Lehigh University

Advanced Technology for Large Structural Systems

CEE 466 – Advanced Finite Element Methods

Project 2A: Stochastic Elastic FE code – Report

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Contents

Introduction

This code is developed to perform a stochastic finite element analysis on a linear 2D structure with 1D elements; e.g. beams and bars. This code takes all the information of the material properties, geometries, loadings, and boundary conditions in the format of an input file, discretizes the system into a finer meshing with desired number of elements, calculates the stiffness matrix and solves the equation of equilibrium for the structure. Furthermore, this code also takes the probabilistic characteristics of the input for the stochastic analysis. It is assumed that the only non-deterministic input parameters are the Young's modulus of the members, thus information regarding the mean values, standard deviations and correlation of these parameters must also be provided in the input file. The output results are the nodal displacements, and reaction forces if the deterministic analysis is selected. For the stochastic analysis, there are several approaches that could be selected to run the stochastic analysis. A summary of the type of output results for different analyses are brought in Table 1. The code is fully automatic in the sense that only the input file must be written in a format of MATLAB script, and by running the FE code, the rest will be done automatically.

Table 1 - Summary of the type of output results provided by each stochastic approach

	MCS	Perturbation	Polynomial – Method B
Mean values, standard deviations, and covariance matrix of the nodal displacements	✓	✓	✓
Mean values, standard deviations, and covariance matrix of the reaction forces	✓	×	×
PDF of the nodal displacements	✓	×	✓
PDF of the reaction forces	✓	×	×

In this code different types of stochastic approaches are run, and their performance are compared in terms of accuracy of the results and efficiency to better see how each of them could be utilized for different purposes.

There are also some additional features included in the code for each of which a brief description is brought here:

- 1D elements with varying cross sections along their length could be defined, and depending on the number of elements used for the analysis the accuracy of the approximations could be enhanced.
- Auto-meshing is included in the code, so that the analyst can decide how fine the meshing should be in order to achieve the accuracy of desire. The location of the nodes and elements before and after the meshing will be displayed as a graphical check for the analyst.
- Four different types of elements are defined that could be selected for the analysis: bar with direct stiffness method, complete Bernoulli beam with direct stiffness method, complete Timoshenko beam with direct stiffness method, complete Timoshenko beam with linear shape functions.
- Dummy stiffness check is embedded in the code to give the analyst warnings for any of the following three cases. i) if there is a node that is not connected to the system, ii) if a diagonal element of the global stiffness matrix is zero for a beam element, iii) and finally checks if there is a concentrated load at that degree of freedom. In any of these cases, it informs the analyst to either just verify or fix the issue.

- For the convenience of the analyst there are multiple ways to define the covariance of the input. The analyst can choose the random variables to be independent, dependent with given correlation matrix defined by the analyst or dependent with certain correlation length and the matrix will be constructed automatically by the code. This means the covariance matrix will be constructed based on the number of elements created by the code with the mesh refinement defined by the analyst.
- If MCS or PRA-B approaches are selected, plots showing the probability distribution of the nodal displacements (along with the plots showing the probability distribution of the reactions forces for MCS) are displayed at the end of the analysis.

There are also 5 test problems for which the results obtained by the code are validated either by basic hand calculations, or by using commercial softwares. For the stochastic analyses, the exact answer is considered to be the one obtained by MCS, and the rest are compared to it.

Challenges

There were a lot of challenges during developing this code. I would like to briefly point out the ones particularly occurred in the stochastic part. The first main challenge was to have the code as efficient as possible. To do so, I had to choose which variables were better to be stored beforehand, and which were better to be calculated on the fly. This was of great importance in the loops over the elements and the loops over the samples in MCS. The second main challenge was to find the best structure and the most efficient way to pass the information through sub functions in Perturbation and PRA-B method. The third and last challenge was to find a way to calculate and organize the polynomial coefficients and to use them to build the equations for the nodal displacements as functions of random variables in PRA-B method.

Test problems

Test problem 1 – Cantilever beam

In this problem we have a beam element of length $L = 10\ m$ with a concentrated load of *1 N* applied to its end tip as shown in the Fig. 1. We are interested in the deflection of the free end of the beam as the output. The model is discretized into 10 Timoshenko beam elements as shown in Fig. 2.

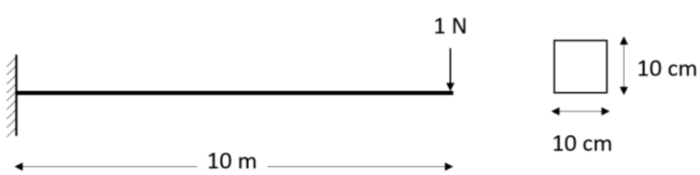


Figure 1 – Test problem 1 - cantilever beam

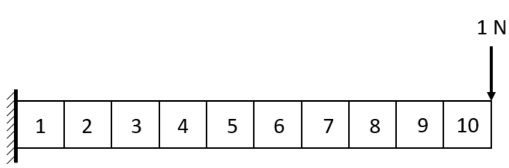


Figure 2 - Discretization of the cantilever beam

The material properties are as follows.

$$E_i = E_0(1 + \alpha_i)$$

in which E_0 = *200 GPa* and α_i is a random variable with mean = *0* and standard deviation of *0.2*. The correlation coefficient among two generic α_i and α_j is given by the equation:

$$\rho_{ij} = exp(-\frac{|\Delta x_{ij}|}{0.8L})$$

in which is the distance between the midpoints of elements *i* and *j*. A deterministic value of G = 76.92 *GPa* is assumed for the shear modulus. The cross section is a square of *10×10 cm* as shown in Fig. 1.

It is shown in Fig. 3 how the code creates a desired number of elements of type Timoshenko and labels the nodes.

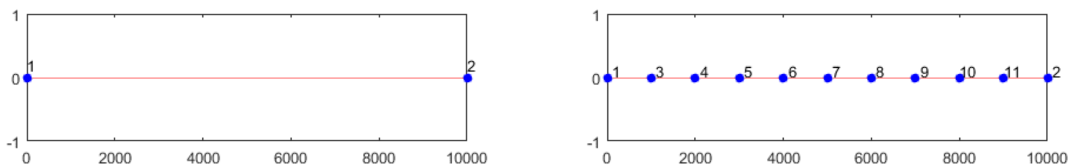


Figure 3 - Discretization of cantilever beam in test problem 1

We expect the mean value of the displacement of the free end to converge to its exact value which can be calculated as below.

$$\Delta_{v,B} = \int_0^L \frac{M(x)m(x)}{EI(x)}dx + \int_0^L \frac{V(x)v(x)}{GA_s(x)}dx = \int_0^L \frac{(10-x)(10-x)}{(2\times10^{11})\times\left(\frac{0.1^4}{12}\right)}dx + \int_0^L \frac{1\times1}{(7.692\times10^{10})\times\left(\frac{2}{3}\times0.1^2\right)}dx$$

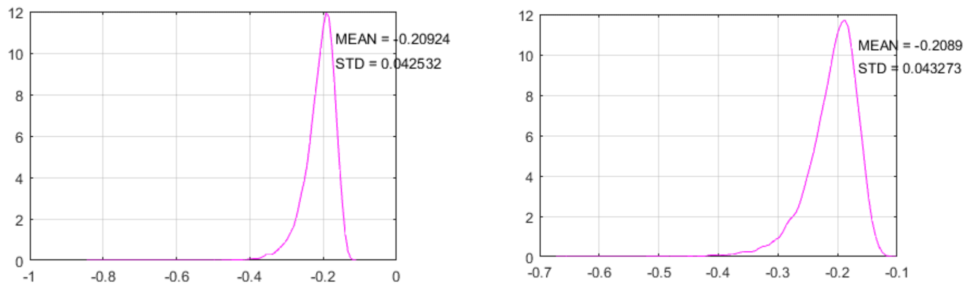
$$= 0.2000\ mm$$

Summary of the results are brought in Table 1. It shows the probabilistic characteristics of the vertical displacement of the free end including the mean value and standard deviation along with the time it took each approach to run.

Table 2 - Summary of the output results for cantilever beam

	Deterministic	MCS (NS = 10000)	Perturbation	PRA-B S1 (NS = 10000)
Mean value	-0.2000 *mm*	-0.2092 *mm*	-0.2000 *mm*	-0.2089 *mm*
Standard deviation	-	0.0436 *mm*	0.0355 *mm*	0.0430 *mm*
Computational time	1.8 *s*	437.3 *s*	1.9 *s*	278.5 *s*

For each of the MCS and PRA-B methods, the probability distribution of the vertical displacement of the free end is brought in Fig. 4.



(a) (b)

Figure 4 – Probability distribution of the vertical displacement of the free end assessed by (a) MCS and (b) PRA-B

Test problem 2 – Clamped beam

In the second test problem we have a clamped beam of length *8 m* with a concentrated load of *150 N* applied at its midpoint. The material properties are the same as in the test problem 1. The cross section is a square of *8×8 cm* as shown in Fig. 5.

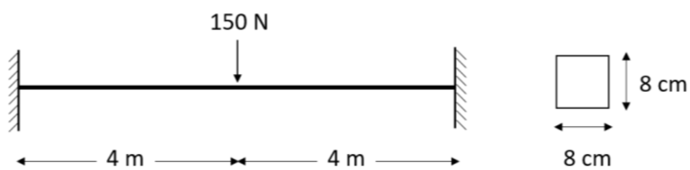


Figure 5 - Test problem 2 - clamped beam

For the maximum element size of 8 *cm*, the code discretizes the structure into 10 Timoshenko beam elements as shown in Fig. 6.

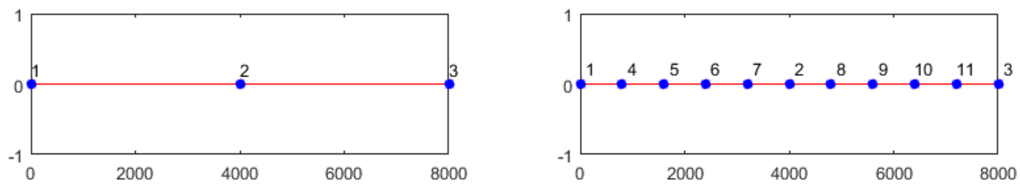


Figure 6 - Discretization of clamped beam in test problem 2

The exact answer can be computed by the formulation below (explanation not brought here).

$$\frac{P}{2} = K_{eqv} \times \Delta_{2,v} = \frac{1}{1+\varphi}\left(\frac{12EI}{(L/2)^3}\right)$$

$$\varphi = \frac{12EI}{G(A/alpha)(L/2)^2}$$

Plugging in the values for $E = 2e11\ Pa$, $G = 7.692e10\ Pa$, $A = 0.8^2\ m^2$, $I = 0.8^4/12\ m^4$, $L = 8m$, $P = 150\ N$. and *alpha =1.2* for rectangular cross sections, we get:

$$\Delta_{2,v} = 0.5867\ mm$$

A summary of results is brought in table 2.

Table 3 - Summary of results for test problem 2

	Deterministic		MCS (NS = 10000)		Perturbation		PRA-B S1 (NS = 10000)	
	$\Delta_{2,v}$ (mm)	θ_2 (rad)	$\Delta_{2,v}$ (mm)	θ_2 (rad)	$\Delta_{2,v}$ (mm)	θ_2 (rad)	$\Delta_{2,v}$ (mm)	θ_2 (rad)
Mean value	-0.5867	-1.2623 e-18	-0.5982	1.9804 e-8	-0.5867	-1.2623 e-18	-0.5977	2.6068 e-8
Standard deviation	-	-	0.0786	1.0188 e-5	0.0727	9.1531 e-6	0.0760	9.872 e-6
Computational time	1.9 s		452.5 s		2.0 s		258.9 s	

The probability distribution of the output results of interest are also brought in Fig. 7.

(a)

(b)

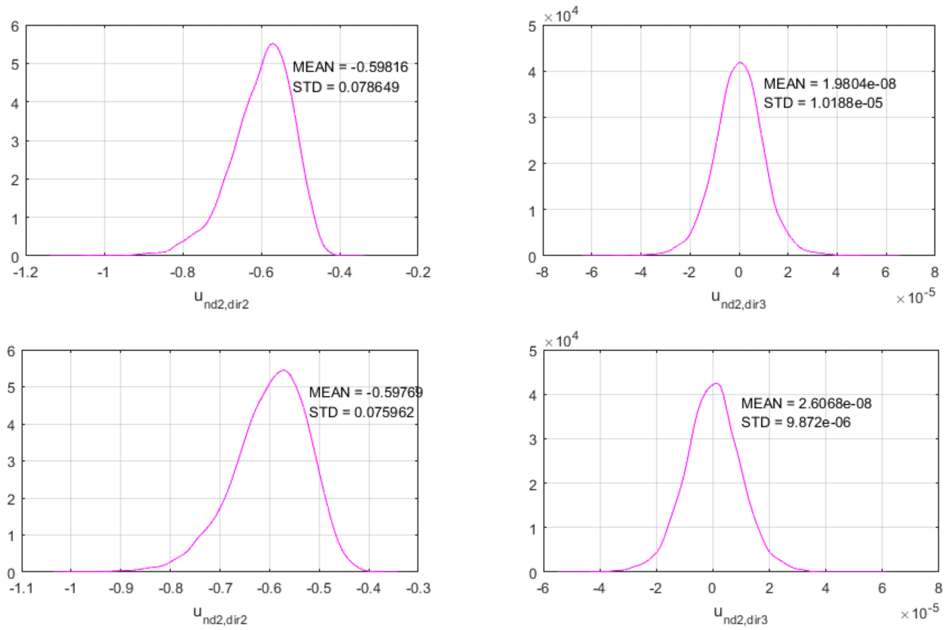


Figure 7 - Probability distribution of the vertical displacement (on the left) and rotation (on the right) of the midpoint of the clamped beam by (a) MCS and (b) PRA-B

Test problem 3 – 3-story frame

In the third test problem we have a 3-story frame as shown in Fig. 8. The cross sections of all the beams, columns and braces are identical with $A = 0.150\ cm^2$, $A_s = 0.125\ cm^2$, *and* $I = 0.002\ m^2$. The random variables are the moduli of elasticity formulated as $E_i = E_0(1 + \alpha_i)$ in which $E_0 = 210\ GPa$, and α_i is a random variable with mean of zero and standard deviation of 0.15. The shear modulus has a deterministic value of 77 GPa. In this problem, the random variables are assumed to be independent.

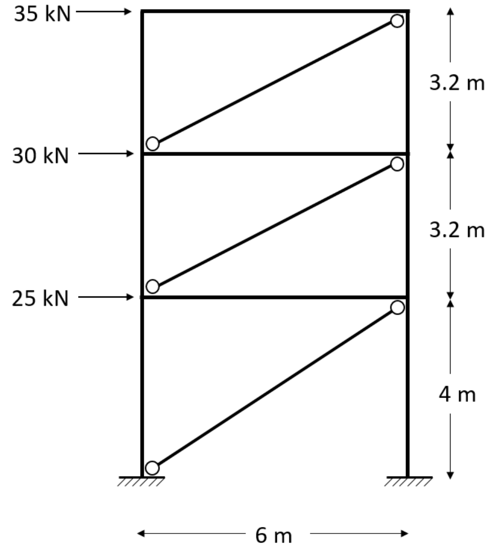


Figure 8 - Test problem 3 - 3-story frame

For complete Timoshenko beam elements (and bar elements if used) of maximum size *8 m*, the code discretizes the structure as shown in Fig. 9. Here for instance, it can be seen that the braces are modeled as bar elements with different colors (could also be modeled as beam elements with end releases for rotation).

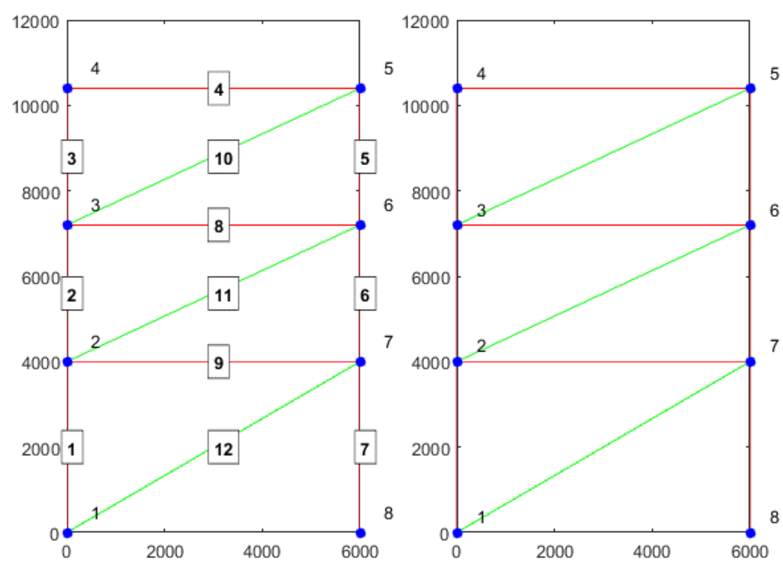


Figure 9 - Model of 3-floor steel frame in test problem 3

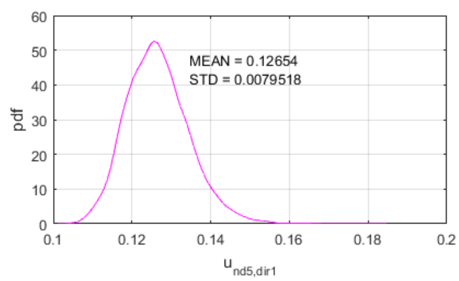
The parameters of interest in this problem are the horizontal displacement of the right top corner (node 8) of the frame, and also the moment at the base of the right column. The summary of the results is brought in Table 4. Notice that the only method that gives the probabilistic characteristics of the reaction forces is MCS. Also for the deterministic value of the reaction forces, only the deterministic analysis should be run.

Table 4 – Summary of the results for test problem 3

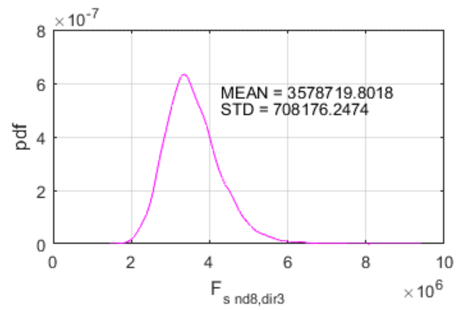
	Deterministic		MCS (NS = 10000)		Perturbation		PRA-B S1 (NS = 10000)	
	$\Delta_{5,h}$ (mm)	M_8 $(kN.m)$	$\Delta_{5,h}$ (mm)	M_8 $(kN.m)$	$\Delta_{5,h}$ (mm)	M_8 $(kN.m)$	$\Delta_{5,h}$ (mm)	M_8 (k$N.m$)
Mean value	0.1237		0.1265	3.5208	0.1237	-	0.1265	-
Standard deviation	-	-	0.0080	7.4044 e-8	0.0072	-	0.0079	-
Computational time	2.1 s		457.4 s		2.5 s		211.2 s	

The probability distributions of the horizontal displacement of the node 8 is assessed by both MCS and PRA-B methods and brought in Fig. 10 along with the probability distribution of the moment of the base of the right column assessed by MCS only.

(a)



(b)



(c)

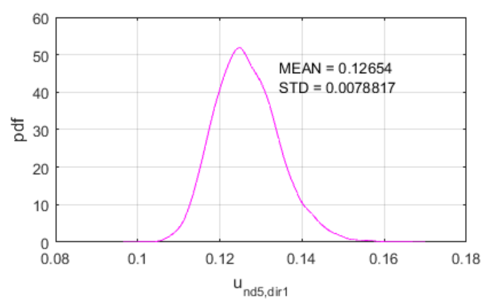


Figure 10 - Probability distribution of the parameters of interest: (a) displacement of the node 8 by MCS, (b) displacement of the node 8 by PRA-B, and (c) moment of the base of the right column by MCS.

Also in the MCS approach we are able to calculate the correlation of the two parameters of interest. Having all the generated samples for each, the covariance can be calculated and then divided by the standard deviation of each of the random variables which gives us $\rho = 0.1055$. This is a very small value and pretty close to zero which shows that there is not much correlation between these two random variables.

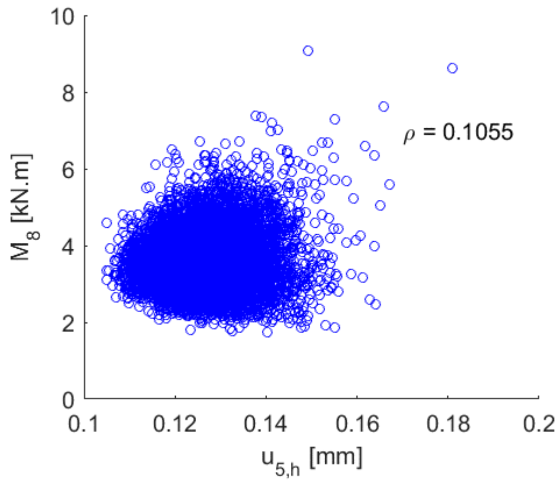


Figure 11 - Correlation between the horizontal displacement of node 8amd moment of the base of right column

Test problem 4 – 2-bay frame with inverted V bracing

In the fourth test problem, we have a 2-bay moment frame with inverted bracing in both bays under both gravity and wind load as shown in Fig. 12. The cross sections are squares of *30×30 cm* for the columns, squares of *20×20 cm* for the beam, and squares of *10×10 cm* for the bracing. The random variables are the moduli of elasticity formulated as $E_i = E_0(1 + \alpha_i)$ in which $E_0 = 200\ GPa$, and α_i is a random variable with mean of zero and standard deviation of 0.3. The Poisson's ratio is assumed to be deterministic and equal to 0.35. The variables are correlated in the same way as in test problems 1 and 2 with correlation length of 4m. Note that the bracings are working in the structure as 1 dimensional element and take no moments. The element types are the same as in previous problem for different members.

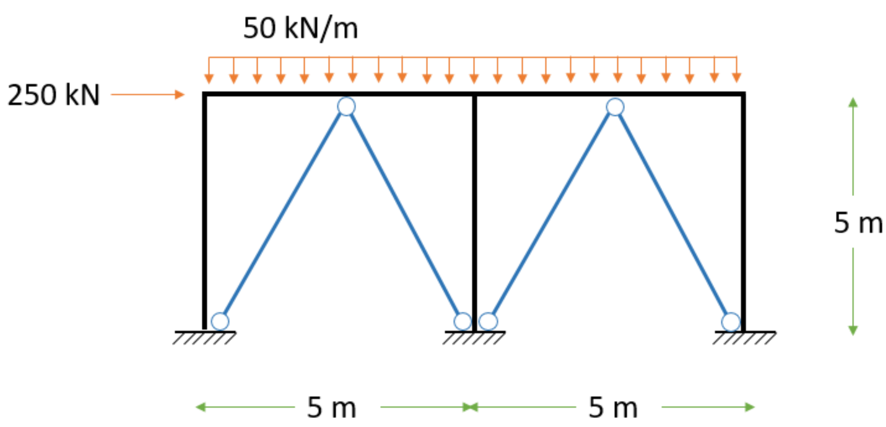


Figure 12 - Test problem – 2-bay frame with inverted V bracing

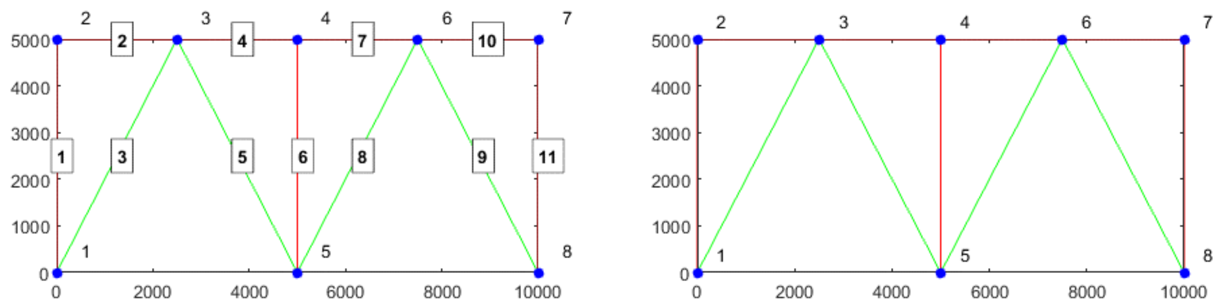


Figure 13 - Test problem 4 modeled by Linear FE code

The results are briefly brought in Table 5 in terms of the horizontal displacement of the top left corner (node 2) in 2 cases, with and without the bracing. It can be observed that how determining the role of the bracing could be in this structure.

Table 5 – Summary of the results for test problem 4

	Deterministic		MCS (NS = 10000)		Perturbation		PRA-B S1 (NS = 10000)	
	$\Delta_{2h,b}$ (mm)	$\Delta_{2h,ub}$ (mm)	$\Delta_{2h,b}$ (mm)	$\Delta_{2h,ub}$ (mm)	$\Delta_{2h,b}$ (mm)	$\Delta_{2h,ub}$ (mm)	$\Delta_{2h,b}$ (mm)	$\Delta_{2h,ub}$ (mm)
Mean value	0.4312	14.309	0.4316	14.315	0.4312	14.309	0.4316	14.312
Standard deviation	-	-	0.0096	0.2823	0.0096	0.2799	0.0096	0.2781
Computational time	2.0 s	1.9 s	458.5 s	451.9 s	2.1 s	2.0 s	160.9 s	102.6 s

The probability distribution obtained by MCS for each case is brought in Fig. 14. It can be seen that the mean value has a considerable increase while not using the bracing and accordingly the standard deviation also increases. As measure of dispersion though, the coefficient of variation must be computed for each case. It can be seen both in the plots and from the equations that there is not a big difference.

$$\rho_{braced} = \frac{0.0096}{0.4316} = 0.022 = \%2.2$$

$$\rho_{unbraced} = \frac{0.2823}{14.315} = 0.020 = \%2.0$$

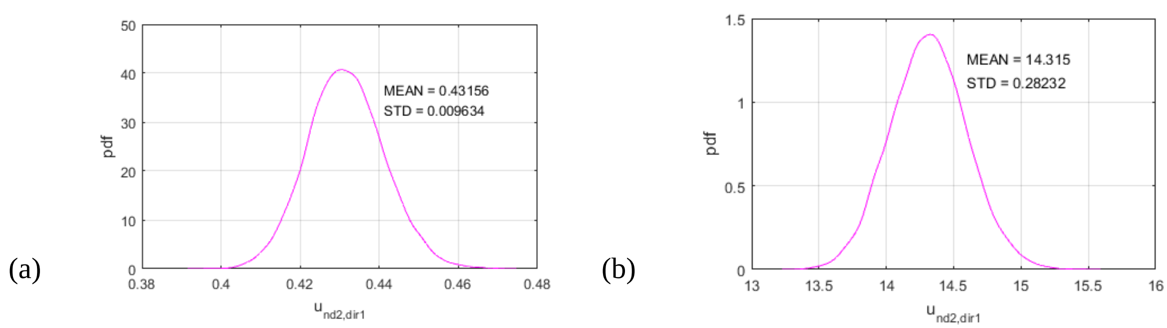


Figure 14 - Probability distribution of the horizontal displacement of the node 2 obtained by MCS (a) with and (b) without bracing

Test problem 5 – Statue of PB

For the last test problem, we have a frame made in the shape of a statue of Dr. Paolo Bocchini's initials. The dimensions are shown in Fig. 15. A cross bracing is used between the two main parts to hold the structure together during the tornados. The cross section of the structure is a circle of diameter *100 cm*, and for the bracing, a circle of diameter *20 cm is used*. The random variables are again the moduli of elasticity formulated as $E_i = E_0(1 + \alpha_i)$ in which $E_0 = 210\ GPa$, and α_i is a random variable with mean of zero and standard deviation of 0.25. The Poisson's ratio is assumed to be deterministic and equal to 0.25. The variables are correlated in the same way as in test problems 1 and 2 with correlation length of 7.5m. Note that the bracings are working in the structure as 1 dimensional element and take no moments. Timoshenko beam element is used for all the elements.

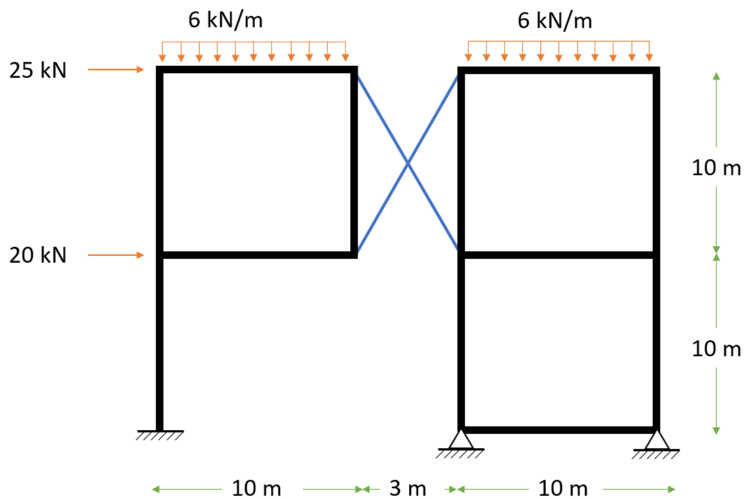


Figure 15 - Test problem 5, statue of PB

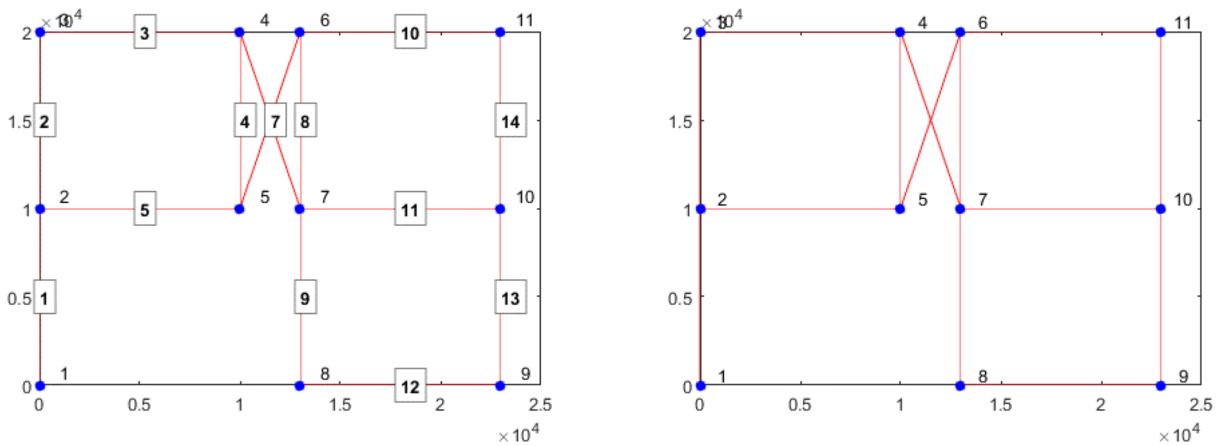


Figure 16 - Test problem 5, modeled by Linear FE code

The parameter of interest is vertical displacement of the top right corner of the letter "P" (node *4*). A summary of the result is brought in Table 6.

Table 6 – Summary of the results for test problem 5

	Deterministic	MCS (NS = 10000)	Perturbation	PRA-B S1 (NS = 10000)
Mean value	-15.426 *mm*	-16.3867 *mm*	-15.426 *mm*	-15.951 *mm*
Standard deviation	-	4.7403 *mm*	3.5647 *mm*	3.9135 *mm*
Computational time	1.9 *s*	464.9 *s*	2.2 *s*	333.0 *s*

It is also interesting to see how the horizontal displacements of the top corners of the structure (nodes *3,4,6,* and *11)* are correlated with each other. This is shown in Fig. 17. First, it is observed that the coefficient is equal for each RV with itself (obvious). Second, it can be seen that the displacements of each two tip belonging to a same letter of the statue is equal to one which means that the axial stiffness of the beams are large enough to make the two nodes move horizontally together. Third, it can be seen that the coefficient between the displacements of the two nodes from different letters of the statue is negative. It could mean that (if there is a relationship to be between them) the more one goes to the right the more the other one goes to the left. It does not make sense to me physically, or does it?

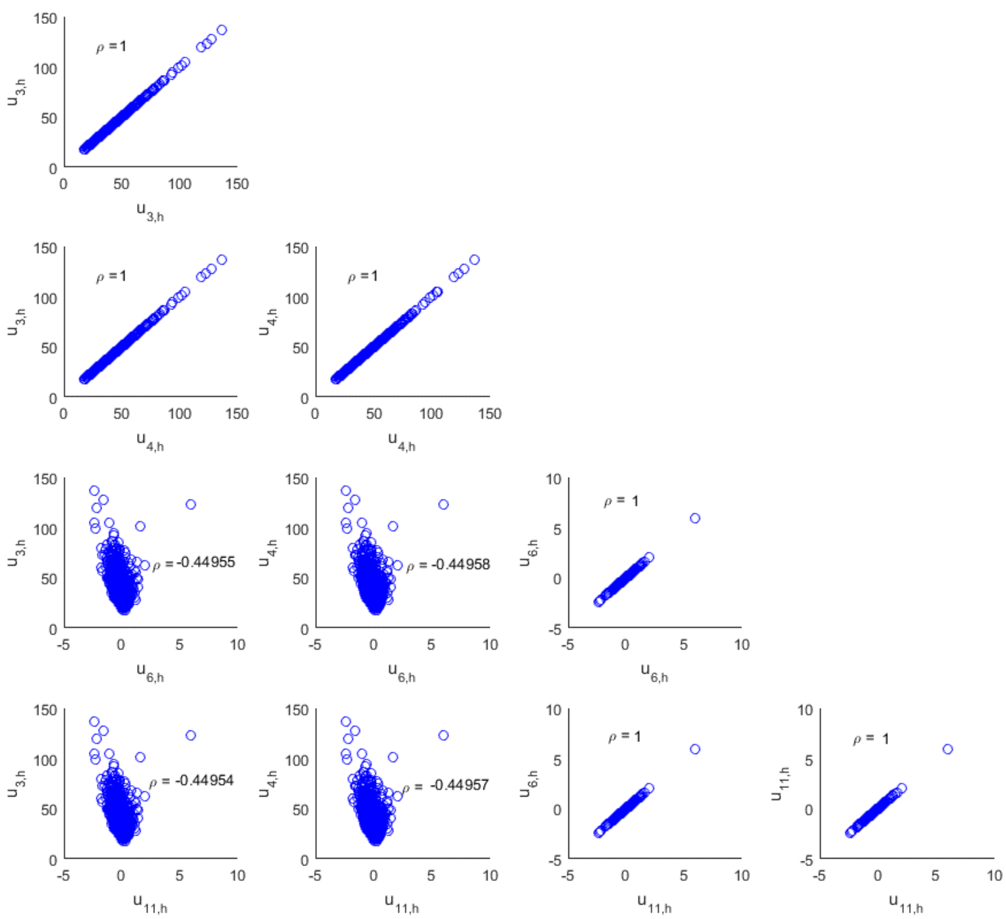


Figure 17- Correlation between the horizontal displacements of the top corners of the statue

Conclusion and Discussion

In this project, I could develop a code that analyzes a linear 2D structure with 1D elements e.g. bars and beams both deterministically and stochastically. It still has a lot of limitations in the FEM part. Yet, it can analyze a considerable number of linear problems including beams, frames and trusses. For the stochastic part, it also has a lot of limitations. There are other characteristics of the structure that could be considered as random, like the dimensions of the cross sections, the location of the nodes, the loads applied to the structure, and so forth. There are also many other distributions that any of the aforementioned random variables could take other than Gaussian.

In summary, in this part of the project, I started with the code I had developed from the first project and I added a new feature to the tool to perform also a stochastic analysis. There were some issues from the previous part that was fixed, e.g. the number of elements by the automated messing, end node releases, etc. For the stochastic part, I wrote the script for three required approaches, and for each, there are different types and amount of information that the code produces in the output. There are some comments on the results and on the comparison of the approaches that are briefly discussed below.

The fist observation is that all the mean values of the random variables of interest obtained by Perturbation method are equal to the value of the parameter obtained by deterministic analysis. They are all highlighted in green. The reason for that is that in Perturbation method the key assumption is that each of the output parameters (nodal displacements) are a linear function of the input random variables (Elasticity moduli). We know also for a fact that *the mean value of the linear combination of random variables is equal to the linear combination of the mean values of the random variables.* This was also observed in all the results.

On the other hand, it can be seen that the results from PRA-B and MCS are slightly different from the previous two. These results are highlighted in yellow. Now it is obvious that the reason is that this time there is no linear relationship between the input and output random variables. Specifically, in polynomial ratios, we have the form of summation of the polynomial ratios, and in MCS as the number of random variables, standard deviation, etc. increases, the degree of nonlinearity also increases. For future study, it is a good idea to study the change in the difference between the results from PRA-B and MCS with the mesh refinement. It seems like by refining the mesh and increasing the number of random variables, the difference will increase.

It can also be observed that there is also some difference between the results by MCS and PRA-B. This difference does not disappear even if we increase the number of trials in PRA-B. This could be due to the fact the PRA-B is also an approximation, since it is constraining the output random variables to be a function of input random variables in a specific predefined form. In other words, this method gets rid of part of the nonlinearity o the problem by simply not considering the cross terms in the equations.

In terms of efficiency, it was observed that Perturbation method was pretty much faster than the other two though losing the accuracy. In this project, due to the computational restrictions, meshing was not refined to a reasonable extent, thus we can expect that by increasing the number of elements, the error increases in the results by Perturbation method. PRA-B also saved some time compared to MCS. Roughly speaking, this code runs with PRA-B 1/3 faster than MCS. At the end, what MCS provides is the probability distributions and correlation among various random variables, and also information regarding the reaction forces. These are certainly valuable information that could not be obtained using other approaches.