Lekhnitskii formulation for stress concentration around a circular hole in a thin orthotropic plate

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Nomenclature

Greek Symbols

α	Real part of complex solution
β	Imaginary part of complex solution
ε	Normal strain
γ	Shear strain
μ	Shear modulus
ν	Poisson's ratio
ω_0	Rigid body rotation
$\overline{\gamma}$	Average shear strain
$\overline{\sigma}$	Average normal stress
$\overline{\tau}$	Average shear stress
ϕ	Potential stress function
σ	Normal stress
τ	Shear stress
Θ	Angle of hole
θ	Angle

Latin Symbols

A	A matrix in ABD matrix
a	a matrix in ABD inverse matrix
B	B matrix in ABD matrix
b	b matrix in ABD inverse matrix
C	Stiffness tensor
c	c matrix in ABD inverse matrix
D	D matrix in ABD matrix
d	d matrix in ABD inverse matrix
M	Transformation matrix
Q	Orthogonal tensor/Reduced stiffness matrix
R	Reuter matrix
S'	Transformed compliance tensor
S	Compliance tensor
$\mathfrak{Im}$	Imaginary
$\mathfrak{Re}$	Real
$\mathfrak{S}$	Sign
$\overline{U}$	Average body force
$\overline{u}$	Average displacement along x-axis
$\overline{u}^*$	Average prescribed displacement along x-axis
$\overline{v}$	Average displacement along y-axis
$\overline{v}^*$	Average prescribed displacement along y-axis
$\overline{X}$	Average prescribed traction along x-axis
$\overline{Y}$	Average prescribed traction along y-axis

A_k	Force Resultant Around Hole
E	Young's modulus
f	Potential stress function solution
$g_n^{(k)}$	Disturbance Field
h	Thickness
h_k	Homogeneous Stress Field
N	Normal resultant load
n	Normal vector
p	Prescribed stress
R	Resultant load
r	Radius
s	Complex variable/Characteristic equation roots/infinitesimal element
T	Tangential resultant load
U	Body force
u	Displacement along x-axis
u_0	Rigid body displacement along x-axis
V	Potential function
v	Displacement along y-axis
v_0	Rigid body displacement along y-axis
w	Displacement along z-axis
X	Prescribed traction along x-axis
x	x-axis
Y	Prescribed traction along y-axis
y	y-axis
z	z-axis

Superscript/Subscript

1	Ply local coordinate (horizontal axis)
2	Ply local coordinate (vertical axis)
3	Ply local coordinate
c	cosine
s	sine
t	Total

Terminology

Analytical function A complex function is called analytical / Holomorphic over a region R is it is differentiable at any point in region R.

Balanced laminate A laminate in which the number of plies with $\pm\theta$ are equal.

Fundamental theorem of Algebra If two series are equal, either finite or infinite, then the coefficient of their like powers are equal.

Abstract

Lekhnitskii formalism is one of the few approaches in analyzing anisotropic elastic bodies which essentially is the generalization of Muskhelishvili formalism for isotropic elastic bodies. Lekhnitskii theory starts by deriving stresses from the equilibrium equation, then follows the compatibility to solve strains and finally calculate displacement fields. In this methodology, a potential function approach, analogous to the Airy stress function, along with complex analysis is invoked. The goal of the present report is to derive the generalized Lekhnitskii theory and then apply it to a thin orthotropic laminate containing a hole.

Contents

1   Introduction

Lekhnitskii theory for anisotropic elastic bodies is a robust tool in analyzing the stretching, bending and stretch-bending coupling of composite laminates. This theory provides an analytical approach to analyze the stress and displacement field in composite laminates with/without cut-off (also known as irregularity as used in [1]) which are very common in the aerospace industry, *e.g*, riveted holes and elliptical or rounded rectangular windows in civilian aircraft.

The present report is divided into three main sections. In the first, we discuss the general theory of Lekhnitskii. The second section narrows the Lekhnitskii theory to thin orthotropic laminates and in the last section, we invoke the Classical Composite Laminate Theory along with the Lekhnitskii theory to find the stress field around an infinite plate with a circular hole under biaxial loading.

Despite the derivation of all equations, in subsequent sections, most of the contents are borrowed from [2] and [3].

2   Plane Problem in Linear Elasticity of An Anisotropic Body

2.1   Generalized Plane Stress Theory

The generalized plane stress was introduced to overcome the peculiar dependency of the displacements on z in the plane stress. The whole idea of the generalized plane stress is upon averaging the stress and displacement over the domain thickness to remove the z dependency. The generalized plane stress assumes

1. The domain is bounded between two parallel planes, here $z = \pm h$ (see Figure. 1) where both planes are either traction-free or symmetrically loaded through the thickness so that their average value over the thickness vanishes.

2. The plate thickness is negligible w.r.t. the other geometrical dimension.

3. Body forces are independent of z or symmetrically distributed through the thickness.

4. The displacement along the z-axis is an odd function that is $w(x,y,z) = -w(x,y,-z)$, in other words, the mid-plane does not undergo bending.
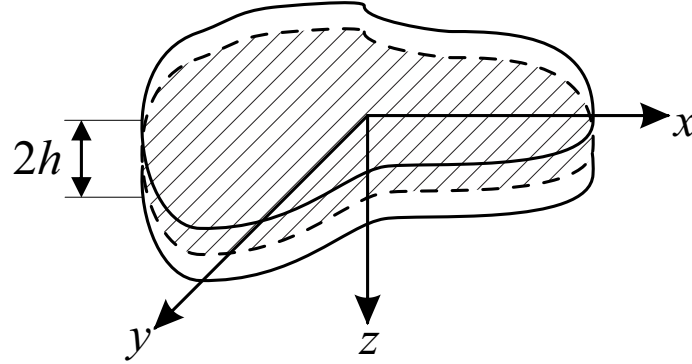
5. Deformations are infinitesimal.

Figure 1: The elastic body representing the generalized plane stress. The shaded area represents the mid-plane.

Figure. 1 illustrates the generalized plane stress. The origin is placed at the mid-plane center and h indicates the plate thickness. As mentioned earlier, the values of stress and displacement are averaged over the plate thickness in this theory; therefore,

$$\overline{\sigma}_x = \frac{1}{2h} \int_{-h}^{h} \sigma_x \mathrm{d}z \quad , \overline{\sigma}_y = \frac{1}{2h} \int_{-h}^{h} \sigma_y \mathrm{d}z \quad , \overline{\tau}_{xy} = \frac{1}{2h} \int_{-h}^{h} \tau_{xy} \mathrm{d}z$$

$$\overline{\sigma}_z = \frac{1}{2h} \int_{-h}^{h} \sigma_z \mathrm{d}z \quad , \overline{u} = \frac{1}{2h} \int_{-h}^{h} u \mathrm{d}z \quad , \overline{v} = \frac{1}{2h} \int_{-h}^{h} v \mathrm{d}z$$

$$(2.1)$$

σ and τ designate the normal and shear stress, u and v are the displacements in the x and y-direction, respectively. The rest of variables with a bar sign indicate the corresponding average value. The subscripts also indicate the coordinate axis. While σ_z is negligible compared to other stress components, it is worth noting that the value of $\overline{\sigma}_z$ is zero as it is assumed to be symmetric w.r.t the mid-plane.

2.2 Field Equations

The equilibrium equation for the generalized plane stress reduces to

$$\frac{\partial \overline{\sigma}_x}{\partial x} + \frac{\partial \overline{\tau}_{xy}}{\partial y} + \overline{U}_x = 0$$

$$\frac{\partial \overline{\tau}_{xy}}{\partial x} + \frac{\partial \overline{\sigma}_y}{\partial y} + \overline{U}_y = 0$$

$$(2.2)$$

where $\overline{U}_x$ and $\overline{U}_y$ are average body force per unit volume along x and y axes and are

$$\overline{U}_x = \frac{1}{2h} \int_{-h}^{h} U_x \mathrm{d}z \quad , \overline{U}_y = \frac{1}{2h} \int_{-h}^{h} U_y \mathrm{d}z$$

$$(2.3)$$

For a composite laminate with fibers oriented in angles other than 0 and/or 90 degrees, some nonzero coupling terms appear in the compliance tensor, S'. For such laminate the stress-strain

$(\sigma - \epsilon)$ relation is [4]

$$
\begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\varepsilon}_z \\ \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \\ \bar{\gamma}_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11}' & S_{12}' & S_{13}' & 0 & 0 & S_{16}' \\ S_{12}' & S_{22}' & S_{23}' & 0 & 0 & S_{26}' \\ S_{13}' & S_{23}' & S_{33}' & 0 & 0 & S_{36}' \\ 0 & 0 & 0 & S_{44}' & S_{45}' & 0 \\ 0 & 0 & 0 & S_{45}' & S_{55}' & 0 \\ S_{16}' & S_{26}' & S_{36}' & 0 & 0 & S_{66}' \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ 0 \\ 0 \\ 0 \\ \bar{\tau}_{xy} \end{Bmatrix}
\tag{2.4}
$$

The components of the compliance tensor are given in Appendix. A. For the generalized stress sate, Eq(2.4) reduces to

$$
\bar{\varepsilon}_x = S_{11}'\bar{\sigma}_x + S_{12}'\bar{\sigma}_y + S_{16}'\bar{\tau}_{xy}
$$
$$
\bar{\varepsilon}_y = S_{12}'\bar{\sigma}_x + S_{22}'\bar{\sigma}_y + S_{26}'\bar{\tau}_{xy}
$$
$$
\bar{\gamma}_{xy} = 2\bar{\varepsilon}_{xy} = S_{16}'\bar{\sigma}_x + S_{26}'\bar{\sigma}_y + S_{66}'\bar{\tau}_{xy}
\tag{2.5}
$$

The strain-displacement can be written as

$$
\bar{\varepsilon}_{\alpha\beta} = \frac{1}{2}(\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha})
\tag{2.6}
$$

where α and β occupy x and y, $\bar{\varepsilon}$ and $\bar{\gamma}$ are the average normal and shear strain over the thickness. The compatibility equation can be calculated by properly differentiating the displacement components in Eq(2.6) (see Appendix. B for the derivation).

$$
\frac{\partial^2 \bar{\varepsilon}_x}{\partial y^2} + \frac{\partial^2 \bar{\varepsilon}_y}{\partial x^2} - \frac{\partial^2 \bar{\gamma}_{xy}}{\partial x \partial y} = 0
\tag{2.7}
$$

Assuming that the body forces can be derived from a potential function, V such that

$$
\bar{U}_x = -\frac{\partial \bar{V}}{\partial x} \quad , \bar{U}_y = -\frac{\partial \bar{V}}{\partial y}
$$

the stress components (see Eq(2.2)) can be expressed via the following set of equations.

$$
\bar{\sigma}_x = \frac{\partial^2 \phi}{\partial y^2} + \bar{V} \quad , \bar{\sigma}_y = \frac{\partial^2 \phi}{\partial x^2} + \bar{V} \quad , \bar{\tau}_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}
\tag{2.8}
$$

where $\phi = \phi(x,y)$ is called the potential stress function. Now, by replacing stress components from Eq(2.8) in Eq(2.5), the following partial differential equation is formed which should be fulfilled by the potential stress function.

$$S'_{22}\frac{\partial^4\phi}{\partial x^4} - 2S'_{26}\frac{\partial^4\phi}{\partial x^3\partial y} + (2S'_{12}+S'_{66})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2S'_{16}\frac{\partial^4\phi}{\partial x\partial y^3} + S'_{11}\frac{\partial^4\phi}{\partial y^4}$$

$$+ (S'_{12}+S'_{22})\frac{\partial^2\overline{V}}{\partial x^2} - (S'_{16}+S'_{26})\frac{\partial^2\overline{V}}{\partial x\partial y} + (S'_{11}+S'_{12})\frac{\partial^2\overline{V}}{\partial y^2} = 0 \quad (2.9)$$

which in the absence of body forces reduces to

$$S'_{22}\frac{\partial^4\phi}{\partial x^4} - 2S'_{26}\frac{\partial^4\phi}{\partial x^3\partial y} + (2S'_{12}+S'_{66})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2S'_{16}\frac{\partial^4\phi}{\partial x\partial y^3} + S'_{11}\frac{\partial^4\phi}{\partial y^4} = 0 \qquad (2.10)$$

See `pde_plane_stress.mw` for the corresponding Maple file. It can be shown that a similar equation is derived in case of plane deformation (plane strain). This is proved in Appendix. C.

In general, the applied boundary conditions on an elastic body lie within three categories: (1) Traction boundary-value problems, (2) Displacement boundary-value problems and (3) Mixed traction boundary-value problems.

Traction boundary-value problems: When an elastic body undergoes the external and body forces, then the boundary conditions can be found from

$$\begin{bmatrix}\overline{\sigma}_x & \overline{\tau}_{xy} \\ \overline{\tau}_{xy} & \overline{\sigma}_y\end{bmatrix}\begin{Bmatrix}\cos(n,x) \\ \cos(n,y)\end{Bmatrix} = \begin{Bmatrix}\overline{\sigma}_x\cos(n,x)+\overline{\tau}_{xy}\cos(n,y) \\ \overline{\tau}_{xy}\cos(n,x)+\overline{\sigma}_y\cos(n,y)\end{Bmatrix} = \begin{Bmatrix}\overline{X} \\ \overline{Y}\end{Bmatrix} \qquad (2.11)$$

n is the surface normal vector, cosine indicates the angle between the normal vector and the x and y axes. $\overline{X}$ and $\overline{Y}$ are the external forces in the x and y-direction, respectively.

Displacement boundary-value problems: When the applied displacements on the surface and body forces are given.

$$\overline{u} = \overline{u}^* \quad ,\overline{v} = \overline{v}^* \qquad (2.12)$$

$\overline{u}^*$ and $\overline{v}^*$ designate the prescribed displacements on the body in x and y-direction.

Mixed traction boundary-value problems: When both external forces and displacements are applied on portions of the surface. This case is not provided in the present report.

2.3   Solve the Compatibility Equation in Terms of Potential Stress Function

In the following, we remove the bar sign from the averaged values of displacement, strain and stress as well as the prime sign in the compliance tensor and neglect the body forces. Eq(2.10) changes to

$$S_{22}\frac{\partial^4\phi}{\partial x^4} - 2S_{26}\frac{\partial^4\phi}{\partial x^3\partial y} + (2S_{12}+S_{66})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2S_{16}\frac{\partial^4\phi}{\partial x\partial y^3} + S_{11}\frac{\partial^4\phi}{\partial y^4} = 0 \qquad (2.13)$$

2.3.1   Step 1: Remove S_{16} and S_{26}

For a composite laminate with fibers oriented in angles $\pm\theta$ under tensile stress, the laminate experiences shear deformation. By transforming the fibers to the principal directions such that the original coordinate system coincides with the ply coordinate system, one is able to nullify S_{16} and S_{26} [4]. This is demonstrated in Figure. 2.

Consequently, Eq(2.13) reduces to

$$S_{22}\frac{\partial^4\phi}{\partial x^4} + (2S_{12}+S_{66})\frac{\partial^4\phi}{\partial x^2\partial y^2} + S_{11}\frac{\partial^4\phi}{\partial y^4} = 0 \tag{2.14}$$

S_{11}, S_{12}, S_{22} and S_{66} are given in Eq(A.2).
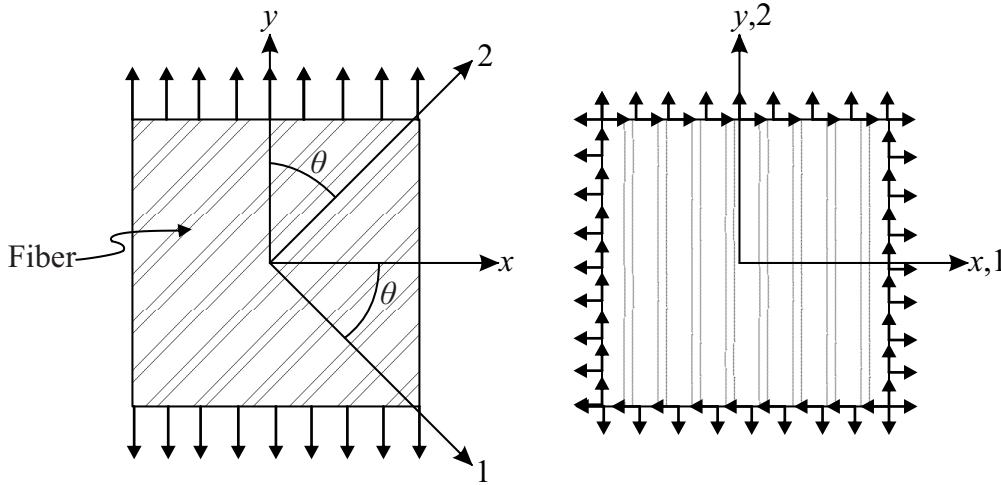

Figure 2: S_{16} and S_{26} in Eq(2.13) can be nullified by matching the original $(x-y)$ and ply coordinate system (1-2) [5]

2.3.2   Step 2: Find the Possible Solutions

Let us assume $\phi(x,y) = f(z)$ where $z = x + sy$ is complex. See Appendix. D for the reason of selecting z as a complex expression. Using the chain role

$$\frac{\partial\phi}{\partial x} = \frac{\mathrm{d}f(z)}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} \rightarrow \frac{\partial^4\phi}{\partial x^4} = \frac{\mathrm{d}^4f(z)}{\mathrm{d}z^4}$$

$$\frac{\partial\phi}{\partial y} = \frac{\mathrm{d}f(z)}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}y} \rightarrow \frac{\partial^4\phi}{\partial x^4} = s^4\frac{\mathrm{d}^4f(z)}{\mathrm{d}z^4}$$

$$\frac{\partial^2\phi}{\partial x^2\partial y^2} = s^2\frac{\mathrm{d}^4f(z)}{\mathrm{d}z^4}$$

Eq(2.14) can be rewritten as

$$S_{22}\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} + (2S_{12} + S_{66})s^2\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} + S_{11}s^4\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} = 0$$

$$\xrightarrow{\times\frac{1}{S_{11}}} \frac{S_{22}}{S_{11}}\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} + \frac{(2S_{12} + S_{66})}{S_{11}}s^2\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} + s^4\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} = 0$$

$$r^2\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} + 2as^2\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} + s^4\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} = 0$$

$$(r^2 + 2as^2 + s^4)\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} = 0 \tag{2.15}$$

in which

$$r^2 = \frac{S_{22}}{S_{11}} \quad , 2a = \frac{(2S_{12} + S_{66})}{S_{11}} \tag{2.16}$$

Seeking for the non-trivial solution, $\dfrac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} \neq 0$; thus,

$$r^2 + 2as^2 + s^4 = 0 \tag{2.17}$$

Eq(2.17) is called the characteristic equation. It is shown by Lekhnitskii [6] and Koussios [5] that the roots of Eq(2.17) for an ideal elastic body (conventional material in industry) with finite and not equal to zero values of S_{11}, $2S_{12} + S_{66}$ and S_{22} are either complex or pure imaginary. For the proof see Appendix. E.

2.3.3   Step 3: The Solution to the Characteristic Equation

Let assume $s = A^2$. As a result, Eq(2.17) becomes

$$r^2 + 2aA + A^2 = 0$$

which is a quadratic equation and can be easily solved.

$$A = \frac{-2a \pm \sqrt{4a^2 - 4r^2}}{2} = -a \pm i\sqrt{r^2 - a^2}$$

Next,

$$A = -a \pm i\sqrt{r^2 - a^2} = -a \pm i\sqrt{r-a}\sqrt{r+a}$$

$$= \left(\frac{r-a}{2}\right) \pm 2i\sqrt{\frac{r-a}{2}}\sqrt{\frac{r+a}{2}} - \left(\frac{r+a}{2}\right) = \left(\sqrt{\frac{r-a}{2}} \pm i\sqrt{\frac{r+a}{2}}\right)^2$$

Thus,

$$s = \pm \left(\sqrt{\frac{r-a}{2}} \pm i \sqrt{\frac{r+a}{2}} \right) \tag{2.18}$$

Eq(2.18) can also be written as

$$s_1 = \sqrt{\frac{r-a}{2}} + i\sqrt{\frac{r+a}{2}}$$

$$s_2 = -\sqrt{\frac{r-a}{2}} + i\sqrt{\frac{r+a}{2}}$$

$$s_3 = \sqrt{\frac{r-a}{2}} - i\sqrt{\frac{r+a}{2}}$$

$$s_4 = -\sqrt{\frac{r-a}{2}} - i\sqrt{\frac{r+a}{2}} \tag{2.19}$$

Therefore, the general solution to Eq(2.14) can be written as the summation of $f(z_k)$.

$$\phi(x,y) = f_1(z_1) + f_2(z_2) + f_3(z_3) + f_4(z_4) \tag{2.20}$$

in which $z_k = x + s_k y$ where $k = 1, 2, 3, 4$. The above solution to the potential stress function implies that $f_k(z_k)$ is complex because s_k is complex which indeed leads to the fact that every $f_k(z_k)$ function which is analytical does satisfy Eq(2.20).

2.3.4   Step 4: The Homogeneous Solution

As mentioned earlier, the roots to the characteristic equation (Eq(2.17)) are either complex or imaginary. Furthermore, it is also shown in Appendix. E that $r + a$ in the set of solutions in Eq(2.19) is positive (see Appendix. E). Hence, we are left with two cases for $r - a$ expression.

Case 1: $r - a \geq 0$

In this case, according to Eq(E.1),

$$s_3 = \overline{s_1} \quad , s_4 = \overline{s_2}$$

where the bar sign indicates the complex conjugate.

Case 2: $r - a < 0$

In this case, according to Eq(E.2),

$$s_3 = \overline{s_2} \quad , s_4 = \overline{s_1}$$

Consequently, Eq(2.20) can be written as

$$\phi(x,y) = f_1(z_1) + f_2(z_2) + f_3(\overline{z_1}) + f_4(\overline{z_2}) \tag{2.21}$$

for case 1 or

$$\phi(x,y) = f_1(z_1) + f_2(z_2) + f_3(\overline{z_2}) + f_4(\overline{z_1}) \tag{2.22}$$

for case 2. Both Eq(2.21) and Eq(2.22) imply that $\phi(x,y)$ must always be real due to the existence of two complex conjugates. The general solution can be written in the form of Eq(2.23). It is noteworthy to mention that choosing either pair (case 1 and case 2) does not affect the final solution as the imaginary parts cancel out each other [5].

$$\phi(x,y) = f_1(z_1) + f_2(z_2) + \overline{f_1(z_1)} + \overline{f_2(z_2)} = 2\Re[f_1(z_1) + f_2(z_2)] \tag{2.23}$$

$\Re$ indicates the real part of the expression.

2.3.5 Summary

The next few lines summarize all the required information for the selection of the potential stress function to solve Eq(2.14).

1. $\phi(x,y)$ should satisfy the equilibrium equation (Eq(2.2)).

2. $\phi(x,y)$ should satisfy the compatibility equation (Eq(2.2)).

3. $\phi(x,y)$ is an analytical function.

4. $\phi(x,y)$ is real.

5. $\phi(x,y)$ can be found from Eq(2.23) where $z_1 = x + s_1 y$ and $z_2 = x + s_2 y$ and s_1 and s_2 can be derived from Eq(2.19).

6. $\phi(x,y)$ should also satisfy the boundary condition (Eq(2.11) and Eq(2.12)) We will further discuss this topic in Section. 2.5.

The above list is also applicable to $f_1(z_1)$ and $f_2(z_2)$ as they both can be chosen interchangeably instead of $\phi(x,y)$.

2.4   Stress, Strain and Displacement

Stresses can be calculated from Eq(2.8) and Eq(2.23).

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left\{ \frac{\partial f_1(z_1)}{\partial y} + \frac{\partial f_2(z_2)}{\partial y} + \frac{\partial \overline{f_1(z_1)}}{\partial y} + \frac{\partial \overline{f_2(z_2)}}{\partial y} \right\}$$

$$= \frac{\partial}{\partial y}\left\{ \frac{\mathrm{d}f_1(z_1)}{\mathrm{d}z_1}\frac{\partial z_1}{\partial y} + \frac{\mathrm{d}f_2(z_2)}{\mathrm{d}z_2}\frac{\partial z_2}{\partial y} + \frac{\mathrm{d}\overline{f_1(z_1)}}{\mathrm{d}z_1}\frac{\partial \overline{z_1}}{\partial y} + \frac{\mathrm{d}\overline{f_2(z_2)}}{\mathrm{d}z_2}\frac{\partial \overline{z_2}}{\partial y} \right\} \xrightarrow{\frac{\partial z_k}{\partial y}=s_k}$$

$$= \frac{\partial}{\partial y} \left\{ s_1 \frac{\mathrm{d} f_1(z_1)}{\mathrm{d} z_1} + s_2 \frac{\mathrm{d} f_2(z_2)}{\mathrm{d} z_2} + \overline{s_1} \frac{\mathrm{d} \overline{f_1(z_1)}}{\mathrm{d} z_1} + \overline{s_2} \frac{\mathrm{d} \overline{f_2(z_2)}}{\mathrm{d} z_2} \right\}$$

$$= \frac{\partial}{\partial y} \left\{ s_1 \Phi_1(z_1) + s_2 \Phi_2(z_2) + \overline{s_1} \overline{\Phi_1(z_1)} + \overline{s_2} \overline{\Phi_2(s_2)} \right\}$$

In above equation, $\dfrac{\mathrm{d} f_k(z_k)}{\mathrm{d} z_k} = \Phi_k(z_k)$ and $\dfrac{\mathrm{d} \overline{f_k(z_k)}}{\mathrm{d} z_k} = \overline{\Phi_k(z_k)}$ where $k = 1, 2$.

$$\frac{\partial}{\partial y}(s_1 \Phi_1(z_1)) = \overset{0}{\cancel{\frac{\partial s_1}{\partial y}}} \Phi_1(z_1) + s_1 \frac{\partial \Phi_1(z_1)}{\partial y} = s_1 \frac{\mathrm{d} \Phi_1(z_1)}{\mathrm{d} z_1} \frac{\partial z_1}{\partial y} = s_1^2 \frac{\mathrm{d} \Phi_1(z_1)}{\mathrm{d} z_1} = s_1^2 \Phi_1'(z_1)$$

Repeating the above procedure for all the terms gives

$$\sigma_x = s_1^2 \Phi_1'(z_1) + s_2^2 \Phi_2'(z_2) + \overline{s_1}^2 \overline{\Phi_1'(z_1)} + \overline{s_2}^2 \overline{\Phi_2'(z_2)} = 2\mathfrak{Re}[s_1^2 \Phi_1'(z_1) + s_2^2 \Phi_2'(z_2)]$$

The stress components become

$$\sigma_x = 2\mathfrak{Re}[s_1^2 \Phi_1'(z_1) + s_2^2 \Phi_2'(z_2)]$$
$$\sigma_y = 2\mathfrak{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)]$$
$$\tau_{xy} = -2\mathfrak{Re}[s_1 \Phi_1'(z_1) + s_2 \Phi_2'(z_2)] \tag{2.24}$$

Strain can be found from Eq(2.5). For instance,

$$\varepsilon_x = S_{11}\sigma_x + S_{12}\sigma_y = 2S_{11}\mathfrak{Re}[s_1^2 \Phi_1'(z_1) + s_2^2 \Phi_2'(z_2)] + 2S_{12}\mathfrak{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)]$$

The strains become

$$\varepsilon_x = 2\mathfrak{Re}[(S_{11}s_1^2 + S_{12})\Phi_1'(z_1)] + 2\mathfrak{Re}[(S_{11}s_2^2 + S_{12})\Phi_2'(z_2)]$$
$$\varepsilon_y = 2\mathfrak{Re}[(S_{12}s_1^2 + S_{22})\Phi_1'(z_1)] + 2\mathfrak{Re}[(S_{12}s_2^2 + S_{22})\Phi_2'(z_2)]$$
$$\gamma_{xy} = -2S_{66}\mathfrak{Re}[s_1 \Phi_1'(z_1) + s_2 \Phi_2'(z_2)] \tag{2.25}$$

The displacement components can be easily found by integrating the strains. The displacement along the x-axis, u, can be found by integrating ε_x over the x axis.

$$u = \int \varepsilon_x \mathrm{d}x + p(y)$$

$$= 2 \int \mathfrak{Re}[(S_{11}s_1^2 + S_{12})\Phi_1'(z_1) + (S_{11}s_2^2 + S_{12})\Phi_2'(z_2)]\mathrm{d}x + p(y)$$

$$= 2 \int \mathfrak{Re}\left[(S_{11}s_1^2 + S_{12})\frac{\mathrm{d}\Phi_1(z_1)}{\mathrm{d}z_1} + (S_{11}s_2^2 + S_{12})\frac{\mathrm{d}\Phi_2'(z_2)}{\mathrm{d}z_2}\right]\mathrm{d}x + p(y)$$

$$= 2 \int \mathfrak{Re}\left[(S_{11}s_1^2 + S_{12})\frac{\mathrm{d}\Phi_1(z_1)}{\mathrm{d}x} \underset{\overset{\nearrow 1}{\mathrm{d}z_1}}{\mathrm{d}x} + (S_{11}s_2^2 + S_{12})\frac{\mathrm{d}\Phi_2(z_2)}{\mathrm{d}x} \underset{\overset{\nearrow 1}{\mathrm{d}z_2}}{\mathrm{d}x}\right]\mathrm{d}x + p(y)$$

$$= 2\mathfrak{Re}[(S_{11}s_1^2 + S_{12})\Phi_1(z_1) + (S_{11}s_2^2 + S_{12})\Phi_2(z_2)] + p(y)$$

Following the same steps for v, the displacement along the y-axis becomes

$$v = \int \varepsilon_y \mathrm{d}y + q(x)$$

$$= 2 \int \mathfrak{Re}[(S_{12}s_1^2 + S_{22})\Phi_1'(z_1) + (S_{12}s_2^2 + S_{22})\Phi_2'(z_2)]\mathrm{d}y + q(x)$$

$$= 2 \int \mathfrak{Re}\left[(S_{12}s_1^2 + S_{22})\frac{\mathrm{d}\Phi_1(z_1)}{\mathrm{d}z_1} + (S_{12}s_2^2 + S_{22})\frac{\mathrm{d}\Phi_2'(z_2)}{\mathrm{d}z_2}\right]\mathrm{d}y + q(x)$$

$$= 2 \int \mathfrak{Re}\left[(S_{12}s_1^2 + S_{22})\frac{\mathrm{d}\Phi_1(z_1)}{\mathrm{d}y} \underset{\overset{\nearrow \frac{1}{s_1}}{\mathrm{d}z_1}}{\mathrm{d}y} + (S_{12}s_2^2 + S_{22})\frac{\mathrm{d}\Phi_2(z_2)}{\mathrm{d}y} \underset{\overset{\nearrow \frac{1}{s_2}}{\mathrm{d}z_2}}{\mathrm{d}y}\right]\mathrm{d}y + q(x)$$

$$= 2\mathfrak{Re}\left[\left(\frac{S_{12}s_1^2 + S_{22}}{s_1}\right)\Phi_1(z_1) + \left(\frac{S_{12}s_2^2 + S_{22}}{s_2}\right)\Phi_2(z_2)\right] + q(x)$$

Plugging in the expressions for u and v in γ_{xy} (Eq(2.25)$_3$), one is able to calculate $p(y)$ and $q(y)$.

$$p(y) = \omega_0 y + u_0$$
$$q(x) = -\omega_0 x + v_0$$

The above expressions represent the rigid-body motion where ω_0 is the rotation about the z-axis, u_0 and v_0 are the translation along the x and y axes. Then, u and v can be written as

$$u = 2\mathfrak{Re}[u_1\Phi_1(z_1) + u_2\Phi_2(z_2)] + \omega_0 y + u_0$$
$$v = 2\mathfrak{Re}[v_1\Phi_1(z_1) + v_2\Phi_2(z_2)] - \omega_0 x + v_0 \qquad (2.26)$$

where

$$\begin{cases} u_1 = S_{11}s_1^2 + S_{12} \\ u_2 = S_{11}s_2^2 + S_{12} \end{cases} , \begin{cases} v_1 = \dfrac{S_{12}s_1^2 + S_{22}}{s_1} \\ \\ v_2 = \dfrac{S_{12}s_2^2 + S_{22}}{s_2} \end{cases} \tag{2.27}$$

2.5   Boundary-Value Problems Formulation

In this section, we formulate the first two boundary-value problems in Eq(2.12) and Eq(2.12) for the orthotropic body under study.

2.5.1   Traction Boundary-Value Problem

Let S be an area in the xy-plane. To include the general case, we also assume an opening in the domain (there might be several other openings); in other words, S is a simply-connected domain (Figure. 3(a)). Select a starting point on the external boundary and start moving on the boundary in the clockwise direction (the hatched area is on your left). Cut out an infinitesimal element of the external boundary, ds, and plot its normal vector n (Figure. 3(b)). Since ds is very small, we can assume it as a straight line and use the vector summation to plot dx and dy as depicted in Figure. 3(b). We can write

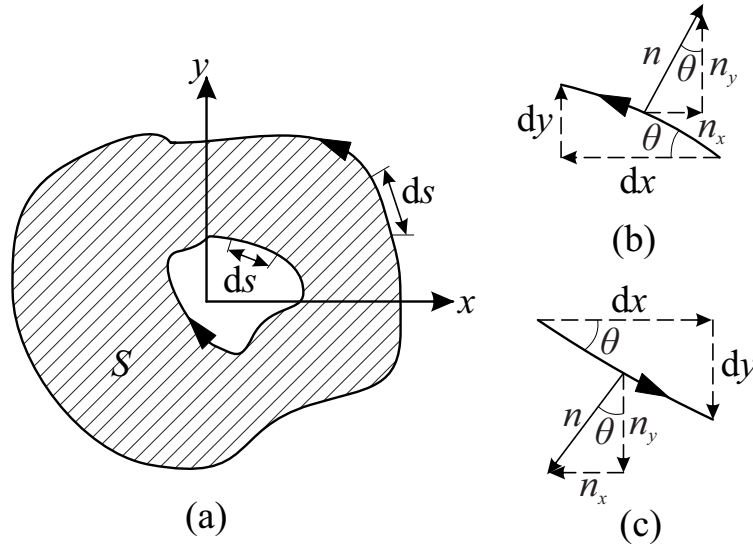$$x = x(s) \quad , y = y(s) \tag{2.28}$$

Figure 3: (a) Schematic representation of the simply-connected domain. (b) Infinitesimal element on the external boundary. (c) Infinitesimal element on the internal boundary.

On the external boundary,

$$\cos(n,y) = \cos\theta = \frac{n_y}{n} = \frac{-\mathrm{d}x}{\mathrm{d}s}$$

$$\cos(n,x) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \frac{n_x}{n} = \frac{\mathrm{d}y}{\mathrm{d}s} \qquad (2.29)$$

Analogous to the external, we select another infinitesimal element on the internal boundary. This time, we need to move clockwise (Figure. 3(c)), as it is required to follow the same rule (material on the left-hand side) for the external boundary.

$$\cos(n,y) = \cos\theta = \frac{-n_y}{n} = \frac{\mathrm{d}x}{\mathrm{d}s}$$

$$\cos(n,x) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \frac{-n_x}{n} = \frac{-\mathrm{d}y}{\mathrm{d}s} \qquad (2.30)$$

Let us remove the bar sign from Eq(2.11) and replace stress components on the left-hand side with their equivalent values from Eq(2.8).

$$\begin{cases} \left(\dfrac{\partial^2\phi}{\partial y^2} + V\right)\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial^2\phi}{\partial x\partial y}\dfrac{\mathrm{d}x}{\mathrm{d}s} = X \\[3mm] -\dfrac{\partial^2\phi}{\partial x\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}s} + \left(\dfrac{\partial^2\phi}{\partial x^2} + V\right)\dfrac{-\mathrm{d}x}{\mathrm{d}s} = Y \end{cases}$$

$$\begin{cases} \dfrac{\partial^2\phi}{\partial y^2}\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial^2\phi}{\partial x\partial y}\dfrac{\mathrm{d}x}{\mathrm{d}s} = X - V\dfrac{\mathrm{d}y}{\mathrm{d}s} \\[3mm] \dfrac{\partial^2\phi}{\partial x\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial^2\phi}{\partial x^2}\dfrac{\mathrm{d}x}{\mathrm{d}s} = -Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial\phi}{\partial y}\right)\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial\phi}{\partial y}\right)\dfrac{\mathrm{d}x}{\mathrm{d}s} = X - V\dfrac{\mathrm{d}y}{\mathrm{d}s} \\[3mm] \dfrac{\partial}{\partial y}\left(\dfrac{\partial\phi}{\partial x}\right)\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial\phi}{\partial x}\right)\dfrac{\mathrm{d}x}{\mathrm{d}s} = -Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s} \end{cases}$$

From Eq(2.28), one can write

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\partial}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} \qquad (2.31)$$

therefore,

$$\begin{cases} \dfrac{\partial}{\partial s}\left(\dfrac{\partial \phi}{\partial y}\right) = X - V\dfrac{\mathrm{d}y}{\mathrm{d}s} \\[2ex] \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right) = -Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s} \end{cases}$$

$$\begin{cases} \dfrac{\partial \phi}{\partial y} = \displaystyle\int_{s_0}^{s_1}\left(X - V\dfrac{\mathrm{d}y}{\mathrm{d}s}\right)\mathrm{d}s + c_1 \\[3ex] \dfrac{\partial \phi}{\partial x} = \displaystyle\int_{s_0}^{s_1}\left(-Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s}\right)\mathrm{d}s + c_2 \end{cases}$$

where s_0 and s_1 are the lower and upper limits on the boundary. Substitute ϕ from Eq(2.23) in the above equation gives

$$\frac{\partial \phi}{\partial x} = \frac{\mathrm{d}\phi}{\mathrm{d}z}\frac{\partial z}{\partial x} = \Phi_1(z_1) + \Phi_2(z_2) + \overline{\Phi_1(z_1)} + \overline{\Phi_2(z_2)} = 2\mathfrak{Re}[\Phi_1(z_1) + \Phi_2(z_2)]$$

$$\frac{\partial \phi}{\partial y} = \frac{\mathrm{d}\phi}{\mathrm{d}z}\frac{\partial z}{\partial y} = s_1\Phi_1(z_1) + s_2\Phi_2(z_2) + \overline{s_1}\overline{\Phi_1(z_1)} + \overline{s_2}\overline{\Phi_2(z_2)} = 2\mathfrak{Re}[s_1\Phi_1(z_1) + s_2\Phi_2(z_2)]$$

Finally, the resultant force on the external boundary become

$$\begin{cases} 2\mathfrak{Re}[s_1\Phi_1(z_1) + s_2\Phi_2(z_2)] = \displaystyle\int_{s_0}^{s_1}\left(X - V\dfrac{\mathrm{d}y}{\mathrm{d}s}\right)\mathrm{d}s + c_1 \\[3ex] 2\mathfrak{Re}[\Phi_1(z_1) + \Phi_2(z_2)] = \displaystyle\int_{s_0}^{s_1}\left(-Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s}\right)\mathrm{d}s + c_2 \end{cases} \tag{2.32}$$

and on the internal boundary, we get

$$\begin{cases} 2\mathfrak{Re}[s_1\Phi_1(z_1) + s_2\Phi_2(z_2)] = \displaystyle\int_{s_0}^{s_1}\left(-X - V\dfrac{\mathrm{d}y}{\mathrm{d}s}\right)\mathrm{d}s + c_3 \\[3ex] 2\mathfrak{Re}[\Phi_1(z_1) + \Phi_2(z_2)] = \displaystyle\int_{s_0}^{s_1}\left(Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s}\right)\mathrm{d}s + c_4 \end{cases} \tag{2.33}$$

In Eq(2.32) and Eq(2.33), c_1 to c_4 are integral constants. See Appendix. F for the derivation of Eq(2.33).

2.5.2 Displacement Boundary-Value Problems

The prescribed displacement has already been derived in Eq(2.26)

$$2\Re[u_1\Phi_1(z_1) + u_2\Phi_2(z_2)] = u^*$$
$$2\Re[v_1\Phi_1(z_1) + v_2\Phi_2(z_2)] = v^* \tag{2.34}$$

See Eq(2.27) for u_1, u_2, v_1 and v_2 coefficients.

3 Stress Concentration Around A Circular Hole in A Thin Orthotropic Plate

Lekhnitskii formulation for the anisotropic plates, especially with irregularities such as holes and/or discontinuities in the plate domain (not on the plate edge) is quite useful [1]. Nevertheless, one should know that Lekhnitskii formulation is built upon two principal assumptions: (1) The domain requires to be simply or multiply-connected. (2) The size of the irregularity should be incomparable with other geometrical dimensions [2]. The aforementioned lines explain the reason behind the effectiveness and attractiveness of Lekhnitskii theory in analyzing stress around cut-offs in anisotropic plates, *e.g.*, pin-hole structures and elliptical/rectangular (with rounded corners) windows in an aeroplane fuselage. In this section, we use Lekhnitskii theory for a thin orthotropic elastic body to formulate the stress distribution around a circular hole located at the origin in an infinite domain. We restrict our study to a two-dimensional case with prescribed tractions and zero body forces. The prescribed displacement can be formulated in an analogous procedure. The contents of the section are obtained from [1] and [5].

3.1 Laurent Series

As mentioned in Section. 2.3.5, the potential stress function, $\phi(x,y)$, in is an analytical function; thus, it can be written in a form of a Laurent series (Laurent's Theorem). Furthermore, $\phi(x,y)$ should satisfy the boundary conditions explained in Section. 2.5. We keep on the solution with Φ instead if ϕ. Remember that these two can be correlated using Eq(2.23) and Eq(3.1).

$$\frac{\mathrm{d}f_k(z_k)}{\mathrm{d}z_k} = \Phi_k(z_k) \quad \text{where} \quad k = 1,2 \tag{3.1}$$

Φ can be written as a Laurent series with both positive and negative powers [5].

$$\Phi_1(z_1) = \sum_{n=-\infty}^{+\infty} g_n^{(1)} z_1^n \quad , \Phi_2(z_2) = \sum_{n=-\infty}^{+\infty} g_n^{(2)} z_2^n \quad \text{where} \quad \begin{cases} z_1 = x + s_1 y \\ \\ z_2 = x + s_2 y \end{cases} \tag{3.2}$$

$g_n^{(1)}$ and $g_n^{(2)}$ are unknown coefficients and are required to be determined. Substitute Eq(3.2) in Eq(2.24)$_{1,2}$

$$\Phi_k'(z_k) = ... + g_{-2}^{(k)}z_k^{-2} + g_{-1}^{(k)}z_k^{-1} + g_0^{(k)} + g_1^{(k)}z_k^1 + g_2^{(k)}z_k^2 + ... \text{ where } k = 1,2$$

Due to the bounded stress fields, the terms with positive powers should be remove from series; therefore, $\Phi_k'(z_k)$ reduces to

$$\Phi_k'(z_k) = g_0^{(k)} + g_{-1}^{(k)}z_k^{-1} + g_{-2}^{(k)}z_k^{-2} + ... \text{ where } k = 1,2$$

Integrating $\Phi_k'(z_k)$ w.r.t z_k gives

$$\Phi_k(z_k) = g_0^{(k)}z_k + g_{-1}^{(k)}\ln z_k + \sum_{n=-1}^{+\infty} g_n^{(k)}z_k^{-n} + c \text{ where } k = 1,2$$

where c is the integral constant. It is preferable to rewrite the above equation as

$$\Phi_k(z_k) = h_k z_k + A_k \ln z_k + \sum_{n=-1}^{+\infty} g_n^{(k)}z_k^{-n} + c \text{ where } k = 1,2 \tag{3.3}$$

Eq(3.3) includes three types of terms.

1. h_k, which characterizes the force resultant for a domain without a hole(s) (homogeneous stress fields), see Eq(2.24).

2. A_k, which characterizes the force resultant around the hole, see Eq(2.32) and Eq(2.33).

3. $g_n^{(k)}$, which characterizes the disturbances due to the hole.

Since Lekhnitskii solution is developed for linear elastic composite laminates, we treat each term separately and the final solution is the superposition of all the terms.

3.2   Homogeneous Stress Field, h_k

To find h_k coefficients, we use the homogeneous boundary conditions for a domain without a hole. Here we assume a rectangular domain. The general boundary condition in the form of pre-scribed traction for a two-dimensional domain is illustrated in Figure. 4.

From Eq(2.32), one can write the following equations, where h_k is unknown.

$$2\Re\mathfrak{e}[s_1\Phi_1(z_1) + s_2\Phi_2(z_2)] = \int_0^s X\mathrm{d}s = \int_0^s p_x\mathrm{d}y + p_{xy}(-\mathrm{d}x) = -p_{xy}x + p_x y$$

$$2\Re\mathfrak{e}[\Phi_1(z_1) + \Phi_2(z_2)] = -\int_0^s Y\mathrm{d}s = -\int_0^s p_{xy}\mathrm{d}y + p_y(-\mathrm{d}x) = p_y x - p_{xy}y \tag{3.4}$$

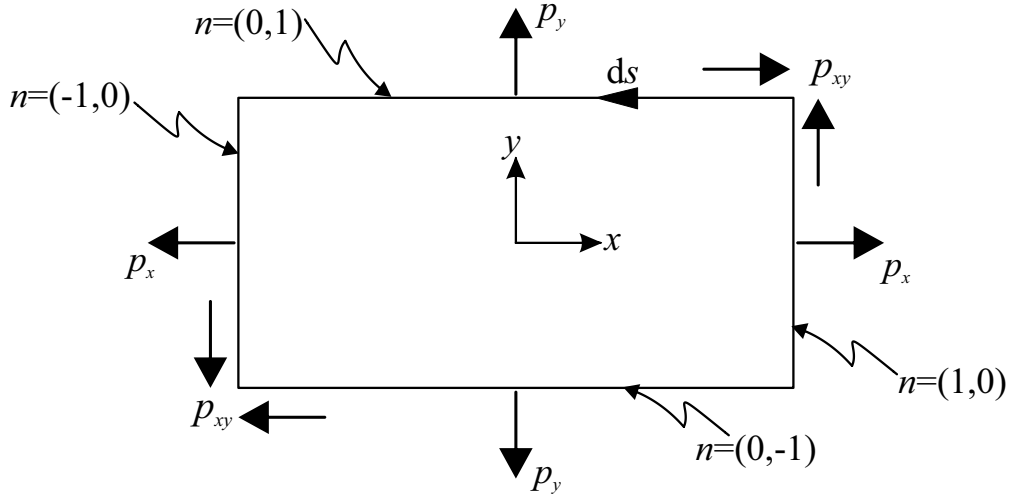For the case $r - a \geq 0$, it is shown that (see Eq(E.1))

Figure 4: The two-dimensional domain without a hole under in-plane loads [1]. n designates the normal vector.

$$s_1 = \sqrt{\frac{r-a}{2}} + i\sqrt{\frac{r+a}{2}} = \alpha + i\beta$$

$$s_2 = -\sqrt{\frac{r-a}{2}} + i\sqrt{\frac{r+a}{2}} = -\alpha + i\beta$$

Plug in s_1 and s_2 from above and replace $\Phi_k(z_k)$ with $h_k z_k$ in Eq(3.4) result in four new equations in which the first and last are the same.

$$2\alpha h_1 - 2\alpha h_2 = -p_{xy}$$
$$2\alpha^2 h_1 + 2\alpha^2 h_2 - 2\beta^2 h_1 - 2\beta^2 h_2 = p_x$$
$$2h_1 + 2h_2 = p_y$$
$$2\alpha h_1 - 2\alpha h_2 = -p_{xy}$$

Jong [3] showed that the fourth equation can be replaced by restricting the rotation. The rotation about z-axis, ω_z, is

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right)$$

with u and v given in Eq(2.26). Solving four equations simultaneously gives [3]

$$h_k = \frac{p_x - p_y s_l^2}{2(s_k^2 - s_l^2)} - \frac{p_{xy}}{4s_k} \qquad \text{where} \quad \begin{cases} k = 1, 2 \\ \\ l = 2, 1 \end{cases} \tag{3.5}$$

See the Maple file, h__k.mw for the case $r - a < 0$ which takes the same form as Eq(3.5).

3.3   Force Resultant Around Hole, A_k

Figure. 5 provides a schematic of the two-dimensional domain with a circular hole under resultant load, R. Let us assume the hole radius, r, is unity (see Appendix. G).
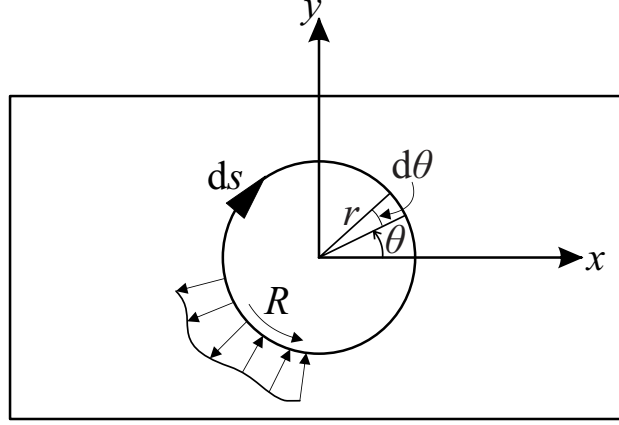

Figure 5: The two-dimensional domain with a hole under resultant load, R.

To derive A_k, we use Eq(2.33), as we solve a traction boundary-value problem on the internal boundary.

$$\begin{cases} 2\mathfrak{Re}[s_1 A_1 \ln z_1 + s_2 A_2 \ln z_2]_{s_0}^{s_1} = -\int_{s_0}^{s_1} X \mathrm{d}s \\[2em] 2\mathfrak{Re}[A_1 \ln z_1 + A_2 \ln z_2]_{s_0}^{s_1} = \int_{s_0}^{s_1} Y \mathrm{d}s \end{cases} \xrightarrow{\mathrm{d}s = r\mathrm{d}\theta \overset{r=1}{=} \mathrm{d}\theta} \begin{cases} 2\mathfrak{Re}[s_1 A_1 \ln z_1 + s_2 A_2 \ln z_2]_{s_0}^{s_1} = -\int_{s_0}^{s_1} X \mathrm{d}s \\[2em] 2\mathfrak{Re}[A_1 \ln z_1 + A_2 \ln z_2]_{s_0}^{s_1} = \int_{s_0}^{s_1} Y \mathrm{d}s \end{cases}$$

$$(3.6)$$

where we also need to integrate the left-hand side of equations on the contour from s_0 to s_1. We follow the solution by integrating $\ln z$.

$$\ln(z)\Big|_{s_0}^{s_1} \overset{z=re^{i\theta}}{=} \ln(re^{i\theta})\Big|_{\theta_0}^{\theta_0 + 2m\pi} = [\ln r + i\theta]_{\theta_0}^{\theta_0 + 2m\pi}$$

$$= \ln r + i(\theta_0 + 2m\pi) - \ln r - i\theta_0 = i(2m\pi) \overset{m=1}{=} i(2\pi) \qquad (3.7)$$

Substitute $\ln z$ from above in Eq(3.6)

$$\begin{cases} 2\mathfrak{Re}[s_1 A_1 i(2\pi) + s_2 A_2 i(2\pi)] = -\int_{s_0}^{s_1} X \mathrm{d}s = -R_x \\[2em] 2\mathfrak{Re}[A_1 i(2\pi) + A_2 i(2\pi)] = \int_{s_0}^{s_1} Y \mathrm{d}s = R_y \end{cases}$$

or

$$\begin{cases} s_1 A_1 + s_2 A_2 - \overline{s_1}\overline{A_1} - \overline{s_2}\overline{A_2} = -\dfrac{R_x}{i(2\pi)} \\[3mm] A_1 + A_2 - \overline{A_1} - \overline{A_2} = \dfrac{R_y}{i(2\pi)} \end{cases} \qquad (3.8)$$

Assuming that the material during deformation remains continuous and is free of any discontinuity and/or defect [1], the prescribed displacement around the hole is zero. From Eq(2.34)

$$2\Re[u_1 A_1 \ln z_1 + u_2 A_2 \ln z_2] = 0$$
$$2\Re[v_1 A_1 \ln z_1 + v_2 A_2 \ln z_2] = 0$$

which can be rewritten in form of complex conjugates.

$$u_1 A_1 + u_2 A_2 - \overline{u_1}\overline{A_1} - \overline{u_2}\overline{A_2} = 0$$
$$v_1 A_1 + v_2 A_2 - \overline{v_1}\overline{A_1} - \overline{v_2}\overline{A_2} = 0 \qquad (3.9)$$

Solving the four linear equations from Eq(3.8) and Eq(3.9) gives [1]

$$A_k = i \frac{R_y u_l + R_x w_k}{4\pi C_{11}(s_k^2 - s_l^2)} \quad \text{where} \quad w_k = \frac{C_{12}}{s_k} + C_{11} s_k \quad, \qquad \begin{cases} k = 1,2 \\[2mm] l = 2,1 \end{cases} \qquad (3.10)$$

C_{11}, C_{12} are the compliance matrix components and are

$$C_{11} = \frac{1}{E_{11}} \quad C_{12} = -\frac{v_{21}}{E_{11}}$$

E_{11} is the longitudinal Young's modulus and v_{12} is in-plane Poisson's ratio. u_l is given in Eq(2.27).

During the development of equations for the determination of A_k, a few interesting properties have been observed.

1. *A_k is multi-valued*: From Eq(3.7), one may notices that for various integers of m, A_k exploits different values. This is also mentioned by Lekhnitskii [2] during his discussion on the uniqueness of Φ_k.

2. *Stress singularity*: Using Eq(2.24), the stress components for $A_k \ln z_k$ are

$$\frac{\mathrm{d}}{\mathrm{d}z_k}(A_k \ln z_k) = \frac{A_k}{z_k}$$
$$\sigma_x = 2\Re[s_1^2 \frac{A_2}{z_2} + s_2^2 \frac{A_2}{z_2}]$$

$$\sigma_y = 2\mathfrak{Re}[\frac{A_1}{z_1} + \frac{A_2}{z_2}]$$

$$\tau_{xy} = -2\mathfrak{Re}[s_1 \frac{A_1}{z_1} + s_2 \frac{A_2}{z_2}]$$

which means $A_k \ln z_k$ causes singular stress fields at the hole center. However, due the absence of material at the hole center, this stress singularity does not affect the solution.

3. A_k triggers when the force resultant is not equal to zero on openings, even if one opening/hole is loaded, A_k is considered in the Laurent series.

3.4 Disturbance Field, $g_n^{(k)}$
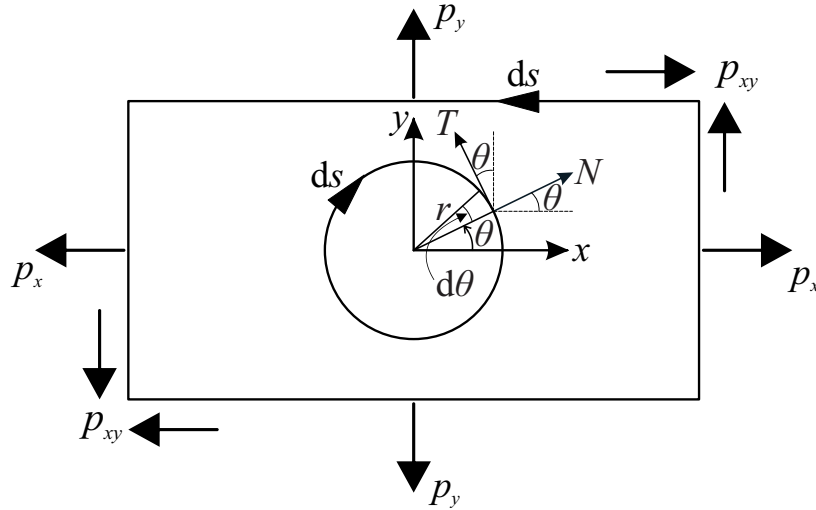
Figure 6: The two-dimensional domain with a hole undergoing in-plane loading at infinity, radial, N, and tangential, T, load on the hole.

Due to the nature of $g_n^{(k)}$ constants, we consider the whole domain with the hole and applied boundary condition at infinity and on the edge hole (see Figure. 6). To determine $g_n^{(k)}$, we assume a harmonic boundary condition in the form of prescribed traction on the hole edge. Provided that the prescribed boundary condition on the hole edge can be represented by a Fourier series [1], one can write the radial, N and tangential, T resultant loads as

$$N(m,\theta) = N_1 + \sum_{n=1}^{m}[\cos(n\theta)N_{n+1}^{(c)} + \sin(n\theta)N_{n+1}^{(s)}]$$

$$T(m,\theta) = T_1 + \sum_{n=1}^{m}[\cos(n\theta)T_{n+1}^{(c)} + \sin(n\theta)T_{n+1}^{(s)}] \tag{3.11}$$

where N and T are real constants and $0 \leq \theta \leq 2\pi$, superscript c and s designate the cosine and sine terms in the Fourier series and N_1 and T_1 are the first terms in the cosine Fourier series.

Let us decompose N and T on the hole edge in the x and y-direction so that we are able to write the load resultant equations in both x and y axes.

$$
\begin{aligned}
X(m,\theta) &= N(m,\theta)\cos\theta - T(m,\theta)\sin\theta \\
Y(m,\theta) &= N(m,\theta)\sin\theta + T(m,\theta)\cos\theta
\end{aligned}
\tag{3.12}
$$

Next, we invoke Eq(2.33) to constitute the required equations for the traction boundary-value problem.

$$
2\Re\mathfrak{e}\left[\sum_{k=1}^{2} s_k(h_k z_k + A_k \ln z_k + \sum_{n=-1}^{+\infty} g_n^{(k)} z_k^{-n})\right] = -\int_0^s X \mathrm{d}s
$$

$$
2\Re\mathfrak{e}\left[\sum_{k=1}^{2} (h_k z_k + A_k \ln z_k + \sum_{n=-1}^{+\infty} g_n^{(k)} z_k^{-n})\right] = \int_0^s Y \mathrm{d}s
$$

Substitute the equivalent expression from Eq(3.4) and $\ln z = i\theta$ from Section. 3.3, the above set of equations becomes

$$
2\Re\mathfrak{e}\left[\sum_{k=1}^{2} s_k(\sum_{n=-1}^{+\infty} g_n^{(k)} z_k^{-n})\right] = -\int_0^s X \mathrm{d}s + p_{xy}x - p_x y + \frac{R_x \theta}{2\pi}
$$

$$
2\Re\mathfrak{e}\left[\sum_{k=1}^{2} (\sum_{n=-1}^{+\infty} g_n^{(k)} z_k^{-n})\right] = \int_0^s Y \mathrm{d}s - p_y x + p_{xy}y - \frac{R_y \theta}{2\pi}
$$

Next, we replace $x = r\cos\theta \overset{r=1}{=} \cos\theta$, $y = r\sin\theta \overset{r=1}{=} \sin\theta$, X and Y from Eq(3.12). Notice that $\dfrac{R_x \theta}{2\pi}$ and $\dfrac{R_y \theta}{2\pi}$ are vanished (see Appendix. H); therefore, we conclude

$$
2\Re\mathfrak{e}\left[\sum_{k=1}^{2} s_k(\sum_{n=-1}^{+\infty} g_n^{(k)} z_k^{-n})\right] = -(p_x + N_1)\sin\theta + (p_{xy} - T_1)\cos\theta
$$

$$
- \int_0^s \left\{ \sum_{n=1}^{m} [\cos(n\theta)N_{n+1}^{(c)} + \sin(n\theta)N_{n+1}^{(s)}]\cos\theta \right.
$$

$$
\left. - \sum_{n=1}^{m} [\cos(n\theta)T_{n+1}^{(c)} + \sin(n\theta)T_{n+1}^{(s)}]\sin\theta \right\} \mathrm{d}\theta
\tag{3.13}
$$

$$2\mathfrak{Re}[\sum_{k=1}^{2} s_k (\sum_{n=-1}^{+\infty} g_n^{(k)} z_k^{-n})] = -(p_y + N_1)\cos\theta + (p_{xy} + T_1)\sin\theta$$

$$+ \int_0^s \left\{ \sum_{n=1}^{m} [\cos(n\theta)N_{n+1}^{(c)} + \sin(n\theta)N_{n+1}^{(s)}]\sin\theta \right.$$

$$\left. + \sum_{n=1}^{m} [\cos(n\theta)T_{n+1}^{(c)} + \sin(n\theta)T_{n+1}^{(s)}]\cos\theta \right\} \mathrm{d}\theta \qquad (3.14)$$

Looking at Eq(3.13) and Eq(3.14), one may notice that the left-hand side is expressed via z_k while the right-hand side includes θ terms. The two variables can be correlated through $z = e^{i\theta}$ and a suitable mapping (see Eq(G.1) for an elliptical and Eq(G.2) for a circular hole). Consequently, we get a mapped expression, $\sum_{n=-1}^{+\infty} c_n^{(k)} \zeta_k^{-n}$, introducing a new constant $c_n^{(k)}$ and mapped complex variable ζ_k^{-n}. According to the fundamental theorem of Algebra, the coefficients of two series can be extracted by equating the two series [7]. Let us assume [5] the prescribed traction on the hole edge as a Fourier series of order 1 (only the first term in the series are included).

$$X(\theta) = X_c \cos\theta + X_s \sin\theta$$
$$Y(\theta) = Y_c \cos\theta + Y_s \sin\theta \qquad (3.15)$$

where the subscripts c and s represent the cosine and sine terms, respectively. Substitute $X(\theta)$ and $Y(\theta)$ from above into the left-hand side of Eq(2.33)

$$\int Y(\theta)\mathrm{d}s = \int Y_c \cos\theta + Y_s \sin\theta \mathrm{d}\theta = Y_c \sin\theta - Y_s \cos\theta$$

$$-\int X(\theta)\mathrm{d}s = -\int X_c \cos\theta + X_s \sin\theta \mathrm{d}\theta = -X_c \sin\theta + X_s \cos\theta \qquad (3.16)$$

On the right-hand side of Eq(2.33)

$$2\mathfrak{Re}[s_1\Phi_1(z_1) + s_2\Phi_2(z_2)] = 2\mathfrak{Re}[h_1 z_1 + \cancelto{0}{A_1 \ln z_1} + g_1^{(1)} z_1^{-1} + h_2 z_2 + \cancelto{0}{A_2 \ln z_2} + g_1^{(2)} z_2^{-1}]$$

$$2\mathfrak{Re}[\Phi_1(z_1) + \Phi_2(z_2)] = 2\mathfrak{Re}[s_1 h_1 z_1 + \cancelto{0}{s_1 A_1 \ln z_1} + s_1 g_1^{(1)} z_1^{-1} + s_2 h_2 z_2 + \cancelto{0}{s_2 A_2 \ln z_2} + s_2 g_1^{(2)} z_2^{-1}]$$

$$(3.17)$$

where we only included the power of ± 1 in the series. $A_k \ln z$ is zero due to the harmonic assumption of the resultant load on the hole edge (see Appendix. H). We then continue with Eq(3.16)$_1$ and Eq(3.17)$_1$.

$$2\mathfrak{Re}[h_1 z_1 + h_2 z_2 + g_1^{(1)} z_1^{-1} + g_1^{(2)} z_2^{-1}] = Y_c \sin\theta - Y_s \cos\theta$$

$$2\mathfrak{Re}[g_1^{(1)} z_1^{-1} + g_1^{(2)} z_2^{-1}] = Y_c \sin\theta - Y_s \cos\theta - 2\mathfrak{Re}[h_1 z_1 + h_2 z_2]$$

$$2\mathfrak{Re}[g_1^{(1)} z_1^{-1} + g_1^{(2)} z_2^{-1}] = Y_c \sin\theta - Y_s \cos\theta - (p_y x - p_{xy} y) \tag{3.18}$$

Using the following properties ($r = 1$)

$$z = e^{i\theta} \qquad ,z = x + iy \quad ,z^{-1} = x - iy \qquad ,x = \cos\theta = \mathfrak{Re}[z]$$

$$z^{-1} = re^{-i\theta} \quad ,iz = ix - y \quad ,iz^{-1} = ix + y \quad ,y = \sin\theta = \mathfrak{Re}[iz^{-1}] \tag{3.19}$$

Eq(3.18)$_3$ can be rewritten as

$$2\mathfrak{Re}[g_1^{(1)} z_1^{-1} + g_1^{(2)} z_2^{-1}] = Y_c \mathfrak{Re}[iz^{-1}] - Y_s \mathfrak{Re}[z^{-1}]] - (p_y \mathfrak{Re}[z^{-1}] - p_{xy} \mathfrak{Re}[iz^{-1}])$$

The left-hand side of the above equation can be written as

$$Y_c \left(\frac{iz^{-1} - iz}{2} \right) - Y_s \left(\frac{z^{-1} + z}{2} \right) - \left[p_y \left(\frac{z^{-1} + z}{2} \right) - p_{xy} \left(\frac{iz^{-1} - iz}{2} \right) \right] \tag{3.20}$$

and the left-hand side can also be written as

$$g_1^{(1)} z_1^{-1} + g_1^{(2)} z_2^{-1} + \overline{g_1^{(1)}} \, \overline{z_1^{-1}} + \overline{g_1^{(2)}} \, \overline{z_2^{-1}} = g_1^{(1)} z_1^{-1} + g_1^{(2)} z_2^{-1} + \overline{g_1^{(1)}} z_1 + \overline{g_1^{(2)}} z_2 \tag{3.21}$$

As it is seen, Eq(3.20) is a function of z while Eq(3.21) is a function of z_1 and z_2. Utilizing a correct mapping, it can be shown that

$$\sum_{n=1}^{\infty} g_n^{(k)} z_k^{-n} = \sum_{n=1}^{\infty} c_n^{(k)} \zeta_k^{-n} = \sum_{n=1}^{\infty} c_n^{(k)} z^{-n} \tag{3.22}$$

where $c_n^{(k)} \zeta_k^{-n}$ are the mapped version of $g_n^{(k)} z_k^{-n}$ term. The proof for a circular hole of unit radius is provided in Appendix. I. Notice that we use ζ_k instead of z_k to avoid misinterpreting with z_1 and z_2 in Eq(2.20) to Eq(2.23). Exploiting the mapping from Eq(3.22), Eq(3.21) becomes (only available on the hole edge)

$$g_1^{(1)} z_1^{-1} + g_1^{(2)} z_2^{-1} + \overline{g_1^{(1)}} z_1 + \overline{g_1^{(2)}} z_2 = c_1^{(1)} z^{-1} + c_1^{(2)} z^{-1} + \overline{c_1^{(1)}} z + \overline{c_1^{(2)}} z$$

Consequently, from above equation and Eq(3.20)

$$c_1^{(1)}z^{-1} + c_1^{(2)}z^{-1} + \overline{c_1^{(1)}}z + \overline{c_1^{(2)}}z =$$
$$Y_c\left(\frac{iz^{-1}-iz}{2}\right) - Y_s\left(\frac{z^{-1}+z}{2}\right) - \left[p_y\left(\frac{z^{-1}+z}{2}\right) - p_{xy}\left(\frac{iz^{-1}-iz}{2}\right)\right]$$

By collecting identical coefficients of z and z^{-1}, we get

$$c_1^{(1)} + c_1^{(2)} = \frac{1}{2}(iY_c - Y_s - p_y + p_{xy})$$
$$\overline{c_1^{(1)}} + \overline{c_1^{(2)}} = \frac{1}{2}(-iY_c - Y_s - p_y - ip_{xy}) \tag{3.23}$$

The second equation in Eq(3.23) is the complex conjugate of the first equation. We need to derive another equation to find $c_1^{(1)}$ and $c_1^{(2)}$. Let us continue with Eq(3.16)$_2$ and Eq(3.17)$_2$.

$$2\Re\mathfrak{e}[s_1h_1z_1 + s_2h_2z_2 + s_1g_1^{(1)}z_1^{-1} + s_2g_1^{(2)}z_2^{-1}] = -X_c\sin\theta + X_s\cos\theta$$
$$2\Re\mathfrak{e}[s_1g_1^{(1)}z_1^{-1} + s_2g_1^{(2)}z_2^{-1}] = -X_c\sin\theta + X_s\cos\theta - 2\Re\mathfrak{e}[s_1h_1z_1 + s_2h_2z_2]$$
$$2\Re\mathfrak{e}[s_1g_1^{(1)}z_1^{-1} + s_2g_1^{(2)}z_2^{-1}] = -X_c\sin\theta + X_s\cos\theta - (-p_{xy}x + p_xy) \tag{3.24}$$

Invoking Eq(3.19), the right-hand side of Eq(3.24)$_3$ becomes

$$-X_c\Re\mathfrak{e}[iz^{-1}] + X_s\Re\mathfrak{e}[z^{-1}]] - (-p_{xy}\Re\mathfrak{e}[z^{-1}] + p_x\Re\mathfrak{e}[iz^{-1}]) =$$
$$-X_c\left(\frac{iz^{-1}-iz}{2}\right) + X_s\left(\frac{z^{-1}+z}{2}\right) - \left[-p_{xy}\left(\frac{z^{-1}+z}{2}\right) + p_x\left(\frac{iz^{-1}-iz}{2}\right)\right] \tag{3.25}$$

and the left-hand side can be written as

$$s_1g_1^{(1)}z_1^{-1} + s)2g_1^{(2)}z_2^{-1} + \overline{s_1}\overline{g_1^{(1)}}\overline{z_1^{-1}} + \overline{s_2}\overline{g_1^{(2)}}\overline{z_2^{-1}} = \overline{s_1}g_1^{(1)}z_1^{-1} + \overline{s_2}g_1^{(2)}z_2^{-1} + \overline{s_1}\overline{g_1^{(1)}}z_1 + \overline{s_2}\overline{g_1^{(2)}}z_2 \tag{3.26}$$

Following the same mapping strategy from Eq(3.22), Eq(3.26) becomes

$$s_1g_1^{(1)}z_1^{-1} + s_1g_1^{(2)}z_2^{-1} + \overline{s_1}\overline{g_1^{(1)}}z_1 + \overline{s_2}\overline{g_1^{(2)}}z_2 = s_1c_1^{(1)}z^{-1} + s_2c_1^{(2)}z^{-1} + \overline{s_1}\overline{c_1^{(1)}}z + \overline{s_2}\overline{c_1^{(2)}}z$$

By equating the above equation and Eq(3.25)

$$s_1 c_1^{(1)} z^{-1} + s_1 c_1^{(2)} z^{-1} + \overline{s_1} \overline{c_1^{(1)}} z + \overline{s_2} \overline{c_1^{(2)}} z =$$
$$-X_c \left(\frac{iz^{-1} - iz}{2} \right) + X_s \left(\frac{z^{-1} + z}{2} \right) - \left[-p_{xy} \left(\frac{z^{-1} + z}{2} \right) + p_x \left(\frac{iz^{-1} - iz}{2} \right) \right]$$

and collecting identical terms, we get

$$s_1 c_1^{(1)} + s_2 c_1^{(2)} = \frac{1}{2} (-iX_c + X_s - p_{xy} - ip_x)$$
$$\overline{s_1} \overline{c_1^{(1)}} + \overline{s_2} \overline{c_1^{(2)}} = \frac{1}{2} (iX_c + X_s + p_{xy} + ip_x) \tag{3.27}$$

By solving Eq(3.23)$_1$ and Eq(3.27)$_1$ simultaneously, one is able to find $c_1^{(1)}$ and $c_1^{(2)}$.

3.4.1 The sign of the Mapped Function, ζ_k

The remaining challenge is the determination of the sign of ζ_k (see Eq(G.2)) as it is required to avoid the multi-valued solution caused by different signs of ζ_k. For this purpose, we use a pseudocode developed by Koussios [8] (see Algorithm 1) presented in [5] which indeed determines the sign of Δ portion in ζ_k.

$$\zeta_k = \frac{z_k \pm \sqrt{z_k^2 - s_k^2 - 1}}{1 - is_k} = \frac{z_k \pm \Delta}{1 - is_k} \quad \text{where} \quad \Delta = \sqrt{z_k^2 - s_k^2 - 1} \tag{3.28}$$

The presented algorithms [5] is modified by the author of the current report to handle the case $x = 0 (z_k = x + s_k y)$ as well.

In Algorithm 1, k includes 1 and 2 and is the subscript in Eq(2.23), Θ is the angle of the hole where $0 \leq \Theta \leq 2\pi$, α and β are also the real and imaginary part of s_k given in Eq(2.19) and $\mathfrak{S}$ is the sign output.

3.5 Stress Derivation

Eq(3.3) for a harmonic resultant load on the edge hole reduces to

$$\Phi_k(z_k) = h_k z_k + A_k \ln z_k + \sum_{n=-1}^{+\infty} c_1^{(k)} \zeta_k^{-1} \tag{3.29}$$

where $z_k = x + s_k y$ and ζ_k can be determined from Eq(3.28). To get stresses from Eq(2.24), we just need to find $\Phi_k'(z_k)$.

Algorithm 1 Δ sign selection pseudocode.

Ensure: $\Delta = \sqrt{z_k^2 - s_k^2 - 1}, \alpha = \sqrt{\dfrac{r-a}{2}}, \beta = \sqrt{\dfrac{r+a}{2}}$

  if $x = 0$ **then**
    for k **in** $[1, 2]$ **do**
      if $\mathrm{sign}(y) = \mathrm{sign}(\mathfrak{Im}(\Delta))$ **then**
        $1 \leftarrow \mathfrak{S}$
      else
        $-1 \leftarrow \mathfrak{S}$
      end if
    end for
  else
    for k **in** $[1, 2]$ **do**
      if $\mathrm{mode}(\Theta, \pi/2) = 0$ **then**
        if $\mathrm{sign}(y - \alpha x) = \mathrm{sign}(\mathfrak{Im}(\Delta))$ **then**
          $1 \leftarrow \mathfrak{S}$
        else
          $-1 \leftarrow \mathfrak{S}$
        end if
      else
        if $\mathrm{sign}(\beta x) = \mathrm{sign}(\mathfrak{Re}(\Delta))$ **then**
          $1 \leftarrow \mathfrak{S}$
        else
          $-1 \leftarrow \mathfrak{S}$
        end if
      end if
    end for
  end if

$$\frac{\mathrm{d}\Phi_k(z_k)}{\mathrm{d}z_k} = h_k + \frac{A_k}{z_k} + c_1^{(k)} \frac{\mathrm{d}\zeta_k^{-1}}{\mathrm{d}z_k} \tag{3.30}$$

where

$$\frac{\mathrm{d}\zeta_k^{-1}}{\mathrm{d}z_k} = -\frac{(1 - is_k)\left(1 + \dfrac{z}{\mathfrak{S}\sqrt{z_k^2 - s_k^2 - 1}}\right)}{(z + \mathfrak{S}\sqrt{z_k^2 - s_k^2 - 1})^2} = -\frac{(1 - is_k)\left(1 + \dfrac{z}{\mathfrak{S}\Delta}\right)}{(z + \mathfrak{S}\Delta)^2}$$

3.6   Discussion on the Uniqueness of the Solution

As it is discussed by Lekhnitskii [2], we also categorize the final solution (Eq(3.3)) into three cases. It has been shown in Eq(3.7) that the only source of a multi-valued solution is A_k coefficient.

1. For the simply-connected domain S (see Figure. 3), due to the absence of A_k coefficient, the solution is single-valued.

2. For the multiply-connected domain S with either zero traction on the internal contours (irregularities) or in the case of the harmonic traction (see Appendix. H), the solution is single-valued.

3. For the multiply-connected domain S, with even one non-zero traction on an internal contour, the solution is multi-valued as A_k coefficients are non-zero.

4 Solution to An Infinite Orthotropic Plate with A Circular Hole

In this section, we utilize Lekhnitskii solution to solve the problem of a thin infinite orthotropic plate with a circular hole under biaxial loading. Henceforth, we assume the tractions on the hole in form of Eq(3.11) to get a single-value solution.

4.1 Problem Statement

A symmetric balanced square composite laminate with a stacking sequence, $\theta = [45/-45/90/0/0]_s$ (subscript s designates a symmetric layup) is biaxially-loaded on the outer edges, $p_x = p_y = p$ and $p_{xy} = 0$, while the circular hole at the center of the plate undergoes a uniform radial stress, $p/2$ as illustrated in Figure. 7. The ply mechanical properties are summarized in Table. 1. The hole radius is assumed 1 and is incomparable with other geometrical dimensions.

Table 1: The ply mechanical properties.

Property	Symbol	Unit	Value
Longitudinal Young's modulus	E_{11}	[GPa]	145
Transverse Young's modulus	E_{22}	[GPa]	7
In-plane shear modulus	μ_{12}	[GPa]	3.5
In-plane Poisson's ratio	ν_{12}	[-]	0.34

4.2 Compliance Tensor

The first step is to calculate the laminate stiffness tensor from the so-called ABD matrix. This can be achieved through the Classical Laminate Plate Theory and a tool developed by the author of the present report[1]. The ABD matrix pseudocode is given in Algorithm 2.
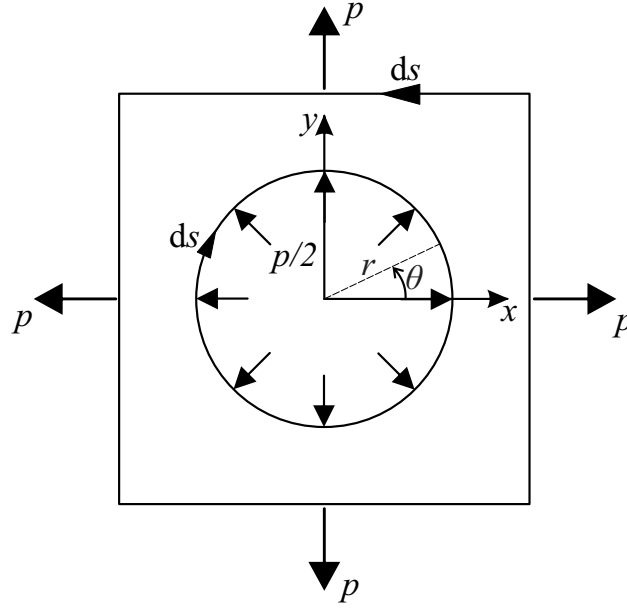
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[1]Available on `github.com/amirbaharvand66/CLTPLUS.git`

Figure 7: The square composite laminate geometry, coordinate system and boundary condition (the hole is largely-sketched for convenience).

Algorithm 2 ABD matrix pseudocode.

Ensure: $A = B = D = 0$

   $v_{12} \leftarrow v_{12}E_{22}/E_{11}$

   $Q \leftarrow$ Eq(4.2)

   for θ **in** stacking sequence **do**

      $M \leftarrow$ Eq(4.1)

      $R \leftarrow$ Eq(4.3)

      $C_p \leftarrow M^{-1}QRMR^{-1}$

      $A \leftarrow C_p(z_i - z_{i-1}) + A$

      $B \leftarrow \dfrac{1}{2}C_p(z_i^2 - z_{i-1}^2) + B$

      $D \leftarrow \dfrac{1}{3}C_p(z_i^3 - z_{i-1}^3) + D$

   end for

In Algorithm 2, C_p is an intermediate variable, z is the coordinate along the laminate thickness, i is the ply number, M is the transformation matrix (Eq(4.1)), Q is the reduced stiffness matrix (Eq(4.2)) and R is the Reuter matrix [9].

$$M = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & s^2 - c^2 \end{bmatrix} \quad , \quad c = \cos\theta, \quad s = \sin\theta \tag{4.1}$$

$$Q = \begin{bmatrix} \dfrac{E_{11}}{\lambda} & \dfrac{v_{21}E_{11}}{\lambda} & 0 \\[3mm] \dfrac{v_{21}E_{11}}{\lambda} & \dfrac{E_{22}}{\lambda} & 0 \\[3mm] 0 & 0 & \mu_{12} \end{bmatrix} \quad , \quad \lambda = 1 - v_{12}v_{21} \tag{4.2}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{4.3}$$

The compliance tensor can be found by inverting the laminate stiffness tensor which is achieved through the inversion of the ABD matrix [10]. This matrix is known by small letters abcd.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1}$$

$$
\begin{aligned}
a &= A^{-1} - (\mathfrak{B}\mathfrak{D}^{-1})\mathfrak{C} \\
b &= \mathfrak{B}\mathfrak{D}^{-1} \\
c &= -\mathfrak{D}^{-1}\mathfrak{C} \\
d &= \mathfrak{D}^{-1}
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
A^{-1} &= \text{Inverse of A matrix in the ABD matrix} \\
\mathfrak{B} &= -A^{-1}B \\
\mathfrak{C} &= BA^{-1} \\
\mathfrak{D} &= D - (BA^{-1})B
\end{aligned}
$$

Finally, the compliance tensor is

$$S = \frac{1}{h}a \tag{4.5}$$

where h is the laminate thickness. Consequently, one is able to calculate r and a from Eq(2.16).

4.3   Laurent Series

The solution can be assumed in form of a Laurent series, Eq(3.3).

4.3.1   Homogeneous Stress Field, h_k

The homogeneous Stress Field Constant, h_k, can be calculated from Eq(3.5).

$$h_1 = \frac{p_x - p_y s_2^2}{2(s_1^2 - s_2^2)} - \frac{p_{xy}}{4s_1} = \frac{p - p s_2^2}{2(s_1^2 - s_2^2)}$$

$$h_2 = \frac{p_x - p_y s_1^2}{2(s_2^2 - s_1^2)} - \frac{p_{xy}}{4s_2} = \frac{p - p s_1^2}{2(s_2^2 - s_1^2)} \tag{4.6}$$

where s_1 and s_2 are given in Eq(2.19).

4.3.2   Force Resultant around Hole, A_k

Due to the constant load on the hole edge (internal pressure $p/2$), we may replace the radial stress with a harmonic function. We assume the same function from Eq(3.15). Table. 2 shows that the replaced harmonic function satisfies the prescribed boundary condition on the hole edge.

Table 2: Equivalent prescribed traction in form of a harmonic function.

θ	$X(\theta)$	$Y(\theta)$	$\cos\theta$	$\sin\theta$	X_c	Y_c	Y_s	X_s
0	$p/2$	0	1	0	$p/2$	-	-	-
0	$p/2$	0	1	0	-	0	-	-
$\pi/2$	0	$p/2$	0	1	-	-	$p/2$	-
$\pi/2$	0	$p/2$	0	1	-	-	-	0

As previously mentioned in Section. 3.3 and proved in Appendix. H, A_k vanishes for a harmonic function; hence, $A_k \ln z_k$ is zero.

4.3.3   Disturbance Field, $g_n^{(k)}$

As already discussed in the previous section, we assume the boundary condition on the hole edge with a harmonic function from Eq(3.15); therefore, $c_1^{(1)}$ and $c_1^{(2)}$ can be easily solved from Eq(3.23)$_1$ and Eq(3.27)$_1$.

The implemented algorithm is given in Figure. 8.

4.4   Result and Discussion

Figure. 9a and Figure. 9b show the normalize stress on the hole edge in the Cartesian (Eq(2.24)) and polar coordinate system, respectively. The stress in the Cartesian and polar coordinate systems can be correlated through the transformation below.

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{4.7}$$

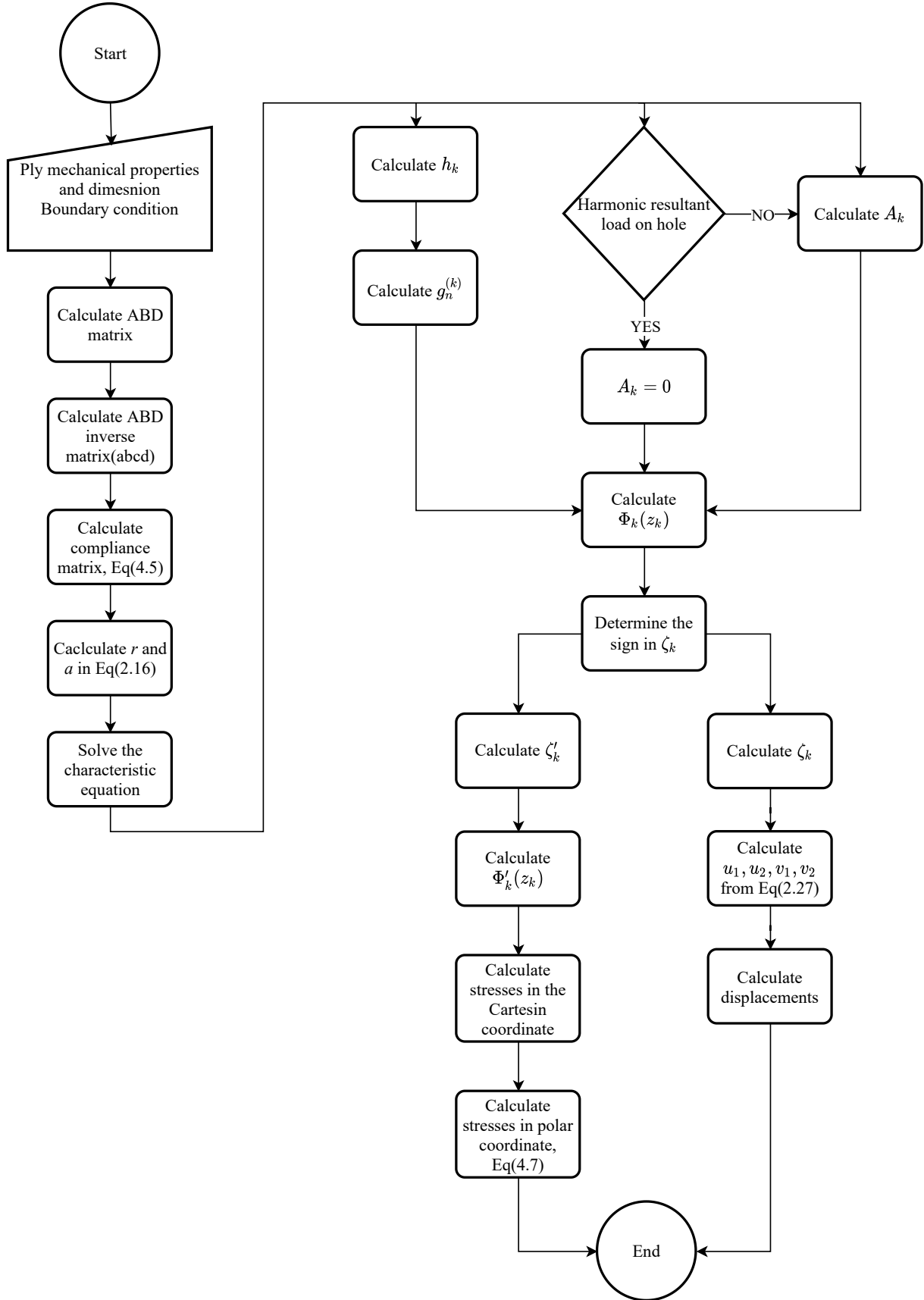**Figure 8:** The implemented algorithm from Lekhnitskii theory.

while σ_x, σ_y and τ_{xy} are the stress in the Cartesian coordinate, σ_r, σ_θ and $\tau_{r\theta}$ are the polar stress components.
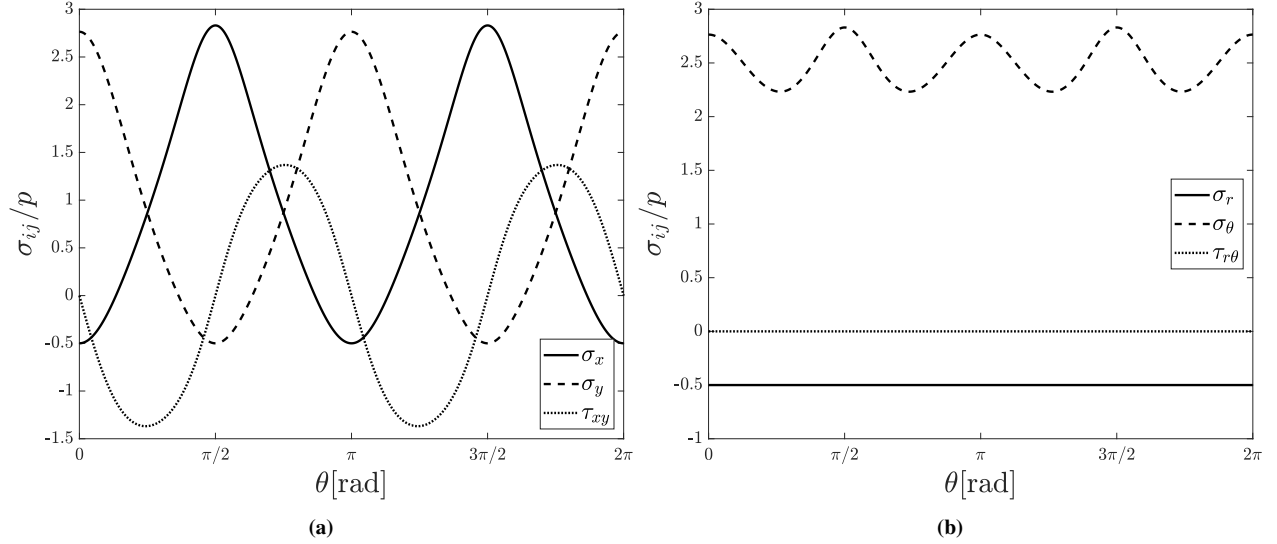

Figure 9: Normalized stress distributions on the hole edge in (a) Cartesian (b) Polar coordinate system.

Despite the sinusoidal behavior of the Cartesian stress components in Figure. 9a, the polar stress provides a better insight into stress distribution on the hole edge (Figure. 9b). The radial stress, σ_r, and shear stress, $\tau_{r\theta}$, coincide with their prescribed traction value on the hole boundary (see Figure. 10). The hoop stress, σ_θ, various sinusoidally on the hole edge and has the highest stress value (almost $5\sigma_r$ on average).
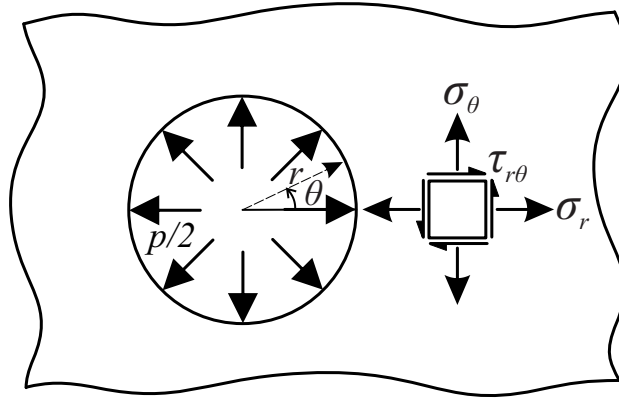
Figure 10: Stress components on the hole edge.

The stress field around the hole are plotted in Figure. 11a and Figure. 11b. As is seen, all the stress components converge to their value at infinity. Approaching the hole edge, σ_x and σ_y go to infinity and τ_{xy} changes sign from $+\infty$ to $-\infty$ which does not affect our solution as they are inside the hole and do not represent any physical meaning.
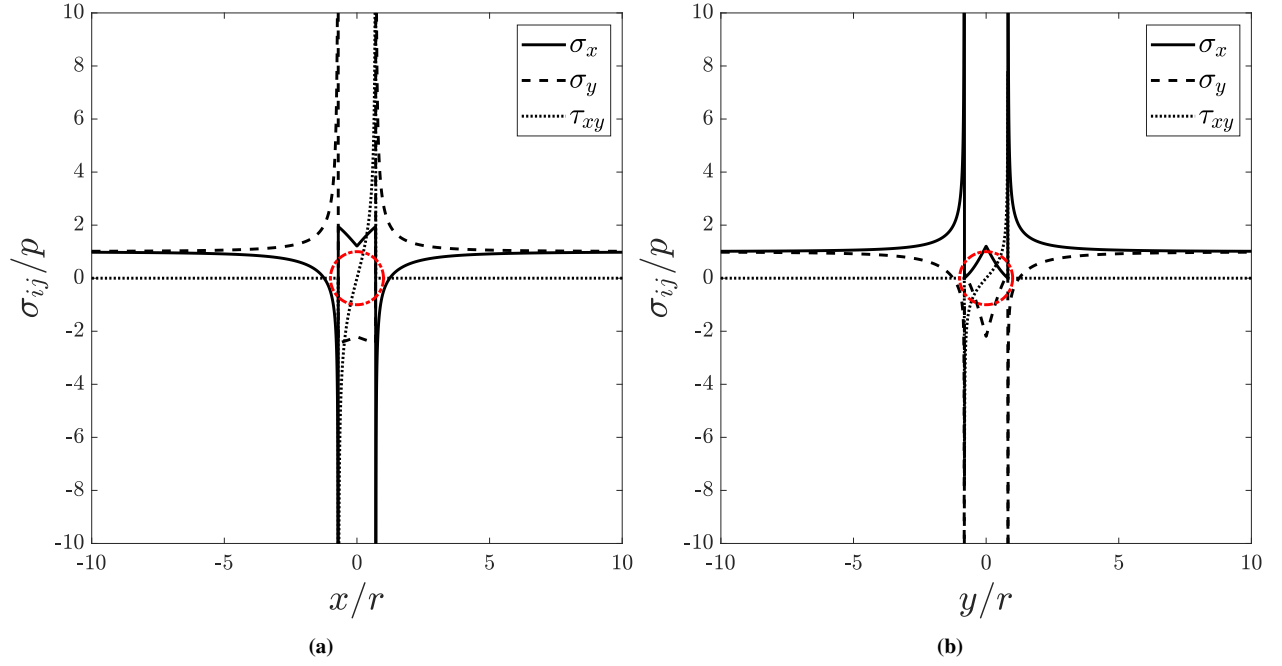
Figure 11: Normalized stress distributions along (a) *x*-axis (b) *y*-axis. The circle designates the hole.

In order to get an overview of which stress field plays a key role around the hole, the plot of the polar stress components at distances of 5 and 10 times the hole radius are also added and compared in Figure. 12. σ_θ is the highest on the hole edge and reduces as we move away from the hole. The absolute value of σ_r increases by moving in the opposite direction towards the external boundary as it converges to its value at infinity. For this loading condition, $\tau_{r\theta}$ is zero everywhere as expected.

The deformation around the whole can be calculated from Eq(2.27). Figure. 13b shows the resultant hole deformation calculated from

$$u_t = \sqrt{v^2 + v^2}$$

The deformed hole is shown in Figure. 13b. Unlike an isotropic material which deforms uniformly, the composite laminate undergoes more deformation in the *y*-axis than the *x*-axis which is due to the stacking sequence (the majority (40%) of the layers are 0) and smaller transverse Young's modulus compared to the longitudinal. The separate plots of normalized *u* and *v* around the hole are also given in Figure. 13a.

5 Conclusion

Lekhnitskii formalism is a powerful tool in analyzing the anisotropic elastic bodies and has many applications in aerospace. In this report, we revisited Lekhnitskii theory and applied it to a thin phototropic laminate with a circular hole. During the development, we utilized the two-dimensional general stress and assumed that the external boundary condition does not influence the hole. The result show that the hole disturbs the stress field in the body inducing a high stress region
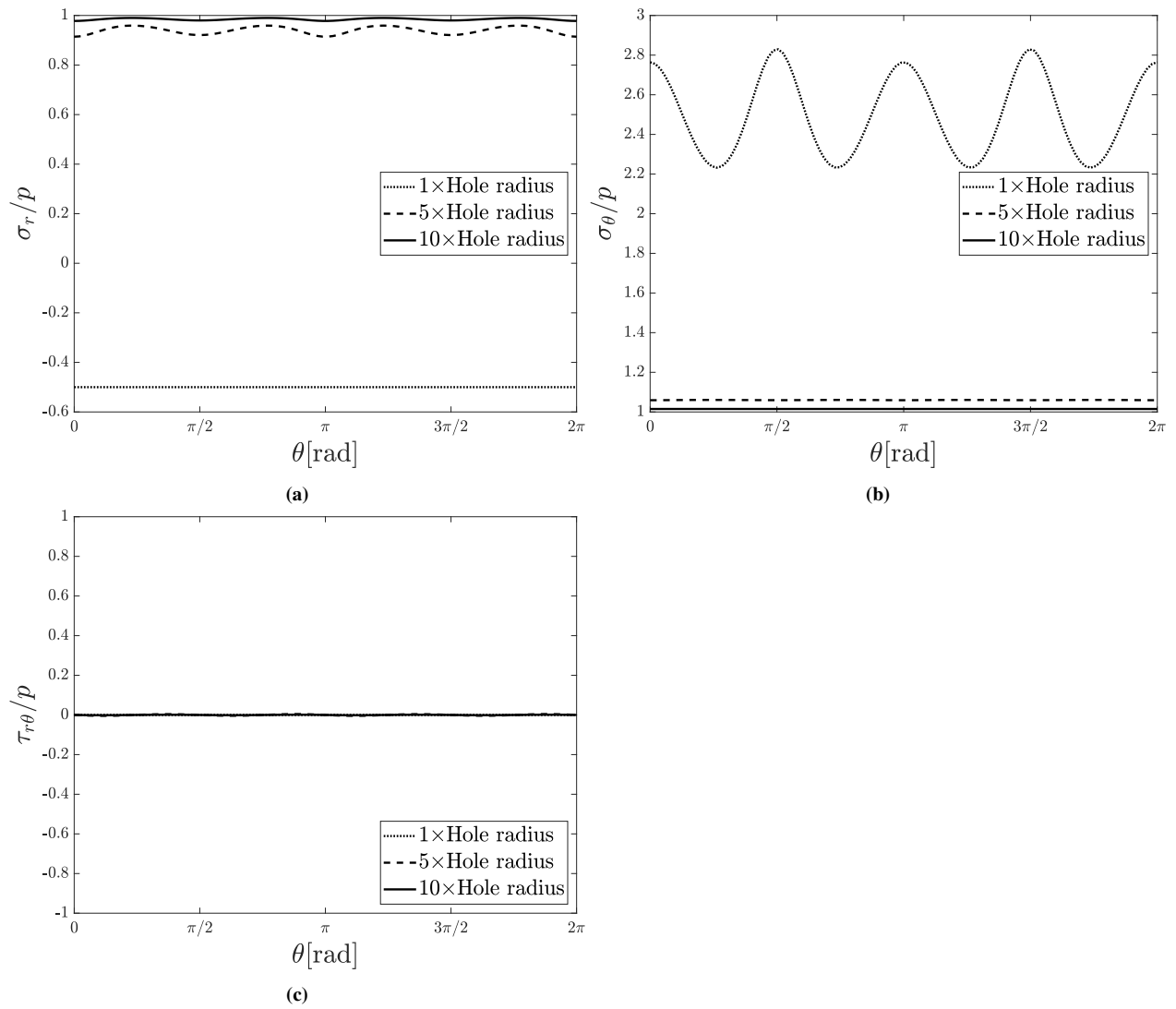
in the vicinity of the hole.

Figure 12: Normalized (a) radial (b) hoop (c) shear stress distributions at various distance from the hole edge.
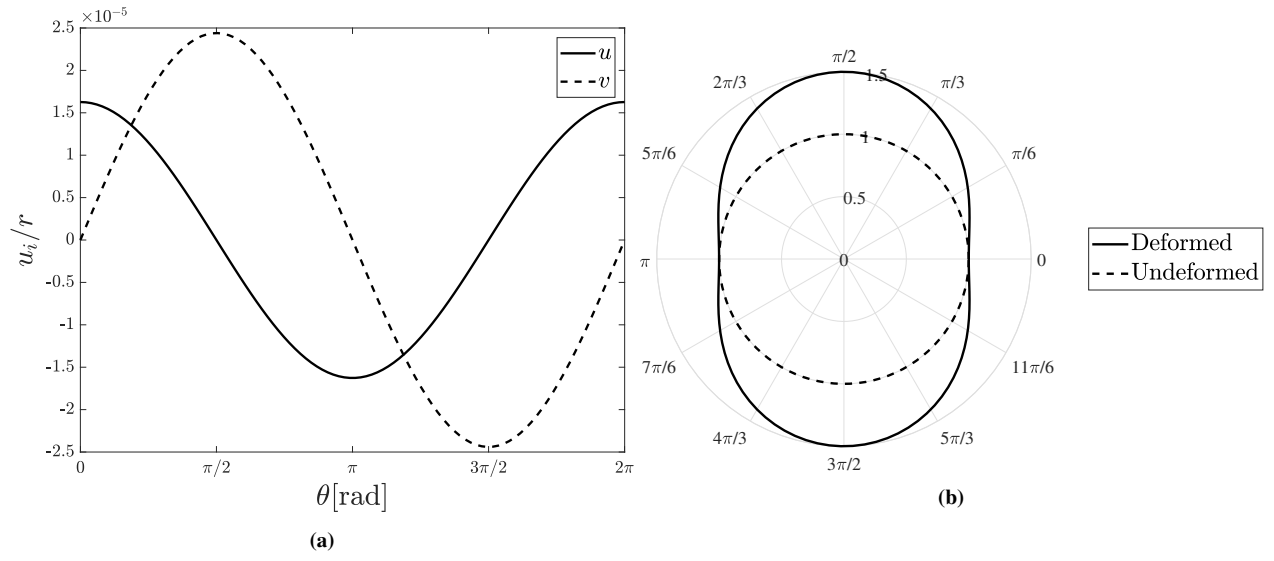
Figure 13: (a) Normalized displacement distribution around the hole. (b) Total deformation of circular hole.

Appendix A   Components of the Compliance Tensor

The compliance tensor for a general laminate can be calculated from the tensor transformation rule. As the compliance is a fourth-order tensor, the compliance tensor in the transformed coordinate, $\boldsymbol{S}'$, is

$$S'_{ijkl} = Q_{pi}Q_{qj}Q_{rk}Q_{sl}S_{pqrs}$$

where Q is the proper orthogonal tensor .The compliance tensor components become [4]

$$S'_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4$$
$$S'_{12} = S_{12}(c^4 + s^4) + (S_{11} + S_{22} - S_{66})s^2c^2$$
$$S'_{13} = S_{13}c^2 + S_{23}s^2$$
$$S'_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 + (2S_{12} - 2S_{22} + S_{66})cs^3$$
$$S'_{22} = S_{11}s^4 + S_{22}c^4 + (2S_{12} + S_{66})s^2c^2$$
$$S'_{23} = S_{13}s^2 + S_{23}c^2$$
$$S'_{26} = (2S_{12} - 2S_{22} + S_{66})sc^3 + (2S_{11} - 2S_{12} - S_{66})cs^3$$
$$S'_{33} = S_{33}$$
$$S'_{36} = 2(S_{13} - S_{23})sc$$
$$S'_{44} = S_{44}c^2 + S_{55}s^2$$
$$S'_{45} = (S_{55} - S_{44})sc$$
$$S'_{55} = S_{44}s^2 + S_{55}c^2$$
$$S'_{66} = 4(S_{11} - 2S_{12} + S_{22})c^2s^2 + (s^2 - c^2)^2 S_{66} \tag{A.1}$$

where

$$S_{11} = \frac{1}{E_{11}} \quad , S_{12} = -\frac{\nu_{12}}{E_{11}} \quad , S_{13} = -\frac{\nu_{13}}{E_{11}} \quad , S_{22} = \frac{1}{E_{22}} \quad , S_{23} = -\frac{\nu_{23}}{E_{22}} \quad , S_{33} = \frac{1}{E_{33}} \quad , S_{44} = \frac{1}{\mu_{23}}$$
$$S_{55} = \frac{1}{\mu_{13}} \quad , S_{66} = \frac{1}{\mu_{12}} \quad , c = \cos(\theta) \quad , s = \sin(\theta)$$

E_{11} is the longitudinal Young's modulus, E_{22} and E_{33} are transverse Young's moduli, μ_{12} and μ_{13} are in-plane shear moduli, μ_{23} is the out-of-plane shear modulus, ν_{12} and ν_{13} are in-plane Poisson's ratios and ν_{23} is the out-of-plane Poisson's ratio. The compliance tensor for an orthotropic laminate is

$$S = \begin{bmatrix} \dfrac{1}{E_{11}} & -\dfrac{\nu_{12}}{E_{11}} & -\dfrac{\nu_{13}}{E_{11}} & 0 & 0 & 0 \\[2mm] -\dfrac{\nu_{12}}{E_{11}} & \dfrac{1}{E_{22}} & -\dfrac{\nu_{23}}{E_{22}} & 0 & 0 & 0 \\[2mm] -\dfrac{\nu_{13}}{E_{11}} & -\dfrac{\nu_{23}}{E_{22}} & \dfrac{1}{E_{33}} & 0 & 0 & 0 \\[2mm] 0 & 0 & 0 & \dfrac{1}{\mu_{23}} & 0 & 0 \\[2mm] 0 & 0 & 0 & 0 & \dfrac{1}{\mu_{31}} & 0 \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1}{\mu_{12}} \end{bmatrix} \qquad (A.2)$$

Appendix B   Derivation of the Compatibility Equation

The strain-displacement relation is

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad , \varepsilon_y = \frac{\partial v}{\partial y} \quad , \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Differentiating ε_x and ε_y w.r.t y and x twice together with differentiating γ_{xy} w.r.t x and then y give

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial y^2 \partial x} \quad , \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y} \quad , \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

Summing the first two terms and then subtracting the third term yield the compatibility equation.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

Appendix C   Derivation of Compatibility Equation in Terms of Stress Function for the Plane Deformation Case

Let us assume

$$u = u(x,y) \quad , v = v(x,y) \quad , w = 0$$

thus we have

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad , \varepsilon_y = \frac{\partial v}{\partial y} \quad , \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad , \varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$$

The stress and strain can be correlated using the inverse of the compliance matrix in Eq(2.4) or the so-called reduced transformation matrix, Q.

$$\begin{Bmatrix} \overline{\varepsilon}_x \\ \overline{\varepsilon}_y \\ 0 \\ \overline{\gamma}_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{16} \\ Q_{12} & Q_{22} & Q_{23} & Q_{26} \\ Q_{13} & Q_{23} & Q_{33} & Q_{36} \\ Q_{16} & Q_{26} & Q_{36} & Q_{66} \end{bmatrix} \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\sigma}_z \\ \overline{\tau}_{xy} \end{Bmatrix} \tag{C.1}$$

where [2]

$$Q_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}} \qquad (i,j=1,2,6)$$

$\overline{\sigma}_z$ can be solved from Eq(C.1).

$$\sigma_z = -\frac{Q_{13}\sigma_x + Q_{23}\sigma_y + Q_{36}\tau_{xy}}{Q_{33}}$$

Invoking Eq(2.8) and Eq(C.1), we get an analogous equation to Eq(2.9) with different constants.

$$Q_{22}\frac{\partial^4\phi}{\partial x^4} - 2Q_{26}\frac{\partial^4\phi}{\partial x^3\partial y} + (2Q_{12}+Q_{66})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2Q_{16}\frac{\partial^4\phi}{\partial x\partial y^3} + Q_{11}\frac{\partial^4\phi}{\partial y^4}$$
$$+ (Q_{12}+Q_{22})\frac{\partial^2\overline{V}}{\partial x^2} - (Q_{16}+Q_{26})\frac{\partial^2\overline{V}}{\partial x\partial y} + (Q_{11}+Q_{12})\frac{\partial^2\overline{V}}{\partial y^2} = 0$$

which reduces to below, in case of zero body forces.

$$Q_{22}\frac{\partial^4\phi}{\partial x^4} - 2Q_{26}\frac{\partial^4\phi}{\partial x^3\partial y} + (2Q_{12}+Q_{66})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2Q_{16}\frac{\partial^4\phi}{\partial x\partial y^3} + Q_{11}\frac{\partial^4\phi}{\partial y^4} = 0$$

See `pde_plane_strain.mw` for the corresponding Maple file.

Appendix D   Proof of $\phi(x,y) = f(z)$

The proof of this section is provided in [5]. The compatibility equation in terms of potential stress function reduces to below based on the assumptions in Section. 2.3.

$$r^2\frac{\partial^4\phi(x,y)}{\partial x^4} + 2a\frac{\partial^4\phi(x,y)}{\partial x^2\partial y^2} + \frac{\partial^4\phi(x,y)}{\partial y^4} = 0$$

Using the "," operator instead of partial differentiation, the above equation becomes

$$r^2\phi(x,y)_{,xxxx} + 2a\phi(x,y)_{,xxyy} + \phi(x,y)_{,yyyy} = 0$$

Replace r^2 and $2a$ with $\xi^2\lambda^2$ and $-(\xi^2+\lambda^2)$, where both ξ and λ are independent constants.

$$\xi^2\lambda^2\phi(x,y)_{,xxxx} - (\xi^2+\lambda^2)\phi(x,y)_{,xxyy} + \phi(x,y)_{,yyyy} = 0$$
$$(\xi^2\phi(x,y)_{,xx} - \phi(x,y)_{,yy})(\lambda^2\phi(x,y)_{,xx} - \phi(x,y)_{,yy}) = 0$$
$$(\xi\phi(x,y)_{,x} + \phi(x,y)_{,y})(\xi\phi(x,y)_{,x} - \phi(x,y)_{,y})(\lambda\phi(x,y)_{,x} + \phi(x,y)_{,y})(\lambda\phi(x,y)_{,x} - \phi(x,y)_{,y}) = 0$$

As a result, each term in the last line of the above set of equations should be zero. Let us continue with $\xi\phi(x,y)_{,x} - \phi(x,y)_{,y}$. The same strategy is applicable to the rest of the terms.

$$\xi\phi(x,y)_{,x} - \phi(x,y)_{,y} = 0$$
$$\{\phi(x,y)_{,x} \quad \phi(x,y)_{,y}\}\begin{Bmatrix} \xi \\ -1 \end{Bmatrix} = 0$$

for the above equation to be valid,

$$\begin{Bmatrix} \phi(x,y)_{,x} \\ \phi(x,y)_{,y} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \xi \end{Bmatrix}$$

The function that satisfies the above is $\phi(x,y) = f(x+\xi y)$ which is a complex function.

Appendix E   Determination of the Characteristic Equation Roots

For an ideal elastic body $r+a$ is always positive.

$$r = \sqrt{\frac{S_{22}}{S_{11}}} = \sqrt{\frac{E_{11}}{E_{22}}} > 0$$

For an ideal elastic laminate, $0 < \nu_{12} < \dfrac{1}{2}$.

$$2a = \frac{2S_{12}+S_{66}}{S_{11}} = \frac{-2\dfrac{\nu_{12}}{E_{11}} + \dfrac{1}{\mu_{12}}}{\dfrac{1}{E_{11}}} = \frac{E_{11} - 2\mu_{12}\nu_{12}}{\mu_{12}} \xrightarrow{E_{11}>\mu_{12}>0} \frac{E_{11}-2\mu_{12}\nu_{12}}{\mu_{12}} > 0$$

Next, for

Case 1: $r - a \geq 0$

$$s_1 = \sqrt{\frac{r-a}{2}} + i\sqrt{\frac{r+a}{2}}$$

$$s_2 = -\sqrt{\frac{r-a}{2}} + i\sqrt{\frac{r+a}{2}}$$

$$s_3 = \sqrt{\frac{r-a}{2}} - i\sqrt{\frac{r+a}{2}}$$

$$s_4 = -\sqrt{\frac{r-a}{2}} - i\sqrt{\frac{r+a}{2}} \tag{E.1}$$

then, s_1 to s_4 are all complex.

Case 2: $r - a < 0$

$$s_1 = i\left(\sqrt{\frac{a-r}{2}} + \sqrt{\frac{r+a}{2}}\right)$$

$$s_2 = i\left(-\sqrt{\frac{a-r}{2}} + \sqrt{\frac{r+a}{2}}\right)$$

$$s_3 = -i\left(-\sqrt{\frac{a-r}{2}} + \sqrt{\frac{r+a}{2}}\right)$$

$$s_4 = -i\left(\sqrt{\frac{a-r}{2}} + \sqrt{\frac{r+a}{2}}\right) \tag{E.2}$$

then, s_1 to s_4 are all imaginary.

Appendix F   Derivation of Traction Equation on the Internal Boundary Due to An Opening

$$\begin{cases} \left(\dfrac{\partial^2 \phi}{\partial y^2} + V\right)\dfrac{-\mathrm{d}y}{\mathrm{d}s} - \dfrac{\partial^2 \phi}{\partial x \partial y}\dfrac{\mathrm{d}x}{\mathrm{d}s} = X \\[4mm] \dfrac{\partial^2 \phi}{\partial x \partial y}\dfrac{\mathrm{d}y}{\mathrm{d}s} + \left(\dfrac{\partial^2 \phi}{\partial x^2} + V\right)\dfrac{-\mathrm{d}x}{\mathrm{d}s} = Y \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial y^2}\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial^2 \phi}{\partial x \partial y}\dfrac{\mathrm{d}x}{\mathrm{d}s} = -X - V\dfrac{\mathrm{d}y}{\mathrm{d}s} \\[4mm] \dfrac{\partial^2 \phi}{\partial x \partial y}\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial^2 \phi}{\partial x^2}\dfrac{\mathrm{d}x}{\mathrm{d}s} = Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial y}\right)\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi}{\partial y}\right)\dfrac{\mathrm{d}x}{\mathrm{d}s} = -X - V\dfrac{\mathrm{d}y}{\mathrm{d}s} \\[4mm] \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right)\dfrac{\mathrm{d}y}{\mathrm{d}s} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi}{\partial x}\right)\dfrac{\mathrm{d}x}{\mathrm{d}s} = Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s} \end{cases}$$

From Eq(2.31)

$$\begin{cases} \dfrac{\partial}{\partial s}\left(\dfrac{\partial \phi}{\partial y}\right) = -X - V\dfrac{\mathrm{d}y}{\mathrm{d}s} \\[4mm] \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right) = Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s} \end{cases}$$

$$\begin{cases} \dfrac{\partial \phi}{\partial y} = \displaystyle\int_{s_0}^{s_1}\left(-X - V\dfrac{\mathrm{d}y}{\mathrm{d}s}\right)\mathrm{d}s + c_3 \\[4mm] \dfrac{\partial \phi}{\partial x} = \displaystyle\int_{s_0}^{s_1}\left(Y - V\dfrac{\mathrm{d}x}{\mathrm{d}s}\right)\mathrm{d}s + c_4 \end{cases}$$

Appendix G   On the Selection of Hole Radius

Take the general case of an ellipse in polar coordinate system where

$$x = d\cos\theta \quad, y = b\sin\theta$$

z in its complex form can be written as

$$z = x + iy = d\cos\theta + ib\sin\theta \overset{(1)}{=} d\cos\theta - id\sin\theta + ib\sin\theta + id\sin\theta \overset{(2)}{=} de^{i\theta} + i(b-d)\sin\theta$$

$$\overset{(3)}{=} de^{i\theta} + \left(\dfrac{b-d}{2}\right)(e^{i\theta} - e^{-i\theta}) = \left(\dfrac{b+d}{2}\right)e^{i\theta} - \left(\dfrac{b-d}{2}\right)e^{-i\theta}$$

where we

$$\text{added and subtract} \quad id\sin\theta \qquad \text{in (1)}$$
$$\text{utilized Euler's formula} \quad e^{-i\theta} = \cos\theta - i\sin\theta \qquad \text{in (2)}$$

$$\text{replaced} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \quad \text{in (3)}$$

Based on the above equation [3]

$$z_k = \left(\frac{d - is_k b}{2}\right) e^{i\theta} + \left(\frac{d + is_k b}{2}\right) e^{-i\theta}$$

$$\overset{z = e^{i\theta}}{=} \left(\frac{d - is_k b}{2}\right) z + \left(\frac{d + is_k b}{2}\right) \frac{1}{z}$$

$$\Rightarrow (d - is_k b) z^2 - 2z_k z + (d + is_k b) = 0$$

Solving for the roots, ζ_k, of the above equation (we use ζ_k instead of z_k to avoid possible misinterpretation caused by z_k in Eq(2.20) to Eq(2.23))

$$\zeta_k = \frac{2z_k \pm \sqrt{4z_k^2 - 4(d + is_k b)(d - is_k b)}}{d - is_k b} = \frac{z_k \pm \sqrt{z_k^2 - s_k^2 b^2 - d^2}}{d - is_k b} \quad \text{where} \quad k = 1, 2 \quad \text{(G.1)}$$

For a circular opening ($d = b = 1$) with a radius r, to be on the edge, $|z| = r$, and

$$\zeta_k = \frac{z_k/r \pm \sqrt{(z_k/r)^2 - s_k^2 - 1}}{1 - is_k} \quad \text{(G.2)}$$

The unitless parameter z_k/r, on the circle, can be replaced with unity; hence, z_k/r can be simply written as z_k and the hole radius can be assumed as one.

Appendix H   Derivation of A_k constants for A Harmonic Load on the Hole Edge

For the general case of harmonic load, Eq(3.11), we replace X and Y in Eq(3.6) by Eq(3.12).

$$2\Re\mathfrak{e}[s_1 A_1 \ln z_1 + s_2 A_2 \ln z_2]_{s_0}^{s_1} = -\int_{s_0}^{s_1} N(m, \theta) \cos\theta - T(m, \theta) \sin\theta \, \mathrm{d}s$$

$$2\Re\mathfrak{e}[A_1 \ln z_1 + A_2 \ln z_2]_{s_0}^{s_1} = \int_{s_0}^{s_1} N(m, \theta) \cos\theta + T(m, \theta) \sin\theta \, \mathrm{d}s$$

Integrating on the hole ($0 \leq \theta \leq 2\pi$)

$$s_1 A_1 + s_2 A_2 - \overline{s_1 \overline{A_1}} - \overline{s_2 \overline{A_2}} = -\int_0^{2\pi} N(m,\theta)\cos\theta - T(m,\theta)\sin\theta \mathrm{d}\theta = 0$$

$$A_1 + A_2 - \overline{A_1} - \overline{A_2} = \int_0^{2\pi} N(m,\theta)\cos\theta + T(m,\theta)\sin\theta \mathrm{d}\theta = 0$$

Due to the absence of the resultant load, $R_x = R_y = 0$ on the hole edge, as is the case for the harmonic traction, then $A_1 = A_2 = \overline{A_1} = \overline{A_2} = 0$; thus, $\sum_{k=1}^{2} A_k \ln z_k$ vanishes.

Appendix I   Mapping $g_n^{(k)} z_k^{-n}$ to $c_n^{(k)} \zeta_k^{-n}$

For a circular hole ($r = 1$),

$$x = \cos\theta \quad , y = \sin\theta$$

$$
\begin{aligned}
\zeta_k &= \frac{z_k \pm \sqrt{z_k^2 - s_k^2 - 1}}{1 - is_k} = \frac{x + s_k y \pm \sqrt{(x + s_k y)^2 - s_k^2 - 1}}{1 - is_k} \\
&= \frac{\cos\theta + s_k \sin\theta \pm \sqrt{(\cos\theta + s_k \sin\theta)^2 - s_k^2 - 1}}{1 - is_k} \\
&= \frac{\cos\theta + s_k \sin\theta \pm \sqrt{\cos^2\theta + s_k^2 \sin^2\theta + 2s_k \cos\theta \sin\theta - s_k^2 - 1}}{1 - is_k} \\
&= \frac{\cos\theta + s_k \sin\theta \pm \sqrt{-(1 - \cos^2\theta) - s_k^2(1 - \sin^2\theta) + 2s_k \cos\theta \sin\theta}}{1 - is_k} \\
&= \frac{\cos\theta + s_k \sin\theta \pm \sqrt{-\sin^2\theta - s_k^2 \cos^2\theta + 2s_k \cos\theta \sin\theta}}{1 - is_k} \\
&= \frac{\cos\theta + s_k \sin\theta \pm \sqrt{(\sin^2\theta - s_k^2 \cos^2\theta)^2}}{1 - is_k} \\
&= \frac{\cos\theta + s_k \sin\theta \pm i(\sin\theta - s_k \cos\theta)}{1 - is_k}
\end{aligned}
$$

Let us replace $\pm$ with $\mathfrak{S}$ as it is required to determine the proper sign during the solution. The sign selection is given in Algorithm 1.

$$\zeta_k = \frac{\cos\theta + s_k\sin\theta + \mathfrak{S}i(\sin\theta - s_k\cos\theta)}{1 - is_k} \overset{\mathfrak{S}=1}{=} \frac{\cos\theta(1 - is_k) + \sin\theta(s_k + i)}{1 - is_k}$$

$$= \frac{\cos\theta(1 - is_k) + i\sin\theta(1 - is_k)}{1 - is_k} = \cos\theta + i\sin\theta = e^{i\theta} = z$$

which is true when we are on the hole edge. In fact, on the hole edge $\zeta_1 = \zeta_2 = z$.

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pde_plane_stress.mw

This Maple file includes the required relations and equation which give the final PDE derived from the compatibility equation for the plane stress case of an anisotropic body.

N.B. the stress and strain vectors are all averaged through the thickness of the body, i.e.,

$$\bar{\sigma} = \frac{1}{2\,h} \int_{-h}^{h} \sigma(x,y)\,\mathrm{d}z$$

```
> restart:
```

Stress vector

```
> stress := Vector[column]([ sigma__x ,
                    sigma__y ,
                    0 ,
                    0 ,
                    0 ,
                    tau__xy ]);
```

$$stress := \begin{bmatrix} \sigma_x \\ \sigma_y \\ 0 \\ 0 \\ 0 \\ \tau_{xy} \end{bmatrix}$$

(1)

Non-balanced laminate transformed compliance tensor

```
> S := Matrix([[ S__11 , S__12 , S__13 , 0 , 0 , S__16 ],
          [ S__12 , S__22 , S__23 , 0 , 0 , S__26 ],
          [ S__13 , S__23 , S__33 , 0 , 0 , S__36 ],
          [ 0 , 0 , 0 , S__44 , S__45 , 0 ],
          [ 0 , 0 , 0 , S__45 , S__55 , 0 ],
          [ S__16 , S__26 , S__36 , 0 , 0 , S__66 ]]);
```

$$S := \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}$$

(2)

Strain vector

```
> strain := S . stress
```

(3)

$$strain := \begin{bmatrix} S_{11}\,\sigma_x + S_{12}\,\sigma_y + S_{16}\,\tau_{xy} \\[4pt] S_{12}\,\sigma_x + S_{22}\,\sigma_y + S_{26}\,\tau_{xy} \\[4pt] S_{13}\,\sigma_x + S_{23}\,\sigma_y + S_{36}\,\tau_{xy} \\[4pt] 0 \\[4pt] 0 \\[4pt] S_{16}\,\sigma_x + S_{26}\,\sigma_y + S_{66}\,\tau_{xy} \end{bmatrix} \tag{3}$$

Calculating stress from a potential function, $\phi(x,y)$

V(x,y) is the body force
```
> sigma__x := diff(phi(x, y), y$2) + V(x, y)
```
$$\sigma_x := \frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y) \tag{1.1}$$

```
> sigma__y := diff(phi(x, y), x$2) + V(x, y)
```
$$\sigma_y := \frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y) \tag{1.2}$$

```
> tau__xy := -diff(phi(x, y), x, y)
```
$$\tau_{xy} := -\frac{\partial^2}{\partial x \partial y}\,\phi(x,y) \tag{1.3}$$

Stress components

```
> epsilon__x := strain[1]
```
$$\epsilon_x := S_{11}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + S_{12}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right) - S_{16}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right) \tag{2.1}$$

```
> epsilon__y := strain[2]
```
$$\epsilon_y := S_{12}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + S_{22}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right) - S_{26}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right) \tag{2.2}$$

```
> gamma__xy := strain[6]
```
$$\gamma_{xy} := S_{16}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + S_{26}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right) - S_{66}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right) \tag{2.3}$$

Compatibility Equation (body force included)
```
> ce := diff(epsilon__x, y$2) + diff(epsilon__y, x$2) - diff
  (gamma__xy, x, y) = 0
```
$$ce := S_{11}\left(\frac{\partial^4}{\partial y^4}\,\phi(x,y) + \frac{\partial^2}{\partial y^2}\,V(x,y)\right) + S_{12}\left(\frac{\partial^4}{\partial x^2 \partial y^2}\,\phi(x,y) + \frac{\partial^2}{\partial y^2}\,V(x,y)\right) \tag{4}$$
$$- S_{16}\left(\frac{\partial^4}{\partial x \partial y^3}\,\phi(x,y)\right) + S_{12}\left(\frac{\partial^4}{\partial x^2 \partial y^2}\,\phi(x,y) + \frac{\partial^2}{\partial x^2}\,V(x,y)\right) + S_{22}\left(\frac{\partial^4}{\partial x^4}\,\phi(x,y)\right.$$

$$+ \frac{\partial^2}{\partial x^2} V(x, y) \Bigg) - S_{26} \left(\frac{\partial^4}{\partial x^3 \partial y} \, \phi(x, y) \right) - S_{16} \left(\frac{\partial^4}{\partial x \partial y^3} \, \phi(x, y) + \frac{\partial^2}{\partial x \partial y} V(x, y) \right)$$

$$- S_{26} \left(\frac{\partial^4}{\partial x^3 \partial y} \, \phi(x, y) + \frac{\partial^2}{\partial x \partial y} V(x, y) \right) + S_{66} \left(\frac{\partial^4}{\partial x^2 \partial y^2} \, \phi(x, y) \right) = 0$$

Compatibility Equation (body force excluded)

```
> eval(ce, V(x, y) = 0)
```

$$S_{11} \left(\frac{\partial^4}{\partial y^4} \, \phi(x, y) \right) + 2 \, S_{12} \left(\frac{\partial^4}{\partial x^2 \partial y^2} \, \phi(x, y) \right) - 2 \, S_{16} \left(\frac{\partial^4}{\partial x \partial y^3} \, \phi(x, y) \right) + S_{22} \left(\frac{\partial^4}{\partial x^4} \, \phi(x, \quad \textbf{(5)}$$

$$y) \Bigg) - 2 \, S_{26} \left(\frac{\partial^4}{\partial x^3 \partial y} \, \phi(x, y) \right) + S_{66} \left(\frac{\partial^4}{\partial x^2 \partial y^2} \, \phi(x, y) \right) = 0$$

pde_plane_strain.mw

This Maple file includes the required relations and equation which give the final PDE derived from the compatibility equation for the plane strain case of an anisotropic body.

N.B. the stress and strain vectors are all averaged through the thickness of the body, i.e.,

$$\bar{\sigma} = \frac{1}{2\,h}\int_{-h}^{h} \sigma(x,y)\,\mathrm{d}z$$

Assuming $u = u(x,y),\ v = v(x,y),\ w = 0$

```
> restart:
```

Stress vector

```
> stress := Vector[column]([ sigma__x ,
                sigma__y ,
                sigma__z ,
                0 ,
                0 ,
                tau__xy ]);
```

$$stress := \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ 0 \\ 0 \\ \tau_{xy} \end{bmatrix} \tag{1}$$

Non-balanced laminate transformed elastic matrix

```
> Q := Matrix([[ Q__11 , Q__12 , Q__13 , 0 , 0 , Q__16 ],
        [ Q__12 , Q__22 , Q__23 , 0 , 0 , Q__26 ],
        [ Q__13 , Q__23 , Q__33 , 0 , 0 , Q__36 ],
        [ 0 , 0 , 0 , Q__44 , Q__45 , 0 ],
        [ 0 , 0 , 0 , Q__45 , Q__55 , 0 ],
        [ Q__16 , Q__26 , Q__36 , 0 , 0 , Q__66 ]]);
```

$$Q := \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & Q_{26} \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & Q_{36} \\ 0 & 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & 0 & Q_{45} & Q_{55} & 0 \\ Q_{16} & Q_{26} & Q_{36} & 0 & 0 & Q_{66} \end{bmatrix} \tag{2}$$

Strain vector

```
> strain := Q . stress
```

$$strain := \begin{bmatrix} Q_{11}\,\sigma_x + Q_{12}\,\sigma_y + Q_{13}\,\sigma_z + Q_{16}\,\tau_{xy} \\ Q_{12}\,\sigma_x + Q_{22}\,\sigma_y + Q_{23}\,\sigma_z + Q_{26}\,\tau_{xy} \\ Q_{13}\,\sigma_x + Q_{23}\,\sigma_y + Q_{33}\,\sigma_z + Q_{36}\,\tau_{xy} \\ 0 \\ 0 \\ Q_{16}\,\sigma_x + Q_{26}\,\sigma_y + Q_{36}\,\sigma_z + Q_{66}\,\tau_{xy} \end{bmatrix} \tag{3}$$

Solve for σ_z

```
> epsilon__z := strain[3]
```
$$\epsilon_z := Q_{13}\,\sigma_x + Q_{23}\,\sigma_y + Q_{33}\,\sigma_z + Q_{36}\,\tau_{xy} \tag{1.1}$$

```
> sigma__z := solve(epsilon__z = 0, sigma__z)
```
$$\sigma_z := -\frac{Q_{13}\,\sigma_x + Q_{23}\,\sigma_y + Q_{36}\,\tau_{xy}}{Q_{33}} \tag{1.2}$$

Calculating stress from a potential function, $\phi(x,y)$

V(x,y) is the body force
```
> sigma__x := diff(phi(x, y), y$2) + V(x, y)
```
$$\sigma_x := \frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y) \tag{2.1}$$

```
> sigma__y := diff(phi(x, y), x$2) + V(x, y)
```
$$\sigma_y := \frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y) \tag{2.2}$$

```
> tau__xy := -diff(phi(x, y), x, y)
```
$$\tau_{xy} := -\frac{\partial^2}{\partial x \partial y}\,\phi(x,y) \tag{2.3}$$

Stress components

```
> epsilon__x := strain[1]
```
$$\epsilon_x := Q_{11}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + Q_{12}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right) \tag{3.1}$$
$$-\frac{1}{Q_{33}}\left(Q_{13}\left(Q_{13}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + Q_{23}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right)\right)\right)$$

$$-Q_{36}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right)\Bigg)\Bigg) - Q_{16}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right)$$

```
> epsilon__y := strain[2]
```

$$\epsilon_y := Q_{12}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + Q_{22}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right) \tag{3.2}$$

$$- \frac{1}{Q_{33}}\left(Q_{23}\left(Q_{13}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + Q_{23}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right)\right.\right.$$

$$\left.\left. - Q_{36}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right)\right)\right) - Q_{26}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right)$$

```
> gamma__xy := strain[6]
```

$$\gamma_{xy} := Q_{16}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + Q_{26}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right) \tag{3.3}$$

$$- \frac{1}{Q_{33}}\left(Q_{36}\left(Q_{13}\left(\frac{\partial^2}{\partial y^2}\,\phi(x,y) + V(x,y)\right) + Q_{23}\left(\frac{\partial^2}{\partial x^2}\,\phi(x,y) + V(x,y)\right)\right.\right.$$

$$\left.\left. - Q_{36}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right)\right)\right) - Q_{66}\left(\frac{\partial^2}{\partial x \partial y}\,\phi(x,y)\right)$$

Compatibility Equation (body force included)

```
> ce := diff(epsilon__x, y$2) + diff(epsilon__y, x$2) - diff
  (gamma__xy, x, y) = 0: simplify(%)
```

$$\frac{1}{Q_{33}}\left(\left(\left(Q_{66} + 2\,Q_{12}\right)Q_{33} - Q_{36}{}^2 - 2\,Q_{23}\,Q_{13}\right)\left(\frac{\partial^4}{\partial x^2 \partial y^2}\,\phi(x,y)\right) + \left(Q_{22}\,Q_{33}\right.\right. \tag{4}$$

$$\left. - Q_{23}{}^2\right)\left(\frac{\partial^4}{\partial x^4}\,\phi(x,y)\right) + \left(2\,Q_{23}\,Q_{36} - 2\,Q_{26}\,Q_{33}\right)\left(\frac{\partial^4}{\partial x^3 \partial y}\,\phi(x,y)\right) + \left(2\,Q_{13}\,Q_{36}\right.$$

$$\left. - 2\,Q_{16}\,Q_{33}\right)\left(\frac{\partial^4}{\partial x \partial y^3}\,\phi(x,y)\right) + \left(Q_{11}\,Q_{33} - Q_{13}{}^2\right)\left(\frac{\partial^4}{\partial y^4}\,\phi(x,y)\right) + \left(\left(Q_{12}\right.\right.$$

$$\left. + Q_{22}\right)Q_{33} - Q_{23}\left(Q_{13} + Q_{23}\right)\right)\left(\frac{\partial^2}{\partial x^2}\,V(x,y)\right) + \left(\left(-Q_{16} - Q_{26}\right)Q_{33} + Q_{36}\left(Q_{13}\right.\right.$$

$$\left. + Q_{23}\right)\right)\left(\frac{\partial^2}{\partial x \partial y}\,V(x,y)\right) + \left(\frac{\partial^2}{\partial y^2}\,V(x,y)\right)\left(\left(Q_{11} + Q_{12}\right)Q_{33} - Q_{13}\left(Q_{13} + Q_{23}\right)\right)\Bigg)$$

$$= 0$$

Compatibility Equation (body force excluded)

```
> eval(ce, V(x, y) = 0): simplify(%)
```

$$\frac{1}{Q_{33}}\left(\left(\left(Q_{66} + 2\,Q_{12}\right)Q_{33} - Q_{36}{}^2 - 2\,Q_{23}\,Q_{13}\right)\left(\frac{\partial^4}{\partial x^2 \partial y^2}\,\phi(x,y)\right) + \left(Q_{22}\,Q_{33}\right.\right. \tag{5}$$

$$\left. - Q_{23}{}^2\right)\left(\frac{\partial^4}{\partial x^4}\,\phi(x,y)\right) + \left(2\,Q_{23}\,Q_{36} - 2\,Q_{26}\,Q_{33}\right)\left(\frac{\partial^4}{\partial x^3 \partial y}\,\phi(x,y)\right) + \left(2\,Q_{13}\,Q_{36}\right.$$

$$\left. - 2\,Q_{16}\,Q_{33}\right)\left(\frac{\partial^4}{\partial x \partial y^3}\,\phi(x,y)\right) + \left(Q_{11}\,Q_{33} - Q_{13}{}^2\right)\left(\frac{\partial^4}{\partial y^4}\,\phi(x,y)\right)\Bigg) = 0$$

h__k.mw

This Maple file shows that $h_k = \dfrac{p_x - p_y s_l^2}{2\left(s_k^2 - s_l^2\right)} - \dfrac{p_{xy}}{4\,s_k}$ is the solution to the homogeneous part of the Laurent series that is the solution to the potential function in the Lekhnitskii formulation for an elastic anisotropic body.

```
> restart:
```

Case 1: r-a ≥ 0

s_k
```
> s[1] := I * (gamma + beta);
  s[2] := I * (-gamma + beta)
```
$$s_1 := I\,(\gamma + \beta)$$
$$s_2 := I\,(-\gamma + \beta) \tag{1.1}$$

h_k
```
> h[1] := (p__x - p__y * s[2]^2) / (2 * (s[1]^2 - s[2]^2)) - p__xy
  / (4 * s[1]);
  h[2] := (p__x - p__y * s[1]^2) / (2 * (s[2]^2 - s[1]^2)) - p__xy
  / (4 * s[2])
```
$$h_1 := \frac{p_y\,(-\gamma + \beta)^2 + p_x}{-2\,(\gamma + \beta)^2 + 2\,(-\gamma + \beta)^2} + \frac{I\,p_{xy}}{4\,(\gamma + \beta)}$$
$$h_2 := \frac{p_y\,(\gamma + \beta)^2 + p_x}{2\,(\gamma + \beta)^2 - 2\,(-\gamma + \beta)^2} + \frac{I\,p_{xy}}{4\,(-\gamma + \beta)} \tag{1.2}$$

Load resultant in x-direction
```
> 2 * evalc(Re(s[1]^2 * h[1] + s[2]^2 * h[2])): simplify(%)
```
$$p_x \tag{1.3}$$

Load resultant in y-direction
```
> 2 * evalc(Re(h[1] + h[2])): simplify(%)
```
$$p_y \tag{1.4}$$

Shear resultant
```
> -2 * evalc(Re(s[1] * h[1] + s[2] * h[2])): simplify(%)
```
$$p_{xy} \tag{1.5}$$

Rotation

u_k
```
> u[1] := S__11 * s[1]^2 + S__12;
  u[2] := S__11 * s[2]^2 + S__12
```
$$u_1 := -S_{11}\,(\gamma + \beta)^2 + S_{12}$$
$$u_2 := -S_{11}\,(-\gamma + \beta)^2 + S_{12} \tag{1.1.1}$$

v_k

```
> v[1] := (S__12 * s[1]^2 + S__22) / s[1];
  v[2] := (S__12 * s[2]^2 + S__22) / s[2]
```

$$v_1 := \frac{-\mathrm{I}\left(-S_{12}\,(\gamma+\beta)^2 + S_{22}\right)}{\gamma+\beta}$$

$$v_2 := \frac{-\mathrm{I}\left(-S_{12}\,(-\gamma+\beta)^2 + S_{22}\right)}{-\gamma+\beta} \tag{1.1.2}$$

u

```
> u_ := 2 * evalc(Re(u[1] * h[1] + u[2] * h[2]))
```

$$u\_ := \frac{2\left(p_y\,(-\gamma+\beta)^2 + p_x\right)\left(-S_{11}\,(\gamma+\beta)^2 + S_{12}\right)}{-2\,(\gamma+\beta)^2 + 2\,(-\gamma+\beta)^2} \tag{1.1.3}$$

$$+\frac{2\left(p_y\,(\gamma+\beta)^2 + p_x\right)\left(-S_{11}\,(-\gamma+\beta)^2 + S_{12}\right)}{2\,(\gamma+\beta)^2 - 2\,(-\gamma+\beta)^2}$$

v

```
> v_ := 2 * evalc(Re(v[1] * h[1] + v[2] * h[2]))
```

$$v\_ := \frac{p_{xy}\left(-S_{12}\,(\gamma+\beta)^2 + S_{22}\right)}{2\,(\gamma+\beta)^2} + \frac{p_{xy}\left(-S_{12}\,(-\gamma+\beta)^2 + S_{22}\right)}{2\,(-\gamma+\beta)^2} \tag{1.1.4}$$

ω_z

```
> omega__z := 1/2 * (diff(V, x) - diff(U, y)): simplify(%)
```

$$0 \tag{1.1.5}$$

```
> restart:
```

Case 1: r-a < 0

s_k

```
> s[1] := alpha + I * beta;
  s[2] := -alpha + I * beta
```

$$s_1 := \alpha + \mathrm{I}\,\beta$$

$$s_2 := -\alpha + \mathrm{I}\,\beta \tag{2.1}$$

h_k

```
> h[1] := (p__x - p__y * s[2]^2) / (2 * (s[1]^2 - s[2]^2)) - p__xy
  / (4 * s[1]);
  h[2] := (p__x - p__y * s[1]^2) / (2 * (s[2]^2 - s[1]^2)) - p__xy
  / (4 * s[2])
```

$$h_1 := \frac{-p_y\,(-\alpha+\mathrm{I}\,\beta)^2 + p_x}{2\,(\alpha+\mathrm{I}\,\beta)^2 - 2\,(-\alpha+\mathrm{I}\,\beta)^2} - \frac{p_{xy}}{4\,(\alpha+\mathrm{I}\,\beta)}$$

$$\tag{2.2}$$

$$h_2 := \frac{-p_y\,(\alpha + \mathrm{I}\,\beta)^2 + p_x}{-2\,(\alpha + \mathrm{I}\,\beta)^2 + 2\,(-\alpha + \mathrm{I}\,\beta)^2} - \frac{p_{xy}}{4\,(-\alpha + \mathrm{I}\,\beta)} \tag{2.2}$$

Load resultant in x-direction
```
> 2 * evalc(Re(s[1]^2 * h[1] + s[2]^2 * h[2])): simplify(%)
```
$$p_x \tag{2.3}$$

Load resultant in y-direction
```
> 2 * evalc(Re(h[1] + h[2])): simplify(%)
```
$$p_y \tag{2.4}$$

Shear resultant
```
> -2 * evalc(Re(s[1] * h[1] + s[2] * h[2])): simplify(%)
```
$$p_{xy} \tag{2.5}$$

Rotation

u_k
```
> u[1] := S__11 * s[1]^2 + S__12;
  u[2] := S__11 * s[2]^2 + S__12
```
$$u_1 := S_{11}\,(\alpha + \mathrm{I}\,\beta)^2 + S_{12}$$
$$u_2 := S_{11}\,(-\alpha + \mathrm{I}\,\beta)^2 + S_{12} \tag{2.1.1}$$

v_k
```
> v[1] := (S__12 * s[1]^2 + S__22) / s[1];
  v[2] := (S__12 * s[2]^2 + S__22) / s[2]
```
$$v_1 := \frac{S_{12}\,(\alpha + \mathrm{I}\,\beta)^2 + S_{22}}{\alpha + \mathrm{I}\,\beta}$$
$$v_2 := \frac{S_{12}\,(-\alpha + \mathrm{I}\,\beta)^2 + S_{22}}{-\alpha + \mathrm{I}\,\beta} \tag{2.1.2}$$

u
```
> u_ := 2 * evalc(Re(u[1] * h[1] + u[2] * h[2]))
```
$$u_- := 2\left(\frac{p_y}{4} - \frac{p_{xy}\,\alpha}{4\,(\alpha^2 + \beta^2)}\right)\left(S_{11}\,(\alpha^2 - \beta^2) + S_{12}\right) - 4\left(-\frac{-p_y\,(\alpha^2 - \beta^2) + p_x}{8\,\alpha\,\beta}\right. \tag{2.1.3}$$
$$+ \left.\frac{p_{xy}\,\beta}{4\,(\alpha^2 + \beta^2)}\right)S_{11}\,\alpha\,\beta + 2\left(\frac{p_y}{4} + \frac{p_{xy}\,\alpha}{4\,(\alpha^2 + \beta^2)}\right)\left(S_{11}\,(\alpha^2 - \beta^2) + S_{12}\right)$$
$$+ 4\left(\frac{-p_y\,(\alpha^2 - \beta^2) + p_x}{8\,\alpha\,\beta} + \frac{p_{xy}\,\beta}{4\,(\alpha^2 + \beta^2)}\right)S_{11}\,\alpha\,\beta$$

v
```
> v_ := 2 * evalc(Re(v[1] * h[1] + v[2] * h[2]))
```

$$v_- := \frac{1}{\alpha^2 + \beta^2} \left(2 \left(\left(\frac{p_y}{4} - \frac{p_{xy}\,\alpha}{4\left(\alpha^2 + \beta^2\right)} \right) \left(S_{12}\left(\alpha^2 - \beta^2\right) + S_{22} \right) - 2 \left(\right. \right.$$

$$\left. - \frac{-p_y\left(\alpha^2 - \beta^2\right) + p_x}{8\,\alpha\,\beta} + \frac{p_{xy}\,\beta}{4\left(\alpha^2 + \beta^2\right)} \right) S_{12}\,\alpha\,\beta \right) \alpha \right) + \frac{1}{\alpha^2 + \beta^2} \left(2 \left(\left(\right. \right.$$

$$\left. - \frac{-p_y\left(\alpha^2 - \beta^2\right) + p_x}{8\,\alpha\,\beta} + \frac{p_{xy}\,\beta}{4\left(\alpha^2 + \beta^2\right)} \right) \left(S_{12}\left(\alpha^2 - \beta^2\right) + S_{22} \right) + 2 \left(\frac{p_y}{4} \right. \right.$$

$$\left. - \frac{p_{xy}\,\alpha}{4\left(\alpha^2 + \beta^2\right)} \right) S_{12}\,\alpha\,\beta \right) \beta \right) - \frac{1}{\alpha^2 + \beta^2} \left(2 \left(\left(\frac{p_y}{4} + \frac{p_{xy}\,\alpha}{4\left(\alpha^2 + \beta^2\right)} \right) \left(S_{12}\left(\alpha^2 \right. \right. \right.$$

$$\left. - \beta^2\right) + S_{22} \right) + 2 \left(\frac{-p_y\left(\alpha^2 - \beta^2\right) + p_x}{8\,\alpha\,\beta} + \frac{p_{xy}\,\beta}{4\left(\alpha^2 + \beta^2\right)} \right) S_{12}\,\alpha\,\beta \right) \alpha \right)$$

$$+ \frac{1}{\alpha^2 + \beta^2} \left(2 \left(\left(\frac{-p_y\left(\alpha^2 - \beta^2\right) + p_x}{8\,\alpha\,\beta} + \frac{p_{xy}\,\beta}{4\left(\alpha^2 + \beta^2\right)} \right) \left(S_{12}\left(\alpha^2 - \beta^2\right) + S_{22} \right) \right. \right.$$

$$\left. - 2 \left(\frac{p_y}{4} + \frac{p_{xy}\,\alpha}{4\left(\alpha^2 + \beta^2\right)} \right) S_{12}\,\alpha\,\beta \right) \beta \right) \tag{2.1.4}$$

ω_z

```
> omega__z := 1/2 * (diff(V, x) - diff(U, y)): simplify(%)
                              0                                           (2.1.5)
```