ریاضیات مهندسی

جلسه ۱۵

نکته: اگر $\mathcal{F}(f(x)) = F(\omega)$ آنگاه $\mathcal{F}(x f(x)) = -i \dfrac{d}{d\omega} F(\omega)$

مثال: می‌دانیم $\mathcal{F}(e^{-|x|}) = \sqrt{\dfrac{2}{\pi}} \dfrac{1}{1+\omega^2}$ ، درنتیجه:

$$\mathcal{F}(x e^{-|x|}) = -i \frac{d}{d\omega}\left(\sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}\right) = -i \sqrt{\frac{2}{\pi}} \frac{-2\omega}{(1+\omega^2)^2} =$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{i\omega}{(1+\omega^2)^2}$$

مثال: مطلوب است $\mathcal{F}(e^{-kx^2})$ $(k>0)$

$$f(x) = e^{-kx^2} \Rightarrow f'(x) = -2kx\, e^{-kx^2} = -2kx\, f(x)$$

$$\Rightarrow f'(x) = -2kx\, f(x) \Rightarrow \mathcal{F}(f'(x)) = -2k\, \mathcal{F}(x f(x))$$

$$\Rightarrow -i\omega F(\omega) = -2k\left(-i \frac{d}{d\omega} F(\omega)\right)$$

$$\omega F(\omega) = -2k F'(\omega) \qquad \frac{F'(\omega)}{F(\omega)} = -\frac{\omega}{2k}$$

$$\int \frac{F'(\omega)\, d\omega}{F(\omega)} = \int -\frac{\omega}{2k}\, d\omega \rightarrow \ln F(\omega) = -\frac{\omega^2}{4k} + C_1$$

$$F(\omega) = C e^{-\frac{\omega^2}{4k}} \qquad \omega = 0 \Rightarrow C = F(0)$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt$$

$$\omega=0 \Rightarrow F(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\,dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-kt^2}\,dt=$$

$$\sqrt{k}\,t=z \rightarrow dt=\frac{1}{\sqrt{k}}\,dz$$

$$=\frac{1}{\sqrt{2k\pi}}\int_{-\infty}^{+\infty} e^{-z^2}\,dz=\frac{1}{\sqrt{2k\pi}}\int_{-\infty}^{+\infty} e^{-z^2}\,dz=$$

$$=\frac{1}{\sqrt{2k\pi}}\sqrt{\pi}=\frac{1}{\sqrt{2k\pi}}$$

$$\mathcal{F}\left(e^{-kx^2}\right)=\frac{1}{\sqrt{2k}}\,e^{-\frac{\omega^2}{4k}}$$

نتیجه:

$$\mathcal{F}^{-1}\left(e^{-\frac{\omega^2}{4k}}\right)=\sqrt{2k}\,e^{-kx^2}$$

$$\mathcal{F}^{-1}\left(e^{-t\omega^2}\right)=\sqrt{\frac{1}{2t}}\,e^{-\frac{x^2}{4t}}$$

مثال: معادله‌ی زیر را به کمک تبدیل فوریه حل کنید:

$$\begin{cases} u_t=u_{xx} & -\infty<x<+\infty \quad t>0 \\ u(x,0)=\sqrt{\frac{\pi}{2}}\,e^{-|x|} \end{cases}$$

فرض کنیم $U(\omega,t)$، تبدیل فوریه‌ی $u(x,t)$ نسبت به x باشد.

$$U_t = U_{xx} \Rightarrow \mathcal{F}(U_t) = \mathcal{F}(U_{xx})$$

$$\Rightarrow U_t(\omega,t) = -\omega^2 U(\omega,t)$$

$$\begin{cases} U_t(\omega,t) = -\omega^2 U(\omega,t) \\ U(\omega,0) = \dfrac{1}{1+\omega^2} \end{cases}$$

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$$U(\omega,t) = Ce^{-\omega^2 t} \quad \Rightarrow U(\omega,t) = \frac{1}{1+\omega^2} e^{-\omega^2 t}$$

$$C = U(\omega,0)$$

$$u(x,t) = \mathcal{F}^{-1}\left(\frac{1}{1+\omega^2} e^{-t\omega^2}\right) = \mathcal{F}^{-1}\left(\frac{1}{1+\omega^2}\right) * \mathcal{F}^{-1}\left(e^{-t\omega^2}\right) =$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-|x|} * \sqrt{\frac{1}{2t}}\, e^{-\frac{x^2}{4t}} =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\frac{\pi}{2}}\, e^{-|x-z|} \sqrt{\frac{1}{2t}}\, e^{-\frac{z^2}{4t}}\, dz$$

$$u(x,t) = \frac{1}{2\sqrt{2}\,t} \int_{-\infty}^{+\infty} e^{-|x-z| - \frac{z^2}{4t}}\, dz$$