Orthogonal Tensor

Transformation Matrix between two Rectangular Cartesian Coordinates

Transformation law for Cartesian Components of a Vector/Tensor

A topic from earlier sessions:

$$T_{ij} = e_i \cdot Te_j$$

$$T_{12} = e_1 \cdot Te_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} T_{11} & T_{12} & T_{13} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = T_{12}$$

Orthogonal Tensor (Q):

An orthogonal tensor is a tensor which transform a vector while preserves its length and angle.

$$\left.\begin{array}{l} |Qa| = |a| \\ |Qb| = |b| \end{array}\right\} \text{preserves length}$$

$$\cos(a,b) = \cos(Qa, Qb)$$
preserves angle

$$Qa \cdot Qb = a \cdot b$$

$$|Qa|\,|Qb| \cos(Qa, Qb) = |a|\,|b| \cos(a,b)$$

$$Qa \cdot Qb = \underbrace{Qb}_{bQ^T} \cdot Qa = \underbrace{bQ^T Qa = b \cdot Ia}_{} \Rightarrow Q^T Q = I$$

tensorial form: $$QQ^T = Q^TQ = I$$

$$Q^{-1} = Q^T$$

Matrix form: $$[Q][Q]^T = [Q]^T[Q] = [I]$$

Indicial notation: $$Q_{im}Q_{jm} = Q_{mi}Q_{mj} = \delta_{ij}$$

Determinant of an orthogonal tensor:

$$[Q][Q]^T = I$$

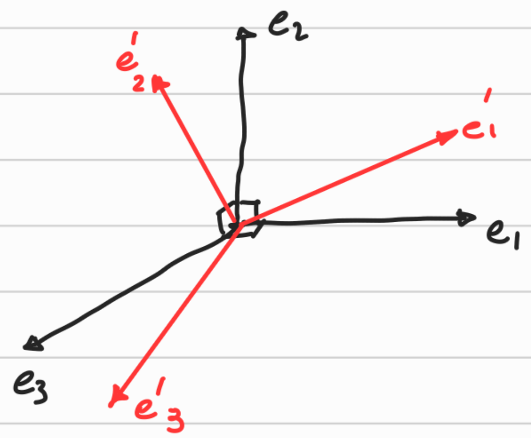
$$|[Q][Q]^T| = |I|$$

$$|Q||Q^T| = 1$$

$$|Q|^2 = 1 \Rightarrow |Q| = \pm 1$$

$$\begin{cases} \text{if } |Q| = 1, \text{ we call } Q \text{ a rotation tensor} \\ \text{if } |Q| = -1, \text{ '' '' '' '' reflection tensor} \end{cases}$$

Transformation Matrix between two Rectangular Cartesian Coordinates:



$$e_i' = Qe_i = Q_{mi}e_m \quad ①$$

$$Q_{11} = e_1 \cdot \underbrace{Qe_1}_{Q_{m1}e_m} \overset{①}{=} e_1 \cdot Q_{m1}e_m = e_1 \cdot (Q_{11}e_1 + Q_{21}e_2 + Q_{31}e_3)$$

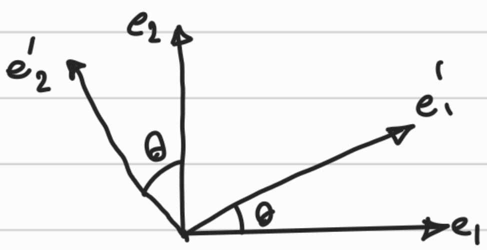
$$= Q_{11} \underbrace{\cancel{e_1 \cdot e_1}}_{\delta_{11}}^{1} + Q_{21} \underbrace{\cancel{e_1 \cdot e_2}}_{\delta_{12}}^{0} + Q_{31} \underbrace{\cancel{e_1 \cdot e_3}}_{\delta_{13}}^{0} = Q_{11}$$

$\therefore Q_{11} = e_1 \cdot \underbrace{Qe_1}_{e_1'} = e_1 \cdot e_1'$ $\equiv$ which means the cosine angle between e_1 and e_1'

$Q_{13} = e_1 \cdot \underbrace{Qe_3}_{e_3'} = e_1 \cdot e_3'$ $\equiv$ " " " " " " e_1 and e_3'

$$\therefore Q = \begin{array}{c c} & \begin{array}{ccc} e_1' & e_2' & e_3' \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \end{array}$$

Example 2.16.1



$$Q = \begin{array}{c c} & \begin{array}{ccc} e_1' & e_2' & e_3' \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{bmatrix} \cos\theta & \cos(\frac{\pi}{2}+\theta) & \cos(\frac{\pi}{2}) \\ \cos(\frac{\pi}{2}-\theta) & \cos\theta & \cos(\frac{\pi}{2}) \\ \cos(\frac{\pi}{2}) & \cos\frac{\pi}{2} & \cos(0) \end{bmatrix} \end{array} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation Law for Cartesian Components of a Vector:

$a_i = a \cdot e_i$

$a_i' = a \cdot e_i'$, $e_i' = Q_{mi}e_m$ ②

$$a'_i \overset{②}{=} a \cdot Q_{mi} e_m = Q_{mi} \underbrace{(a \cdot e_m)}_{a_m} = Q_{mi} a_m$$

Indicial notation : $a'_i = Q_{mi} a_m$

Matrix form : $[a]' = [Q]^T [a]$

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$a_i = Q_{im} a'$, $[a] = [Q][a']$

Transformation Law for Cartesian Components of a Tensor:

$$T_{ij} = e_i \cdot T e_j$$

$$T'_{ij} = e'_i \cdot T e'_j \quad , \quad e'_i = Q_{mi} e_m \ ③ \quad , \quad e'_j = Q_{nj} e_n \ ④$$

$$\overset{③}{=} Q_{mi} e_m \cdot T e'_j \overset{④}{=} Q_{mi} e_m \cdot T Q_{nj} e_n$$

$$= Q_{mi} Q_{nj} \underbrace{e_m \cdot T e_n}_{T_{mn}} = Q_{mi} Q_{nj} T_{mn} = Q_{mi} T_{mn} Q_{nj}$$

$$\therefore T'_{ij} = Q_{mi} T_{mn} Q_{nj} = Q_{mi} Q_{nj} T_{mn}$$ indicial notation

$$[T]' = [Q]^T [T] [Q]$$ tensor form

$[T] = [Q][T][Q]^T$, $T_{ij} = Q_{im} T_{mn} Q_{jn}$

$= Q_{im} Q_{nj} T_{mn}$

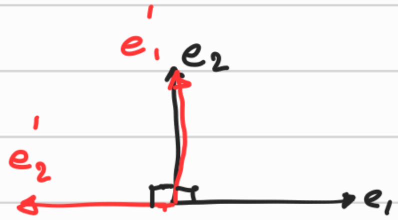
You should differentiate between (1) $[T]' = [Q]^T[T][Q]$

(2) $T' = Q^T T Q$

(1)	(2)
matrix form	tensorial form
to calculate T components in the primed coordinate system	the multiplication of three tensor

Example 2.18.1 and 2.18.3

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$Q = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} e_1' & e_2' & e_3' \end{array} \\ \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & \cos(\pi) & \cos\left(\frac{\pi}{2}\right) \\ \cos(0) & \cos\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \\ \cos\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & \cos(0) \end{bmatrix} \end{array} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T' = [Q]^T[T][Q] = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T'_{12} = ?$ $\qquad$ Eq (2.18.9) : $T'_{ij} = [e'_i]^T [T] [e'_j]$

$$T'_{12} = e'_1 \cdot T e'_2 \overset{6,7}{=} e_2 \cdot T(-e_1) = -e_2 \cdot T e_1 = -T_{21} = -1$$

$$e'_1 = Q e_1 \overset{3}{=} Q_{j1} e_j = Q_{11} e_1 + Q_{21} e_2 + Q_{31} e_3$$

$$\overset{5}{=} 0 + e_2 + 0 = e_2 \quad 6$$

$$Q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 5$$

$$e'_2 = Q e_2 = Q_{j2} e_j = Q_{12} e_1 + Q_{22} e_2 + Q_{32} e_3$$

$$\overset{5}{=} -e_1 + 0 + 0 = -e_1 \quad 7$$