Summation, Product, Transpose

Dyadic Product, Horizontal / Vertical dot product

Trace, Identity Tensor, Inverse of a Tensor

Summation:

$$\underbrace{(T+S)}_{W} a = Ta + Sa = Wa$$

$$W_{ij} = e_i \cdot (T+S) e_j = e_i \cdot T e_j + e_i \cdot S e_j = T_{ij} + S_{ij}$$

$$[W] = [T] + [S]$$

Product of two tensor:

$$\left.\begin{aligned} (TS)a &= T(Sa) \\ \text{and also } (ST)a &= S(Ta) \end{aligned}\right\} ①$$

$$(TS)_{ij} = e_i \cdot (TS) e_j \overset{①}{=} e_i \cdot T(S e_j) \overset{②}{=} e_i \cdot T(S_{nj} e_n)$$

② $S e_j = S_{nj} e_n$

$$= S_{nj}\, e_i \cdot T e_n = S_{nj} T_{in}$$

$$(ST)_{ij} = e_i \cdot (ST) e_j \overset{①}{=} e_i \cdot S(T e_j) \overset{③}{=} e_i \cdot S(T_{nj} e_n)$$

③ $T e_j = T_{nj} e_n$

$$= T_{nj}\, e_i \cdot S e_n = T_{nj} S_{in} = S_{in} T_{nj}$$

$[TS] = [T][S]$

$[ST] = [S][T]$ BUT $[ST] \neq [TS]$, $ST \neq TS$

$$(\underbrace{T(SV)}_{A})a \equiv \underbrace{T((SV}_{A})a) \equiv T(S(Va))$$

$T^3 = TTT$ $T^2 = TT$ $T^n = TTT\ldots$

Transpose:

$$a.Tb = b.T^T a \quad (4)$$

$$(4): e_i.Te_j = e_j.T^T e_i \Rightarrow T_{ij} = T^T_{ji}$$

$$[T]^T = [T^T]$$

$(T^T)^T = T$

$(AB\ldots Z)^T = Z^T \ldots B^T A^T$

Dyadic Product:

$$(a \otimes b)c = a(b.c) \quad (5)$$

$$(a\otimes b)(\alpha c + \beta d) \overset{(5)}{=} a(b.(\alpha c + \beta d)) = a(\alpha b.c + \beta b.d)$$

α, β: scalar

$$= \alpha a(b.c) + \beta a(b.d) \overset{(5)}{=} \alpha(a\otimes b)c + \beta(a\otimes b)d$$

$$a \otimes b \neq b \otimes a$$

Define a tensor using dyadic product:

$$T_{ij} = e_i \cdot T e_j = e_i \cdot (a \otimes b) e_j \overset{(5)}{=} \underbrace{e_i\, a}_{a_i} \underbrace{(b \cdot e_j)}_{b_j} = a_i b_j$$

we assume that $T = a \otimes b$

$$T_{ij} = a_i b_j$$

$$T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

$$e_1 \otimes e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_1 \otimes e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T = T_{11} e_1 \otimes e_1 + T_{12} e_1 \otimes e_2 + \dots = T_{ij} e_i \otimes e_j$$

Trace of a tensor:

$$\left\{ \begin{aligned} tr(T+S) &= tr\,T + tr\,S \\ tr(\alpha T) &= \alpha\, tr(T) \\ tr(a \otimes b) &= a \cdot b \end{aligned} \right. \qquad (6)$$

$$tr(T) = tr(T_{ij} e_i \otimes e_j) = T_{ij}\, tr(e_i \otimes e_j) \overset{(6.1)}{=} T_{ij}\, e_i \cdot e_j$$

$$= T_{ij} \delta_{ij} = T_{ii} = T_{jj}$$

$$tr(T) = T_{ii} = T_{jj} = T_{11} + T_{22} + T_{33}$$

$$tr\,T^T = tr\,T$$

$$\underbrace{tr(AB)}_{C} \qquad C = AB \ , \ C_{ij} = A_{im}B_{mj}$$

$$tr(C) = C_{ii} = A_{im}B_{mi}$$

$$\underbrace{tr(ABCD)}_{E} = A_{im}B_{mn}C_{nj}D_{ji}$$

$$tr(E) = E_{ii}$$

Identity Tensor (I) and Inverse of a Tensor:

$$Ia = a \quad (7)$$

$$TI = IT = T$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. $ST = I$ that means $S = T^{-1}$

2. $[T][T]^{-1} = [T]^{-1}[T] = [I]$

3. $(T^{-1})^{T} = (T^{T})^{-1}$

4. $(TS)^{-1} = S^{-1}T^{-1}$, $(ABC\cdots D)^{-1} = D^{-1}\cdots C^{-1}B^{-1}A^{-1}$

$$Ta = b \longrightarrow \underbrace{T^{-1}T}_{I}a = T^{-1}b \Rightarrow Ia = T^{-1}b \Rightarrow a = T^{-1}b$$

$Ta = b$:
- if T is invertible $\therefore$ there is a one-to-one mapping T that can transform a into b.
- if T is not invertible $(\det T = 0)$ $\therefore$ there is more than one mapping like T that transform a into b.

Example 2.14.2

$T = c \otimes d$

(a) $$T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} = \begin{bmatrix} c_1 d_1 & c_1 d_2 & c_1 d_3 \\ c_2 d_1 & c_2 d_2 & c_2 d_3 \\ c_3 d_1 & c_3 d_2 & c_3 d_3 \end{bmatrix}$$

$\det(T) = 0$

(b) if $Ta = b$ then $T(a+h) = b$ wher $h \perp d$ $(h \cdot d = 0)$ $\blacksquare$

$$T(a+h) = (c \otimes d)(a+h) \overset{5}{=} c\big(d \cdot (a+h)\big) = c(d \cdot a + d \cdot h)$$

$$\underset{d \cdot h = 0}{=} c(d \cdot a) \overset{5}{=} (c \otimes d)\, a = Ta = b$$

Horizontal / Vertical dot product:

$F \cdot\cdot G = F_{ij} G_{ji}$ Horizontal dot product

$F : G = F_{ij} G_{ij}$ Vertical " "

$$F : G = F_{11} G_{11} + F_{12} G_{12} + F_{13} G_{13} + F_{21} G_{21} + \cdots$$

$$F \cdot\cdot G = F_{11} G_{11} + F_{12} G_{21} + F_{13} G_{31} + F_{21} G_{12} + \cdots$$