Subject: جلسه فوق العاده یکشنبه خانم نوری (سری سیزدهم)

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1_ الف) در ~~[illegible]~~ تابعی که در معادله لاپلاس صدق کند همساز است

و با فرض $f = u + iv$ و u همساز است اگر با استفاده از روابط کشی ریمان

v برای آن بدست بیاوریم به آن مزدوج همساز می گویند.

$u = \theta + r^2\cos 2\theta \Rightarrow$ روابط کشی ریمان $\begin{cases} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta \end{cases}$

$u_r = 2r\cos 2\theta \Rightarrow v_\theta = r u_r = 2r^2\cos 2\theta$

~~[illegible]~~ $v = r^2\sin 2\theta + h(r)$

$\xRightarrow{\text{مشتق نسبت به } r} 2r\sin 2\theta + h'(r) = -\frac{1}{r}u_\theta = -\frac{1}{r} + 2r\sin 2\theta$

$\Rightarrow h'(r) = -\frac{1}{r} \Rightarrow h(r) = -\ln r + c$

$\Rightarrow v = r^2\sin 2\theta - \ln r + c$

ب) $xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow u_x = \frac{1}{x}$ روابط کشی ریمان $\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$

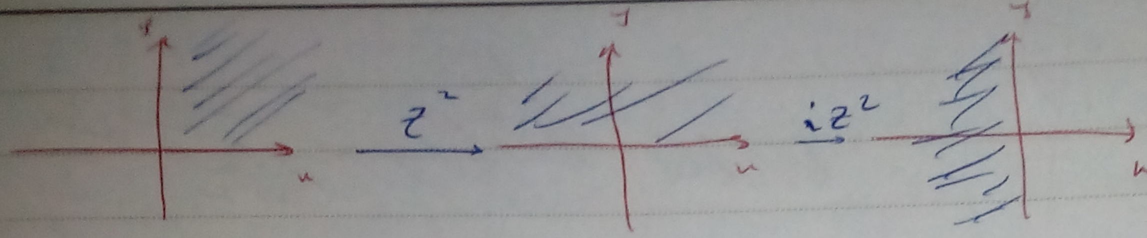
$v_y = y$

$\Rightarrow u_x = \frac{1}{x} = y = v_y$, $u_y = -v_x = 0$

$\Rightarrow u = \ln x$ $\quad v = \frac{y^2}{2}$ $\quad f = u + iv = \ln x + i\frac{y^2}{2}$

$f' = u_x + iv_x = \frac{1}{x} = y$

2.



$w = iz^2 \,,\; w = x + iy \Rightarrow e^{w} = e^{x+iy} = e^{x}(\cos y + i\sin y)$

$e^{iz^2} = u + iv \rightarrow u = e^{a}\cos y \quad v = e^{a}\sin y$

حال با توجه به شکل 3 تصویر $x=a$ برای $a<0$ بصورت زیر خواهد بود

تصویر خط $x=a$ برای $a<0$ تحت نگاشت فوق ⇐ $u^2 + v^2 = e^{2a}$

دایره ای به مرکز مبدا و شعاع e^{a} است. از آنجایی که $a<0$ پس $e^{a}<1$ پس

تصویر خط $x=a$ داخل دیسک واحد است.

همدیس بودن !!

$f = e^{iz^2} \rightarrow f'(z) \neq 0 \Rightarrow f'(z) = i2z e^{iz^2} \neq 0 \longrightarrow z \neq 0$

که با توجه به ناحیه تعریف شده $z\neq 0$ در ناحیه وجود ندارد

یک به یک بودن : $f(z_1) = f(z_2) \Rightarrow z_1 = z_2 \Rightarrow e^{iz_1^2} = e^{iz_2^2}$

$z_1 = x_1 + iy_1 \Rightarrow z_1^2 = x_1^2 - y_1^2 + 2x_1y_1 i$

$z_2 = x_2 + iy_2 \Rightarrow z_2^2 = x_2^2 - y_2^2 + 2x_2y_2 i$

$\Rightarrow e^{(x_1^2-y_1^2)i - 2x_1y_1} = e^{(x_2^2-y_2^2)i - 2x_2y_2}$

یک به یک نیست

$\Rightarrow$ موهومی: $x_1^2 - y_1^2 = x_2^2 - y_2^2 + 2k\pi$

حقیقی: $x_1y_1 = x_2y_2$

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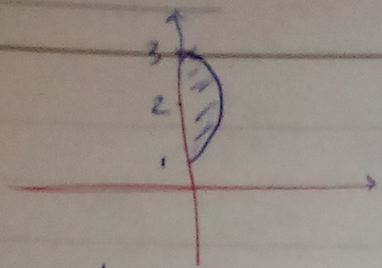
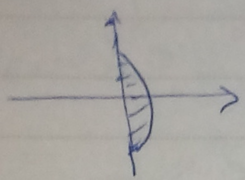
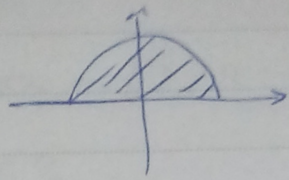


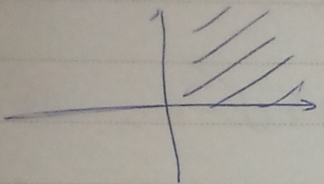
$w_1 = z - 2i$



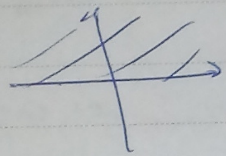
$w_2 = i w_1$



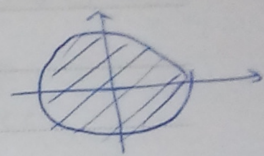
$w_3 = \dfrac{1 + w_2}{1 - w_2}$



$w_4 = w_3^2$



$w_5 = \dfrac{w_4 - i}{w_4 + i}$



$$\Rightarrow w = \frac{\dfrac{[i(z-2i)+1]^2}{[1-i(z-2i)]^2} - i}{\dfrac{[i(z-2i)+1]^2}{[1-i(z-2i)]^2} + i}$$

4- الف) $I_1 = \oint_{|z|=1} y^{51}\, dx$ $\qquad z\bar{z} = 1 \rightarrow \bar{z} = \dfrac{1}{z}$

$z = \cos\theta + i\sin\theta = e^{i\theta}$

$$y = \frac{z + \bar{z}}{2i} = \frac{z - \frac{1}{z}}{2i} = \sin\theta$$

$$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$I = \int y^{51}\,dx = \int (\sin\theta)^{51}(-\sin\theta)\,d\theta$$

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$$I = -\int \left(\frac{z - \frac{1}{z}}{2i}\right)^{52} \frac{dz}{iz} = i\int \frac{1}{z}\, \frac{\left(z - \frac{1}{z}\right)^{52}}{2^{52}} \times dz$$

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تغییر متغیرها

$$= \frac{i}{2^{52}} \oint \frac{1}{z}\left(z - \frac{1}{z}\right)^{52} dz \overset{\text{تغییر متغیرها}}{=} \frac{i}{2^{52}}\, 2\pi i \operatorname{Res}\left(\frac{1}{z}\left(z-\frac{1}{z}\right)^{52}, 0\right)$$

$$\left(z - \frac{1}{z}\right)^{52} = \sum_{k=0}^{52} \binom{52}{k} z^{52-k} \times \frac{1}{z^{k}} = \sum \binom{52}{k} z^{52-2k}$$

$$\Rightarrow \frac{1}{z}\left(z - \frac{1}{z}\right)^{52} = \sum \binom{52}{k} z^{51-2k} \rightarrow$$ با قرار دادن $k=26$ ضریب $\frac{1}{z}$ را بدست می‌آوریم.

ب)

$$I = \oint_{|z|=1/2} \frac{\sin\frac{1}{z}}{1+z}\, dz = 2\pi i \operatorname{Res}\left(\frac{\sin\frac{1}{z}}{1+z}, z=0\right)$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + z^4 - \cdots$$

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \frac{1}{7!\,z^7} + \cdots$$

$$\Rightarrow \left(\frac{1}{z+1}\right)\sin\frac{1}{z} = \left(1 - z + z^2 + \cdots\right)\left(\frac{1}{z} - \frac{1}{3!\,z^3} - \cdots\right)$$

$$\Rightarrow \text{ضریب } \frac{1}{z} = \operatorname{Res} = \left(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots\right) = \sin 1$$

$$\rightarrow I = 2\pi i \sin 1$$

5) الف)

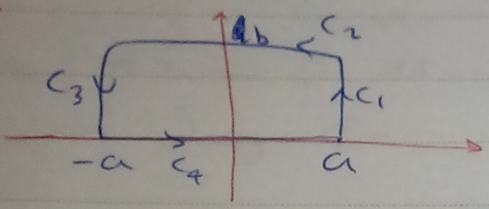
$$f(z) = \frac{3z^2 - 6z + 2}{z^3 - 3z^2 + 2z} = \frac{1}{z-1} + \frac{1}{z-2} + \frac{1}{z}$$

$$1 < |z| \longrightarrow \frac{1}{z\left(1 - \frac{1}{z}\right)} = \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} \overset{\left|\frac{1}{z}\right| < 1}{=\!=} \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} \cdots$$

ب) $\dfrac{1}{z-2} = \dfrac{-1/2}{1-z/2} = -\dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n \xrightarrow[\text{از طرفین}]{\text{مشتق}}$

$$\frac{1}{(z-2)^2} = \frac{1}{4}\sum_{n=1}^{\infty} n\left(\frac{z}{2}\right)^{n-1}$$

6-



الف) $\oint_C e^{-z^2}\,dz = 0$

طبق قضیه کوشی

$$\text{ب)}\ \int e^{-z^2}dz = \overset{①}{\int_0^b e^{-(a+iy)^2}\, i\,dy} + \int_a^{-a} e^{-(x+ib)^2}dx$$

$$+ \overset{②}{\int_b^0 e^{-(-a+iy)^2}\, i\,dy} + \int_{-a}^{a} e^{-(x)^2} = 0$$

$$a \to +\infty \Rightarrow ①, ② = 0 \Rightarrow -\int_{-\infty}^{\infty} e^{-(x+ib)^2}dx + \int_{-\infty}^{\infty} e^{-x^2}dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-(x^2-b^2+2xbi)}dx = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\Rightarrow e^{b^2}\int_{-\infty}^{\infty} e^{-(x^2+2xbi)}dx = \sqrt{\pi} \Rightarrow$$

$$e^{b^2}\int_{-\infty}^{\infty} e^{-x^2}\big(\underbrace{\cos 2xb}_{\text{زوج}} + \underbrace{i\sin 2xb}_{\text{فرد} \Rightarrow 0}\big)dx = \sqrt{\pi}$$

$$\Rightarrow e^{b^2} \times 2\int_0^{\infty} e^{-x^2}\cos 2xb\,dx = \sqrt{\pi}$$

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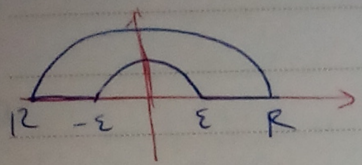
7- $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x(x^2-2x+2)}$ $\qquad f(z)=\dfrac{1}{z(z^2-2z+2)}$

$\to z=0,\ 1\pm i$



$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} |f(z)dz|$

$= \displaystyle\int_{C_R} \frac{dz}{|z||z^2-2z+2|} = \int_{C_R}\frac{dz}{R|z^2-2z+2|}$

$|a|-|b| \le |a-b| \le |a|+|b|$

$\Rightarrow R^2 - \underbrace{|2z-2|} \le |z^2-2z+2|$

$\hookrightarrow |2z-2| \le 2|z|+2 \Rightarrow -|2z-2| \ge -2|z|-2 = -2R-2$

$\Rightarrow R^2-2R+2 \le |z^2-2z+2|$

$\Rightarrow \left|\int_{C_R} f(z)dz\right| \le \dfrac{\pi}{R(R^2-2R-2)}$ if $R\to\infty$ $=0$

$\int_{C_\varepsilon} f dz = -\pi i\, Res\{f(z),0\} = -\frac{\pi}{2}i$

$\displaystyle\int_{-R}^{-\varepsilon} f + \int_{\varepsilon}^{R} f + 0 - \frac{i\pi}{2}$ $\Leftarrow \varepsilon\to 0,\ R\to\infty$ حال با توجه به اینکه

$= 2\pi i\, Res(f, 1+i) \Rightarrow \displaystyle\int_{-\infty}^{0} f + \int_{0}^{\infty} f = \frac{i\pi}{2}$

$+ 2\pi i\, Res(f,1+i) \Rightarrow \displaystyle\int_{-\infty}^{\infty} f = \frac{i\pi}{2} + 2\pi i\, Res\left(\frac{1}{z(z-i+1)}\right)\Big|_{z=1+i}$

$= \dfrac{i\pi}{2} + 2\pi i\, \dfrac{1}{2(1+i)}$

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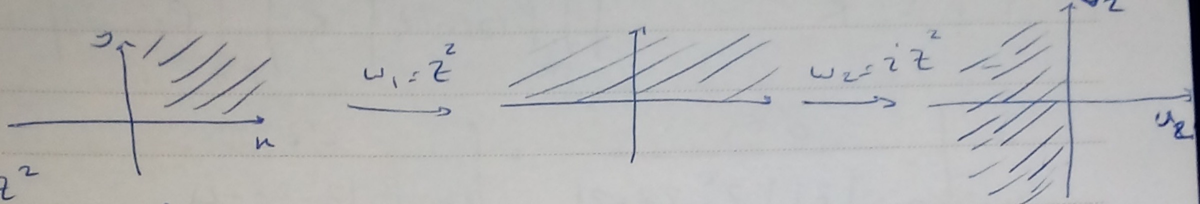
جلسه فوق العاده دوشنبه آقای مظفری (سری 13)

1 - ب) $xy = 1 \to y = \frac{1}{x}$ $\left\{ \begin{array}{l} u_x = \frac{1}{x} \\ v_y = y \end{array} \right. \Rightarrow \left\{ \begin{array}{l} u = \ln x \\ v = \frac{y^2}{2} \end{array} \right.$

طبق روابط کشی ریمان

$f'(z) = u_x + i v_x = \frac{1}{x} = y$

2 - الف)



$w = e^{iz^2}$

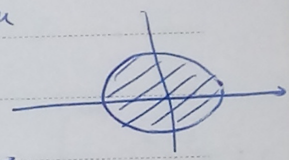
$\Rightarrow \quad e^{iz^2} = \cos(z^2) + i\sin(z^2)$

در نظر بگیریم

$x = a$ $\quad$ if $w = e^{z} \Rightarrow w = e^{x+iy} = \underbrace{e^{x}\cos y}_{u} + i\,\underbrace{e^{x}\sin y}_{v}$

$(a<0)$

$\left\{ \begin{array}{l} u = e^{a}\cos y \\ v = e^{a}\sin y \end{array} \right. \Rightarrow u^2 + v^2 = e^{2a} \Rightarrow r = e^{a}$



$e^{iz^2} \Rightarrow |w| = e^{u_2} \to -\infty < u_2 < 0$

راه دوم : $w_3 = e^{iz^2}$

$Arg(w) = v_2 \qquad -\pi < v_2 < \pi$

ب) $f(z) = e^{iz^2} \Rightarrow f'(z) = 2ize^{iz^2} \Rightarrow f'(z) = 0 \Rightarrow z = 0$

هرجا به جز این نقطه همدیس است

$z_1 = x_1 + iy_1 \qquad f(z_1) = e^{iz_1^2} = e^{i(x_1^2 - y_1^2) - 2x_1y_1}$

$\Rightarrow$

$z_2 = x_2 + iy_2 \qquad f(z_2) = e^{iz_2^2} = e^{i(x_2^2 - y_2^2) - 2x_2y_2}$

$x_1y_1 = x_2y_2$

$\Rightarrow$

$x_1^2 - y_1^2 = x_2^2 - y_2^2 + 2k\pi \longrightarrow$ پس یک به یک نیست

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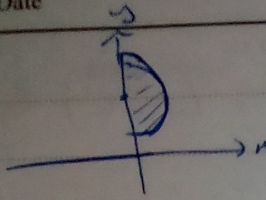
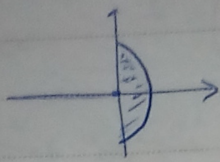
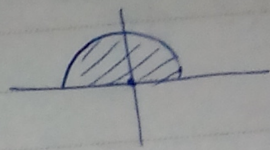


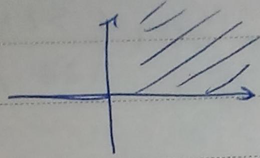
$\xrightarrow{w_1 = z-2i}$



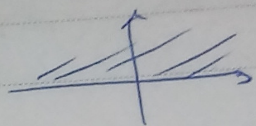
$\xrightarrow{w_2 = e^{\frac{\pi}{2}i} w_1}$



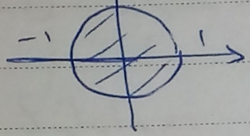
$\xrightarrow{w_3 = \frac{1+w_2}{1-w_2}}$



$\xrightarrow{w_4 = w_3^2}$



$\xrightarrow{w_5 = \frac{w_4 - i}{w_4 + i}}$



$$w = \frac{\dfrac{\left(i(z-2i)-1\right)^2}{\left(1-i(z-2i)\right)^2} - i}{\dfrac{\left(i(z-2i)+1\right)^2}{\left(1-i(z-2i)\right)^2} + i}$$

4- الف) $I = \oint_{|z|=1} y^{51}\, dx = ?$ $\quad z = re^{i\theta} = e^{i\theta} \Rightarrow \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$

$z = x + iy \rightarrow dx = -d\theta \sin\theta$

$\Rightarrow I = \int_0^{2\pi} (\sin\theta)^{51}(-\sin\theta)\, d\theta = -\int_0^{2\pi} (\sin\theta)^{52}\, d\theta$

$y = \frac{z - \bar{z}}{2i} \overset{|z|=1}{\Longrightarrow} \frac{z - 1/z}{2i} \qquad dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{ie^{i\theta}}$

$\Rightarrow I = -\oint_{|z|=1} \frac{(z - \frac{1}{z})^{52}}{2^{52}\, i^{52}} \frac{dz}{iz} = \frac{i}{2^{52}} \oint_{|z|=1} \frac{1}{z}\left(z - \frac{1}{z}\right)^{52} dz$

[illegible]: $I = 2\pi i\, a_{-1} \quad , \quad a_{-1} = \operatorname{Res}\left[\frac{(z-\frac{1}{z})^{52}}{z}, z = 0\right]$

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$$\left(z-\frac{1}{z}\right)^{52} = \sum_{k=0}^{\infty} \binom{52}{k} z^{2k-52} \Rightarrow \frac{\left(z-\frac{1}{z}\right)^{52}}{z} = \sum \binom{52}{k} z^{2k-53}$$

در بسط لوران مانده‌ی صفر می‌شود ضریب $\frac{1}{z}$ ⇐ ضریب $\frac{1}{z}$ = a_{-1}

$$\Rightarrow k = 26 \quad \Rightarrow I = \frac{i}{2^{52}} \times 2\pi i \binom{52}{26}$$

ب) $I = \int_{|z|=\frac{1}{2}} \frac{\sin\frac{1}{z}}{1+z}\,dz = ?$ ⇐ در این ناحیه فقط نقطه‌ی تکین اساسی $z=0$ را داریم

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!\,z^3} + \dots$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - \dots \Rightarrow \frac{\sin\frac{1}{z}}{1+z} = \dots$$

$$\Rightarrow \operatorname{Res}(\ \ , z=0) = \text{ضریب } \frac{1}{z} = 1 - \frac{1}{3!} + \frac{1}{5!} + \dots = \sin 1$$

$$\Rightarrow I = 2\pi i \sin 1$$

5 _ الف _ $f(z) = \dfrac{3z^2 - 6z + 2}{z^3 - 3z^2 + 2z} = \dfrac{1}{z-1} + \dfrac{1}{z-2} + \dfrac{1}{z}$

$$\frac{1}{z-1} = \frac{1}{z \times \left(1-\frac{1}{z}\right)} \overset{\frac{1}{|z|}<1}{=} \frac{1}{z} \sum \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{n+1}$$

$$\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = \frac{-1}{2\left(1-\frac{z}{2}\right)} = -\frac{1}{2} \sum \left(\frac{z}{2}\right)^n$$

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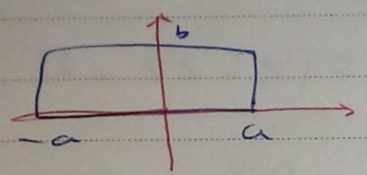
—) $\textcircled{b}$ $|z|<2 \Rightarrow \frac{1}{z-2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \longrightarrow$ $\cancel{\textcircled{d}}$

$$\frac{-1}{(z-2)^2} = -\frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{z}{2}\right)^{n-1} \Rightarrow \frac{1}{(z-2)^2} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{2}\right)^n$$

6—



$$\oint_C e^{-z^2}dz = 0 \quad \leftarrow$$ طبق قضیه انتگرال کشی

$$\oint_C e^{-z^2}dz = \underbrace{\int_0^b e^{-(a+iy)^2}\,i\,dy}_{①} + \int_{-a}^{a} e^{-(x+ib)^2}dx + \underbrace{\int_b^0 e^{-(-a+iy)^2}\,i\,dy}_{②}$$

$$+\int_{-a}^{a} e^{-(x+0)^2}dx \qquad \text{if } a \to \infty \quad ①, ② = 0$$

$$= \int_{-\infty}^{\infty} e^{-(x^2-b^2+2ixb)}dx + \int_{-\infty}^{\infty} e^{-x^2}dx = 0$$

$$\Rightarrow e^{b^2}\left(\int_{-\infty}^{\infty} e^{-x^2}(\underbrace{\cos 2bx}_{\text{زوج}} + i\underbrace{\sin 2bx}_{\text{فرد}=0})\,dx = \int_{-\infty}^{\infty} e^{-x^2}dx\right.$$ که تابع

$$\Rightarrow \int_0^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

7— $$I = \int_{-\infty}^{\infty}\frac{dx}{x(x^2-2x+2)} \qquad f_{(z)} = \frac{1}{z(z^2-2z+2)} \Rightarrow \begin{cases} z=0 \\ z = 1 \pm i \end{cases}$$

PAPCO ما باید انتگرال را بازای قطب های بالای محور حقیقی یا

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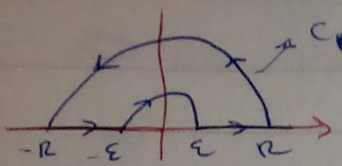

$\int f(z)dz = ?$

$|z^2 - 2z+2|$

$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} \frac{1}{R|z^2-2z+1|} dz \qquad R^2 - |2z-2| \le$

$|2z-2| \le 2R+2 \rightarrow R^2-2R-2 \le |z^2-2z+2|$

$\Rightarrow = \int \frac{a\, dz}{R^2(R^2-2R+2)} = \frac{\pi R}{R(R^2-2R+2)} \xrightarrow{R\to\infty} 0$

$\int_{C_\varepsilon} f(z)dz = \int_{C_\varepsilon}\frac{dz}{z(z^2-2z+2)} = -\pi i a_{-1} = -\pi i/2$

$\int_\varepsilon^R + \int_{-R}^{-\varepsilon} = \int_{-\infty}^0 f(z)dz + \int_0^\infty f(z)dz = \int_{-\infty}^\infty f(z)dz$

$\oint_C f(z)dz = 2\pi i \, Res(f(z), z=1+i) = \frac{\pi(1-i)}{2}$

$\Rightarrow \int_{-\infty}^\infty f(z)dz = \frac{\pi}{2}(1-i) + i\pi/2 = \pi/2$

$\Rightarrow \int_{-\infty}^\infty \frac{1}{x(---)}dx = \pi/2$

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