Rate of heat flow

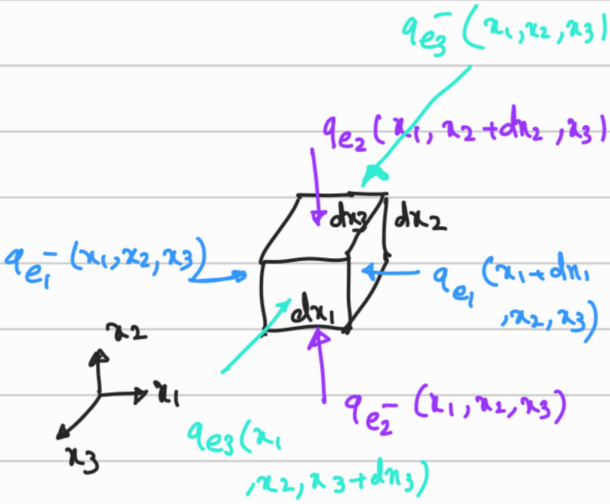
Energy Equation (The first law of thermodynamics)

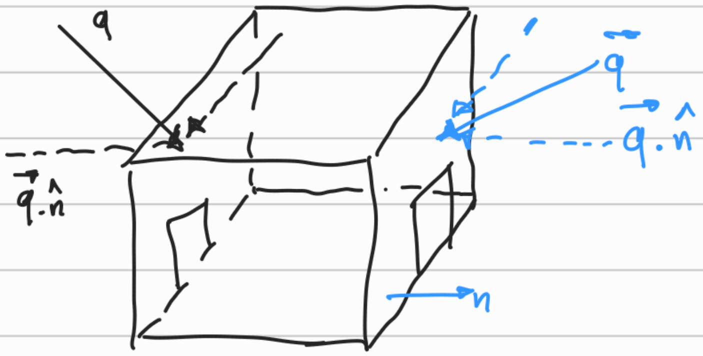
Entropy Inequality (The second law of thermodynamics)

Rate of Heat Flow:

$\vec{t}$ (force/area)

$\vec{q}$ (heat flow/area) by conduction





$$\left\{-(q\cdot e_1)_{\{x_1+dx_1,\, x_2,\, x_3\}} + (q\cdot e_1)_{\{x_1, x_2, x_3\}}\right\} dx_2\, dx_3$$

$$= \left\{-\frac{\partial(q\cdot e_1)}{\partial x_1} dx_1\right\} dx_2\, d_3 = \left\{-\frac{\partial q}{\partial x_1}\cdot e_1 - q\cdot \cancel{\frac{\partial e_1}{\partial x_1}}\right\} dx_1\, dx_2\, d_3$$

$$= \frac{-\partial q_1}{\partial x_1} dx_1\, dx_2\, dx_3 \quad (1)$$

$$\left\{-(q\cdot e_2)_{\{x_1,\,x_2+dx_2,\,x_3\}} + (q\cdot e_2)_{\{x_1,x_2,x_3\}}\right\} dx_1\, dx_3$$

$$= \left\{-\frac{\partial(q\cdot e_2)}{\partial x_2}\, dx_2\right\} dx_1\, d_3 = \left\{-\frac{\partial q}{\partial x_2}\cdot e_2 - q\cdot \cancel{\frac{\partial e_2}{\partial x_2}}\right\} dx_1 dx_2 d_3$$

$$= \frac{-\partial q_2}{\partial x_2}\, dx_1\, dx_2\, dx_3 \qquad (2)$$

$$\left\{-(q\cdot e_3)_{\{x_1,\,x_2,\,x_3+dx_3\}} + (q\cdot e_3)_{\{x_1,x_2,x_3\}}\right\} dx_1\, dx_2$$

$$= \left\{-\frac{\partial(q\cdot e_3)}{\partial x_3}\, dx_3\right\} dx_1\, dx_2 = \left\{-\frac{\partial q}{\partial x_3}\cdot e_3 - q\cdot \cancel{\frac{\partial e_3}{\partial x_3}}\right\} dx_1 dx_2 d_3$$

$$= \frac{-\partial q_3}{\partial x_3}\, dx_1\, dx_2\, dx_3 \qquad (3)$$

(1) + (2) + (3) : $Q_c = -\left(\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3}\right) dV = -\left(\frac{\partial q_i}{\partial x_i}\right) dV$

net heat flow by conduction $\quad = -(\nabla\cdot q)\, dV = -(\mathrm{div}\, q)\, dV$

Example 4.14.1

Fourier heat conduction law: $\vec{q} = -\kappa \nabla\theta$

$$Q_c = -(\mathrm{div}\, q)\, dV$$

$$\mathrm{div}\, q = \frac{\partial}{\partial x_1}(-\kappa\nabla\theta) + \frac{\partial}{\partial x_2}(-\kappa\nabla\theta) + \frac{\partial}{\partial x_3}(-\kappa\nabla\theta)$$

$$= \frac{\partial}{\partial x_1}\left(-\kappa\frac{\partial\theta}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(-\kappa\frac{\partial\theta}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(-\kappa\frac{\partial\theta}{\partial x_3}\right)$$

$$= -K\left(\frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2} + \frac{\partial^2\theta}{\partial x_3^2}\right) = -K\nabla^2\theta$$

$\begin{cases} \text{steady-state} & \nabla^2\theta = 0 \quad \text{Laplace equation} \\ \text{transient} & \end{cases}$

Energy Equation:

U: Internal energy

KE: Kinetic Energy

Q_c: net rate of heat inflow by conduction

P: rate of work done (Power) by body force and surface force)

Q_s: the rate of heat input

Energy equation: $\dfrac{D}{Dt}(U + KE) = P + Q_c + Q_s$

The rate of increase of internal and kinetic energy of a particle is equal to the net sum of the rate of work done by external forces (body force + surface force on the boundary) and rate of heat inflow from the boundary and the rate of heat genetrated inside the volume.

$$P = \frac{D}{Dt}(KE) + T_{ij}\frac{\partial v_i}{\partial x_j}dV \quad (4)$$

$$Q_c = -\frac{\partial q_i}{\partial x_i}dV \quad (5)$$

(4) and (5) in Energy Equation:

$$\frac{DU}{Dt} + \cancel{\frac{D(KE)}{Dt}} = \cancel{\frac{D(KE)}{Dt}} + T_{ij}\frac{\partial v_i}{\partial x_j}dV - \frac{\partial q_i}{\partial x_i}dV + Q_s \quad (6)$$

u: Internal energy per unit mass $\quad u = U/m$

conservation of mass

$$\frac{DU}{Dt} = \frac{D(u\,dm)}{Dt} = \frac{D(u\rho dV)}{Dt} = \frac{Du}{Dt}\rho dV + u\underbrace{\cancel{\frac{D(\rho dV)}{Dt}}}_{0}$$

(6)

$$\Rightarrow \rho \cancel{dV} \frac{Du}{Dt} = T_{ij}\frac{\partial v_i}{\partial x_j}\cancel{dV} - \frac{\partial q_i}{\partial x_i}\cancel{dV} + q_s \rho \cancel{dV}$$

$$\rho\frac{Du}{Dt} = T_{ij}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho q_s$$

Energy equation in differential form

$$\boxed{\rho\frac{Du}{Dt} = \mathrm{tr}(TD) - \mathrm{div}\, q + \rho q_s}$$

Entropy Inequality:

let $\eta(x,t)$: Entropy/mass $\qquad \eta(x,t)\rho dV$: Entropy

$$\underbrace{\frac{D(\eta\rho dV)}{Dt}}_{\text{rate of increase of entropy}} = \frac{D\eta}{Dt}\rho dV + \eta\underbrace{\cancel{\frac{D(\rho dV)}{Dt}}}_{0} = \rho dV\frac{D\eta}{Dt}$$

Entropy Inequality:

$$\rho\frac{D\eta}{Dt} \geqslant -\mathrm{div}\left(\frac{q}{\theta}\right) + \frac{\rho q_s}{\theta}$$

rate of increase of entropy in a particle

entropy inflow across the surface boundary

entropy generated inside

Example 4.16.1

$\vec{q} = -\kappa \nabla\theta$

temperature distribution:

$$\frac{\partial q_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(-\kappa\nabla\theta\right)$$

$$\underset{\substack{\text{Example} \\ 4.14.1}}{=} -\kappa\left(\frac{\partial^2\theta}{\partial x_1^2} + \cancelto{0}{\frac{\partial^2\theta}{\partial x_2^2}} + \cancelto{0}{\frac{\partial^2\theta}{\partial x_3^2}}\right) = -\kappa\left(\frac{\partial^2\theta}{\partial x_i^2}\right) = 0 \quad \text{steady state}$$

$$\frac{\partial^2\theta}{\partial x_1^2} = 0 \Rightarrow \frac{\partial\theta}{\partial x_1} = C_1 \Rightarrow \theta = C_1 x + C_2$$

$$\begin{cases} @\, x=0 : \theta = \theta_1 \Rightarrow \theta_1 = C_2 \\ @\, x=l : \theta = \theta_2 \rightarrow \theta_2 = C_1 l + \cancelto{\theta_1}{C_2} \Rightarrow C_1 = \dfrac{\theta_2-\theta_1}{l} \end{cases}$$

$$\theta = \left(\frac{\theta_2-\theta_1}{l}\right)x + \theta_1$$

$\kappa \geq 0$:

$$\rho \cancelto{0}{\frac{D\eta}{Dt}} \geq -\operatorname{div}\left(\frac{q}{\theta}\right) + \cancelto{0}{\frac{\rho q_s}{\theta}}$$

(assumption)

$$0 \geq -\operatorname{div}\left(\frac{q}{\theta}\right)$$

$$0 \geq -\operatorname{div}\left(\frac{-\kappa\nabla\theta}{\theta}\right) = \kappa\frac{\partial}{\partial x_i}\left(\frac{\nabla\theta}{\theta}\right) = \kappa\frac{\partial}{\partial x_i}\left(\frac{1}{\theta}\frac{\partial\theta}{\partial x_i}\right)$$

$$0 \geq \kappa\left\{\frac{\partial(1/\theta)}{\partial x_i}\frac{\partial\theta}{\partial x_i} + \frac{1}{\theta}\frac{\partial^2\theta}{\partial x_i^2}\right\}$$

$$0 \geq \kappa\left\{\frac{-1}{\theta^2}\frac{\partial\theta}{\partial x_1}\frac{\partial\theta}{\partial x_1} + \frac{1}{\theta}\cancelto{0}{\frac{\partial^2\theta}{\partial x_1}}\right\}$$

0 (steady-state)

$$0 \geqslant \frac{-\kappa}{\theta^2}\left(\frac{\partial \theta}{\partial x_1}\right)^2 \Rightarrow \kappa \geqslant 0$$

Entropy Inequality in Terms of Helmholtz Energy Function:

Background: $du = dQ + dW = \theta\, d\eta + T_{ij}\, d\epsilon_{ij} \quad \therefore u(\eta, \epsilon_{ij})$

Helmholtz energy function : $A = u - \theta\eta \Rightarrow u = A + \theta\eta$

$$\begin{cases} du = dA + d\theta\,\eta + \cancel{\theta\, d\eta} \\ du = \cancel{\theta\, d\eta} + T_{ij}\, d\epsilon_{ij} \end{cases}$$

$$dA + d\theta\,\eta = T_{ij}\, d\epsilon_{ij} \Rightarrow dA = T_{ij}\, d\epsilon_{ij} - d\theta\,\eta$$

$$\therefore A(\epsilon_{ij}, \theta)$$

$$\rho \frac{D\eta}{Dt} \geqslant -\mathrm{div}\left(\frac{q_i}{\theta}\right) + \frac{\rho q_s}{\theta}$$

To remove $\frac{D\eta}{Dt}$ we would like to replace "u" in energy equation with the Helmholtz energy function.

$$\rho \frac{Du}{Dt} = \mathrm{tr}(TD) - \mathrm{div}\, q + \rho q_s$$

$$\rho \frac{D(A+\theta\eta)}{Dt} = \rho \frac{DA}{Dt} + \rho \frac{D\theta}{Dt}\eta + \rho\theta \frac{D\eta}{Dt}$$

$$\rho \frac{DA}{Dt} + \rho \frac{D\theta}{Dt}\eta + \rho\theta \frac{D\eta}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho q_s$$

$$\rho\theta \frac{D\eta}{Dt} = -\left(\rho \frac{DA}{Dt} + \rho \frac{D\theta}{Dt}\eta\right) + T_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho q_s$$

$$\rho\theta \frac{D\eta}{Dt} \geq -\theta \operatorname{div}\left(\frac{q_i}{\theta}\right) + \cancel{\theta} \frac{\rho q_s}{\cancel{\theta}}$$

$$-\left(\rho \frac{DA}{Dt} + \rho \frac{D\theta}{Dt}\eta\right) + T_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \cancel{\rho q_s} \geq -\theta \operatorname{div}\left(\frac{q_i}{\theta}\right) + \cancel{\rho q_s} \quad (6)$$

$$\theta \operatorname{div}\left(\frac{q_i}{\theta}\right) = \theta \frac{\partial (q_i/\theta)}{\partial x_i} = \cancel{\theta} \frac{1}{\cancel{\theta}} \frac{\partial q_i}{\partial x_i} - \cancel{\theta} \frac{q_i}{\theta^{\cancel{2}}} \frac{\partial \theta}{\partial x_i}$$

$$= \frac{\partial q_i}{\partial x_i} - \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i} \quad (7)$$

(7) in (6):

$$-\left(\rho \frac{DA}{Dt} + \rho \frac{D\theta}{Dt}\eta\right) + T_{ij} \frac{\partial v_i}{\partial x_j} - \cancel{\frac{\partial q_i}{\partial x_i}} \geq -\cancel{\frac{\partial q_i}{\partial x_i}} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i}$$

$$\boxed{-\left(\rho \frac{DA}{Dt} + \rho \frac{D\theta}{Dt}\eta\right) + T_{ij} \frac{\partial v_i}{\partial x_j} - \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i} \geq 0}$$

Example 4.17.1

linear thermo-elasticity $\quad A = A(E_{ij}, \theta)$

$$\frac{DA}{Dt} = \frac{\partial A}{\partial E_{ij}} \frac{DE_{ij}}{Dt} + \frac{\partial A}{\partial \theta} \frac{D\theta}{Dt}$$

$$\frac{DE_{ij}}{Dt} = \frac{D}{Dt}\left(\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)\right) = \frac{1}{2}\left\{\frac{D}{Dt}\left(\frac{\partial u_i}{\partial X_j}\right) + \frac{D}{Dt}\left(\frac{\partial u_j}{\partial X_i}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{\partial}{\partial X_j}\left(\overset{v_i}{\cancel{\frac{Du_i}{Dt}}}\right) + \frac{\partial}{\partial X_i}\left(\overset{v_j}{\cancel{\frac{Du_j}{Dt}}}\right)\right\}$$

$$= \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right) \xrightarrow[\text{infinitesimal deformation}]{\text{linear elasticity}}$$

$$\simeq \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = D_{ij}$$

$$-\left(\rho \frac{\partial A}{\partial E_{ij}} D_{ij} + \rho \frac{\partial A}{\partial \theta}\frac{D\theta}{Dt} + \rho \frac{D\theta}{Dt}\eta\right) + T_{ij}D_{ij} - \frac{q_i}{\theta}\frac{\partial \theta}{\partial x_i} \geq 0$$

$$\left(-\rho \frac{\partial A}{\partial E_{ij}} + T_{ij}\right) D_{ij} - \left(\rho \frac{\partial A}{\partial \theta} + \rho\eta\right)\frac{D\theta}{Dt} - \frac{q_i}{\theta}\frac{\partial \theta}{\partial x_i} \geq 0$$

in case $\frac{D\theta}{Dt} = 0$ and $\frac{\partial \theta}{\partial x_i} = 0$: $\left(-\rho \frac{\partial A}{\partial E_{ij}} + T_{ij}\right) D_{ij} = 0$

$$\rightarrow T_{ij} = \rho \frac{\partial A}{\partial E_{ij}}$$

in case $D_{ij} = 0$ and $\frac{\partial \theta}{\partial x_i} = 0$: $\left(\rho \frac{\partial A}{\partial \theta} + \rho\eta\right)\frac{D\theta}{Dt} = 0$

$$\eta = -\frac{\partial A}{\partial \theta}$$

in case $D_{ij} = 0$ and $\frac{D\theta}{Dt} = 0$: $-\frac{q_i}{\theta}\frac{\partial \theta}{\partial x_i} \geq 0$

Fourier heat conduction law: $q = -\kappa \nabla \theta$

$$-\frac{(-K\nabla\theta)}{\theta}\frac{\partial\theta}{\partial x_i} \Longrightarrow \frac{K\nabla\theta}{\theta}\frac{\partial\theta}{\partial x_i} \Longrightarrow \frac{K}{\theta}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_i} \geq 0$$