Intro to derivative of a tensor

Gradient, Divergence, Curl, Laplacian

Derivative of a tensor:

$$\frac{dT(m)}{dm} = \lim_{\Delta m \to 0} \frac{T(m+\Delta m) - T(m)}{\Delta m}$$

$$\frac{d}{dm}(T+S) = \frac{dT(m)}{dm} + \frac{dS(m)}{dm}$$

$$\frac{d}{dm}(\alpha(m)T) = \frac{d\alpha}{dm}T + \alpha\frac{dT}{dm}$$

$$\frac{d}{dm}(TS) = \frac{dT}{dm}S + T\frac{dS}{dm}$$

$$\frac{d}{dm}(Ta) = \frac{dT}{dm}a + T\frac{da}{dm}$$

$$\frac{d}{dm}(T^T) = \left(\frac{dT}{dm}\right)^T$$

Example 2.26.3

Rotation tensor : $R(t)$

vector : r_0

$r(t) = R(t)r_0$

$$\frac{dr}{dt} = \omega \times r$$

where $\omega = \frac{dR}{dt}R^T$

$$r(t) = R(t)\,r_0 \qquad \frac{dr(t)}{dt} = \frac{dR(t)}{dt} r_0 + R(t)\cancel{\frac{dr_0}{dt}} \quad (1)$$

$$r = Rr_0 \Rightarrow R^T r = R^T R r_0 = I r_0 = r_0 \Rightarrow r_0 = R^T r \quad (2)$$

Substitute (2) in (1) $\quad \dfrac{dr}{dt} = \underbrace{\dfrac{dR}{dt}}_{\text{antisymmetric tensor}} R^T r = \omega \times r \quad$ where ω is the dual vector $\dfrac{dR}{dt} R^T$

from dual vector : $Ta = t^A \times r$

Gradient and Gradient of a Scalar field:

$$\frac{d\varphi(r)}{dr} = \frac{\varphi(r+dr) - \varphi(r)}{dr} = \nabla\varphi \quad (\text{Gradient of } \varphi)$$

$$d\varphi = \nabla\varphi \cdot dr$$

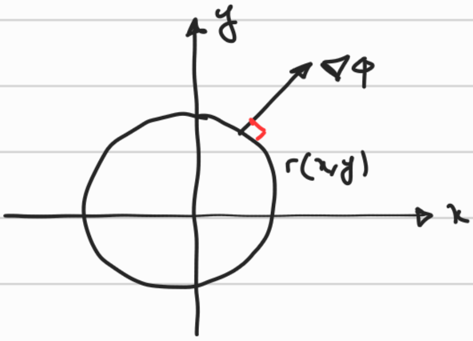
$$\frac{d\varphi}{|dr|} = \nabla\varphi \cdot \frac{dr}{|dr|} = \nabla\varphi \cdot e$$

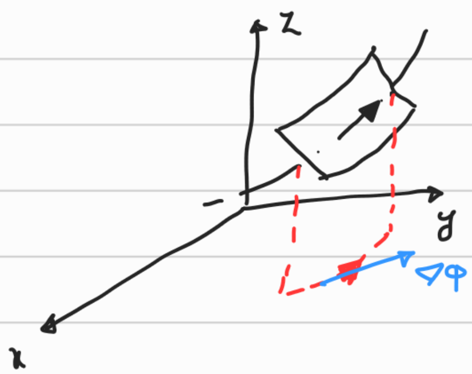
$$\frac{\partial\varphi}{\partial x_i} = \nabla\varphi \cdot e_i$$

$$\nabla\varphi = \frac{\partial\varphi}{\partial x_1}\hat{e}_1 + \frac{\partial\varphi}{\partial x_2}\hat{e}_2 + \frac{\partial\varphi}{\partial x_3}\hat{e}_3$$

$\therefore$ The gradient of a scalar field is a vector field.

Geometrical Meaning of Gradient:





Gradient of a Vector Field:

$$\nabla \equiv \frac{\partial(\,)}{\partial x_i} \otimes \hat{e}_i \quad , \quad \nabla \varphi = \frac{\partial \varphi}{\partial x_i} \otimes \hat{e}_i = \frac{\partial \varphi}{\partial x_i} \hat{e}_i \quad \text{(For a scalar field)}$$

$$\nabla \vec{v} = \frac{\partial(v_j \hat{e}_j)}{\partial x_i} \otimes \hat{e}_i = \frac{\partial v_j}{\partial x_i} \hat{e}_j \otimes \hat{e}_i + v_j \cancelto{0}{\frac{\partial \hat{e}_j}{\partial x_i}} \otimes \hat{e}_i$$

$$= \frac{\partial v_j}{\partial x_i} \hat{e}_j \otimes \hat{e}_i$$

$$= \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Gradient of a tensor:

$$\nabla T = \frac{\partial(T)}{\partial x_i} \otimes \hat{e}_i = \frac{\partial(T_{jk}\, e_j \otimes \hat{e}_k)}{\partial x_i} \otimes \hat{e}_i$$

$$= \frac{\partial T_{jk}}{\partial x_i} \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}_i + T_{jk} \cancelto{0}{\frac{\partial e_j}{\partial x_i}} \otimes \hat{e}_k \otimes \hat{e}_i + T_{jk} \hat{e}_j \otimes \cancelto{0}{\frac{\partial \hat{e}_k}{\partial x_i}} \otimes \hat{e}_i$$

$$= \frac{\partial T_{jk}}{\partial x_i} \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}_i \quad \text{(3rd-order tensor)}$$

Divergence of a Scalar Field:

Divergence of a scalar field does not exist.

Divergence of a vector Field:

$$\operatorname{div} \vec{v} \equiv \operatorname{tr}(\nabla v)$$

$$\operatorname{div} \equiv (\nabla \cdot \ )$$

$$\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = \frac{\partial\vec{v}}{\partial x_i}\cdot\hat{e}_i = \frac{\partial(v_j\hat{e}_j)}{\partial x_i}\cdot\hat{e}_i$$

$$= \frac{\partial v_j}{\partial x_i}\hat{e}_j\cdot\hat{e}_i + v_j\cancel{\frac{\partial\hat{e}_j}{\partial x_i}}^{0}\cdot\hat{e}_i = \frac{\partial v_j}{\partial x_i}\hat{e}_j\cdot\hat{e}_i = \frac{\partial v_j}{\partial x_i}\delta_{ij}$$

$$= \frac{\partial v_i}{\partial x_i} = tr(\nabla v) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \quad (\text{scalar})$$

Divergence of a Tensor:

$$\operatorname{div} T \equiv \nabla\cdot T = \frac{\partial(T)}{\partial x_i}\cdot\hat{e}_i = \frac{\partial(T_{jk}\hat{e}_j\otimes\hat{e}_k)}{\partial x_i}\cdot\hat{e}_i$$

$$= \frac{\partial T_{jk}}{\partial x_i}\hat{e}_j\otimes\underbrace{\hat{e}_k\cdot\hat{e}_i}_{\delta_{ki}} = \frac{\partial T_{ji}}{\partial x_i}\hat{e}_j \quad (\text{vector})$$

Curl of a Scalar Field:

There is no curl in a scalar field.

Curl of a vector:

$$\operatorname{curl}\vec{v} \equiv 2t^A$$

$$\operatorname{curl} \equiv (\nabla\times)$$

$$\operatorname{curl}\vec{v} \equiv \nabla\times\vec{v} = \frac{\partial\vec{v}}{\partial x_i}\times\hat{e}_i = \frac{\partial(v_j\hat{e}_j)}{\partial x_i}\times\hat{e}_i$$

$$= \frac{\partial v_j}{\partial x_i}\underbrace{\hat{e}_j\times\hat{e}_i}_{\epsilon_{jik}\hat{e}_k} = \frac{\partial v_j}{\partial x_i}\epsilon_{jik}\hat{e}_k = -\frac{\partial v_i}{\partial x_j}\epsilon_{ijk}\hat{e}_k \quad (\text{vector})$$

Divergence of a Tensor:

$$\text{curl } T \equiv \nabla \times T = \frac{\partial T}{\partial x_i} \times \hat{e}_i = \frac{\partial (T_{jk}\, \hat{e}_j \otimes \hat{e}_k)}{\partial x_i} \times \hat{e}_i$$

$$= \frac{\partial T_{jk}}{\partial x_i}\, \hat{e}_j \otimes \underbrace{\hat{e}_k \times \hat{e}_i}_{\epsilon_{kim}\, \hat{e}_m} = \frac{\partial T_{jk}}{\partial x_i}\, \epsilon_{kim}\, \hat{e}_j \otimes \hat{e}_m \quad \text{(tensor)}$$

Laplacian:

$$\nabla^2(\,) = \nabla \cdot \nabla(\,) \equiv \frac{\partial^2(\,)}{\partial x_i\, \partial x_i}$$

Laplacian of a scalar Field:

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \nabla \cdot \frac{\partial \phi}{\partial x_i} \hat{e}_i = \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i} \hat{e}_i\right) \cdot \hat{e}_j$$

$$= \frac{\partial^2 \phi}{\partial x_i \partial x_j} \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\delta_{ij}} = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \quad \text{(scalar)}$$

$$= \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$$

Laplacian of a vector field:

$$\nabla^2 \vec{v} = \nabla \cdot \nabla \vec{v} = \frac{\partial}{\partial x_k}\left(\frac{\partial v_j}{\partial x_i}\, \hat{e}_j \otimes \hat{e}_i\right) \cdot \hat{e}_k$$

$$= \frac{\partial^2 v_j}{\partial x_i \partial x_k}\, \hat{e}_j \otimes \underbrace{\hat{e}_i \cdot \hat{e}_k}_{\delta_{ik}} = \frac{\partial^2 v_j}{\partial x_i \partial x_i}\, \hat{e}_j \quad \text{(vector)}$$

Laplacian of a Tensor:

$$\nabla^2 T = \nabla \cdot \nabla T = \frac{\partial}{\partial x_m}\left(\frac{\partial T_{jk}}{\partial x_i}\hat{e}_j \otimes \hat{e}_k \otimes \hat{e}_i\right) \cdot \hat{e}_m$$

$$= \frac{\partial^2 T_{jk}}{\partial x_i \partial x_m}\hat{e}_j \otimes \hat{e}_k \otimes \underbrace{\hat{e}_i \cdot \hat{e}_m}_{\delta_{im}} = \frac{\partial^2 T_{jk}}{\partial x_i \partial x_i}\hat{e}_j \otimes \hat{e}_k$$