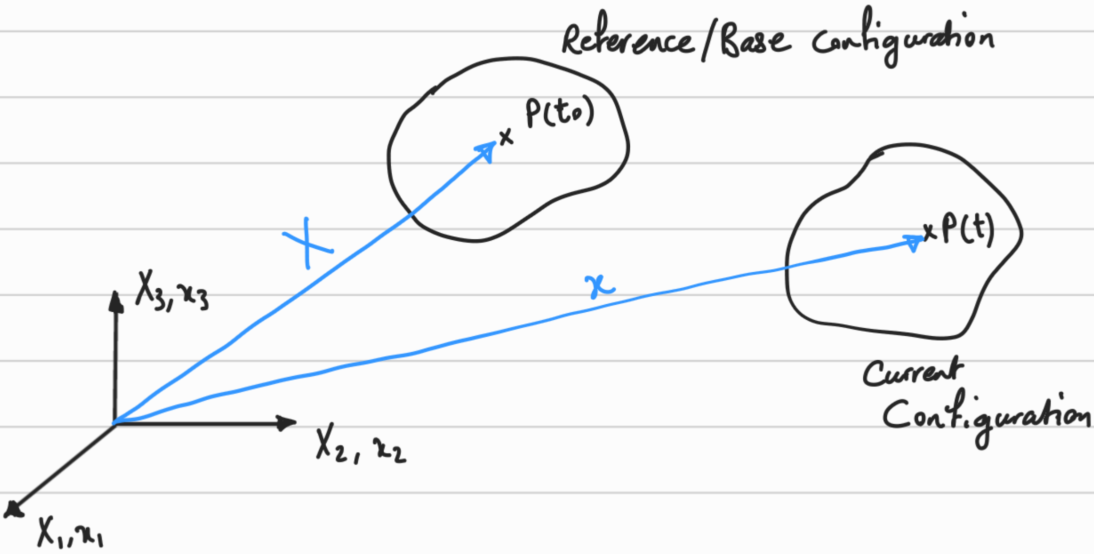
Reference and Current Configuration

Lagrangian / Eulerian Point of view

Material Derivative

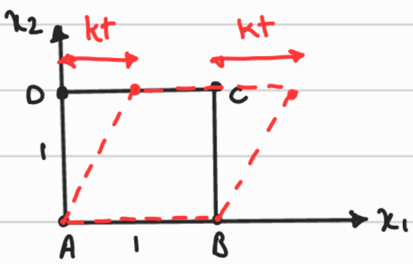
Reference and Current Configuration:



$$\boxed{x = x(X, t) \quad \text{with} \quad X = x(X, t_0)}$$

$$\underbrace{\begin{aligned} x_1 &= x_1(X_1, X_2, X_3, t) \\ x_2 &= x_2(X_1, X_2, X_3, t) \\ x_3 &= x_3(X_1, X_2, X_3, t) \end{aligned}}_{\text{motion of continuum}} \quad , \quad \underbrace{\begin{aligned} X_1 &= x_1(X_1, X_2, X_3, t_0) \\ X_2 &= x_2(X_1, X_2, X_3, t_0) \\ X_3 &= x_3(X_1, X_2, X_3, t_0) \end{aligned}}_{\text{material coordinate}}$$

Example 3.1.1



— Reference configuration

--- Current Configuration

$$\vec{x} = \vec{X} + ktX_2\,\hat{e}_1 = (X_1\hat{e}_1 + X_2\hat{e}_2 + X_3\hat{e}_3) + ktX_2\hat{e}_1 \qquad (*)$$

where $\vec{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

$$\overset{X_1\ X_2\ X_3}{A(0,0,0)} \xrightarrow{(*)} x = 0\hat{e}_1 + 0\hat{e}_2 + 0\hat{e}_3$$

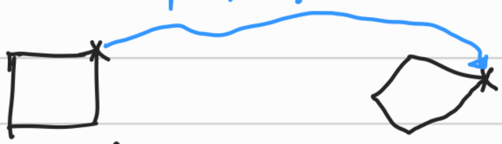
$$\overset{X_1\ X_2\ X_3}{B(1,0,0)} \xrightarrow{(*)} x = 1\hat{e}_1 + 0\hat{e}_2 + 0\hat{e}_3$$

$$\overset{X_1\ X_2\ X_3}{C(1,1,0)} \xrightarrow{(*)} x = 1\hat{e}_1 + 1\hat{e}_2 + 0\hat{e}_3 + kt\,\hat{e}_1 = (1+kt)\hat{e}_1 + 1\hat{e}_2 + 0\hat{e}_3$$

$$\overset{X_1\ X_2\ X_3}{D(0,1,0)} \xrightarrow{(*)} x = (0+kt)\hat{e}_1 + 1\hat{e}_2 + 0\hat{e}_3 = kt\hat{e}_1 + 1\hat{e}_2 + 0\hat{e}_3$$

Lagrangian and Eulerian Point of view:

Lagrangian (material description) you move with a particle

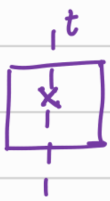


Reference config. t_0 Current config t

$$\theta = \hat{\theta}(X_1, X_2, X_3, t)$$

usually used in Solid Mechanics

Eulerian (spatial description)



$$\theta = \tilde{\theta}(x_1, x_2, x_3, t)$$

usually used in Fluid Mechanics

Example 3.2.1

$$\begin{cases} x_1 = X_1 + kt X_2 \\ x_2 = (1+kt) X_2 \\ x_3 = X_3 \end{cases}$$

$\theta = \alpha(x_1 + x_2)$ spatial description

a) $\theta = \alpha(x_1 + x_2) = \alpha(X_1 + ktX_2 + (1+kt)X_2) = \alpha X_1 + (1+2kt)X_2$

material description

b) $v_i = \left(\dfrac{\partial x_i}{\partial t}\right)_{X_i\text{-fixed}}$ $x_i = x_i(X_j, t)$

$$\begin{cases} v_1 = kX_2 \\ v_2 = kX_2 \\ v_3 = 0 \end{cases} \Rightarrow \vec{v} = kX_2\,\hat{e}_1 + kX_2\,\hat{e}_2 \quad \text{material description } (**)$$

To get $\vec{v}$ in spatial coordinate, we find X_2 in terms of x_i and then substitute it with X_2 in $(**)$ $x_2 = (1+kt)X_2 \Rightarrow X_2 = \dfrac{1}{1+kt}\,x_2$

$$\vec{v} = \frac{k}{1+kt}\,x_2\,\hat{e}_1 + \frac{k}{1+kt}\,x_2\,\hat{e}_2 \quad \text{spatial description}$$

Material Derivative:

For a scalar field: $\theta = \theta(X_1, X_2, X_3, t)$

$\dfrac{D}{Dt} \equiv$ material derivative

$$\frac{D\theta}{Dt} = \left(\frac{\partial\theta(X_1, X_2, X_3, t)}{\partial t}\right)_{X_i\text{-fixed}} \quad \text{material description}$$

But what if we are in the spatial description!

$\theta = \theta(x_1, x_2, x_3, t)$

$$\frac{D\theta}{Dt} = \frac{D\theta(x_1, x_2, x_3, t)}{Dt} = \frac{\partial\theta}{\partial x_1}\cancelto{v_1}{\frac{\partial x_1}{\partial t}} + \frac{\partial\theta}{\partial x_2}\cancelto{v_2}{\frac{\partial x_2}{\partial t}} + \frac{\partial\theta}{\partial x_3}\cancelto{v_3}{\frac{\partial x_3}{\partial t}} + \frac{\partial\theta}{\partial t}$$

$$= \frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial x_1}v_1 + \frac{\partial\theta}{\partial x_2}v_2 + \frac{\partial\theta}{\partial x_3}v_3$$

$$\boxed{\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + v_i\cdot\left(\frac{\partial\theta}{\partial x_i}\right)} \quad \text{spatial description (indicial notation)}$$

$$\boxed{\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + v \cdot (\nabla \theta)}$$

spatial description (tensorial notation)

in cylindrical coordinate system: $\nabla\theta = \text{grad}(\theta) = \frac{\partial \theta}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \theta}{\partial \theta}\hat{e}_\theta + \frac{\partial \theta}{\partial z}\hat{e}_z$

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) \cdot \left\{ \frac{\partial \theta}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \theta}{\partial \theta}\hat{e}_\theta + \frac{\partial \theta}{\partial z}\hat{e}_z \right\}$$

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + v_r \frac{\partial \theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial \theta}{\partial \theta} + v_z \frac{\partial \theta}{\partial z}$$

For a vector field:

$$v = \left(\frac{\partial x}{\partial t}\right)_{X_i\text{-fixed}} = \frac{Dx}{Dt} \quad , \quad a = \left(\frac{\partial v}{\partial t}\right)_{X_i\text{-fixed}} = \frac{Dv}{Dt}$$

material description

But what if you want to get $\frac{D\vec{v}}{Dt}$ in spatial coordinate.

$\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$ and also each v_i is a function of x_j and time, t; therefore,

$$\vec{v} = v_1(x_1, x_2, x_3, t)\hat{e}_1 + v_2(x_1, x_2, x_3, t)\hat{e}_2 + v_3(x_1, x_2, x_3, t)\hat{e}_3$$

$$a = \frac{Dv}{Dt} = \frac{\partial v_1}{\partial x_1}\cancelto{v_1}{\frac{\partial x_1}{\partial t}} + \frac{\partial v_1}{\partial x_2}\cancelto{v_2}{\frac{\partial x_2}{\partial t}} + \frac{\partial v_1}{\partial x_3}\cancelto{v_3}{\frac{\partial x_3}{\partial t}} + \frac{\partial v_1}{\partial t}$$
$$+ \frac{\partial v_2}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial v_2}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial v_2}{\partial x_3}\frac{\partial x_3}{\partial t} + \frac{\partial v_2}{\partial t}$$
$$+ \frac{\partial v_3}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial v_3}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial v_3}{\partial x_3}\frac{\partial x_3}{\partial t} + \frac{\partial v_3}{\partial t}$$

$$\boxed{a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}}$$

spatial description (indicial notation)

$$\boxed{a = \frac{\partial v}{\partial t} + (\nabla v)\, v}$$ spatial description(tensorial notation)

There fore, in general (spatial description)

For a scalar field (■) $\dfrac{D■}{Dt} = \dfrac{\partial ■}{\partial t} + v \cdot (\nabla ■)$

For a vector field (▲) $\dfrac{D▲}{Dt} = \dfrac{\partial ▲}{\partial t} + (\nabla ▲)\, v$

Example 3.4.2

$$\begin{cases} v_1 = \dfrac{kx_1}{1+kt} \\ v_2 = \dfrac{kx_2}{1+kt} \\ v_3 = \dfrac{kx_3}{1+kt} \end{cases}$$

a) $\dfrac{D\vec{v}}{Dt} = \vec{a} = \dfrac{\partial}{\partial t}\left(v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3\right) + (\nabla v)\, v$

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = \dfrac{-k^2}{(1+kt)^2} x_1 \\ \dfrac{\partial v_2}{\partial t} = \dfrac{-k^2}{(1+kt)^2} x_2 \\ \dfrac{\partial v_3}{\partial t} = \dfrac{-k^2}{(1+kt)^2} x_3 \end{cases}$$

$$\nabla v = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial x_2} & \dfrac{\partial v_1}{\partial x_3} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \dfrac{k}{1+kt} & 0 & 0 \\ 0 & \dfrac{k}{1+kt} & 0 \\ 0 & 0 & \dfrac{k}{1+kt} \end{bmatrix}$$

$$a = \frac{-k^2}{(1+kt)^2} x_1 \hat{e}_1 - \frac{k^2}{(1+kt)^2} x_2 \hat{e}_2 - \frac{k^2}{(1+kt)^2} x_3 \hat{e}_3$$

$$+ \frac{k^2}{(1+kt)^2} x_1 \hat{e}_1 + \frac{k^2}{(1+kt)^2} x_2 \hat{e}_2 + \frac{k^2}{(1+kt)^2} x_3 \hat{e}_3 = 0\hat{e}_1 + 0\hat{e}_2 + 0\hat{e}_3$$

b) $x_i = x_i(X_j, t)$ $\left(\frac{Dv}{Dt}\right)_{X_i\text{-fixed}}$

$$v_1 = \left(\frac{dx_1}{dt}\right)_{X_1\text{-fixed}} = \frac{kx_1}{1+kt}$$

to get the displacement we integrate

$$\frac{dx_1}{dt} = \frac{kx_1}{1+kt} \Rightarrow \int_{X_1}^{x_1} \frac{dx_1}{x_1} = \int_0^t \frac{k}{1+kt}\, dt$$

$$\ln x_1 - \ln X_1 = \ln(1+kt) \Rightarrow \ln \frac{x_1}{X_1} = \ln(1+kt)$$

$$x_1 = (1+kt)X_1$$

By following similar procedures, $x_2 = (1+kt)X_2$

$$x_3 = (1+kt)X_3$$

$$v_i = \left(\frac{\partial x_i}{\partial t}\right)_{X_i\text{-fixed}} = kX_1\,\hat{e}_1 + kX_2\,\hat{e}_2 + kX_3\,\hat{e}_3$$

$$a_i = \left(\frac{\partial v_i}{\partial t}\right)_{X_i\text{-fixed}} = 0\,\hat{e}_1 + 0\,\hat{e}_2 + 0\,\hat{e}_3$$