Lagrange Multiplier and Penalty Method in Contact Mechanics

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1 Problem Statement

A cantilever beam, shown in Figure. 1a, undergoes a point load, F, at its middle point in the x-direction. A rigid wall is located at a distance of 0.1mm in the positive direction of the x-axis. To solve the problem using the finite element method, the beam is divided into four elements as illustrated in Figure. 1b. The numerical model consists of four bar elements with only one degree of freedom in the x-direction as shown in Figure. 1c. The problem can be summarized as

$$\begin{aligned} \min \quad & \Pi \\ \text{s.t.} \quad & u_5 - 0.1e-3 = 0 \end{aligned} \tag{1}$$

where Π is the potential energy and u_5 indicates the displacement of node 5 (see Figure. 1c). Π is the difference between the strain energy and work done by external forces and can be written in the following matrix form.

$$\Pi = \frac{1}{2}\mathbf{D}^T\mathbf{K}\mathbf{D} - \mathbf{D}^T\mathbf{P} \tag{2}$$

$\mathbf{D}$ is the displacement, $\mathbf{K}$ is the tangential stiffness matrix, $\mathbf{P}$ is the external load and the superscript T indicates the transpose. In the following, we solve the problem using the Lagrange multipliers and penalty methods. In penalty method, we also investigate the inherited penetration into the rigid wall due to the algorithm implementation.

2 Lagrange Multiplier

To impose the constraint given in Eq. 1 using the Lagrange multiplier, we add a vector of Lagrange multipliers, λ to Eq. 2 and solve the minimization problem by taking the derivative of the potential energy w.r.t the displacement ($\partial\Pi/\partial\mathbf{D}$) and Lagrange multipliers ($\partial\Pi/\partial\lambda$). As a result, Eq. 2 becomes [1]

$$\Pi = \frac{1}{2}\mathbf{D}^T\mathbf{K}\mathbf{D} - \mathbf{D}^T\mathbf{P} + \lambda^T(\mathbf{C}\mathbf{D} - \mathbf{F}) \tag{3}$$

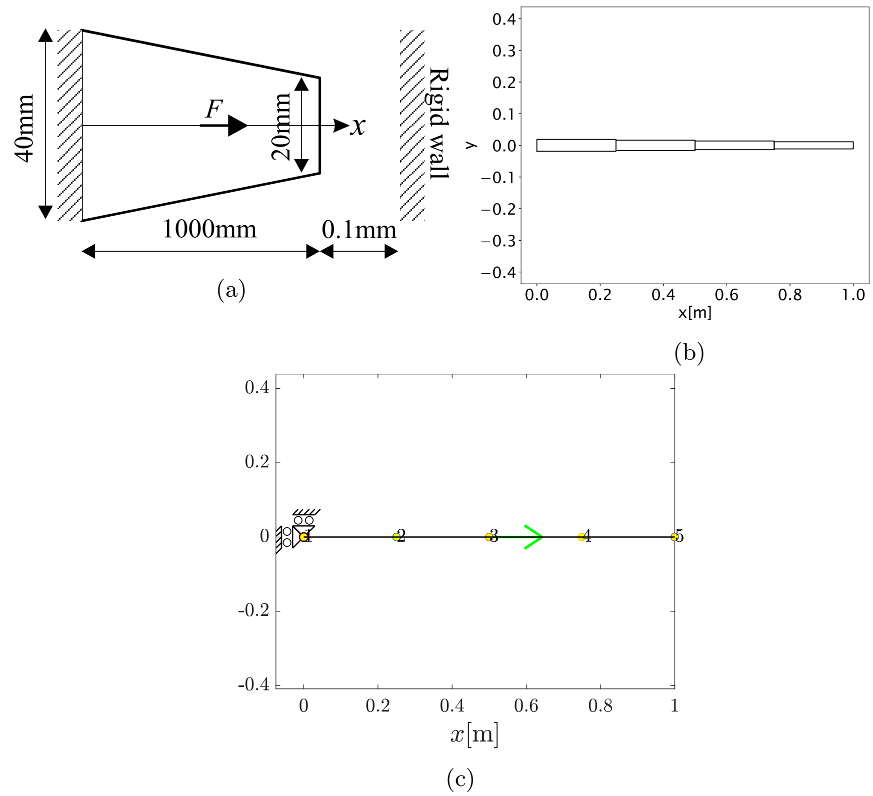


Figure 1: (a) The cantilever tapered beam. (b) The beam model with four elements. (c) The numerical model of the beam with bar elements.

where λ is the Lagrange multiplier, $\mathbf{C}$ is the constraint (the coefficient of u_5 in Eq. 1) and $\mathbf{F}$ is the constraint value (0.1e-3 in Eq. 1). After taking the derivatives, the final equation can be written in the following matrix form [1].

$$\begin{bmatrix} \mathbf{K} & \mathbf{C}^T \\ \mathbf{C} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{D} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{P} \\ \mathbf{F} \end{Bmatrix}$$

A Python script is developed based on algorithm 1.

Algorithm 1 Lagrange multiplier pseudocode.

Ensure: $D = 0, K = 0$
  Compute K
  $K \leftarrow$ Assemble K and C
  Apply B.C on K
  $P \leftarrow$ Assemble P and F
  Solve $D = K^{-1}P$

Figure. 2 shows the undeformed and deformed configuration of the beam using the Lagrange multiplier method. As it is clearly visible the Lagrange multiplier solution is exact and the beam does not penetrate in the rigid wall.

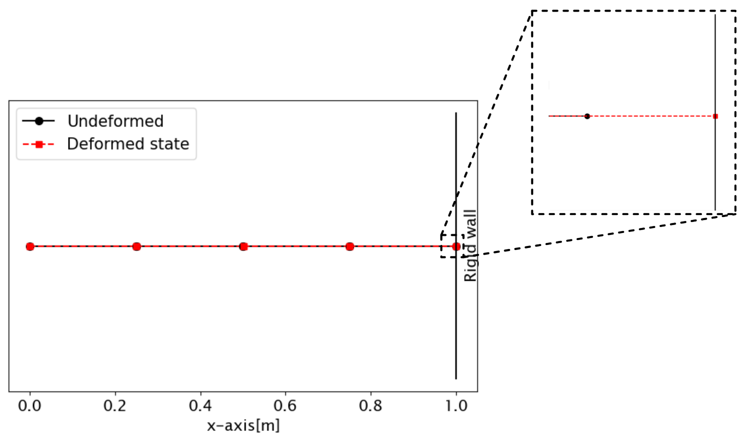


Figure 2: The undeformed and deformed state of the beam using the Lagrange multiplier method.

3 Penalty Method

Unlike the Lagrange multiplier, the penalty method does not introduce any extra variable; however, the accuracy of the solution depends on the penalty coefficient in this method. Therefore, Eq. 2 remains intact while the values of **K** and **P** are manipulated using the penalty coefficient, ϵ_p. algorithm 2 summarizes the penalty method implementation.

Algorithm 2 Penalty method pseudocode.

Ensure: $D = 0, K = 0, \epsilon_p = 1e14$
  Compute K
  $K \leftarrow K + \epsilon_p C^T C$
  Apply B.C on K
  $P \leftarrow P + \epsilon_p F$
  Solve $D = K^{-1}P$

Figure. 3 shows the effect of ϵ_p on the accuracy of the solution using the penalty method. It is seen that by increasing ϵ_p, the solution converges 0.1mm.

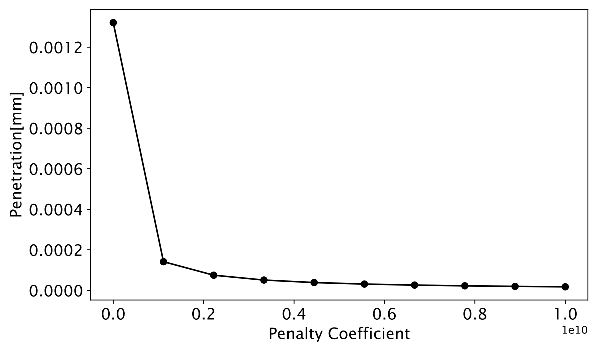


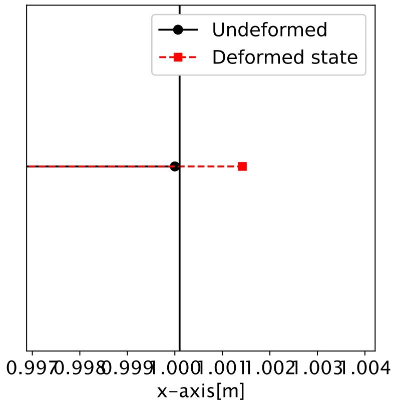
Figure 3: The effect of penalty coefficient on the accuracy of the solution.

Table. 1 shows the penetration values for various ϵ_p. The penetration vanishes at about ϵ_p = 1e14.

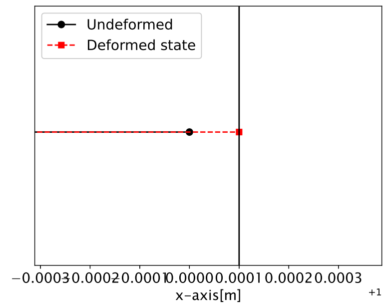
Table 1: Penalty coefficients and their corresponding penetration values in penalty method.

ϵ_p	Penetration[mm]
1	1.32e-03
1e5	1.32e-03
1e6	1.31e-03
1e7	1.23e-03
1e8	7.54e-04
1e9	1.55e-04
1e10	1.73e-05
1e11	1.76e-06
1e12	1.76e-07
1e13	1.76e-08
1e14	1.79e-09

Figure. 4a (maximum penetration) and Figure. 4b (no penetration) shows the penetration into the rigid body.



(a) ϵ_p=1



(b) ϵ_p=1e14

Figure 4: The penetration of beam into the rigid wall in the penalty method.

References

[1] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt, *Concepts and Applications of Finite Element Analysis, 4th Edition*. Wiley, 4 ed., Oct. 2001.