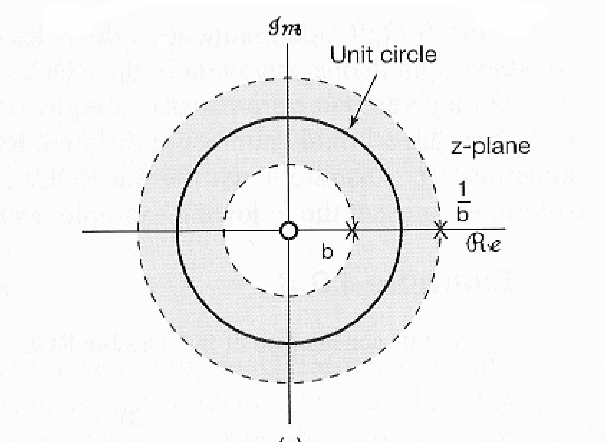
- Example 10.7:

$$X(z) = \left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \quad b < |z| < \frac{1}{b}$$

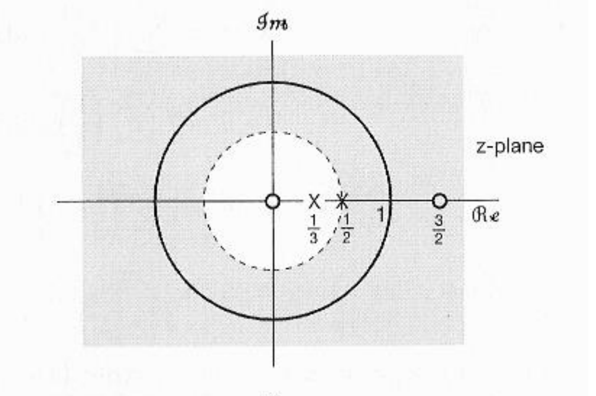


- Properties of ROC:

7. If the z-transform X(z) of x[n] is rational,
then its ROC is bounded by poles or extends to ∞



(e)



(c)

$$X(z) = \left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \quad b < |z| < \frac{1}{b}$$

$$\frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$

- Properties of ROC:

  8. If the z-transform X(z) of x[n] is rational

  – If x[n] is right sided,
  then the ROC is the region in the z-plane outside
  the outermost pole ---
  i.e. outside the circle of radius equal to
  the largest magnitude of the poles of X(z)

  – Furthermore, if x[n] is causal, then the ROC
  also includes $z = \infty$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=0}^{+\infty} x[n] \left(\frac{1}{z}\right)^n$$

- Properties of ROC:

9. If the z-transform $X(z)$ of $x[n]$ is rational and If $x[n]$ is left sided,

then the ROC is the region in the z-plane inside the innermost pole ---

i.e. inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z = 0$

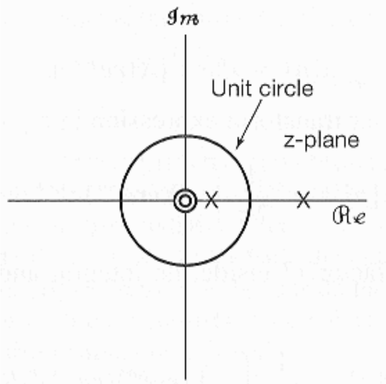
and extending inward and possibly including $z = 0$

In particular, if $x[n]$ is anti-causal,

(i.e., if it is left sided and $= 0$ for $n > 0$), then the ROC also includes $z = 0$

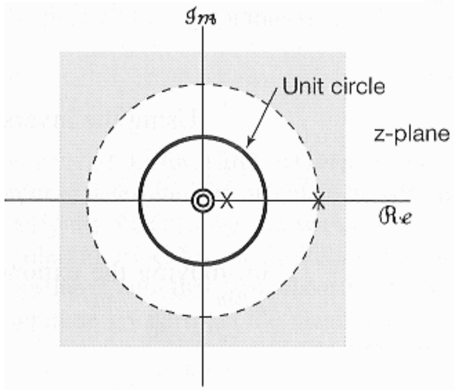
- **Example 10.8:**

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$



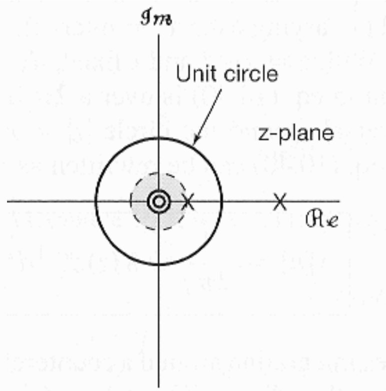
(a)

right-sided



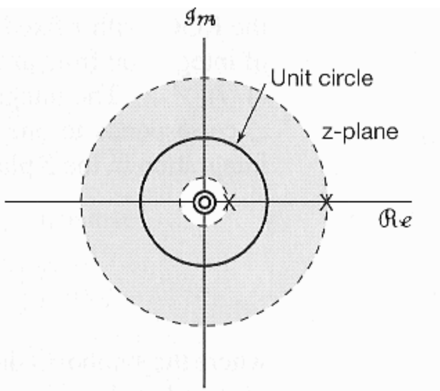
(b)

left-sided



(c)

two sided



(d)

The Inverse z-Transform:

- By the technique of partial fraction expansion

$$X(z) = \frac{A_1}{1 - a_1 z^{-1}} + \frac{A_2}{1 - a_2 z^{-1}} + \cdots + \frac{A_m}{1 - a_m z^{-1}}$$

$$x[n] = A_1\, a_1^n\, u[n] - A_2\, a_2^n\, u[-n-1] + \cdots + x_m[n]$$

(if ROC outside $z = a_1$) (if ROC inside $z = a_2$)

- Example 10.9:

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})} + \frac{2}{(1 - \frac{1}{3}z^{-1})}$$

$$\left(\frac{1}{4}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{(1 - \frac{1}{4}z^{-1})}, \quad |z| > \frac{1}{4}$$

$$2\left(\frac{1}{3}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{2}{(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

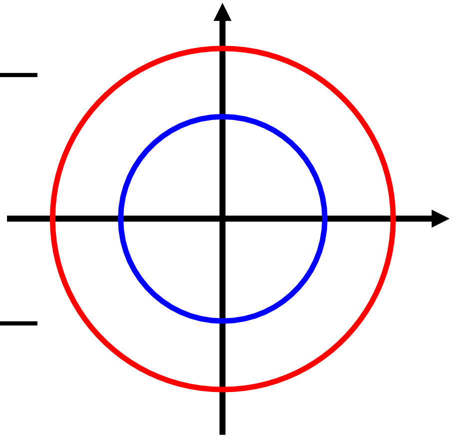
$$\Rightarrow \; x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

- Examples 10.9, 10.10, 10.11:

$$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}} \qquad \text{ROC}|z|>a$$

$$-a^n u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}} \qquad \text{ROC}|z|<a$$

	$\lvert z\rvert < \frac{1}{4}$	$\frac{1}{4} < \lvert z\rvert$
$\frac{1}{(1-\frac{1}{4}z^{-1})}$	$-\left(\frac{1}{4}\right)^n u[-n-1]$	$\left(\frac{1}{4}\right)^n u[n]$



	$\lvert z\rvert < \frac{1}{3}$	$\frac{1}{3} < \lvert z\rvert$
$\frac{1}{(1-\frac{1}{3}z^{-1})}$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$\left(\frac{1}{3}\right)^n u[n]$

	$\lvert z\rvert < \frac{1}{4}$	$\frac{1}{4} < \lvert z\rvert < \frac{1}{3}$	$\frac{1}{3} < \lvert z\rvert$
$\frac{1}{(1-\frac{1}{4}z^{-1})}$	$-\left(\frac{1}{4}\right)^n u[-n-1]$	$\left(\frac{1}{4}\right)^n u[n]$	$\left(\frac{1}{4}\right)^n u[n]$
$\frac{1}{(1-\frac{1}{3}z^{-1})}$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$\left(\frac{1}{3}\right)^n u[n]$

- **Linearity of the z-Transform:**

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad ROC = R_1$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad ROC = R_2$$

$$a x_1[n] + b x_2[n] \quad \overset{z}{\longleftrightarrow} \quad a X_1(z) + b X_2(z),$$

with ROC containing $R_1 \cap R_2$

- Time Shifting:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0} X(z), \quad ROC = R$$

except for the possible addition or deletion of the origin or infinity

Scaling in the z-Domain:

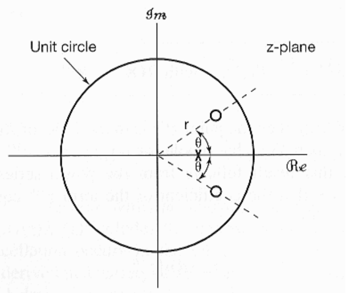
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

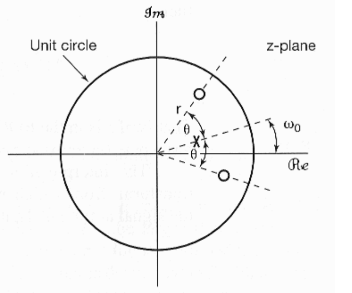
$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z), \quad ROC = R$$

$$z_0^n x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \quad ROC = |z_0| R$$

$$e^{jw_0 n} x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X\left(e^{-jw_0} z\right), \quad ROC = R$$



(a)



(b)

- Time Reversal:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad ROC = R$$

$$x[-n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{1}{z}\right), \quad ROC = \frac{1}{R}$$

- Time Expansion:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = k \cdot m \\ 0, & \text{otherwise} \end{cases}$$

k is a constant
m is a new time variable

$$x_{(k)}[n] \overset{z}{\longleftrightarrow} X(z^k), \quad ROC = R^{1/k}$$

- Conjugation:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \quad ROC = R$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] \ z^{-n}$$

- Convolution Property:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x_1[n] \stackrel{z}{\longleftrightarrow} X_1(z), \quad ROC = R_1$$

$$x_2[n] \stackrel{z}{\longleftrightarrow} X_2(z), \quad ROC = R_2$$

$$x_1[n] * x_2[n] \quad \stackrel{z}{\longleftrightarrow} \quad X_1(z) X_2(z), \quad \text{with } ROC \text{ containing } R_1 \cap R_2$$

$R_1 \cap R_2$ may be larger
if pole-zero cancellation
occurs in the product

- Differentiation in the z-Domain:

$$x[n] \xleftrightarrow{z} X(z), \quad ROC = R$$

$$nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z), \quad ROC = R$$

- The Initial-Value Theorem:

If $x[n] = 0$ for $n < 0$

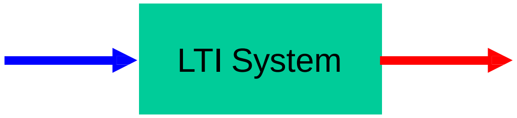
$\Rightarrow x[0] = \lim_{z \to \infty} X(z)$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$

$$= x[0] + x[1]\, z^{-1} + x[2]\, z^{-2}$$

Analysis & Characterization of LTI Systems:

$$H(z) = \mathcal{Z}\{h[n]\}$$



$X(z)$ → LTI System → $Y(z) = H(z)\,X(z)$

$H(z)$: system function

or transfer function

- Causality
- Stability

Causality:

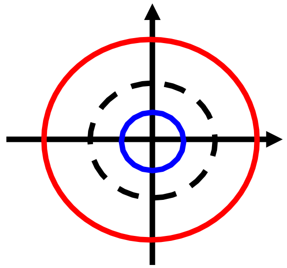
$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

- For a causal LTI system,
  $h[n] = 0$ for $n < 0$, and thus is right-sided

- A DT LTI system is causal if and only if the ROC of the system function $H(z)$ is the exterior of a circle in the z-plane, including infinity

- A DT LTI system with a rational $H(z)$ is causal if and only if
  (a) ROC is exterior of a circle outside outermost pole; and infinity must be in the ROC; and
  (b) order of numerator <= order of denominator

- Example 10.21:



$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$\Rightarrow$ ROC: the exterior of a circle of outside the outermost pole

$\Rightarrow$ the impulse response is right-sided

$$H(z) = \frac{2-\frac{5}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{2z(z-\frac{5}{4})}{(z-\frac{1}{2})(z-2)} = \frac{2z^2-\frac{5}{2}z}{z^2-\frac{5}{2}z+1}$$

$\Rightarrow$ deg of num of $H(z)$ = deg of den of $H(z)$

$\Rightarrow$ the system is causal

$$\Rightarrow \; h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right] u[n] \qquad \Rightarrow \; h[n] = 0, n < 0$$

- Stability:

$$\sum_{k=-\infty}^{+\infty} \left| h[k] \right| < \infty$$

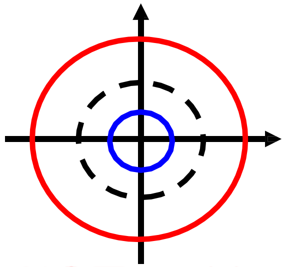
absolutely summable

- An DT LTI system is stable if and only if
  the ROC of H(z) includes the unit circle [i.e., $|z| = 1$]

- A causal LTI system with rational H(z) is stable if and only if
  all of the poles of H(z) lie in the inside the unit circle, i.e.,
  all of the poles have magnitude < 1

- Example 10.22:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$



$\Rightarrow$ ROC does not include the unit circle $\Rightarrow$ NOT stable

$$\Rightarrow \text{ i.e., } h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right] u[n] \rightarrow \infty, \quad \text{as } n \rightarrow \infty$$

- If $ROC = 1/2 < |z| < 2 \quad \Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$

  $\Rightarrow$ the system is NOT causal, but stable

- If $ROC = |z| < 1/2 \quad \Rightarrow h[n] = -\left[\left(\frac{1}{2}\right)^n + 2^n\right] u[-n-1]$

  $\Rightarrow$ the system is neigher causal nor stable

- LTI Systems by Linear Constant-Coef Difference Equations:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$Y(z) = X(z) H(z) \qquad H(z) = \frac{Y(z)}{X(z)}$$

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z)$$

$$\mathcal{Z}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{Z}\left\{y[n-k]\right\} = \sum_{k=0}^{M} b_k \mathcal{Z}\left\{x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

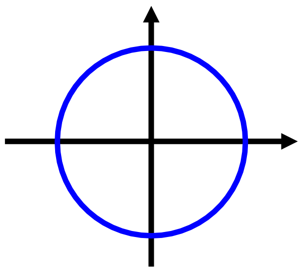
$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

zeros

poles

- Example 10.25:



$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

$$\Rightarrow Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$\Rightarrow H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{z+\frac{1}{3}}{z-\frac{1}{2}} = \frac{z}{z-\frac{1}{2}} + \frac{\frac{1}{3}}{z-\frac{1}{2}}$$

- If $ROC = \{|z| > 1/2\}$, $\Rightarrow h[n]$ is right-sided

$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

- If $ROC = \{|z| < 1/2\}$, $\Rightarrow h[n]$ is left-sided

$$\Rightarrow h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$$

Example 10.6

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \ a > 0 \\ 0, & \text{otherwise} \end{cases}.$$

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} \\ &= \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}. \end{aligned}$$

Example 10.6

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1,\ a > 0 \\ 0, & \text{otherwise} \end{cases}.$$

$$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}.$$

Roots of the Denominator $\quad z^{N-1} = 0, \qquad z = a$

$z^N = a^N \qquad z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N-1.$

Zeros: $\quad z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1.$

Poles: $\quad z^{N-1} = 0,$

Example 10.17

x[n]?

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|,$$

Solution

$$\frac{dX}{dz} = -\frac{-az^{-2}}{1 + az^{-1}}$$

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

$$-z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}},$$

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$$

$$(-a)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 + az^{-1}}$$

$$(-a)^{n-1} u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{1 + az^{-1}}$$

$$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z)$$

Example 10.17

x[n]?

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|,$$

Solution

$$nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz}$$

$$-z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}},$$

$$(-a)^{n-1}u[n-1] \xleftrightarrow{z} \frac{z^{-1}}{1+az^{-1}}$$

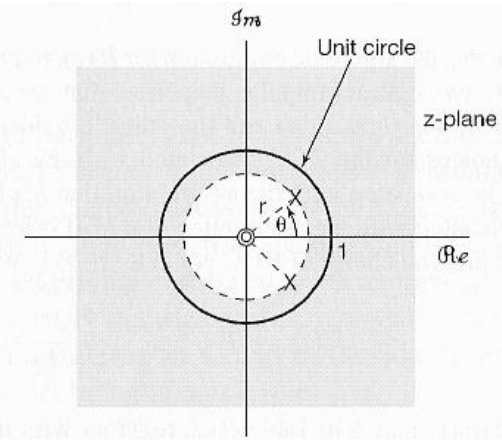
$$x[n] = \frac{-(-a)^n}{n}u[n-1].$$

- Example 10.24:

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}} = \frac{z^2}{z^2-(2r\cos\theta)z+r^2}$$

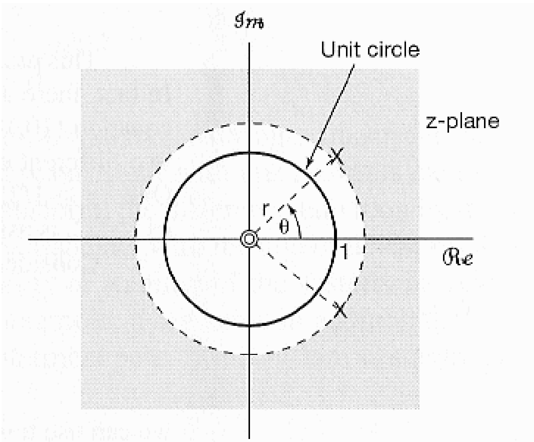
$$z^{-1} = \frac{2r\cos\theta \pm \sqrt{4r^2\cos^2\theta - 4r^2}}{2r^2} = \frac{2r\cos\theta \pm 2r\sqrt{\cos^2\theta - 1}}{2r^2}$$

$$z^{-1} = \frac{1}{r}e^{j\theta}, \frac{1}{r}e^{-j\theta}$$



$r < 1$, stable

If it is causal, $|z| > |r|$



$r > 1$, unstable