Problem 11.19 page 477

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$$\frac{d^2y}{dx^2} = \frac{-a^2b}{a(a^2-x^2)\sqrt{a^2-x^2}} \tag{1}$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2} \tag{2}$$

Substituting 2 in 1 gives

$$\frac{d^2y}{dx^2} = \frac{b^4}{a^2y^3} \tag{3}$$

We also have

$$\frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2-x^2}} = \frac{b^2}{a^2}\frac{x}{y} \tag{4}$$

Plug-in $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from 4 and 1 into R_1

$$R_1 = \frac{(1+y'^2)^{\frac{3}{2}}}{y''} = \frac{(a^4y^2+b^4x^2)\sqrt{a^4y^2+b^4x^2}}{a^4b^4} \tag{5}$$

Using the figure from Problem 11.20 (figure 1) and changing variables, in triangle ABC we have $\tan(\phi) = dy/dx$. Replacing the left-hand side of 4 with $\tan(\phi)$ gives

$$\tan(\phi) = \frac{-bx}{a\sqrt{a^2-x^2}} \tag{6}$$

whereby solving 6

$$x = \pm\frac{\tan(\phi)a^2}{\sqrt{a^2\tan^2(\phi)+b^2}} = \pm\frac{\sin(\phi)a^2}{\sqrt{a^2\sin^2(\phi)+b^2\cos^2(\phi)}} \tag{7}$$

By substituting x from 7 in 4

$$y = \pm\frac{b}{a}\sqrt{a^2 - \frac{\tan^2(\phi)a^4}{a^2\tan^2(\phi)+b^2}} = \pm b^2\frac{\cos(\phi)}{\sqrt{a^2\sin^2(\phi)+b^2\cos^2(\phi)}} \tag{8}$$

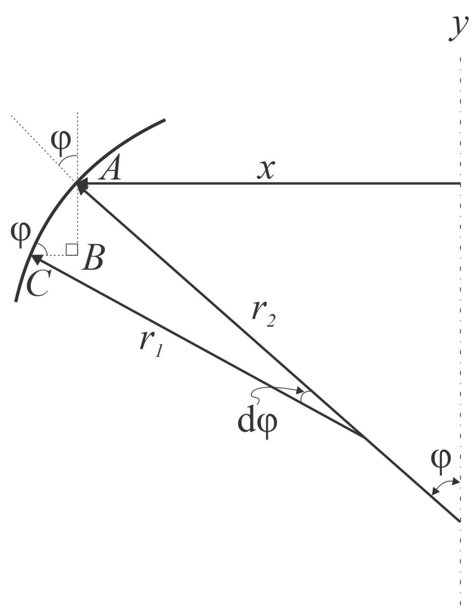

Figure 1: Problem 11.20 figure.

Since, in the following, we alway have an even power (squared) of x and y, the positives values of both x and y will be used in the following. Now by replacing x and y from 7 and 8 in 5

$$\begin{aligned} R_1 = & \frac{(a^4y^2 + b^4x^2)\sqrt{a^4y^2 + b^4x^2}}{a^4b^4} \\ = & \frac{\frac{a^4b^4\cos^2(\phi)}{a^2\sin^2(\phi) + b^2\cos^2(\phi)} + \frac{b^4a^4\sin^2(\phi)}{a^2\sin^2(\phi) + b^2\cos^2(\phi)}}{a^4b^4} \\ = & \frac{a^2b^2}{(a^2\sin^2(\phi) + b^2\cos^2(\phi))^{\frac{3}{2}}} \end{aligned} \tag{9}$$

Also, we know

$$\tan(\phi) = \frac{x}{h} = y'(x) = \frac{b^2}{a^2}\frac{x}{y} \Rightarrow h = \frac{a^2}{b^2}y \tag{10}$$

From Pythagorem theorem,

$$
\begin{aligned}
R_2^2 =& x^2 + h^2 = x^2 + \frac{a^4}{b^4} y^2 \\
=& \frac{a^4 \sin^2(\phi)}{a^2 \sin^2(\phi) + b^2 \cos^2(\phi)} + \frac{a^4}{b^4} \frac{a^4 \cos^2(\phi)}{a^2 \sin^2(\phi) + b^2 \cos^2(\phi)} \\
=& \frac{a^4}{a^2 \sin^2(\phi) + b^2 \cos^2(\phi)}
\end{aligned}
\tag{11}
$$