Some points

A sufficient condition for a system to have frequency response, it to be **stable.**

Solving a differential equation using Fourier transform approach gives only **stable** solution.

$$h(t) * h_i(t) = \delta(t)$$

$$H(jw).H_I(jw) = 1 \rightarrow H_I(jw) = \frac{1}{H(jw)}$$

Provided that H(jw) is not zero for all w

Some points

$$\frac{1}{jw + a} \xrightarrow{\mathcal{F}^{-1}} ?$$

$$\frac{1}{jw+a} \overset{\mathcal{F}^{-1}}{\rightarrow} ? \qquad a > 0 \overset{\mathcal{F}^{-1}}{\rightarrow} e^{-at}u(t)$$

$$x(t) \overset{\mathcal{F}}{\rightarrow} X(jw)$$

$$x(-t) \overset{\mathcal{F}}{\rightarrow} X(-jw)$$

$$x(t) = e^{-at}u(t), \quad a > 0 \overset{\mathcal{F}}{\rightarrow} \frac{1}{jw+a}$$

$$x(-t) = e^{at}u(-t), \quad a > 0 \overset{\mathcal{F}}{\rightarrow} \frac{1}{-jw+a}$$

$$\frac{1}{+jw-a} \overset{\mathcal{F}^{-1}}{\rightarrow} -e^{at}u(-t)$$

Spring 2011

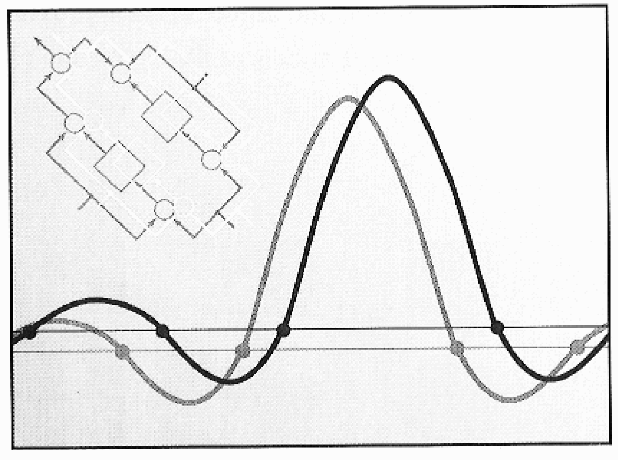
信號與系統
Signals and Systems

Chapter SS-4
The Discrete-Time Fourier Series

Feng-Li Lian

NTU-EE

Feb11 – Jun11



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- **Basic Idea:**
  - To represent signals
    as linear combinations of basic signals

- **Key Properties:**

1. The set of basic signals can be used
   to construct a broad and useful class of signals

2. The response of an LTI system to each signal should be simple enough in structure
   to provide us with a convenient representation for the response of the system to any signals
   constructed as linear combination of basic signals

- Harmonically related complex exponentials

$$\phi_k[n] = e^{jkw_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} e^{jN\left(\frac{2\pi}{N}\right)n}$$

$$\Rightarrow \ \phi_k[n] = \phi_{k+N}[n] = \cdots = \phi_{k+rN}[n]$$

- The Fourier Series Representation:

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- Periodicity of Fourier Coefficients

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\begin{aligned} x[n] &= a_0\phi_0[n] + a_1\phi_1[n] + \cdots + a_{N-1}\phi_{N-1}[n] \\ &= a_1\phi_0[n] + a_1\phi_1[n] + \cdots + a_{N-1}\phi_{N-1}[n] + a_N\phi_N[n] \end{aligned}$$

Since, $\phi_0[n]$ and $\phi_N[n]$ are the same, $\because a_0$ and a_N are identical

- <u>Procedure of Determining the Coefficients:</u>

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$n = 0 \qquad \sum_{k=0}^{N-1} a_k = x[0]$$

$$n = 1 \qquad \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}} = x[1]$$

$$n = 2 \qquad \sum_{k=0}^{N-1} a_k e^{jk\frac{2\times 2\pi}{N}} = x[2]$$

$$\vdots$$

$$n = N-1 \qquad \sum_{k=0}^{N-1} a_k e^{jk\frac{(N-1)\times 2\pi}{N}} = x[N-1]$$

Procedure of Determining the Coefficients:

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)}$$

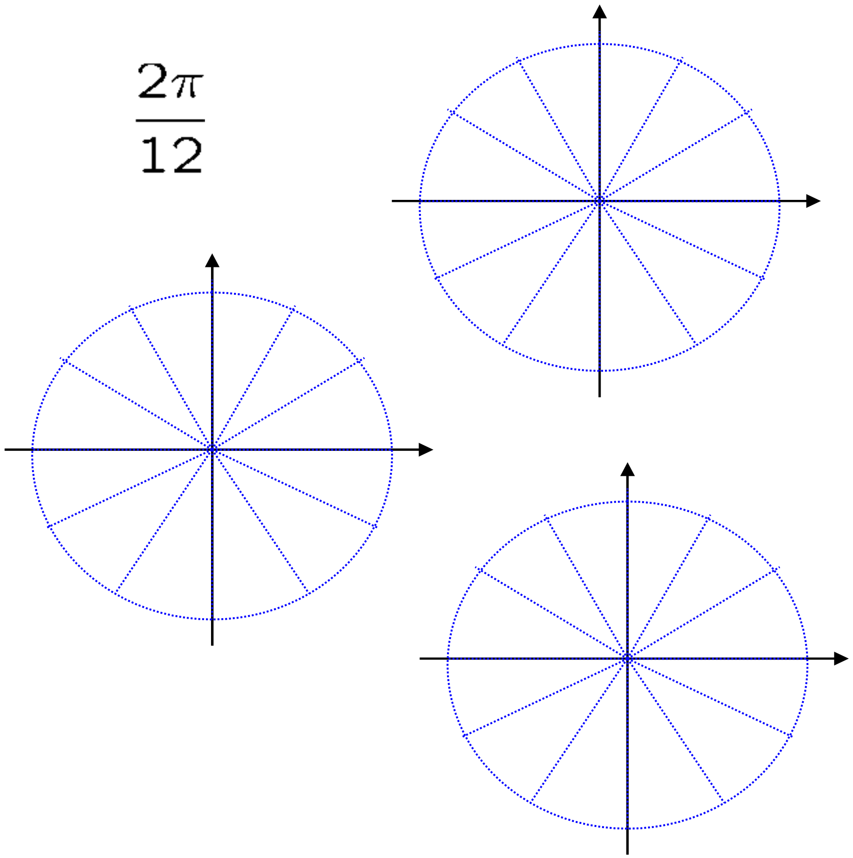
$$x[2] = \sum_{k=<N>} a_k e^{jk2\left(\frac{2\pi}{N}\right)}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{jk(N-1)\left(\frac{2\pi}{N}\right)}$$

$$x[N] = \sum_{k=<N>} a_k e^{jk(N)\left(\frac{2\pi}{N}\right)}$$

$$\frac{2\pi}{12}$$



and $$\sum_{n=<N>} e^{jm\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & m = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

We have a geometric series, where its start term is 1

and the common ratio is equal to $e^{jm\frac{2\pi}{N}}$

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{jm\frac{2\pi}{N}n} = \frac{1}{N}\frac{1-\left(e^{jm\frac{2\pi}{N}}\right)^{N}}{1-e^{jm\frac{2\pi}{N}}} = 0$$

- Procedure of Determining the Coefficients:

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \times \qquad \sum_{n=<N>} e^{-jr\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$= a_r N$$

$$\Rightarrow \ a_r = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n}$$

In Summary:

- The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- The analysis equation:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = a_{k+N}$$

- $x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

- $\{a_k\}$: the Fourier series coefficients

  or the spectral coefficients of $x[n]$

- Example 3.10:

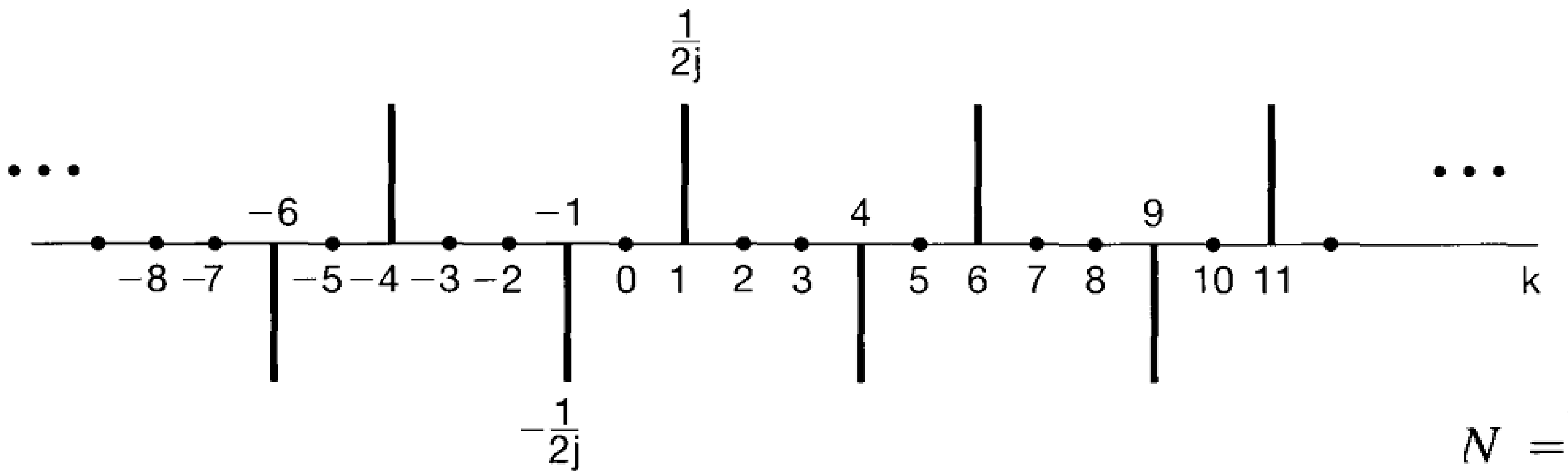
$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n] = \sin \omega_0 n,$$

periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers.

$$\omega_0 = \frac{2\pi}{N}, \qquad x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}.$$

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j},$$



$N = 5$

- Example 3.10:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n] = \sin \omega_0 n,$$

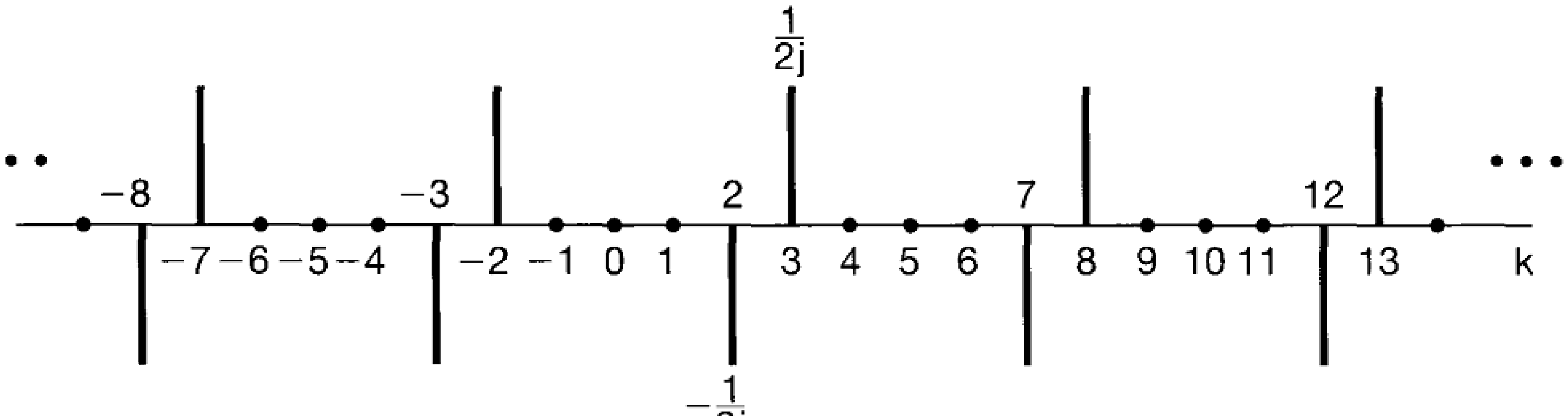
periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers.

M and N do not have any common factors.

$$\omega_0 = \frac{2\pi M}{N}$$

$$x[n] = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n}$$

$$a_M = (1/2j),\ a_{-M} = (-1/2j),$$



$M = 3$ and $N = 5$

- Example 3.11:

$$x[n] \;=\; 1 \;+\; \sin\left(\frac{2\pi}{N}\right)n \;+\; 3\cos\left(\frac{2\pi}{N}\right)n \;+\; \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

$$\Rightarrow\; x[n] = \;1 \;+\; \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{N}\right)n} - e^{-j\left(\frac{2\pi}{N}\right)n}\right] \;+\; \frac{3}{2}\left[e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{N}\right)n}\right]$$

$$+\; \frac{1}{2}\left[e^{j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)} \;+\; e^{-j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)}\right]$$

$$\Rightarrow\; x[n] = \;1 \;+\; \left(\frac{3}{2}+\frac{1}{2j}\right)e^{j\left(\frac{2\pi}{N}\right)n} \;+\; \left(\frac{3}{2}-\frac{1}{2j}\right)e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$+\; \frac{1}{2}\,e^{j\left(\frac{\pi}{2}\right)}\,e^{j2\left(\frac{2\pi}{N}\right)n} \;+\; \frac{1}{2}\,e^{-j\left(\frac{\pi}{2}\right)}\,e^{-j2\left(\frac{2\pi}{N}\right)n}$$

- Example 3.11:

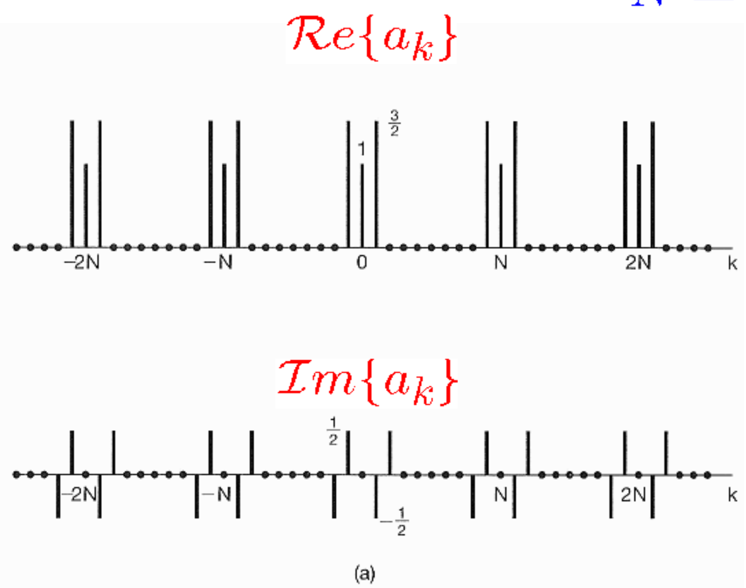
$$\Rightarrow \begin{cases} a_0 = 1 \\ a_1 = \left(\frac{3}{2} + \frac{1}{2j}\right) = \frac{3}{2} - \frac{1}{2}j \\ a_{-1} = \left(\frac{3}{2} - \frac{1}{2j}\right) = \frac{3}{2} + \frac{1}{2}j \\ a_2 = \frac{1}{2}j \\ a_{-2} = -\frac{1}{2}j \\ a_k = 0, \quad \text{others in} < N > \end{cases}$$

$$\boxed{a = |a| e^{j \sphericalangle a}}$$

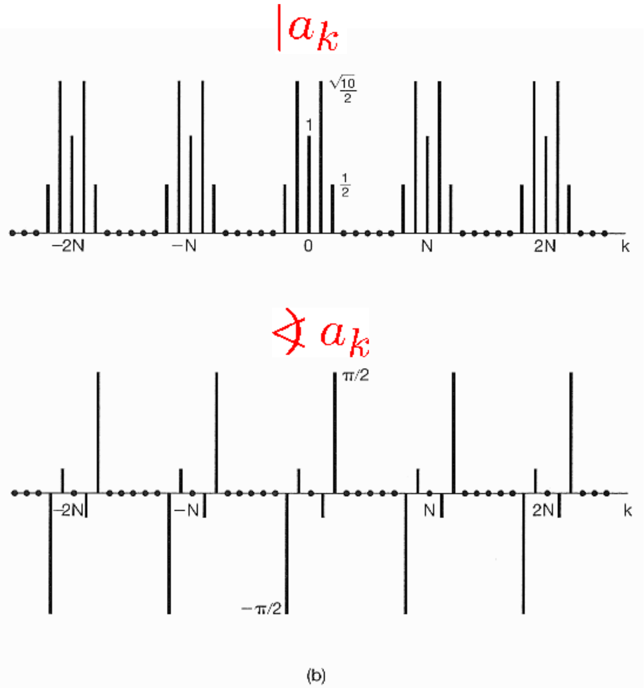
$$\boxed{a = |a| \left[\cos(\sphericalangle a) + j \sin(\sphericalangle a) \right]}$$

$$\boxed{a = b + jc = \sqrt{b^2+c^2} \left[\frac{b}{\sqrt{b^2+c^2}} + j \frac{c}{\sqrt{b^2+c^2}} \right]}$$

$N = 10$



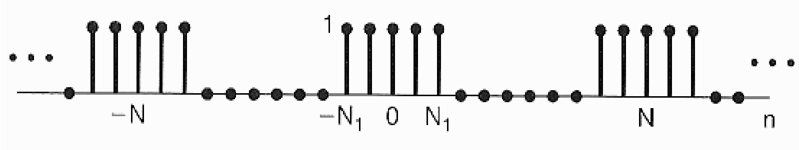
(a)



(b)

- Example 3.12:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$



$$x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & \text{others in } <N> \end{cases}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1 \cdot e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1}\left(e^{-jk\left(\frac{2\pi}{N}\right)}\right)^n$$

$$= \frac{1}{N}\left[\left(\cdot\right)^{-N_1} + \left(\cdot\right)^{-N_1+1} + \cdots + \left(\cdot\right)^{N_1}\right]$$

$$= \frac{1}{N}\left(\cdot\right)^{-N_1}\left[\frac{1-\left(\cdot\right)^{(2N_1+1)}}{1-\left(\cdot\right)}\right] \qquad \left(\cdot\right) \ne 1$$

$$= \frac{1}{N}\left(\cdot\right)^{-N_1}\left[1 + \left(\cdot\right)^{1} + \cdots + \left(\cdot\right)^{2N_1}\right]$$

- Let $m = n + N_1$ or $n = m - N_1$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} \qquad = \frac{1}{N}e^{jk\left(\frac{2\pi}{N}\right)N_1}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

Example 3.12:

$$1 - e^{-j\theta}$$
$$= e^{-j\theta/2} e^{j\theta/2} - e^{-j\theta/2} e^{-j\theta/2}$$
$$= e^{-j\theta/2} \left(e^{j\theta/2} - e^{-j\theta/2} \right)$$

- $k = 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{2N_1 + 1}{N}$$

- $k \neq 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \left(\frac{1 - e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}} \right)$$

$$= \frac{1}{N} \frac{e^{-jk\left(\frac{2\pi}{2N}\right)} \left[e^{jk\left(\frac{2\pi}{2N}\right)(2N_1+1)} - e^{-jk\left(\frac{2\pi}{2N}\right)(2N_1+1)} \right]}{e^{-jk\left(\frac{2\pi}{2N}\right)} \left[e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)} \right]}$$

$$= \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k(N_1 + \frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$

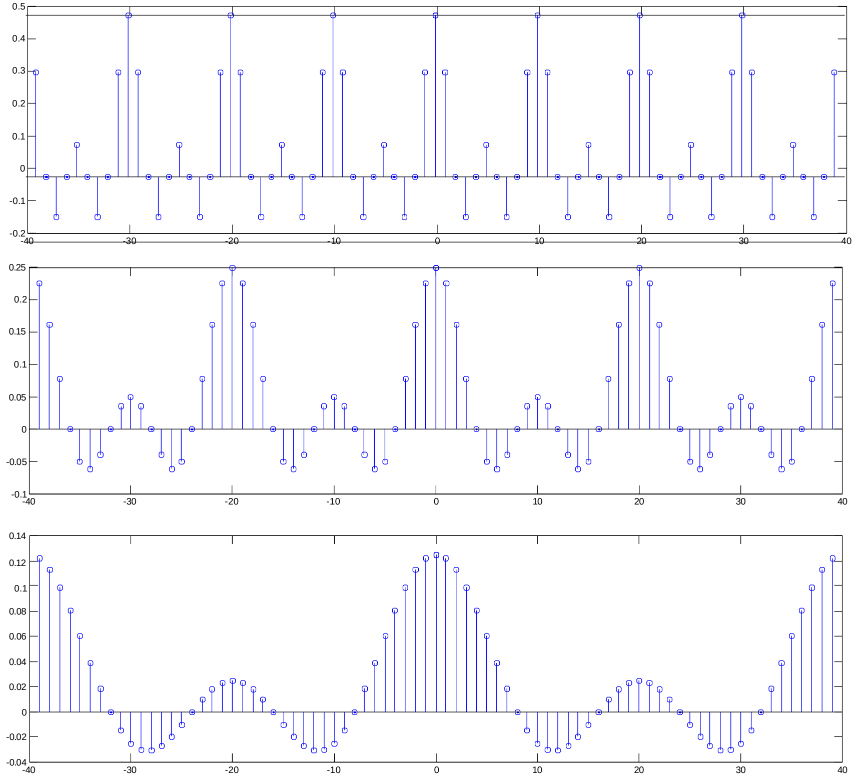
- Example 3.12:

$$a_k = \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k (N_1 + \frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$

- $2N_1 + 1 = 5$

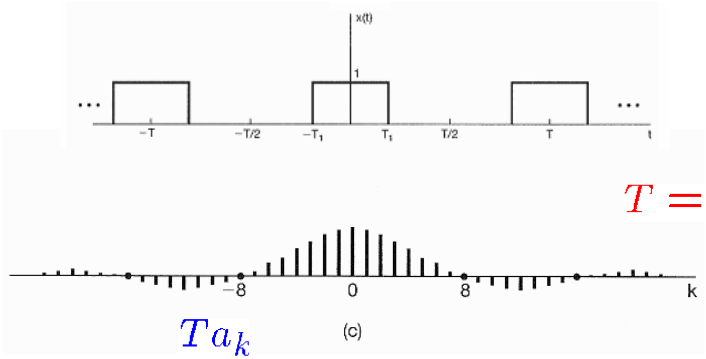
a_k

- $N = 10$
- $N = 20$
- $N = 40$



- Examples 3.5 (CT) & 3.12 (DT):

$$T a_k = T \frac{\sin(k\frac{\pi}{8})}{k\pi}$$

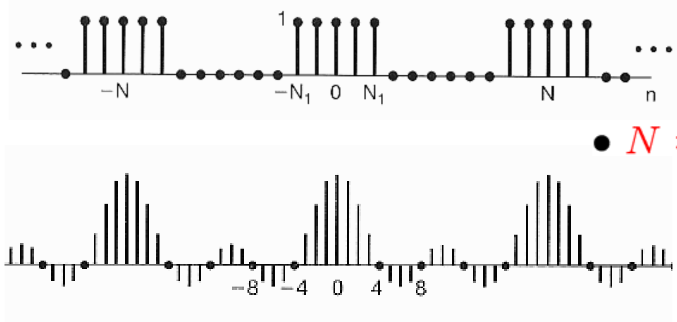


$T = 16T_1$

$T a_k$

$$a_k = \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k (N_1 + \frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$

$$a_k = \frac{2N_1 + 1}{N}$$



- $N = 20$

Partial Sum:

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

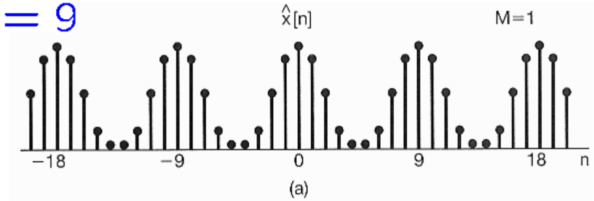
- If N is odd

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

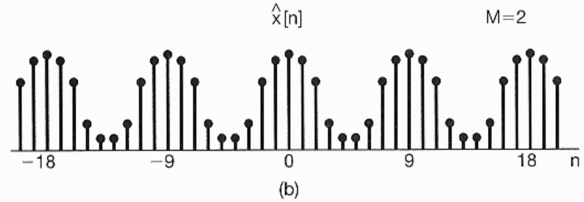
- If N is even

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

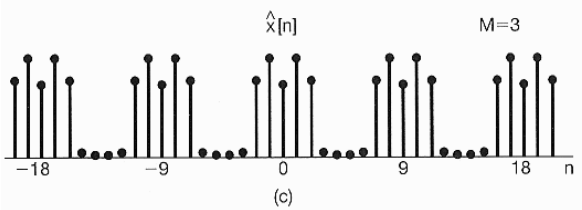
$N = 9$



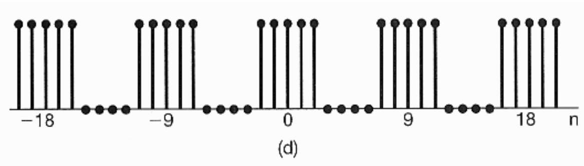
$M = 1$



$M = 2$



$M = 3$



$M = 4$

Outline

Section	Property
	Linearity
	Time Shifting
	Frequency Shifting
	Conjugation
	Time Reversal
	Time Scaling
	Periodic Convolution
3.7.1	Multiplication
3.7.2	First Difference
	Running Sum
	Conjugate Symmetry for Real Signals
	Symmetry for Real and Even Signals
	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.7.3	Parseval's Relation for Periodic Signals

Properties of DT Fourier Series

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

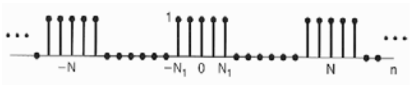
Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

- In Summary:

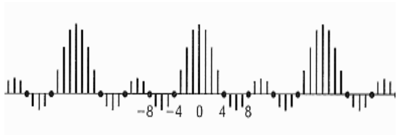
- The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$



- The analysis equation:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$



$$a_k = a_{k+N}$$

- $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

Linearity:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$$

- $x[n], y[n]$: periodic signals with period N

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$\Rightarrow z[n] = Ax[n] + By[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$$

Time Shifting:

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x[n-n_0] \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

- Multiplication:

- $x[n], y[n]$: periodic signals with period N

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x[n] = \sum_{l=<N>} a_l e^{jlw_0 n}$$

$$y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y[n] = \sum_{m=<N>} b_m e^{jmw_0 n}$$

$\Rightarrow$ $x[n]y[n]$: also periodic with N

$$x[n]y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=<N>} a_l \, b_{k-l}$$

$\Rightarrow$ a periodic convolution

- First Difference:

$$x[n] = \sum_{l=<N>} a_k e^{jkw_0 n}$$

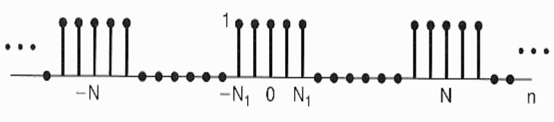
$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

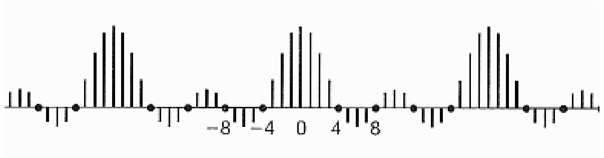
$$\Rightarrow \; x[n-n_0] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0} a_k \; = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

$$\Rightarrow \; x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0} a_k \; = e^{-jk\left(\frac{2\pi}{N}\right)} a_k$$

$$x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} \left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k$$

- Parseval's relation for DT periodic signals:
  - As shown in Problem 3.57:





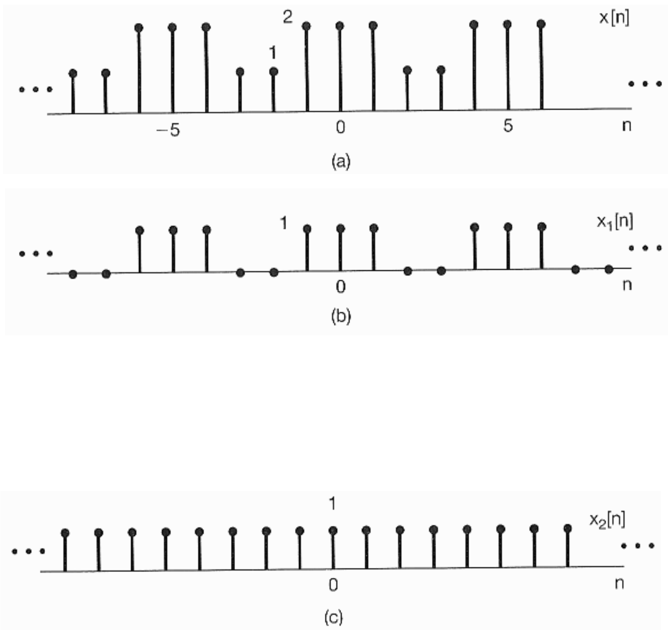
$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

$$\frac{1}{N} \sum_{k=<N>} \left| x[n] \right|^2 = \sum_{k=<N>} \left| a_k \right|^2$$

- Parseval's relation states that
  the total average power in a periodic signal equals
  the sum of the average powers in all of its harmonic components
  (only N distinct harmonic components in DT)

- Example 3.13:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n}$$



$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$x_1[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$x_2[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$

$$x[n] = x_1[n] + x_2[n]$$

$$\Rightarrow \quad a_k = b_k + c_k$$

$$\Rightarrow \; b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

$$\Rightarrow \; c_k = \begin{cases} 0, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ 1, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

- Example 3.14:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0 n}$$

Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

- Example 3.14:

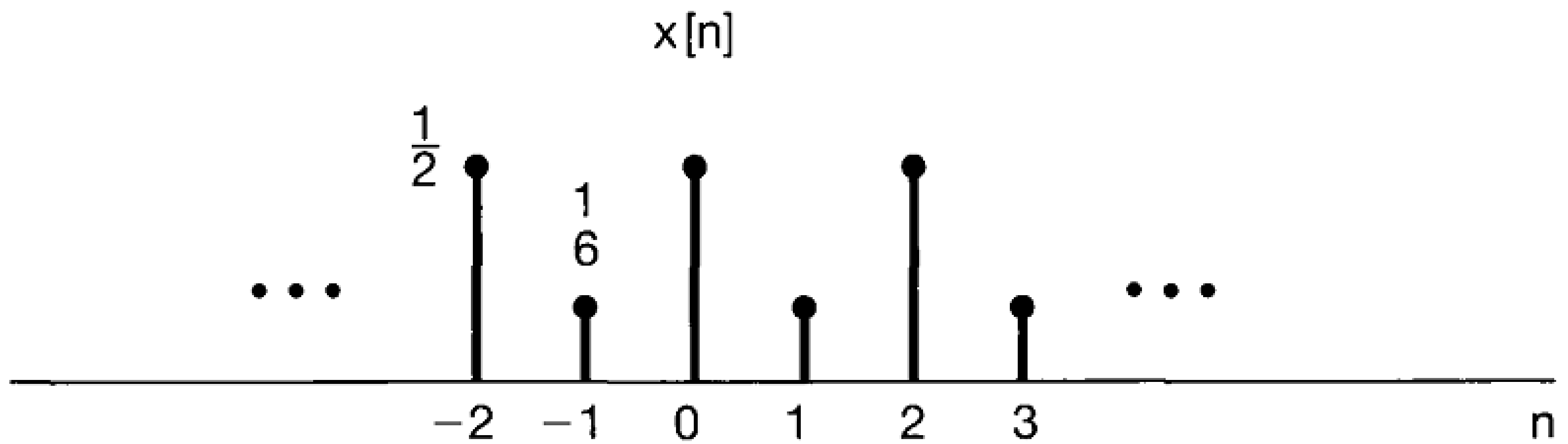
$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n}$$

$$\sum_{n=0}^{5} x[n] = 2 \quad \Rightarrow \quad a_0 = 1/3$$

$$(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n}$$

$$\sum_{n=2}^{7} (-1)^n x[n] = 1 \quad \Rightarrow \quad a_3 = 1/6$$

$$P = \sum_{k=0}^{5} |a_k|^2. \quad \Rightarrow \quad x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n,$$



The Response of an LTI System:

– On pages 12-14

$$in \longrightarrow \boxed{\text{LTI}} \longrightarrow out \qquad \begin{cases} \text{CT:} & e^{st} \longrightarrow H(s)e^{st} \\ \text{DT:} & z^{n} \longrightarrow H(z)z^{n} \end{cases}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt \qquad \Rightarrow \text{ the impulse response}$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \qquad \Rightarrow \text{ the system functions}$$

- If $s = jw$ or $z = e^{jw}$:

$$H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$$

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-jwn}$$

$\Rightarrow$ the frequency response

- In Summary:

$in \longrightarrow$ LTI H(*s/z/w*) $\longrightarrow out$

$\boxed{a = |a| e^{j \measuredangle a}}$

$\boxed{H = |H| e^{j \measuredangle H}}$

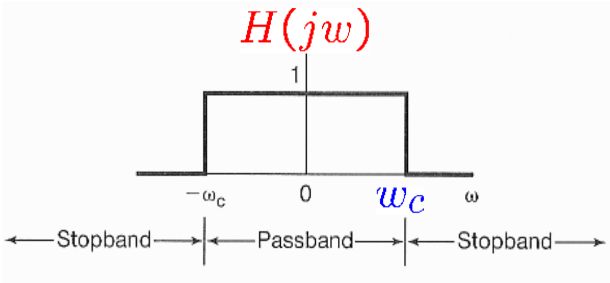
$(\ s_i = jw_i$ or $z_i = e^{jw_i} \ )$ $\begin{cases} \text{CT:} \quad e^{s_i t} \longrightarrow H(s_i) e^{s_i t} \\ \text{DT:} \quad z_i^n \longrightarrow H(z_i) z_i^n \end{cases}$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \ e^{jkw_0 t} \qquad \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k \ H(jkw_0) \ e^{jkw_0 t}$$

$$x[n] = \sum_{k=<N>} a_k \ e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \longrightarrow y[n] = \sum_{k=<N>} a_k \ H(e^{j\left(\frac{2\pi}{N}\right)k}) \ e^{jk\left(\frac{2\pi}{N}\right)n}$$

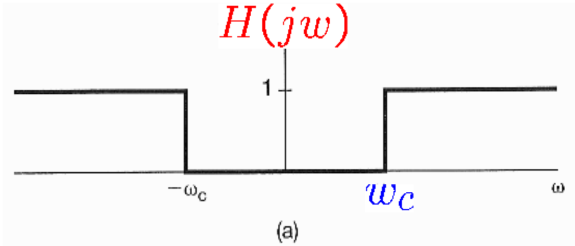
Frequency-Selective Filters:

- Select some bands of frequencies and reject others



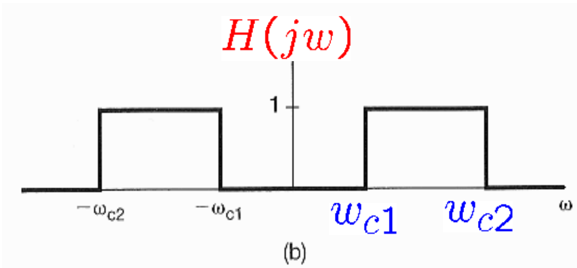
CT ideal lowpass filter

$$H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & |w| > w_c \end{cases}$$



CT ideal highpass filter

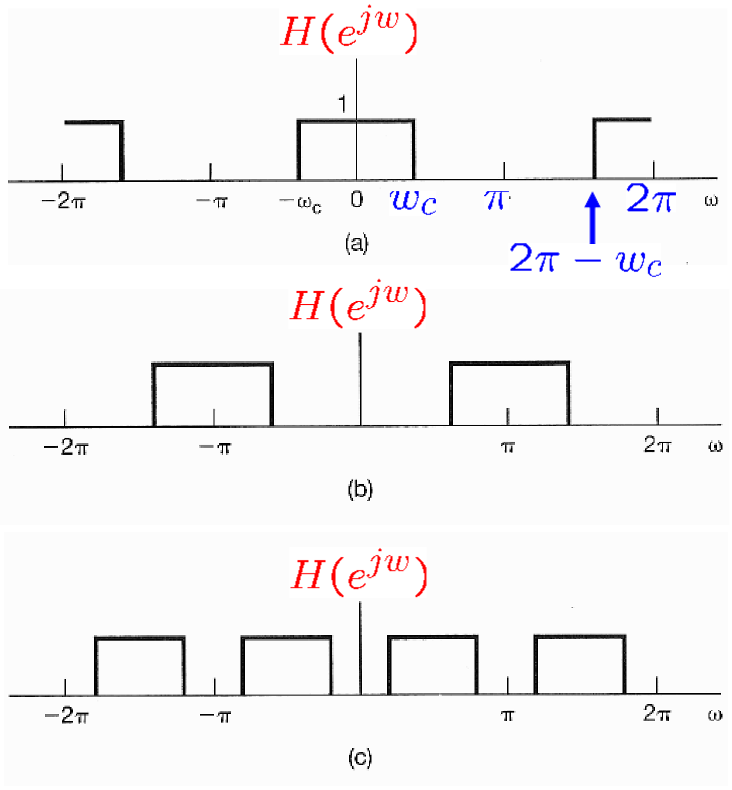
$$H(jw) = \begin{cases} 0, & |w| < w_c \\ 1, & |w| \ge w_c \end{cases}$$



CT ideal bandpass filter

$$H(jw) = \begin{cases} 1, & w_{c1} \le |w| \le w_{c2} \\ 0, & \text{otherwise} \end{cases}$$

- Frequency-Selective Filters:
  - Select some bands of frequencies and reject others



DT ideal lowpass filter

DT ideal highpass filter

DT ideal bandpass filter

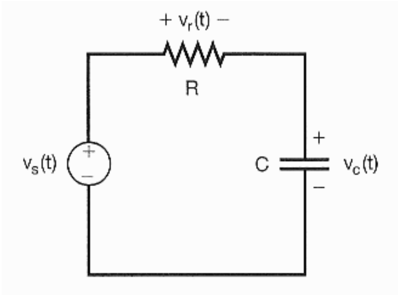
- A Simple RC Lowpass Filter:

Input signal:

$v_s(t) = e^{jwt}$

$\delta(t)$

$u(t)$



Output signal:

$v_c(t) = H(jw)e^{jwt}$

$h(t)$

$s(t)$

$$\Rightarrow \; RC\,\frac{d}{dt}v_c(t) + v_c(t) = v_s(t)$$

$$\Rightarrow \; RC\,\frac{d}{dt}\left[H(jw)e^{jwt}\right] + H(jw)e^{jwt} = e^{jwt}$$

$$\Rightarrow \; RC\;jw\;H(jw)e^{jwt} + H(jw)e^{jwt} = e^{jwt}$$

$$\Rightarrow \; H(jw)e^{jwt} = \frac{1}{1+RCjw}e^{jwt}$$

First-Order Recursive DT Filters:

$$y[n] - ay[n-1] = x[n]$$

- If $x[n] = e^{jwn}$, then $y[n] = H(e^{jw})e^{jwn}$

where $H(e^{jw})$: the frequency response

$$\Rightarrow \; H(e^{jw})\, e^{jwn} - a\, H(e^{jw})\, e^{jw(n-1)} = e^{jwn}$$

$$\Rightarrow \; \left[1 - a\, e^{-jw}\right] H(e^{jw})\, e^{jwn} = e^{jwn}$$

$$\Rightarrow \; H(e^{jw}) = \frac{1}{1 - a\, e^{-jw}}$$

- First-Order Recursive DT Filters:

$$H(e^{jw}) = \frac{1}{1 - a\, e^{-jw}}$$

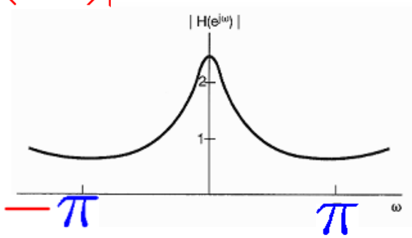
$$y[n] = ay[n-1] + x[n]$$

lowpass filter: $0 < a < 1$

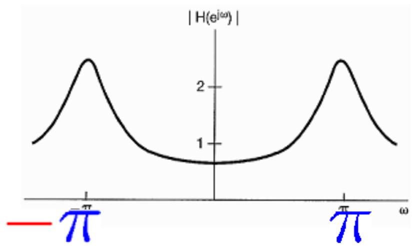
highpass filter: $-1 < a < 0$

$|H(e^{jw})|$

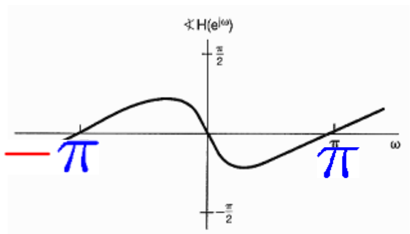
$a = 0.6$

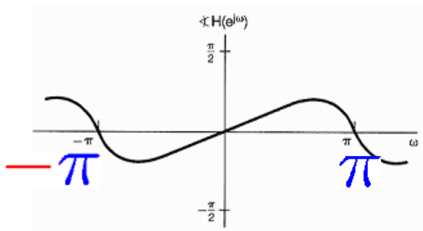


$a = -0.6$



$\measuredangle H(e^{jw})$



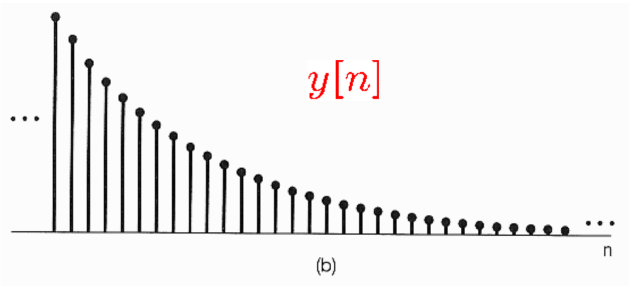


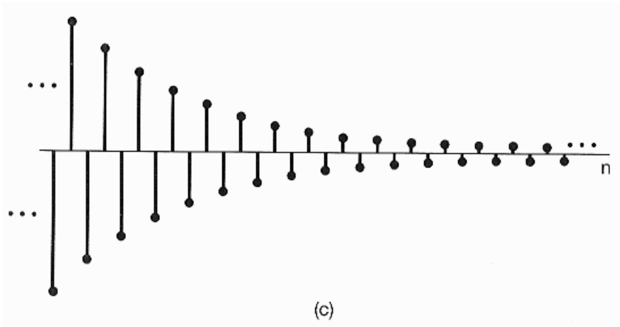
▪ First-Order Recursive DT Filters:

$$y[n] = ay[n-1] + x[n]$$

lowpass filter: $0 < a < 1$

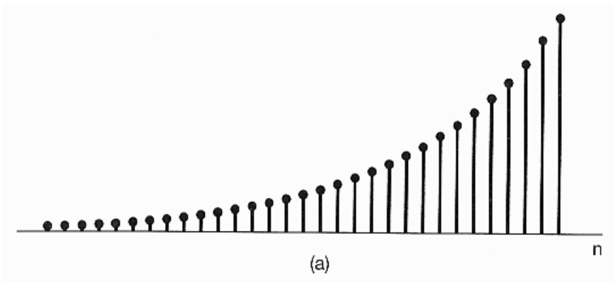
highpass filter: $-1 < a < 0$

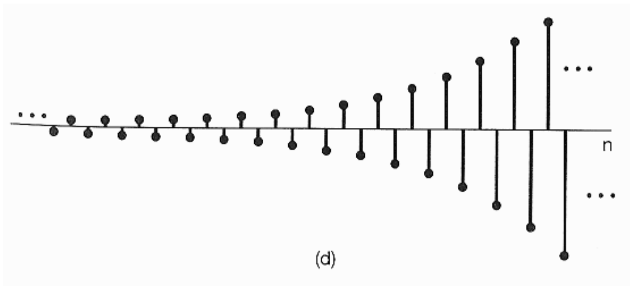




$a > 1$

$a < -1$





- Modulation