# Aircraft Longitudinal Dynamics (draft)

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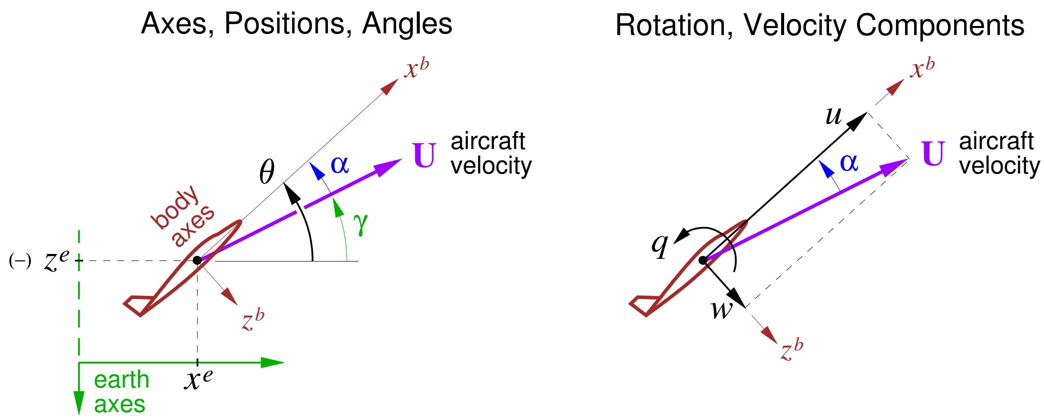
## 1 Variables and Axis Systems

### 1.1 Independent variables

Longitudinal dynamics addresses the motion of the aircraft in the vertical $xz$ plane. This is described by the six dynamical variables diagrammed in the figure and defined in the table below. Both diagrams in the figure show the same aircraft moving with velocity vector $\mathbf{U}$ and having pitch rate $q$, as seen by a ground observer.




| | | |
|---|---|---|
| $u(t)$ | component of aircraft velocity $\mathbf{U}$ along the body $x^b$ axis (positive forward) | |
| $w(t)$ | component of aircraft velocity $\mathbf{U}$ along the body $z^b$ axis (positive down) | |
| $q(t)$ | aircraft pitch rotation rate about its body $y^b$ axis (positive nose up) | (1) |
| $\theta(t)$ | elevation angle of aircraft's $x^b$ axis above the earth's horizontal plane | |
| $x^e(t)$ | horizontal position of aircraft's reference point relative to earth origin | |
| $z^e(t)$ | vertical position of aircraft's reference point relative to earth origin (positive down) | |

The location of the reference point ($x^b, z^b$ origin) on the aircraft is arbitrary. Here we will assume it is at the aircraft's center of gravity, which will give the simplest possible equations of motion.

The aircraft motion depends on any number of control variables. Most aircraft have

| | | |
|---|---|---|
| $\delta_e(t)$ | elevator | |
| $\delta_f(t)$ | flap | (2) |
| $\delta_T(t)$ | "throttle" | |

and there may be others. The "throttle" variable may literally be the power-lever position, or it can alternatively be the engine power, propeller rotational speed, or some other propulsion-related variable.

All the above variables, the notation, and the sign conventions are standard in modern flight dynamics and control theory. The first four variables $u, w, q, \theta$ are collectively referred to as the *longitudinal state vector*. The position coordinates $x^e, z^e$ are considered *navigation* variables, which are typically treated separately from the others, although here it will be convenient to treat all six variables simultaneously. The control variables $\delta_e, \delta_f \ldots$ are collectively referred to as the *longitudinal control vector*.

### 1.2 Dependent variables

The following aerodynamic variables are not independent, but are explicit functions of the state vector, specifically the aircraft velocity components $u, w$, and the vertical position $z^e$ which determines the ambient air density $\rho$.

$$\begin{aligned} \alpha &\equiv \arctan(w/u) && \text{angle of attack} \\ Q &\equiv \tfrac{1}{2}\rho\,(u^2+w^2) && \text{dynamic pressure} \\ V &\equiv \sqrt{u^2+w^2} && \text{true airspeed} \end{aligned} \tag{3}$$

It is also convenient to define the following variables associated with the aircraft trajectory, which depend explicitly on the state vector.

$$\begin{aligned} u^e &\equiv u\,\cos\theta \,+\, w\,\sin\theta && \text{ground speed} \\ w^e &\equiv w\,\cos\theta \,-\, u\,\sin\theta && \text{descent speed (positive down)} \end{aligned} \tag{4}$$

$$\gamma \equiv \arctan(-w^e/u^e) \qquad \text{climb angle} \tag{5}$$

# 2 Force and Moment Characterization

## 2.1 Force components in stability axes

The lift, drag, and pitching moment are quantified via their corresponding coefficients and the associated reference wing area $S$ and mean aerodynamic chord $\bar{c}$,

$$\begin{aligned} L &= QS\,C_L(\alpha,\bar{q},\delta_e,\delta_f,Re,Ma) \\ D &= QS\,C_D(\alpha,\bar{q},\delta_e,\delta_f,Re,Ma) \\ \mathcal{M} &= QS\bar{c}\,C_m(\alpha,\bar{q},\delta_e,\delta_f,Re,Ma) \\ \bar{q} &\equiv q\bar{c}/2V \end{aligned} \tag{6}$$

where $\bar{q}$ is a non-dimensionalized pitch rate. An alternative parameterization for the drag coefficient is $C_D(C_L,\bar{q},\delta_e,\delta_f,Re,Ma)$ where the $C_L$ dependence naturally arises from the induced drag term in a drag build-up.

The thrust force will typically be specified as some function of the throttle variable $\delta_T$. If the aircraft has a propeller, a natural choice for this parameter is the propeller rotation rate, $\delta_T \equiv \Omega$. Propeller theory then gives the thrust as

$$\begin{aligned} T &= \tfrac{1}{2}\rho\,(\Omega R)^2\,\pi R^2\,C_T(\lambda,\text{geometry}) \\ \lambda &= V/\Omega R \end{aligned} \tag{7}$$

where $R$ is the prop radius. The thrust coefficient $C_T$ is a function mainly of the advance ratio $\lambda$, and can be experimentally determined for any given propeller geometry. Note that it is defined with the prop disk area $\pi R^2$, and with the tip dynamic pressure $\frac{1}{2}\rho(\Omega R)^2$ rather than $Q$, so $C_T$ is well defined even at the static-thrust case with zero airspeed.

The streamwise force of a blown wing cannot be easily separated into drag and thrust components. In this case it may be convenient to treat the net streamwise force directly, via a "net drag" coefficient $C_X$.

$$D-T = QS\,C_X(C_L,\delta_T,\bar{q},\delta_f,Re,Ma) \tag{8}$$

If the static case is to be considered, then the following alternative normalization with the prop tip dynamic pressure can be used, like in equation (7).

$$D-T = \tfrac{1}{2}\rho\,(\Omega R)^2\,\pi R^2\,C_X'(C_L,\delta_T,\bar{q},\delta_f,Re,Ma) \tag{9}$$

This will fail, however, if the stopped-propeller case where $\Omega = 0$ is to be considered.

Note that if $C_X$ or $C_X'$ are used to represent a blown wing, there will still be an additional separate $C_D$ term for the profile drag of the other aircraft components, and also for the induced drag. An additional $C_T$ term may also be used if there are propulsors separate from the blown wing system.

## 2.2 Force linearization

For flight dynamics analysis, the coefficient functions in the force and moment formulas (6) are typically approximated via linearization about some *trim state*. Specifically,

$$\begin{aligned} C_L &\simeq C_{L_0} + C_{L_\alpha}\,\Delta\alpha + C_{L_q}\,\Delta\bar{q} + C_{L_{\delta_f}}\,\Delta\delta_f + C_{L_{\delta_e}}\,\Delta\delta_e \\ C_m &\simeq C_{m_0} + C_{m_\alpha}\,\Delta\alpha + C_{m_q}\,\Delta\bar{q} + C_{m_{\delta_f}}\,\Delta\delta_f + C_{m_{\delta_e}}\,\Delta\delta_e \end{aligned} \tag{10}$$

where $(\,)_0$ denotes the trim state value, and $\Delta(\,)$ denotes a change from the trim state. The subscripted coefficients denote *stability derivatives* and *control derivatives*, which are partial derivatives of the functions in (6), evaluated at the trim state, e.g.

$$\begin{aligned} \Delta\alpha &\equiv \alpha - \alpha_0 \\ C_{L_\alpha} &\equiv \left.\frac{\partial C_L}{\partial\alpha}\right|_0 \\ C_{L_q} &\equiv \left.\frac{\partial C_L}{\partial\bar{q}}\right|_0 \qquad \text{etc.} \end{aligned} \tag{11}$$

Additional derivative terms for the Reynolds and Mach number dependence can be included in (10), although these are usually neglected. For dynamics analysis, the $C_D$ is usually not linearized, but is simply assumed constant, i.e. $C_D \simeq C_{D_0}$.

Theoretically, all stability and control derivatives depend on the particular trim state being assumed. For example, $C_{L_\alpha}$ decreases as stall is approached, and passes through zero at the maximum $C_L$ value. $C_{m_\alpha}$ will also typically display some irregular behavior at stall. However, for trim states away from stall the derivatives are generally well behaved, and can be assumed to be constant across most of the aircraft's $\alpha$ range and for modest elevator and flap deflections. The stability and control derivatives in this well-behaved flight regime are typically computed using a Vortex-Lattice model of the aircraft configuration.

## 2.3 Blown lift effects

In the case of blown lift, the coefficients in (6) will also depend on the blowing coefficient $\Delta C_J$ (not to be confused with its deviation from the trim state), which may be tied to the power control variable $\delta_T$, or it may be a separate independent parameter.

$$\begin{aligned} L &= QS\; C_L(\alpha, \bar{q}, \delta_e, \delta_f, \Delta C_J, Re, Ma) \\ D &= QS\; C_D(\alpha, \bar{q}, \delta_e, \delta_f, \Delta C_J, Re, Ma) \\ \mathcal{M} &= QS\bar{c}\; C_m(\alpha, \bar{q}, \delta_e, \delta_f, \Delta C_J, Re, Ma) \end{aligned} \tag{12}$$

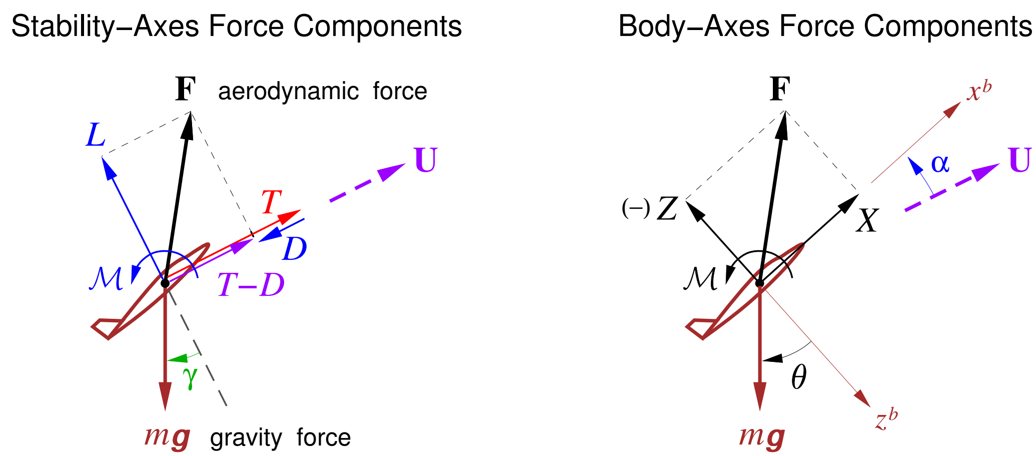
Because the dependence of the coefficients on $\Delta C_J$ is fairly nonlinear, it is generally not very practical to capture its effects by adding another derivative term to each linearized form in (10). Instead, it is more effective to consider all the coefficients be nonlinear functions of $\Delta C_J$. Specifically, we assume,

$$\begin{aligned} C_L &\simeq C_{L_0}(\Delta C_J) + C_{L_\alpha}(\Delta C_J)\,\Delta\alpha + C_{L_q}(\Delta C_J)\,\Delta\bar{q} + C_{L_{\delta_f}}(\Delta C_J)\,\Delta\delta_f + C_{L_{\delta_e}}(\Delta C_J)\,\Delta\delta_e \\ C_m &\simeq C_{m_0}(\Delta C_J) + C_{m_\alpha}(\Delta C_J)\,\Delta\alpha + C_{m_q}(\Delta C_J)\,\Delta\bar{q} + C_{m_{\delta_f}}(\Delta C_J)\,\Delta\delta_f + C_{m_{\delta_e}}(\Delta C_J)\,\Delta\delta_e \end{aligned} \tag{13}$$

where the $C_{L_0}(\Delta C_J), C_{L_\alpha}(\Delta C_J) \ldots$ functions are nonlinear. However, they are fairly smooth and hence can be approximated with simple polynomials fitted to numerical values obtained from aerodynamic simulation calculations. This then allows using the dynamics analysis described here over a wide range of blowing power settings.

### 2.4 Force components in body axes

The $L, D, T$ treated above are the components of the overall aerodynamic force vector $\mathbf{F}$ in stability axes, which are along and normal to the unset flow which is opposite to the aircraft velocity vector $\mathbf{U}$. Referring to the force diagrams below, the alternative components $X, Z$ in the aircraft body axes are defined as follows.




$$
\begin{aligned}
X(t) &\equiv (T-D)\cos\alpha + L\sin\alpha && \mathbf{F} \text{ component along body } x^b \text{ axis (positive forward)} \\
Z(t) &\equiv (T-D)\sin\alpha - L\sin\alpha && \mathbf{F} \text{ component along body } z^b \text{ axis (positive down)}
\end{aligned}
\tag{14}
$$

The pitching moment $\mathcal{M}$ is the same in both axis systems.

# 3 Equations of Motion

### 3.1 Full longitudinal dynamics

The time evolution of the state vector is given by the following six equations of motion, which give the state time rates of change as functions of the current state vector and current control vector.

$$
\boxed{
\begin{aligned}
\dot{u} &= \tfrac{1}{m} X - g\sin\theta - w\,q \\
\dot{w} &= \tfrac{1}{m} Z + g\cos\theta + u\,q \\
\dot{q} &= \tfrac{1}{I_y}\mathcal{M} \\
\dot{\theta} &= q \\
\dot{x}^e &= u^e \\
\dot{z}^e &= w^e
\end{aligned}
}
\tag{15}
$$

These assume that the $x^b, z^b$ origin, which is also the point about which the moment $\mathcal{M}$ is defined, is at the aircraft center of gravity.

The first two equations in (15) above, involving the aircraft mass $m$, express the conservation of linear momentum along $x^b$ and $z^b$. The last terms $-wq$ and $uq$ are d'Alembert forces per mass resulting from the $x^b, z^b$ axes being a non-inertial rotating frame. The third equation in (15), involving the pitch inertia $I_y$, expresses the conservation of angular momentum about $y^b$. The last three equations in (15) are kinematic time-rate definitions.

### 3.2 Numerical solution

Equation system (15) forms six coupled first-order ODEs. Given some initial values for the state vector at $t=0$,

$$u(0) \quad , \quad w(0) \quad , \quad q(0) \quad , \quad \theta(0) \quad , \quad x^e(0) \quad , \quad z^e(0)$$

and control variables time histories $\delta_e(t), \delta_f(t) \ldots$ are also be provided, these ODEs can be numerically integrated in time to determine the resulting aircraft motion. Example applications are:

- Dynamic behavior analysis, where the control variables are either held fixed, or specified to have simple "doublet" inputs
- Flight simulation, where the control variable values are provided by the pilot in real time
- Autopilot behavior analysis, where the control variable values are provided by feedback control laws. These typically have the general forms

$$\begin{aligned} \delta_e &= \delta_{e_{\text{spec}}} + f_e(u, w, q, \theta, x^e, z^e) \\ \delta_T &= \delta_{T_{\text{spec}}} + f_T(u, w, q, \theta, x^e, z^e) \quad \ldots \end{aligned}$$

  where $\delta_{e_{\text{spec}}}$ and $\delta_{T_{\text{spec}}}$ are the pilot's commanded elevator and throttle, and $f_e$ and $f_T$ are the autopilot corrections which are functions of the current state as indicated. An example is a landing autopilot, where $f_e$ and $f_T$ adjust elevator and throttle so as to maintain a required landing approach path as closely as possible.

## 4 Simplified dynamics

If only the aircraft trajectory $x^e(t), z^e(t)$ is required, the equation system (15) can be simplified by assuming that the aircraft is always in pitching-moment equilibrium, with $\alpha$ being specified, either directly or via a specified $C_L$ and the assumed $C_L(\alpha)$ dependence, i.e.

$$\begin{aligned} \alpha &= \alpha_{\text{spec}} \quad \text{or} \\ C_L(\alpha) &= C_{L_{\text{spec}}} \quad \rightarrow \quad \alpha \end{aligned} \tag{16}$$

It is then more convenient to time-integrate the earth-axis velocities $u^e, w^e$ directly. Combining

$$\begin{aligned} &[\dot{u} \text{ equation}] \cos\theta + [\dot{w} \text{ equation}] \sin\theta \\ \text{and} \quad &[\dot{w} \text{ equation}] \cos\theta - [\dot{u} \text{ equation}] \sin\theta \end{aligned}$$

gives the necessary ODEs.

$$\boxed{\begin{aligned} \dot{u}^e &= \frac{1}{m} F_{x^e} \\ \dot{w}^e &= \frac{1}{m} F_{z^e} + g \\ \dot{x}^e &= u^e \\ \dot{z}^e &= w^e \end{aligned}} \tag{17}$$

The earth-axis forces above are comnputed by rotating $L, D, T$,

$$\begin{aligned} F_{x^e} &= (T-D) \cos\gamma + L \sin\gamma \\ F_{z^e} &= (T-D) \sin\gamma - L \sin\gamma \end{aligned} \tag{18}$$

in which the flight-path angle $\gamma$ is obtained directly from its definition (5). The aircraft pitch angle

$$\theta = \gamma + \alpha \tag{19}$$

can also be computed if needed.