Capacitated Vehicle Routing Problem (CVRP)

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(I) Discussion and Problem Statement

Delivery companies every day need to deliver packages to many different clients. The deliveries are accomplished using an available fleet of vehicles from a central warehouse. The goal of is to design a route for each vehicle so that all customers are served, the total traveled cost by all the vehicles are minimized. In addition, the capacity of each vehicle should not be exceeded.

The intuition behind Our proposed solution

Our solution aims to serve the maximum number possible of customers with each vehicle at the minimum cost, considering the maximum capacity of each vehicle. The intuition behind our approach is as follows: We have a set of customers C, where each customer c_i has a demand d_i. The goal is to meet these demands using the available vehicles. Since each vehicle can serve multiple customers in a single route before returning to the depot, the sum of the demands of the selected customers should not exceed the vehicle's maximum capacity Q.

To achieve this, we first identify all possible subsets of customers whose total demand does not exceed the vehicle's capacity. For each feasible subset, we calculate the cost of the route required to serve these customers. Among these routes, we select the one with the minimum cost. This process assigns the optimal route for the first vehicle. The same procedure is then repeated for subsequent vehicles in an iterative way, ensuring an efficient and cost-effective distribution of services.

(II) General Mathematical Formulation

The Capacitated Vehicle Routing Problem (CVRP) involves determining the optimal set of routes for a fleet of vehicles to deliver goods to a given set of customers. The problem can be formulated as follows:

Sets and Indices

- V: Set of nodes, including the depot (node 0) and customers $1, 2, \ldots, n$.
- K: Set of vehicles.
- C: Set of customers.
- A: Set of arcs (i, j) where $\{i, j \in N$ and $i \neq j\}$.

Parameters

- c_{ij}: Cost of traveling from node i to node j.
- d_i: Demand of customer i.
- Q: Capacity of each vehicle.

Decision Variables

- x_{ijk}: Binary variable that is 1 if vehicle k travels directly from node i to node j, and 0 otherwise.
- q_ik: Auxiliary variable representing the load of the vehicle k after visiting node i.

Objective Function

Minimize the total cost of the routes for all the vehicles:

$$\min \sum_{k \in K} \sum_{(i,j) \in A} c_{ij} x_{ijk}$$

Constraints

(1) Each customer is visited exactly once:

$$\sum_{k\in K}\sum_{j\in V, j\neq i} x_{ijk} = 1, \quad \forall i \in C, i \neq 0 \tag{1}$$

(2) Vehicle capacity constraints:

$$\sum_{i\in C} d_i y_{ijk} \leq q, \quad \forall k \in K \tag{2}$$

(3) Depot departure and return constraints:

$$\sum_{k\in K}\sum_{j\in V} x_{0jk} = 1 \tag{3}$$

$$\sum_{k\in K}\sum_{i\in V} x_{i0k} = 1 \tag{4}$$

(4) Binary and non negativity constraints:

$$x_{ijk} \in \{0,1\} \quad \forall (i,j) \in A, \forall k \in K \tag{5}$$

$$q_{ik} \geq 0 \quad \forall i \in C, \forall k \in K \tag{6}$$

$$q_{ik} \leq Q, \quad \forall i \in C, \forall k \in K \tag{7}$$

(III)Complexity of the Capacitated Vehicle Routing Problem (CVRP)

The Capacitated Vehicle Routing Problem (CVRP) is a well-known combinatorial optimization problem that is recognized as NP-hard. This classification of complexity has significant implications for the study and solution of the problem. Here's a detailed look at the complexity of the CVRP:

NP-Hardness

- **Definition of NP-Hard:**
  - A problem is NP-hard if every problem in NP can be reduced to it in polynomial time. This means that if there were a polynomial-time algorithm to solve an NP-hard problem, then every problem in NP could also be solved in polynomial time.
  - The CVRP is a generalization of the Traveling Salesman Problem (TSP), which is also NP-hard. Since the TSP can be reduced to the CVRP by setting vehicle capacities high enough to accommodate all demands, the CVRP inherits the NP-hardness of the TSP.

Implications of NP-Hardness

- **No Known Polynomial-Time Solution:**

  - For NP-hard problems, no polynomial-time algorithms are known. As a result, solving CVRP exactly for large instances is computationally infeasible.
  - Exact algorithms (e.g., branch and bound, branch and cut, dynamic programming) can solve CVRP only for relatively small instances.

- **Exponential Growth of Solution Space:**
  - The number of possible routes grows exponentially with the number of customers. This combinatorial explosion makes it impractical to enumerate and evaluate all possible solutions for large instances.
  - Specifically, the number of possible solutions is factorial in the number of customers (e.g., for n customers, there are $n!$ possible ways to visit them).

- **Approximation and Heuristics:**
  - Given the intractability of finding exact solutions for large instances, researchers have developed numerous heuristic and metaheuristic methods to find good approximate solutions within reasonable time frames.
  - Metaheuristics such as Genetic Algorithms, Simulated Annealing, Tabu Search, and Ant Colony Optimization are also widely used to tackle large instances of CVRP.

- **Complexity of Subproblems:**
  - Many subproblems and variations of CVRP (e.g., Vehicle Routing Problem with Time Windows, Multi-Depot VRP) are also NP-hard, adding further complexity to their solution approaches.

Intractability in Practice

- **Large-Scale Instances:**
  - For real-world applications with hundreds or thousands of customers, exact algorithms become impractical. Heuristics and metaheuristics are essential for providing feasible solutions in a reasonable time.
  - Large-scale instances often require parallel computing and advanced optimization techniques to handle the computational burden.

- **Quality of Solutions:**
  - While heuristics and metaheuristics do not guarantee optimal solutions, they can often produce high-quality solutions that are close to optimal.
  - The performance of these algorithms is typically evaluated based on the quality of the solution and the computational time required.

(VI) Graph Representation

The CVRP can be represented as a directed graph where nodes represent the depot and customers, and edges represent the possible routes between them with associated costs.

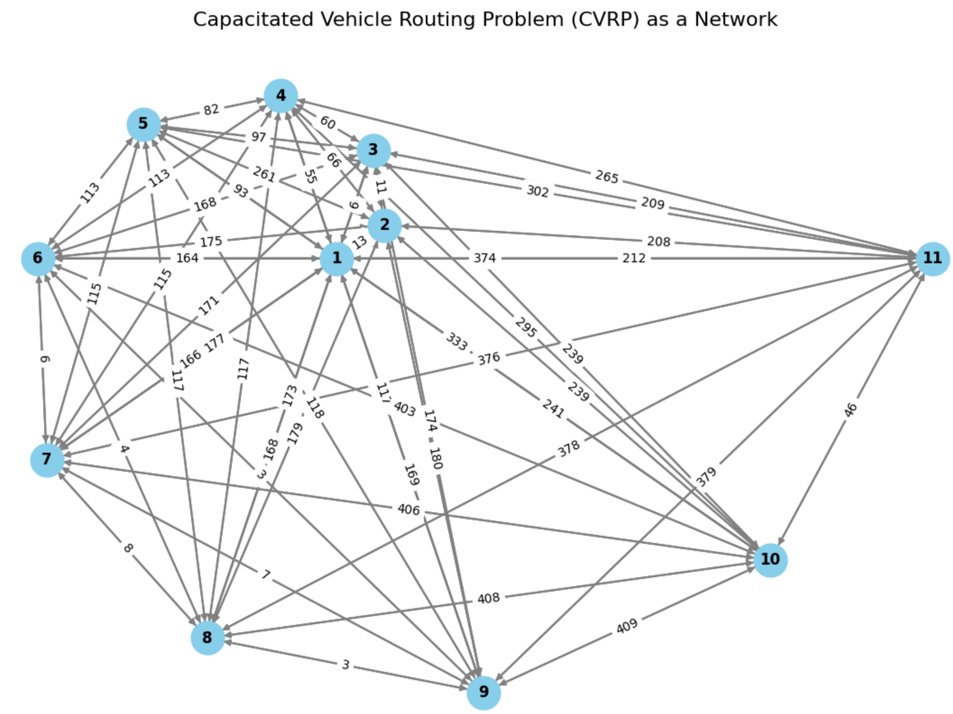

Figure 1: Graph Representation of CVRP