POLITECNICO
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Financial Engineering

Rating migrations

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Name	**CP**	**MAT**
Amirreza Khajouei	11023276	274892
Federico Favali	10712580	258314

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0. Introduction

The credit portfolio value at risk (VaR), as discussed in Schönbucher (Section 10.4), is highly sensitive to the average correlation of the return of assets (ρ) between the standardized returns. However, this correlation is not directly observable and calibrating it from historical data has remained a challenging and unresolved problem over the past two decades.
Our objective in this project is to calibrate the correlation parameter on firms initially rated "A" and "BBB", so that the model's joint credit rating migration probabilities align with historical quarterly transitions. By experimenting with various loss functions, we find that a *weighted mean squared error* yields the most accurate calibration results.
Furthermore, we extend the elementary single-factor Markov chain model for the calculation of the VaR of the Credit Portfolio, using the calibrated correlation parameter (ρ). We compute the 99.9% one-year VaR, taking into account migration risk at seven rating levels: AAA, AA, A, BBB, BB, B, and CCC, in addition to default.
Our sample portfolio consisting of 100 exposures:

- 50 zero-coupon bonds with a 2-year original maturity (face value \$1), issued by different A-rated firms;
- 50 zero-coupon bonds with a 2-year original maturity (face value \$1), issued by different BBB-rated firms.

We compare our results against the VaR calculation from Basel II regulatory correlations. The correlations in Basel II are derived from asset correlation functions using the default probabilities and adjusted for firm size (annual sales of €50 million).
The results confirm the findings originally presented by Frye [**frye**]: asset correlations tend to significantly overstate actual default correlations. Although regulatory correlations are more refined than a single constant, they still rely on asset-based models that overestimate portfolio credit risk. This suggests that traditional and regulatory methods may lead to an overestimated portfolio credit risk.

1. Theoretical Framework

1.1. Asset Value and Credit Quality Thresholds (Merton)

Merton model(refrence) establishes a relationship between the credit quality of a firm's asset and its rating. In this framework, a firm defaults when its asset value falls below a certain threshold; additionally, in the asset return space, there are thresholds that can define the credit ratings of a firm. These thresholds can be derived from the historical transition probabilities using the inverse of the standard normal CDF:

$$Z_j = \Phi^{-1}(P_j) \tag{1}$$

- P_j: Cumulative probability of migrating to rating j or lower
- Φ^{-1}: Inverse of the standard normal cumulative distribution function (quantile function)
- Z_j: Threshold corresponding to rating j

For a firm currently rated A, or BBB, we define thresholds $Z_{AAA}, Z_{AA}, Z_A, Z_{BBB}, \ldots, Z_{Default}$ such that:

- If $v_i > Z_{AAA}$, the firm upgrades to AAA
- If $Z_{AA} < v_i \leq Z_{AAA}$, the firm upgrades to AA
- If $Z_A < v_i \leq Z_{AA}$, the firm upgrades to A
- If $Z_{BBB} < v_i \leq Z_A$, the firm remains at BBB
- If $v_i \leq Z_{Default}$, the firm defaults

and so on for the remaining rating categories.

1.2. Joint Rating Migrations

The joint probability of rating migrations for two firms depends on their asset correlation. For a firm currently rated A and a firm currently rated BBB, the probability that the firm rated A migrates to rating k and the firm rated BBB migrates to rating l is given by:

$$P(\text{firm rated A} \rightarrow k,\ \text{firm rated BBB} \rightarrow l) = \int_{Z_{k-1}}^{Z_k} \int_{Z_{l-1}}^{Z_l} \phi_2(x, y; \rho)\, dx\, dy \tag{2}$$

where:

- $\phi_2(x, y; \rho)$ is the bivariate standard normal density function with correlation ρ,
- Z_i are the threshold values corresponding to the rating migrations for firms in the same category.

This integral can be calculated using the bivariate normal cumulative distribution function Φ_2 as follows:

$$\begin{aligned}\int_{Z_{k-1}}^{Z_k} \int_{Z_{l-1}}^{Z_l} \phi_2(x, y; \rho)\, dx\, dy = \Phi_2(Z_k, Z_l; \rho) - \Phi_2(Z_{k-1}, Z_l; \rho) \\ - \Phi_2(Z_k, Z_{l-1}; \rho) + \Phi_2(Z_{k-1}, Z_{l-1}; \rho)\end{aligned} \tag{3}$$

where:

- $\Phi_2(Z_i, Z_j; \rho)$ is the bivariate normal cumulative distribution function with correlation parameter ρ.

This can be seen as the probability mass within a rectangle formed by rating thresholds in the bivariate normal space. For each category, returns are segmented by these thresholds, creating a grid where each rectangle shows a specific pair of rating migrations. The formula uses inclusion-exclusion to calculate the probability in each rectangle. The correlation parameter ρ shapes the distribution. A higher correlation concentrates the mass along the diagonal, that shows similar rating changes at the same time. See the following plot for better understanding:

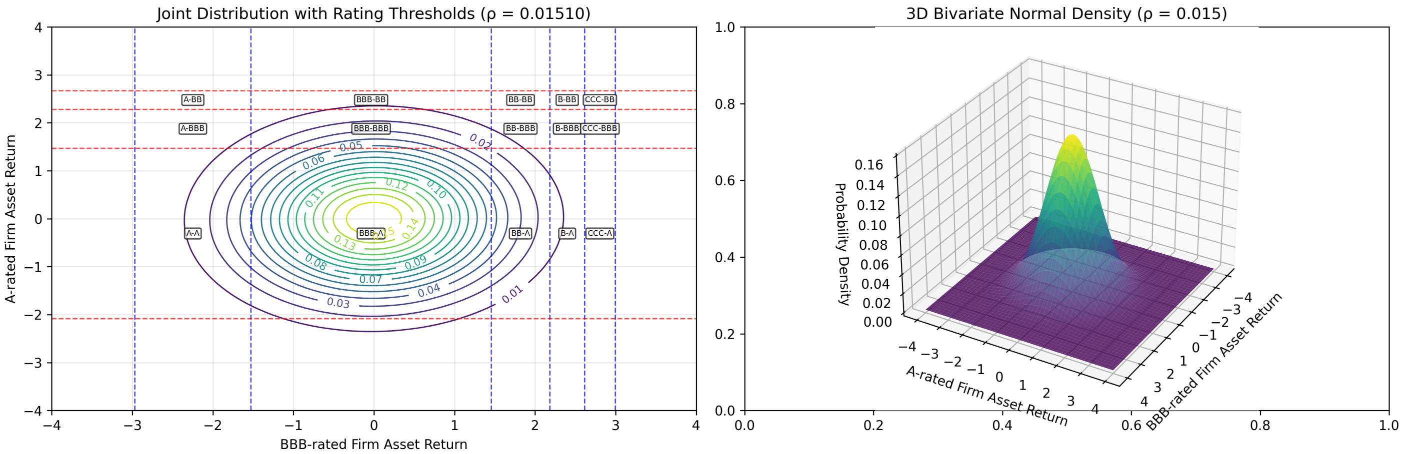

Figure 1: Theoretical joint distribution plot for $\rho = 0.01510$

1.3. Single-Factor Vasicek Model

The fundamental framework used in modeling the credit quality of a portfolio is the Single-factor Vasicek Model. In this framework the asset return of a firm is driven by a common systematic factor and an idiosyncratic component. The asset returns of different firms are correlated with each other, with correlation ρ:

$$v_i = \sqrt{\rho} y + \sqrt{1-\rho}\, \epsilon_i \tag{4}$$

- v_i: Standardized asset return of firm i
- y: Systematic market factor (common to all firms), assumed $\mathcal{N}(0,1)$
- ϵ_i: Idiosyncratic factor specific to firm i, assumed $\mathcal{N}(0,1)$
- ρ: Asset correlation between firms
- $v_i \sim \mathcal{N}(0,1)$ due to the standard normality of y and ϵ_i

1.4. Transition Matrix

The transition matrix shows the probabilities of credit migrations over a specific time horizon (yearly in our case). It has the following properties:

- It includes 8 rating categories, including the default scenario.
- Each row sums to 1, representing all possible outcomes for a given initial rating.
- The diagonal elements are typically the largest, indicating that ratings tend to be stable over time.
- Default is an absorbing state (represented by the last column), meaning once a firm defaults, it remains in default.
- The matrix has Markovian properties, allowing multi-period transition probabilities to be calculated by raising the one-period transition matrix to the corresponding power.

1.5. Portfolio Credit Value at Risk

The loss distribution conditional on the systematic factor y is given by:

$$L(y) = \sum_{i=1}^{N} \mathrm{LGD}_i \cdot \mathrm{EAD}_i \cdot \mathbf{1}_{\{v_i \le Z_{\mathrm{Default}}\}} \tag{5}$$

Where:

- LGD_i is the loss given default for exposure i, equal to $1 - \text{Recovery Rate}$
- EAD_i is the exposure at default for exposure i
- $\mathbf{1}_{\{v_i \le Z_{\mathrm{Default}}\}}$ is an indicator function that equals 1 if firm i defaults, and 0 otherwise

The conditional default probability given y is:

$$p_i(y) = \Phi\left(\frac{Z_{\mathrm{Default}} - \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \tag{6}$$

Where:

- $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution
- ρ is the asset correlation
- Z_{Default} is the default threshold

The Value-at-Risk (VaR) at confidence level α can be calculated by finding the loss threshold that is exceeded with probability $1-\alpha$.

2. Correlation Calibration Methodology

2.1. Calibration Framework

We estimate ρ by minimizing the difference between theoretical joint migration probabilities (derived from the Merton model in equation (3)) and empirically observed joint migration probabilities from historical rating data. The calibration process involves solving the following optimization problem:

$$\hat{\rho} = \arg\min_{\rho \in [0,1]} \mathcal{L}(P^{theoretical}(\rho), P^{observed}) \tag{7}$$

where $\mathcal{L}(\cdot)$ is the loss function measuring the distance between the theoretical and observed probabilities.

2.2. Loss Function Specifications

We implement multiple loss functions and compare their performance:

1. Mean Squared Error (MSE):

$$\mathcal{L}_{MSE} = \sum_{i=1}^{8} \sum_{j=1}^{8} (P_{ij}^{theoretical}(\rho) - P_{ij}^{observed})^2 \tag{8}$$

2. Mean Absolute Error (MAE):

$$\mathcal{L}_{MAE} = \sum_{i=1}^{8} \sum_{j=1}^{8} |P_{ij}^{theoretical}(\rho) - P_{ij}^{observed}| \tag{9}$$

3. Maximum Likelihood:

$$\mathcal{L}_{ML} = - \sum_{i,j:P_{ij}^{observed}>0} P_{ij}^{observed} \log(P_{ij}^{theoretical}(\rho) + \epsilon) \tag{10}$$

where $\epsilon = 10^{-10}$ is added for numerical stability.

4. Kullback-Leibler Divergence: The KL divergence measures the information lost when using the theoretical distribution to approximate the observed distribution:

$$\mathcal{L}_{KL} = \sum_{i,j:P_{ij}^{observed}>0} P_{ij}^{observed} \log\left(\frac{P_{ij}^{observed}}{P_{ij}^{theoretical}(\rho) + \epsilon}\right) \tag{11}$$

5. Jensen-Shannon Divergence: The JSD provides a symmetric and bounded measure of divergence between probability distributions:

$$\mathcal{L}_{JSD} = \frac{1}{2} D_{KL}(P||M) + \frac{1}{2} D_{KL}(Q||M) \tag{12}$$

where $M = \frac{1}{2}(P^{observed} + P^{theoretical})$ is the average distribution, and D_{KL} denotes the Kullback-Leibler divergence.

6. Weighted Loss Functions: To account for the varying importance of different migration scenarios, we implement weighted versions of MSE and MAE:

Weighted MSE:

$$\mathcal{L}_{Weighted\ MSE} = \sum_{i=1}^{8} \sum_{j=1}^{8} w_{ij} \cdot (P_{ij}^{theoretical}(\rho) - P_{ij}^{observed})^2 \tag{13}$$

Weighted MAE:

$$\mathcal{L}_{Weighted\ MAE} = \sum_{i=1}^{8}\sum_{j=1}^{8} w_{ij} \cdot |P_{ij}^{theoretical}(\rho) - P_{ij}^{observed}| \tag{14}$$

The following shows the plot of all loss functions for correlation between 0 and 1 Note that after the optimum point, all the correlations are increasing in the range of the correlation.

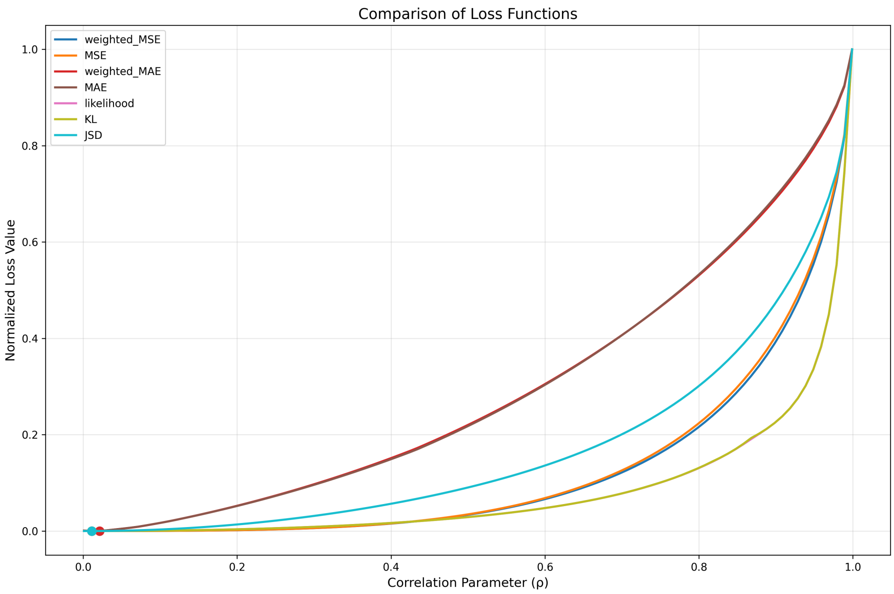

Figure 2: Loss function plot

2.3. Optimization Algorithm

The calibration employs a hybrid optimization approach combining grid search with local optimization:

1. **Grid Search Phase:**
   - Initialize a coarse grid of ρ values: $\rho \in \{0.01, 0.02, \ldots, 0.99\}$ with 20 equally spaced points
   - Evaluate the loss function at each grid point
   - Identify the grid point with minimum loss: $\rho^{grid} = \arg\min_{\rho \in grid} \mathcal{L}(\rho)$
2. **Local Optimization Phase:**
   - Use ρ^{grid} as the starting point for constrained optimization
   - Apply `fmincon` with bounds $\rho \in [0, 1]$ to find the solution
   - Final calibrated parameter: $\hat{\rho} = \arg\min_{\rho \in [0,1]} \mathcal{L}(\rho)$ subject to $\rho_0 = \rho^{grid}$
3. **Quality Check:**
   - Compare the local optimization result with the grid search result
   - If local optimization fails to improve upon the grid search, retain the grid search result
   - This ensures robustness against potential issues in local optimization

The calibration methodology addresses several key challenges that ensures a robust estimation of asset correlation:

1. **Starting Point Sensitivity:** The grid search method tests many different starting values automatically, so our results do not depend on where we begin the search.
2. **Loss Function Comparison:** We use several different ways to measure how well our model fits the data. This helps us to check whether our results are reliable and find the best approach for our specific data.

2.4. Basel Correlation

The Basel II framework introduces a specific approach to correlation in credit risk modeling through the Internal Ratings-Based (IRB) methodology. Unlike constant correlation assumptions, Basel II prescribes a functional relationship between correlation and the probability of default (PD).

Basel II Correlation Function

For corporate exposures, the asset correlation function is defined as:

$$\rho(PD) = 0.12 \times \frac{1 - e^{-50 \times PD}}{1 - e^{-50}} + 0.24 \times \left(1 - \frac{1 - e^{-50 \times PD}}{1 - e^{-50}}\right) \tag{15}$$

This formulation exhibits key features: correlation decreases as PD increases, ranging from approximately 24% for very low-risk obligors to 12% for high-risk obligors. For firms with annual sales between €5 million and €50 million, a size adjustment can be applied:

$$\rho_{adjusted} = \rho(PD) - 0.04 \times \left(1 - \frac{S - 5}{45}\right) \tag{16}$$

where S is measured in millions of euros.

The Basel correlation function reflects empirical observations: flight to quality during downturns increases correlations among high-quality credits, while lower-quality credits exhibit higher idiosyncratic risk. The function derives from Gordy's (2003) single risk factor model, providing the theoretical foundation for the IRB approach.

3. Model Implementation

This section describes the methodology for implementing the Single-Factor Vasicek model to calculate portfolio credit VaR using Monte Carlo simulation. The implementation captures the default and migration behavior of bonds in the portfolio through correlated asset returns. Setting up the key model parameters we have:

- Asset correlation ρ: We set the asset correlation once with the calibrated model and another time from the one year PD associated to each rating class according to Basel II.
- Recovery rate π
- Monte Carlo simulations: 1,000,000 scenarios for convergence
- Confidence level: 99.9% for VaR calculation

The portfolio structure consists of 100 zero-coupon bonds with 2-year original maturity:

- 50 bonds issued by A-rated firms (Face Value = \$1 each)
- 50 bonds issued by BBB-rated firms (Face Value = \$1 each)

For each initial rating class, we determine the credit quality thresholds by applying the inverse cumulative distribution function (CDF) of credit migration probabilities derived from the transition matrix:

$$Z_{j,k} = \Phi^{-1}\left(\sum_{h=k}^{8} \pi_{j,h}\right) \tag{17}$$

where $(\pi_{j,h})$ is the transition probability from rating (j) to rating (h) for a specified rating.

Portfolio Mark-to-Market Formula

For A-rated bonds:

$$V_A = \sum_{j=1}^{7} P_{A,j}^{1y} \times B(0,1) \times \left[(1 - PD_j^{1y}) \times B(0;1,2) + PD_j^{1y} \times B(0;1,1.5) \times \pi\right] + P_{A,\text{default}}^{1y} \times B(0,0.5) \times \pi \tag{18}$$

For BBB-rated bonds:

$$V_{BBB} = \sum_{j=1}^{7} P_{BBB,j}^{1y} \times B(0,1) \times \left[(1 - PD_j^{1y}) \times B(0;1,2) + PD_j^{1y} \times B(0;1,1.5) \times \pi\right] + P_{BBB,\text{default}}^{1y} \times B(0,0.5) \times \pi$$

Portfolio Total Value:

$$\text{Portfolio MTM} = 50 \times V_A + 50 \times V_{BBB} \tag{19}$$

- $P_{i,j}^{1y}$ = 1-year transition probability from rating i to rating j
- PD_j^{1y} = 1-year default probability for rating class j
- $B(0,t)$ = discount factor from time 0 to time t
- $B(0;t_1,t_2) = \frac{B(0,t_2)}{B(0,t_1)}$ = forward discount factor from t_1 to t_2
- π = recovery rate (40%)

3.1. Correlated Asset Return Generation for Bond Valuation

The the methodology involves generating correlated asset returns following the Single-Factor Vasicek structure. For each simulation (s) and each bond (i):

$$v_{i,s} = \sqrt{\rho} \cdot y_s + \sqrt{1-\rho} \cdot \epsilon_{i,s} \tag{20}$$

The implementation steps are:

1. Generate a single systematic factor $y_s \sim \mathcal{N}(0,1)$ for each simulation.

2. Generate independent idiosyncratic factors $\epsilon_{i,s} \sim \mathcal{N}(0,1)$ for each bond.
3. Combine these factors according to the correlation structure.
4. Ensure the same systematic factor affects all bonds to create the desired correlation.
5. **Rating Assignment**: Compare each bond's asset return against the thresholds to determine its new rating category
6. **Forward Price Calculation**: Calculate forward bond prices for each rating outcome using:

$$\text{Forward Price}_r = \begin{cases} 1 \times \text{B(1,2)} & \text{if no default} \\ \pi \times \text{B(1,3/2)} & \text{if default} \end{cases} \tag{21}$$

7. **Portfolio Aggregation**: Sum individual bond values to obtain total portfolio value for each simulation
8. **Loss Distribution Construction**:

$$\text{Losses}_A = \frac{\text{Bond\_MTM}^A}{B(0,1)} \times \text{Price}_A^{1y} \tag{22}$$

$$\text{Losses}_{BBB} = \frac{\text{Bond\_MTM}^{BBB}}{B(0,1)} \times \text{Price}_{BBB}^{1y} \tag{23}$$

$$\text{Losses}_{\text{total}} = \left(\text{Losses}_A \times N_{\text{issuers}}^A\right) + \left(\text{Losses}_{\text{BBB}} \times N_{\text{issuers}}^{\text{BBB}}\right) \tag{24}$$

Sorting these portfolio losses, we can find the loss value corresponding to the 99.9th percentile that represents our portfolio Value at Risk. This approach captures the full distributional characteristics of credit losses, including fat tails and skewness that are typical in credit portfolios.

3.2. Key Remarks

The implementation methodology accounts for several important aspects:

- **Correlation Structure**: The asset correlation ρ calibrated from quarterly joint rating migrations is assumed to remain constant over the one-year horizon used in the Vasicek model simulations.
- **Migration vs. Default**: The model captures both rating migrations and defaults, providing a comprehensive view of credit risk
- **Time Horizon**: The 1-year horizon aligns with standard credit risk measurement practices and regulatory requirements
- **Optimization starting point**: The plots of the loss functions in correlation calibration show that the loss increases after the minimum point for all cases. Since the minimizer in MATLAB or Python returns a local minimum within a given search range, starting the optimization after the true minimum typically leads it to converge to a point on the left side of the search interval.

4. Interpretation of the Results

4.1. Correlation Calibration

The calibrated correlation from the joint transition matrix reflects the observed co-movement in credit quality changes between firms rated A and BBB. We estimate this correlation, denoted by ρ, using various loss functions and a grid of 20 evenly spaced starting points:

Table 1: Correlation result with different methods

Method	Calibrated Rho
MSE	0.01383
MAE	0.01771
likelihood	0.00707
KL	0.00707
JSD	0.00717
weighted MAE	0.02023
weighted MSE	0.01510

We select the correlation obtained using the *weighted MSE* loss function. In this method, the weights $w_{ij} = i + j$ give more importance to larger rating changes. This ensures that extreme upgrades (to higher ratings) or downgrades (to lower ratings), which have a greater impact on Value-at-Risk (VaR) despite being less frequent, are emphasized during optimization.

This raises two important questions:

- **1.** Does the low estimated correlation imply that rating migrations and default events for A and BBB-rated firms are mostly independent?
- **2.** Is there a statistically significant difference between the correlation from the weighted MSE and those from other loss functions?

To answer these questions, we implemented statistical bootstrapping by repeatedly resampling from the S&P joint probability matrix. For each resampled matrix, we perform the calibration process again—repeating this at least 200 times. This generates a distribution of the calibrated correlation ρ for each method, allowing us to construct confidence intervals and assess the variability of the estimates. (See Appendix B for a detailed discussion and results.)

- To answer Question 1, we test the null hypothesis that there is no correlation between the rating migrations:

$$H_0 : \rho = 0 \quad \text{(no correlation between rating migrations)} \tag{25}$$

$$H_1 : \rho \neq 0 \quad \text{(significant correlation existss)} \tag{26}$$

  In all cases, the null hypothesis H_0 is rejected, indicating that the estimated correlations are statistically significant—even if their values are small.

- For Question 2, we compare the correlation estimates from different optimization methods using a two-sample bootstrap hypothesis test:

$$H_0 : \mu_{\text{ref}} = \mu_{\text{alt}} \quad \text{(no difference in correlation estimates)} \tag{27}$$

$$H_1 : \mu_{\text{ref}} \neq \mu_{\text{alt}} \quad \text{(significant difference exists)} \tag{28}$$

  Here, $\mu_{\text{ref}} = E[\hat{\rho}_{\text{ref}}]$ and $\mu_{\text{alt}} = E[\hat{\rho}_{\text{alt}}]$ represent the expected values of the correlation estimates from two different methods. (See Appendix B for more details.)

The results show **no significant differences** in the estimated correlations ($p \geq 0.05$). This suggests that the methods yield consistent and reliable estimates. We can therefore trust the results from our chosen approach.

Empirical vs. Regulatory Correlation

In contrast, the Basel IRB framework uses fixed asset correlation values: 0.27 for A-rated firms and 0.23 for BBB-rated firms. These represent theoretical correlations of firms' asset values, based on the assumption that companies tend to move together in terms of fundamentals more than their credit ratings would suggest.

As noted by Frye (2008), empirical correlations—such as those derived from default or transition data—are typically lower than regulatory values. This gap lies in model simplifications, unobserved liability dynamics, and conditional effects. However, these differences have a major impact on portfolio VaR and capital requirements.

Comment on Calibration Results Across Loss Functions

The calibrated values of the asset correlation parameter ρ are consistently low, ranging from approximately 0.0071 (Likelihood, KL) to 0.0202 (Weighted MAE), suggesting weak average asset correlation implied by the 3-month joint migration data for A and BBB rated firms.
Methods like Likelihood, KL and JSD yield the lowest and most stable estimates ($\rho \approx 0.0071$), with narrower confidence intervals. In contrast, error-based losses (MSE, MAE) produce moderately higher values ($\rho \approx 0.0138$–0.0177), with MSE showing greater uncertainty. Weighted loss functions, which emphasize adverse credit events, yield the highest ρ values, with Weighted MSE at $\rho \approx 0.0151$ and Weighted MAE at $\rho \approx 0.0202$.
Conclusion: Weighted MSE is retained as the preferred criterion due to its focus on adverse joint events, aligning with credit risk modeling objectives. The low estimated ρ values indicate a modest level of systematic risk, which directly impacts subsequent portfolio risk metrics.

4.2. VaR Impact and Risk Management

We calculate the VaR for our portforlio of 50 bonds rated A and 50 bonds rated BBB using the Vasicek model. The following table shows the result using different correlation methodology.

Table 2: VaR Analysis Results for Point 2

Scenario	**A-rated Correlation**	**BBB-rated Correlation**	**VaR (99.9%)**
Point 2.A: Constant Correlation	$\rho = 0.015$	$\rho = 0.015$	\$1.24
Point 2.B: Basel II Rating-Dependent	$\rho_A = 0.2359$	$\rho_{BBB} = 0.2275$	\$3.45

In point 2.a, using the calibrated correlation in the single-factor Vasicek model, we obtain a 99.9% one-year VaR of \$1.24. In contrast, in point 2.b, employing the Basel II correlations, the VaR increases to \$3.45.

Correlation Level Impact: The Basel II correlations ($\rho_A = 0.2359$, $\rho_{BBB} = 0.2275$) are significantly higher than the empirically calibrated constant correlation ($\rho = 0.015$). This demonstrates that regulatory correlations, which are conservative by design, lead to substantially higher capital requirements.

Magnitude of Difference: The Basel II rating-dependent correlation approach yields a VaR that is approximately 2.77 times higher (\$3.45 vs. \$1.24) than the constant correlation method. This substantial difference of \$2.21 represents a 177% increase in estimated risk. This shows that IRB curves include a considerable safety margin: they treat asset correlation as a proxy for default correlation and increase it with credit risk.

By contrast, our empirical estimate (based on historical joint rating migrations between A and BBB firms) captures the actual default correlation, which is much lower, and therefore results in significantly reduced capital requirements. The following plots show the distribution of the losses under the two correlation assumptions.

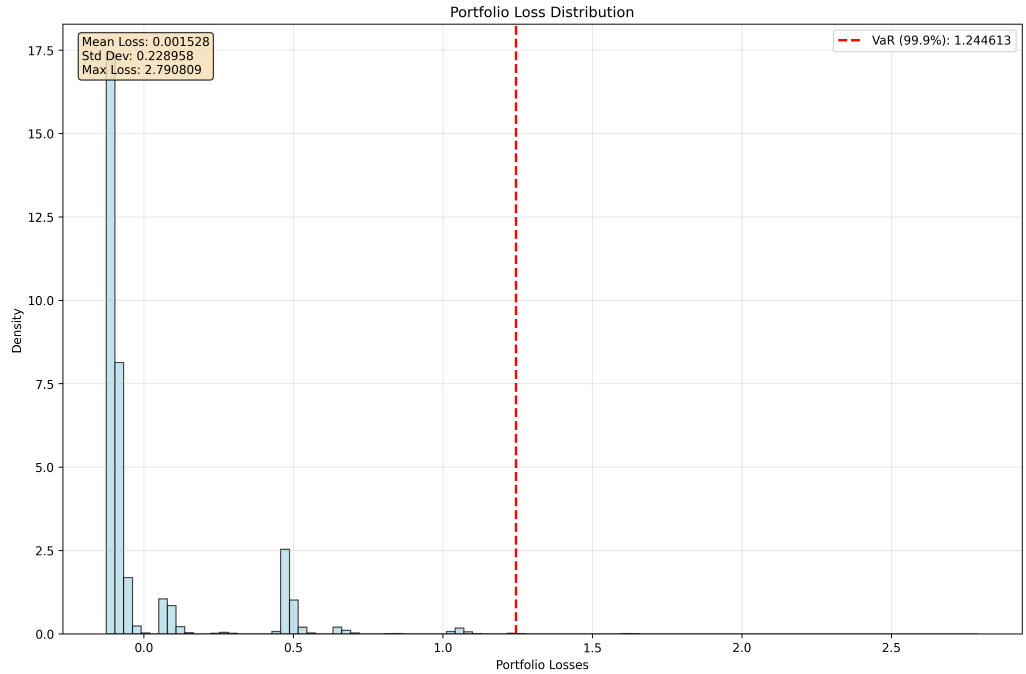

Figure 3: Portfolio Loss Distribution for $\rho = 0.01510$, Monte Carlo simulated loss distributions for the 100-bond portfolio.

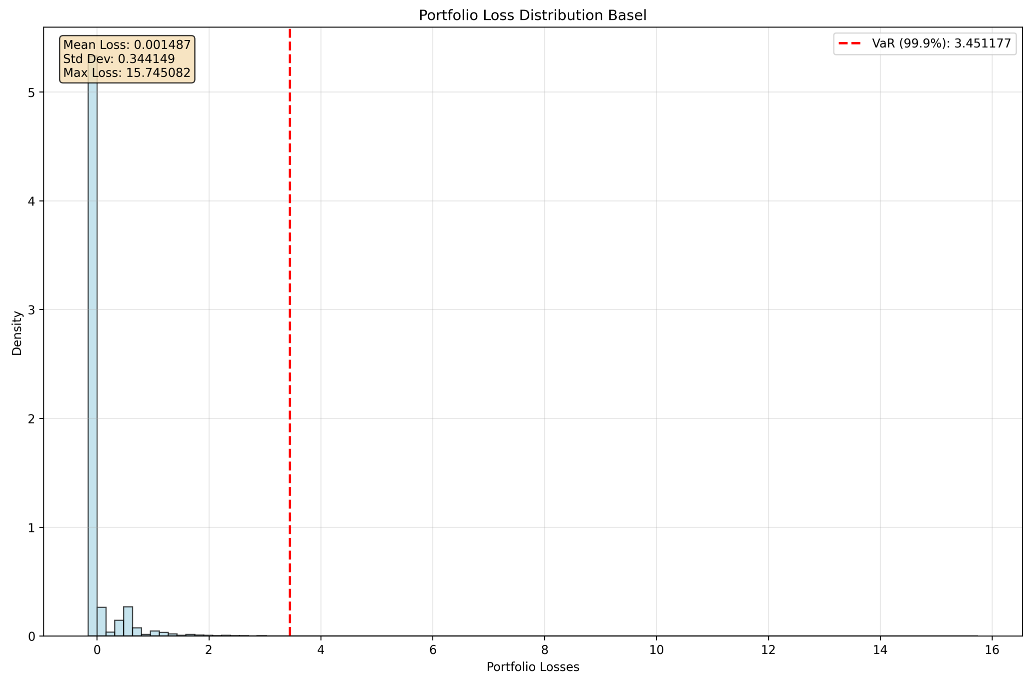

Figure 4: Portfolio Loss Distribution using Basel II correlations

Risk Management Consequences: The large gap between the two methods has major effects on:

- **Capital allocation:** Banks using Basel II correlations would need to hold significantly more capital.
- **Portfolio optimization:** Risk-adjusted returns appear lower under the regulatory approach.
- **Model validation:** Picking a different correlation model can greatly change the accuracy of credit risk evaluations.

These findings are consistent with Frye's evidence that asset correlations from regulatory models tend to overstate actual default correlations observed in historical data, potentially leading to overly conservative risk estimates.

5. Limitations and Future Research

This analysis focused on a simplified two-rating class model using historical S&P data. Future research could extend this work by:

- Analyzing time-varying correlation patterns
- Comparing results across different rating agencies and time periods
- Investigating the impact of economic cycles on correlation estimates

6. Conclusion

In this project we calibrated asset correlation parameters using joint rating migration data and evaluated their impact on credit portfolio VaR calculations. The analysis provides several important insights for credit risk management and regulatory capital assessment.

Key Findings

The correlation parameters were estimated using S&P's joint rating migration data, resulting in a constant value of $\rho = 0.015$. This low correlation suggests that defaults and rating changes among firms are related, but not as strongly as many theoretical models assume.

The VaR analysis revealed substantial differences between correlation approaches:

- The calibrated constant correlation ($\rho = 0.015$) produced a 99.9% VaR of \$ 1.24
- Basel II rating-dependent correlations yielded a significantly higher VaR of \$ 3.45
- This represents a 177% difference, highlighting the critical impact of correlation assumptions on risk estimates

Practical Consequences According to Frye, the difference in VaR arises from structural changes in the relationship between asset values and defaults, primarily due to the increase in firm-specific (idiosyncratic) risk as firms near default. The large gap between our calibrated correlation and the Basel II correlations show a few important points.

First, Basel II correlations are conservative by design, leading to higher capital requirements. Second, the way correlations are calculated has a great impact on how portfolio risk is measured. Therefore, if real-world correlations are lower than what Basel II assumes, banks might be holding more capital than needed. The project demonstrates that while the single-factor Vasicek model provides a robust framework for credit portfolio analysis, the calibration of correlation parameters remains crucial for accurate risk assessment and effective capital management.

A. Raw Data

Table 3: Joint Probability Matrix for firms rated A and BBB

AAA	AA	A	BBB	BB	B	CCC	Default
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00002	0.00140	0.00007	0.00001	0.00000	0.00000	0.00000
0.00000	0.00124	0.05638	0.00356	0.00052	0.00028	0.00000	0.00000
0.00000	0.01575	0.78700	0.05139	0.00643	0.00317	0.00000	0.00000
0.00000	0.00106	0.05288	0.00370	0.00041	0.00024	0.00000	0.00000
0.00000	0.00022	0.00897	0.00067	0.00010	0.00006	0.00000	0.00000
0.00000	0.00007	0.00282	0.00016	0.00002	0.00002	0.00000	0.00000
0.00000	0.00004	0.00124	0.00008	0.00001	0.00000	0.00000	0.00000

Table 4: Seven rating classes (i.e. AAA, AA, A, BBB, BB, B, CCC) prior to default (Yearly)

AAA	AA	A	BBB	BB	B	CCC	Default
0.90050	0.09180	0.00550	0.00050	0.00080	0.00030	0.00050	0.00000
0.00570	0.90060	0.08610	0.00580	0.00060	0.00070	0.00020	0.00020
0.00030	0.01960	0.91650	0.05750	0.00370	0.00150	0.00020	0.00070
0.00010	0.00130	0.03830	0.90960	0.04080	0.00630	0.00140	0.00220
0.00020	0.00040	0.00170	0.05760	0.84500	0.07850	0.00760	0.00890
0.00000	0.00030	0.00120	0.00250	0.06210	0.83680	0.05050	0.04650
0.00000	0.00000	0.00180	0.00270	0.00810	0.15830	0.51400	0.31520

Table 5: Discount Factors (Semi-Annual)

Date	Discount Factor
2023-08-02 00:00:00	0.98559
2024-02-02 00:00:00	0.96830
2024-08-02 00:00:00	0.95260
2025-02-03 00:00:00	0.93889

B. Statistical Methodology for Correlation Analysis

B.1. Bootstrap Validation Framework

To assess the statistical significance of our calibrated correlation parameters, we implement a bootstrap methodology that addresses two key questions:

1. Are the low correlation values statistically significant or due to random variation?
2. Do different optimization methods yield statistically equivalent results?

B.2. Bootstrap Procedure

Sample Generation: We generate bootstrap samples using multinomial resampling from the original joint probability matrix $\mathbf{P}$:

$$\mathbf{C}^{(b)} \sim \text{Multinomial}(N = 789{,}683, \text{vec}(\mathbf{P})) \tag{29}$$

where $\mathbf{C}^{(b)}$ represents the b-th bootstrap count vector, reshaped back into probability matrices.

Correlation Re-estimation: For each bootstrap sample $\mathbf{P}^{(b)}$, we re-calibrate the correlation parameter $\hat{\rho}^{(b)}$ using the same optimization procedure as the original analysis.
After $B = 200$ bootstrap iterations, we obtain the empirical distribution $\{\hat{\rho}^{(1)}, \hat{\rho}^{(2)}, \ldots, \hat{\rho}^{(B)}\}$.

B.3. Hypothesis Testing

Test 1 - Significance of Correlation:

$$H_0 : \rho = 0 \quad \text{(no correlation between rating migrations)} \tag{30}$$
$$H_1 : \rho \neq 0 \quad \text{(significant correlation exists)} \tag{31}$$

The bootstrap p-value is calculated as:

$$p\text{-value} = 2 \times \min\left\{\frac{1}{B}\sum_{b=1}^{B}\mathbf{1}(\hat{\rho}^{(b)} \leq 0), \frac{1}{B}\sum_{b=1}^{B}\mathbf{1}(\hat{\rho}^{(b)} \geq 0)\right\} \tag{32}$$

Test 2 - Method Comparison: To compare different optimization methods, we test whether the correlation estimates from weighted MSE (reference method) differ significantly from alternative methods:

$$H_0 : \mu_{\text{ref}} = \mu_{\text{alt}} \quad \text{(methods yield equivalent estimates)} \tag{33}$$
$$H_1 : \mu_{\text{ref}} \neq \mu_{\text{alt}} \quad \text{(methods yield different estimates)} \tag{34}$$

Using paired differences $D^{(b)} = \hat{\rho}^{(b)}_{\text{alt}} - \hat{\rho}^{(b)}_{\text{ref}}$, the test statistic is:

$$t = \frac{\bar{D}}{SE(D)} \sim t_{2B-2} \tag{35}$$

B.4. Results Summary

Confidence Intervals: Bootstrap 95% confidence intervals are constructed using the percentile method:

$$CI_{0.95} = \left[Q_{0.025}(\hat{\rho}^{(b)}), Q_{0.975}(\hat{\rho}^{(b)})\right] \tag{36}$$

Key Findings:

- All calibrated correlations are statistically significant (confidence intervals exclude zero)
- Different optimization methods yield statistically equivalent results ($p \geq 0.05$)
- This validates the robustness of our correlation estimates across different loss functions

The statistical framework confirms that the relatively low correlation values (around 1.5%) are genuine empirical findings rather than statistical noise, supporting the conclusion that actual default correlations are substantially lower than regulatory asset correlation assumptions.

Table 6: Correlation result full table

Method	Calibrated ρ	Bootstrap Mean	Bootstrap SE	CI Lower	CI Upper	Loss Value
weighted MSE	0.01510	0.01066	0.00286	0.00697	0.01712	0.00000
MSE	0.01383	0.01001	0.00000	0.01001	0.01001	0.00000
weighted MAE	0.02023	0.01040	0.00255	0.00470	0.01903	0.01432
MAE	0.01771	0.01035	0.00192	0.00846	0.01746	0.00185
likelihood	0.00707	0.00863	0.00215	0.00398	0.01109	0.93196
KL	0.00707	0.00987	0.00075	0.00765	0.01001	0.00007
JSD	0.00717	0.00987	0.00085	0.00832	0.01001	0.00002

References

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