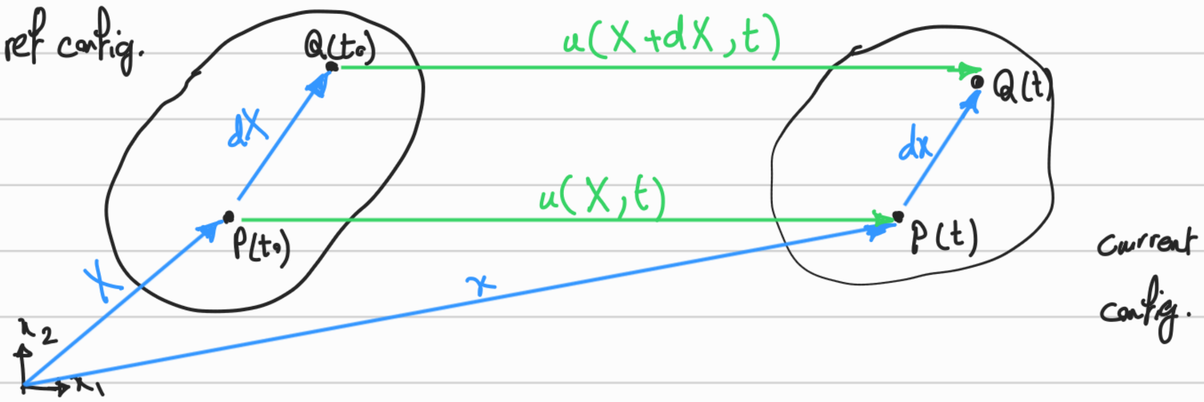
Deformation gradient $\begin{cases} \text{local rigid body motion} \\ \text{pure stretch} \end{cases}$

Deformation Gradient:



$$dx = x(X+dX,t) - x(X,t) = (\nabla x)\,dX \quad , \quad \boxed{dx = F\,dX}$$

$$F = \nabla x$$

$$\nabla x = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

$$\left.\begin{aligned} x &= X + u(X,t) \\ x + dx &= X + dX + u(X+dX,t) \end{aligned}\right\} \quad dx = dX + u(X+dX,t) - u(X,t)$$

$$dx = dX + \nabla u\, dX$$

$$\boxed{dx = (I + \nabla u)\,dX}$$

1. There's this transformation of each node from reference to current configuration and vice-versa. Therefore, F^{-1} exists.

2. When the reference configuration is deformed, it deforms such that the initial unit vectors elongate and rotate. If we have $e_1 . (e_2 \times e_3) = e_2 . (e_3 \times e_1) = e_3 . (e_1 \times e_2)$ then it remains the same in the current configuration; therefore, for the transformation F

$$Fe_1 . (Fe_2 \times Fe_3) = Fe_2 . (Fe_3 \times Fe_1) = Fe_3 . (Fe_1 \times Fe_2)$$
$$= \det(F) > 0$$

If $\det(F) < 0$ then it shows that the particle does not exist in the current configuration. If $\det(F) < 0$ then we cannot simply move back from current to the reference because the particle does not exist.

(Assumption: We use right-handed basis)

Local Rigid Body Motion:

Neither length nor angle of element are changed.

$\det(F) > 0$ and F^{-1} exists.

$FF^T = I$ and $\det(F) = 1$

Pure Stretch:

Pure Stretch is a deformation tensor that is positive definit.

$\forall \vec{a} : \vec{a} \cdot U\vec{a} \geq 0$ where U: pure stretch

It can be shown that if a tensor is positive definite and symmetric, all of its eigenvalues are positive.

If we rewrite U in the form of its eigenvalues,

$$U = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$dx = U\,dX \Rightarrow \begin{Bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{Bmatrix} dX^1 \\ dX^2 \\ dX^3 \end{Bmatrix}$$

$$\begin{Bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{Bmatrix} = \begin{Bmatrix} \lambda_1 dX^1 \\ \lambda_2 dX^2 \\ \lambda_3 dX^3 \end{Bmatrix}$$

$$\lambda_1 = \frac{|dx^1|}{|dX^1|} , \quad \lambda_2 = \frac{|dx^2|}{|dX^2|} , \quad \lambda_3 = \frac{|dx^3|}{|dX^3|}$$