Numerical Matrix Calculation PROJECT

Modelling the evolution of predators and prey taking into account pollution and ecological factors

Group Members:
- Amira BOUDAOUD
- Ghofrane BEN RHAIEM

Supervisor:
CHAHINE Elie

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1 Introduction

The Lotka-Volterra equations are widely used to model the dynamics of biological systems, specifically to capture the interactions between predators and prey. These interactions are represented by a pair of first-order nonlinear differential equations, independently introduced by Alfred Lotka in 1925 and Vito Volterra in 1926. Volterra initially developed the model to explain observed fluctuations in fish populations in the Adriatic Sea by classifying species into prey and predators and accounting for factors like birth rates, mortality rates, and initial population sizes. To simplify, the model assumes that prey have unlimited food sources and no competition, resulting in a fundamental equation that describes the relationship between the predator and prey populations under these conditions which is the following:

$$\begin{cases} x^{'}(t) = x(t)(a - by(t)) \\ y^{'}(t) = y(t)(dx(t) - c) \\ x(0) = x_0, y(0) = y_0 \end{cases} \tag{S1}$$

which can be writting as follow:

$$\begin{cases} x^{'}(t) = ax(t) - bx(t)y(t)) \\ y^{'}(t) = dy(t)x(t) - cy(t) \\ x(0) = x_0, y(0) = y_0 \end{cases} \tag{S1}$$

where :

- t is the time,
- $x(t)$ is the number of prey as a function of time,
- $y(t)$ is the number of predators as a function of time,
- $a > 0$ characterizes the intrinsic reproduction rate of prey,
- $b > 0$ characterizes the mortality rate of prey due to predators encountered,
- $c > 0$ characterizes the intrinsic mortality rate of predators, regardless of the number of prey,
- $d > 0$ characterizes the reproduction rate of predators according to the prey encountered and eaten.

The positive sign of parameters a, b, c, and d is necessary for the model to have biological significance.

The goal is to apply this model to study the dynamics between lizards (prey) and a bird species (predators) within an ecological system. Initially, the simplified model is examined, followed by a modified version that incorporates the effects of pesticide use on the species' environment. Finally, an additional modification is introduced to account for resource limitations that may arise from deforestation.

2 Part A : Understanding the Simplified Model $(S1)$

1. Explaination of the meaning of the term $-bx(t)y(t)$:

   The term $-bx(t)y(t)$ in the model represents the impact of predators on the prey population, with each component having a specific role. The negative sign indicates a reduction in the prey population, showing how encounters with predators contribute to this decline. In this term, b is the mortality rate of prey due to predator encounters, $x(t)$ represents the number of prey at time t, and $y(t)$ represents the number of predators at the same time. Thus, $-bx(t)y(t)$ reflects the loss in the prey population due to the presence of predators, illustrating that as the number of predators y increases, the rate of decline in the prey population also increases proportionally. In the absence of predators, this term would vanish, allowing for unlimited prey growth without any checks from predator interactions.

2. Explaination of the negative sign in front of the term $cy(t)$:

   The negative sign in front of the term $-cy(t)$ in the second equation represents the natural decrease in the predator population due to intrinsic mortality, independent of prey availability. Here, c is the intrinsic mortality rate of predators, and $y(t)$ is the number of predators at time t. This term indicates that the predator population decreases at a rate c solely due to natural causes, ensuring that even without prey, the predator population will gradually decline over time, emphasizing that predator mortality is an inherent aspect of the system.

3. Prove that the number of prey in this model increases exponentially in the absence of predators and that the number of predators decreases exponentially towards 0 in the absence of prey.

   -To demonstrate that the prey population grows exponentially in the absence of predators, we start by setting $y(t) = 0$ in the first equation of the system $(S1)$. This leads to the following differential equation:

   $$x^{'}(t) = ax(t) \Longrightarrow \frac{dx(t)}{dt} = ax(t)$$

   Integrating both sides, we find the solution for the prey population:

   $$x(t) = C\exp(at)$$

   where C is a constant determined by the initial condition $x_0 = x(0)$.This solution $x(t) = Cexp(at)$ shows that, without predators, the prey population grows exponentially over time at a rate determined by the intrinsic growth rate a. and

   $$\lim_{t\to\infty} x(t) = \infty,$$

   indicating that the prey population will grow without bound in the absence of predators.

   - To prove that the predator population decreases exponentially in the absence of prey, we start by setting $x(t) = 0$ in the second equation of the system $(S1)$. This leads to the following differential equation:

   $$y^{'}(t) = -cy(t) \Longrightarrow \frac{dy(t)}{dt} = -cy(t)$$

Integrating both sides, we find the solution for the predator population:

$$y(t) = D\exp(-ct)$$

where D is a constant determined by the initial condition $y(0)$. This solution $y(t) = D\exp(-ct)$ confirms that in the absence of prey, the predator population declines exponentially toward zero over time, with c representing the natural mortality rate. and

$$\lim_{t\to\infty} y(t) = 0,$$

showing that the predator population will asymptotically approach zero as time goes to infinity in the absence of prey.

3 Part B : Numerical solving the simplified Model $(S1)$

1. Rewrite the model (S1) in the matrix form

   we have :

$$\begin{cases} x^{'}(t) = ax(t) - bx(t)y(t)) \\ y^{'}(t) = dy(t)x(t) - cy(t) \\ x(0) = x_0, y(0) = y_0 \end{cases} \tag{$S1$}$$

   and

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}^{'} = F\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f_1(x(t), y(t)) \\ f_2(x(t), y(t)) \end{pmatrix}$$

   which gives us :

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}^{'} = \begin{pmatrix} ax(t) - bx(t)y(t)) \\ dx(t)y(t) - cy(t) \end{pmatrix}$$

   thus the expressions are :

$$\begin{cases} f_1(x(t), y(t)) = ax(t) - bx(t)y(t)) \\ f_2(x(t), y(t)) = dx(t)y(t) - cy(t) \end{cases} \tag{1}$$

2. Cite the theorem and assumptions about F that allow to assert that the matrix equation obtained admits a unique solution for a given initial condition.

   The existence of a unique solution for the system can be assured by the Cauchy-Lipschitz theorem (also known as the Existence and Uniqueness Theorem for ODEs or Picard-Lindelöf's). This theorem applies to systems of first-order ordinary differential equations under certain conditions:

   Continuity of F: The vector function

$$F(X) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$$

should be continuous in both variables x and y. In this case,

$$f_1(x,y) = x(a-by) \quad \text{and} \quad f_2(x,y) = y(dx-c)$$

are continuous for all $x, y \in \mathbb{N}$.

Lipschitz Condition: There should exist a constant $L > 0$ such that for all X_1, X_2 in a neighborhood of a point, the following inequality holds:

$$\|F(X_1) - F(X_2)\| \leq L\|X_1 - X_2\|$$

This guarantees the uniqueness of the solution. Both f_1 and f_2 are linear in x and y, so they satisfy the Lipschitz condition.

Therefore, given that F is continuous and satisfies the Lipschitz condition, the Cauchy-Lipschitz theorem asserts that the system of equations has a unique solution for a given initial condition

$$X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

3. Calculate in terms of a, b, c, and d the points for which $x^{'}(t)$ and $y^{'}(t)$ vanish, these points are called the equilibrium points.

   To find the equilibrium points, we need to set the derivatives to zero:

$$\begin{cases} x^{'}(t) = ax(t) - bx(t)y(t) = 0 \\ \\ y^{'}(t) = dy(t)x(t) - cy(t) = 0 \end{cases} \tag{E1}$$

   Solving these equations, we have:

   1. From the first equation:

$$x(t)(a - by(t)) = 0$$

   This gives us two cases:

   - Case 1: $x(t) = 0$. - Case 2: $a - by(t) = 0 \Rightarrow y(t) = \frac{a}{b}$.

   2. From the second equation:

$$y(t)(dx(t) - c) = 0$$

   This also leads to two cases:

   - Case 1: $y(t) = 0$. - Case 2: $dx(t) - c = 0 \Rightarrow x(t) = \frac{c}{d}$.

   Thus, the equilibrium points are:

   (a) $(x^*, y^*) = (0, 0)$

(b) $(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right)$

- significance of the equilibrium points :

- **Point** $(0, 0)$**:** This represents a situation where both the prey and predator populations are extinct. It's the trivial equilibrium, indicating that without any populations, there is no interaction.
- **Point** $\left(\frac{c}{d}, \frac{a}{b}\right)$**:** This is a more interesting equilibrium point where both populations coexist. Here, the prey population stabilizes at $\frac{c}{d}$, and the predator population stabilizes at $\frac{a}{b}$. This means the system reaches a balance where both species persist over time, with no further changes in population size at this point.

- for this case : $x(t) = 0$ and $a - by(t) = 0$, i.e., $y(t) = \frac{a}{b}$
Here, the prey population is zero, but the predator population stabilizes at $\frac{a}{b}$:

$$(x, y) = \left(0, \frac{a}{b}\right)$$

Interpretation: This point represents a situation where the prey is extinct, but the predator population remains at a constant value. However, biologically, this equilibrium is not realistic since predators can't survive long-term without prey.

- for this case: $y(t) = 0$ and $dx(t) - c = 0$, i.e., $x(t) = \frac{c}{d}$
Here, the predator population is zero, but the prey population stabilizes at $\frac{c}{d}$:

$$(x, y) = \left(\frac{c}{d}, 0\right)$$

Interpretation: This equilibrium represents a scenario where the predators are extinct, but the prey population remains constant at $\frac{c}{d}$. This situation could be biologically possible since prey can survive without predators, at least temporarily.

Why Didn't We Include These Two Points Initially?

The points $(0, \frac{a}{b})$ and $(\frac{c}{d}, 0)$ were omitted in the earlier discussion because they represent biologically unlikely scenarios:

- In $(0, \frac{a}{b})$, predators would be able to sustain themselves without any prey, which is not realistic biologically.
- In $(\frac{c}{d}, 0)$, the prey population would grow without bound in the absence of predators, though in reality, other limiting factors would restrict their growth.

Moreover, **mathematically**, these points do not qualify as valid equilibrium points in this model, since the parameters a, b, c, and d were defined to be positive constants (> 0). Therefore, neither population can actually reach zero in these specific forms.

4. the Euler explicit scheme that allows to solve system $(S1)$

$$\begin{cases} x_{n+1}(t) = x_n(t) + h.f_1(x_n(t), y_n(t)) \\ y_{n+1}(t) = y_n(t) + h.f_2(x_n(t), y_n(t)) \\ x(0) = x_0, y(0) = y_0 \end{cases} \tag{2}$$

which can be reformulated into :

$$\begin{cases} x_{n+1}(t) = x_n(t) + h.(ax_n(t) - bx_n(t)y_n(t)) \\ y_{n+1}(t) = y_n(t) + h.(dx_n(t)y_n(t) - cy_n(t)) \\ x(0) = x_0, y(0) = y_0 \end{cases} \tag{3}$$

after reformulation we will get the following equation (the t is taken into consideration but for simplification we removed it from the equation)

$$\begin{cases} x_{n+1} = x_n(1 + h.(a - by_n)) \\ y_{n+1} = y_n(1 - h.(c + dx_n)) \\ x(0) = x_0, y(0) = y_0 \end{cases} \tag{4}$$

- Write then a Matlab code to solve system (S1) with the Euler explicit method.

for the MATLAB code and the results, check the file:

`[PartB_question4.m]`

5. Plotting the Dynamics of Prey and Predator:

To visualize the behavior of the predator and prey populations over time, we will plot graphs for different parameter values.

- **Case 1:** Initial time $t_0 = 0$, final time $T = 8$, subdivisions number $N = 10000$, $x_0 = 4$, $y_0 = 10$, $a = 3$, $b = 0.5$, $c = 4$, $d = 0.5$.

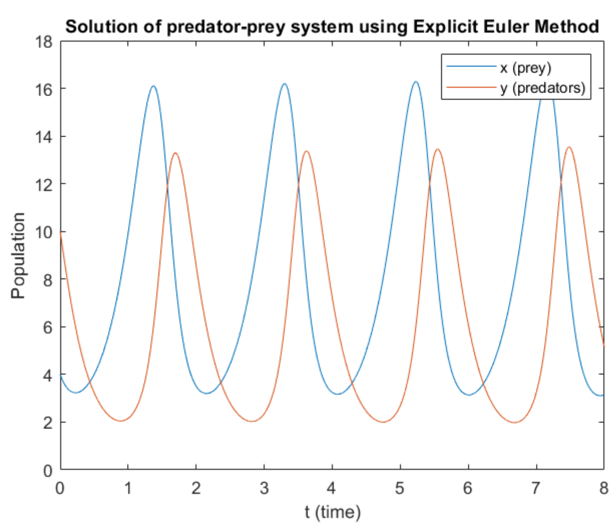


Figure 1: screenshot of the results.

Interpretation of the Results:

- **Initial Conditions:**
  $x_0 = 4$, $y_0 = 10$
- **Parameters:**
  $a = 3$, $b = 0.5$, $c = 4$, $d = 0.5$.

- **Parameter Analysis:**

  The prey has a strong intrinsic growth rate $a = 3$, while the predator has a significant mortality rate $c = 4$. The predation rate $b = 0.5$ means the predators can impact the prey population, but not excessively. The predator reproduction rate $d = 0.5$ indicates they reproduce moderately in the presence of prey.

- **Interpretation:**

  Initially, the prey population grows, but as predators increase due to the availability of prey, they begin consuming more of the prey. The predator population declines at first because of high mortality; however, as prey becomes more abundant, the predator population begins to recover. The system exhibits oscillations, with the prey population peaking and then decreasing as predators grow, leading to a subsequent decline in predators. These oscillations represent the classic predator-prey cycles, with both populations alternating between highs and lows over time.

- **Case 2:** Initial time $t_0 = 0$, final time $T = 8$, subdivisions number $N = 10000$, $x_0 = 10$, $y_0 = 2$, $a = 3$, $b = 0.5$, $c = 4$, $d = 0.5$.

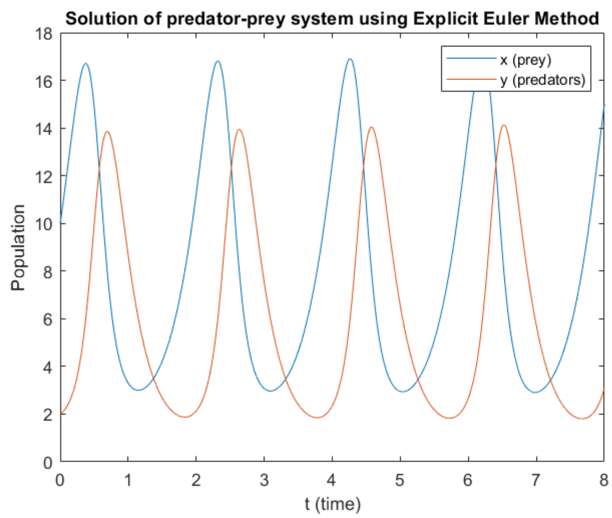


Figure 2: Dynamics for Case 2

Interpretation of the Results:

- **Initial Conditions:**
  $x_0 = 10$, $y_0 = 2$
- **Parameters:**
  $a = 3$, $b = 0.5$, $c = 4$, $d = 0.5$.

- **Parameter Analysis:**

  The initial prey population is much higher than in Case 1, while the predator population starts much lower. The dynamics should start with a rapid increase in predators due to the high prey availability. However, since the predator mortality rate is still significant, the predator population will be constrained by this.
- **Interpretation:**

  The predator population rapidly increases, followed by a gradual decline as the prey population becomes low due to predation. Initially, the prey population increases due to fewer predators, but as the predator population grows, the prey population begins to decrease. The system likely exhibits oscillations again, though they may take longer to stabilize because of the initially low predator population.

- **Case 3:** Initial time $t_0 = 0$, final time $T = 8$, subdivisions number $N = 10000$, $x_0 = 4$, $y_0 = 10$, $a = 3$, $b = 0.5$, $c = 4$, $d = 2$.

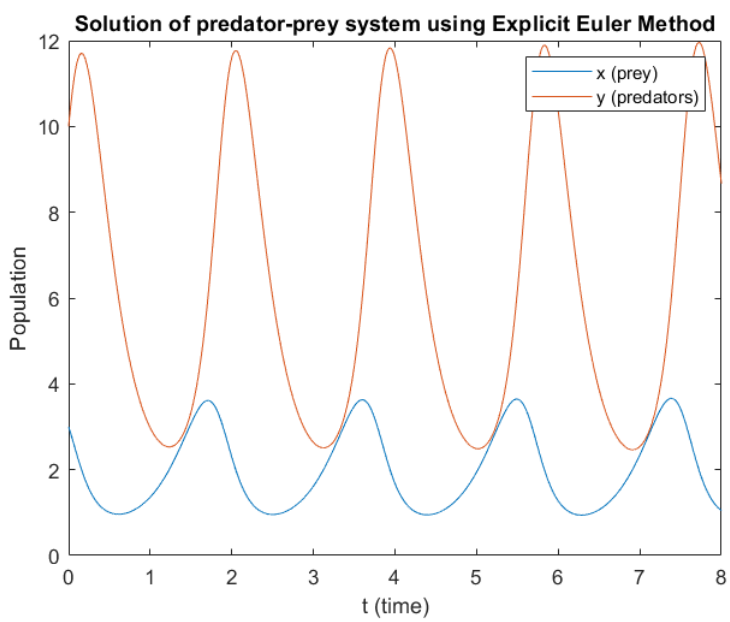


Figure 3: Dynamics for Case 3

Interpretation of the Results:

– **Initial Conditions:**
  $x_0 = 4$, $y_0 = 10$
– **Parameters:**
  $a = 3$, $b = 0.5$, $c = 4$, $d = 2$.

– **Parameter Analysis:**

  The reproduction rate of predators $d = 2$ is now significantly higher, meaning predators reproduce much more efficiently when consuming prey. With a high initial predator population and a lower initial prey population, the prey may experience a sharp decline early on.

– **Interpretation:**

  The prey population declines rapidly as predators consume them more efficiently. The predator population initially rises but may experience a sharp decline if prey becomes scarce. Larger oscillations could occur compared to previous cases, with more pronounced fluctuations in both populations. The system may take longer to reach equilibrium or could exhibit chaotic behavior, depending on the balance between prey growth and predation rates.

- **Case 4:** Initial time $t_0 = 0$, final time $T = 8$, subdivisions number $N = 10000$, $x_0 = 4$, $y_0 = 10$, $a = 8$, $b = 0.5$, $c = 4$, $d = 0.5$.

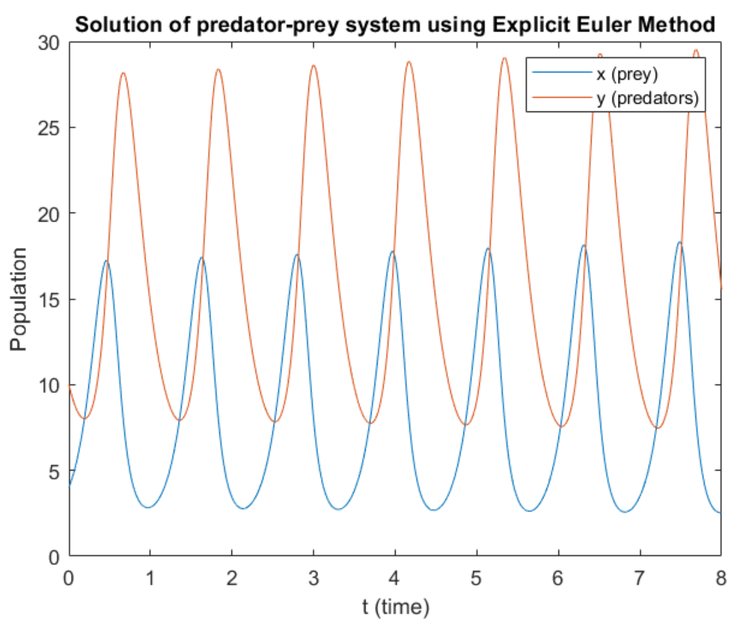


Figure 4: Dynamics for Case 4

Interpretation of the Results:

- **Initial Conditions:**
  $x_0 = 4$, $y_0 = 10$
- **Parameters:**
  $a = 8$, $b = 0.5$, $c = 4$, $d = 0.5$.

- **Parameter Analysis:**

  The prey growth rate is significantly higher $a = 8$, which should lead to a strong increase in the prey population, even in the presence of predators. Predator mortality remains high, but with a decent reproduction rate, the predator population will still be able to grow.

- **Interpretation:**

  The prey population grows rapidly at first, potentially overshooting. The predator population rises in response, though it may struggle to keep pace with the prey's rapid growth. Oscillations are likely, but the prey population may dominate in this case due to its much higher intrinsic growth rate. The system might stabilize with a larger prey population and a relatively smaller predator population compared to other cases.

General Results of the Analysis and Interpretations:

For all cases:

- **Oscillations** are characteristic of the Lotka-Volterra model, representing cycles of predator-prey interactions.
- The prey population typically increases when predators are low and then decreases as predators increase.
- The predator population rises as prey become more abundant and declines when prey become scarce.

The **frequency** and **amplitude** of these oscillations will depend on the specific parameters and initial conditions, as well as the balance between predator growth and mortality, and prey reproduction and predation pressure.

6. Taking as initial time $t_0 = 0$, final time $T = 8$, subdivisions number $N = 10000$, and parameters $a = 3$, $b = 0.5$, $c = 4$, $d = 0.5$, we plot on the same coordinate system the evolution of predators according to prey for different initial values of x_0 and y_0.

   for the MATLAB code and the results, check the file:

   `[PartB_question6.m]`

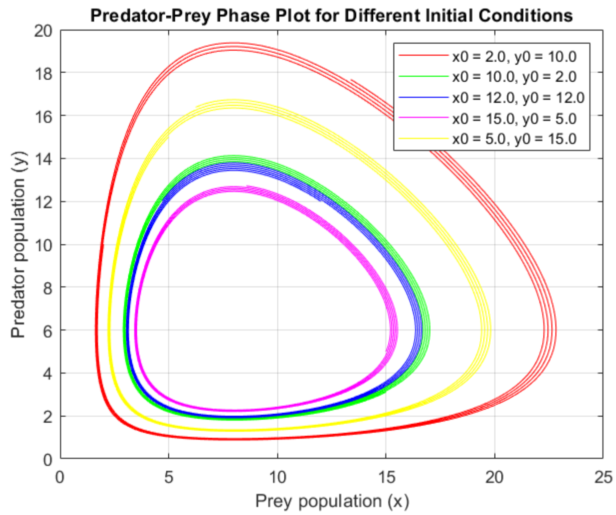


Figure 5: screenshot of the results.

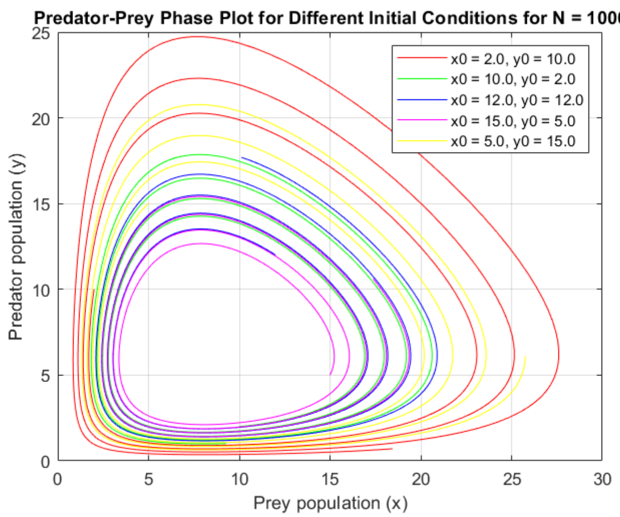
- Key Observation and Analysis:

- **Phase plot:** This Phase plot shows the evolution of predator and prey populations over time for different initial conditions. Each curve represents a trajectory of the system, starting from a specific initial point (x_0, y_0) and evolving according to the Lotka-Volterra equations.
- **Closed trajectories:** In the classical Lotka-Volterra system, populations often trace closed curves in phase space. These indicate oscillations where populations of predators and prey fluctuate periodically without converging to a fixed point or diverging to infinity.
- **Impact of initial conditions:** The initial populations of prey and predators influence the specific trajectory followed in the phase plot. For example, starting with a large predator population and a small prey population (e.g., $x_0 = 2$, $y_0 = 10$) leads to a cycle with a large amplitude and a longer period. Conversely, starting with a balanced population (e.g., $x_0 = 12$, $y_0 = 12$) results in a smaller, more tightly wound cycle. However, for this type of system, different initial conditions can still result in qualitatively similar behavior (oscillations), with the magnitude and starting phase being the only differences.
- **Oscillations:** In all cases,we see oscillatory behavior, with prey population increases leading to predator population growth, and predator population growth leading to prey population declines. These cycles will continue as long as there is no external stabilization mechanism.
- **Stability and Limit Cycles:** The system does not have a stable equilibrium point. Instead, it tends to converge to a limit cycle, which is a closed curve in the phase plane. The limit cycle represents a stable periodic solution, where the predator and prey populations oscillate indefinitely.

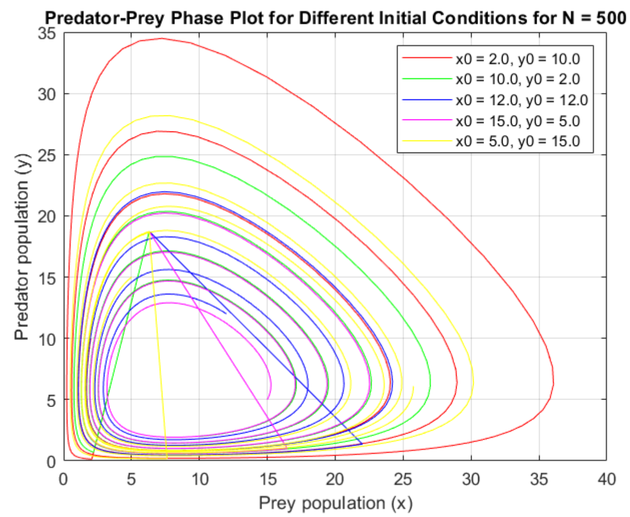
- Final Result

The results show the typical predator-prey dynamics predicted by the Lotka-Volterra equations. The predator and prey populations are interdependent, and their populations will exhibit periodic oscillations around certain equilibrium values. Different initial populations simply shift the magnitude and phase of these oscillations, but the overall dynamics remain consistent across all initial conditions.

7. now we modify the number of subdivisions : represent the results for $N = 1000$ then $N = 500$ for the MATLAB code and the results, check the file: [PartB_question7p1.m] and [PartB_question7p2.m]



(a) screenshot of the results.



(b) screenshot of the results.

Figure 6: screenshot of results

- interpretation of the different Results:

1. Case 1: $N = 1000$
Step Size: $h = \frac{T-t_0}{N} = \frac{8-0}{1000} = 0.008$

- Analysis:

- **Moderate Step Size:** When $N = 1000$, the time step $h = 0.008$ is moderately small, giving a reasonable balance between computational efficiency and accuracy.
- **Phase Plot Behavior:** For this value of N, the trajectories in the phase plot (prey population vs. predator population) are expected to still exhibit smooth oscillations that correspond to the predator-prey cycles. The Euler method with this step size should give a decent approximation of the true Lotka-Volterra dynamics.

  We see oscillating closed trajectories similar to those obtained with finer subdivisions. However, the trajectory might show slight deviations due to the moderate step size, causing some numerical error. The orbits could drift slightly over time, but the system should still capture the periodic predator-prey cycles reasonably well.

2. Case 2: $N = 500$
Step Size: $h = \frac{T-t_0}{N} = \frac{8-0}{500} = 0.016$

- Analysis:

- **Larger Step Size:** For $N = 500$, the time step doubles to $h = 0.016$. This larger step size will increase the numerical error, leading to potentially more noticeable issues in the simulation results.
- **Phase Plot Behavior:** With a larger step size, the explicit Euler method is less accurate, and you may begin to see **more visible deviations** in the predator-prey trajectories. The system may still oscillate, but the paths might no longer follow smooth closed orbits. Instead, you might observe:
  - **Increased oscillation damping:** The oscillations might lose energy faster, and the trajectories may shrink towards the origin or diverge.
  - **Numerical instability:** If the step size becomes too large for the explicit Euler method, the oscillations may become unstable. Instead of smooth, continuous curves, the trajectories could become erratic or even diverge from the expected path.

General Results:

The accuracy of the Euler method is directly related to the size of the time step h. The smaller the time step, the more accurate the solution, but the trade-off is increased computational cost. Larger time steps lead to larger numerical errors, which in the case of the explicit Euler method can result in:

- **Phase errors:** Shifts in the oscillations due to inaccurate time stepping.
- **Damping or instability:** The system might lose or gain energy artificially, resulting in unrealistic behavior such as spiraling towards zero or exploding populations.

4 Part C : Consideration of Pesticide Use :

Now, consider the impact of pesticide use in the ecosystem, where it causes mortality in both prey and predators. For prey, pesticides lead to direct deaths and reduced reproductive success, while predators are affected by a decreased prey base and direct exposure through consumption of contaminated prey. This added mortality disrupts natural population cycles and risks destabilizing the ecosystem.

4.1 Proportional Death Explanation

The introduction of pesticide use in the ecological model accounts for the additional mortality of both prey and predators. It is reasonable to assume that the number of deaths due to pesticide exposure is proportional to the population size of each group because:

- The larger the population, the more individuals will be exposed to pesticides.
- The coefficient ϵ represents the rate at which pesticides affect both populations.

Population Size Effect: The larger the prey or predator population, the more animals will come into contact with pesticides. So, the number of animals dying from pesticides will depend on how big the population is.

Proportional Coefficient (ϵ): The coefficient ϵ represents how quickly the pesticides kill prey and predators. A higher ϵ means more animals die from the pesticides, no matter how the prey and predators interact with each other.

4.2 Analysis of $\epsilon > a$

When $\epsilon > a$, the reproduction rate of the prey population (a - ϵ) becomes negative. This implies that pesticide use is so intense that the prey population will decrease continuously, even without the presence of predators. Essentially, in this case:

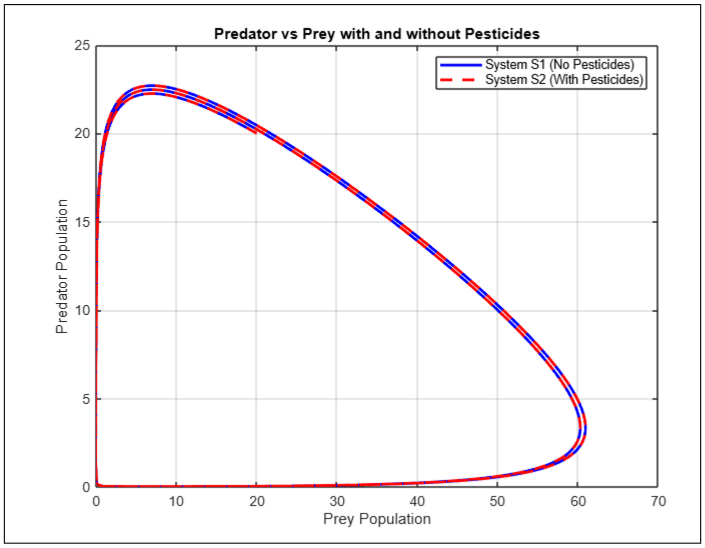
(a) **Prey Population Decline:** Even without predators, the prey population will shrink because the pesticides are killing them faster than they can reproduce.

(b) **Ecosystem Collapse:** Over time, if the prey population collapses, the predators will also decline because they won't have enough food to survive. This could destabilize the entire ecosystem.

4.3 Euler Scheme for the Modified System

To solve the system (S2) using the explicit Euler method, we discretize the equations as follows :

$$\begin{cases} x_{n+1} = x_n + \Delta t \cdot \left[x_n \left((a-\epsilon) - b y_n\right)\right], \\ y_{n+1} = y_n + \Delta t \cdot \left[y_n \left(d x_n - (c+\epsilon)\right)\right], \\ x(0) = x_0, y(0) = y_0, \Delta t = h \end{cases} \tag{5}$$



for the MATLAB code and the results, check the file:

`[PartC_question3p2.m]`

Analysis of the Predator-Prey Phase Plot:

- **System S(1) Red Line:**

This curve represents the classical Lotka-Volterra model without the impact of pesticides. The oscillations form a loop, indicating cyclical dynamics between the prey and predator populations. As prey numbers rise, predator numbers also increase due to the abundance of food. Eventually, as the predator population becomes too large, the prey population is reduced, which leads to a decline in predator numbers. This feedback cycle continues, generating the classic predator-prey loop.

- **System S(2) Blue Dashed Line:**

This curve includes the effect of pesticides, represented by ϵ =0.001. Compared to the red curve, the blue dashed curve System S(2) exhibits similar dynamics but slightly dampened. The loops for System S(2) are slightly smaller than those for System S(1), which indicates that the populations of both predators and prey are somewhat reduced due to the pesticide mortality. However, the effect of ϵ =0.001 is not very pronounced, as the difference is subtle.

→ System S1 (no pesticides): Shows regular, stable cycles of predator-prey interaction typical of the Lotka-Volterra model.

→ System S2 (with pesticides): The cycles shrink as pesticides reduce both prey and predator populations, leading to faster declines and smaller oscillations depending on ϵ.

Impact of Pesticides: The use of pesticides reduces both the prey and predator populations slightly, which is reflected in the smaller loops of System (S2). However, since the pesticide effect ϵ is relatively small (0.001), the system still exhibits cyclical behavior, just with lower overall population levels.

Convergence: The system doesn't show a dramatic collapse in predator or prey populations for this specific choice of ϵ. If ϵ were increased, the difference between Systems (S1) and (S2) would become more evident, and may even observe population decline or extinction in extreme cases.

5 Part D : Addressing Resource Limitations

The objective of this part is to enhance the classical Lotka-Volterra predator-prey model by incorporating a resource limitation term to more accurately reflect real-world environmental constraints on prey population growth. Unlike the traditional model, which assumes unlimited resources and thus allows for infinite prey growth, this modified model introduces a carrying capacity K, that limits prey population size based on available resources. This adjustment is particularly relevant for studying long-term population dynamics under the impact of deforestation and habitat loss, which reduce the availability of green spaces and essential resources for prey species. Through simulations, we aim to examine how these resource limitations, representing environmental degradation, influence the stability of predator-prey interactions and explore the potential consequences of further reductions in carrying capacity due to ongoing deforestation

1. Theoretical Explanation of the $-\frac{ax^2(t)}{K}$ Term

- **Purpose**: This term limits the growth rate of the prey population $x(t)$ as it approaches the carrying capacity K of the ecosystem.
- **Mechanism**:
  - **Growth Regulation**: As $x(t)$ increases, $-\frac{ax^2(t)}{K}$ becomes more significant, reducing the net growth rate.
  - **Stabilization**: When $x(t)$ approaches K, the growth rate slows down and eventually stops when $x(t) = K$, preventing population overshoot.
  - **Negative Feedback**: If $x(t)$ exceeds K, this term turns negative and reduces the population size, pulling it back towards the equilibrium.

we have

$$x^{'}(t) = x(t)(a - (ax(t)/K) - by(t))$$

we can reformulate it as :

$$x^{'}(t) = x(t)(a - (ax(t)/K)) - by(t)x(t)$$

we assume that $y(t) = 0$ to ensure a maximal growth of preys, it results in :

$$x^{'}(t) = x(t)(a - (ax(t)/K)) = ax(t)(1 - (x(t)/K))$$

- thus if x approaches K which will leads to (x/K) will approaches 1, it will lead to slowing down the growth of prey population till x'(t) approaches 0.

- if $x(t) > K$ it will leads to $(x/K) > 1$ resulting in $x'(t) < 0$ which is a decrease in population.

Thus the term K ensures that x(t) does not exceed K. The term $-\frac{ax^2(t)}{K}$ is introduced in the prey population equation to model the effect of limited resources:

2. Equilibrium Points Analysis

To find the equilibrium points of the system:

System Equations:

$$\begin{cases} x' = x\left(a - \frac{ax}{K} - by\right) \\ y' = y(dx - c) \\ x(0) = x_0,\, y(0) = y_0 \end{cases}$$

Equilibrium Conditions: Set $x' = 0$ and $y' = 0$ to solve for the equilibrium points.

- **Prey Equation** ($x' = 0$) implies:

$$x\left(a - \frac{ax}{K} - by\right) = 0 \implies x = 0 \quad \text{or} \quad a - \frac{ax}{K} - by = 0$$

- **Predator Equation** ($y' = 0$) implies:

$$y(dx - c) = 0 \implies y = 0 \quad \text{or} \quad dx = c \implies x = \frac{c}{d}$$

Equilibrium Points:

(a) $(x, y) = (0, 0)$ — : This represents a scenario where both prey and predators go extinct. This is often considered an unstable or non-realistic state because populations rarely die out completely in natural systems without external disruptions. However, it is a valid mathematical solution.

(b) $(x, y) = (K, 0)$ — : Here, the prey population grows to its maximum capacity K without predators present. This represents a stable prey population when there are no predators, but it might be unstable if predators are introduced into the system. This point is biologically meaningful because prey can potentially thrive in the absence of predators, but their growth is still limited by environmental carrying capacity.

(c) $(x^*, y^*) = \left(\frac{c}{d}, \frac{a - \frac{ac}{dK}}{b}\right)$: This point represents a coexistence equilibrium, where both prey and predators have stable populations. This is the most ecologically realistic situation because it reflects the balance between prey population growth and predator pressure. However, if $c > Kd$, this equilibrium becomes meaningless because it predicts a negative predator population, which is not possible. In such a case, the predator population cannot sustain itself, leading to extinction of the predators and only the prey population remaining at K.

Biological Relevance: The non-trivial equilibrium point (x^*, y^*) is valid if both $x^* > 0$ and $y^* > 0$. When $c > Kd$, the prey's carrying capacity is insufficient to support a sustainable predator population, making y^* negative or non-existent, which lacks biological significance.

Condition $c > Kd$ and Biological Sense of the Equilibrium Points :
Let's now explore the condition $c > Kd$, which is important for determining whether all equilibrium points make biological sense.

For Equilibrium Point 3, we found that the predator population at equilibrium is:

$$y = \frac{a}{b}\left(1 - \frac{c}{Kd}\right)$$

For this equilibrium to make biological sense, the value of y must be non-negative, as a negative predator population has no biological meaning.

To ensure $y \geq 0$, we require that:

$$1 - \frac{c}{Kd} \geq 0 \Rightarrow \frac{c}{Kd} \leq 1 \Rightarrow c \leq Kd$$

Thus, if $c > Kd$, then $y = \frac{a}{b}\left(1 - \frac{c}{Kd}\right)$ becomes negative, which is biologically meaningless. In this case, Equilibrium Point 3 does not make sense biologically.

3. Result Interpretation and Further Simulations

for the MATLAB code and the results, check the file: `[PartD_question3.m]`

1. The Original Lotka-Volterra Model (Without the Term $\frac{ax^2(t)}{K}$):
In the original Lotka-Volterra model, the prey population can theoretically grow indefinitely if not controlled by predators. There are no environmental limits like resource scarcity or space limitations that prevent the prey from growing beyond a certain threshold. Therefore, the prey population is regulated only by the predator population.

- Key behaviors :

Prey population growth : Unrestricted as long as predators aren't abundant enough to control it. This means the prey population can potentially grow without bounds, which is unrealistic in long-term ecological systems.
Predator-prey interaction : The predator population depends entirely on the availability of prey. When prey are abundant, the predator population increases, but when prey are scarce, predators die off. There is a cyclical dynamic between both populations, and the populations exhibit oscillations.

2. The Improved Model (With the Term $-\frac{ax^2(t)}{K}$) :
The improved model adds a carrying capacity K to the prey population, which represents the maximum population that the environment can sustain. This term accounts for resource limitations like food, space, or habitat that constrain the growth of the prey population.

- Key changes :

Prey population growth : Limited by the carrying capacity K. As the prey population approaches K, the growth rate slows down, and the prevents indefinite exponential growth and introduces a more realistic ceiling for prey populations.

Predator-prey interaction : The predator population still depends on the availability of prey, but now the prey population's growth is constrained. If the prey population approaches K, predators may find it more difficult to grow their population due to the limited resources. The prey population won't grow beyond the carrying capacity, which adds an extra dynamic to the oscillatory nature of the predator-prey cycles seen in the original Lotka-Volterra model.

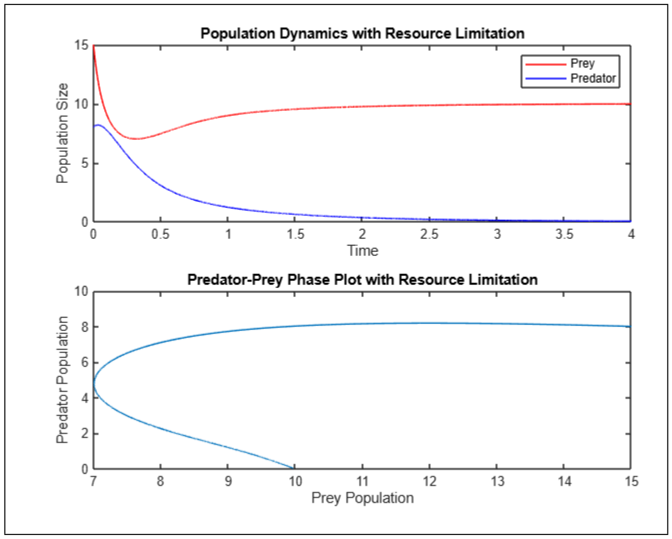


Figure 7: Scenario 1

Case 1 :$t0 = 0, T = 4, N = 20000, x0 = 15, y0 = 8, K = 10, a = 8, b = 0.5, c = 6, d = 0.5.$

Population Dynamics Over Time: In this graph, we see an initial spike in prey (red) and predator (blue) populations. The prey population rapidly increases initially, likely due to favorable conditions and low predator impact. However, as the predator population grows, it limits prey population growth. Over time, both populations approach a steady state where they stabilize. This reflects the carrying capacity constraint for prey, where the population cannot grow indefinitely due to resource limitations (e.g., food scarcity or environmental capacity).
Predator-Prey Phase Plot: This phase plot illustrates a classic trajectory where the predator and prey populations converge toward a stable equilibrium point. The trajectory is a decaying spiral, indicating that the populations undergo initial fluctuations before reaching a stable coexistence. The spiral signifies diminishing oscillations until both predator and prey populations settle at steady levels.

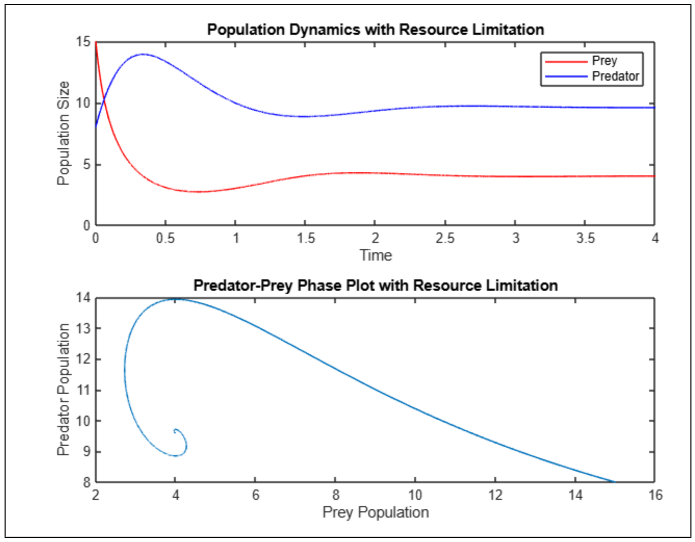


Figure 8: Scenario 2

Case 2 :$t0 = 0, T = 4, N = 20000, x0 = 15, y0 = 8, K = 10, a = 8, b = 0.5, c = 2, d = 0.5.$

Population Dynamics Over Time: In this case, the initial conditions or parameter values seem to differ, as the predator population (blue) starts off relatively high, causing a rapid decline in the prey population (red). This higher initial predator population results in an immediate negative impact on the prey, leading to significant prey reduction. However, the prey population does not collapse; it stabilizes at a lower level as predator numbers also decrease over time, finding a balance. After some oscillations, the populations again stabilize at lower values than in Plot Set 1.

Predator-Prey Phase Plot: The phase plot in this set has a more pronounced spiral pattern, indicating a different dynamic where the system undergoes larger oscillations before stabilizing. The initial loop suggests that the system initially experiences significant predator-prey population fluctuations, likely due to the high initial predator numbers, before it gradually settles into equilibrium.

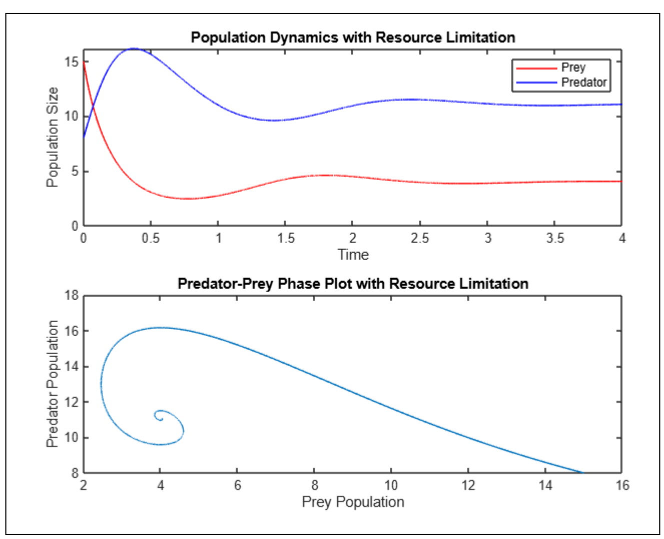


Figure 9: Scenario 3

Case 3 :$t0 = 0, T = 4, N = 20000, x0 = 15, y0 = 8, K = 13, a = 8, b = 0.5, c = 2, d = 0.5.$

Population Dynamics Over Time: The predator population (blue) starts relatively high, causing a quick decline in the prey population (red). Both populations undergo some oscillations but eventually stabilize at lower, balanced values, limited by resource constraints.
Predator-Prey Phase Plot: The spiral pattern shows the system's initial large fluctuations, which gradually decrease as the predator and prey populations settle into equilibrium. This reflects an adjustment phase before reaching a stable balance.

Aspect	**Without Resource Limitation (Classic Lotka-Volterra Model)**	**With Resource Limitation (Modified Model)**
Population Behavior	• Exhibits sustained oscillations with no inherent damping. • Populations cycle indefinitely without settling.	• Shows damped oscillations or converges to a steady state. • Populations stabilize over time due to resource constraints.
Prey Growth	• Assumes unlimited resources; prey population can grow infinitely. • Growth is only checked by predation.	• Introduces a carrying capacity K that limits prey growth. • Growth rate decreases as prey population approaches K.
Predator Population	• Cyclical behavior tightly linked to prey population cycles. • No intrinsic limits aside from prey availability.	• Predator population may decline if prey is limited by K. • Risk of predator extinction if K is too low to sustain them.

Realism	• Less realistic as it ignores environmental limitations on prey. • Doesn't account for factors like food scarcity, habitat space, etc.	• More realistic by incorporating environmental carrying capacity. • Reflects real-world constraints such as limited resources and habitat loss.
Ecological Insight	• Demonstrates basic predator-prey interactions without external constraints. • Limited insight into how environmental changes affect populations.	• Highlights the impact of resource limitations on population dynamics. • Illustrates how habitat loss (e.g., deforestation) can destabilize ecosystems.
Population Stability	• Populations do not reach a stable equilibrium; they keep fluctuating.	• Populations tend to stabilize around equilibrium points influenced by K. • Reduces the amplitude of population swings, leading to more predictable dynamics.
Long-Term Dynamics	• Predicts perpetual cycles without consideration of environmental degradation.	• Accounts for long-term sustainability by enforcing population limits. • Can model the effects of environmental stressors like deforestation over time.

Table 1: Comparison of Predator-Prey Dynamics: Classic Lotka-Volterra Model vs. Modified Model with Resource Limitation

Case 1: Without Resource Limitation (Classic Lotka-Volterra Model)

Prey:

$$\frac{dx}{dt} = ax - bxy \tag{6}$$

Predator:

$$\frac{dy}{dt} = dxy - cy \tag{7}$$

Dynamics

- **Oscillatory Behavior:** The prey and predator populations typically exhibit sustained oscillations. When prey increases, predators also grow due to increased food availability, leading to a rise and fall in both populations.
- **No Carrying Capacity:** Without the term $-\frac{ax^2}{K}$, there is no limit to prey growth except predator pressure, so prey can grow rapidly when predators are low, leading to large oscillations.
- **Population Cycles:** The model predicts endless cycles, with the population sizes continually rising and falling in a repeating pattern.

Ecological Implications

- **Unrealistic Growth:** This model assumes unlimited resources for the prey, which is less realistic in natural ecosystems.
- **No Steady State:** Populations do not settle at stable values, implying a constant predator-prey chase dynamic that doesn't align with most real ecosystems where resources are finite.

Case 2: With Resource Limitation (Added Term $-\frac{ax^2(t)}{K}$)

Prey:

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right) - bxy \tag{8}$$

Predator:

$$\frac{dy}{dt} = dxy - cy \tag{9}$$

Dynamics

- **Stabilization of Prey:** The prey population stabilizes around the carrying capacity K, as the term $-\frac{ax^2}{K}$ prevents unchecked growth. This reduces oscillations, as prey cannot grow indefinitely.
- **Predator Decline:** With limited prey due to K, the predator population lacks sufficient resources, leading it to decline or approach extinction in some cases.
- **Damped Oscillations or Steady State:** Rather than constant cycles, populations move toward a steady state where prey stabilizes below K, and predators stabilize at a low level or near extinction.

Ecological Implications

- **Realistic Population Limits:** This model reflects natural systems better by accounting for finite resources, leading to stable populations.
- **Conservation Insight:** It shows that habitat loss (lower K) could lead to predator decline, emphasizing the importance of preserving environments to maintain ecosystem balance.

Interpretation of Results with Equilibrium Points

Equilibrium Points

From the analysis of the system with the resource limitation term $-\frac{ax^2}{K}$, we have an equilibrium point where both prey and predator populations stabilize. Mathematically, this equilibrium occurs when the growth rates for both prey $\frac{dx}{dt} = 0$ and predator $\frac{dy}{dt} = 0$ are zero.

- **Prey Equilibrium:** The prey population stabilizes at approximately $x = K$ due to the carrying capacity, meaning that even without predator pressure, prey cannot exceed K.
- **Predator Equilibrium:** For the predator, stability is reached when $c = \frac{dxy}{y}$ (assuming sufficient prey). The predator population depends on prey availability, which is capped by K.

Damped Oscillations and Stability

The plots show damped oscillations, with the populations approaching stability over time. This differs from the classic Lotka-Volterra model, where populations oscillate indefinitely. The resource limitation term ensures that the prey population doesn't grow beyond the carrying capacity K, which in turn limits the predator population. This leads to a stable, balanced ecosystem where both populations coexist at sustainable levels.

Proposing Further Simulations with Parameter Variations

Deforestation Impact (Reducing K)

Deforestation or habitat destruction reduces K, the carrying capacity for prey, which can disrupt the balance.

- **Simulation Scenario:** Gradually reduce K to observe the effect on both populations. For instance, try values of K such as $K = 8$, $K = 5$, and eventually $K < \frac{c}{d}$, where the carrying capacity becomes insufficient to support the predator population.
- **Expected Result:** As K decreases, the prey population stabilizes at a lower level, which in turn limits the predator population. If K becomes too small, the prey population may no longer sustain the predators, leading to predator extinction.

Threshold Condition: $K < \frac{c}{d}$

When K falls below the threshold $\frac{c}{d}$, the predator population is unsustainable regardless of prey population. This is a critical tipping point, as predators cannot survive even with a sufficient prey base if their natural growth needs cannot be met.

Ecological Implications

This scenario illustrates a potential collapse of the predator population due to resource scarcity, highlighting the risk that deforestation poses to species at higher trophic levels.

6 Conclusion

In this project, we explored the dynamics of predator-prey interactions using the Lotka-Volterra model, considering both ecological factors and the impact of pollution on species populations.

Through the analysis of the simplified model, we demonstrated the inherent cyclical nature of predator-prey dynamics, highlighting the exponential growth of prey in the absence of predators and the decline of predators in the absence of prey. The equilibrium points identified, particularly $(\frac{c}{d}, \frac{a}{b})$, reveal the potential coexistence and balance between predator and prey populations.

We employed numerical methods, specifically the Euler explicit scheme, to simulate and visualize population dynamics under various initial conditions and parameter values. Our results illustrated classic oscillatory behavior indicative of predator-prey systems, with notable variations in population sizes under differing parameters.

Overall, our findings underscore the complexity of ecological interactions and the importance of understanding both biological and environmental influences on these populations. This work sets a foundation for further research into more intricate models, possibly incorporating additional factors such as habitat changes, resource depletion, and human impacts on ecosystems.