Center of Gravity and Inertia Calculation

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Single-Axis Case

Consider a collection of n point masses m_i, each at location x_i, where $i = 1 \ldots n$ is the mass index. The x-axis origin location relative to the masses is arbitrary.

These define a total mass m, a total mass-moment p, and a total moment of inertia I. The latter two depend on some arbitrary reference point $\bar{x}$.

$$m = \sum m_i \tag{1}$$

$$p(\bar{x}) = \sum m_i(x_i - \bar{x}) = \sum m_i\, x_i - m\,\bar{x} \tag{2}$$

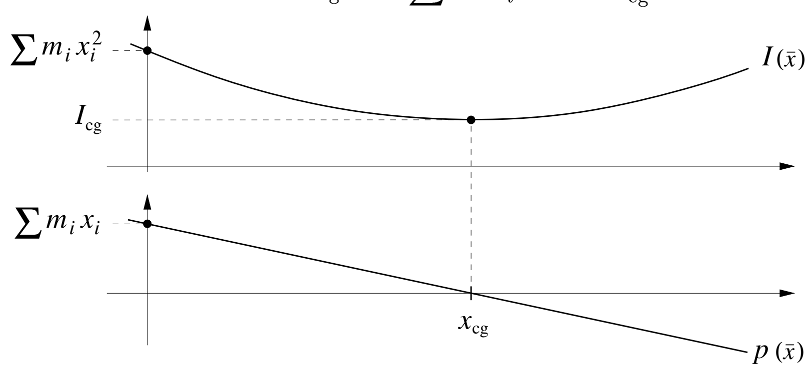
$$I(\bar{x}) = \sum m_i(x_i - \bar{x})^2 = \sum m_i\, x_i^2 - 2p\,\bar{x} - m\,\bar{x}^2 \tag{3}$$

Each sum runs over the point index $i = 1 \ldots n$.

The center of gravity $x_{\rm cg}$, or center of mass, is defined as the particular value of $\bar{x}$ where the $p(\bar{x})$ mass-moment function vanishes. At this location, $I(\bar{x})$ has the minimum value $I_{\rm cg}$.

$$p(x_{\rm cg}) = 0 \qquad \rightarrow \qquad x_{\rm cg} = \frac{1}{m}\sum m_i\, x_i \tag{4}$$

$$I_{\rm cg} = \sum m_i\, x_i^2 - m\, x_{\rm cg}^2 \tag{5}$$



The calculation of $x_{\rm cg}$ and $I_{\rm cg}$ is readily implemented in the spreadsheet in Table 1.

Table 1: Spreadsheet for total mass, center of gravity, and moment of inertia calculation. The unboxed items are input data. The boxed items are formulas. Heavy boxes are the final spreadsheet outputs.

$\boxed{\sum m_i}$		$\boxed{\sum m_i x_i}$	$\boxed{\sum m_i x_i^2}$	$\boxed{m}$	$\boxed{x_{\rm cg}}$	$\boxed{I_{\rm cg}}$
m_1	x_1	$\boxed{m_1 x_1}$	$\boxed{m_1 x_1^2}$			
m_2	x_2	$\boxed{m_2 x_2}$	$\boxed{m_2 x_2^2}$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
m_n	x_n	$\boxed{m_n x_n}$	$\boxed{m_n x_n^2}$			

The inertia $I(\bar{x})$ for any other $\bar{x}$ value is given by definition (3) above. It can be recast into the following form, known as the *parallel axis theorem*.

$$I(\bar{x}) = I_{\rm cg} + m\,(\bar{x} - x_{\rm cg})^2 \tag{6}$$

General 3D Axis Case (Point Masses)

For a general 3D configuration of point masses in space we are also interested in the x, y, z location of the center of gravity, and also the moments of inertia and products of inertia about the three x, y, z axes. The spreadsheet is readily extended to this general case as follows.

Table 2: Spreadsheet for total mass, center of gravity, and moments of inertia of point-mass collection in 3D. Heavy boxes are the final spreadsheet outputs.

m				x_{cg}	y_{cg}	z_{cg}	$(I_{xx})_{cg}$	$(I_{yy})_{cg}$	$(I_{zz})_{cg}$	$(I_{xy})_{cg}$	$(I_{xz})_{cg}$	$(I_{yz})_{cg}$
$\sum m_i$				$\sum m_i x_i$	$\sum m_i y_i$	$\sum m_i z_i$	$\sum m_i x_i^2$	$\sum m_i y_i^2$	$\sum m_i z_i^2$	$\sum m_i x_i y_i$	$\sum m_i x_i z_i$	$\sum m_i y_i z_i$
m_1	x_1	y_1	z_1	$m_1 x_1$	$m_1 y_1$	$m_1 z_1$	$m_1 x_1^2$	$m_1 y_1^2$	$m_1 z_1^2$	$m_1 x_1 y_1$	$m_1 x_1 z_1$	$m_1 y_1 z_1$
m_2	x_2	y_2	z_2	$m_2 x_2$	$m_2 y_2$	$m_2 z_2$	$m_2 x_2^2$	$m_2 y_2^2$	$m_2 z_2^2$	$m_2 x_2 y_2$	$m_2 x_2 z_2$	$m_2 y_2 z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m_n	x_n	y_n	z_n	$m_n x_n$	$m_n y_n$	$m_n z_n$	$m_n x_n^2$	$m_n y_n^2$	$m_n z_n^2$	$m_n x_n y_n$	$m_n x_n z_n$	$m_n y_n z_n$

The 3D center of gravity location and all the moments of inertia and products of inertia, which are shown in the first line of Table 2, contain the following formulas in terms of the various column sums.

$$m = \sum m_i \tag{7}$$

$$x_{cg} = \frac{1}{m} \sum m_i \, x_i \tag{8}$$

$$y_{cg} = \frac{1}{m} \sum m_i \, y_i \tag{9}$$

$$z_{cg} = \frac{1}{m} \sum m_i \, z_i \tag{10}$$

$$(I_{xx})_{cg} = \sum m_i \, y_i^2 + \sum m_i \, z_i^2 - m \, (y_{cg}^2 + z_{cg}^2) \tag{11}$$

$$(I_{yy})_{cg} = \sum m_i \, x_i^2 + \sum m_i \, z_i^2 - m \, (x_{cg}^2 + z_{cg}^2) \tag{12}$$

$$(I_{zz})_{cg} = \sum m_i \, x_i^2 + \sum m_i \, y_i^2 - m \, (x_{cg}^2 + y_{cg}^2) \tag{13}$$

$$(I_{xy})_{cg} = \sum m_i \, x_i y_i - m \, x_{cg} y_{cg} \tag{14}$$

$$(I_{xz})_{cg} = \sum m_i \, x_i z_i - m \, x_{cg} z_{cg} \tag{15}$$

$$(I_{yz})_{cg} = \sum m_i \, y_i z_i - m \, y_{cg} z_{cg} \tag{16}$$

The parallel axis theorem in the 3D case has the following form.

$$I_{xx(\bar{y},\bar{z})} = (I_{xx})_{cg} + m \left[(\bar{y} - y_{cg})^2 + (\bar{z} - z_{cg})^2 \right] \tag{17}$$

$$I_{yy(\bar{x},\bar{z})} = (I_{yy})_{cg} + m \left[(\bar{x} - x_{cg})^2 + (\bar{z} - z_{cg})^2 \right] \tag{18}$$

$$I_{zz(\bar{x},\bar{y})} = (I_{zz})_{cg} + m \left[(\bar{x} - x_{cg})^2 + (\bar{y} - y_{cg})^2 \right] \tag{19}$$

$$I_{xy(\bar{x},\bar{y})} = (I_{xy})_{cg} + m \, (\bar{x} - x_{cg})(\bar{y} - y_{cg}) \tag{20}$$

$$I_{xz(\bar{x},\bar{z})} = (I_{xz})_{cg} + m \, (\bar{x} - x_{cg})(\bar{z} - z_{cg}) \tag{21}$$

$$I_{yz(\bar{y},\bar{z})} = (I_{yz})_{cg} + m \, (\bar{y} - y_{cg})(\bar{z} - z_{cg}) \tag{22}$$

General 3D Axis Case (Finite-size Masses)

In many applications, any individual mass i may have one or more significant moment of inertia components about its own center of gravity x_i, y_i, z_i, in which case it is not accurate to represent it as a point mass in the overall inertia calculation. This effect of such finite-size masses is accounted for by keeping track of their individual inertias. The previous spreadsheet table is now extended to the following form by adding the six moment of inertia data columns.

Table 3: Spreadsheet for total mass, center of gravity, and moments of inertia of finite-mass collection in 3D. The "$\cdots$" on the right denotes the rightmost quantities in Table 2. Heavy boxes are the final spreadsheet outputs.

$\boxed{m}$										$\boxed{x_{\rm cg}}$	$\boxed{y_{\rm cg}}$	$\boxed{z_{\rm cg}}$	$\boxed{(I_{xx})_{\rm cg}}$	$\cdots$
$\boxed{\sum m_i}$				$\boxed{\sum I_{xx_i}}$	$\boxed{\sum I_{yy_i}}$	$\boxed{\sum I_{zz_i}}$	$\boxed{\sum I_{xy_i}}$	$\boxed{\sum I_{xz_i}}$	$\boxed{\sum I_{yz_i}}$	$\boxed{\sum m_i x_i}$	$\boxed{\sum m_i y_i}$	$\boxed{\sum m_i z_i}$	$\boxed{\sum m_i x_i^2}$	$\cdots$
m_1	x_1	y_1	z_1	I_{xx_1}	I_{yy_1}	I_{zz_1}	I_{xy_1}	I_{xz_1}	I_{yz_1}	$\boxed{m_1 x_1}$	$\boxed{m_1 y_1}$	$\boxed{m_1 z_1}$	$\boxed{m_1 x_1^2}$	$\cdots$
m_2	x_2	y_2	z_2	I_{xx_2}	I_{yy_2}	I_{zz_2}	I_{xy_2}	I_{xz_2}	I_{yz_2}	$\boxed{m_2 x_2}$	$\boxed{m_2 y_2}$	$\boxed{m_2 z_2}$	$\boxed{m_2 x_2^2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
m_n	x_n	y_n	z_n	I_{xx_n}	I_{yy_n}	I_{zz_n}	I_{xy_n}	I_{xz_n}	I_{yz_n}	$\boxed{m_n x_n}$	$\boxed{m_n y_n}$	$\boxed{m_n z_n}$	$\boxed{m_n x_n^2}$	$\cdots$

The total mass m and the center of gravity $x_{\rm cg}, y_{\rm cg}, z_{\rm cg}$ formulas in Table 3 are the same as in the point-mass case, given by (7)–(10). Only the inertia expressions (11)–(16) are modified by the addition of the individual-inertia sums.

$$(I_{xx})_{\rm cg} = \sum m_i\, y_i^2 + \sum m_i\, z_i^2 - m\,(y_{\rm cg}^2 + z_{\rm cg}^2) + \sum I_{xx_i} \tag{23}$$
$$(I_{yy})_{\rm cg} = \sum m_i\, x_i^2 + \sum m_i\, z_i^2 - m\,(x_{\rm cg}^2 + z_{\rm cg}^2) + \sum I_{yy_i} \tag{24}$$
$$(I_{zz})_{\rm cg} = \sum m_i\, x_i^2 + \sum m_i\, y_i^2 - m\,(x_{\rm cg}^2 + y_{\rm cg}^2) + \sum I_{zz_i} \tag{25}$$
$$(I_{xy})_{\rm cg} = \sum m_i\, x_i y_i - m\, x_{\rm cg} y_{\rm cg} + \sum I_{xy_i} \tag{26}$$
$$(I_{xz})_{\rm cg} = \sum m_i\, x_i z_i - m\, x_{\rm cg} z_{\rm cg} + \sum I_{xz_i} \tag{27}$$
$$(I_{yz})_{\rm cg} = \sum m_i\, y_i z_i - m\, y_{\rm cg} z_{\rm cg} + \sum I_{yz_i} \tag{28}$$

The inertia expressions reduce to the point-mass case when the individual mass inertias I_{xx_i}, I_{yy_i}, etc. are negligible.

Component Inertia Estimation

The moments of inertia of any single component can often be adequately estimated by modeling that component as a simple shape with a uniform mass distribution. Some examples are below.

Spherical Ball of radius R:

$$I_{xx_i} = I_{yy_i} = I_{zz_i} = \tfrac{2}{5}\, m_i R^2 \qquad \text{(solid)} \tag{29}$$
$$I_{xx_i} = I_{yy_i} = I_{zz_i} = \tfrac{2}{3}\, m_i R^2 \qquad \text{(thin shell)} \tag{30}$$

Circular Disk of radius R, with axis along x:

$$I_{xx_i} = \tfrac{1}{2}\, m_i R^2 \tag{31}$$
$$I_{yy_i} = I_{zz_i} = \tfrac{1}{4}\, m_i R^2 \tag{32}$$

Slender Stick of length ℓ along x, and radius R in yz plane:

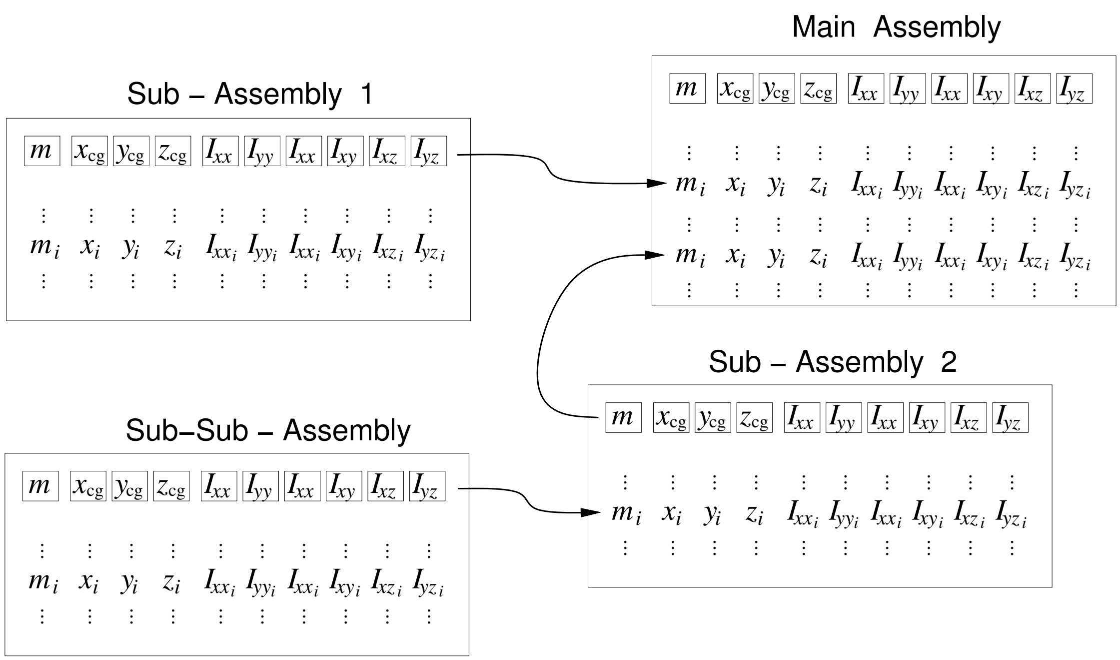
$$I_{xx_i} = \tfrac{1}{2}\, m_i R^2 \tag{33}$$
$$I_{yy_i} = I_{zz_i} = \tfrac{1}{12}\, m_i \ell^2 \tag{34}$$

Rectangular Plate of dimensions $\ell_x \times \ell_y$ lying in xy plane.

$$I_{xx_i} = \tfrac{1}{12}\, m_i\, \ell_y^2 \tag{35}$$
$$I_{yy_i} = \tfrac{1}{12}\, m_i\, \ell_x^2 \tag{36}$$
$$I_{zz_i} = \tfrac{1}{12}\, m_i\, (\ell_x^2 + \ell_y^2) \tag{37}$$

Hierarchical Mass and Inertia Build-Up

Note that the top row in the Table 3 spreadsheet, which is the output of the spreadsheet, has precisely the same form as one of the input data lines in the spreadsheet. This naturally allows a hierarchical build-up of the mass and inertia properties of a very complex collection of masses. Specifically, any one input data line in a spreadsheet can be the output of another lower-level spreadsheet. That lower-level spreadsheet's data lines can in turn be the outputs of still lower-level spreadsheets, and so on. This is illustrated in the figure below.



Moment of Inertia Tensor

The overall symmetric moment of inertia tensor is defined in terms of its six components as follows.

$$\bar{\bar{\mathrm{I}}} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{38}$$

The components can be about the object's center of gravity location $x_{\rm cg}, y_{\rm cg}, z_{\rm cg}$, in which case the components are given by the spreadsheet in Tables 2 or 3. It can also be about some other location $\bar{x}, \bar{y}, \bar{z}$, in which case the components are given by equations (17)–(22).