The Calculus of Chance: Mastering Combined Events

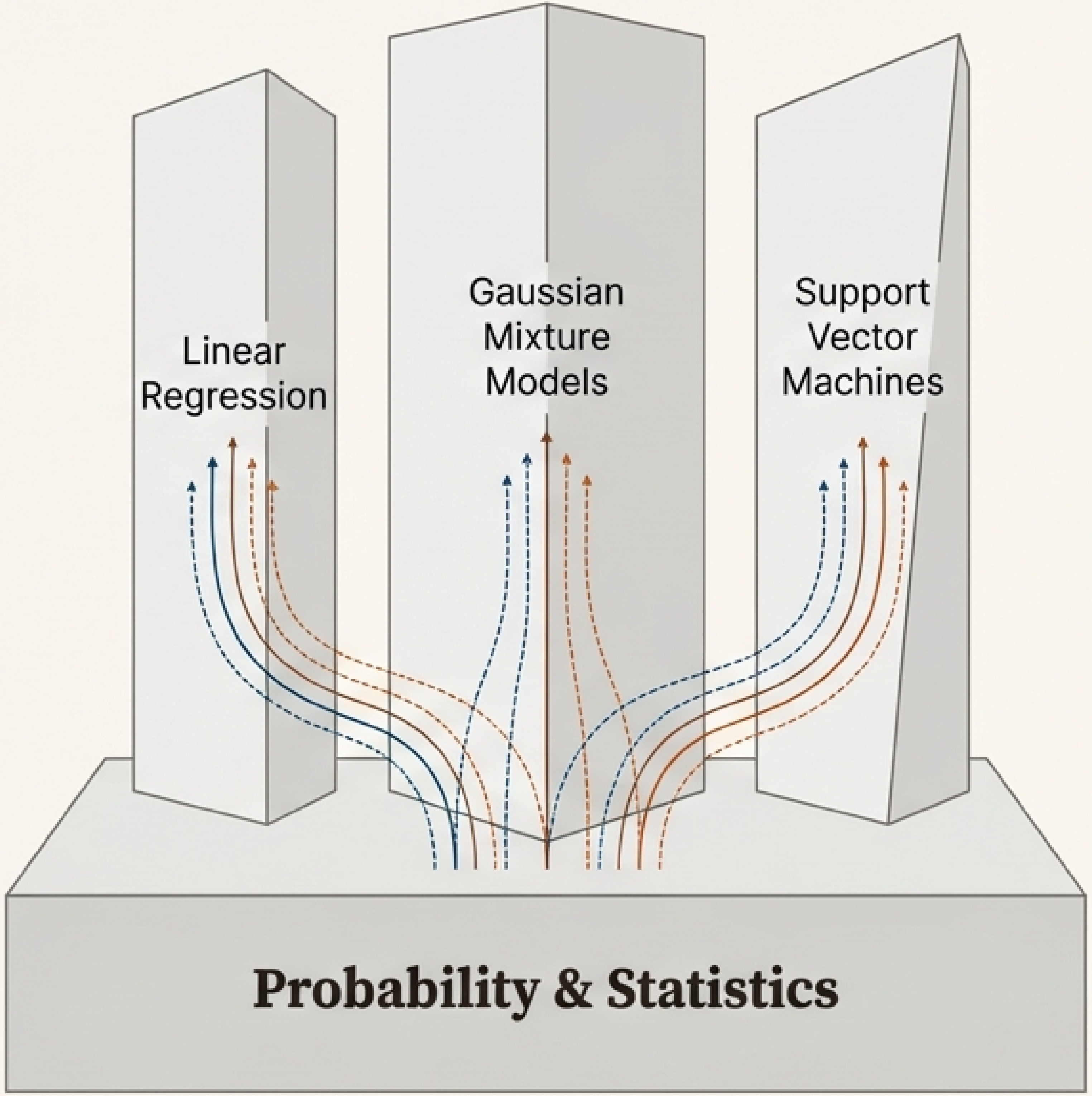
A visual framework for understanding the core logic of probability in machine learning.

Machine Learning is Built on the Bedrock of Probability

The foundational mathematics for machine learning includes “linear algebra, analytic geometry, matrix decompositions, vector calculus, optimization, probability and statistics.” Understanding how events combine is not an academic exercise; it’s a prerequisite for building and interpreting intelligent systems.

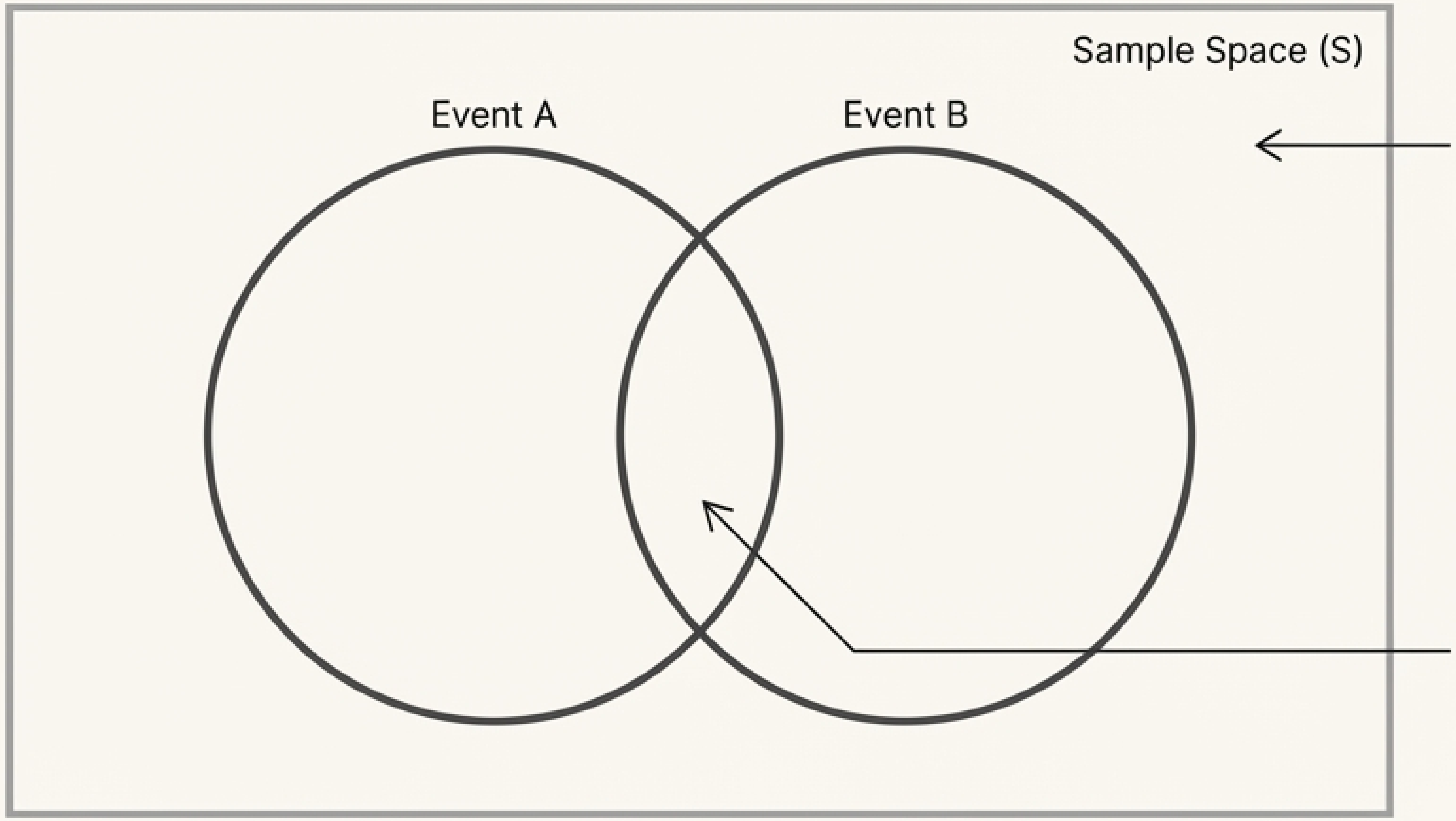
Application Example

Consider a practical ML problem: identifying valuable customers. We might need to model the probability of a user being both “high-spending” (Event A) AND “at-risk-of-churn” (Event B). Answering this requires a formal language for combining events. This deck provides the language and the calculus for doing so.

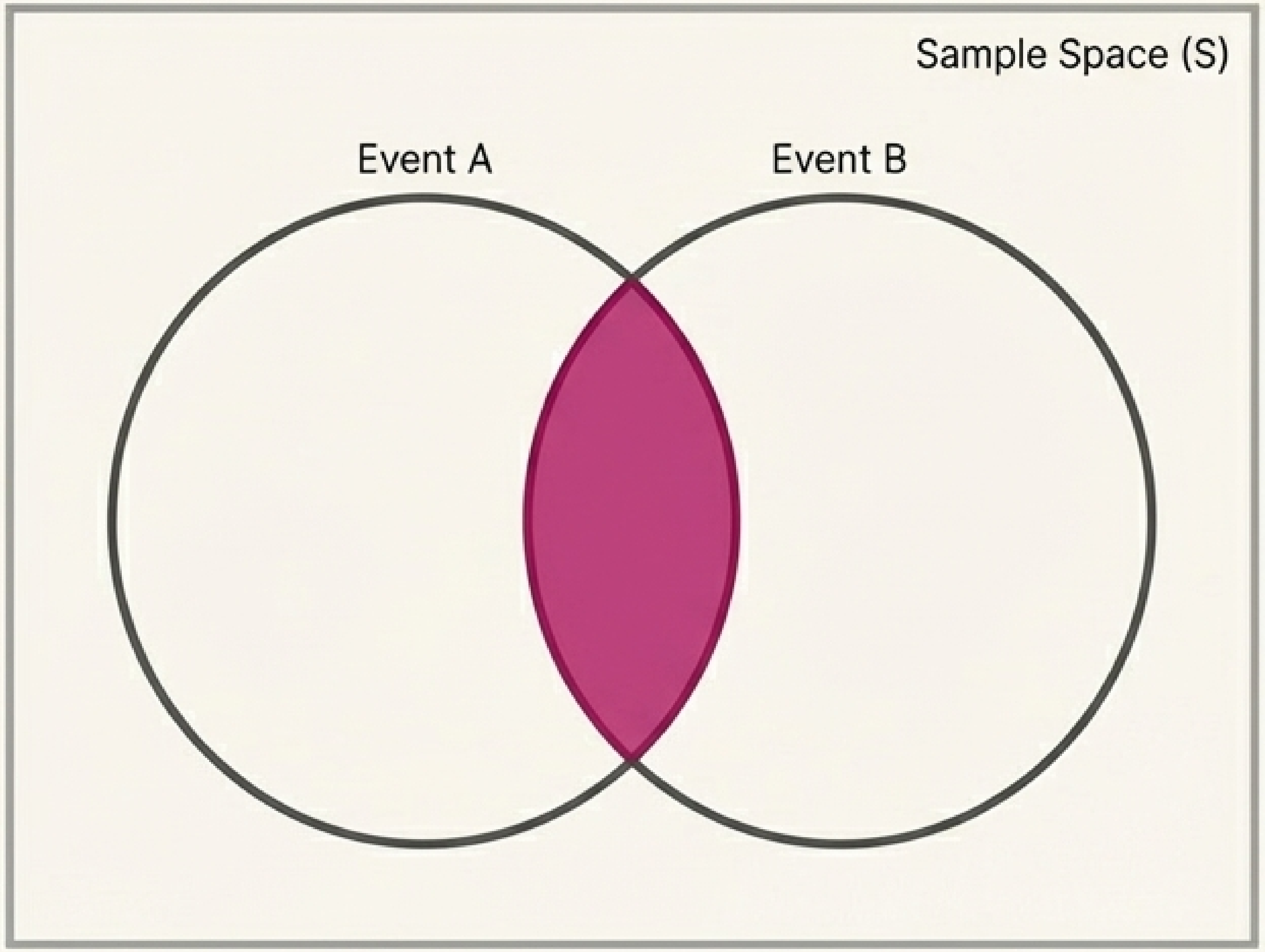


A Visual Language for Events

To reason about how events combine, we use a simple but powerful tool: the Venn diagram. It provides a clear, intuitive map of the relationships between different outcomes.



The Intersection: When Events Occur Together (A AND B)



Definition

The intersection of two events, A and B, is the set of outcomes that are in **both** A and B. We denote this as $\mathbf{A \cap B}$.

Example

Experiment: Rolling a standard six-sided die.
Sample Space = {1, 2, 3, 4, 5, 6}.

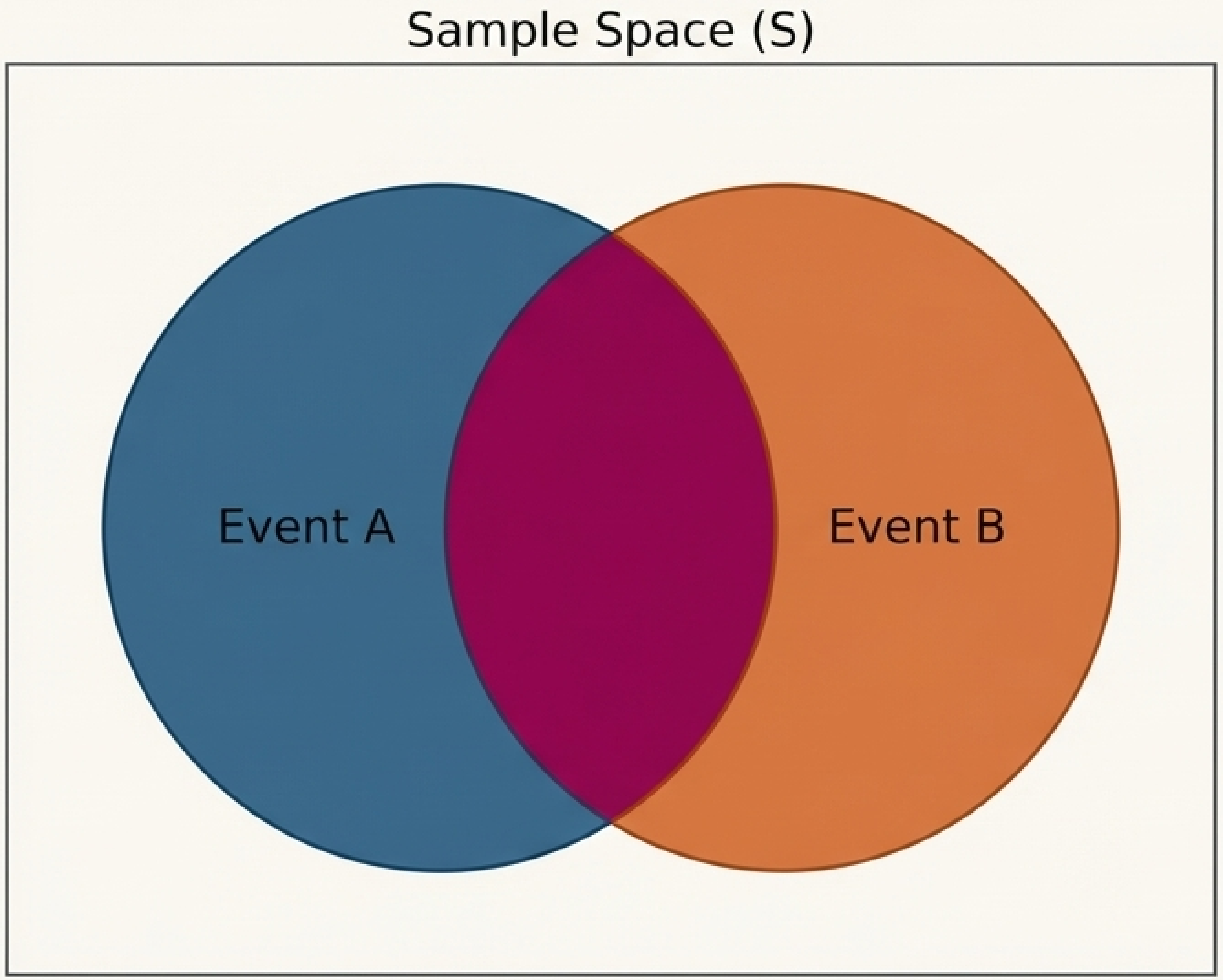
Event A: The number is less than five.
A = {1, 2. The number is less than five. A = {1, 2, 3, 4}.

Event B: The number is odd. B = {1, 3, 5}.

Intersection ($\mathbf{A \cap B}$): The number is *less than five AND odd*. $\mathbf{A \cap B = \{1, 3\}}$.

NotebookLM

The Union: When at Least One Event Occurs (A OR B)



Definition

The union of two events, A and B, is the set of outcomes that are in **A or in B** (or in both). We denote this as **A ∪ B**.

Example

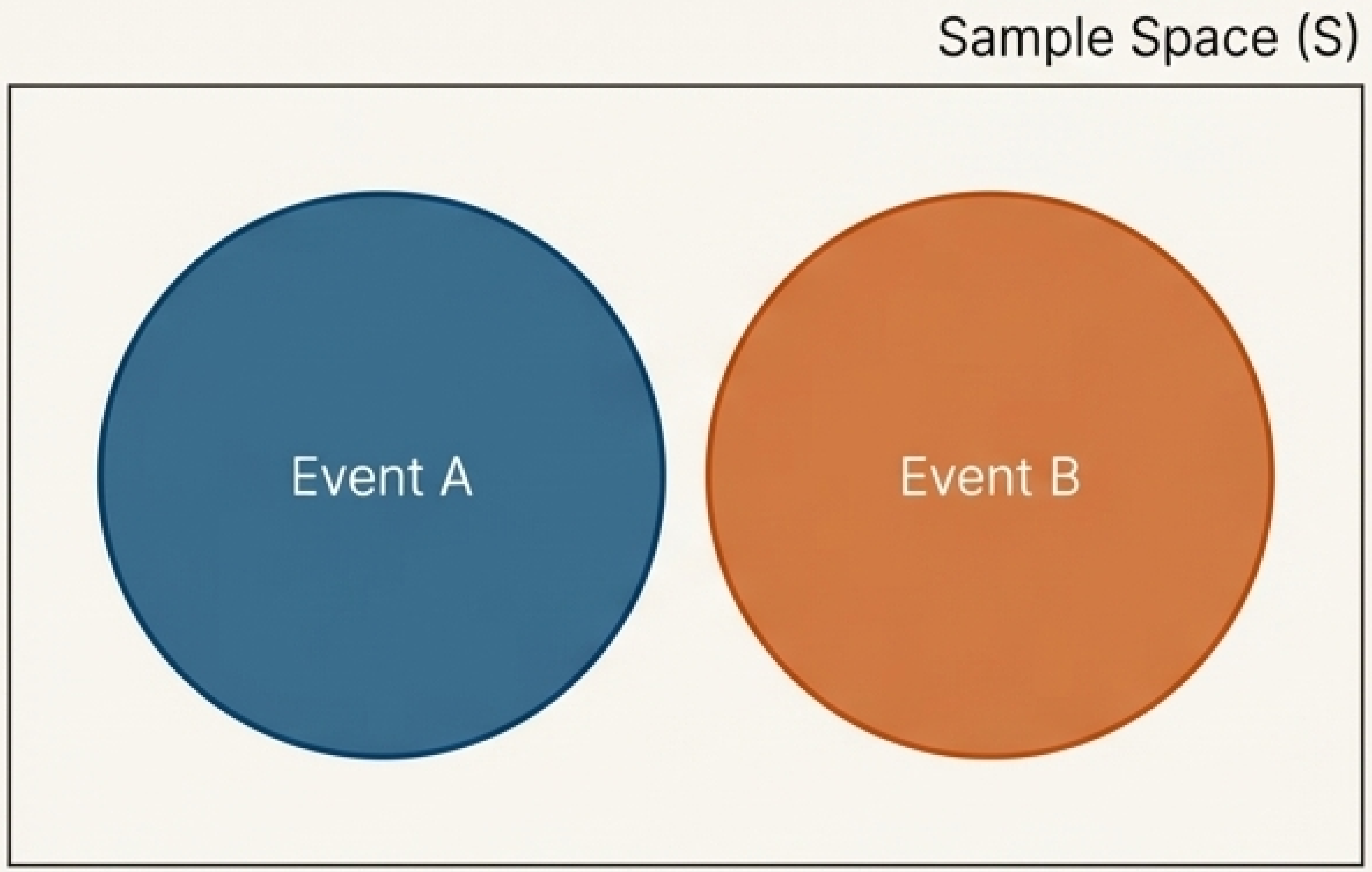
Let's consider two sets of outcomes.

Event A: {1, 2}.
Event B: {2, 3, 4}.

Intersection (A ∩ B) for comparison: {2}.

Union (A ∪ B): The outcomes that appear in either A or B. **A ∪ B = {1, 2, 3, 4}**.

Mutually Exclusive Events: When Events Cannot Occur Together



Definition

Two events are mutually exclusive if they have no outcomes in common. Their intersection is the empty set (∅).

Example

Experiment: Rolling a standard six-sided die.

Event A: The number is even.
$A = \{2, 4, 6\}$.

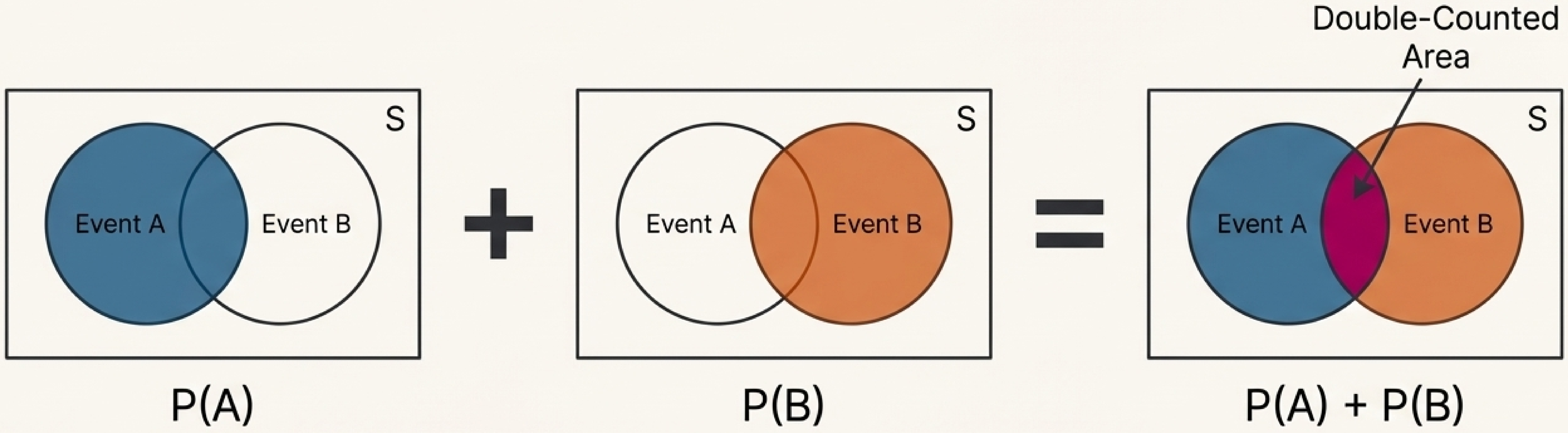
Event B: The number is odd.
$B = \{1, 3, 5\}$.

Intersection ($A \cap B$): No number can be both even and odd. Therefore, $A \cap B = \varnothing$.

Events A and B are mutually exclusive.

Calculating the Probability of a Union

If we want to find P(A U B), we can't simply add P(A) and P(B). Why? Because we would count the intersection twice.



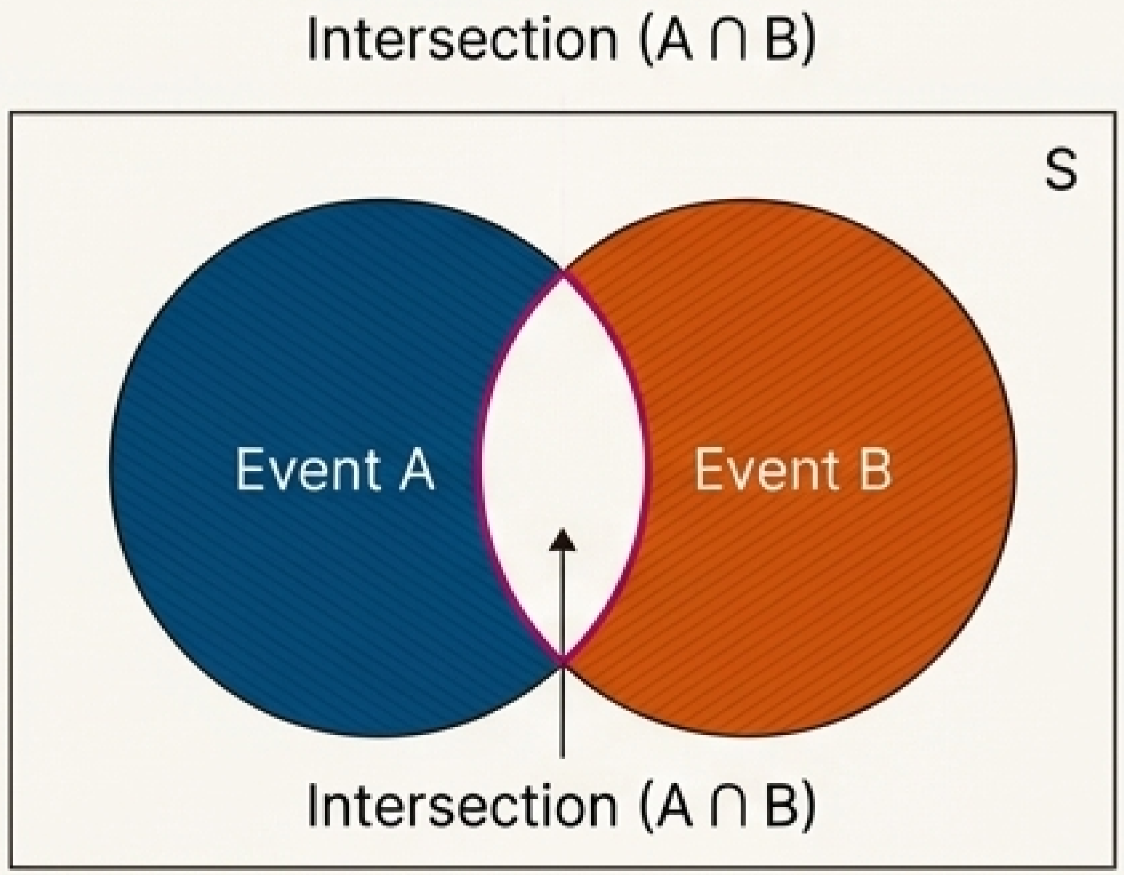
To correct for this overcounting, we must subtract the probability of the intersection.

The Addition Rule for Probabilities

The General Formula (Inclusion-Exclusion Principle)

The probability of the union of two events A and B is:

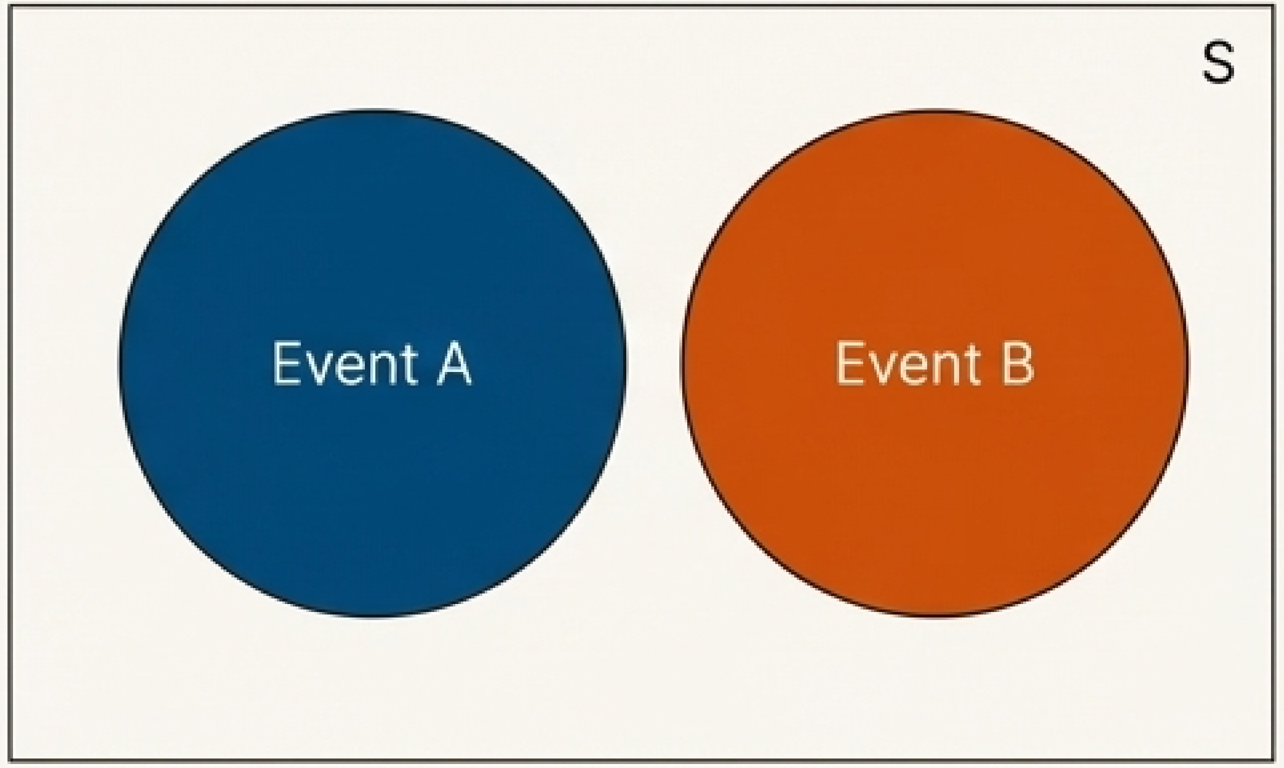
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

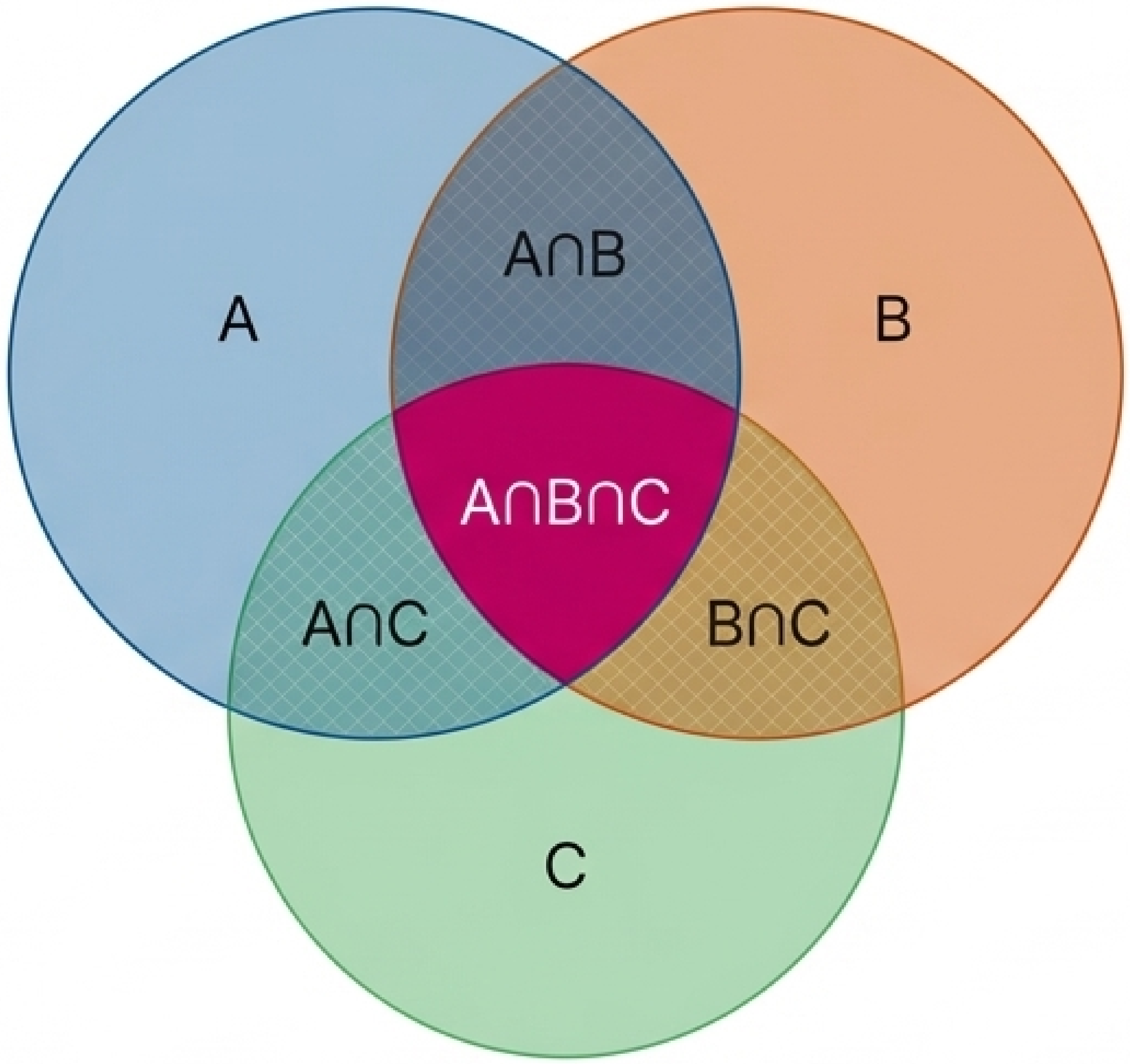


The Special Case: Mutually Exclusive Events

If A and B are mutually exclusive, their intersection is empty, meaning $P(A \cap B) = P(\varnothing) = 0$. The formula simplifies significantly:

$$P(A \cup B) = P(A) + P(B)$$





Extending the Principle to Three Events

The same principle of adding, subtracting overcounts, and adding back undercounts extends to more events.

$P(A \cup B \cup C) =$
$P(A) + P(B) + P(C)$ (Add the individuals)
$- P(A \cap B) - P(A \cap C) - P(B \cap C)$ (Subtract the pairwise overlaps)
$+ P(A \cap B \cap C)$ (Add back the triple overlap)

Mastery Example: Drawing from a 52-Card Deck

We will draw one card from a standard 52-card deck. What is the probability that the card is a **Two**, a **Heart**, or a **Spade**?

• **Values:**	Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King (13 values)
• **Suits:**	Hearts (♥), Diamonds (♦), Clubs (♣), Spades (♠) (4 suits)
• **Total Cards:**	13 values × 4 suits = 52 cards.

- **Event A**: The card is a Two.
- **Event B**: The card is a Heart.
- **Event C**: The card is a Spade.

Find $\mathbf{P}(A \cup B \cup C)$.

Step 1: Calculate Individual Probabilities

We apply our formula: $P(A \cup B \cup C) = [P(A) + P(B) + P(C)] - \ldots$

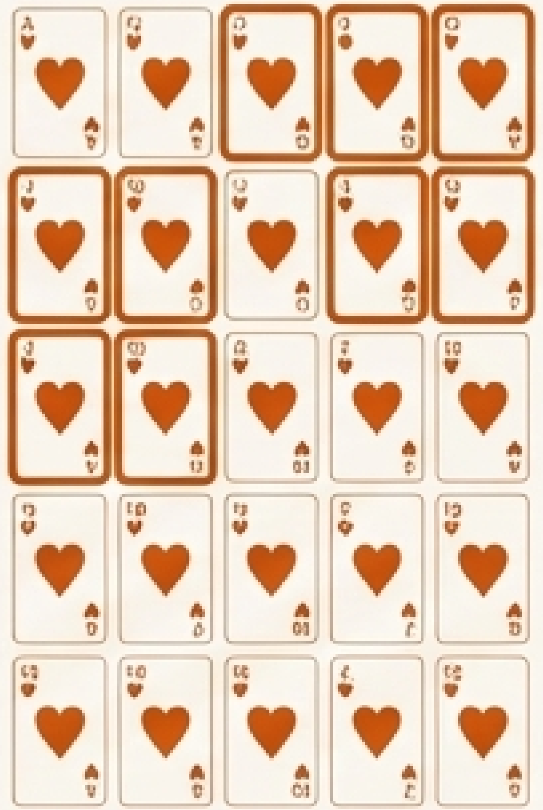
P(A) - Probability of drawing a Two:

There are 4 Twos in the deck (one for each suit).

$P(A) = 4/52$
$= \mathbf{1/13}$

P(B) - Probability of drawing a Heart:

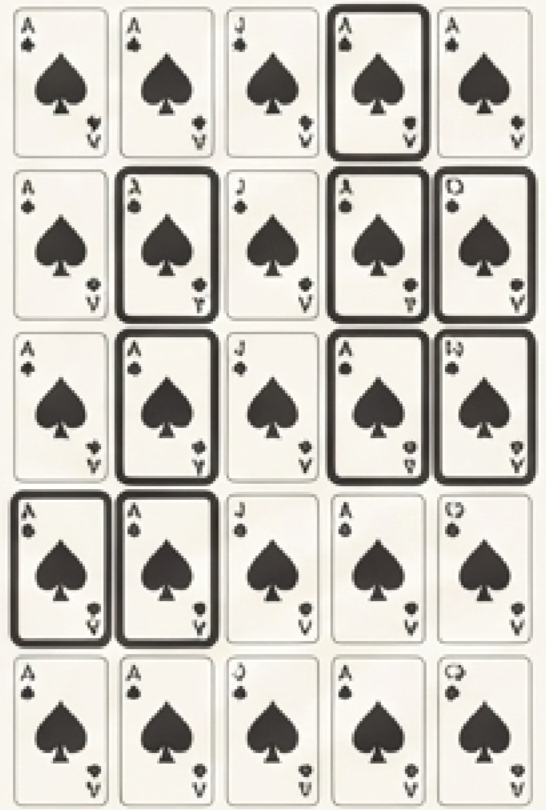
There are 13 Hearts in the deck.



$P(B) = 13/52$
$= \mathbf{1/4}$

P(C) - Probability of drawing a Spade:

There are 13 Spades in the deck.



$P(C) = 13/52$
$= \mathbf{1/4}$

Step 2: Calculate the Pairwise Intersections

Next, we find the probabilities of the overlaps to subtract:

$$\ldots - [P(A \cap B) + P(A \cap C) + P(B \cap C)] + \ldots$$

$P(A \cap B)$ - Probability of a Two AND a Heart:

There is only one such card: the Two of Hearts.

$P(A \cap B) =$ **1/52**

$P(A \cap C)$ - Probability of a Two AND a Spade:

There is only one such card: the Two of

$P(A \cap C) =$ **1/52**

$P(B \cap C)$ - Probability of a Heart AND a Spade:

A card cannot be both a Heart and a Spade. The events are mutually.

$P(B \cap C) =$ **0**

Step 3: Assemble the Final Probability

Finally, we add back the triple intersection and combine all terms.

$P(A \cap B \cap C)$ - Probability of a Two AND a Heart AND a Spade:

A single card cannot have two suits. This is impossible.

$P(A \cap B \cap C) = \mathbf{0}$

Putting It All Together

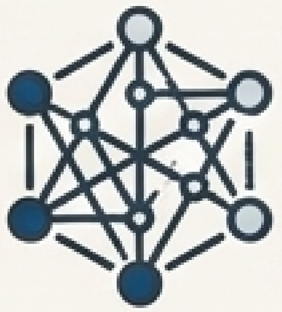
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$P(A \cup B \cup C) = (4/52 + 13/52 + 13/52) - (1/52 + 1/52 + 0) + 0$$
$$P(A \cup B \cup C) = 30/52 - 2/52$$

$$\mathbf{P(A \cup B \cup C) = 28/52}$$

Verification: We can check this by counting: 13 Hearts + 13 Spades + the Two of Clubs + the Two of Diamonds = 26 + 2 = 28 cards. The formula works.

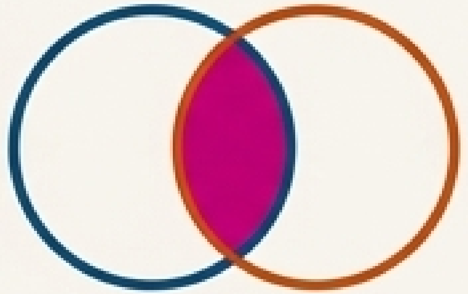
From Visual Intuition to Principled Calculation

WHY



The need to model complex, real-world scenarios in fields like machine learning.

WHAT



A visual language of events (Intersection, Union, Mutual Exclusivity) grounded in Venn diagrams.

HOW

$\sum$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A formal calculus (the Addition Rule) to precisely quantify the probability of combined events.

The Takeaway: Mastering this framework moves us beyond simple probability. It provides a systematic and extensible method for analyzing and quantifying the overlapping, interconnected conditions that define complex systems.