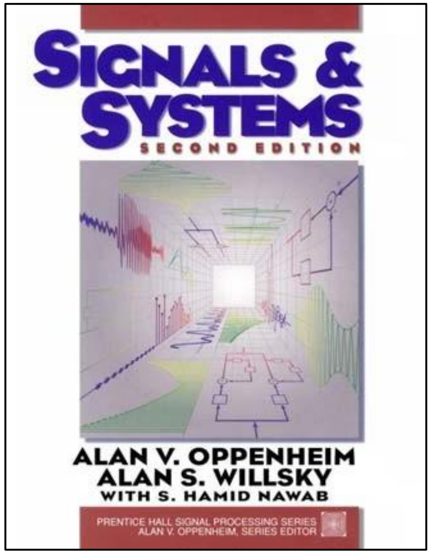
Spring 2010

信號與系統
Signals and Systems

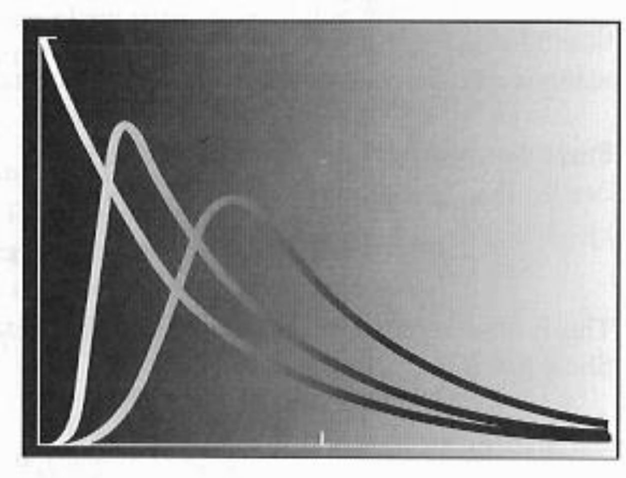
Chapter SS-2
Linear Time-Invariant Systems



Feng-Li Lian

NTU-EE

Feb10 – Jun10



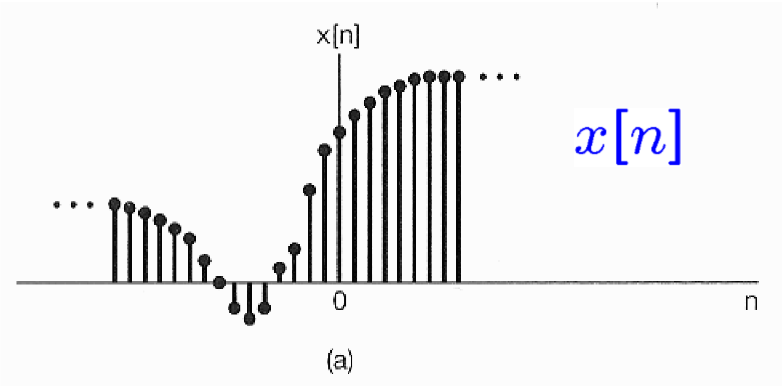
Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

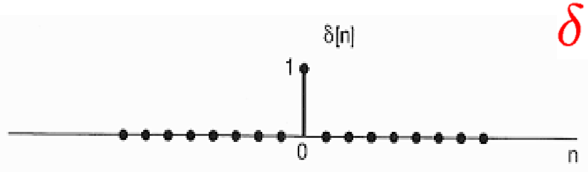
$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] \qquad x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

- <u>Sample by Unit Impulse</u>
  - For x[n]

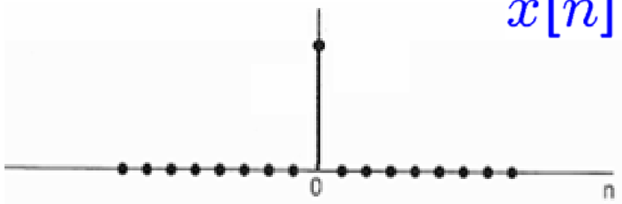
$$x[n]\delta[n] = x[0]\delta[n]$$



$x[n]$



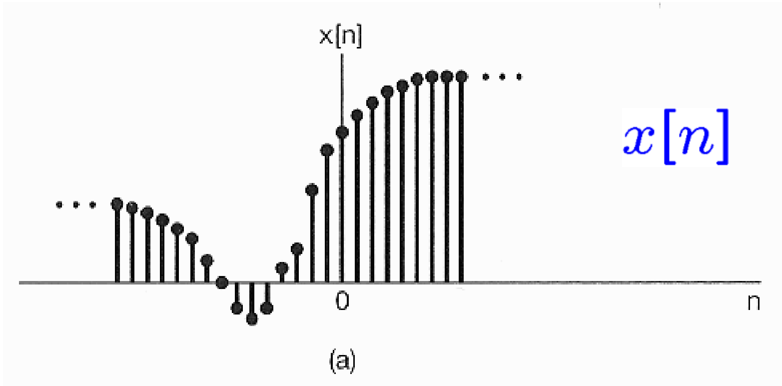
$\delta[n]$



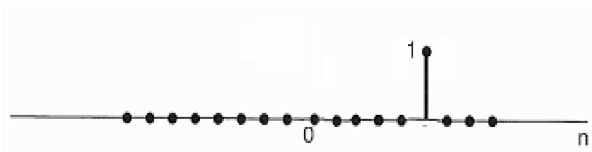
$x[n]\delta[n]$

  - More generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

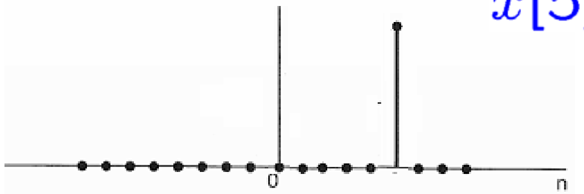


$x[n]$



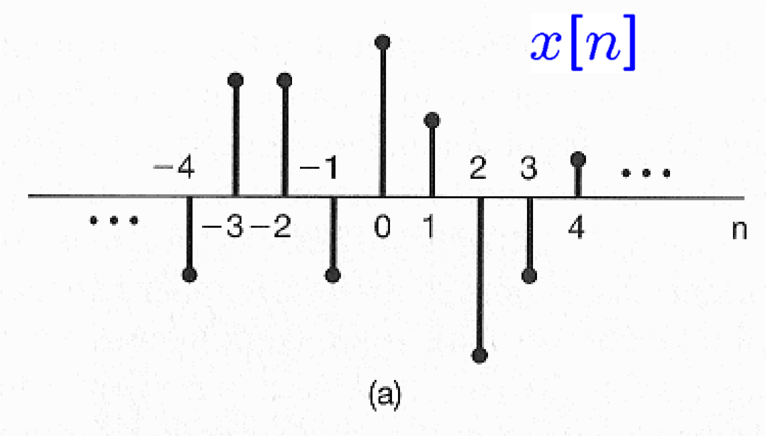
$\delta[n-5]$

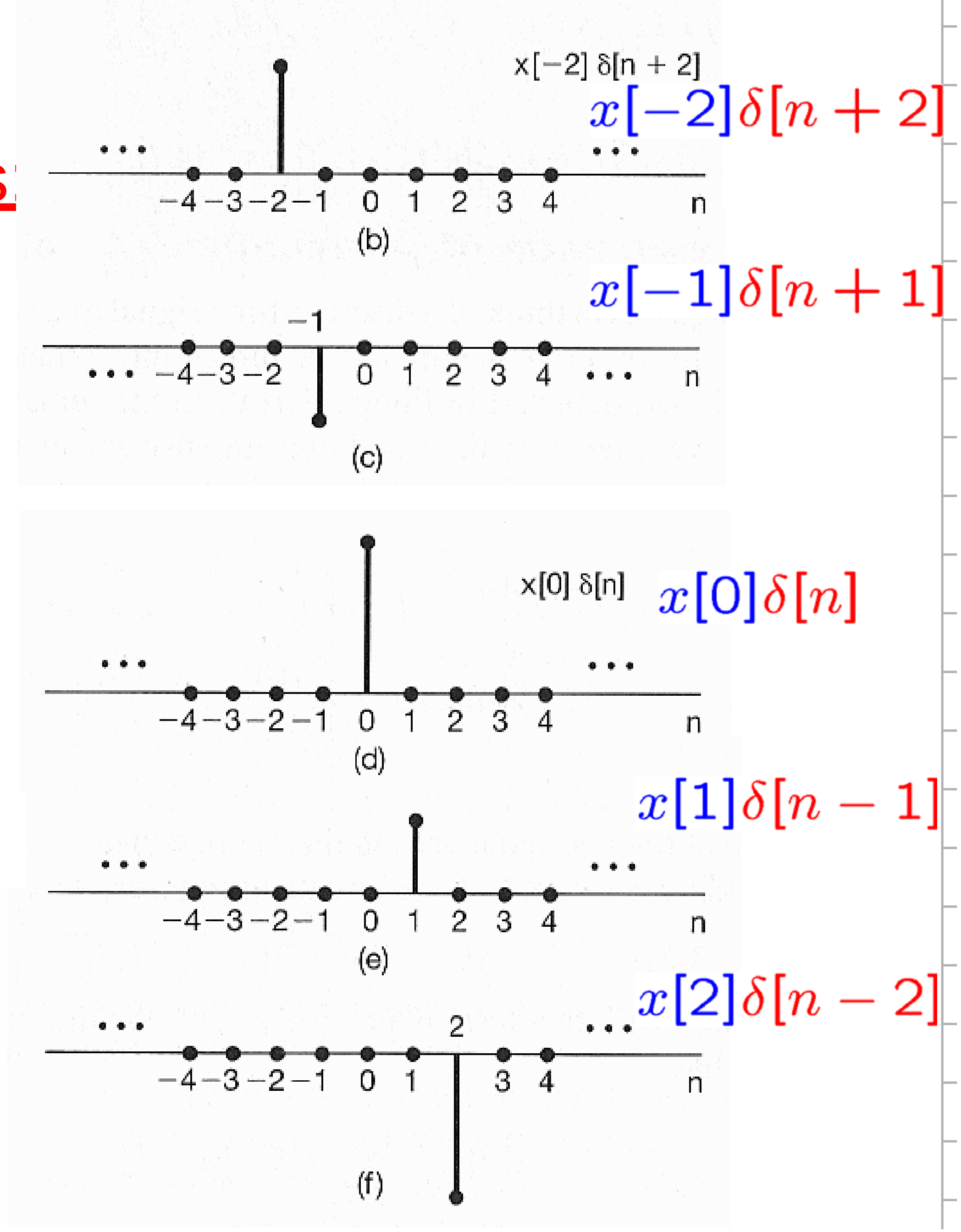
5



$x[5]\delta[n-5]$

- **Representation of DT Signals by Impulses:**





Representation of DT Signals by Impulses:

- More generally,

$$
\begin{aligned}
x[n] = \cdots &+ x[-3]\delta[n+3] + x[-2]\delta[n+2] \\
&+ x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] \\
&+ x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots \\
= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]
\end{aligned}
$$

- The sifting property of the DT unit impulse
- x[n] = a superposition of scaled versions of shifted unit impulses δ[n-k]

DT Unit Impulse Response & Convolution Sum:

$$\text{input} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \text{output}$$

$$\delta[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_0[n]$$

$$\delta[n-1] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_1[n]$$

$$\delta[n-2] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_2[n]$$

$$\vdots$$

$$\delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n]$$

▪ DT Unit Impulse Response & Convolution Sum:

$$x[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow y[n]$$

$$x[0] \cdot \delta[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_0[n] \cdot x[0]$$

$$x[1] \cdot \delta[n-1] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_1[n] \cdot x[1]$$

$$x[2] \cdot \delta[n-2] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_2[n] \cdot x[2]$$

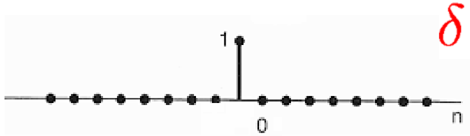
$$\vdots$$

$$x[k] \cdot \delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n] \cdot x[k]$$

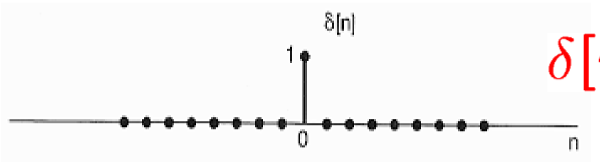
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

DT LTI Systems: Convolution Sum

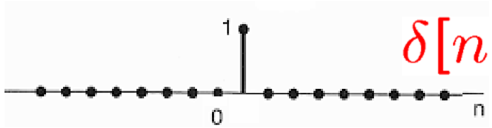
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

$x[-1] \times$ 

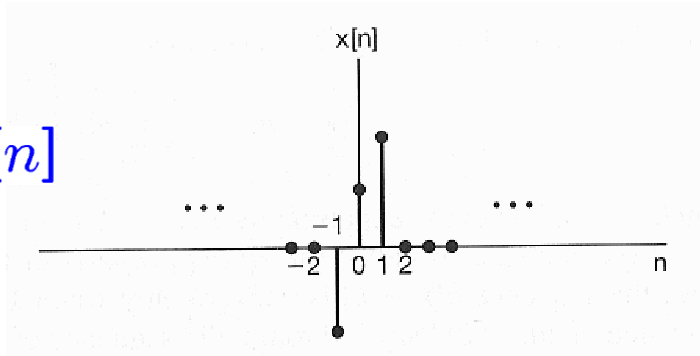
$\delta[n+1]$

$x[0] \times$ 

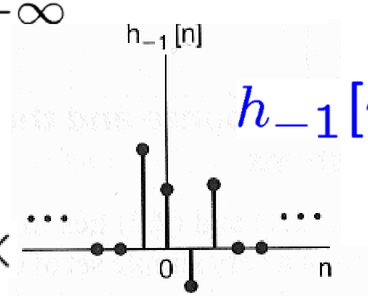
$\delta[n]$

$x[1] \times$ 

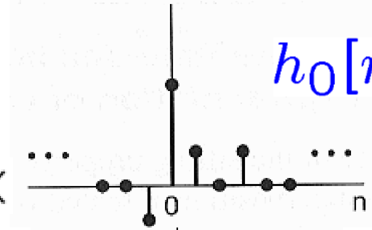
$\delta[n-1]$

$x[n]$ 

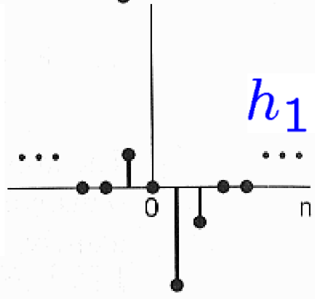
(a)

$x[-1] \times$ 

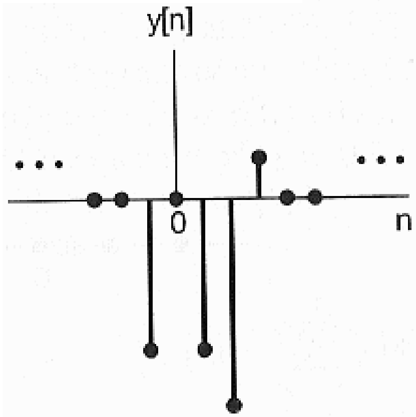
$h_{-1}[n]$

$x[0] \times$ 

$h_0[n]$

$x[1] \times$ 

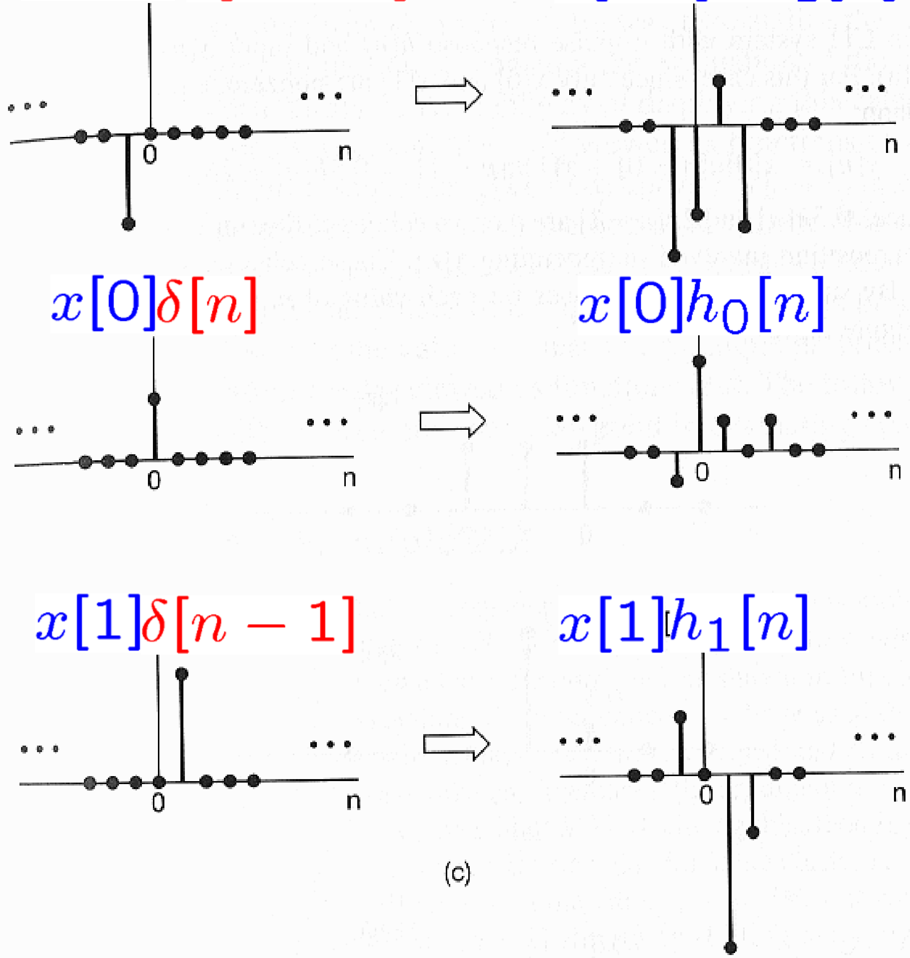
$h_1[n]$



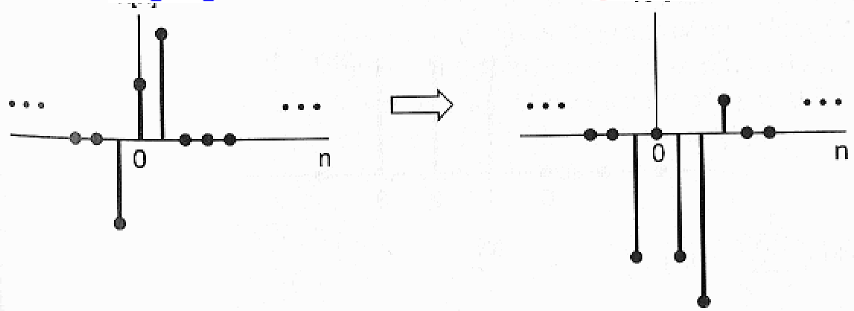
$y[n]$

DT LTI Systems: Convolution Sum

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \Longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$



(c)



(d)

$x[n] \longrightarrow$ Linear System $\longrightarrow y[n]$

- If the linear system (L) is also time-invariant (TI)
  - Then,

$$h_k[n] = h_0[n-k] = h[n-k]$$

- Hence, for an LTI system,

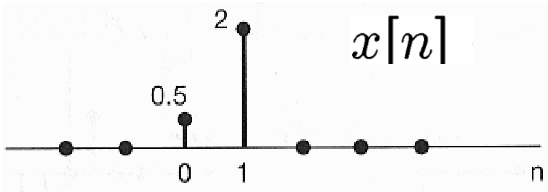
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

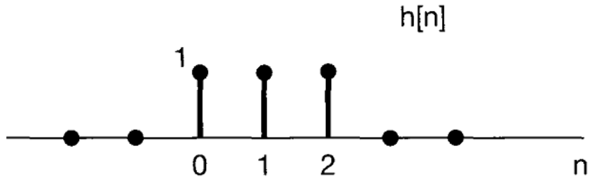
  - Known as the convolution of x[n] & h[n]
  - Referred as the convolution sum or superposition sum

- Symbolically, $y[n] = x[n] * h[n] = h[n] * x[n]$

- **Example 2.1:**

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

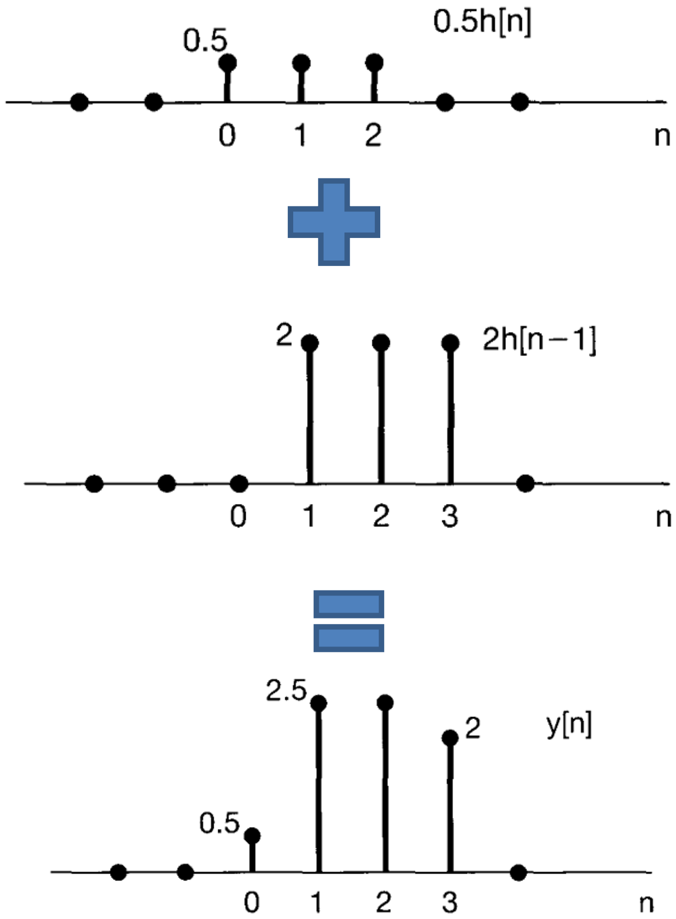




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

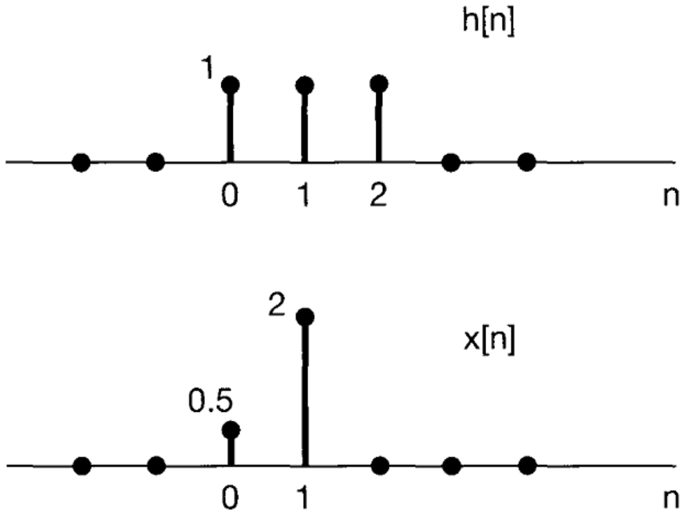
$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

$$= 0.5h[n] + 2h[n-1]$$



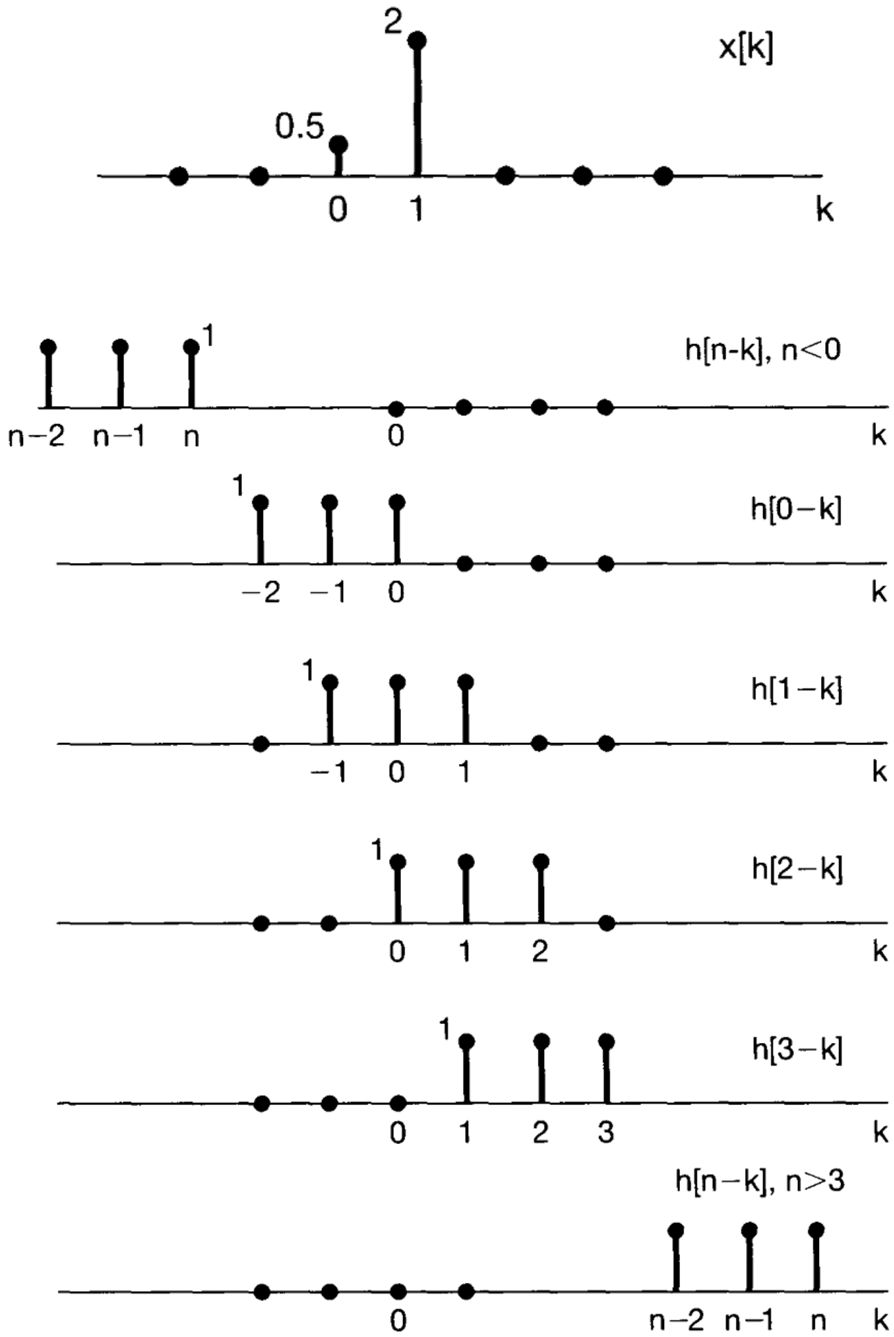
- **Example 2.2:**

The problem is as Example 2.1, but , we consider k as the independent variable!



(a)

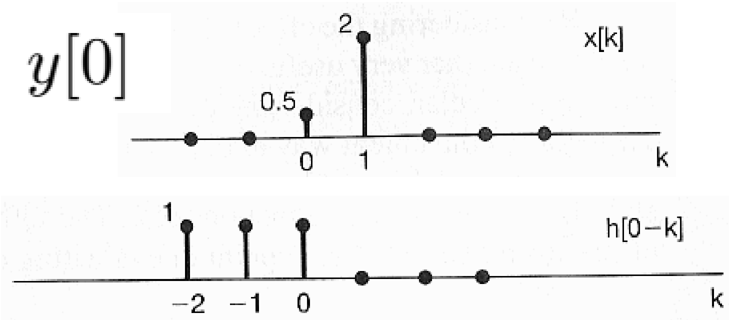
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



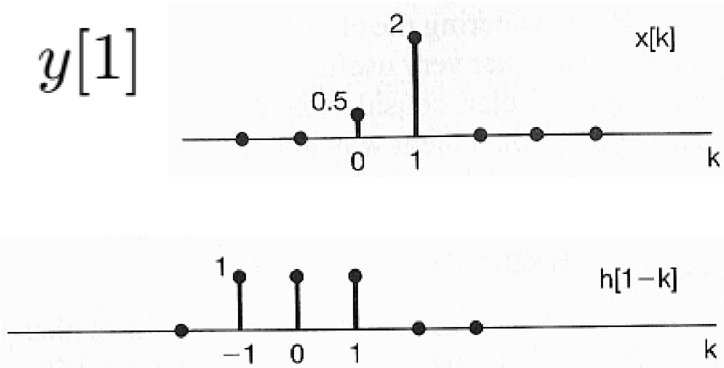
- **Example 2.2:**

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

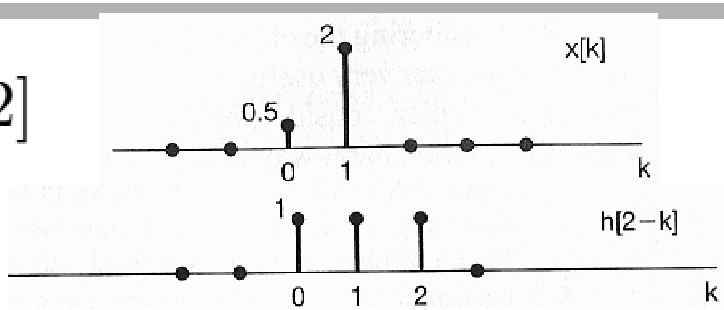
$y[0]$



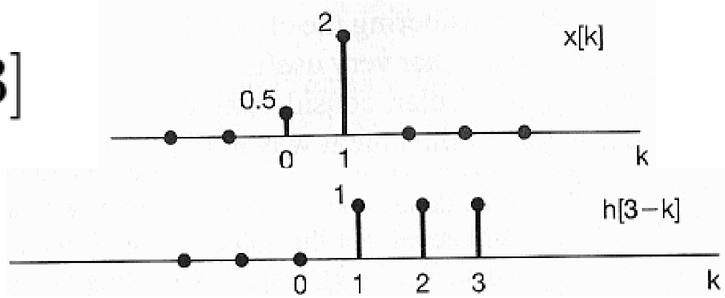
$y[1]$



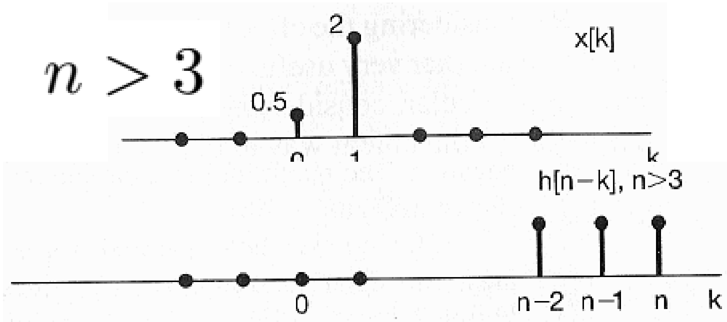
$y[2]$



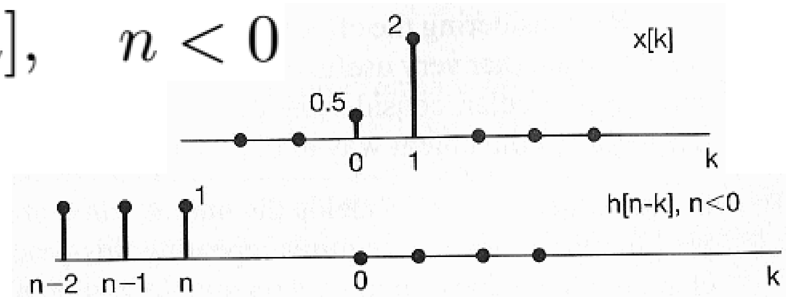
$y[3]$



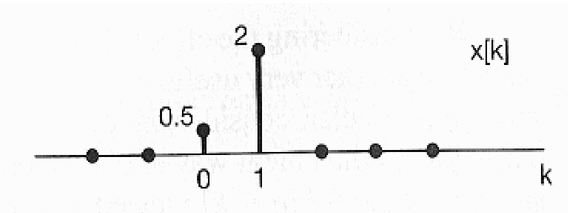
$y[n], \quad n > 3$

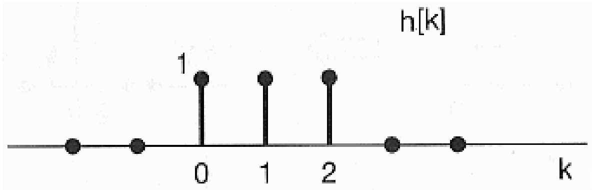


$y[n], \quad n < 0$



- **Example 2.2:** $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$





$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$

$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k] = 2.5$$

$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 2.5$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 2.5$$

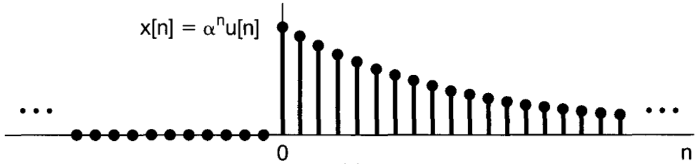
$$y[n] = 0 \text{ for } n < 0$$

$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 2.0$$

$$y[n] = 0 \text{ for } n > 3$$

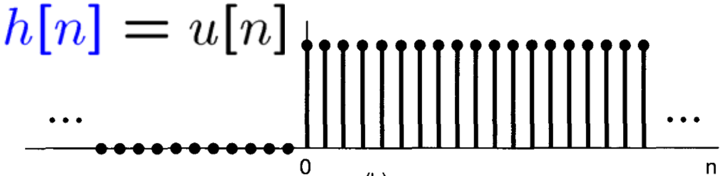
- **Example 2.3:**

$$x[n] = \alpha^n \, u[n], \qquad 0 < \alpha < 1$$



(a)

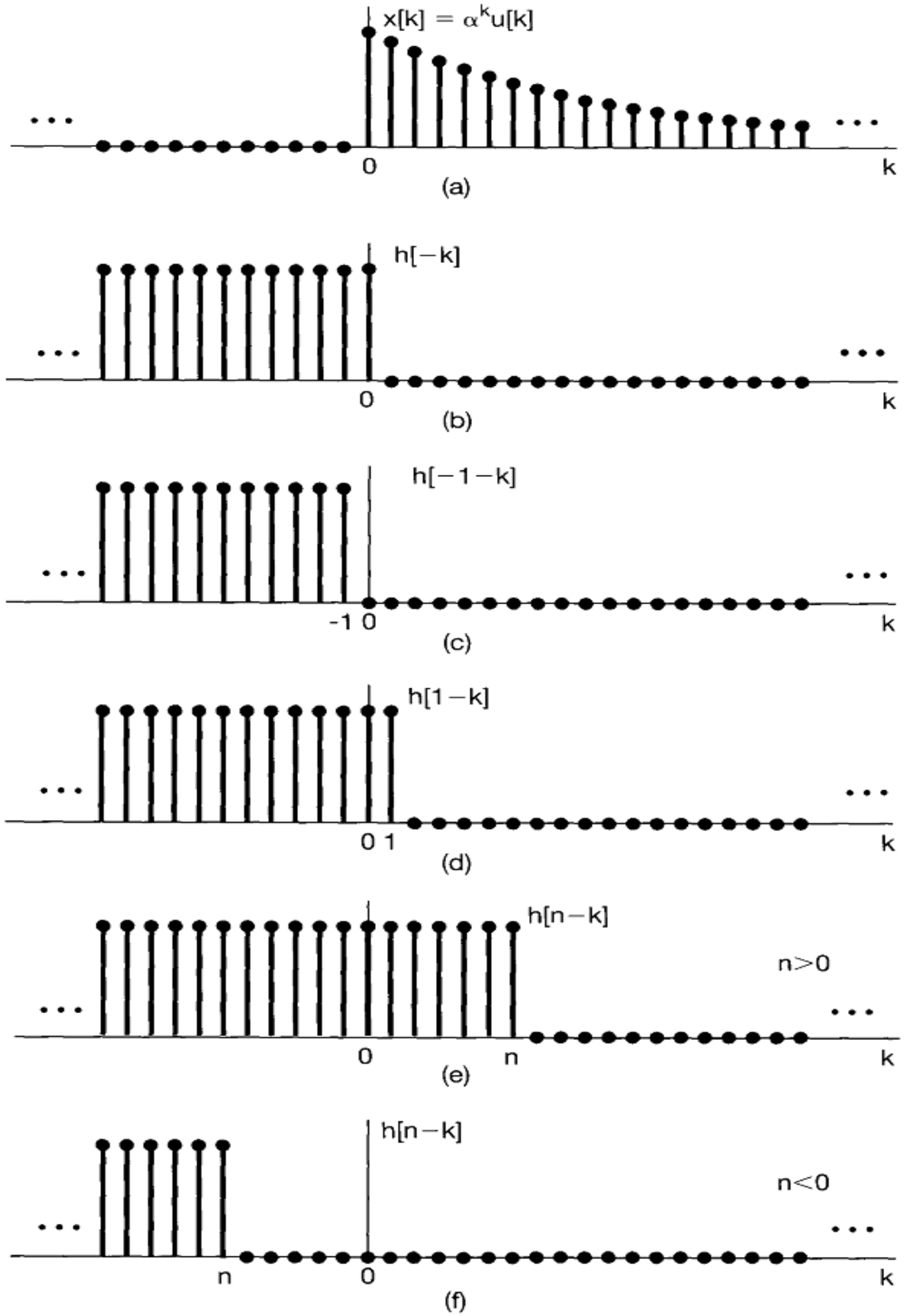
$$h[n] = u[n]$$



(b)

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

We consider k as
the independent variable!

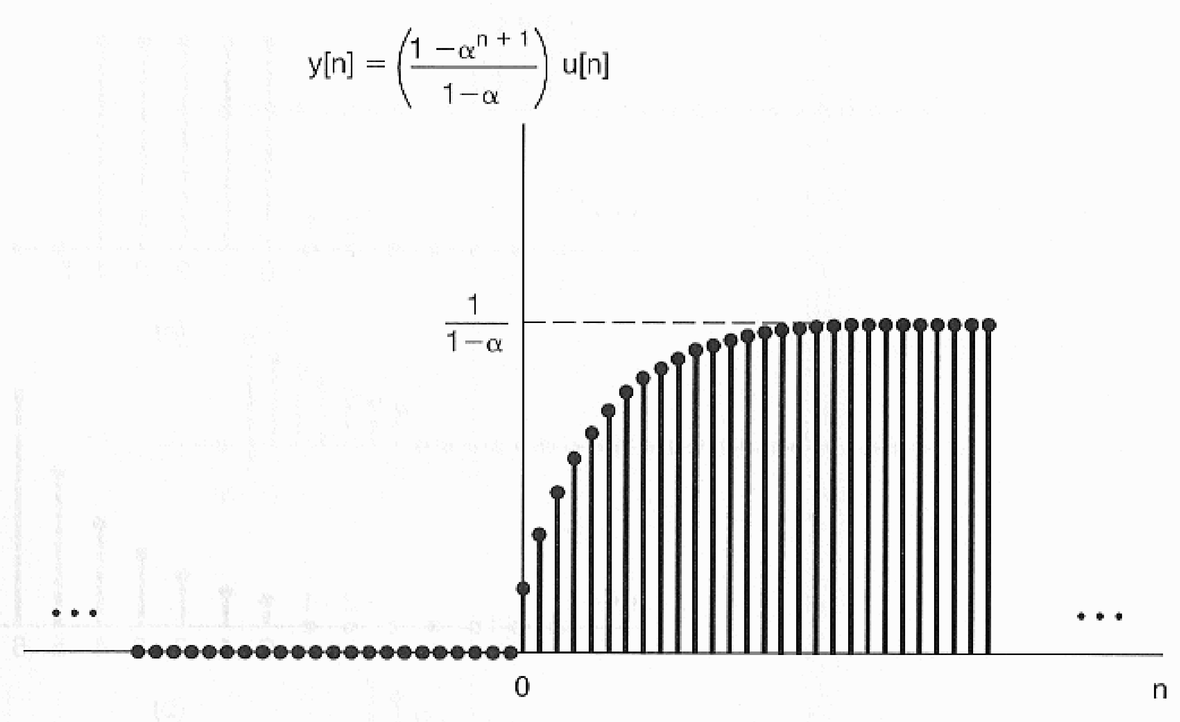


- Example 2.3:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

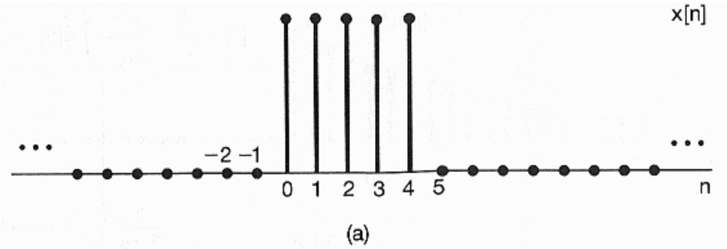
for $n \geq 0$,

$$x[k]\,h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases} \Rightarrow y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

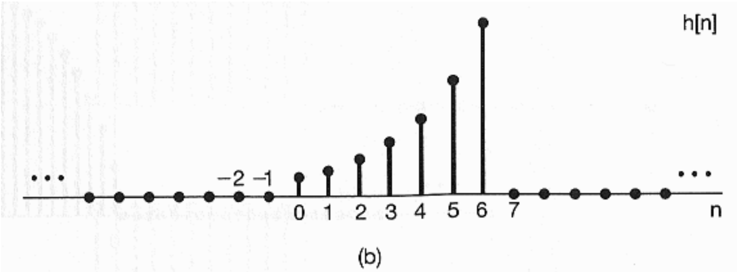


- Example 2.4:

$$x[n] = 1, \ 0 \leq n \leq 4$$



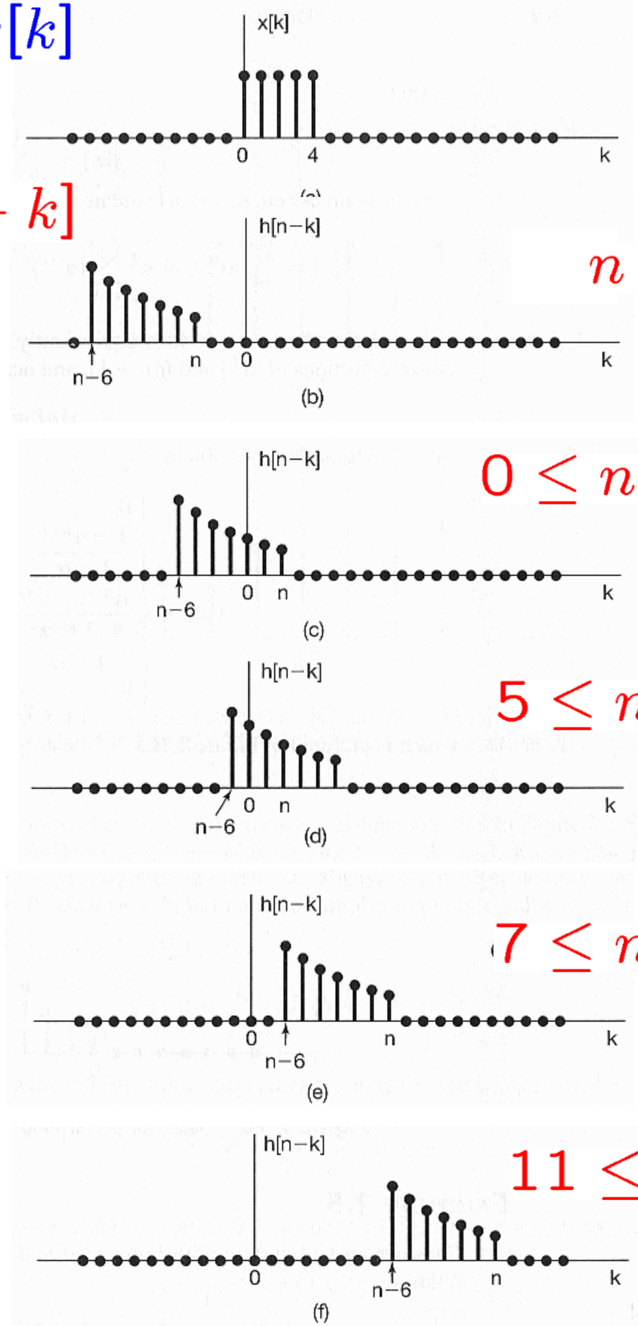
$$h[n] = \alpha^n, \ 0 \leq n \leq 6$$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$x[k]$

$h[n-k]$



$n \leq -1$

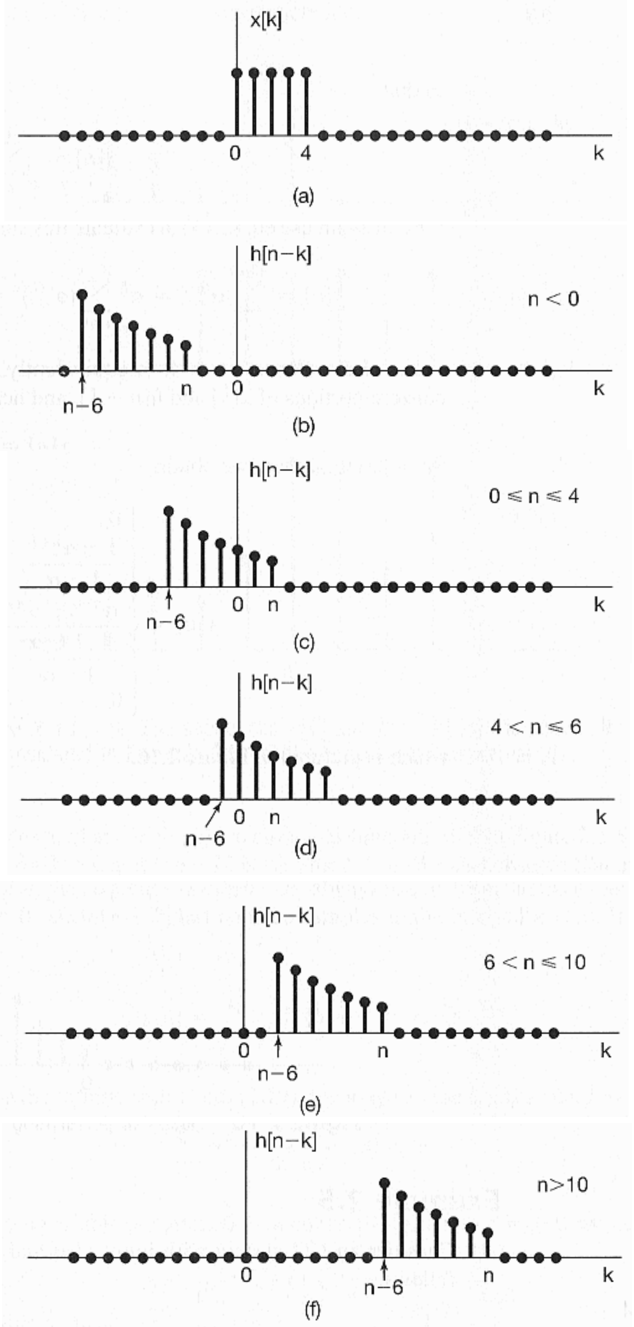
$0 \leq n \leq 4$

$5 \leq n \leq 6$

$7 \leq n \leq 10$

$11 \leq n$

DT LTI Systems: Convolution Sum



for $n < 0$, $\quad x[k]\, h[n-k] = 0 \quad \Rightarrow \quad y[n] = 0$

for $0 \le n \le 4$, $\quad x[k]\, h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{n-k} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for $4 < n \le 6$, $\quad x[k]\, h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

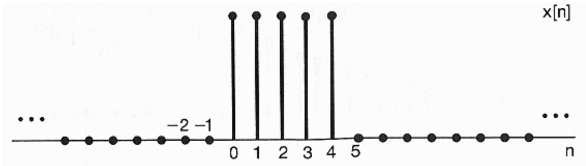
for $6 < n \le 10$, $\quad x[k]\, h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$$

for $n > 10$, $\quad y[n] = 0$

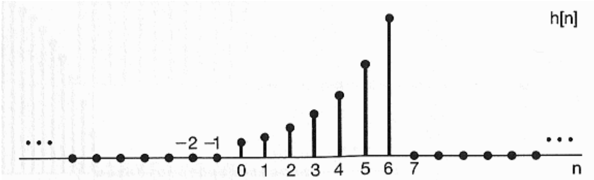
DT LTI Systems: Convolution Sum

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



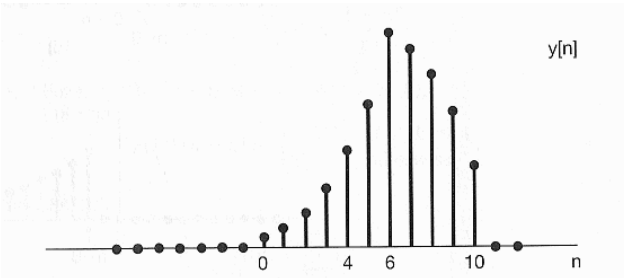
(a)

$$x[n] = 1, \; 0 \le n \le 4$$



(b)

$$h[n] = \alpha^n, \; 0 \le n \le 6$$



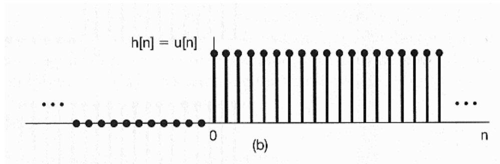
$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases}$$

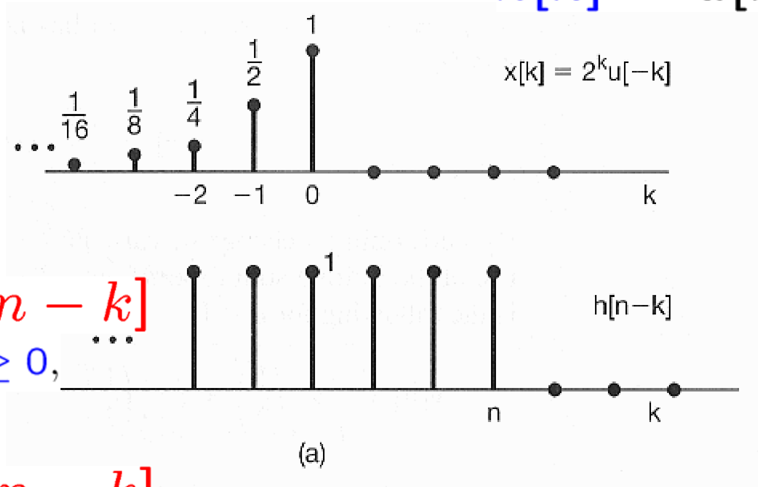
- **Example 2.5:**

$$x[n] = 2^n u[-n]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] = u[n]$$



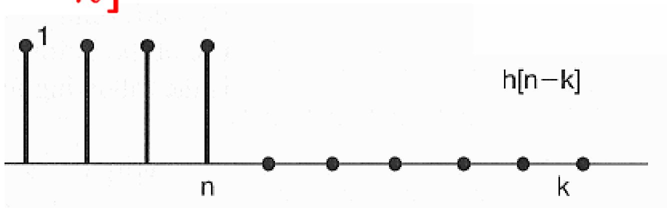


$h[n-k]$

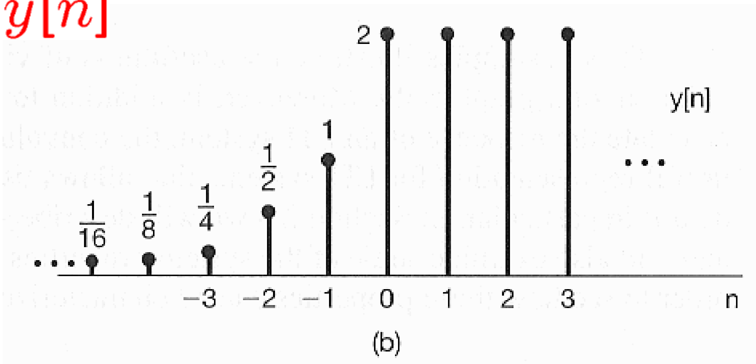
$n \geq 0$,

$h[n-k]$

$n < 0$



$y[n]$



for $n \geq 0$, $y[n] = \sum_{k=-\infty}^{0} x[k]\, h[n-k] = \sum_{k=-\infty}^{0} 2^k$

$$= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1-(1/2)} = 2$$

for $n < 0$, $y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k] = \sum_{k=-\infty}^{n} 2^k$

$$= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n}$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}$$

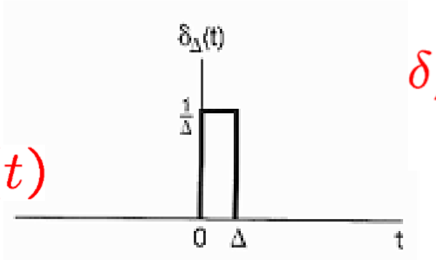
- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $y[n] = x[n] * h[n]$

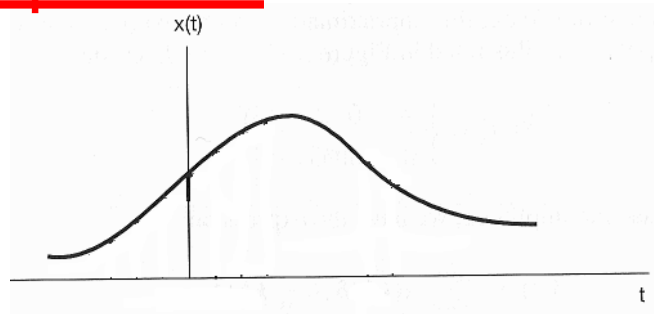
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

▪ Representation of CT Signals by Impulses:

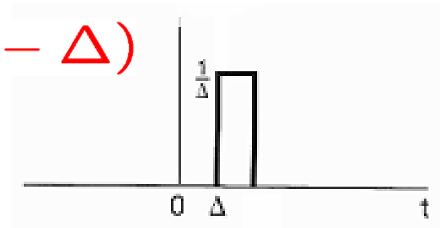
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

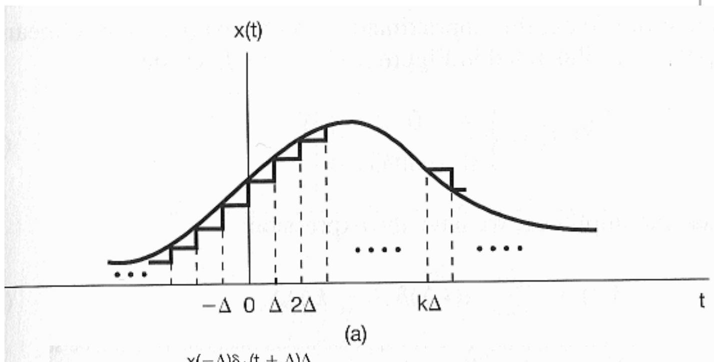
$\delta_\Delta(t)$



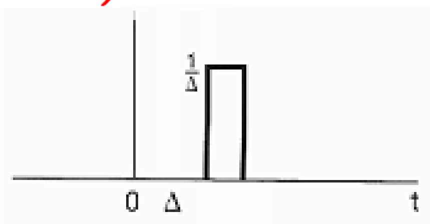


$\delta_\Delta(t-\Delta)$

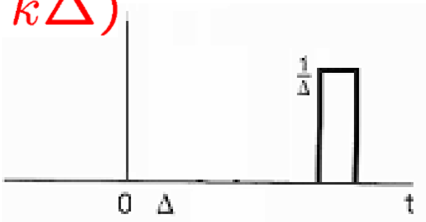


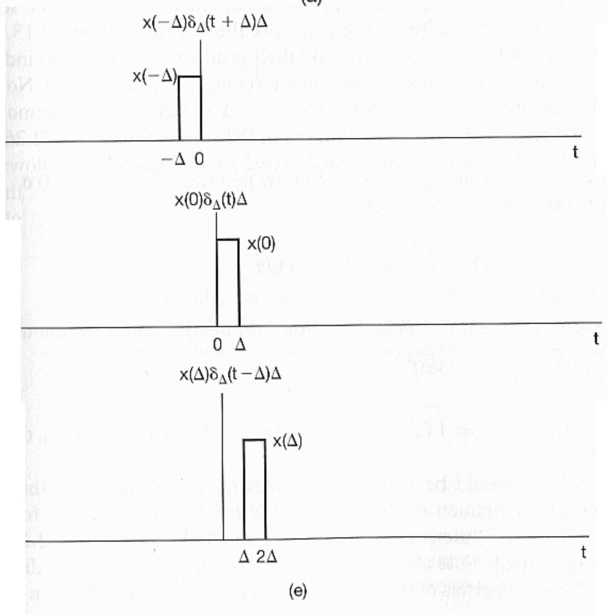


$\delta_\Delta(t-2\Delta)$



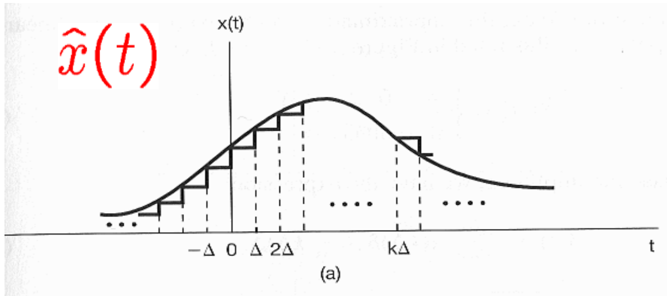
$\delta_\Delta(t-k\Delta)$



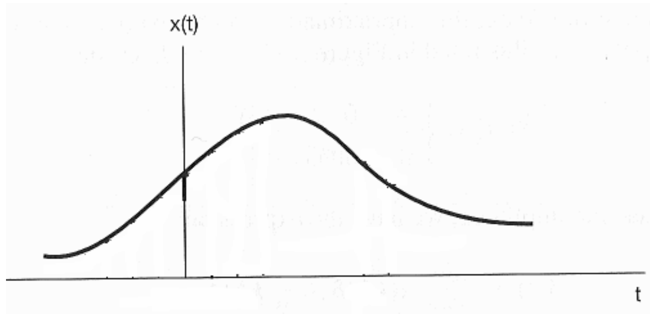


▪ Representation of CT Signals by Impulses:

$\hat{x}(t)$



$x(t)$



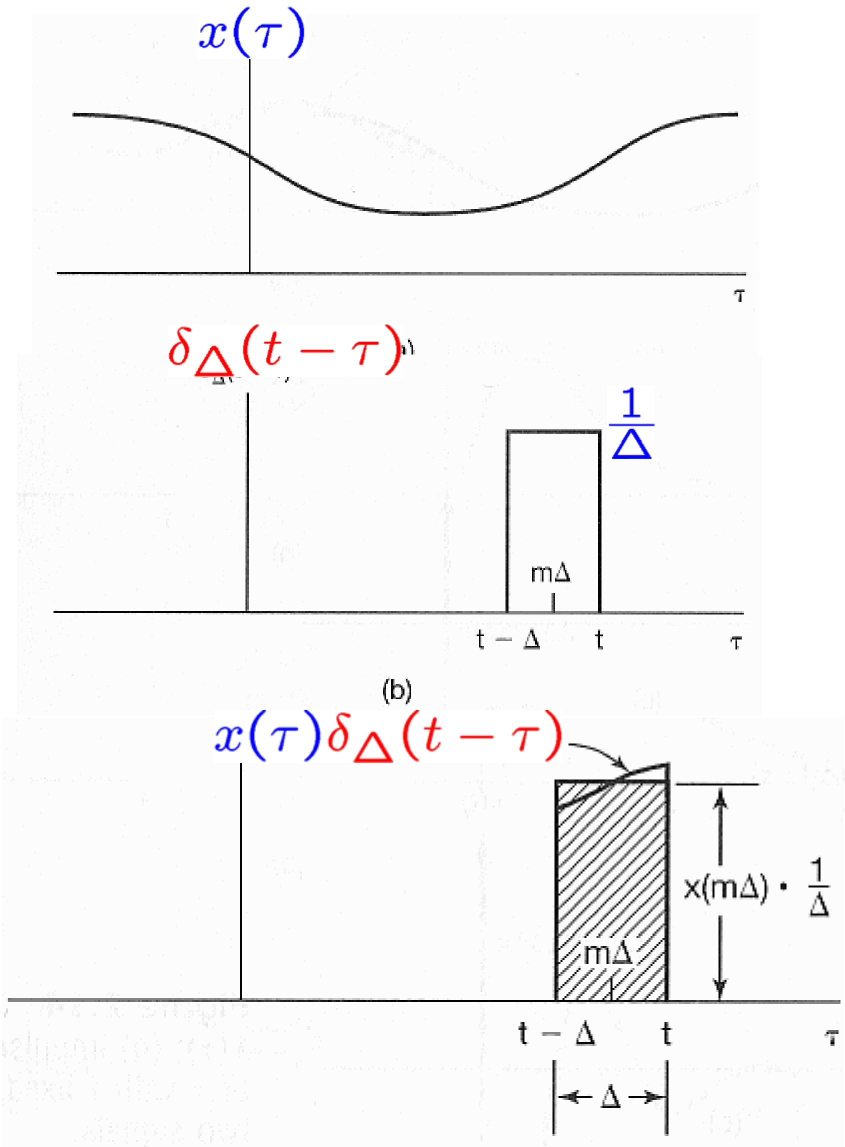
$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta \qquad x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

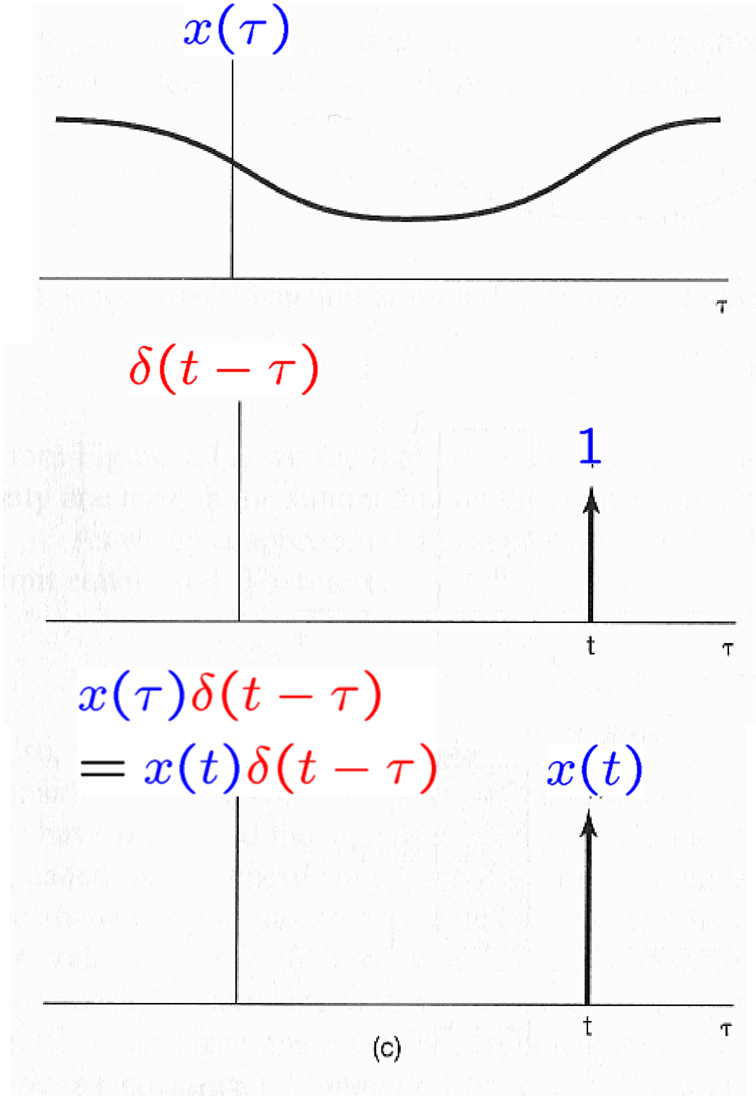
$$\int_{-\infty}^{+\infty} f(\tau)d\tau = \lim_{\Delta\to 0} \sum_{k=-\infty}^{k=+\infty} f(k\Delta)\Delta \Rightarrow x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

the sifting property of CT impulse

$x(t)$ = an integral of weighted, shifted impulses

- Graphical interpretation:





▪ CT Impulse Response & Convolution Integral:

$$\text{input} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \text{output}$$

$$\delta_\Delta(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{0\Delta}(t)$$

$$\delta_\Delta(t - 1\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{1\Delta}(t)$$

$$\delta_\Delta(t - 2\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{2\Delta}(t)$$

$$\vdots$$

$$\delta_\Delta(t - k\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{h}_{k\Delta}(t)$$

▪ CT Impulse Response & Convolution Integral:

input $\longrightarrow$ Linear System $\longrightarrow$ output

$x(0\Delta)$	$\delta_{\Delta}(t) \longrightarrow$	Linear System	$\longrightarrow \hat{h}_{0\Delta}(t)$	$x(0\Delta)$
$x(1\Delta)$	$\delta_{\Delta}(t-1\Delta) \longrightarrow$	Linear System	$\longrightarrow \hat{h}_{1\Delta}(t)$	$x(1\Delta)$
$x(2\Delta)$	$\delta_{\Delta}(t-2\Delta) \longrightarrow$	Linear System	$\longrightarrow \hat{h}_{2\Delta}(t)$	$x(2\Delta)$
		⋮		
$x(k\Delta)$	$\delta_{\Delta}(t-k\Delta) \longrightarrow$	Linear System	$\longrightarrow \hat{h}_{k\Delta}(t)$	$x(k\Delta)$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta \Longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\,\Delta$$

CT Unit Impulse Response & Convolution Integral:

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \longrightarrow \boxed{\text{Linear System}} \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\,\Delta$$

$$x(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\,\Delta$$

$$\delta(t-\tau) \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_\tau(t)$$

$$x(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y(t)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \qquad \Longrightarrow \qquad y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau$$

- If the linear system (L) is also time-invariant (TI)
  - Then,

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$h_\tau(t) = h_0(t-\tau) = h(t-\tau)$$

- Hence, for an LTI system,

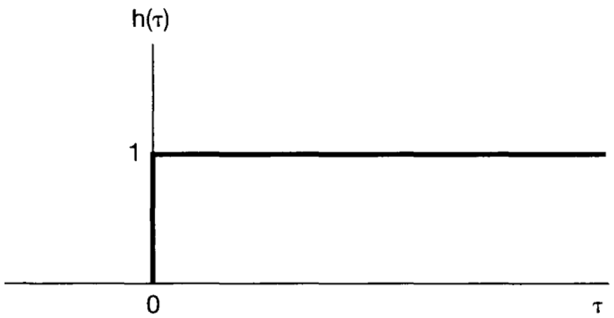
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

  - Known as the convolution of x(t) & h(t)
  - Referred as the convolution integral or the superposition integral

- Symbolically, $y(t) = x(t) * h(t) \qquad = h(t) * x(t)$

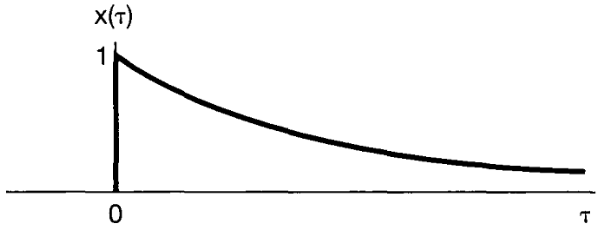
- **Example 2.6:** $y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$

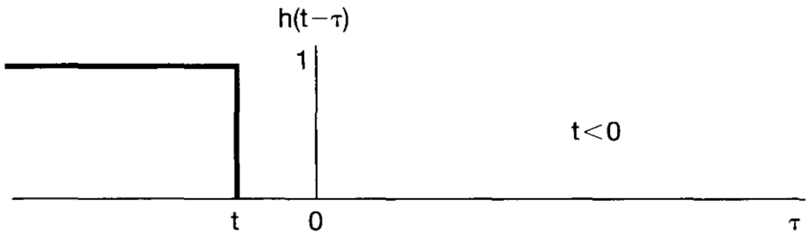
$h(\tau)$

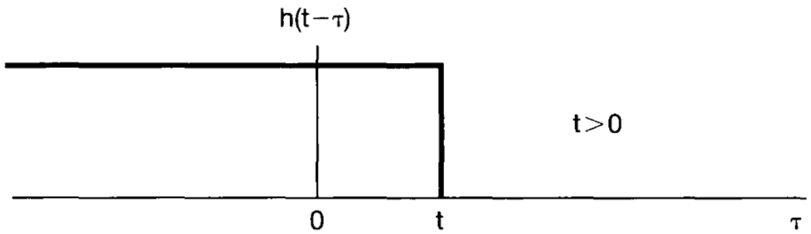


$x(t) = e^{-at}\, u(t)$

$a > 0$







for $t < 0$, $\quad x(\tau)\, h(t-\tau) = 0$

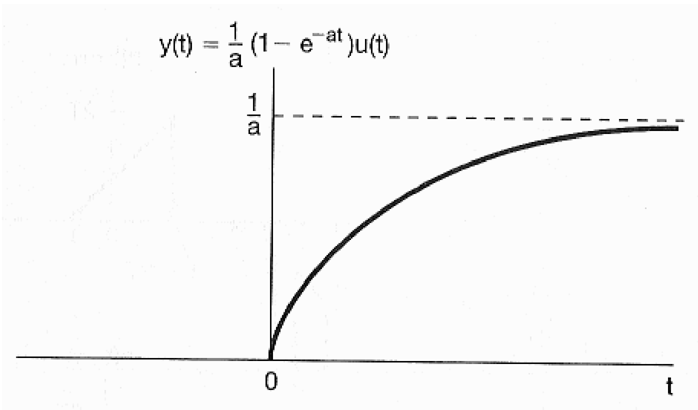
$$\Rightarrow \quad y(t) = \int_{-\infty}^{t} 0 \, d\tau = 0$$

for $t \geq 0$, $\quad x(\tau)\, h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \quad y(t) = \int_{0}^{t} e^{-a\tau} d\tau$$

$$= -\frac{1}{a} e^{-a\tau} \Big|_{0}^{t}$$

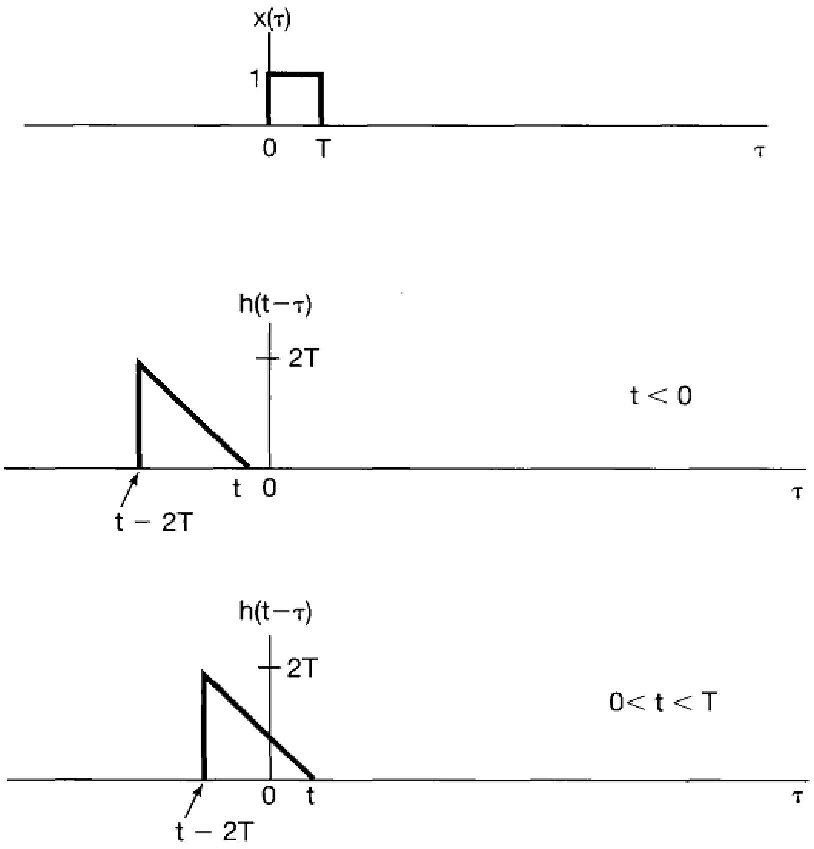
$$= \frac{1}{a}(1 - e^{-at})$$

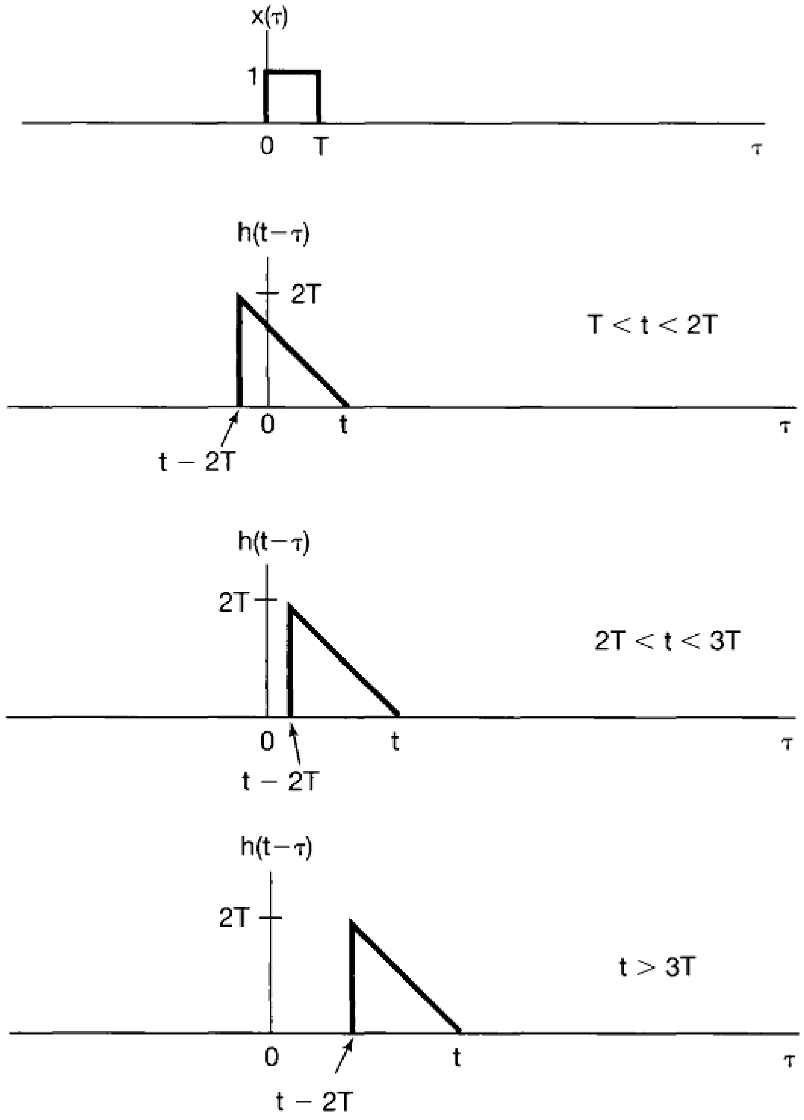


- **Example 2.7:** $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$





- **Example 2.7:** $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

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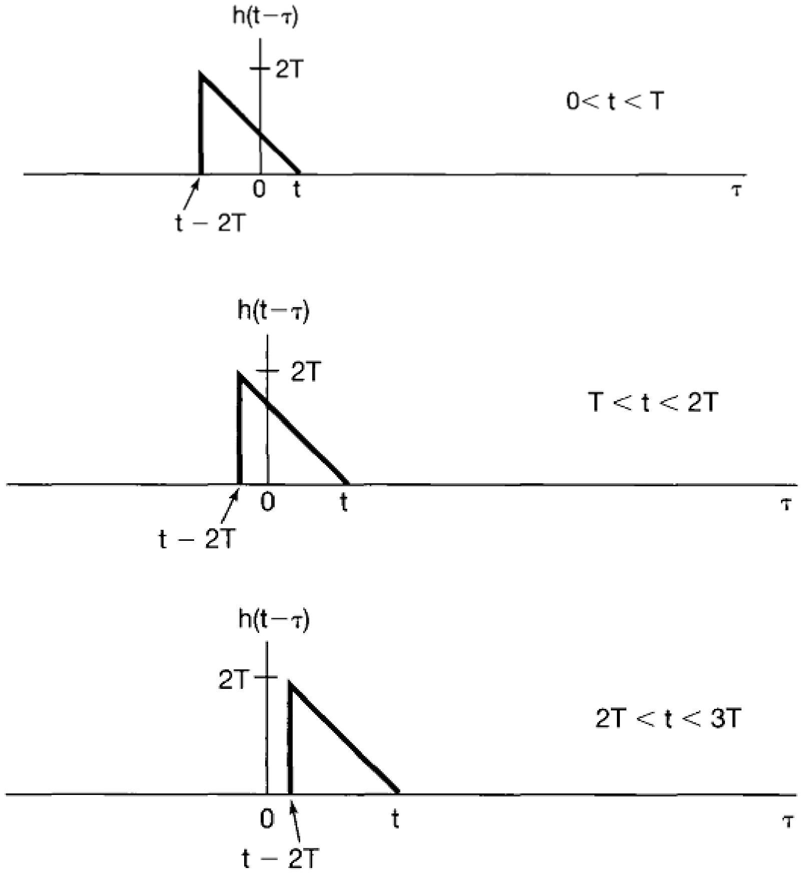
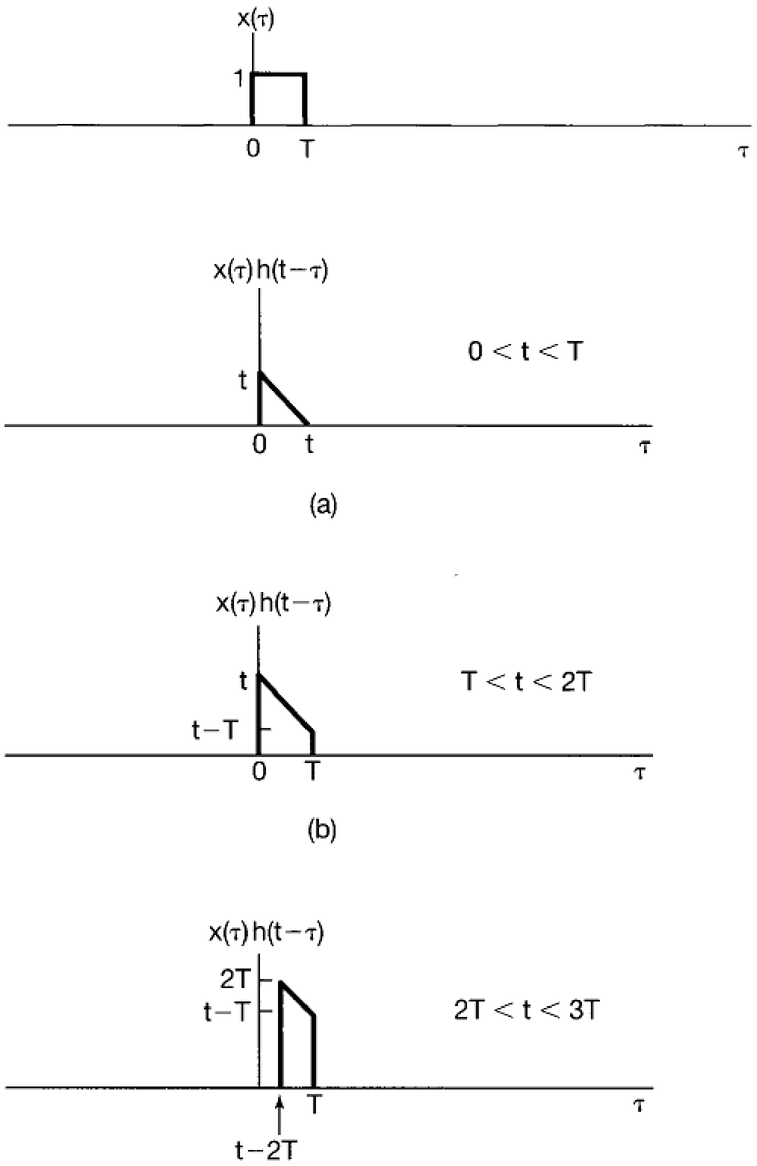


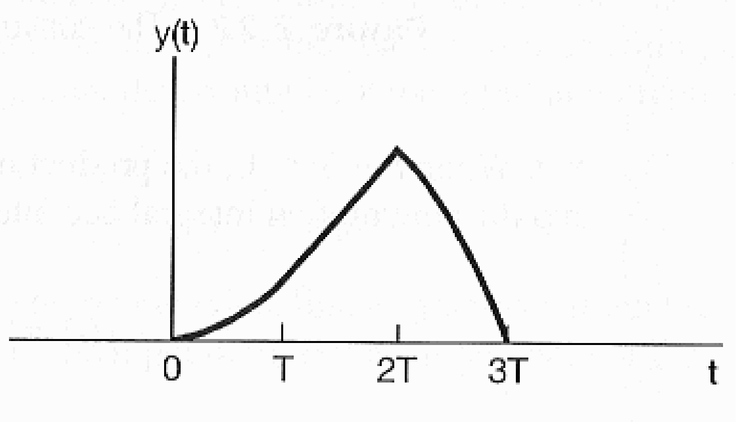


- Example 2.7:

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

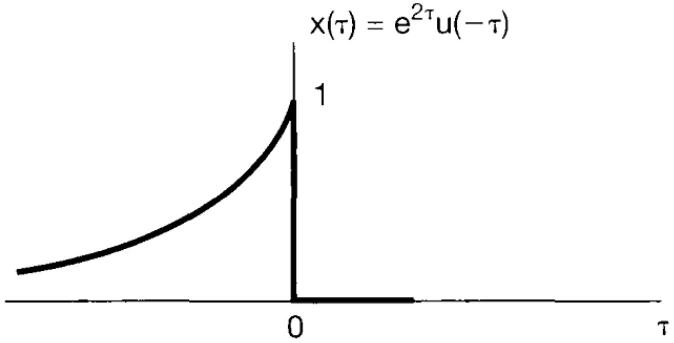
$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$

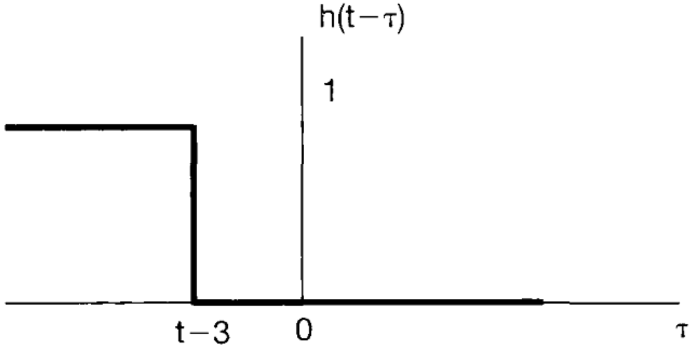


- Example 2.8:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x(t) = e^{2t}u(-t) \qquad h(t) = u(t-3)$$



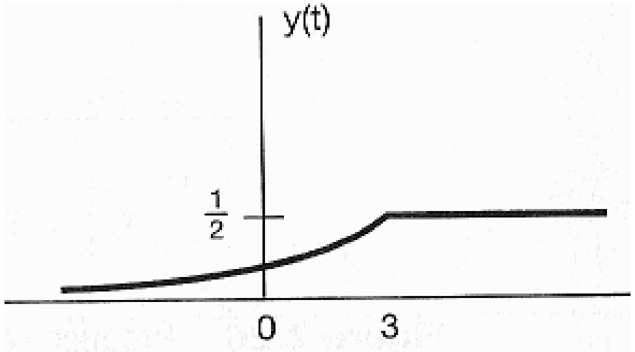


for $t-3 \le 0$, $y(t) = \int_{-\infty}^{t-3} e^{2\tau}d\tau$

$$= \frac{1}{2}e^{2(t-3)}$$

for $t-3 \ge 0$, $y(t) = \int_{-\infty}^{0} e^{2\tau}d\tau$

$$= \frac{1}{2}$$



Signal and System:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = x(t) * h(t)$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

All the following systems have h[n] as their impulse response

$$h[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$y[n] = (x[n] + x[n-1])^2,$$

$$y[n] = \max(x[n], x[n-1]).$$

$$y[n] = x[n] + x[n-1].$$

Only the third system is LTI and h[n] describes its complete characteristics!

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $\quad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

1. Commutative property

$$\begin{aligned} a \times b &= b \times a \\ a + b &= b + a \end{aligned}$$

2. Distributive property

$$a \times (b + c) = a \times b + a \times c$$

3. Associative property

$$\begin{aligned} a \times (b \times c) &= (a \times b) \times c \\ &= \cdots = a \times b \times c \end{aligned}$$

4. With or without memory

5. Invertibility

6. Causality

7. Stability

8. Unit step response

$$\boxed{a \times b \;=\; b \times a}$$

- Commutative Property:

$$n - k = r$$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \qquad = \sum_{r=+\infty}^{-\infty} x[n-r]h[r]$$

$$= \sum_{r=-\infty}^{+\infty} h[r]x[n-r] \qquad = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad \begin{aligned} t - \tau &= \sigma \\ -d\tau &= d\sigma \end{aligned}$$

$$= \int_{+\infty}^{-\infty} x(t-\sigma)h(\sigma)(-d\sigma)$$

$$= \int_{-\infty}^{+\infty} x(t-\sigma)h(\sigma)d\sigma$$

$$= \int_{-\infty}^{+\infty} h(\sigma)x(t-\sigma)d\sigma \;= h(t) * x(t)$$

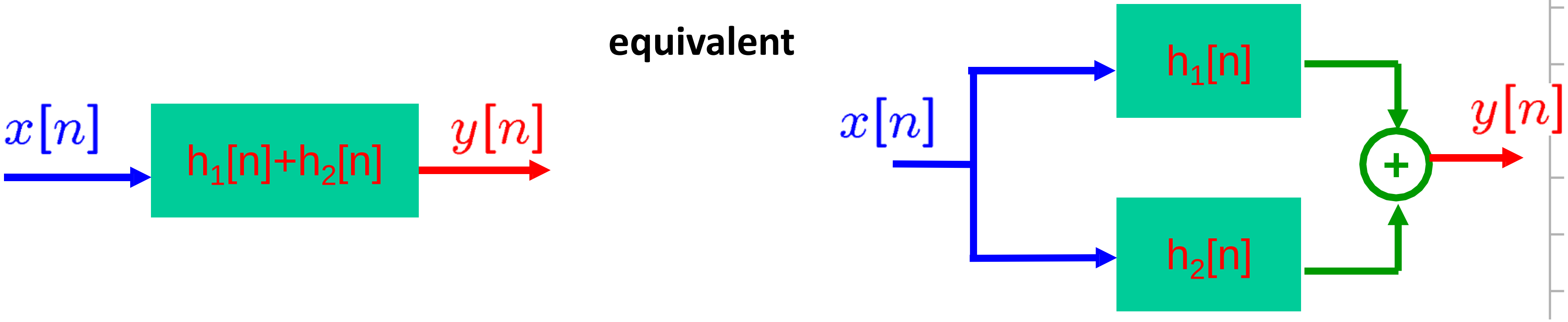
- # Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$



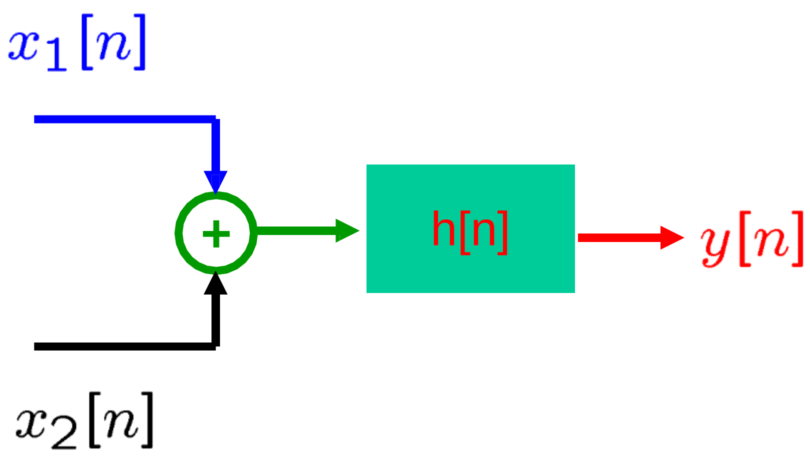
- **Distributive Property:**

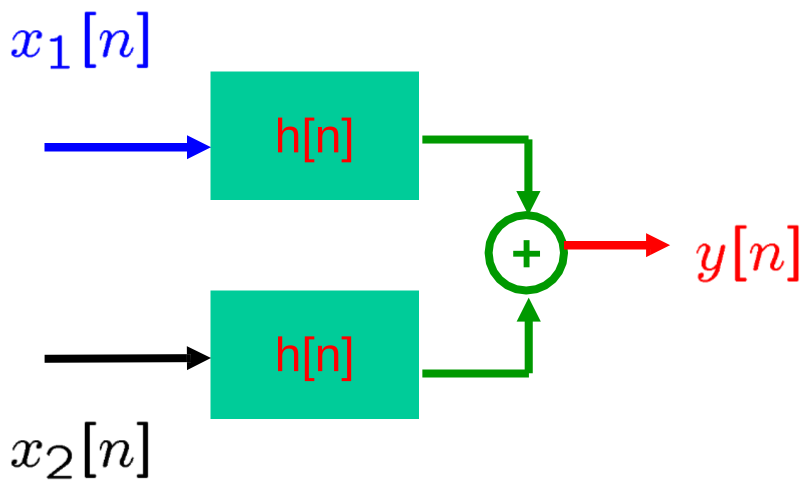
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Big(x_1[n] + x_2[n]\Big) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$\Big(x_1(t) + x_2(t)\Big) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

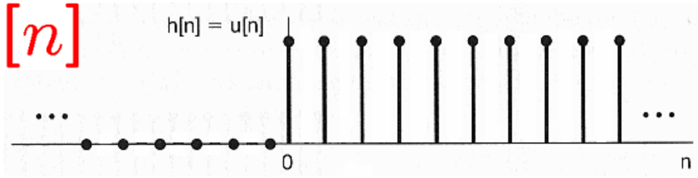




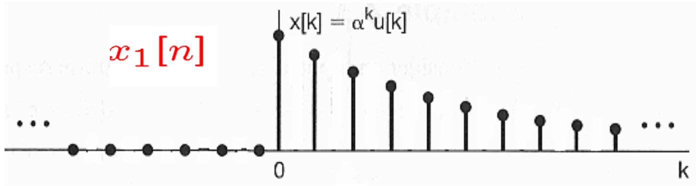
- Example 2.10

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

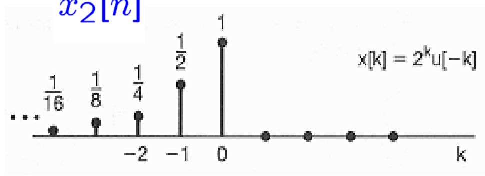
$h[n]$

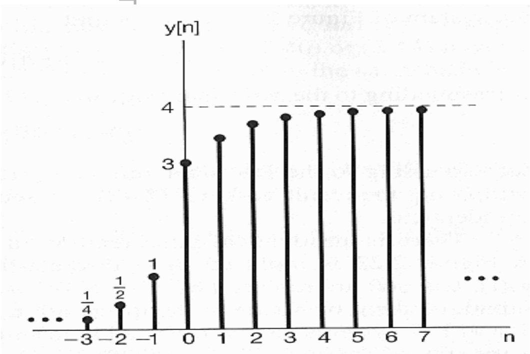


$x_1[n]$

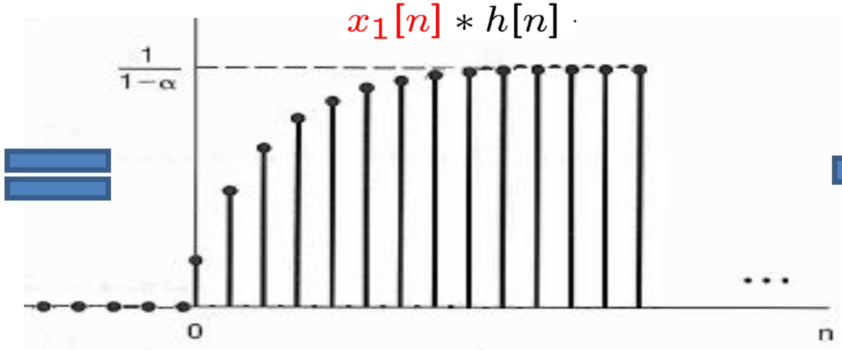


$x_2[n]$

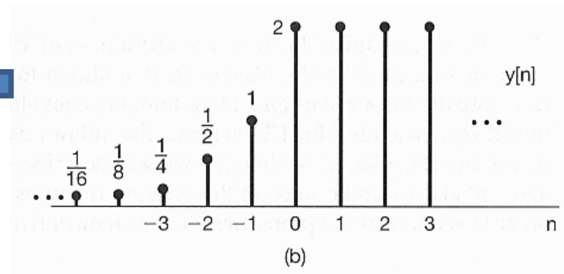




$x_1[n] * h[n]$



$x_2[n] * h[n]$



$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \left(x_1[n] + x_2[n]\right) * h[n] = x_1[n] * h[n] + x_2[n] * h[n] \end{aligned}$$

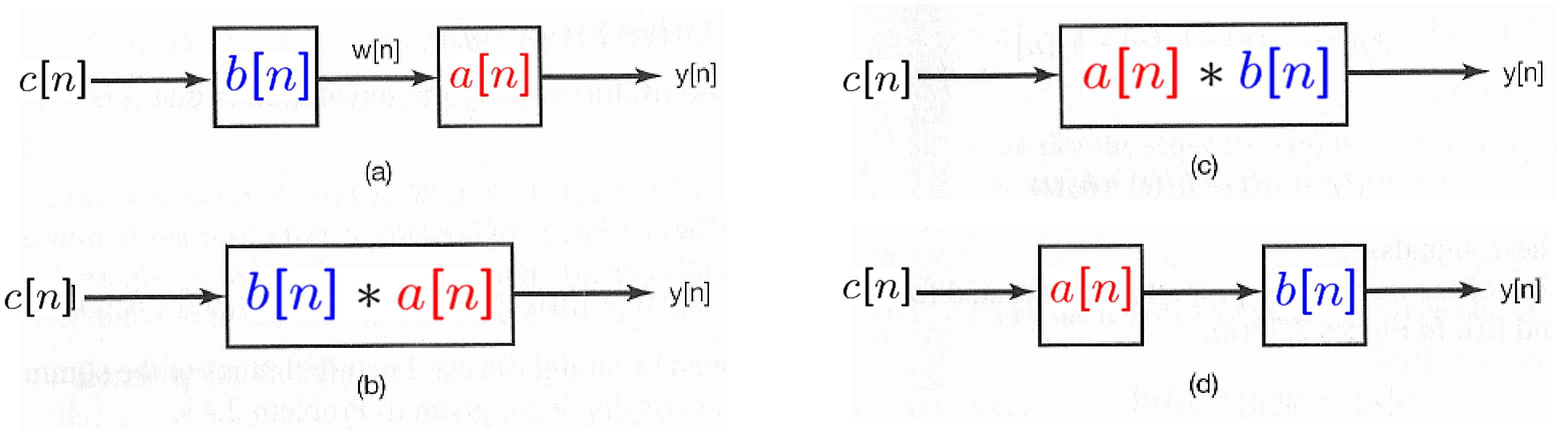
- **Associative Property:**

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$a[n] * \Big(b[n] * c[n]\Big) = \Big(a[n] * b[n]\Big) * c[n]$$

$$a(t) * \Big(b(t) * c(t)\Big) = \Big(a(t) * b(t)\Big) * c(t)$$



- Systems with or without memory
  - Memoryless systems
    - Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t)$$ (resistor)

  - Systems with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ (accumulator)

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

$$y[n] = x[n-1]$$ (delay)

Memoryless:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- A DT LTI system is memoryless if $h[n] = 0$ for $n \neq 0$

- The impulse response: $h[n] = K\delta[n], \quad K = h[0]$

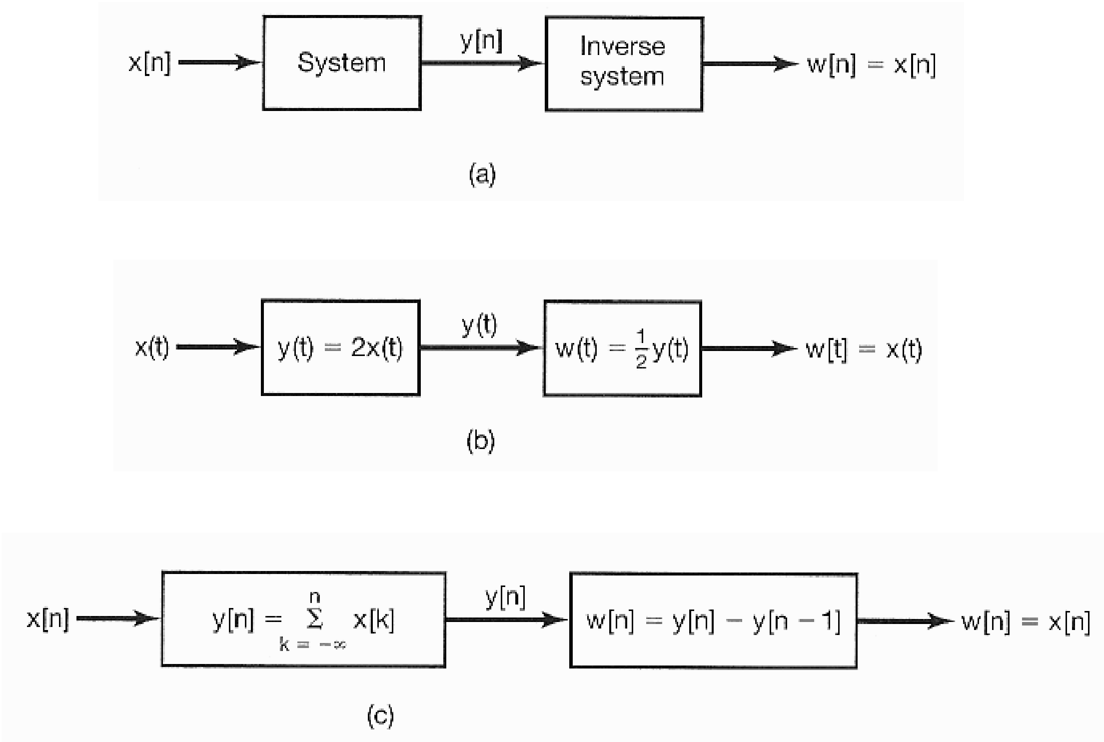
- The convolution sum: $y[n] = x[n] * h[n] = Kx[n]$

- Similarly, for CT LTI system: $y(t) = x(t) * h(t) = Kx(t)$

Invertibility & Inverse Systems

Invertible systems

- Distinct inputs lead to distinct outputs



$y(t) = x(t)^2$ is not invertible

- Invertibility:

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x(t) \rightarrow \boxed{\text{h1(t)}} \rightarrow y(t) \rightarrow \boxed{\text{h2(t)}} \rightarrow w(t)$$

$$y(t) = x(t) * h_1(t) \qquad w(t) = y(t) * h_2(t)$$

$$\Rightarrow w(t) = x(t) * h_1(t) * h_2(t)$$

$$x(t) \rightarrow \boxed{\text{Identity System } \delta\text{(t)}} \rightarrow x(t)$$

$$x(t) = x(t) * \delta(t)$$

$$\Longrightarrow h_2(t) * h_1(t) = \delta(t)$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- Example 2.11: Pure time shift

$$x(t) \rightarrow \boxed{\text{h1(t)}} \rightarrow y(t) \rightarrow \boxed{\text{h2(t)}} \rightarrow w(t)$$

- $y(t) = x(t - t_0)$
  - delay if $t_0 > 0$
  - advance if $t_0 < 0$

$$\Rightarrow \quad h_1(t) = \delta(t - t_0) \quad \Rightarrow \quad \boxed{x(t) * \delta(t - t_0) = x(t - t_0)}$$

- $w(t) = x(t) = y(t + t_0)$

$$\Rightarrow \quad h_2(t) = \delta(t + t_0) \quad \Rightarrow \quad y(t) * \delta(t + t_0) = y(t + t_0)$$

$$\Rightarrow \quad h_1(t) * h_2(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

- Example 2.12

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x[n] \rightarrow \boxed{\text{h1[n]}} \rightarrow y[n] \rightarrow \boxed{\text{h2[n]}} \rightarrow w[n]$$

$$h_1[n] = u[n]$$

$$\Rightarrow \; y[n] = \sum_{k=-\infty}^{\infty} x[k]\, u[n-k] \; = \sum_{k=-\infty}^{n} x[k]$$

$\Rightarrow$ a running-sum operation

- Its inverse is a first difference operation:

$$w[n] = y[n] - y[n-1] \;\Rightarrow\; h_2[n] = \delta[n] - \delta[n-1]$$

$$\Rightarrow \; h_1[n] * h_2[n] = u[n] - u[n-1] = \delta[n]$$

- **Causality:**

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

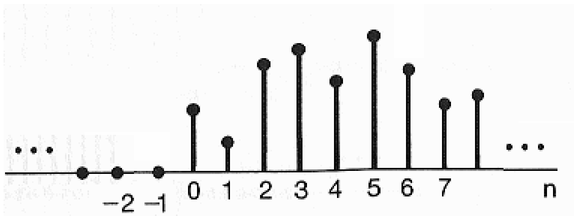
  - The output of a causal system depends only on the present and past values of the input to the system
  - Specifically, y[n] must not depend on x[k], for k > n

$$h[n-k] = 0, \qquad \text{for } k > n$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$h[n] = 0, \qquad \text{for } n < 0$$

  - It implies that the system is initially rest



  - A CT LTI system is causal if $h(t) = 0, \qquad \text{for } t < 0$

- Convolution Sum & Integral

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \; h[n-k]$$

$$= \sum_{k=-\infty}^{n} x[k] \; h[n-k]$$

$$= \sum_{k=0}^{\infty} h[k] \; x[n-k]$$

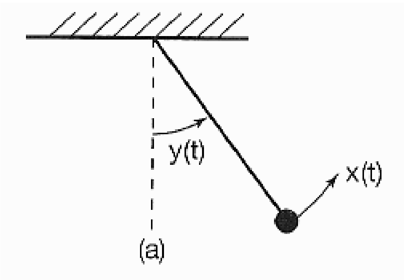
$$y(t) = \int_{-\infty}^{\infty} x(\tau) \; h(t-\tau) \; d\tau = \int_{0}^{\infty} h(\tau) \; x(t-\tau) \; d\tau$$

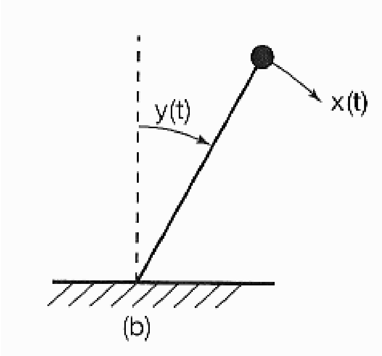
$$= \int_{-\infty}^{t} x(\tau) \; h(t-\tau) \; d\tau$$

Stability

Stable systems

- Small inputs lead to responses that do not diverge
- Every bounded input excites a bounded output
  - Bounded-input bounded-output stable (BIBO stable)
  - For all $|x(t)| < a$, then $|y(t)| < b$, for all t





- Balance in a bank account? $y[n] = 1.01y[n-1] + x[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- **Stability:**
  - A system is stable
    if every bounded input produces a bounded output

$$x[n] \rightarrow \boxed{\text{Stable LTI}} \rightarrow y[n]$$

$\left|x[n]\right| < B$ for all n

$$\left|y[n]\right| = \left|\sum_{k=-\infty}^{+\infty} h[k]x[n-k]\right|$$

$$\Rightarrow \left|y[n]\right| \leq \sum_{k=-\infty}^{+\infty} \left|h[k]\right| \left|x[n-k]\right|$$

$$\Rightarrow \left|y[n]\right| \leq B \left(\sum_{k=-\infty}^{+\infty} \left|h[k]\right|\right)$$

if $\sum_{k=-\infty}^{+\infty} \left|h[k]\right| < \infty$,i.e., absolutely summable then, $y[n]$ is bounded

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- **Stability:**
  - For CT LTI stable system:

$$x(t) \rightarrow \boxed{\text{Stable LTI}} \rightarrow y(t)$$

$$\left|x(t)\right| < B \quad \text{for all } t \qquad \left|y(t)\right| = \left|\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right|$$

$$\Rightarrow \left|y(t)\right| \leq \int_{-\infty}^{+\infty} \left|h(\tau)\right|\left|x(t-\tau)\right|d\tau$$

$$\Rightarrow \left|y(t)\right| \leq B \left(\int_{-\infty}^{+\infty} \left|h[(\tau)\right|d\tau\right)$$

if $\int_{-\infty}^{+\infty} \left|h(\tau)\right|d\tau < \infty$,i.e., absolutely integrable then, $y(t)$ is bounded

Example 2.13: Pure time shift

- $y[n] = x[n - n_0]$ & $h[n] = \delta[n - n_0]$
- $y(t) = x(t - t_0)$ & $h(t) = \delta(t - t_0)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \left|h[n]\right| = \sum_{n=-\infty}^{+\infty} \left|\delta[n - n_0]\right| = 1$$ absolutely summable

$$\Rightarrow \int_{-\infty}^{+\infty} \left|h(\tau)\right| = \int_{-\infty}^{+\infty} \left|\delta(\tau - t_0)\right| d\tau = 1$$ absolutely integrable

$\Rightarrow$ A (CT or DT) pure time shift is stable

Example 2.13: Accumulator

- $y[n] = \sum_{k=-\infty}^{n} x[k]$ & $h[n] = u[n]$
- $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ & $h(t) = u(t)$

$\Rightarrow \quad \sum_{n=-\infty}^{+\infty} \left| h[n] \right| = \sum_{n=0}^{+\infty} \left| u[n] \right| = \infty$ NOT absolutely summable

$\Rightarrow \quad \int_{-\infty}^{+\infty} \left| h(\tau) \right| = \int_{0}^{\infty} \left| u(\tau) \right| d\tau = \infty$ NOT absolutely integrable

$\Rightarrow$ A accumulator or integrator is NOT stable

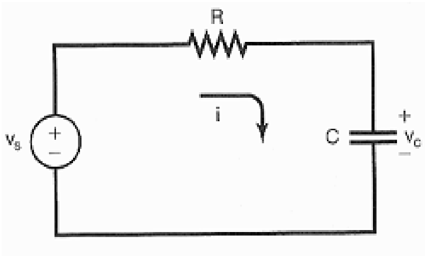
- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $\quad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $x(t) * \Big(h_1(t) + h_2(t)\Big) = x(t) * h_1(t) + x(t) * h_2(t)$
  3. Associative property $a(t) * \Big(b(t) * c(t)\Big) = \Big(a(t) * b(t)\Big) * c(t)$
  4. With or without memory $h(t) = 0$ for $t \neq 0$
  5. Invertibility $h_2(t) * h_1(t) = \delta(t)$
  6. Causality $h(t) = 0,\quad$ for $t < 0$
  7. Stability if $\int_{-\infty}^{+\infty} \big|h(\tau)\big| d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions $u_k(t) * u_r(t) = u_{k+r}(t)$

Linear Constant-Coefficient Differential Equations

- Describe an important class of continuous systems.
- In Diff Eqn, we have learn a way to analyze their solutions.
- In subsequent chapters, additional tools for the analysis of systems described by such equations are developed.

- Provide an implicit specification of the system

Input signal: $v_s(t)$



Output signal: $v_c(t)$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

$x(t) \rightarrow$ RC Circuit $\rightarrow y(t)$

$$\Rightarrow \frac{d}{dt}y(t) + a\, y(t) = b\, x(t)$$

Example 2.14

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$
$$y(0) = 0$$
$$x(t) = Ke^{3t}u(t) \Longrightarrow y(t) = ?$$

$$y(t) = y_h(t) + y_p(t)$$

$y_h(t)$ solution of homogeneous equation, $x(t) = 0$

$y_p(t)$ particular response, $x(t) = Ke^{3t}u(t)$

Example 2.14

homogeneous response

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$y(0) = 0$$

$$x(t) = Ke^{3t}u(t) \Longrightarrow y(t) = ?$$

$$\frac{dy_h(t)}{dt} + 2y_h(t) = 0$$

$$y_h(t) = Ae^{st} \Longrightarrow sAe^{st} + 2Ae^{st} = 0$$

$$\Longrightarrow (s+2)Ae^{st} = 0 \Longrightarrow s = -2$$

$$y_h(t) = Ae^{-2t}$$

Example 2.14

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$y(0) = 0$$

$$x(t) = Ke^{3t}u(t) \Longrightarrow y(t) = ?$$

Particular response for t>0

$$\frac{dy_p(t)}{dt} + 2y_p(t) = Ke^{3t} \qquad t \geq 0$$

$$y_p(t) = Be^{3t} \Longrightarrow 3Be^{3t} + 2Be^{3t} = Ke^{3t} \Longrightarrow B = K/5$$

$$y_p(t) = Ke^{3t}/5$$

$$y(t) = Ae^{-2t} + Ke^{3t}/5$$

$$y(0) = A + K/5 = 0 \Longrightarrow A = -K/5$$

Example 2.14

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$y(0) = 0$$

$$x(t) = Ke^{3t}u(t) \Longrightarrow y(t) = ?$$

Particular response for t<0

$$\frac{dy_p(t)}{dt} + 2y_p(t) = 0 \qquad t < 0$$

$$y_p(t) = 0$$

$$y(t) = Ae^{-2t}$$

$$y(0) = 0 = A$$

Linear Constant-Coefficient Differential Equations

- The response of a linear system to zero input is zero !

$$x_1(t) = 0 \Longrightarrow y_1(t) = a$$
$$x_2(t) = Kx_1(t) = 0 \Longrightarrow y_2 = Ky_1(t) = Ka \quad K \neq 1$$
$$Ka = a \to a = 0.$$

- So, when the auxiliary condition is not zero, the system is not linear

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$
$$y(0) = \alpha \neq 0$$

Linear Constant-Coefficient Differential Equations

- For a causal system, we have

$$x_1(t) = x_2(t) \quad \forall t < t_0$$

$$\Longrightarrow$$

$$y_1(t) = y_2 \quad \forall t < t_0$$

- So, a causal linear system should be initially at rest!

$$x_1(t) = 0 \quad \forall t \Longrightarrow y_1(t) = 0 \quad \forall t$$

$$x_2(t) = 0 \quad \forall t < t_0 \Longrightarrow y_2 = 0 \quad \forall t < t_0$$

Linear Constant-Coefficient Differential Equations

- For a general CT LTI system, with N-th order,

$$x(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow y(t)$$

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$

$$= b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

To completely determine the answer $y(t_0), \ldots, y^{N-1}(t_0)$, should be given

Linear Constant-Coefficient Differential Equations

- The linear ODEs without specifying initial conditions, do not declare a unique system.
- Consider

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

If $y(t_0), \ldots, y^{N-1}(t_0)$ Is not declared, $\Rightarrow$ No specific system is implied

If initial condition $y(t_0), \ldots, y^{N-1}(t_0) \neq 0$ $\Rightarrow$ The system is not linear

If t_0 in $y(t_0)$ Is fixed $\Rightarrow$ The system is neither causal nor time invariant

- The linear Constant Coefficient Differential Equation Which is initially at rest specifies a causal LTI system

Linear Constant-Coefficient Differential Equations

- The linear constant coefficient difference equation without specifying initial conditions, does not declare a unique system.
- Consider

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

If $y[n_0-1], \dots, y[n_0-N]$ Is not declared, $\Rightarrow$ No specific system is implied

If initial condition $y[n_0-1], \dots, y[n_0-N] \neq 0$ $\Rightarrow$ The system is not linear

If n_0 in $y[n_0]$ Is fixed $\Rightarrow$ The system is neither causal nor time invariant

- The linear Constant Coefficient Difference Equation Which is initially at rest specifies a causal LTI system

Linear Constant-Coefficient Difference Equations

- For a general DT LTI system, with N-th order,

$$x[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow y[n]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\Rightarrow \sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$$\Rightarrow h[n] = ?$$

- Recursive Equation:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$$\Rightarrow \; y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k \, x[n-k] - \sum_{k=1}^{N} a_k \, y[n-k] \right\}$$

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$
$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

Recursive Equation:

- For example, (Example 2.15)

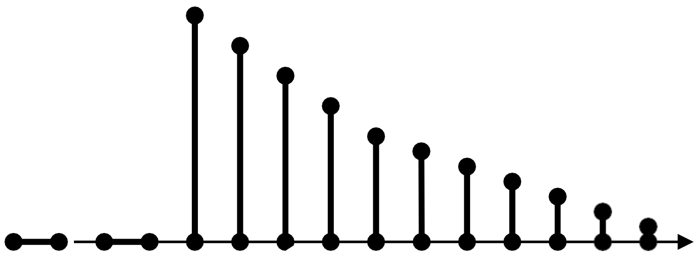
$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

$$y[n] = 0, \quad \text{for } n \leq -1$$

$$x[n] = K\ \delta[n]$$

$$\Rightarrow \begin{cases} y[0] = x[0] + \frac{1}{2}y[-1] & = K \\ y[1] = x[1] + \frac{1}{2}y[0] & = K\ \frac{1}{2} \\ y[2] = x[2] + \frac{1}{2}y[1] & = K\ \left(\frac{1}{2}\right)^2 \\ \vdots & \\ y[n] = x[n] + \frac{1}{2}y[n-1] & = K\ \left(\frac{1}{2}\right)^n \end{cases}$$

$$\Rightarrow\ h[n] = \left(\frac{1}{2}\right)^n u[n]$$



$\Rightarrow$ an Infinite Impulse Response (IIR) system

- **Nonrecursive Equation:**

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

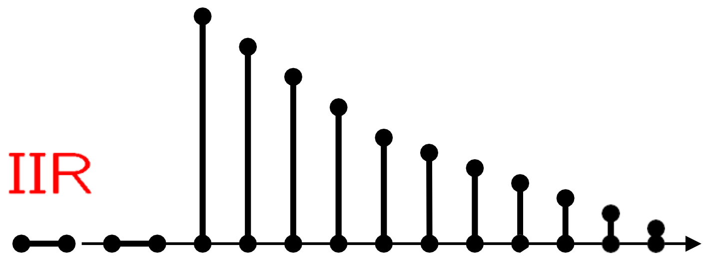
  - When $N = 0$,

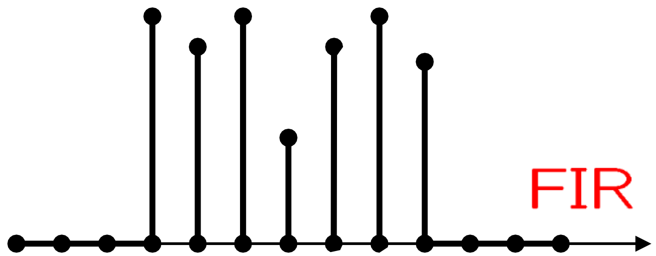
$$\Rightarrow \quad y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0} \right) x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$$

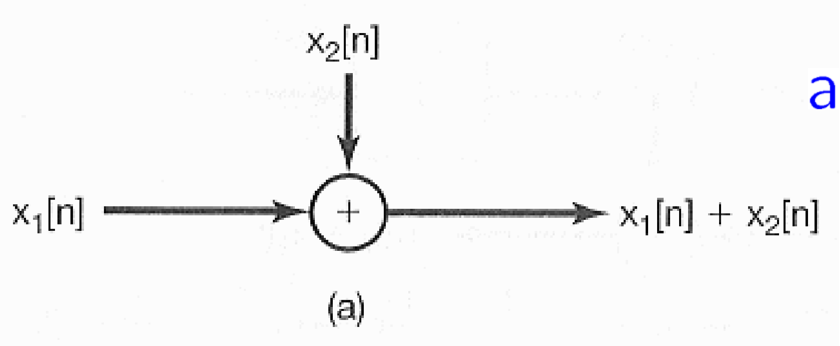
$$\Rightarrow \quad h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ a Finite Impulse Response (FIR) system



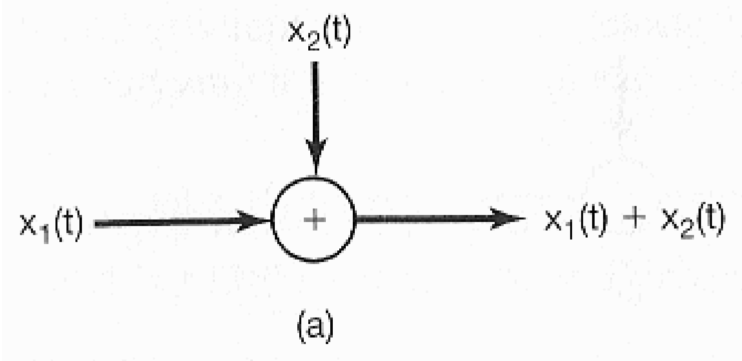


▪ Block Diagram Representations:



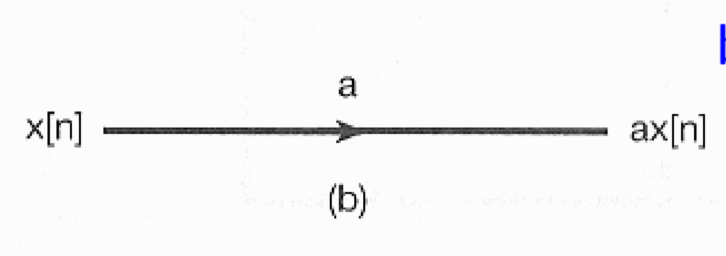
(a)

an adder

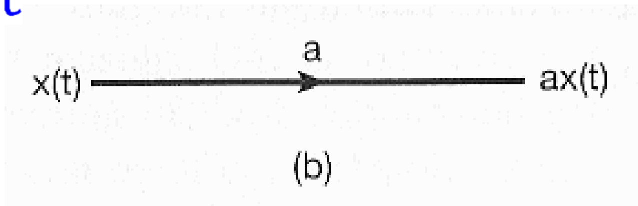


(a)

multiplication
by a coefficient

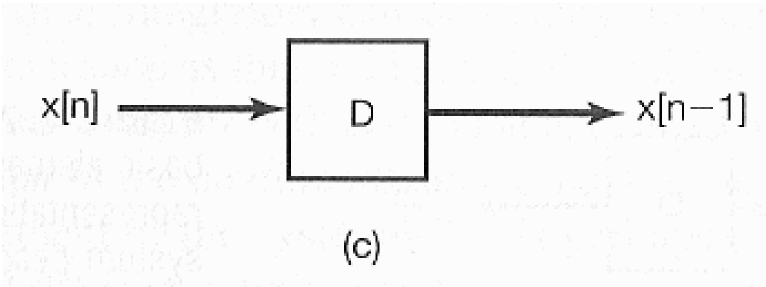


(b)

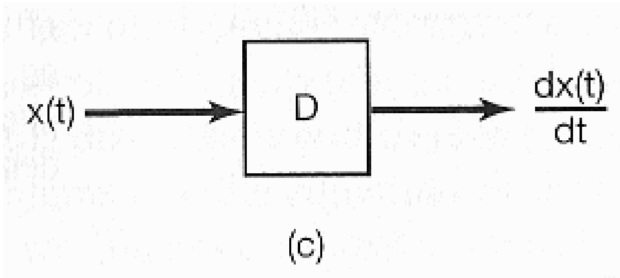


(b)

a unit delay/
differentiator



(c)



(c)

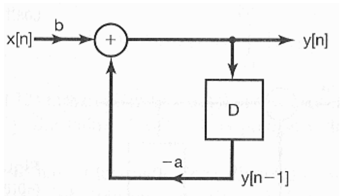
Block Diagram Representations:

$$y[n] + a y[n-1] = b x[n]$$

$$\frac{d}{dt} y(t) + a y(t) = b x(t)$$

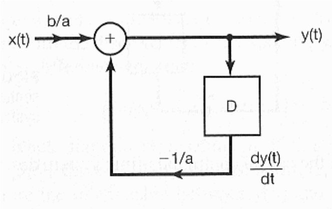
$$y[n] = -a y[n-1] + b x[n]$$

$$y(t) = -\frac{1}{a}\frac{d}{dt} y(t) + \frac{b}{a} x(t)$$



$D \Longleftrightarrow z^{-1}$

$D \Longleftrightarrow s$



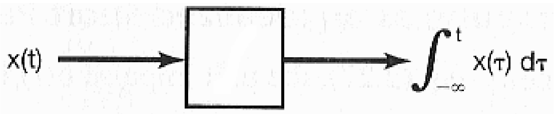
Block Diagram Representations:

- In practice, implementation of derivative block is difficult and it would be sensitive to noise.
- So, alternatively, we can use integral block

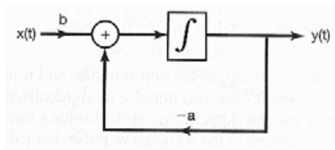
$$\frac{d}{dt}y(t) = bx(t) - ay(t)$$

$$\Rightarrow \ y(t) = \int_{-\infty}^{t} \left[bx(\tau) - ay(\tau) \right] d\tau$$

$$\Rightarrow \ y(t) = y(t_0) \ + \ \int_{t_0}^{t} \left[bx(\tau) - ay(\tau) \right] d\tau$$



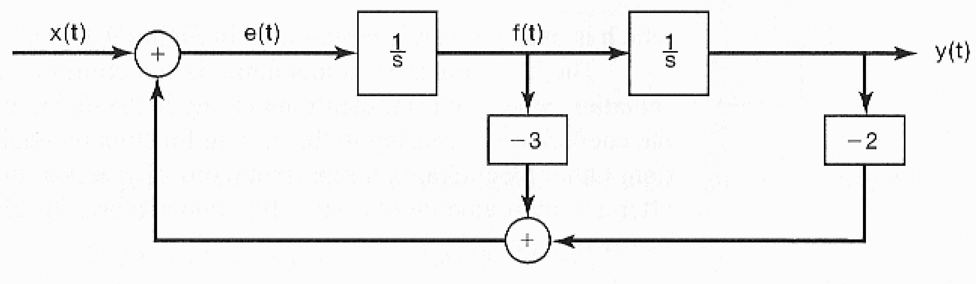
$$\int \Longleftrightarrow \frac{1}{s}$$



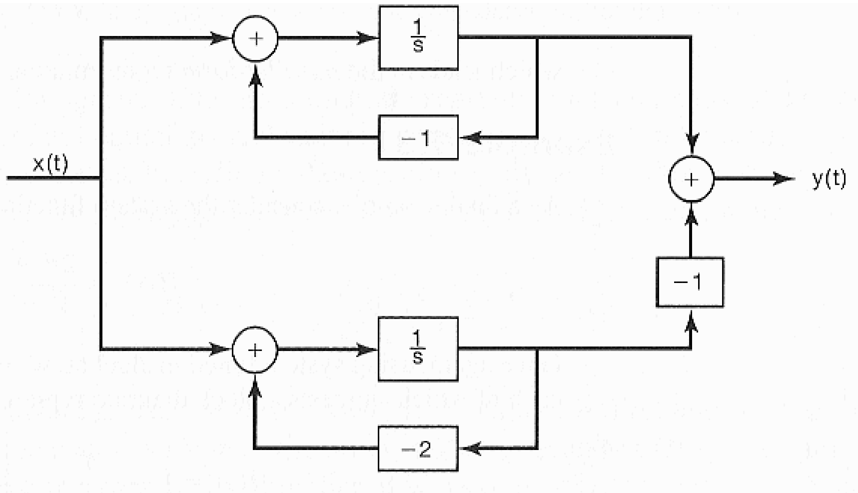
- Block Diagram Representations:

$$\int \Longleftrightarrow \frac{1}{s}$$

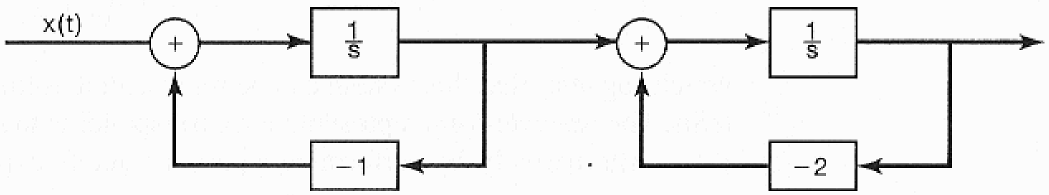
$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$



(a)



(c)



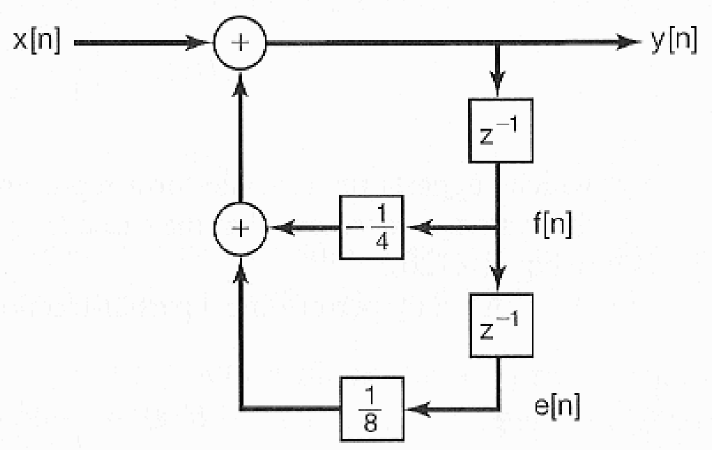
(b)

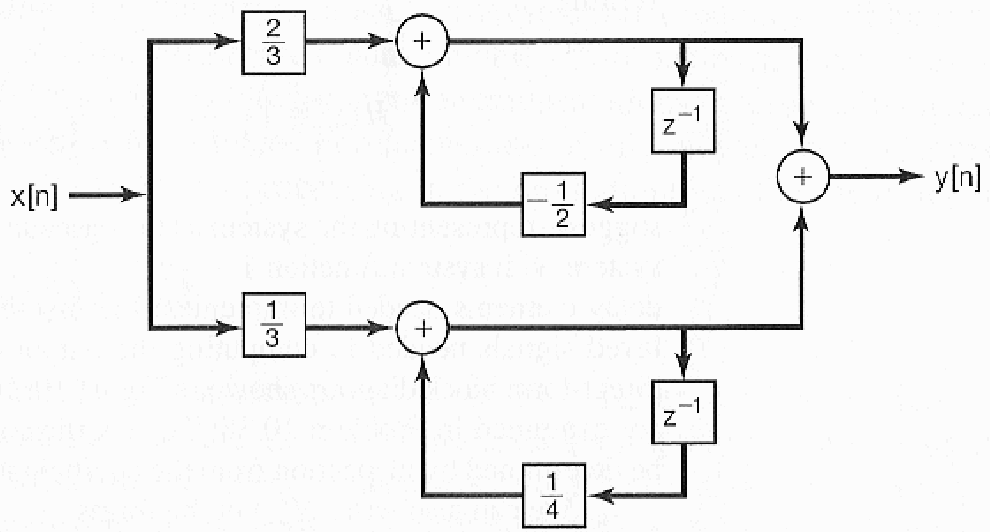
- Example 9.30 (pp.711)

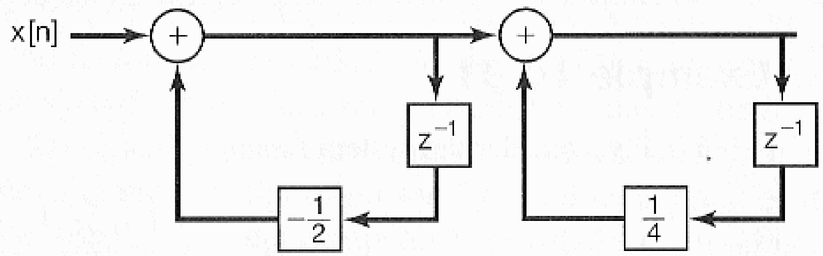
$$H(s) = \frac{1}{(s+1)(s+2)}$$

Block Diagram Representations:

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$







(b)

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

- Example 10.30 (pp.786)

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

- Properties of Linear Time-Invariant Systems
  1. Commutative property
  2. Distributive property
  3. Associative property
  4. With or without memory
  5. Invertibility
  6. Causality
  7. Stability
  8. Unit step response

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$$

$h[n] = 0$ for $n \neq 0$

$h(t) = 0,$ for $t < 0$

$h_2(t) * h_1(t) = \delta(t)$

if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

- Singularity Functions

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

$$h[n] = \delta[n] * h[n]$$

Unit Step Response:

- For an LTI system, its impulse response is:

$$\delta[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow h[n] \qquad \qquad \delta(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow h(t)$$

- Its unit step response is:

$$u[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow s[n] \qquad \qquad u(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow s(t)$$

$$\Rightarrow \quad s[n] = u[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} u[n-k]h[k]$$

$$= \sum_{k=-\infty}^{n} h[n]$$

$$\Rightarrow \quad h[n] = s[n] - s[n-1]$$

$$\Rightarrow \quad s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{+\infty} n(t-\tau)h(\tau)d\tau$$

$$= \int_{-\infty}^{t} h(\tau)d\tau$$

$$\Rightarrow \quad h(t) = \frac{ds(t)}{dt}$$

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $\quad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$
  3. Associative property $a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$
  4. With or without memory $h[n] = 0$ for $n \neq 0$ $\quad h(t) = 0,\quad$ for $t < 0$
  5. Invertibility
  6. Causality
  7. Stability $h_2(t) * h_1(t) = \delta(t)$ $\quad$ if $\int_{-\infty}^{+\infty} \left|h(\tau)\right| d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions $h[n] = \left(\frac{1}{2}\right)^n u[n]$

Singularity Functions

- CT unit impulse function is one of singularity functions

$$\delta(t) = u_0(t) \qquad\qquad u(t) = u_{-1}(t)$$

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = u(t)$$

$$\frac{d}{dt}\delta(t) = u_1(t) \qquad\qquad \int_{-\infty}^{t} u(\tau) d\tau = u_{-2}(t)$$

$$\frac{d^2}{dt^2}\delta(t) = u_2(t) \qquad\qquad \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} u(\sigma) d\sigma \right) d\tau = u_{-3}(t)$$

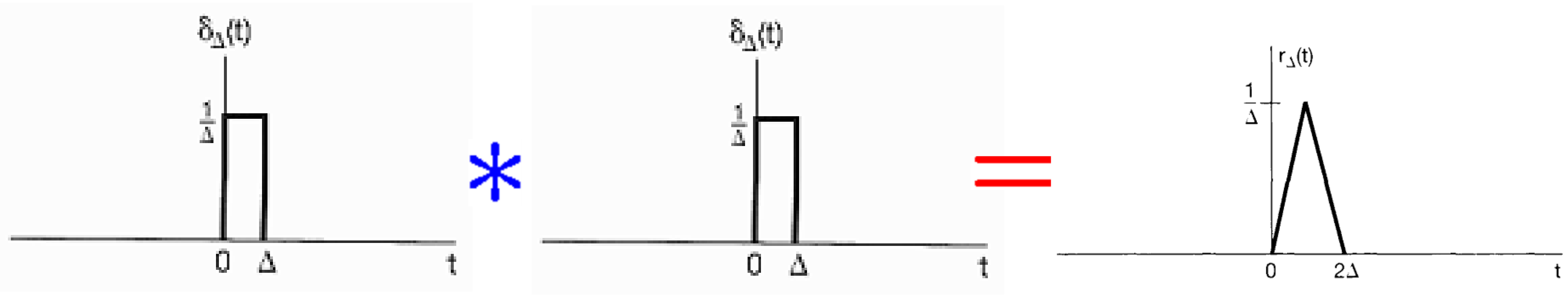
$$\frac{d^k}{dt^k}\delta(t) = u_k(t) \qquad\qquad \int_{-\infty}^{t} \cdots \left(\int_{-\infty}^{\tau} u(\sigma) d\sigma \right) \cdots d\tau = u_{-k}(t)$$

- **Singularity Functions**

$$x(t) = x(t) * \delta(t)$$

$$\delta(t) = \delta(t) * \delta(t)$$

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$$



$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t) \qquad \Rightarrow \quad \lim_{\Delta \to 0} r_\Delta(t) = \delta(t)$$

- **Example 2.16** $\frac{d}{dt}y(t) + 2\,y(t) = x(t)$

with initial-rest condition

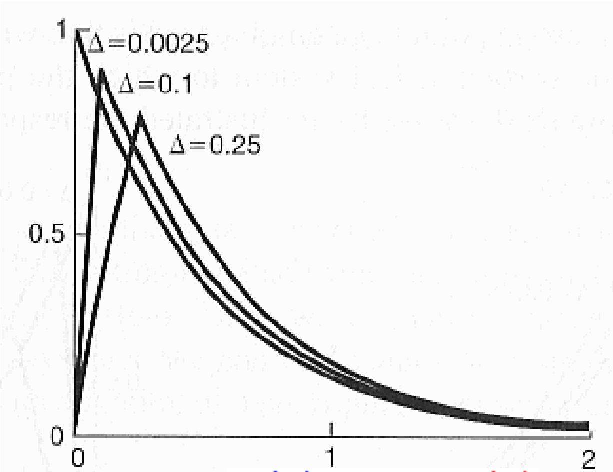
$x(t) = \delta_\triangle(t)$

$x(t) = r_\triangle(t)$

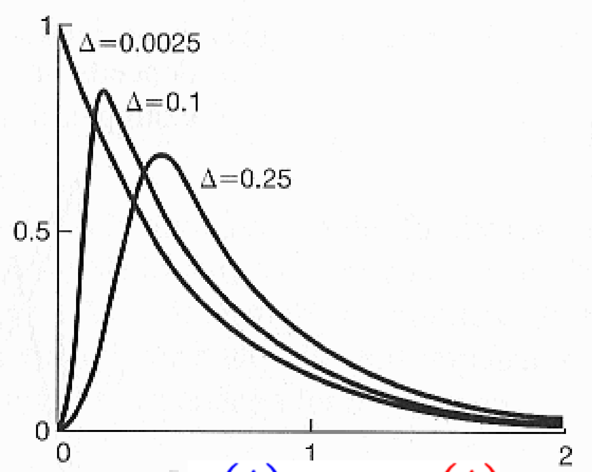
$x(t) = \delta_\triangle(t) * r_\triangle(t)$

$x(t) = r_\triangle(t) * r_\triangle(t)$

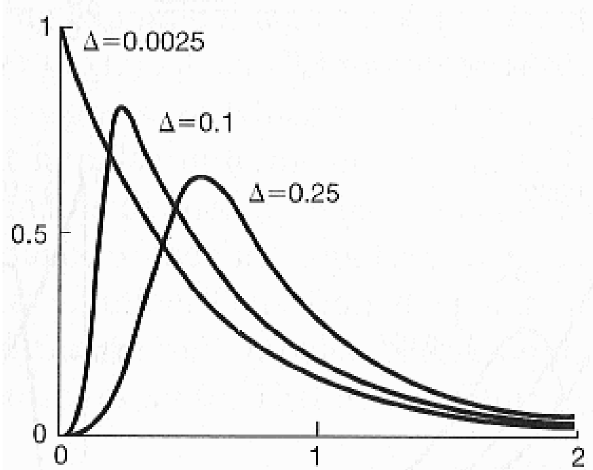
$x(t) = \delta(t)$



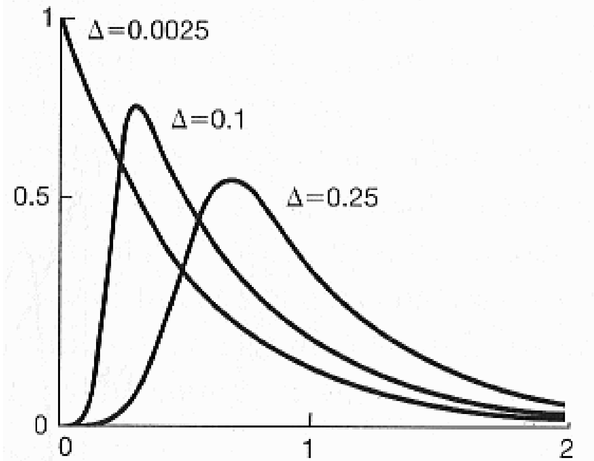
$x(t) = \delta_\triangle(t)$



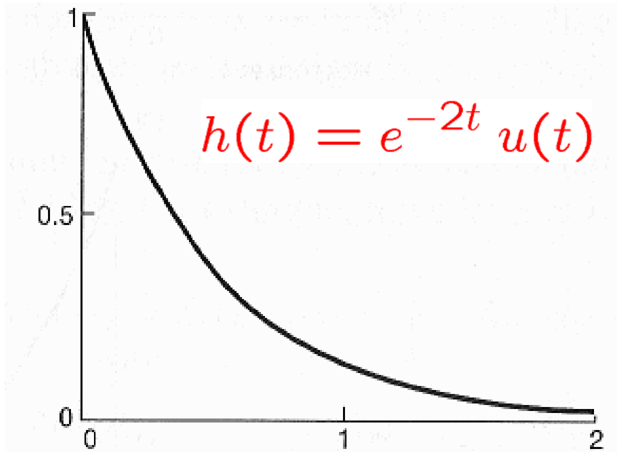
$x(t) = r_\triangle(t)$



$x(t) = \delta_\triangle(t) * r_\triangle(t)$



$x(t) = r_\triangle(t) * r_\triangle(t)$



$x(t) = \delta(t)$

(e)

- Example 2.16 $\frac{d}{dt}y(t) + 20\, y(t) = x(t)$

with initial-rest condition

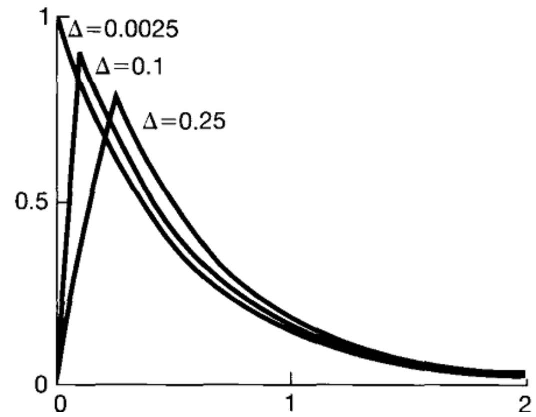
$x(t) = \delta_{\triangle}(t)$

$x(t) = r_{\triangle}(t)$

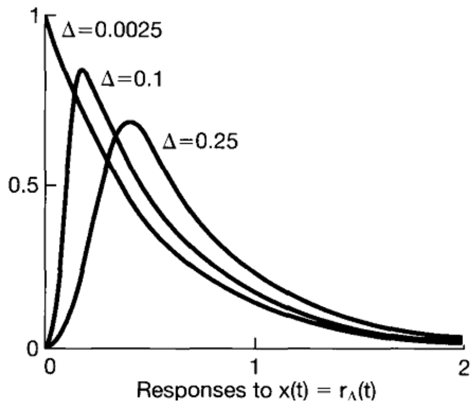
$x(t) = \delta_{\triangle}(t) * r_{\triangle}(t)$

$x(t) = r_{\triangle}(t) * r_{\triangle}(t)$

$x(t) = \delta(t)$

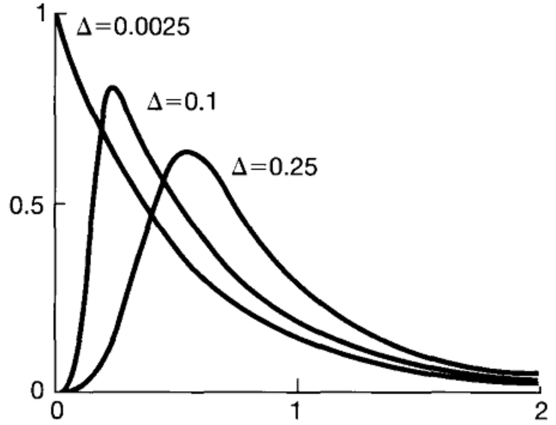


$x(t) = \delta_{\triangle}(t)$

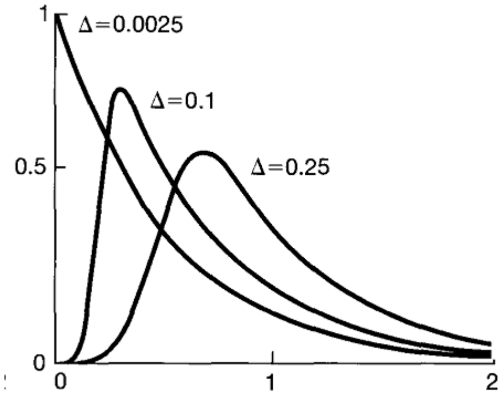


$x(t) = r_{\triangle}(t)$

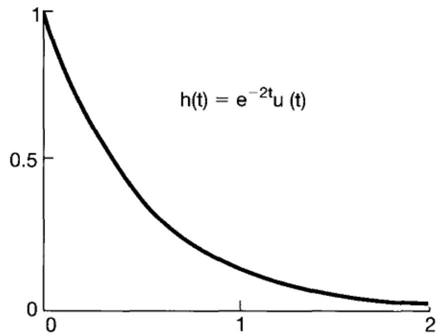
$h(t) = e^{-20t}\, u(t)$



$x(t) = \delta_{\triangle}(t) * r_{\triangle}(t)$



$x(t) = r_{\triangle}(t) * r_{\triangle}(t)$



$x(t) = \delta(t)$

Defining the Unit Impulse through Convolution:

$$x(t) = x(t) * \delta(t)$$

- Let $x(t) = 1$,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- So that the unit impulse has unit area

Unit Doublets of Derivative Operation:

- A system: Output is the derivative of input

$$y(t) = \frac{d}{dt} x(t)$$

$\Rightarrow$ The unit impulse response of the system

is the derivative of the unit impulse,

which is called the unit doublet $u_1(t)$

- That is, from $x(t) = x(t) * \delta(t)$, we have

$$\frac{d}{dt} x(t) = x(t) * u_1(t)$$

Unit Doublets of Derivative Operation:

- Similarly,

$$\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$$

- But,

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{d}{dt}x(t)\right) = \left(x(t) * u_1(t)\right) * u_1(t)$$

- Therefore,

$$u_2(t) = u_1(t) * u_1(t)$$

- In general,

$$u_k(t),\ k > 0, \text{ the } k\text{th derivative of } \delta(t)$$

$$u_k(t) = u_1(t) * \cdots * u_1(t)$$

Unit Doublets of Integration Operation:

- A system: Output is the integral of input

$$y(t) = \int_{-\infty}^{\infty} x(\tau) d\tau$$

- Therefore,

$$u(t) = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- Hence, we have

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

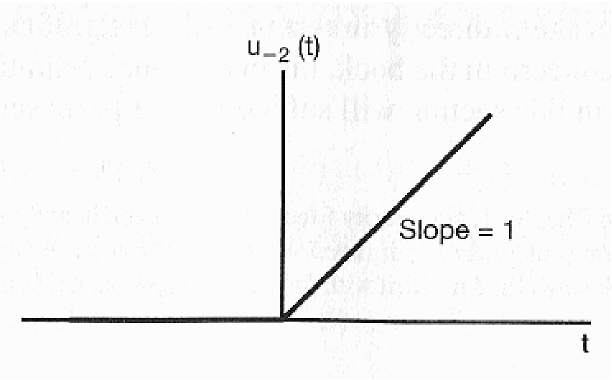
Unit Doublets of Integration Operation:

- Similarly,

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

- That is,

$$u_{-2}(t) = t\, u(t)$$ the unit ramp function



Unit Doublets of Integration Operation:

- Moreover,

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$$

$$= \left(\int_{-\infty}^{t} x(\sigma) d\sigma \right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau$$

- In general,

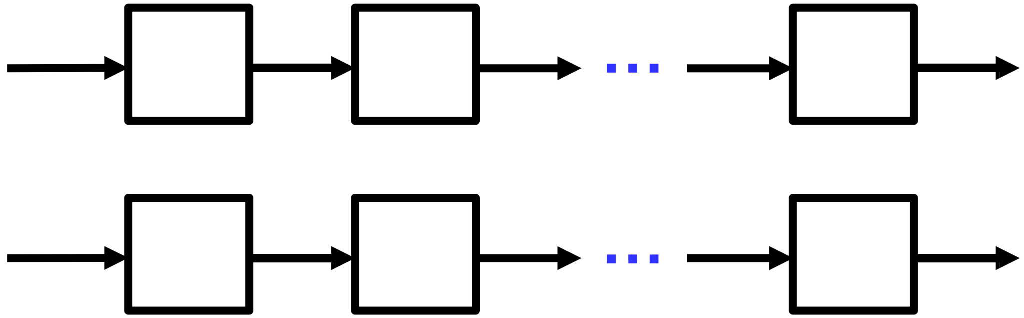
$$u_{-k}(t) = u(t) * \cdots * u(t) = \int_{-\infty}^{t} u_{-(k-1)}(\tau) d\tau$$

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t)$$

In Summary

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$



$u_k(t)$

$k > 0,$

Impulse response of a cascade of k differentiators

$k < 0,$

Impulse response of a cascade of $|k|$ integrators

$$u(t) * u_1(t) = \delta(t) \quad \text{or,} \quad u_{-1}(t) * u_1(t) = u_0(t)$$

$$\Rightarrow \quad u_k(t) * u_r(t) = u_{k+r}(t)$$