ARIEL UNIVERSITY

MASTER THESIS PROPOSAL

# Ramsey Theory in Random Graphs

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# Declaration of Authorship

I, Amir Meir SABAG, hereby declare that this thesis proposal entitled, "Ramsey Theory in Random Graphs" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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# *Abstract*

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**Ramsey Theory in Random Graphs**

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A celebrated result of R¨odl and Ruci´nski states that for every graph $F$, which is not a forest of stars and paths of length 3, and fixed number of colours $r2$ there exist positive constants $c, C$ such that for $p \leq cn^{1/m_(F)}$ the probability that every colouring of the edges of the random graph $\mathbb{G}(n, p)$ contains a monochromatic copy of $F$ is $o(1)$ (the "0-statement"), while for $p \geq Cn1/m_2(F)$ it is $1o(1)$ (the "1-statement"). Here $m_2(F)$ denotes the 2-density of F. On the other hand, the case where F is a forest of stars has a coarse threshold which is determined by the appearance of a certain small subgraph in $\mathbb{G}(n, p)$.

Bucić and Khamseh proved that the restriction of how many colours is permitted in place of one in copy of $F$ is crucial. They showed it is changes the Ramsey's number of for mono-chromatic paths from exponential to linear for (r-1)-chromatic paths.

In our research we will consider the settings of random graphs and randomly perturbed graphs combined with the relaxation of being able to colour copies of subgraphs by more than one colour alone.

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# Chapter 1

# Introduction

Given graphs $G$, $F$ and a positive integer $r$, write $G \to (F)_r$ in order to denote the property that every edge-colouring of $G$ with $r$ colours admits a monochromatic copy of $F$. Ramsey's Theorem [6] asserts that for every two integers $t, r \geq 2$ there exists an integer $N(t,r)$ such that $K_{N(t,r)} \to (K_t)_r$. In particular, Ramsey's Theorem asserts that given $t, r \in \mathbb{N}$, there exists an integer $R_r(t)$ which is the smallest natural number $n$ such that any edge-colouring of $K_n$ with $r$ colours admits a monochromatic copy of $K_t$. The integer $R_r(t)$ is referred to as the *(symmetric) Ramsey number* of $K_t$ with $r$ colours.

The first lower bound for Ramsey's Numbers established by Erdős [1] and with Szekeres they established an upper bound as well [15]

$$2^{t/2} \leq R_2(t) = O\left(\frac{4^t}{\sqrt{t}}\right). \tag{1.1}$$

The lower bound seen in (1.1), was improved using the (symmetric) *Lovász local lemma* [1, 7] as to yield $R_2(t) \geq (1+o(1))\frac{t\sqrt{2}}{e}2^{t/2}$. More recently, Campos, Griffiths, Morris and Sahasrabudhe [14] proved that $R_2(t) \leq (4-\varepsilon)^t$ with $\varepsilon \leq 2^{-7}$.

**Ramsey properties of random graphs.** The study of Ramsey properties of random graphs was initiated by Łuczak, Ruciński and Voigt [13] and continued in [11, 17, 18, 20, 21] (see references therein for additional details). In this venue, the threshold for the property $G \to (F)_r$ is sought after, where $G \sim \mathbb{G}(n,p)$, $F$ is a fixed graph and, $r \in \mathbb{N}$; we write $\mathbb{G}(n,p) \to (F)_r$ in the sequel.

Rödl and Ruciński [20, 21] determined the threshold for the property $\mathbb{G}(n,p) \to (F)_r$ for the general case of graph $F$. The remaining cases were handled by Friedgut and Krivelevich which determined the threshold for graph $F$ as forest of stars [8] (see Theorem 1.1). To state their results, we require some additional notation.

For a graph $G$ set

$$d_2(G) := \begin{cases} 0 & \text{if } e(G) = 0, \\ 1/2 & \text{if } e(G) = 1, v(G) = 2, \\ \frac{e(G)-1}{v(G)-2} & otherwise; \end{cases}$$

and

$$m_2(G) := \max_{H \subseteq G} d_2(H).$$

The notation $m_2(G)$ is referred to as the *m*aximum 2-*density* of $G$. If $d_2(G) = m_2(G)$, then $G$ is called 2-*balanced*, and is referred to as *s*trictly 2-balanced if all proper subgraphs of $G$ have smaller 2-density than $G$ itself, this is, $d_2(G') < d_2(G)$ whenever $G' \subsetneq G$.

We are now in a position to state the threshold result [8, 20, 21].

**Theorem 1.1.** *Let $2 \leq r \in \mathbb{N}$ and let $F$ be a graph with at least one edge.*

- *If $F$ is a forest of stars, then*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{G}(n,p) \to (F)_r] = \begin{cases} 1 & \text{if } p \gg n^{-1-1/(r(\Delta(F)-1)+1)} \\ 0 & \text{if } p \ll n^{-1-1/(r(\Delta(F)-1)+1)} \end{cases} \tag{1.2}$$

- *If $r = 2$ and $F$ is a forest of stars and at least one path with exactly 3 edges, then there exists a constant $C$ such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{G}(n,p) \to (F)_r] = \begin{cases} 1 & \text{if } p \geq Cn^{-1/m_2(F)} \\ 0 & \text{if } p \ll n^{-1/m_2(F)} \end{cases} \tag{1.3}$$

- *In all other cases there exist constants $c = c(F,r)$ and $C = (F,r)$ such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{G}(n,p) \to (F)_r] = \begin{cases} 1 & \text{if } p \geq Cn^{-1/m_2(F)} \\ 0 & \text{if } p \leq cn^{-1/m_2(F)}. \end{cases} \tag{1.4}$$

The expected number of copies of graph $F$ in $G \sim \mathbb{G}(n,p)$ is $\Theta(n^{v(F)}p^{e(F)})$ and the expected number of edges in $G$ is $\Theta(n^2p)$. Theorem 1.1 asserts that for a typical host graph $G$ the transition from the 0-statement to the 1-statement occurs when the value of $p$ is such that, these two quantities are roughly equal. This can be explained by the following intuition: if every copy of $F$ contains an edge which does not belong to any other copy of $F$, then by colouring all such edges red and every other edge with another colour, clearly yields a colouring without a monochromatic copy of $F$ in $G$. If on the other hand, each edge is contained in many copies of $F$, then these copies overlap heavily and a monochromatic copy is unavoidable.

**Ramsey properties of random hypergraphs.** Consider the random hypergraph model $\mathbb{H}_{n,p;k}$. A random $k$-uniform hypergraph on vertex set $[n]$ and $m$ edges of size $k$. Where each of the $\binom{n}{k}$ possibles edge occurs independently with probability $p$. For comparison every simple graph is 2-uniform hypergraph.

Following this intuition of Theorem 1.1, Rödl and Ruciński [22] conjectured that a monochromatic copy of $F$ appears in every colouring (with a fixed number of colours) whenever the expected number of copies of $F$ per hyperedge exceeds a large constant. Rödl, Ruciński and Schacht [19] proved that for $k$-partite $k$-uniform hypergraphs. Friedgut, Rödl and Schacht [9] proved the conjectured 1-statement for all $k$-uniform hypergraphs. Similar results were obtained independently by Conlon and Gowers [4]. To state their result, we require additional notation.

For a k-uniform hypergraph $H$ set

$$d_k(H) := \begin{cases} 0 & \text{if } e(H) = 0, \\ 1/k & \text{if } e(H) = 1, v(H) = k, \\ \frac{e(H)-1}{v(H)-k} & otherwise; \end{cases}$$

and

$$m_k(H) := \max_{H' \subseteq H} d_k(H').$$

**Theorem 1.2** ([9]). *Let $F$ be a $k$-uniform hypergraph with maximum degree at least 2 and let $r$ be an integer. There exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{H}_{n,p;k} \to (F)_r] = 1$$

*whenever $p \geq Cn^{-1/m_k(F)}$.*

A complete characterisation of the threshold, as in Theorem 1.1 is still not known. Moreover, Nenadov and Person [10] proved that there exist hypergraphs for which the threshold is neither the conjectured $n^{-1/m_k(F)}$ nor is it determined by the appearance of a small sub-hypergraph.

**Theorem 1.3.** *For every $k \geq 4$ there exist a $k$-uniform hypergraph $F$ and positive constants $1 < \Theta < m_k(F)$ and $c, C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{H}_{n,p;k} \to (F)_2] = \begin{cases} 1 & \text{if } p \geq Cn^{-1/\Theta} \\ 0 & \text{if } p \leq cn^{-1/\Theta} \end{cases}. \tag{1.5}$$

**Ramsey properties of randomly perturbed graphs.** A random *perturbation* of a fixed graph $G$ on $n$ vertices, denoted by $G \cup \mathbb{G}(n,p)$, is a distribution over the $n$-vertex supergraphs of $G$. The elements of such distribution are generated via the addition of randomly sampled edges to $G$. The fixed graph $G$ being *perturbed* or *augmented* in this manner is referred to as the *seed* of the *perturbation* (or *augmentation*) $G \cup \mathbb{G}(n,p)$. The model of randomly perturbed graphs was introduced by Bohman, Frieze and Martin [23], who allowed the seed $G$ to range over the family of graphs on $n$ vertices with minimum degree at least $\delta n$, which we denote here by $\mathcal{G}_{\delta,n}$. In particular, they discovered the phenomenon that for every $\delta > 0$, there exists a constant $C(\delta) > 0$ such that $G \cup \mathbb{G}(n,p)$ a.a.s. admits a Hamiltonian cycle, whenever $p := p(n) \geq C(\delta)/n$ and $G \in \mathcal{G}_{\delta,n}$.

Note that the value of $p$ attained by their result is smaller by a logarithmic factor than that required for the emergence of Hamilton cycles in $\mathbb{G}(n,p)$. That is, while $G$ itself might not be Hamiltonian, making it Hamiltonian requires far fewer random edges than the number of random edges which typically form Hamilton cycle by themselves. The notation $\mathcal{G}_{\delta,n} \cup \mathbb{G}(n,p)$ then suggests itself to mean the collection of perturbations arising from the members of $\mathcal{G}_{\delta,n}$ for a prescribed $\delta > 0$.

Another prominent line of research regarding random perturbations concerns Ramsey properties of $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$, where here $\mathscr{G}_{d,n}$ stands for the family of graphs of with $n$ vertices and with edge-density at least $d > 0$, where $d$ is a constant. This strand stems from the work of Krivelevich, Sudakov, and Tetali [12]. This line of research is heavily influenced by the now fairly mature body of results regarding the thresholds of various Ramsey properties in random graphs.

As previously noted, Krivelevich, Sudakov, and Tetali [12] were the first to study Ramsey properties of random perturbations. In particular, they proved that for every real $d > 0$, integer $t \geq 3$, and graph $G \in \mathscr{G}_{d,n}$, the perturbation $G \cup \mathbb{G}(n,p)$ a.a.s. satisfies the property $G \cup \mathbb{G}(n,p) \to (K_3, K_t)$, whenever $p := p(n) = \omega(n^{-2/(t-1)})$; moreover, this bound on $p$ is asymptotically best possible. Here, the notation $G \to (H_1, ..., H_r)$ is used to denote that $G$ has the asymmetric Ramsey property asserting that any $r$-edge-colouring of $G$ admits a colour $i \in [r]$ such that $H_i$ appears with all its edges assigned the colour $i$.

Additionally, they studied the so-called containment problem of small prescribed graphs in such perturbations. The emergence of complete graphs of fixed size in this model was studied earlier by Bohman, Frieze, Krivelevich and Martin [24] who determined the relevant thresholds.

Recently, the aforementioned result of Krivelevich, Sudakov, and Tetali [12] has been significantly extended by Das and Treglown [5] and also by Powierski [16]. In particular, there is now a significant body of results pertaining to the property $G \cup \mathbb{G}(n,p) \to (K_r, K_s)$ for any pair of integers $r, s \geq 3$, whenever $G \in \mathscr{G}_{d,n}$ for constant $d > 0$.

# Chapter 2

# The t-chromatic Ramsey number

Given graphs $G$ and $F$, and let $t, r$ be an integers, $r > t \geq 1$. We denote by

$$G \to (F)^t_r$$

the property that every colouring of edges of $G$ with $r$ colours contains copy of $F$ coloured at most by $t$ colours. And say that there is $t$-chromatic copy of $F$ in $G$.

Note that $G \to (F)^1_r$ is exactly $G \to (F)_r$ from Theorem 1.1. And for $G \to (F)^r_r$, it is the threshold for instance of at least one copy of $F$, as there is no constraints regarding the colouring (see [2]).

The Ramsey's Number for paths graphs where $t = r-1$ has been studied by Bucić and Khamesh [3]. They found that the least $n$ for $\ell, r \geq 2$ such that $K_n \to (P_\ell)^{r-1}_r$ which is $n = \ell + \lfloor \frac{\ell-2}{2^r-2} \rfloor$. Where $P_\ell$ is the path graph size $\ell$.

For every $1 < t \leq r-1$ these numbers has not been studied for random graphs, which we would expand on, specifically the cases:

- $t = 2$ and fixed $k$ for which $\mathbb{G}(n,p) \to (K_k)^2_r$.
- $t = r-2$ and fixed $\ell$ for which $\mathbb{G}(n,p) \to (P_\ell)^t_r$

Given (fixed) graph $G$ with n vertexes, smaller graph $F$, and $0 < p < 1$. $G \cup \mathbb{G}(n,p)$ is a distribution over the supergraphs of $G$. Let $t, r$ be an integers. And $r \geq t \geq 1$. denote by

$$G \cup \mathbb{G}(n,p) \xrightarrow{t} (F)^i_r$$

the property that $G \cup \mathbb{G}(n,p)$ a.a.s assimilate a copy of $F$ coloured by not more than $t$ colours for any $r$-edge-colouring.

Our first goal is to determine a sharp threshold for the following problem:

**Problem 1.** *Given $r, t, k$ be an integers, $r > t > 1$, find $p := p(n)$ and $c, C \in \mathbb{R}$ constants such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{G}(n,q) \to (K_k)^t_r] = \begin{cases} 1 & \text{if } q \geq Cp \\ 0 & \text{if } q \leq cp. \end{cases}$$

The case $t = 2$ will be our starting point before attempting to solve the general case. In order to tackle the above problem, the following vertex version of it is of use.

Let $t, r$ be an integers, and let $G, F$ graphs. Denote $G \xrightarrow{v} (F)^t_r$ if $G$ assimilate $t$-chromatic copy of $F$ for any $r$-vertex-colouring.

**Problem 2.** *Given $r, t, k$ be an integers, $r > t > 1$, find $p := p(n)$ and $c, C \in \mathbb{R}$ constants such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{G}(n,q) \to (K_k)^t_r] = \begin{cases} 1 & \text{if } q \geq Cp \\ 0 & \text{if } q \leq cp. \end{cases}$$

Another natural generalization of Problem 1 is its statement for a general graph $H$ and not for cliques alone. Formally,

**Problem 3.** *Given $r, t$ be an integers, $r > t > 1$, and graph $F$. Find $p := p(n)$ and $c, C \in \mathbb{R}$ constants such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{G}(n,q) \to (F)_r^t] = \begin{cases} 1 & \text{if } q \geq Cp \\ 0 & \text{if } q \leq cp. \end{cases}$$

In order to tackle Problem 3, a generalization for graph $H$ of Problem 2 to handle general graphs is needed.

We sketch our methodology for solving Problem 1.

1. Sample $G \sim \mathbb{G}(n,p)$ and fix a colouring of $G$.
2. Define a typical vertex in $G$ as every vertex $v \in V(G)$ such that $deg(v) \leq (1 \pm \varepsilon)np$.
3. For every typical vertex $v_i$ denote it as *good* if $G[N_G(v_i)] \overset{v}{\not\to} (K_{k-1})_r^2$ and otherwise denote it as *bad*.
4. Given any sort of the vertices of $G$ such that $V(G) := \{v_1, ..., v_n\}$. For every $i \in [n]$ for which $v_i$ is *good* colour the edges of $G[N_G(v_i)]$ according to the next procedure:

$$\forall j \in [N_G(v_i)] \quad c(v_j, u) := \begin{cases} c(u) \text{ if } v_j, u \text{ not colored yet} \\ \text{no change if } v_j, u \text{ already colored} \end{cases} \tag{2.1}$$

5. Denote $B$ :=$\{v \in V(G) \mid v$ is *bad vertex*$\}$. Show that $\mathbb{E}[|B|] \leq C_B$ for some constant $C_B \in \mathbb{N}$, And the density of $G[B]$, is not more than $O(1)$.
6. Show that a graph with a density smaller or equal to the one of $G[B]$ cannot contain any copy of $K_k$, and definitely not a 2-chromatic copy.

In order to use the proof sketch above, the following main claims shall be proven.

1. Step 2 of proof sketch is directly holds from Chernoff inequality.
2. Prove that the colouring procedure 2.1 of Step 4 of the proof sketch ensures that no *good* vertex can be part of any $K_k$ that is $r$-edge-colored.
3. Prove the following claim.
   Let $k \geq 3, t = 2$, and $r > 1$ there exist $a > 0, b := b(k,p,t,r)$ such that if $p = an^{-2/k}$ then:

   $$\mathbb{P}[\mathbb{G}(n,p) \xrightarrow{v} (K_k)_r^2] = O(n^{-b})$$

   where $b$ is big enough such that step 4 will leave only $O(1)$ bad vertices in *expectation* that not coloured by the colouring Procedure 2.1. Markov inequality ensure that the amount of bad vertices bounded by $\omega(1)$ slow function.
4. Prove that given a graph $F$ that satisfies: $\chi(F) \geq 3$ if $G \to (F)_r^2$ with $v(G) = \omega(1)$ vertices then the density of $G$ is lower bounded by constant $C \in \mathbb{N}$. Which should be the density of $G[B]$ of Step 6.

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