Spring 2011

信號與系統
Signals and Systems

Chapter SS-4
The Continuous-Time Fourier Transform

Feng-Li Lian

NTU-EE

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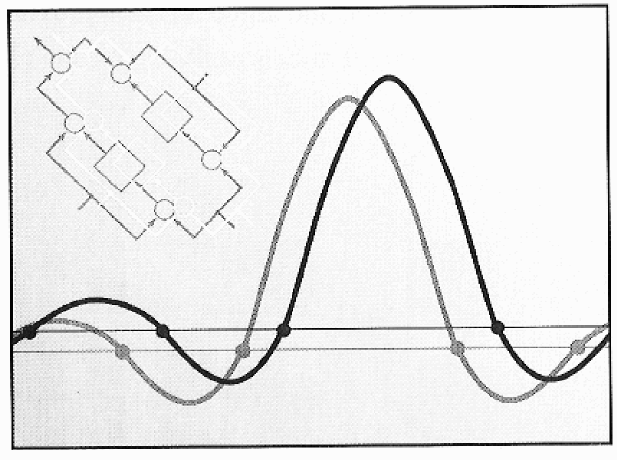
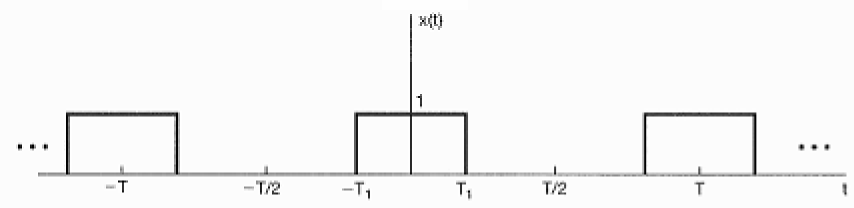


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties
  of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by
  Linear Constant-Coefficient Differential Equations

- Example 3.5: $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$k = 0$ $\quad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$

$k \neq 0$ $\quad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t}dt = \frac{1}{T}\frac{1}{(-jkw_0)}e^{-jkw_0 t}\Big|_{-T_1}^{T_1}$

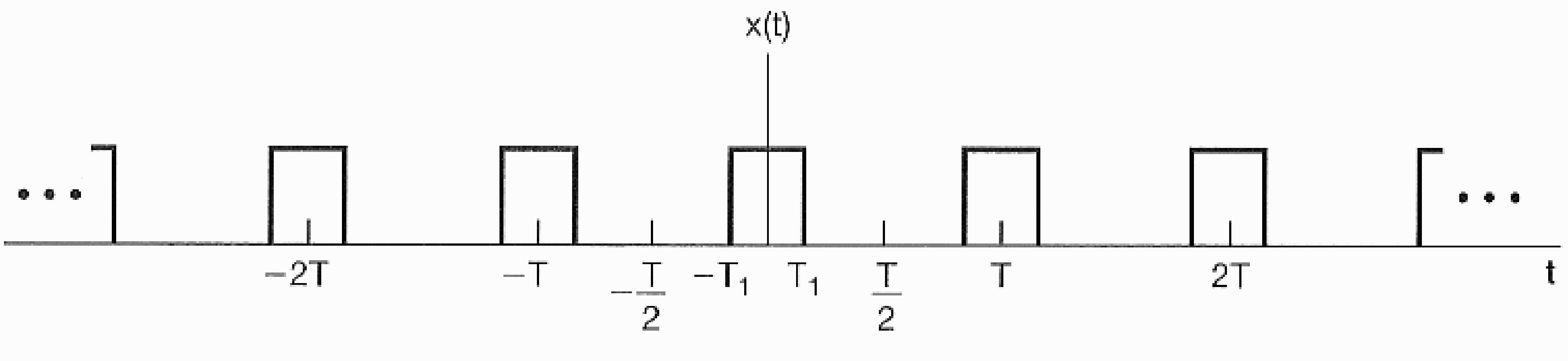
$$= \frac{1}{jkw_0 T}\left[e^{jkw_0 T_1} - e^{-jkw_0 T_1}\right] /$$

$$w_0 = \frac{2\pi}{T}$$

$$= \frac{2\sin(kw_0 T_1)}{kw_0 T} = \frac{\sin(kw_0 T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

Page 193, Ex 3.5

▪ CT Fourier Transform of an Aperiodic Signal:



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_k = \frac{2\sin(k w_0 T_1)}{k w_0 T}$$

$\frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right) = d\,\mathrm{sinc}(kd)$

Fourier series coefficients

$$T a_k = \left.\frac{2\sin(w T_1)}{w}\right|_{w = k w_0}$$

$k\omega_0 \rightarrow \boxed{\omega}$

w as a continuous variable

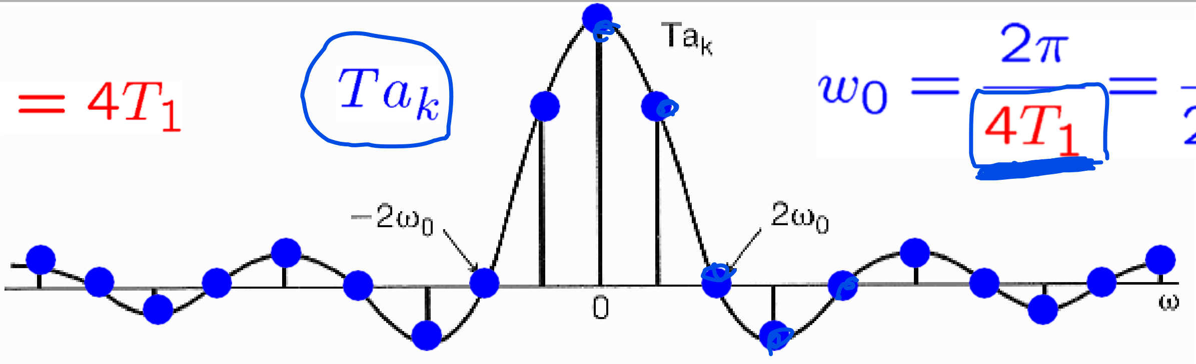
$$Ta_k = \frac{2\sin(wT_1)}{w}$$

$$w = kw_0 = k\frac{2\pi}{T}$$

$$w_0 = \frac{2\pi}{T}$$

$w_0 \propto \frac{1}{T}$

$T = 4T_1$ $\quad Ta_k$ $\quad w_0 = \frac{2\pi}{4T_1} = \frac{\pi}{2T_1}$

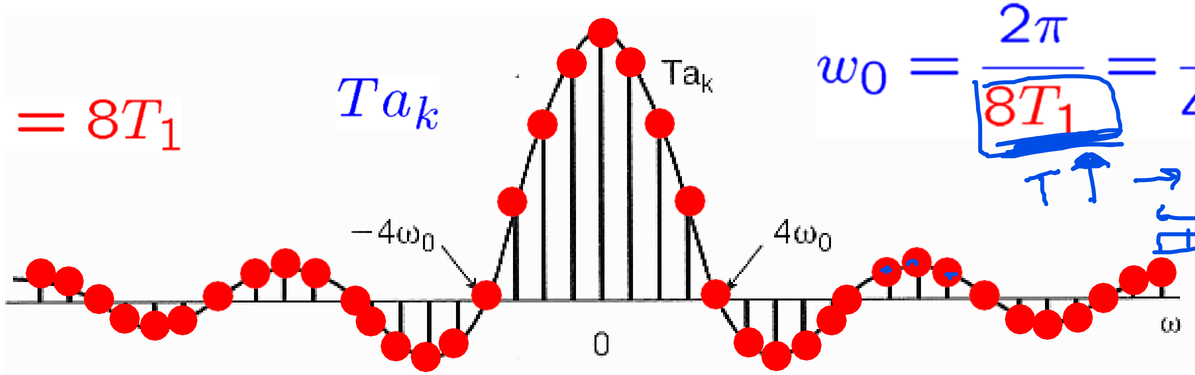


(a)

x(t)

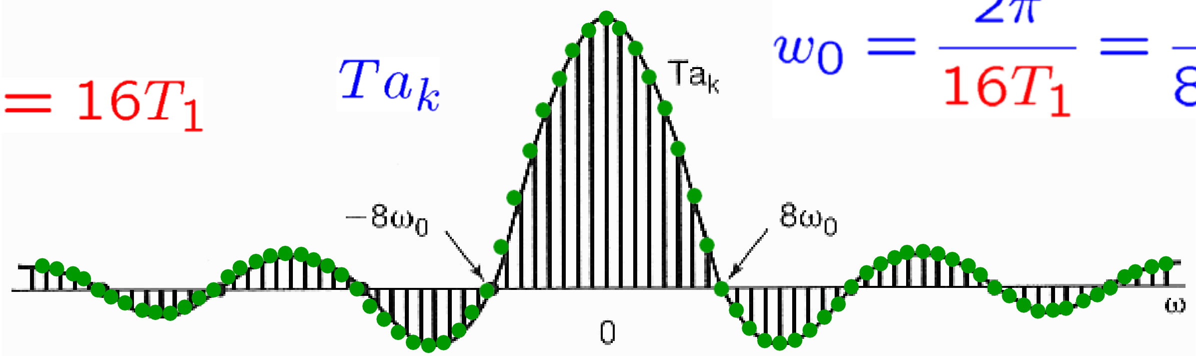
$T = 8T_1$ $\quad Ta_k$ $\quad w_0 = \frac{2\pi}{8T_1} = \frac{\pi}{4T_1}$

$T\uparrow \rightarrow w\downarrow$



(b)

$T = 16T_1$ $\quad Ta_k$ $\quad w_0 = \frac{2\pi}{16T_1} = \frac{\pi}{8T_1}$

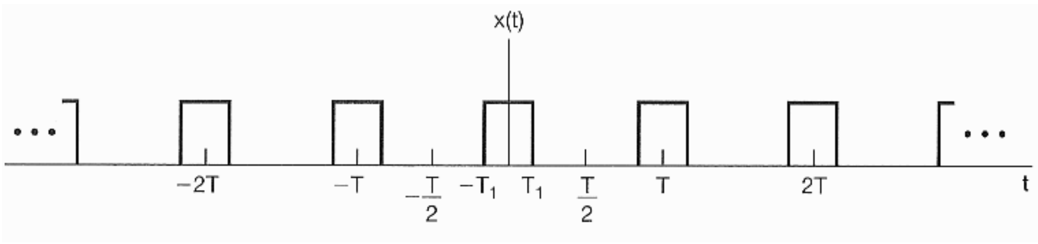


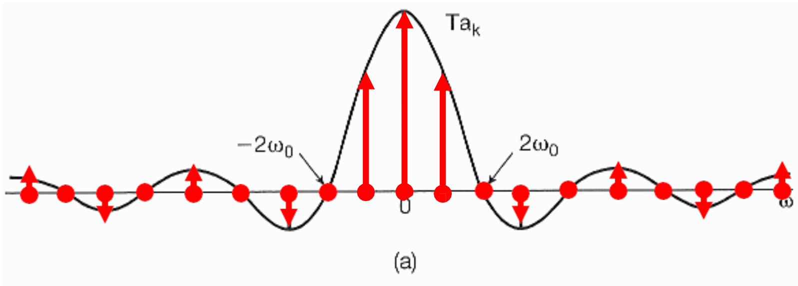
(c)

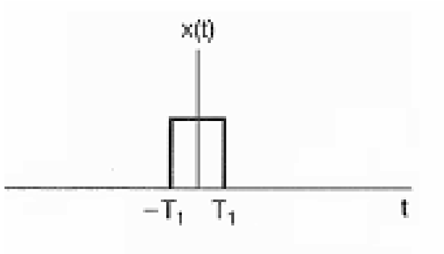
$$\boxed{w = kw_0 = k\frac{2\pi}{T}}$$

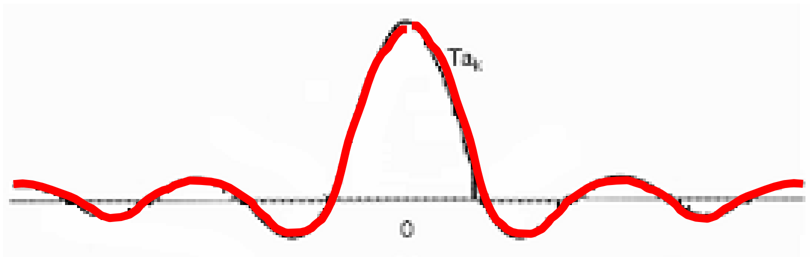
$$T \rightarrow \infty \;\Rightarrow\; \{Ta_k\} \rightarrow \left.\frac{2\sin(wT_1)}{w}\right|_{w=kw_0}$$

$T \rightarrow \infty$
$w \rightarrow 0$

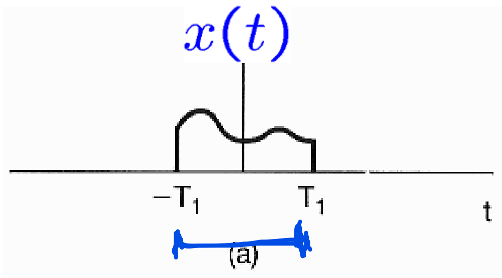




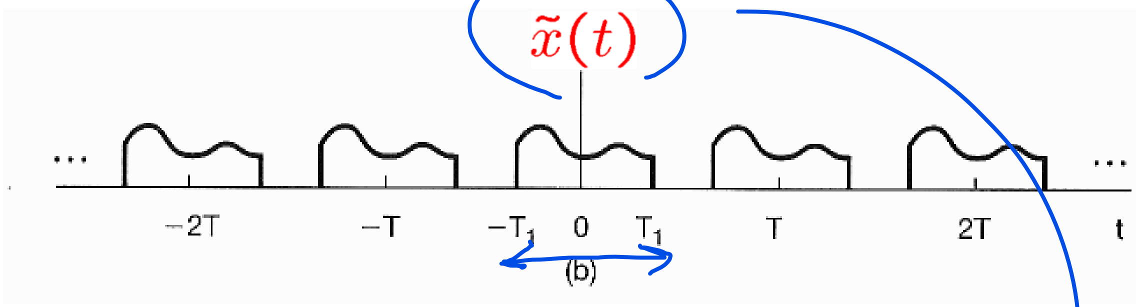




Representation of Aperiodic Signals: CT Fourier Transform



an aperiodic signal



a periodic signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

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$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

T

$x(t) = \tilde{x}(t)$

$$\Rightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jkw_0 t} dt = \frac{1}{T} \underbrace{\int_{-\infty}^{+\infty} x(t) e^{-jkw_0 t} dt}_{X(j\omega)}$$

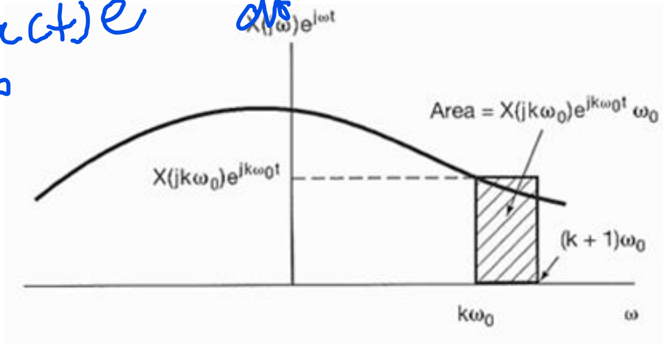
Representation of Aperiodic Signals: CT Fourier Transform

$$T a_k = \frac{2\sin(wT_1)}{w}$$

- Define the envelope $X(jw)$ of Ta_k as

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jkw_0 t} dt$$



- Then,

$$a_k = \frac{1}{T} X(jkw_0)$$

- Hence,

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \overbrace{\frac{1}{T} X(jkw_0)}^{a_k} e^{jkw_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0$$

$$w_0 = \frac{2\pi}{T}$$

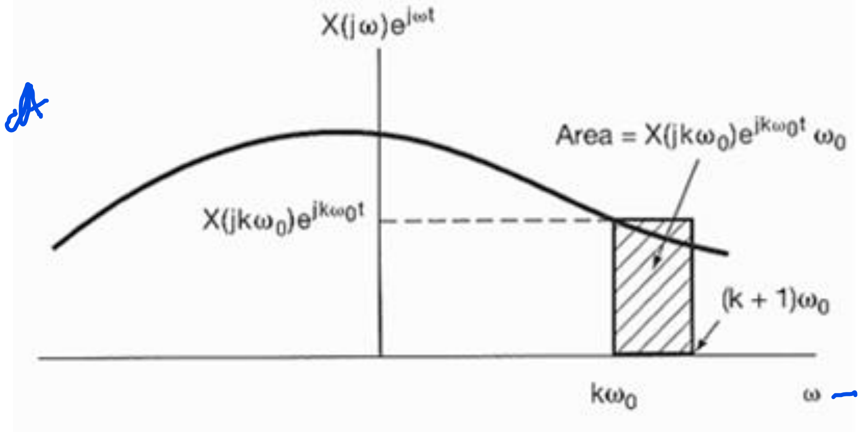
$$\frac{1}{T} = \frac{1}{2\pi} w_0$$

Representation of Aperiodic Signals: CT Fourier Transform

- As $T \to \infty$, $\tilde{x}(t) \to x(t)$

  also $w_0 \to 0$

$\int_{-\infty}^{+\infty} x(t) e^{-jkwt} dt$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

- inverse Fourier transform eqn

- synthesis eqn

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- $X(jw)$: Fourier Transform of $x(t)$

  spectrum

- analysis eqn

$$a_k = \frac{1}{T} X(jw) \Big|_{w=kw_0}$$

Sufficient conditions for the convergence of FT

$$x(t) \xrightarrow{\mathcal{CTFT}} X(jw)$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$\hat{x}(t) \xleftarrow{\mathcal{CTIFT}} X(jw)$$

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$e(t) = \hat{x}(t) - x(t)$$

- If $x(t)$ has finite energy

i.e., square integrable, $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

$\Rightarrow \quad X(jw)$ is finite

$$\Rightarrow \quad \int_{-\infty}^{+\infty} |e(t)|^2 dt = 0$$

- Sufficient conditions for the convergence of FT
  - Dirichlet conditions:

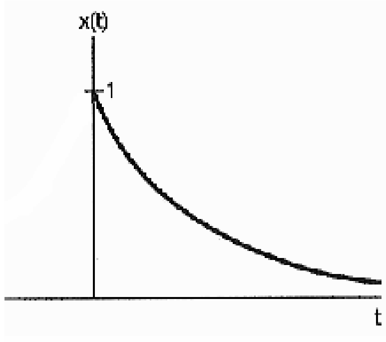
1. x(t) be absolutely integrable; that is, $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$

2. x(t) have a finite number of maxima and minima within any finite interval

3. x(t) have a finite number of discontinuities within any finite interval
   Furthermore, each of these discontinuities must be finite

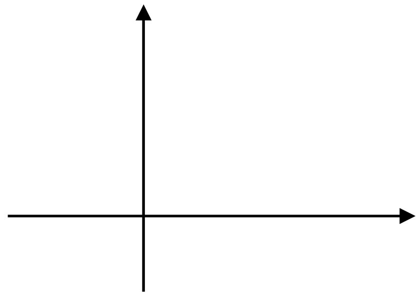
- Example 4.1:

$$x(t) = e^{-at}\, u(t), \quad a > 0$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$



$$\Rightarrow \; X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-jwt} dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-jwt} dt$$

$$= \int_{0}^{\infty} e^{-(a+jw)t}\, dt$$

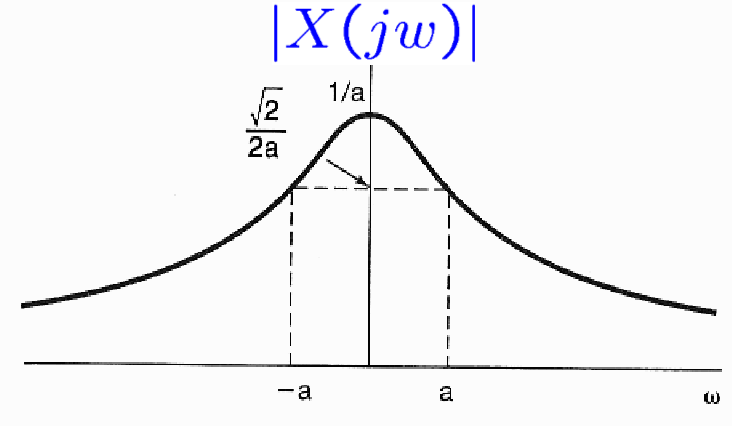
$$= -\frac{1}{a+jw} e^{-(a+jw)t} \Big|_{0}^{\infty}$$

$$= 0 - \left(-\frac{1}{a+jw} e^{-(a+jw)0}\right)$$

$$= \frac{1}{a+jw}, \quad a > 0$$

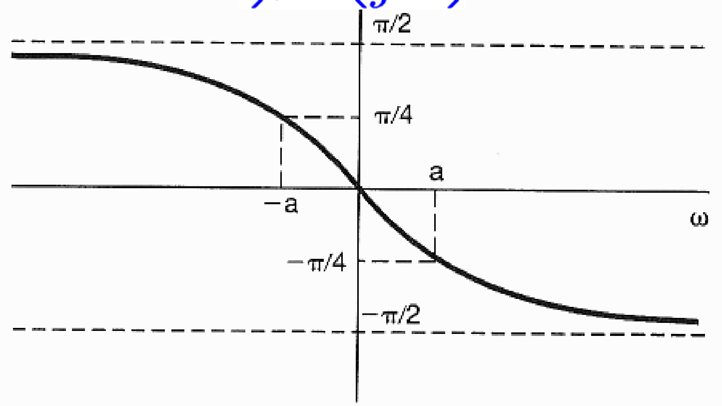
Example 4.1:

$$\Rightarrow \; X(jw) = \frac{1}{a + jw}, \quad a > 0$$



(a)

$$\Rightarrow |X(jw)| = \frac{1}{\sqrt{a^2 + w^2}}$$



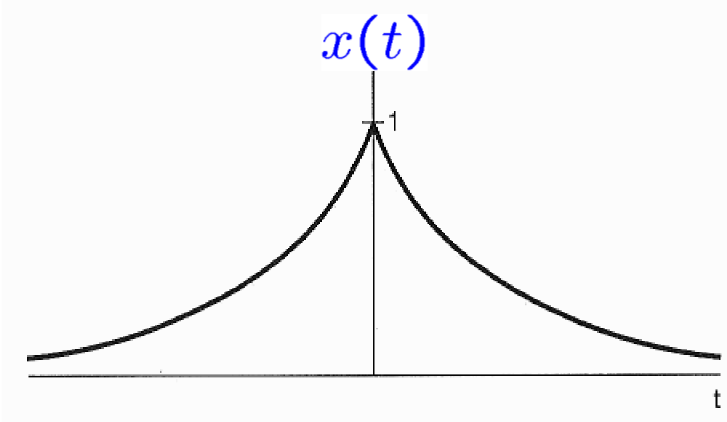
(b)

$$\Rightarrow \sphericalangle X(jw) = -\tan^{-1}\left(\frac{w}{a}\right)$$

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

- Example 4.2:

$$x(t) = e^{-a|t|}, \quad a > 0$$

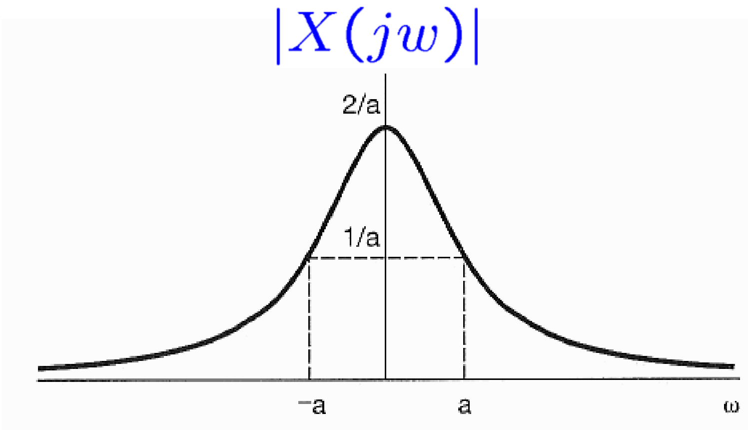


$$\Rightarrow \; X(jw) = \int_{-\infty}^{\infty} e^{-a|t|} \, e^{-jwt} dt$$

$$= \int_{-\infty}^{0} e^{at} \, e^{-jwt} dt \quad + \int_{0}^{\infty} e^{-at} \, e^{-jwt} dt$$

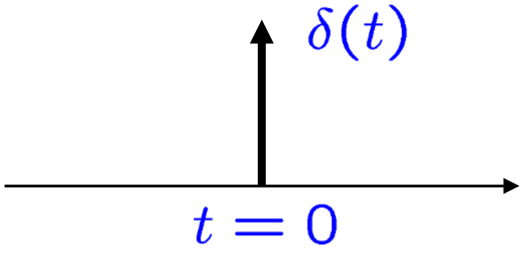
$$= \frac{1}{a - jw} + \frac{1}{a + jw}$$

$$= \frac{2a}{a^2 + w^2}$$



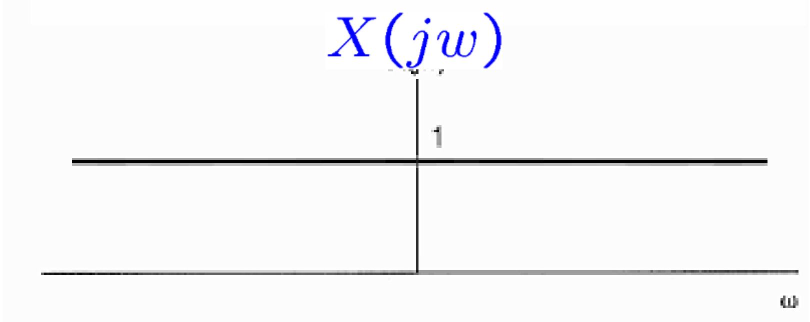
$\boxed{X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt}$

- Example 4.3:



$x(t) = \delta(t)$, i.e., unit impules

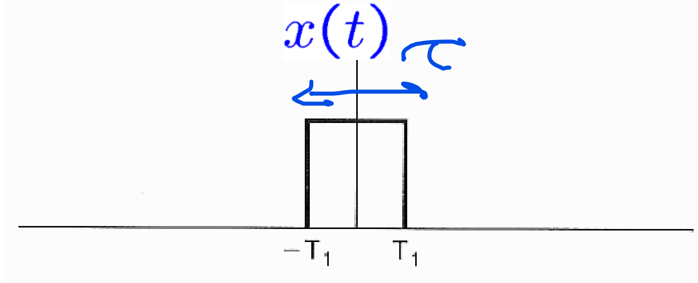
$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} \delta(t)\, e^{-jwt}dt = 1$$



$$\boxed{X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt}$$

- **Example 4.4:**

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



(a)

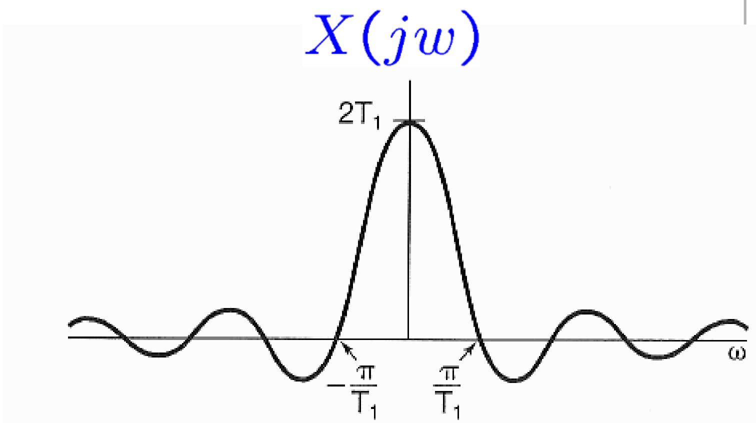
$X(\omega) = \tau \,\mathrm{sinc}(f\tau)$

$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = \int_{-T_1}^{T_1} e^{-jwt}dt$$

$$= \frac{1}{-jw} e^{-jwt}\Big|_{-T_1}^{T_1}$$

$$= \frac{1}{-jw}\left(e^{-jwT_1} - e^{jwT_1}\right)$$

$$= \frac{1}{jw}\left(e^{jwT_1} - e^{-jwT_1}\right)$$



(b)

$$= 2\,\frac{\sin(wT_1)}{w} = 2T_1 \frac{\sin(\pi\, wT_1/\pi)}{\pi\, wT_1/\pi)}$$

- Example 4.5:

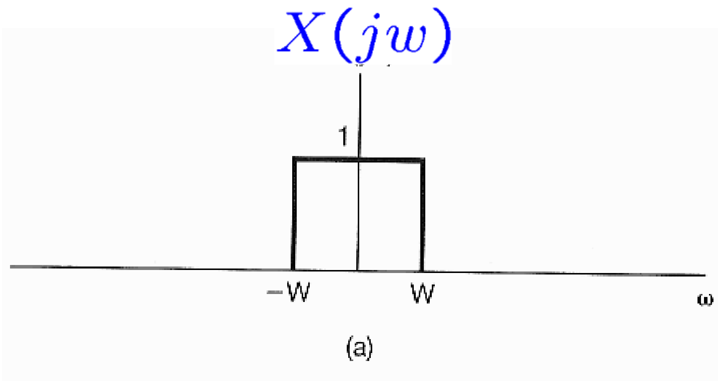
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

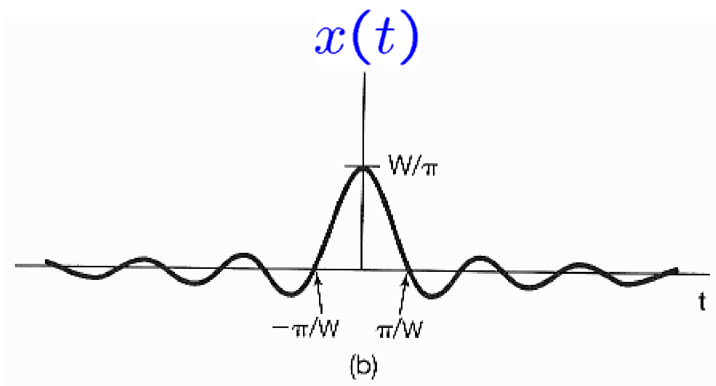
$$X(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{jwt}dw$$

$$= \frac{\sin(Wt)}{\pi t}$$



(a)

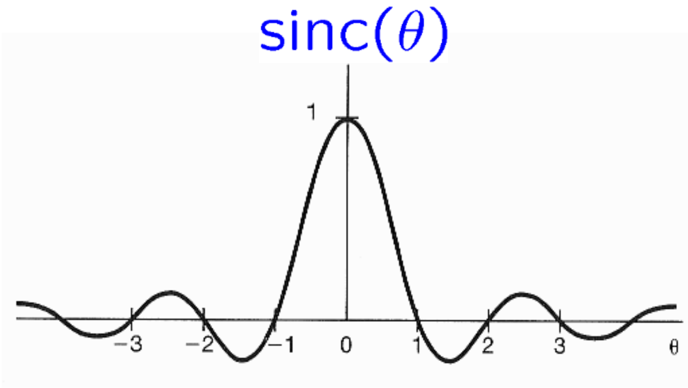


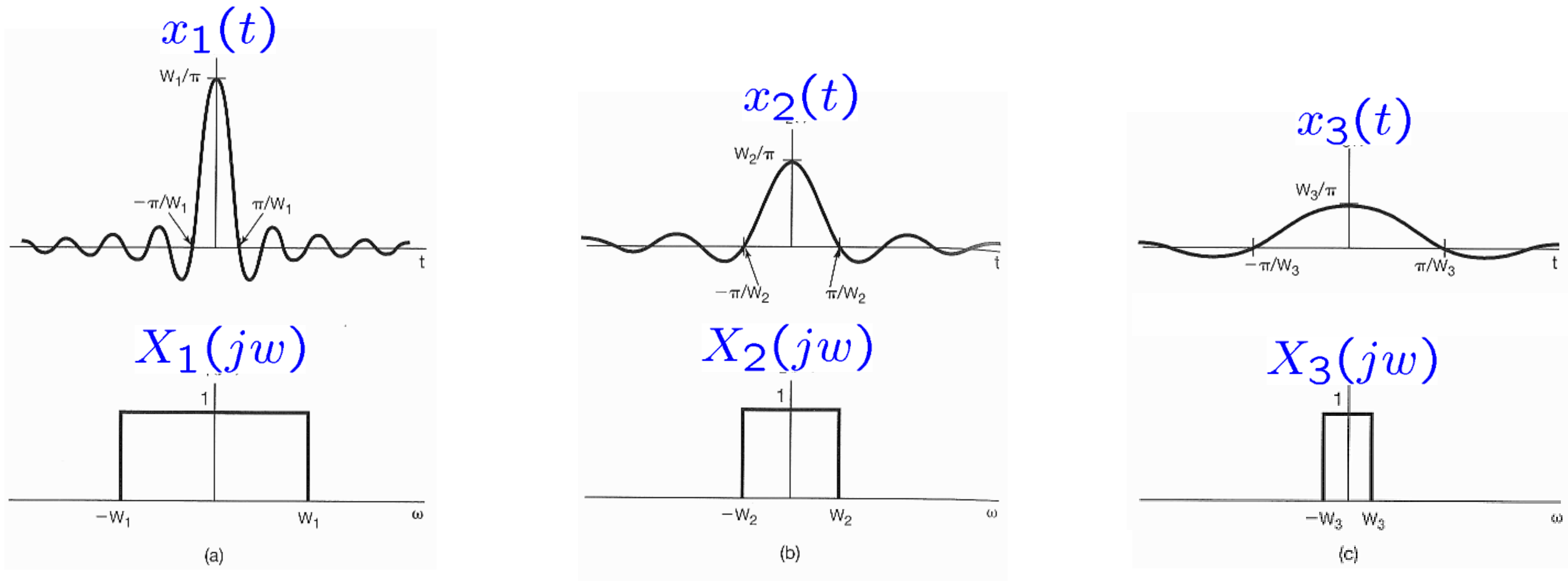
(b)

- sinc functions:

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$





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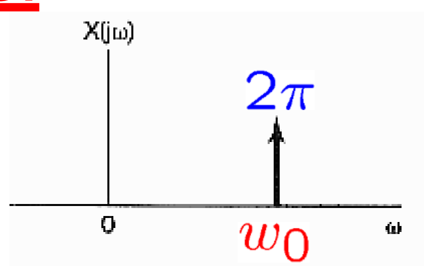
Fourier Transform from Fourier Series:

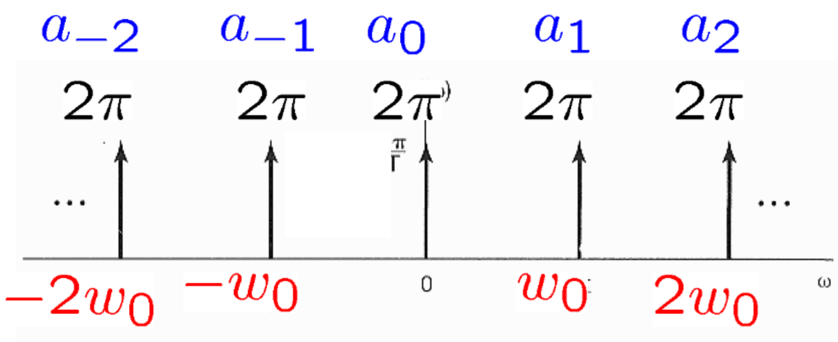
$$X(jw) = 2\pi \quad \delta(\, w \; - \quad w_0)$$



$$\Rightarrow \; x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi \quad \delta(w - \quad w_0) e^{jwt} dw$$

$$= \quad e^{j \quad w_0 t}$$

- more generally,



(b)

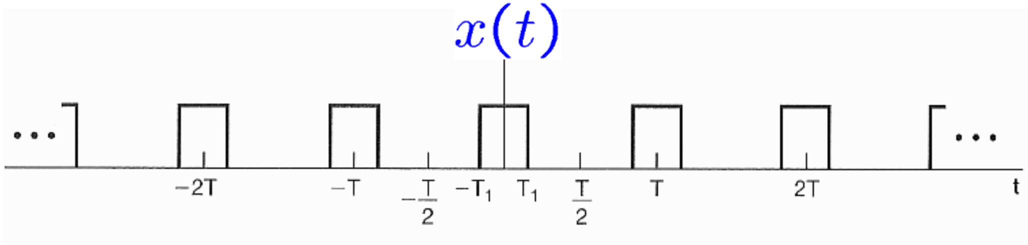
$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - k w_0)$$

$$\Rightarrow \; x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

Fourier series represntation

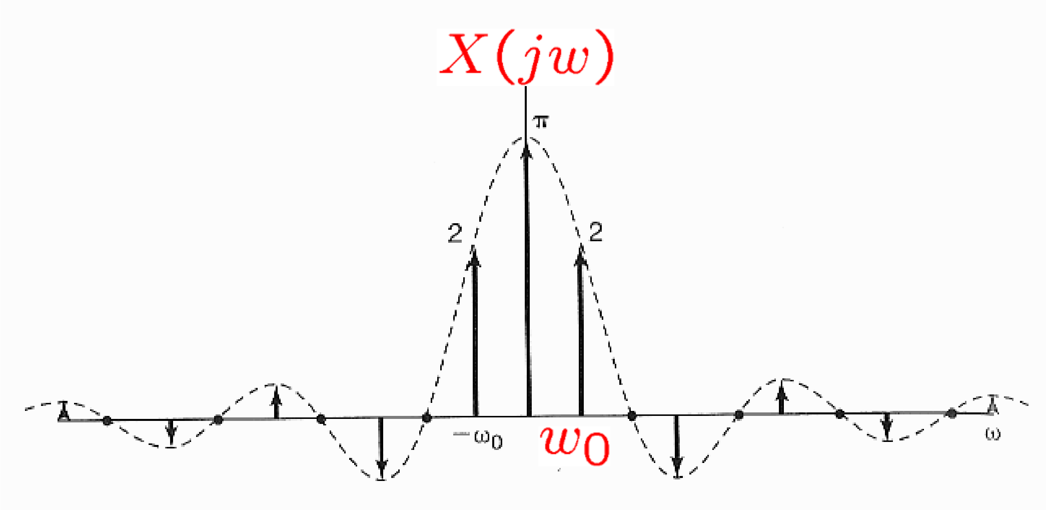
of a periodic signal

Example 4.6:



$$\Rightarrow \quad a_k = \frac{\sin(kw_0T_1)}{\pi k}$$

$$\Rightarrow \; X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0) = \sum_{k=-\infty}^{+\infty} \frac{2\sin(kw_0T_1)}{k} \delta(w - kw_0)$$



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

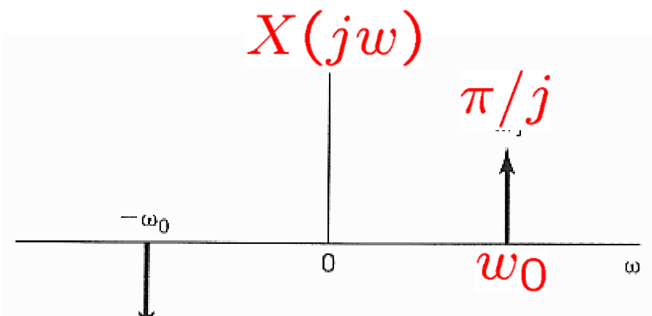
- **Example 4.7:**

$$x(t) = \sin(w_0 t) = \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}$$

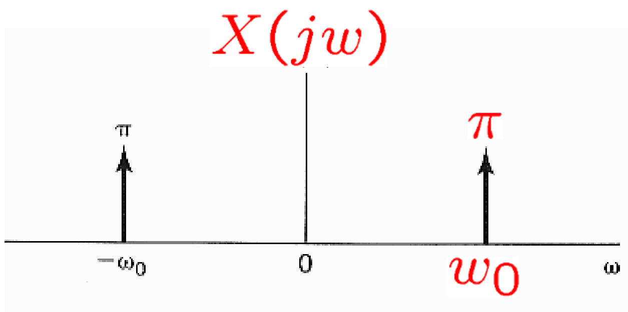
$$\Rightarrow \quad a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_k = 0, \quad k \neq 1, -1$$

$$x(t) = \cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

$$\Rightarrow \quad a_1 = \frac{1}{2} \qquad a_{-1} = \frac{1}{2} \qquad a_k = 0, \quad k \neq 1, -1$$



(a)



(b)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \tilde{x}(t) e^{-jkw_0 t} dt$$

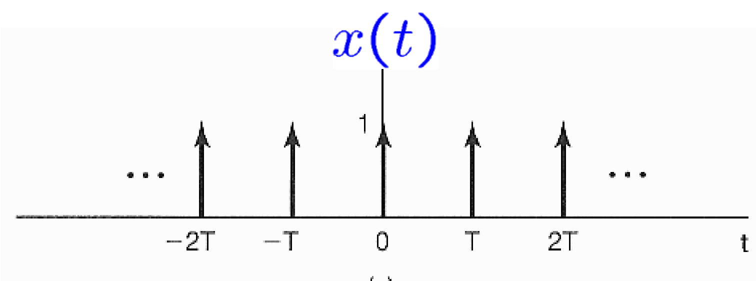
$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

- Example 4.8:

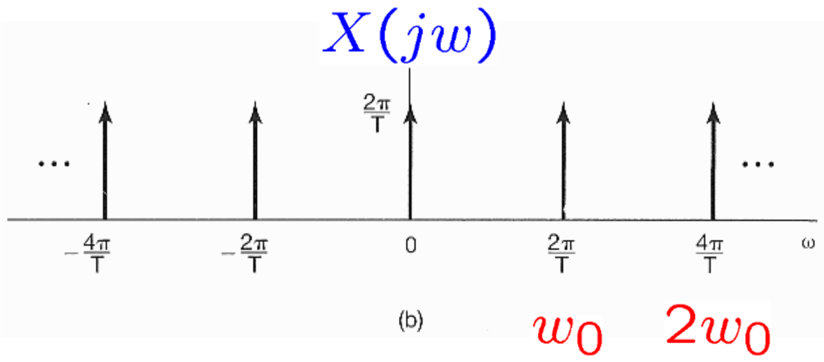
$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$\Rightarrow \quad a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jkw_0 t} dt = \frac{1}{T}$$

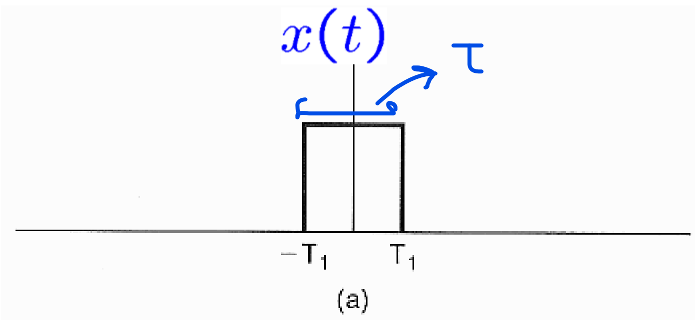
$$\Rightarrow \quad X(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - \frac{2\pi}{T}k)$$



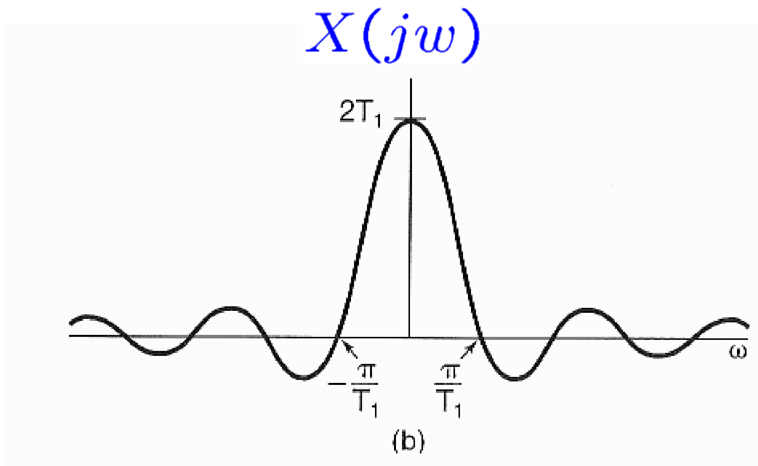
(a)



(b)



(a)



(b)

$$X(jw) = 2T_1 \frac{\sin(\pi \; wT_1 \; /\pi)}{\pi \; wT_1 \; /\pi)} \qquad w = 2\pi f \rightarrow f = \frac{w\pi}{2}$$

$$X(jw) = 2T_1 sinc(2T_1 f) = \boxed{\tau sinc(f\tau)}$$

$$\delta(t) \xrightarrow{F} 1$$

$$e^{-at}u(t) \xrightarrow{F} \frac{1}{a + jw}$$

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- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

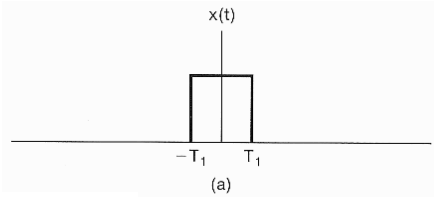
Outline

Section	Property
4.3.1	Linearity
4.3.2	Time Shifting
4.3.6	Frequency Shifting
4.3.3	Conjugation
4.3.5	Time Reversal
4.3.5	Time and Frequency Scaling
4.4	Convolution
4.5	Multiplication
4.3.4	Differentiation in Time
4.3.4	Integration
4.3.6	Differentiation in Frequency
4.3.3	Conjugate Symmetry for Real Signals
4.3.3	Symmetry for Real and Even Signals
4.3.3	Symmetry for Real and Odd Signals
4.3.3	Even-Odd Decomposition for Real Signals
4.3.7	Parseval's Relation for Aperiodic Signals

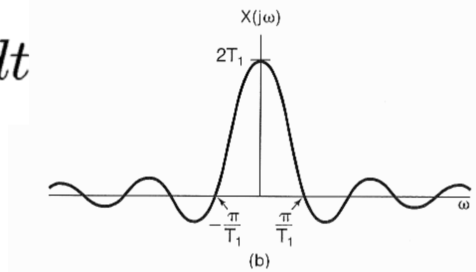
Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

Fourier Transform Pair:



- Synthesis equation: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$
- Analysis equation: $X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$



- Notations:

$$X(jw) = \mathcal{F}\{x(t)\}$$

$$\frac{1}{a+jw} = \mathcal{F}\{e^{-at}u(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(jw)\}$$

$$e^{-at}u(t) = \mathcal{F}^{-1}\{\frac{1}{a+jw}\}$$

$$x(t) \stackrel{\mathcal{CTFT}}{\longleftrightarrow} X(jw)$$

$$e^{-at}u(t) \stackrel{\mathcal{CTFT}}{\longleftrightarrow} \frac{1}{a+jw}$$

- **Linearity:**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw)$$

$$\Rightarrow \quad a\, x(t) + b\, y(t) \overset{\mathcal{F}}{\longleftrightarrow} a\, X(jw) + b\, Y(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- **Time Shifting:**

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\Rightarrow x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jw(t-t_0)}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(e^{-jwt_0}X(jw)\right) e^{jwt}dw$$

$$Y(jw) = \int_{-\infty}^{+\infty} x(t-t_0)e^{-jwt}dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)e^{-jw(\tau+t_0)}d\tau$$

$$= e^{-jwt_0}\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

- Time Shift → Phase Shift:

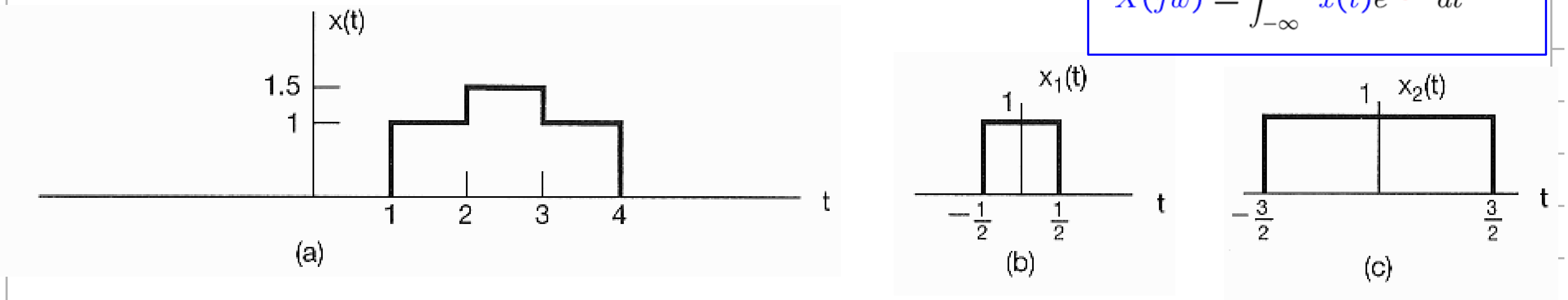
$$\mathcal{F}\{x(t)\} = X(jw) = |X(jw)|e^{j\measuredangle X(jw)}$$

$$\mathcal{F}\{x(t-t_0)\} = e^{-jwt_0}X(jw) = |X(jw)|e^{j[\measuredangle X(jw)-wt_0]}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- Example 4.9:



$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(jw) = \frac{2\sin(w/2)}{w}$$

$$X_2(jw) = \frac{2\sin(3w/2)}{w}$$

$$\Rightarrow \quad X(jw) = e^{-j5w/2}\left\{\frac{\sin(w/2) + 2\sin(3w/2)}{w}\right\}$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- Conjugation & Conjugate Symmetry:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t)^* \overset{\mathcal{F}}{\longleftrightarrow} X^*(-jw)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$x(t)^* = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw)^* e^{-jwt} dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) \; e^{jwt} dw$$

$$= \frac{1}{2\pi} \int_{\infty}^{-\infty} -X(-j\bar{w}) e^{j\bar{w}t} \; d\bar{w}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(-j\bar{w}) e^{j\bar{w}t} d\bar{w}$$

Conjugation & Conjugate Symmetry:

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$

$$\boxed{x(t)^* \xleftrightarrow{\mathcal{F}} X^*(-jw)}$$

- $x(t) = x^*(t) \Rightarrow X(-jw) = X^*(jw)$

  $x(t)$ is real $\Rightarrow$ $X(jw)$ is conjugate symmetric

- $x(t) = x^*(t)$ & $x(-t) = x(t)$

  $\Rightarrow X(-jw) = X^*(jw)$ & $X(-jw) = X(jw)$

  $\Rightarrow X(jw) = X^*(jw)$

  $x(t)$ is real & even $\Rightarrow$ $X(jw)$ are real & even

- $x(t)$ is real & odd $\Rightarrow$ $X(jw)$ are purely imaginary & odd

- Conjugation & Conjugate Symmetry:

If $x(t)$ is a real function

$$x(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\} = x_e(t) + x_o(t)$$

$$\Rightarrow \mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\}$$

$\Rightarrow \mathcal{F}\{x_e(t)\}$: a real function

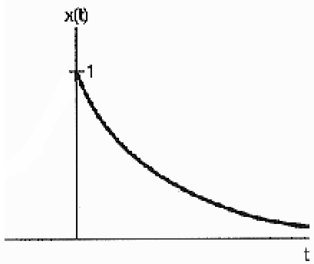
$\Rightarrow \mathcal{F}\{x_o(t)\}$: a purely imaginary function

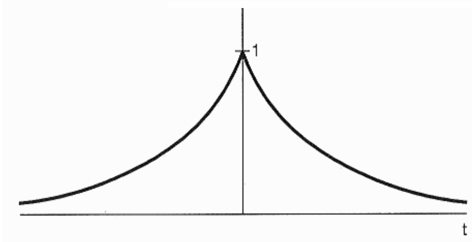
$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$

$$\mathcal{E}v\{x(t)\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(jw)\}$$

$$\mathcal{O}d\{x(t)\} \xleftrightarrow{\mathcal{F}} j\,\mathcal{I}m\{X(jw)\}$$

- Example 4.10:





Ex 4.1 $$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+jw}$$

Ex 4.2 $$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} ?$$

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2\mathcal{E}v\left\{e^{-at}u(t)\right\}$$

$$\mathcal{E}v\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\left\{\frac{1}{a+jw}\right\}$$

$$\mathcal{O}d\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} j\,\mathcal{I}m\left\{\frac{1}{a+jw}\right\}$$

$$X(jw) = 2\mathcal{R}e\left\{\frac{1}{a+jw}\right\} = \frac{2a}{a^2+w^2}$$

- **Differentiation & Integration:**

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$\frac{d}{dt} x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} jwX(jw)$$

$$\frac{d}{dt} x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw)\ jw\ e^{jwt}\ dw$$

$$\int_{-\infty}^{t} x(\tau) d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw} X(jw) + \boxed{\pi X(0) \delta(w)}$$

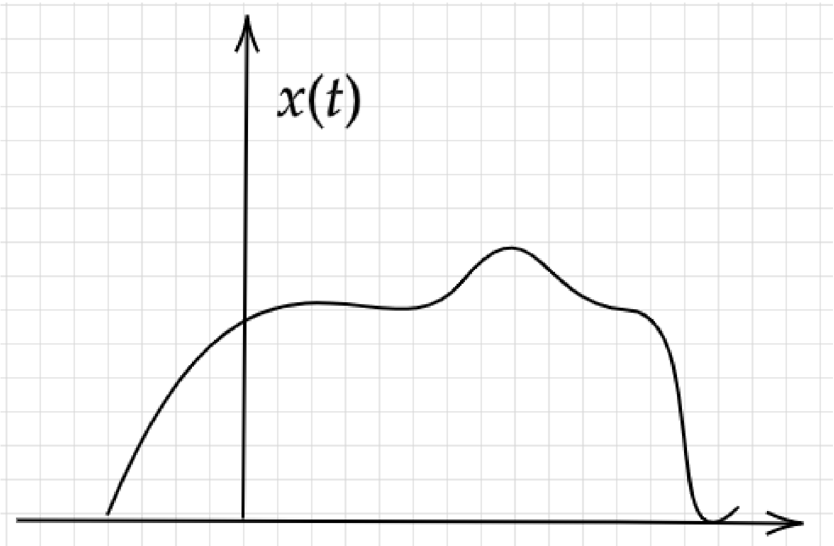
dc or average value

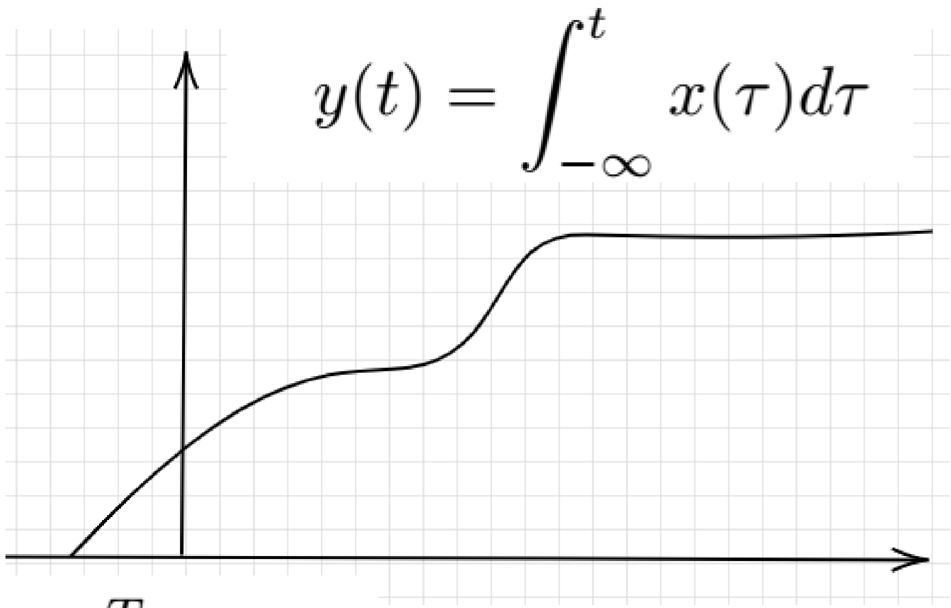
$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

- **Integration:**

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw) \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$\int_{-\infty}^{t} x(\tau) d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw} X(jw) + \pi X(0) \delta(w)$$





$$\mathrm{DC} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} y(\tau) d\tau$$

$$\frac{T X(0)}{2T} = \frac{X(0)}{2}$$

- Example 4.11:

$$x(t) = u(t) \xleftrightarrow{\mathcal{F}} X(jw) = ?$$

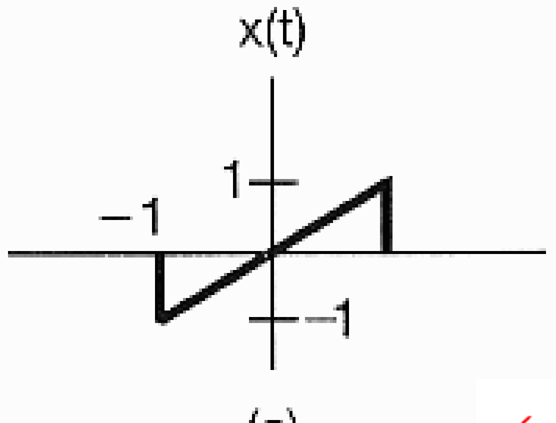
$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(jw) = 1$$

$$x(t) = \int_{-\infty}^{t} g(\tau) d\tau \qquad X(jw) = \frac{1}{jw} G(jw) + \pi G(0) \delta(w)$$

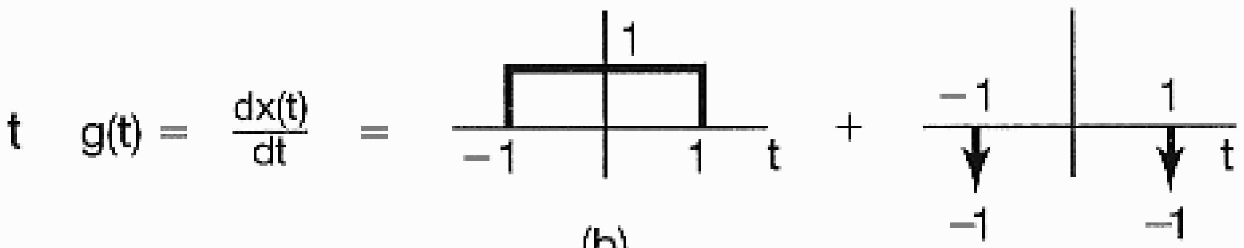
$$= \frac{1}{jw} + \pi \delta(w)$$

$$\delta(t) = \frac{d}{dt} u(t) \xleftrightarrow{\mathcal{F}} jw \left[\frac{1}{jw} + \pi \delta(w) \right] = 1$$

- Example 4.12:



(a)



(b)

$$g(t) = \frac{d}{dt} x(t)$$

$$G(jw) = \frac{2\sin(w)}{w} - e^{jw} - e^{-jw}$$

$$\Rightarrow \quad X(jw) = \frac{G(jw)}{jw} + \pi G(0) \delta(w)$$

$$= \frac{2\sin(w)}{jw^2} - \frac{2\cos(w)}{jw}$$

- **Time & Frequency Scaling:**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{jw}{a}\right)$$

$$\frac{1}{|b|}x\left(\frac{t}{b}\right) \overset{\mathcal{F}}{\longleftrightarrow} X\left(jbw\right)$$

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X\left(-jw\right)$$

$$x(\quad t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jw\quad t}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\quad \bar{w})e^{j\,\bar{w}\,t}\quad d\bar{w}$$

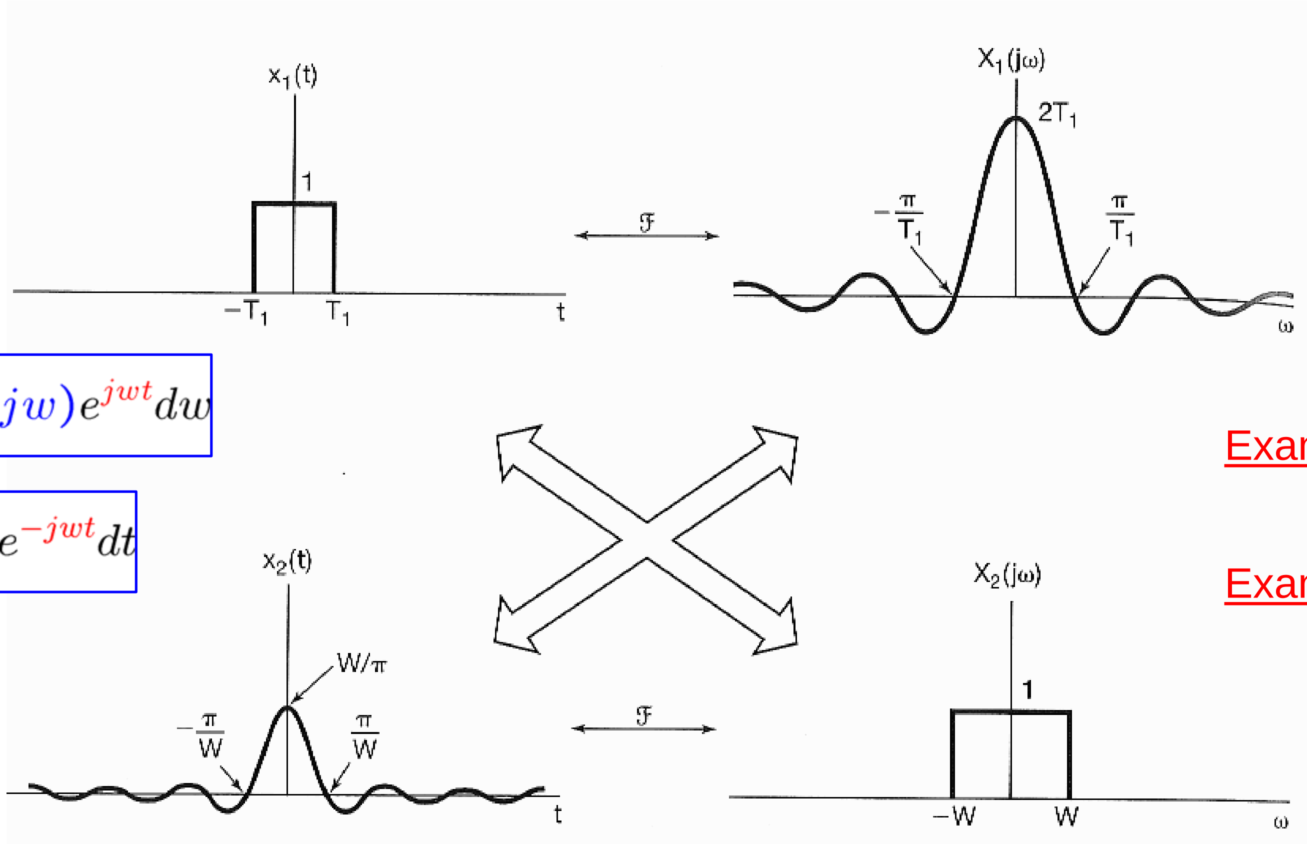
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \quad X(j\quad \bar{w})e^{j\,\bar{w}\,t}d\bar{w}$$

$$= \frac{1}{2\pi}\int_{+\infty}^{-\infty} X(j\quad \bar{w})e^{j\,\bar{w}\,t}\quad d\bar{w}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \quad X(j\quad \bar{w})e^{j\,\bar{w}\,t}d\bar{w}$$

- Duality:

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_1(jw) = \frac{2\sin(wT_1)}{w}$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

Example 4.4

Example 4.5

$$x_2(t) = \frac{\sin(Wt)}{\pi t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_2(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

- **Duality:**

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$\boxed{\begin{array}{l} f(t) \xrightarrow{F} g(jw) \\ g(t) \xrightarrow{F} 2\pi f(-jw) \end{array}}$$

$$B(s) = \int_{-\infty}^{+\infty} A(\tau)e^{-js\tau}d\tau$$

$$B(-s) = \int_{-\infty}^{+\infty} A(\tau)e^{js\tau}d\tau$$

$$A(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(s)e^{js\tau}ds$$

$$A(s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(\tau)e^{js\tau}d\tau$$

$$A(-s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(\tau)e^{-js\tau}d\tau$$

- **Duality:**

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$\frac{d}{dt}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} jwX(jw)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

$$-jtx(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{d}{dw}X(jw)$$

$$e^{jw_0t}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(w-w_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{w} X(\eta)d\eta$$

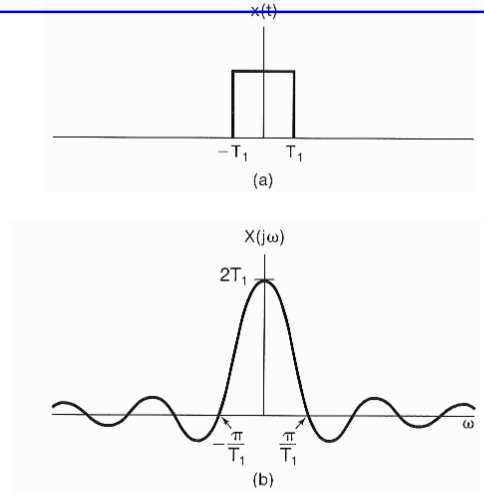
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- **Parseval's relation:**

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$

$$\Rightarrow \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$



$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

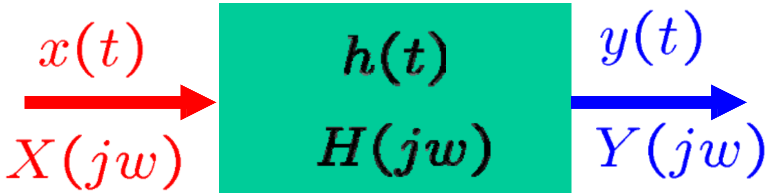
$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)e^{-jwt}dw\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)\left[\int_{-\infty}^{+\infty} x(t)e^{-jwt}dt\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties
  of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by
  Linear Constant-Coefficient Differential Equations

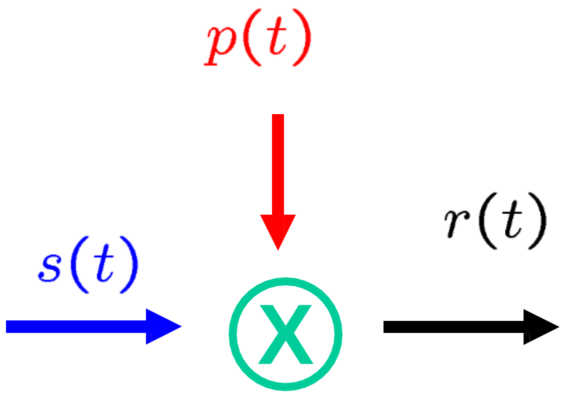
Convolution Property:



$$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Multiplication Property:



$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

- From Superposition (or Linearity): $H(jkw_0) = \int_{-\infty}^{\infty} h(t) e^{-jkw_0 t} dt$

Linear System: $h(t)$, $H(jw)$

$$e^{jkw_0 t} \rightarrow H(jkw_0) e^{jkw_0 t}$$

$$X(jkw_0) e^{jkw_0 t} w_0 \rightarrow X(jkw_0) H(jkw_0) e^{jkw_0 t} w_0$$

$$\sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0 \rightarrow \sum_{k=-\infty}^{+\infty} X(jkw_0) H(jkw_0) e^{jkw_0 t} w_0$$

$$x(t) \rightarrow y(t)$$

$$= \lim_{w_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0 \qquad = \lim_{w_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jkw_0) H(jkw_0) e^{jkw_0 t} w_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw \qquad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) H(jw) e^{jwt} dw$$

- From Superposition (or Linearity):

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)e^{jkw_0 t}w_0 \longrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)H(jw)e^{jwt}dw$$

Since $\quad y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(jw)e^{jwt}dw$

$$\Rightarrow \; Y(jw) = X(jw)H(jw)$$

$$y(t) = x(t) * h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

- From Convolution Integral:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Rightarrow Y(jw) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-jwt}dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-jwt}dt\right]d\tau$$

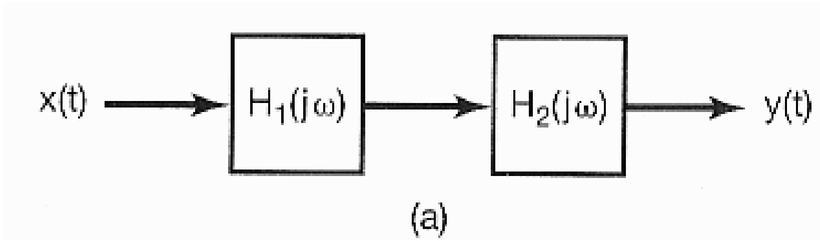
$$= \int_{-\infty}^{+\infty} x(\tau)\left[e^{-jw\tau}\int_{-\infty}^{+\infty} h(\sigma)e^{-jw\sigma}d\sigma\right]d\tau$$

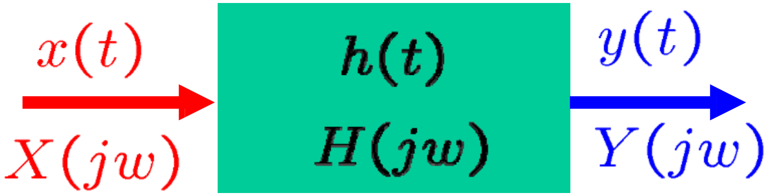
$$= \int_{-\infty}^{+\infty} x(\tau)\left[e^{-jw\tau}H(jw)\right]d\tau$$

$$= H(jw)\int_{-\infty}^{+\infty} x(\tau)e^{-jw\tau}d\tau$$

$$\Rightarrow Y(jw) = H(jw)X(jw)$$

- Equivalent LTI Systems:





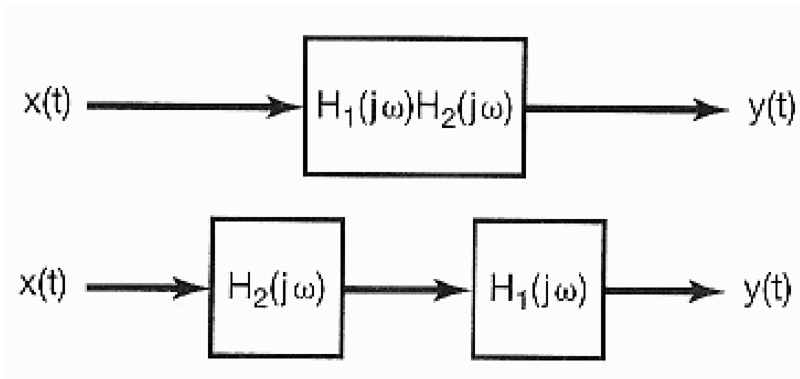
$$h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(jw)$$

impulse response

frequency response

$$y(t) = x(t) * h(t)$$

$$Y(jw) = X(jw)H(jw)$$



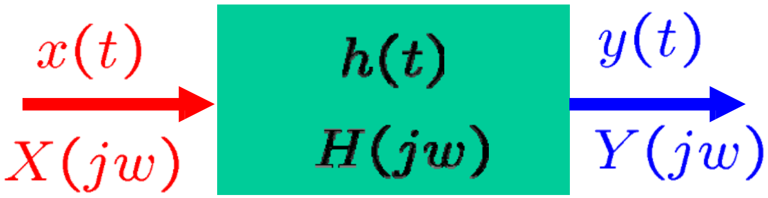
$$\Rightarrow \; Y(jw) = H_1(jw)H_2(jw)X(jw)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

- Example 4.15: Time Shift



$$h(t) = \delta(t-t_0)$$

$$\Rightarrow H(jw) = e^{-jwt_0}$$

$$\begin{aligned} Y(jw) &= H(jw)X(jw) \\ &= e^{-jwt_0}X(jw) \end{aligned}$$

$$\Rightarrow y(t) = x(t-t_0)$$

- Examples 4.16 & 17: Differentiator & Integrator

$$y(t) = \frac{d}{dt}x(t) \quad \Rightarrow \quad Y(jw) = jwX(jw)$$

$$\Rightarrow \quad H(jw) = jw$$

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \quad \Rightarrow \quad h(t) = u(t)$$

impulse response

$$\Rightarrow \quad H(jw) = \frac{1}{jw} + \pi\delta(w)$$

$$\Rightarrow \quad Y(jw) = H(jw)X(jw)$$

$$= \frac{1}{jw}X(jw) + \pi\delta(w)X(jw)$$

$$= \frac{1}{jw}X(jw) + \pi\delta(w)X(0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

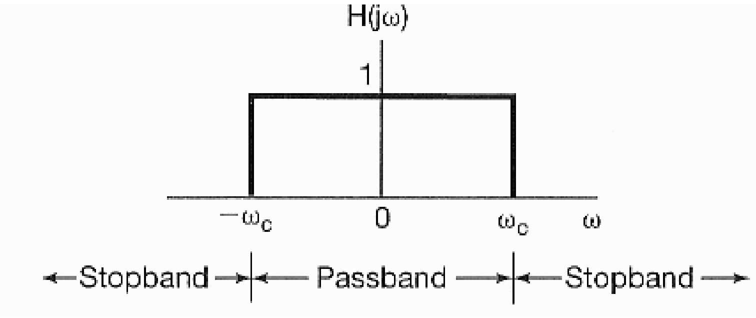
$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

▪ Example 4.18: Ideal Lowpass Filter

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

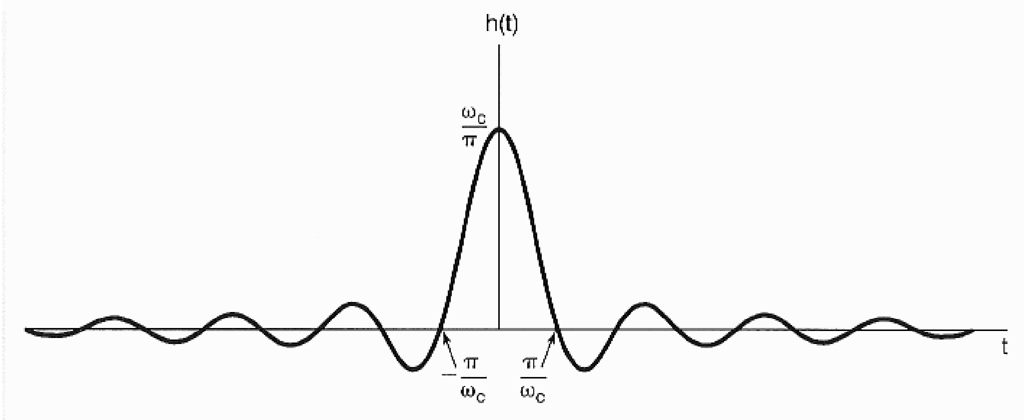
$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$H(jw) = \begin{cases} 1, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$



$$\Rightarrow\ h(t) = \frac{1}{2\pi}\int_{-w_c}^{+w_c} e^{jwt}dw$$

$$= \frac{\sin(w_c t)}{\pi t}$$



- **Filter Design:**

$$H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau) x(\tau) d\tau$$

$$H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$$

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$\Rightarrow Y(jw) = H(jw) X(jw)$$

$$\Rightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(jw) e^{jwt} dw$$

- <u>Filter Design:</u>

$$H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

$\delta(t)$

$x(t)$

$h(t)$

$y(t) = h(t) * x(t)$

$Y(jw) = H(jw)X(jw)$

RC circuit $\quad h(t) = e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} H(jw) = \dfrac{1}{jw+1}$

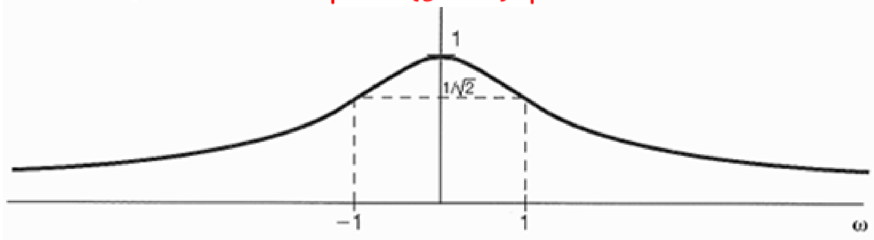
impulse response

frequency response

$h(t)$

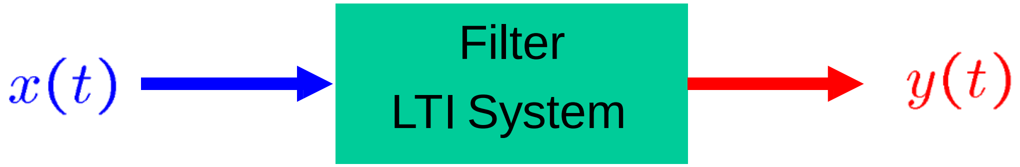
(a)

$|H(jw)|$



(b)

- Example 4.19:



$$h(t) = e^{-at}u(t), \quad a > 0 \qquad \Rightarrow \quad H(jw) = \frac{1}{a+jw}$$

$$x(t) = e^{-bt}u(t), \quad b > 0 \qquad \Rightarrow \quad X(jw) = \frac{1}{b+jw}$$

$$\Rightarrow \quad Y(jw) = H(jw)X(jw) \quad = \frac{1}{a+jw}\,\frac{1}{b+jw}$$

if $a \neq b$

$$= \frac{1}{b-a}\left[\frac{1}{a+jw} - \frac{1}{b+jw}\right]$$

- Example 4.19:

if $a \neq b$ $\quad Y(jw) = \dfrac{1}{b-a}\left[\dfrac{1}{a+jw} - \dfrac{1}{b+jw}\right]$

$$\Rightarrow \quad y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

if $a = b$ $\quad Y(jw) = \dfrac{1}{(a+jw)^2}$

since $\quad e^{-at}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{a+jw}$

and $\quad t\, e^{-at}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} j\,\dfrac{d}{dw}\left[\dfrac{1}{a+jw}\right] = \dfrac{1}{(a+jw)^2}$

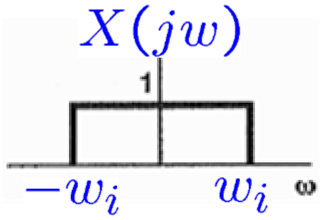
$$\Rightarrow \quad y(t) = te^{-at}u(t)$$

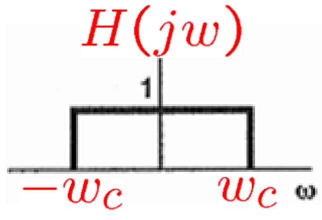
- **Example 4.20:** $h(t) = \dfrac{\sin(w_c t)}{\pi t}$ $H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$

$x(t) = \dfrac{\sin(w_i t)}{\pi t}$ → Filter LTI System → $y(t) = ?$

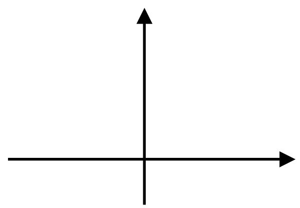
$$\Rightarrow Y(jw) = H(jw) X(jw)$$

$$\Rightarrow X(jw) = \begin{cases} 1, & |w| \le w_i \\ 0, & \text{otherwise} \end{cases}$$





$$\Rightarrow H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & \text{otherwise} \end{cases}$$



$$\Rightarrow Y(jw) = \begin{cases} 1, & |w| \le w_0 \\ 0, & \text{otherwise} \end{cases}$$

$$w_0 = \min(w_c, w_i)$$

$$\Rightarrow y(t) = \frac{\sin(w_0 t)}{\pi t}$$

$$\Rightarrow y(t) = \begin{cases} \dfrac{\sin(w_c t)}{\pi t}, & w_c \le w_i \\ \dfrac{\sin(w_i t)}{\pi t}, & w_c \ge w_i \end{cases}$$

- Representation of Aperiodic Signals:
  the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by
  Linear Constant-Coefficient Differential Equations

Multiplication Property

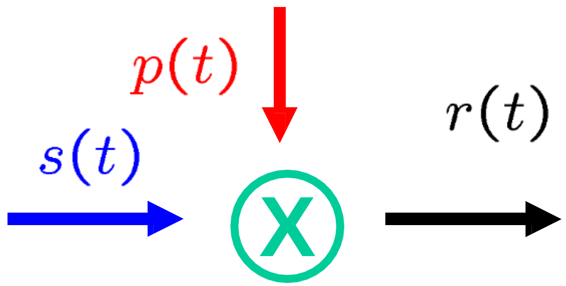
- **Convolution & Multiplication:**

$$y(t) = x(t) * h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$r(t) = s(t)p(t) \stackrel{\mathcal{F}}{\longleftrightarrow} R(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

- **Multiplication of One Signal by Another:**
  - Scale or modulate the amplitude of the other signal
  - Modulation



Multiplication Property

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$r(t) = s(t)p(t)$$

$$\Rightarrow \quad R(jw) = \int_{-\infty}^{\infty} r(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} s(t)p(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} s(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\theta)e^{j\theta t}d\theta\right\}e^{-jwt}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\theta)\left[\int_{-\infty}^{\infty} s(t)e^{-j(w-\theta)t}dt\right]d\theta$$

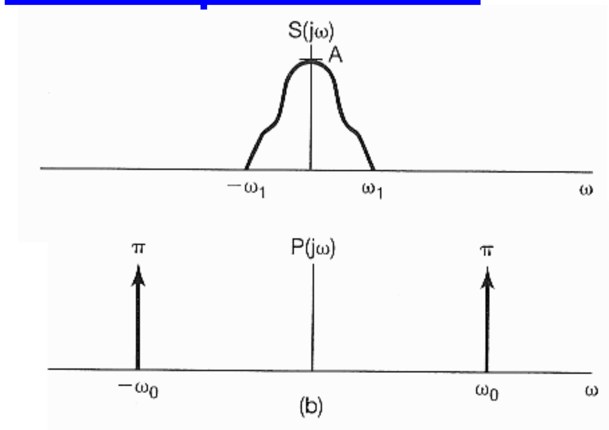
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\theta)S(j(w-\theta))d\theta \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j(w-\theta))S(j\theta)d\theta$$

- Example 4.21:

$$r(t) = s(t)p(t)$$

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(jw)$$

$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(jw)$$



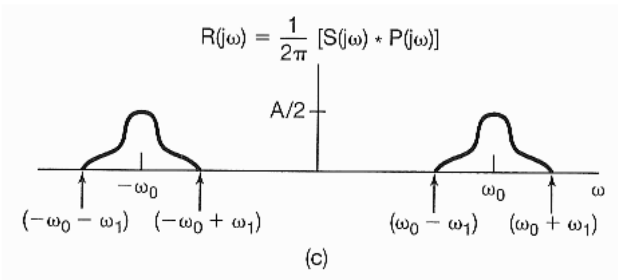
$$p(t) = \cos(w_0 t)$$

$$P(jw) = \pi\delta(w - w_0) + \pi\delta(w + w_0)$$

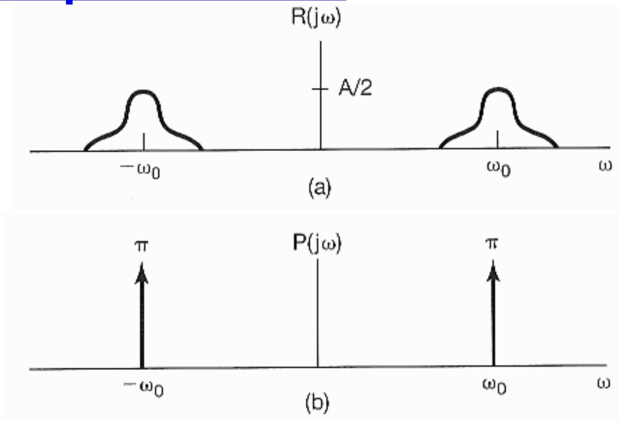
$$R(jw) = \frac{1}{2\pi}\left[S(jw) * P(jw)\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

$$= \frac{1}{2}S\left(j(w - w_0)\right) + \frac{1}{2}S\left(j(w + w_0)\right)$$



- Example 4.22:



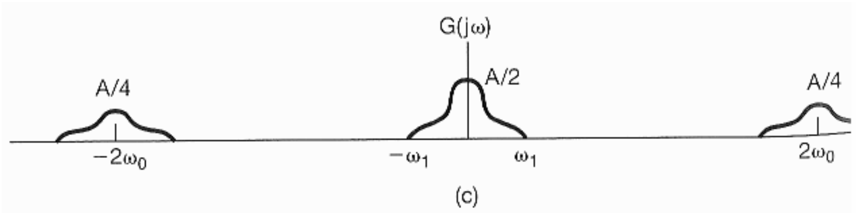
$$g(t) = r(t)p(t)$$

$$r(t) \overset{\mathcal{F}}{\longleftrightarrow} R(jw)$$

$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(jw)$$

$$p(t) = \cos(w_0 t)$$

$$G(jw) = \frac{1}{2\pi}[R(jw) * P(jw)]$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

- Example 4.23:

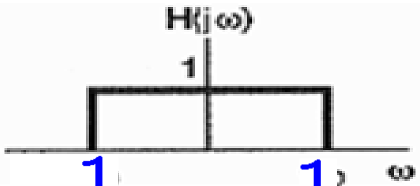
$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

$$X(jw) = \int_{-\infty}^{\infty} \frac{\sin(t)\sin(t/2)}{\pi t^2} e^{-jwt}dt$$

$$= \pi \left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$X(jw) = \frac{1}{2}\mathcal{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

$\frac{1}{2}$



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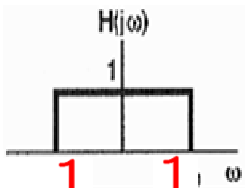
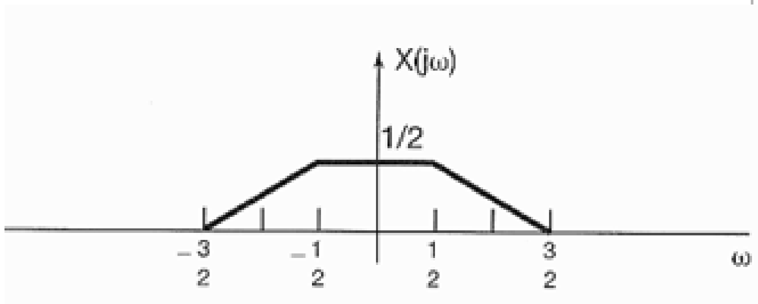
$*$

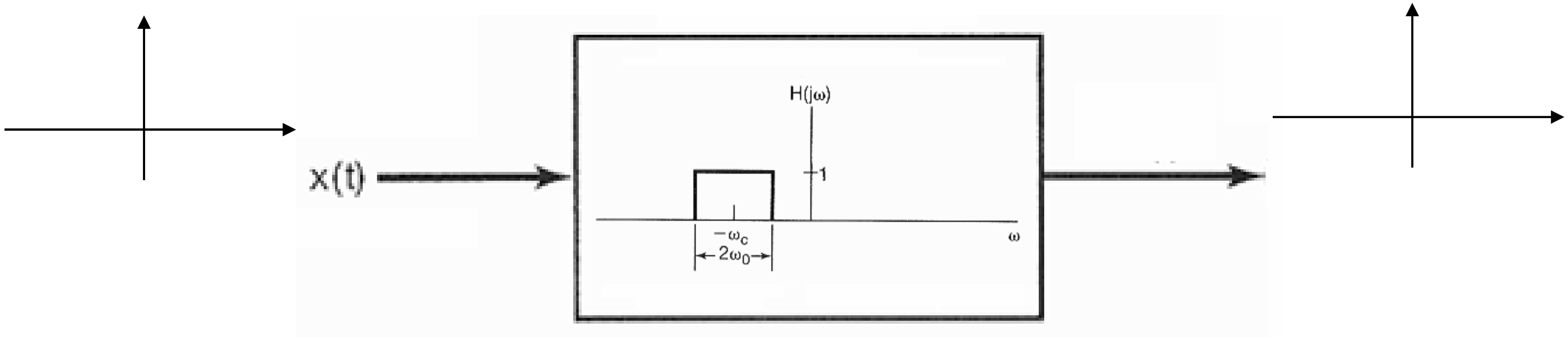


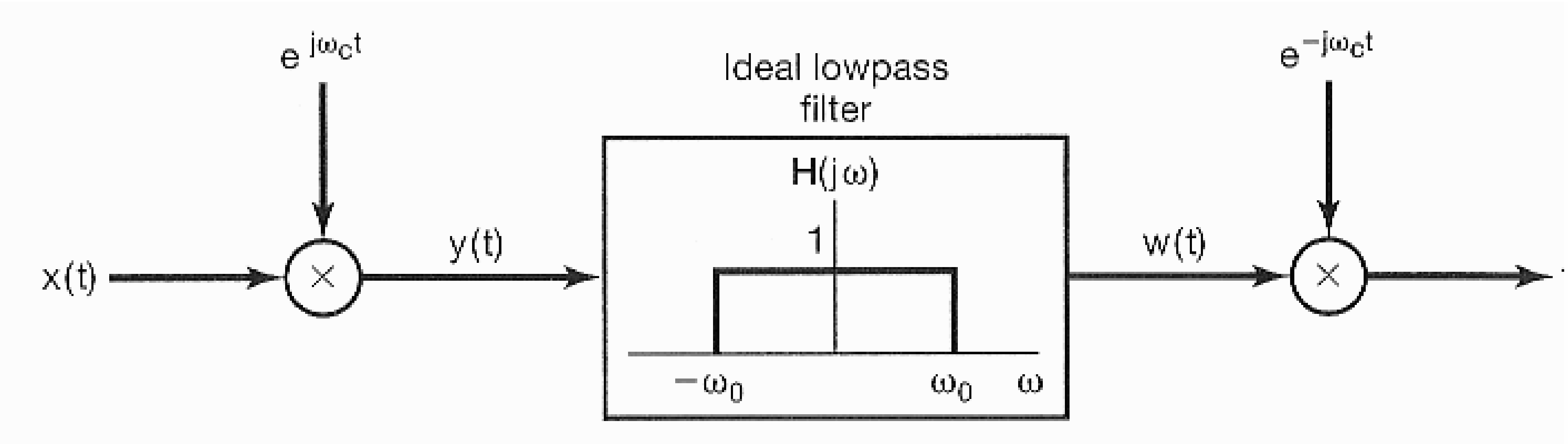
$-\frac{1}{2}$ $\frac{1}{2}$

$=$



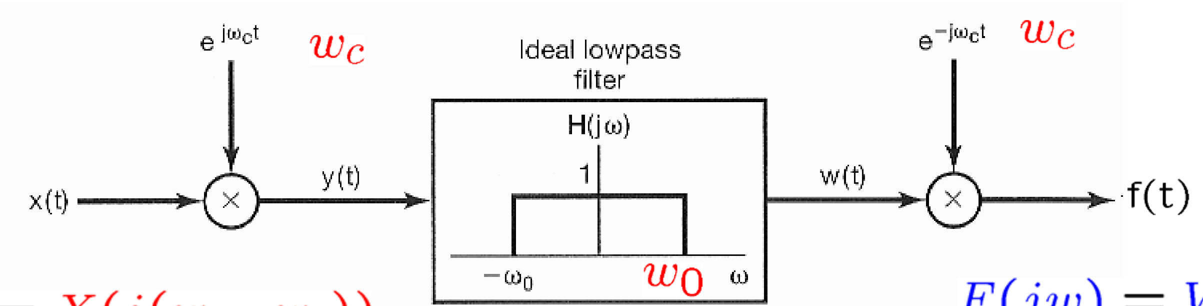
- Bandpass Filter Using Amplitude Modulation:





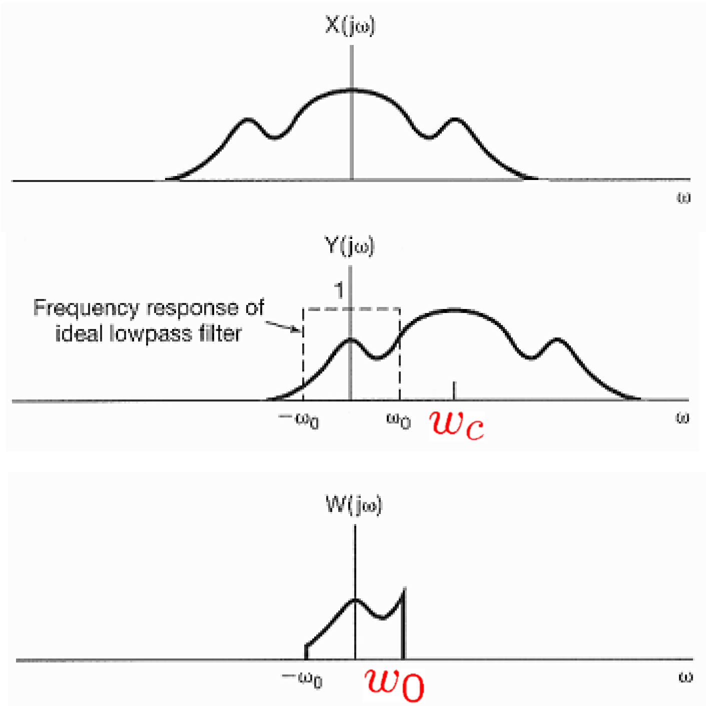
Bandpass Filter Using Amplitude Modulation:

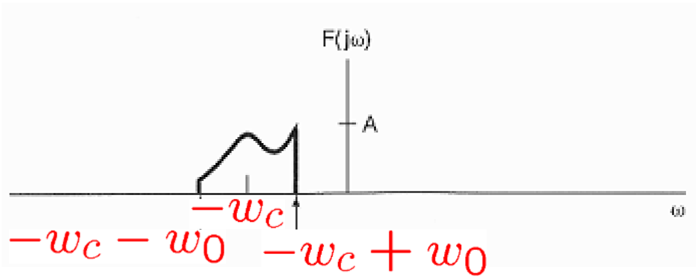
$$e^{jw_ct} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(w - w_c)$$

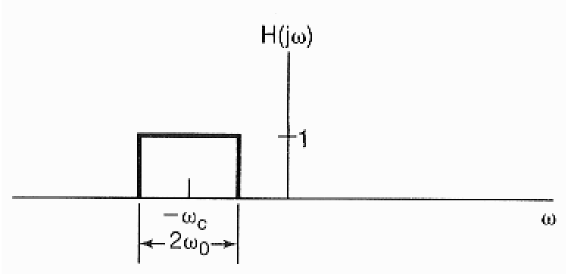


$$Y(jw) = X(j(w - w_c))$$

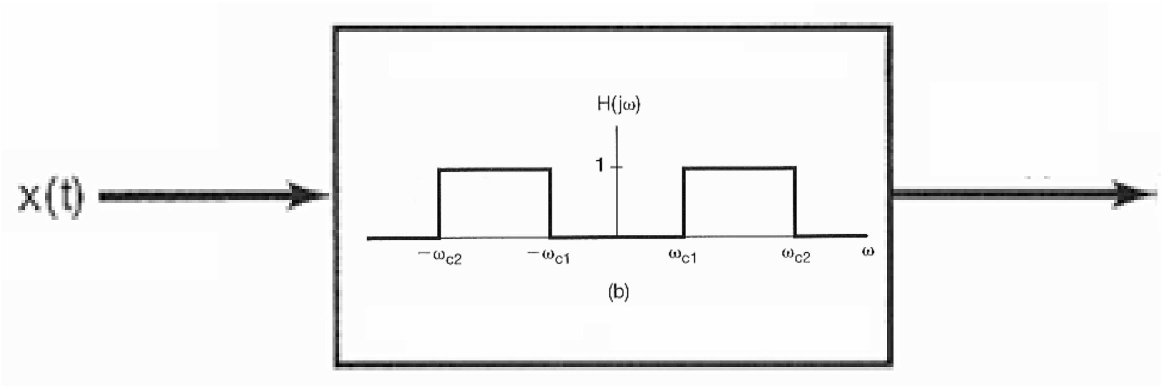
$$F(jw) = W(j(w + w_c))$$



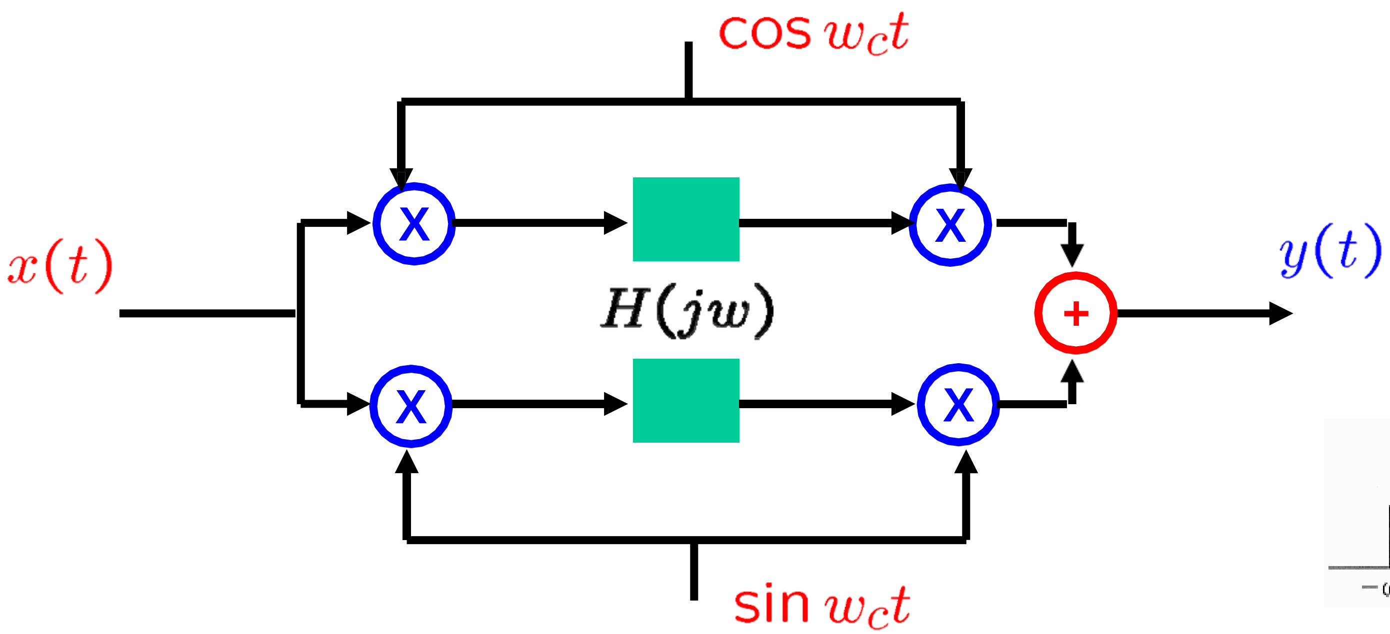




- Bandpass Filter Using Amplitude Modulation:



- On Page 349-350, Problem 4.46



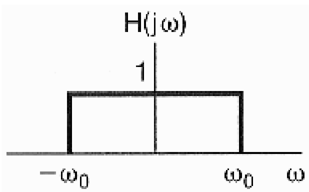


TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

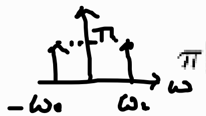
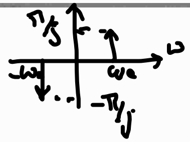
Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\}=-\Im m\{X(-j\omega)\} \\ \|X(j\omega)\|=\|X(-j\omega)\| \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

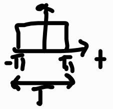
4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

$$\begin{cases} g(t)\longleftrightarrow f(\omega) \\ f(t)\longleftrightarrow 2\pi g(-\omega) \end{cases}$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$ 	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$ 	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k

and $x(t+T) = x(t)$		
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$ 	$\frac{2\sin\omega T_1}{\omega} = T\,\mathrm{sinc}\left(\frac{\omega}{\pi} \times \frac{T}{2}\right)$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—
$e^{-\alpha\lvert t\rvert}$ $(\alpha > 0)$	$\frac{2\alpha}{\omega^2 + \alpha^2}$	

- Representation of Aperiodic Signals: the Continuous-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of the Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

- A useful class of CT LTI systems:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$Y(jw) = X(jw) H(jw) \qquad H(jw) = \frac{Y(jw)}{X(jw)}$$

Systems Characterized by Linear Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k \qquad \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \qquad \frac{d^k x(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k (jw)^k \;\; Y(jw) = \sum_{k=0}^{M} b_k (jw)^k \;\; X(jw)$$

$$Y(jw) \;\; \left[\sum_{k=0}^{N} a_k (jw)^k\right] = X(jw) \;\; \left[\sum_{k=0}^{M} b_k (jw)^k\right]$$

$$\Rightarrow \quad H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{k=0}^{M} b_k (jw)^k}{\sum_{k=0}^{N} a_k (jw)^k} = \frac{b_M (jw)^M + \cdots + b_1 (jw) + b_0}{a_N (jw)^N + \cdots + a_1 (jw) + a_0}$$

- Examples 4.24 & 4.25:

$$H = \frac{Y}{X}$$

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

$$\Rightarrow \quad H(jw) = \frac{1}{jw + a}$$

$$(jw)Y(jw) + aY(jw) = X(jw)$$

$$\Rightarrow \quad h(t) = e^{-at}u(t)$$

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3\,y(t) = \frac{dx(t)}{dt} + 2\,x(t)$$

$$\Rightarrow \quad H(jw) = \frac{(jw)+2}{(jw)^2 + 4(jw) + 3} = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$= \frac{1/2}{jw+1} + \frac{1/2}{jw+3}$$

$$\Rightarrow \quad h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

- Example 4.26:

$$x(t) = e^{-t}u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = ???$$

$$H(jw) = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$\Rightarrow \quad Y(jw) = X(jw)H(jw)$$

$$= \left[\frac{1}{jw+1}\right]\left[\frac{jw+2}{(jw+1)(jw+3)}\right]$$

$$= \frac{jw+2}{(jw+1)^2(jw+3)}$$

$$= \frac{\frac{1}{4}}{jw+1} + \frac{\frac{1}{2}}{(jw+1)^2} - \frac{\frac{1}{4}}{jw+3}$$

$$\Rightarrow \quad y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}\,t\,e^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$

- Representation of Aperiodic Signals: the CT FT
- The FT for Periodic Signals
- Properties of the CT FT
  - Linearity
  - Time Shifting
  - Frequency Shifting
  - Conjugation
  - Time Reversal
  - Time and Frequency Scaling
  - Convolution
  - Multiplication
  - Differentiation in Time
  - Integration
  - Differentiation in Frequency
  - Conjugate Symmetry for Real Signals
  - Symmetry for Real and Even Signals & for Real and Odd Signals
  - Even-Odd Decomposition for Real Signals
  - Parseval's Relation for Aperiodic Signals
- The Convolution Property
- The Multiplication Property
- Systems Characterized by Linear Constant-Coefficient Differential Equations

Questions for Chapter 4

- Why to study FT
  - In order to analyze or represent aperiodic signals
- How to develop FT
  - From FS and let T -> infinity
- Do periodic signals have FT
  - Yes, their FT is function of isolated impulses
- Why to know the properties of FT
  - Avoid using the fundamental formulas of FT to compute the FT
- What the duality of FT and why
  - FT and IFT have almost identical integration formulas
- Why to know the convolution property
  - To analyze system response and/or design proper circuits
  - To simplify computation
- Why to know the multiplication property
  - For signal modulation with different-frequency carriers
  - To simplify computation

X(jw) of Aperiodic Signals and a_k of Periodic Signals

$$a_k = \frac{1}{T} X(jw)\Big|_{w=kw_0}$$

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi \; a_k \; \delta(w - kw_0)$$

$$= \sum_{k=-\infty}^{+\infty} 2\pi \; \frac{1}{T} X(jkw_0) \; \delta(w - kw_0)$$

$$w = mw_0$$

$$X(jmw_0) = \sum_{k=-\infty}^{+\infty} 2\pi \; \frac{1}{T} X(jkw_0) \; \delta(mw_0 - kw_0)$$

$$= 2\pi \; \frac{1}{T} \; X(jmw_0)$$

$$\Rightarrow \; 2\pi = T$$

X(jw) of Aperiodic Signals and a_k of Periodic Signals

$$a_k = \frac{1}{T} X_a(jw) \Big|_{w = kw_0}$$

$$X_p(jw) = \sum_{k=-\infty}^{+\infty} 2\pi \ a_k \ \delta(w - kw_0)$$

$$= \sum_{k=-\infty}^{+\infty} 2\pi \ \frac{1}{T} X_a(jkw_0) \ \delta(w - kw_0)$$

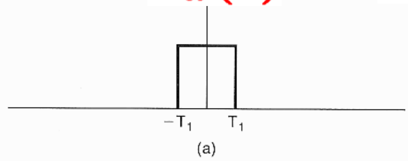
$$w = mw_0$$

$$X_p(jmw_0) = \sum_{k=-\infty}^{+\infty} 2\pi \ \frac{1}{T} X_a(jkw_0) \ \delta(mw_0 - kw_0)$$

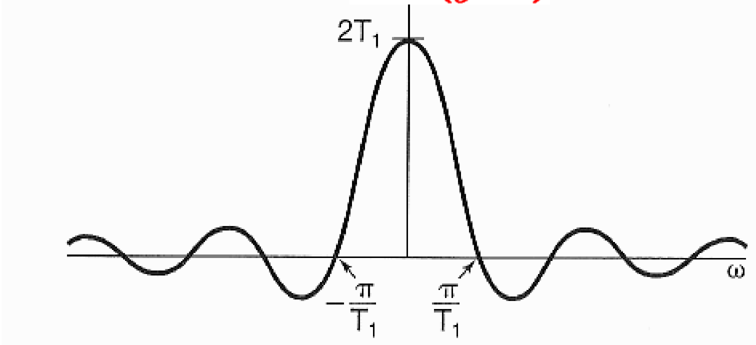
$$= 2\pi \ \frac{1}{T} \ X_a(jmw_0)$$

X(jw) of Aperiodic Signals and a_k of Periodic Signals

$x_a(t)$ $\quad T_1 = 1$

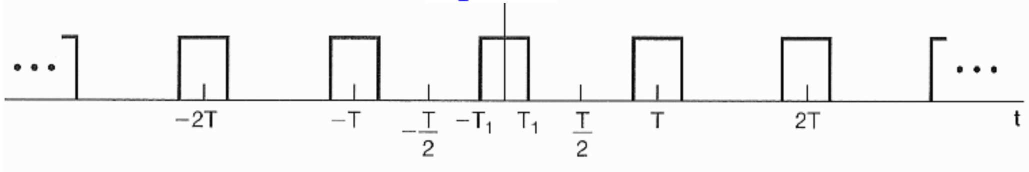


$X_a(jw)$



$T = 4 \quad w_0 = 2\pi/4 = \pi/2 \quad X_a(jw) = 2\dfrac{\sin(wT_1)}{w} = 2\dfrac{\sin(w)}{w}$

$x_p(t)$



$$\Rightarrow \quad a_k = \frac{1}{T} X_a(jw)\Big|_{w=kw_0}$$

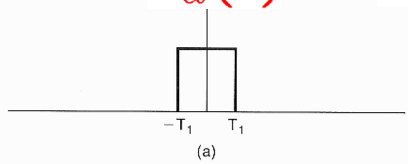
$$= \frac{\sin(k\pi/2)}{\pi k}$$

$$\Rightarrow \quad X_p(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0) = \sum_{k=-\infty}^{+\infty} \frac{2\sin(k\pi/2)}{k} \delta(w - kw_0)$$

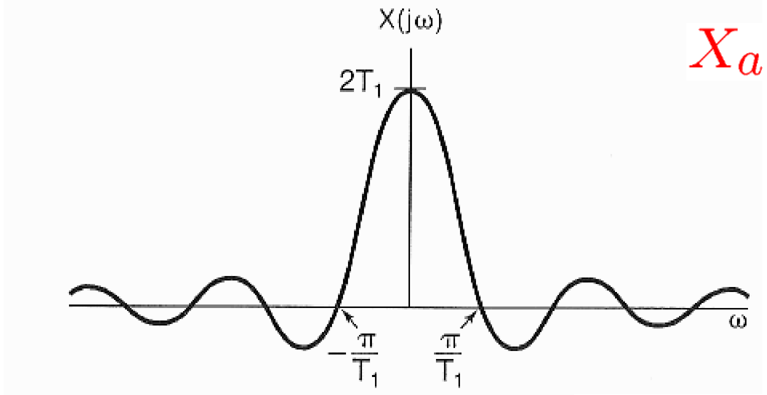
$$\Rightarrow \quad X_p(jmw_0) = \frac{2\sin(m\pi/2)}{m}$$

X(jw) of Aperiodic Signals and a_k of Periodic Signals

$x_a(t)$ $\quad T_1 = 1$

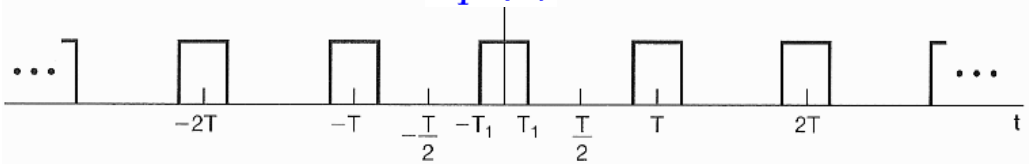


$X_a(jw)$



$T = 4 \qquad w_0 = 2\pi/4 = \pi/2 \qquad X_a(jw) = 2\,\dfrac{\sin(wT_1)}{w} = 2\,\dfrac{\sin(w)}{w}$

$x_p(t)$



$$\Rightarrow \quad a_k = \frac{1}{T} X_a(jw)\Big|_{w=kw_0}$$

$$\Rightarrow \quad X_p(jmw_0) = \frac{2\sin(m\pi/2)}{m} = 2\pi a_m \qquad = \frac{\sin(k\pi/2)}{\pi k}$$

$$= \frac{2\pi}{T} X_a(jmw_0)$$

$$= \frac{\pi}{2} X_a(jmw_0)$$

m	0	1
a_m	$1/2$	$1/\pi$
$2\pi a_m$	π	2
$X_p(jmw_0)$	π	2
$X_a(jmw_0)$	2	$4/\pi$

$X_p(jw)$