Knowledge Representation and Reasoning — Assignment 2

Syntax

Exercise 1 Design a knowledge representation language for the 3-card system, in which each of the three players gets one of three cards coloured red, green, or blue. In other words, choose a set of propositional atoms representing which player gets which card when the cards are dealt. How would you use the language to express the following ideas:

1. Each player gets exactly one card;
2. Player 1 gets the red card if player 2 gets the blue card;
3. Player 1 gets the red card if and only if player 2 gets the blue card;
4. Player 1 gets the red card only if player 2 gets the blue card;
5. Player 1 getting the red card is a sufficient condition for player 2 getting the blue card;
6. Player 1 getting the red card is a necessary condition for player 2 getting the blue card.

Semantics

Exercise 2 Consider the light-fan system and its knowledge-representation language $\mathcal{L}_{\mathcal{P}}$ built up from $\mathcal{P} = \{p, q\}$. We assume its ontology to be $(\mathcal{S}, \mathcal{V})$, where $\mathcal{S} = \{\mathsf{00}, \mathsf{01}, \mathsf{10}, \mathsf{11}\}$ and $\mathcal{V}(\mathsf{00}) = 00$, $\mathcal{V}(\mathsf{01}) = 01$, $\mathcal{V}(\mathsf{10}) = 10$, and $\mathcal{V}(\mathsf{11}) = 11$. Write down a sentence having:

1. Zero models;
2. Exactly one model;
3. Exactly two models;
4. Exactly three models;
5. Four models.

Exercise 3 The light-fan-heater system has three components, and the agent observing the system is interested in which components are on and which are not. A suitable knowledge representation language might have three atoms, say p, q, and r, where p expresses that the light is on, q that the fan is on, and r that the heater is on.

1. Write down the valuations in $\mathcal{U}_{\mathcal{P}}$.
2. What would be an appropriate ontology $(\mathcal{S}, \mathcal{V})$?
3. Pick any state and assume the agent has been able to exclude all the others. Write down a sentence of $\mathcal{L}_{\mathcal{P}}$ to express all the agent's information (i.e., a sentence that has the selected state as its only model).
4. Pick any two states and assume the agent has been able to exclude all the others. Write down a sentence of $\mathcal{L}_{\mathcal{P}}$ to express all the agent's information (i.e., a sentence that has the two selected states as its only models). Now find another sentence of $\mathcal{L}_{\mathcal{P}}$ that has the same models.
5. Suppose the agent's information is expressed by the sentence p. List the states in $\mathcal{M}(p)$.
6. Suppose the agent's information is expressed by the sentence $p \leftrightarrow r$. List the states in $\mathcal{M}(p \leftrightarrow r)$.

7. Give an example of a sentence of $\mathcal{L}_\mathcal{P}$ which is not satisfiable.

8. Give an example of a sentence of $\mathcal{L}_\mathcal{P}$ which is valid.

9. Give an example of a sentence of $\mathcal{L}_\mathcal{P}$ which is contingent.

10. Give an example of a sentence of $\mathcal{L}_\mathcal{P}$ which is a tautology.

Exercise 4 Consider the 3-card system from Exercise 1, where three players are each dealt a different card coloured red, green or blue. Consider $\mathcal{L}_\mathcal{P}$ with $\mathcal{P} = \{r_1, r_2, r_3, g_1, g_2, g_3, b_1, b_2, b_3\}$ and the ontology $(\mathcal{S}, \mathcal{V})$ where $\mathcal{S} = \{\mathsf{rgb}, \mathsf{rbg}, \mathsf{grb}, \mathsf{gbr}, \mathsf{brg}, \mathsf{bgr}\}$ and where $\mathcal{V}$ is the obvious function that maps, say, rgb to the valuation $\mathcal{V}(\mathsf{rgb})$ making r_1 true, g_2 true, b_3 true, and all the other atoms false, and so on.

1. Pick any state and assume the agent has been able to exclude all the others. Write down a sentence of $\mathcal{L}_\mathcal{P}$ to express all the agent's information (i.e., a sentence that has the selected state as its only model).

2. Pick any two states and assume the agent has been able to exclude all the others. Write down a sentence of $\mathcal{L}_\mathcal{P}$ to express all the agent's information (i.e., a sentence that has the two selected states as its only models). Now find another sentence of $\mathcal{L}_\mathcal{P}$ that has the same models.

3. Suppose the agent's information is expressed by the sentence $r_1 \to g_2$. List the states in $\mathcal{M}(r_1 \to g_2)$.

Exercise 5 Consider the light-fan system and its knowledge-representation language $\mathcal{L}_\mathcal{P}$ built up from $\mathcal{P} = \{p, q\}$. As before, we take the ontology to be $(\mathcal{S}, \mathcal{V})$, where $\mathcal{S} = \{\mathsf{00}, \mathsf{01}, \mathsf{10}, \mathsf{11}\}$ and $\mathcal{V}(\mathsf{00}) = 00$, $\mathcal{V}(\mathsf{01}) = 01$, $\mathcal{V}(\mathsf{10}) = 10$, and $\mathcal{V}(\mathsf{11}) = 11$. For each of the following pairs of sentences α and β, work out whether $\alpha \equiv \beta$:

1. $p \vee (p \vee p)$ and $(p \vee p) \vee p$ (associativity of $\vee$);

2. p and $\neg\neg p$ (double negations cancel out);

3. $p \to q$ and $\neg p \vee q$;

4. $p \wedge q$ and $(p \to q) \wedge (q \to p)$;

5. $\neg(p \wedge q)$ and $\neg p \vee \neg q$ (De Morgan identity);

6. $\neg(p \vee q)$ and $\neg p \wedge \neg q$ (another De Morgan identity);

7. $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$.

Exercise 6 Consider the light-fan-heater system from Exercise 3. We take the ontology to be $(\mathcal{S}, \mathcal{V})$, where $\mathcal{S} = \{\mathsf{000}, \mathsf{001}, \mathsf{010}, \mathsf{011}, \mathsf{100}, \mathsf{101}, \mathsf{110}, \mathsf{111}\}$ and $\mathcal{V}$ is the obvious labelling function. For each of the following pairs of sentences α and β, find out whether $\alpha \equiv \beta$:

1. $p \vee q$ and $p \vee q \vee r$;

2. $p \wedge q$ and $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r)$;

3. $\neg p$ and $q \wedge r$.

Exercise 7 Consider the 3-card system from Exercise 4. For each of the following pairs of sentences α and β, find out whether $\alpha \equiv \beta$:

1. $r_1 \wedge g_2$ and $r_1 \leftrightarrow g_2$;

2. $r_1 \to b_3$ and $\neg(r_1 \wedge \neg b_3)$.

Exercise 8 Let $\mathcal{L}_\mathcal{P}$ be a language. Show that for all sentences $\alpha, \beta, \gamma \in \mathcal{L}_\mathcal{P}$ the following equivalences hold:

1. $\alpha \equiv \neg\neg\alpha$;
2. $\alpha \equiv \alpha \vee \alpha$;
3. $\alpha \equiv \alpha \wedge \alpha$;
4. $\alpha \vee \beta \equiv \beta \vee \alpha$;
5. $\alpha \to \beta \equiv \neg\alpha \vee \beta$;
6. $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$;
7. $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$;
8. $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ (a distribution identity);
9. $\alpha \wedge \beta \to \gamma \equiv \neg\alpha \vee \neg\beta \vee \gamma$ (rewriting a conditional sentence as a 'clause');
10. $\alpha \to \beta \wedge \gamma \equiv (\alpha \to \beta) \wedge (\alpha \to \gamma)$.

Exercise 9 (The difference between $\equiv$ and the connective $\leftrightarrow$) Suppose that a language $\mathcal{L}_\mathcal{P}$ and an ontology $(\mathcal{S}, \mathcal{V})$ are given, where $\mathcal{V} : \mathcal{S} \longrightarrow \mathcal{U}_\mathcal{P}$ tells us which valuation is associated with each state. For all sentences α and β, prove that α is equivalent to β if, and only if, the biconditional sentence $\alpha \leftrightarrow \beta$ is satisfied by every state in $\mathcal{S}$. In other words, show that $\alpha \equiv \beta$ is the case if, and only if, the sentence $\alpha \leftrightarrow \beta$ of $\mathcal{L}_\mathcal{P}$ is valid.

Exercise 10 Consider the light-fan system from Exercise 5. For each of the following pairs of sentences α and β, determine whether $\alpha \models \beta$ and justify your answer.

1. $p \wedge q$ and q;
2. p and $p \vee q$;
3. $p \vee q$ and $p \to q$;
4. $p \wedge q$ and $p \vee \neg q$;
5. $p \leftrightarrow q$ and $p \vee \neg q$;
6. p and $p \to q$;
7. p and $q \to p$;
8. $p \wedge (p \to q)$ and q.

Exercise 11 Consider the light-fan-heater system from Exercise 6. For each of the following pairs of sentences α and β, determine whether $\alpha \models \beta$ and justify your answer.

1. $p \wedge q$ and q;
2. $p \wedge q$ and r;
3. $p \vee q \vee r$ and $p \vee q$;
4. $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$;

5. $(p \to q) \to r$ and $p \to (q \to r)$.

Exercise 12 Consider the 3-card system from Exercise 4. For each of the following pairs of sentences α and β, find out whether $\alpha \models \beta$:

1. $r_1 \wedge g_2$ and $r_1 \leftrightarrow g_2$;
2. $r_1 \leftrightarrow g_2$ and $r_1 \wedge g_2$;
3. $r_1 \to b_3$ and $\neg(r_1 \wedge \neg b_3)$.

Exercise 13 Let $\mathcal{L}_\mathcal{P}$ be a language and let $\alpha, \beta, \gamma \in \mathcal{L}_\mathcal{P}$. Is it necessarily the case that:

1. $\alpha \models \alpha$? (Reflexivity of $\models$)
2. If $\alpha \models \beta$, then $\beta \models \alpha$? (Symmetry of $\models$)
3. If $\alpha \models \beta$ and $\beta \models \gamma$, then $\alpha \models \gamma$? (Transitivity of $\models$)
4. If $\alpha \models \beta$, then $\neg\beta \models \neg\alpha$? (Contraposition)
5. If $\alpha \models \beta$, then $\alpha \wedge \gamma \models \beta$? (Monotonicity)
6. If $\alpha \models \beta \vee \gamma$, then $\alpha \models \beta$ or $\alpha \models \gamma$? (Constructivity)
7. If $\alpha \not\models \beta$, then $\alpha \models \neg\beta$? (Completeness)
8. If $\alpha \wedge \beta \models \gamma$, then $\alpha \models \beta \to \gamma$? (the 'hard half' of the Deduction Theorem)
9. If $\alpha \models \beta \to \gamma$, then $\alpha \wedge \beta \models \gamma$? (the 'easy half' of the Deduction Theorem)

Exercise 14 (The difference between $\models$ and the connective $\to$) Suppose that a language $\mathcal{L}_\mathcal{P}$ and an ontology $(\mathcal{S}, \mathcal{V})$ are given, where $\mathcal{V} : \mathcal{S} \longrightarrow \mathcal{U}_\mathcal{P}$ tells us which valuation is associated with each state. For all sentences α and β, prove that α classically entails β if, and only if, the conditional $\alpha \to \beta$ is satisfied by every state in $\mathcal{S}$. In other words, show that $\alpha \models \beta$ is the case if, and only if, the sentence $\alpha \to \beta$ of $\mathcal{L}_\mathcal{P}$ is valid.

Practical exercises

Exercise 15 Using the programming language of your choice, write a program that takes as input a propositional sentence, i.e., a sentence built up from a set of propositional atoms together with the connectives $\neg$, $\wedge$, $\vee$, $\to$ and $\leftrightarrow$, and decides whether the sentence is built according to the grammar rules seen in class. In other words, you are going to construct a *parser* for a propositional language. For the connectives, you can use `not`, `and`, `or`, `if` ... `then`, and `iff`, or any variants thereof that are just as intuitive.

Exercise 16 Using the programming language of your choice, write a program that takes as input a propositional sentence, i.e., a sentence built up from a set of propositional atoms together with the connectives $\neg$, $\wedge$, $\vee$, $\to$ and $\leftrightarrow$, and decides whether the sentence is satisfiable or not.