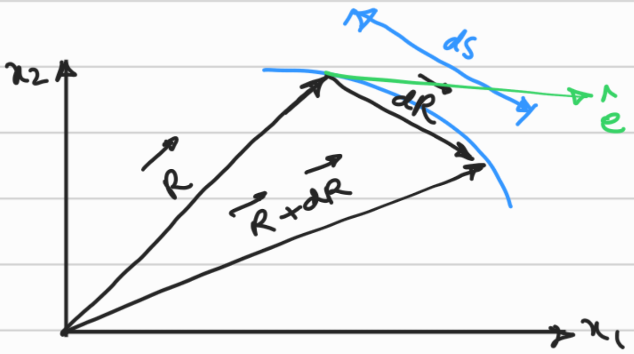
Orthogonal Curvilinear Coordinate System:



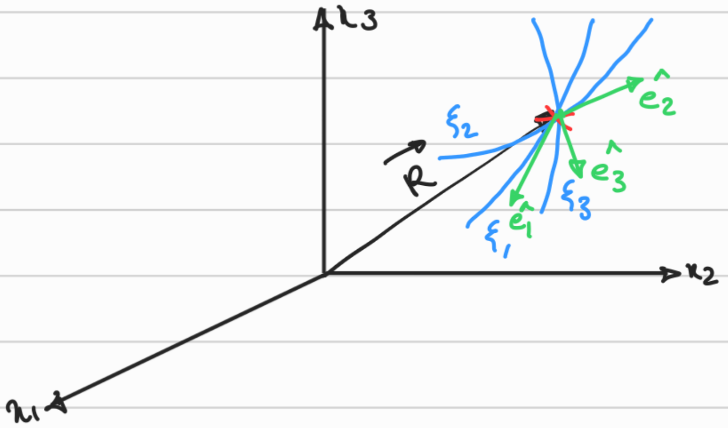
$$|\vec{dR}| = ds$$

If ds goes to zero

$$\hat{e} = \frac{d\vec{R}}{ds}$$

In curvilinear coordinate system, ds is both length and direction-dependent; therefore, $ds = h\, d\xi$ ← direction parameter

length parameter ↗



$$\vec{R} = \vec{R}(\xi_1, \xi_2, \xi_3)$$

$$x_1 = x_1(\xi_1, \xi_2, \xi_3)$$
$$x_2 = x_2(\xi_1, \xi_2, \xi_3)$$
$$x_3 = x_3(\xi_1, \xi_2, \xi_3)$$

$$\hat{e}_\alpha = \frac{\partial \vec{R}}{\partial s} \xrightarrow{ds = h\,d\xi} \boxed{\hat{e}_\alpha = \frac{\partial \vec{R}}{h_\alpha \partial \xi_\alpha} \qquad \alpha = 1,2,3 \ (\text{no sum on } \alpha)}$$

$$\text{OR} \quad \hat{e}_\alpha = \sum_{\alpha=1}^{3} \frac{\partial R}{h_\alpha \partial \xi_\alpha}$$

$$\boxed{h_\alpha = \left|\frac{\partial \vec{R}}{\partial \xi_\alpha}\right| = \sqrt{\frac{\partial R}{\partial \xi_\alpha} \cdot \frac{\partial R}{\partial \xi_\alpha}}}$$

$$\boxed{\vec{dR} = \sum_{\alpha=1}^{3} \frac{\partial \vec{R}}{\partial \xi_\alpha} d\xi_\alpha = \sum_{\alpha=1}^{3} h_\alpha \hat{e}_\alpha \, d\xi_\alpha}$$

$$(ds)^2 = dR \cdot dR = \sum_{\alpha=1}^{3} h_\alpha \hat{e}_\alpha \, d\xi_\alpha \cdot \sum_{\beta=1}^{3} h_\beta \hat{e}_\beta \, d\xi_\beta$$

$$= \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} h_\alpha d\xi_\alpha \, h_\beta d\xi_\beta \, \underbrace{\hat{e}_\alpha \cdot \hat{e}_\beta}_{\delta_{\alpha\beta}} = \boxed{\sum_{\alpha=1}^{3} h_\alpha^2 (d\xi_\alpha)^2 = ds}$$

The remaining question is $\dfrac{\partial \hat{e}_\alpha}{\partial \xi_\beta} = ?$

Let us define $\dfrac{\partial \hat{e}_\alpha}{\partial \xi_\beta} = \displaystyle\sum_{\gamma=1}^{3} \begin{pmatrix} \gamma \\ \alpha \quad \beta \end{pmatrix} \hat{e}_\gamma$ Christoffel Symbol

$$= \begin{pmatrix} 1 \\ \alpha \quad \beta \end{pmatrix} \hat{e}_1 + \begin{pmatrix} 2 \\ \alpha \quad \beta \end{pmatrix} \hat{e}_2 + \begin{pmatrix} 3 \\ \alpha \quad \beta \end{pmatrix} \hat{e}_3$$

Now, we would like to find the Christoffel Symbol.

$\dfrac{\partial \hat{e}_\alpha}{\partial \xi_\beta} = \begin{pmatrix} \gamma \\ \alpha \quad \beta \end{pmatrix} \hat{e}_\gamma$ (no sum on γ)

let us replace γ with σ: $\dfrac{\partial \hat{e}_\alpha}{\partial \xi_\beta} = \begin{pmatrix} \sigma \\ \alpha \quad \beta \end{pmatrix} \hat{e}_\sigma$ (no sum on σ)

$$\xrightarrow{e_\gamma \cdot} \quad \hat{e}_\gamma \cdot \frac{\partial \hat{e}_\alpha}{\partial \xi_\beta} = \hat{e}_\gamma \cdot \begin{pmatrix} \sigma \\ \alpha \quad \beta \end{pmatrix} \hat{e}_\sigma$$

$$= \begin{pmatrix} \sigma \\ \alpha \quad \beta \end{pmatrix} \underbrace{\hat{e}_\gamma \cdot \hat{e}_\sigma}_{\delta_{\gamma\sigma}}$$

$$\boxed{\hat{e}_\gamma \cdot \frac{\partial \hat{e}_\alpha}{\partial \xi_\beta} = \begin{pmatrix} \gamma \\ \alpha \quad \beta \end{pmatrix}} \quad ▨$$

Substituting ▲ in ▨ gives

$$\begin{pmatrix} & \gamma & \\ \alpha & & \beta \end{pmatrix} = \frac{1}{h_\gamma}\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \frac{\partial}{\partial \xi_\beta}\left(\frac{1}{h_\alpha}\frac{\partial \vec{R}}{\partial \xi_\alpha}\right)$$

$$= \frac{1}{h_\gamma}\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \left[\frac{\partial}{\partial \xi_\beta}\left(\frac{1}{h_\alpha}\right)\frac{\partial \vec{R}}{\partial \xi_\alpha} + \frac{1}{h_\alpha}\frac{\partial^2 \vec{R}}{\partial \xi_\beta \partial \xi_\alpha}\right]$$

$$= \frac{1}{h_\gamma}\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \left[\frac{-1}{h_\alpha^2}\left(\frac{\partial h_\alpha}{\partial \xi_\beta}\right)\frac{\partial \vec{R}}{\partial \xi_\alpha} + \frac{1}{h_\alpha}\frac{\partial^2 \vec{R}}{\partial \xi_\beta \partial \xi_\alpha}\right]$$

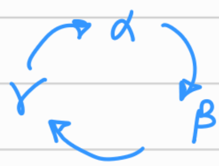
$$= \frac{-1}{h_\gamma h_\alpha^2}\underbrace{\left(\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \frac{\partial \vec{R}}{\partial \xi_\alpha}\right)}_{\substack{(*) \\ h_\gamma \hat{e}_\gamma \cdot h_\alpha \hat{e}_\alpha \\ h_\gamma h_\alpha \delta_{\gamma\alpha}}}\left(\frac{\partial h_\alpha}{\partial \xi_\beta}\right) + \frac{1}{h_\gamma h_\alpha}\underbrace{\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \frac{\partial^2 \vec{R}}{\partial \xi_\beta \partial \xi_\alpha}}$$

(*) $$\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \frac{\partial \vec{R}}{\partial \xi_\alpha} = h_\gamma h_\alpha \delta_{\gamma\alpha}$$

$$\xrightarrow{\frac{\partial}{\partial \xi_\beta}} \quad \frac{\partial}{\partial \xi_\beta}\left(\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \frac{\partial \vec{R}}{\partial \xi_\alpha}\right) = \frac{\partial}{\partial \xi_\beta}\left(h_\gamma h_\alpha \delta_{\gamma\alpha}\right)$$

$$\frac{\partial^2 \vec{R}}{\partial \xi_\beta \partial \xi_\gamma} \cdot \frac{\partial \vec{R}}{\partial \xi_\alpha} + \frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \frac{\partial^2 \vec{R}}{\partial \xi_\beta \partial \xi_\alpha} = \frac{\partial (h_\gamma h_\alpha)}{\partial \xi_\beta}\delta_{\gamma\alpha} + h_\gamma h_\alpha \underbrace{\cancel{\frac{\partial (\delta_{\gamma\alpha})}{\partial \xi_\beta}}}_{0} \qquad (1)$$

Since the order of differentiation doesn't affect our final solution, we cycle over α, β and γ



$$\frac{\partial^2 \vec{R}}{\partial \xi_\gamma \partial \xi_\alpha} \cdot \frac{\partial \vec{R}}{\partial \xi_\beta} + \frac{\partial \vec{R}}{\partial \xi_\alpha} \cdot \frac{\partial^2 \vec{R}}{\partial \xi_\gamma \partial \xi_\beta} = \frac{\partial (h_\alpha h_\beta)}{\partial \xi_\gamma} \delta_{\alpha\beta} \qquad (2)$$

$$\frac{\partial^2 \vec{R}}{\partial \xi_\alpha \partial \xi_\beta} \cdot \frac{\partial \vec{R}}{\partial \xi_\gamma} + \frac{\partial \vec{R}}{\partial \xi_\beta} \cdot \frac{\partial^2 \vec{R}}{\partial \xi_\alpha \partial \xi_\gamma} = \frac{\partial (h_\beta h_\gamma)}{\partial \xi_\alpha} \delta_{\beta\gamma} \qquad (3)$$

(1) + (3) − (2)

$$\frac{\partial \vec{R}}{\partial \xi_\gamma} \cdot \frac{\partial^2 \vec{R}}{\partial \xi_\alpha \partial \xi_\beta} = \frac{1}{2} \left\{ \frac{\partial (h_\gamma h_\alpha)}{\partial \xi_\beta} \delta_{\gamma\alpha} + \frac{\partial (h_\beta h_\gamma)}{\partial \xi_\alpha} \delta_{\beta\gamma} - \frac{\partial (h_\alpha h_\beta)}{\partial \xi_\gamma} \delta_{\alpha\beta} \right\}$$

$$\boxed{\begin{Bmatrix} & \gamma & \\ \alpha & & \beta \end{Bmatrix} = \frac{-1}{h_\alpha} \frac{\partial h_\alpha}{\partial \xi_\beta} \delta_{\gamma\alpha} + \frac{1}{2 h_\gamma h_\alpha} \left\{ \frac{\partial (h_\gamma h_\alpha)}{\partial \xi_\beta} \delta_{\gamma\alpha} + \frac{\partial (h_\beta h_\gamma)}{\partial \xi_\alpha} \delta_{\beta\gamma} - \frac{\partial (h_\alpha h_\beta)}{\partial \xi_\gamma} \delta_{\alpha\beta} \right\}}$$

If $\alpha \neq \beta \neq \gamma \Rightarrow \begin{pmatrix} & \gamma & \\ \alpha & & \beta \end{pmatrix} = 0$

$$\begin{pmatrix} & \alpha & \\ \alpha & & \beta \end{pmatrix} = 0 \qquad \begin{pmatrix} & \alpha & \\ \alpha & & \alpha \end{pmatrix} = 0$$

$$\begin{pmatrix} & \gamma & \\ \alpha & & \alpha \end{pmatrix} = \frac{-1}{h_\gamma} \frac{\partial h_\alpha}{\partial \xi_\gamma}, \quad \begin{pmatrix} & \gamma & \\ \alpha & & \gamma \end{pmatrix} = \frac{1}{h_\alpha} \frac{\partial h_\gamma}{\partial \xi_\alpha}$$

(no sum on repeated indices)

$$\frac{\partial e_\alpha}{\partial e_\beta} = \begin{pmatrix} & \gamma & \\ \alpha & & \beta \end{pmatrix} \hat{e}_\gamma$$

$$\frac{\partial e_1}{\partial \xi_1} = \begin{pmatrix} & \gamma & \\ 1 & & 1 \end{pmatrix} \hat{e}_\gamma = \underbrace{\cancel{\begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix}}}_{0} \hat{e}_1 + \begin{pmatrix} & 2 & \\ 1 & & 1 \end{pmatrix} \hat{e}_2 + \begin{pmatrix} & 3 & \\ 1 & & 1 \end{pmatrix} \hat{e}_3$$

$$= -\frac{1}{h_2} \frac{\partial h_1}{\partial \xi_2} \hat{e}_2 - \frac{1}{h_3} \frac{\partial h_1}{\partial \xi_3} \hat{e}_3$$

$$\frac{\partial \hat{e}_1}{\partial \xi_2} = \begin{pmatrix} & \gamma & \\ 1 & & 2 \end{pmatrix} \hat{e}_\gamma = \underbrace{\cancel{\begin{pmatrix} & 1 & \\ 1 & & 2 \end{pmatrix}}}_{0} \hat{e}_1 + \begin{pmatrix} & 2 & \\ 1 & & 2 \end{pmatrix} \hat{e}_2 + \underbrace{\cancel{\begin{pmatrix} & 3 & \\ 1 & & 2 \end{pmatrix}}}_{0} \hat{e}_3$$

$$= \frac{1}{h_1} \frac{\partial h_2}{\partial \xi_1} \hat{e}_2$$

$$\frac{\partial \hat{e}_1}{\partial \xi_3} = \begin{pmatrix} & \gamma & \\ 1 & & 3 \end{pmatrix} \hat{e}_\gamma = \underbrace{\cancel{\begin{pmatrix} & 1 & \\ 1 & & 3 \end{pmatrix}}}_{0} \hat{e}_1 + \underbrace{\cancel{\begin{pmatrix} & 2 & \\ 1 & & 3 \end{pmatrix}}}_{0} \hat{e}_2 + \begin{pmatrix} & 3 & \\ 1 & & 3 \end{pmatrix} \hat{e}_3$$

$$= \frac{1}{h_1} \frac{\partial h_3}{\partial \xi_1} \hat{e}_3$$

$$\frac{\partial e_2}{\partial \xi_2} = \begin{pmatrix} & \gamma & \\ 2 & & 2 \end{pmatrix} \hat{e}_\gamma = \begin{pmatrix} & 1 & \\ 2 & & 2 \end{pmatrix} \hat{e}_1 + \underbrace{\cancel{\begin{pmatrix} & 2 & \\ 2 & & 2 \end{pmatrix}}}_{0} \hat{e}_2 + \begin{pmatrix} & 3 & \\ 2 & & 2 \end{pmatrix} \hat{e}_3$$

$$= -\frac{1}{h_1} \frac{\partial h_2}{\partial \xi_1} \hat{e}_1 - \frac{1}{h_3} \frac{\partial h_2}{\partial \xi_3} \hat{e}_3$$

$$\frac{\partial \hat{e}_2}{\partial \xi_3} = \begin{pmatrix} & \gamma & \\ 2 & & 3 \end{pmatrix} \hat{e}_\gamma = \underbrace{\cancel{\begin{pmatrix} & 1 & \\ 2 & & 3 \end{pmatrix}}}_{0} \hat{e}_1 + \underbrace{\cancel{\begin{pmatrix} & 2 & \\ 2 & & 3 \end{pmatrix}}}_{0} \hat{e}_2 + \begin{pmatrix} & 3 & \\ 2 & & 3 \end{pmatrix} \hat{e}_3$$

$$= \frac{1}{h_2} \frac{\partial h_3}{\partial \xi_2} \hat{e}_3$$

$$\frac{\partial \hat{e}_2}{\partial \xi_1} = \binom{\gamma}{2\;\;1} \hat{e}_\gamma = \binom{1}{2\;\;1} \hat{e}_1 + \underbrace{\cancel{\binom{2}{2\;\;1}}}_{0} \hat{e}_2 + \underbrace{\cancel{\binom{3}{2\;\;1}}}_{0} \hat{e}_3$$

$$= \frac{1}{h_2} \frac{\partial h_1}{\partial \xi_2} \hat{e}_1$$

$$\frac{\partial \hat{e}_3}{\partial \xi_3} = \binom{\gamma}{3\;\;3} \hat{e}_\gamma = \binom{1}{3\;\;3} \hat{e}_1 + \binom{2}{3\;\;3} \hat{e}_2 + \underbrace{\cancel{\binom{3}{3\;\;3}}}_{0} \hat{e}_3$$

$$= -\frac{1}{h_1} \frac{\partial h_3}{\partial \xi_1} \hat{e}_1 - \frac{1}{h_2} \frac{\partial h_3}{\partial \xi_2} \hat{e}_2$$

$$\frac{\partial \hat{e}_3}{\partial \xi_1} = \binom{\gamma}{3\;\;1} \hat{e}_\gamma = \binom{1}{3\;\;1} \hat{e}_1 + \underbrace{\cancel{\binom{2}{3\;\;1}}}_{0} \hat{e}_2 + \underbrace{\cancel{\binom{3}{3\;\;1}}}_{0} \hat{e}_3$$

$$= \frac{1}{h_3} \frac{\partial h_1}{\partial \xi_3} \hat{e}_1$$

$$\frac{\partial \hat{e}_3}{\partial \xi_2} = \binom{\gamma}{3\;\;2} \hat{e}_\gamma = \underbrace{\cancel{\binom{1}{3\;\;2}}}_{0} \hat{e}_1 + \binom{2}{3\;\;2} \hat{e}_2 + \underbrace{\cancel{\binom{3}{3\;\;2}}}_{0} \hat{e}_3$$

$$= \frac{1}{h_3} \frac{\partial h_2}{\partial \xi_3} \hat{e}_2$$

Second Derivative:

$$\frac{\partial e_\alpha}{\partial \xi_\beta} = \binom{\gamma}{\alpha\;\;\beta} \hat{e}_\gamma \xrightarrow{\frac{\partial}{\partial \xi_\rho}} \frac{\partial^2 e_\alpha}{\partial \xi_\rho \partial \xi_\beta} = \frac{\partial}{\partial \xi_\rho} \left[\binom{\gamma}{\alpha\;\;\beta} \hat{e}_\gamma \right]$$

$$\frac{\partial^2 e_\alpha}{\partial \xi_\rho \partial \xi_\beta} = \frac{\partial}{\partial \xi_\rho} \binom{\gamma}{\alpha\;\;\beta} \hat{e}_\gamma + \binom{\gamma}{\alpha\;\;\beta} \frac{\partial \hat{e}_\gamma}{\partial \xi_\rho}$$

$$\frac{\partial \hat{e}_\alpha}{\partial \xi_\beta} = \begin{pmatrix} \gamma \\ \alpha \quad \beta \end{pmatrix} \hat{e}_\gamma$$

$$= \frac{\partial}{\partial \xi_\rho} \begin{pmatrix} \cancel{\gamma}\ \sigma \\ \alpha \quad \beta \end{pmatrix} \hat{e}_{\cancel{\gamma}\,\sigma} + \begin{pmatrix} \gamma \\ \alpha \quad \beta \end{pmatrix} \begin{pmatrix} \cancel{\gamma}\ \sigma \\ \cancel{\alpha}\,\gamma \quad \cancel{\beta}\,\rho \end{pmatrix} \hat{e}_{\cancel{\gamma}\,\sigma}$$

$$= \left[\frac{\partial}{\partial \xi_\rho} \begin{pmatrix} \sigma \\ \alpha \quad \beta \end{pmatrix} + \begin{pmatrix} \gamma \\ \alpha \quad \beta \end{pmatrix} \begin{pmatrix} \sigma \\ \gamma \quad \rho \end{pmatrix} \right] \hat{e}_\sigma \qquad (4)$$

Swapping ρ and β doesn't affect our solution as differentiation is independent of the order of variables.

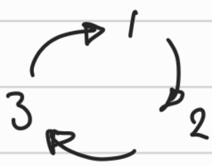
$$\frac{\partial^2 e_\alpha}{\partial \xi_\beta \partial \xi_\rho} = \left[\frac{\partial}{\partial \xi_\beta} \begin{pmatrix} \sigma \\ \alpha \quad \rho \end{pmatrix} + \begin{pmatrix} \gamma \\ \alpha \quad \rho \end{pmatrix} \begin{pmatrix} \sigma \\ \gamma \quad \beta \end{pmatrix} \right] \hat{e}_\sigma \qquad (5)$$

$(5) - (4) = 0$:

$$\begin{pmatrix} \gamma \\ \alpha \quad \rho \end{pmatrix} \begin{pmatrix} \sigma \\ \gamma \quad \rho \end{pmatrix} - \begin{pmatrix} \gamma \\ \alpha \quad \beta \end{pmatrix} \begin{pmatrix} \sigma \\ \gamma \quad \rho \end{pmatrix} + \frac{\partial}{\partial \xi_\beta} \begin{pmatrix} \sigma \\ \alpha \quad \rho \end{pmatrix} - \frac{\partial}{\partial \xi_\rho} \begin{pmatrix} \sigma \\ \alpha \quad \beta \end{pmatrix} = 0$$

$$\begin{cases} \dfrac{\partial}{\partial \xi_3} \left(\dfrac{1}{h_3} \dfrac{\partial h_1}{\partial \xi_3} \right) + \dfrac{\partial}{\partial \xi_1} \left(\dfrac{1}{h_1} \dfrac{\partial h_3}{\partial \xi_1} \right) + \dfrac{1}{h_2^2} \dfrac{\partial h_3}{\partial \xi_2} \dfrac{\partial h_1}{\partial \xi_2} = 0 \\ \dfrac{\partial^2 h_1}{\partial \xi_2 \partial \xi_3} = \dfrac{1}{h_2} \dfrac{\partial h_1}{\partial \xi_2} \dfrac{\partial h_2}{\partial \xi_3} + \dfrac{1}{h_3} \dfrac{\partial h_1}{\partial \xi_3} \dfrac{\partial h_3}{\partial \xi_3} \end{cases}$$

You can find four other equation by replacing subscripts 1, 2, 3



Cylindrical Coordinate System:

$$\vec{R} = (r\cos\theta)\,\hat{e}_1 + (r\sin\theta)\,\hat{e}_2 + z\,\hat{e}_3$$

$$h_r = \left|\frac{\partial R}{\partial \xi_r}\right| = \left|\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$h_\theta = \left|\frac{\partial R}{\partial \xi_\theta}\right| = \left|-r\sin\theta\,\hat{e}_1 + r\cos\theta\,\hat{e}_2\right| = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r$$

$$h_z = \left|\frac{\partial R}{\partial \xi_z}\right| = \left|e_3\right| = 1$$

$$\begin{pmatrix} & r & \\ \theta & & \theta \end{pmatrix} = \frac{-1}{h_r}\,\frac{\partial h_\theta}{\partial \xi_r} = (-1)(1) = -1$$

$$\begin{pmatrix} & r & \\ z & & z \end{pmatrix} = \frac{-1}{h_r}\,\cancelto{0}{\frac{\partial h_z}{\partial \xi_r}} = 0$$

$$\begin{pmatrix} & \theta & \\ r & & \theta \end{pmatrix} = \frac{1}{h_r}\,\frac{\partial h_\theta}{\partial \xi_r} = (1)(1) = 1$$

Spherical Coordinate System:

$$\vec{R} = (r\sin\theta\cos\varphi)\,\hat{e}_1 + (r\sin\theta\sin\varphi)\,\hat{e}_2 + (r\cos\theta)\,\hat{e}_3$$

$$h_r = \left|\frac{\partial R}{\partial \xi_r}\right| = \left|\sin\theta\cos\varphi\,\hat{e}_1 + \sin\theta\sin\varphi\,\hat{e}_2 + \cos\theta\,\hat{e}_3\right|$$

$$= \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta\,\underbrace{\cancel{(\cos^2\varphi + \sin^2\varphi)}}_{1} + \cos^2\theta} = 1$$

$$h_\theta = \left|\frac{\partial R}{\partial \xi_\theta}\right| = \left|r\cos\varphi\cos\theta\,\hat{e}_1 + r\sin\varphi\cos\theta\,\hat{e}_2 - r\sin\theta\,\hat{e}_3\right|$$

$$= \sqrt{r^2\cos^2\varphi\cos^2\theta + r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\theta}$$

$$= \sqrt{r^2\cos^2\theta\,(\underbrace{\cancel{\cos^2\varphi + \sin^2\varphi}}_{1}) + r^2\sin^2\theta} = r$$

$$h_\varphi = \left|\frac{\partial R}{\partial \xi_\varphi}\right| = \left| -r\sin\theta\sin\varphi\,\hat{e}_1 + r\sin\theta\cos\varphi\,\hat{e}_2 \right|$$

$$= \sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi}$$

$$= \sqrt{r^2\sin^2\theta\,(\underbrace{\cancel{\sin^2\varphi + \cos^2\varphi}}_{1})} = r\sin\theta$$

Gradient:

$$\nabla \equiv \frac{\partial()}{\partial x_i}\hat{e}_i = \frac{1}{h_\alpha}\frac{\partial()}{\partial \xi_\alpha}\hat{e}_\alpha$$

$$= \frac{1}{h_1}\frac{\partial()}{\partial \xi_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial()}{\partial \xi_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial()}{\partial \xi_3}\hat{e}_3$$

Divergence:

$$\text{div} \equiv \nabla\cdot e_\alpha = \frac{1}{h_\beta}\frac{\partial(e_\alpha)}{\partial \xi_\beta}\hat{e}_\beta = \frac{1}{h_\beta}\begin{pmatrix} & \gamma & \\ \alpha & & \beta \end{pmatrix}\underbrace{\hat{e}_\gamma\cdot\hat{e}_\beta}_{\delta_{\gamma\beta}}$$

$$= \frac{1}{h_\gamma}\begin{pmatrix} & \gamma & \\ \alpha & & \gamma \end{pmatrix} = \frac{1}{h_\gamma h_\alpha}\frac{\partial h_\gamma}{\partial \xi_\alpha}$$

$$= \frac{1}{h_1 h_\alpha}\frac{\partial h_1}{\partial \xi_\alpha} + \frac{1}{h_2 h_\alpha}\frac{\partial h_2}{\partial \xi_\alpha} + \frac{1}{h_3 h_\alpha}\frac{\partial h_3}{\partial \xi_\alpha} \quad (6)$$

Divergence in Cylindrical Coordinate System:

from (6) $\text{div} = \dfrac{1}{h_r h_r}\dfrac{\partial h_r}{\partial \xi_r} + \dfrac{1}{h_\theta h_r}\dfrac{\partial h_\theta}{\partial \xi_r} + \dfrac{1}{h_z h_r}\dfrac{\partial h_z}{\partial \xi_r}$

$$+ \frac{1}{h_r h_\theta}\frac{\partial h_r}{\partial \xi_\theta} + \frac{1}{h_\theta h_\theta}\frac{\partial h_\theta}{\partial \xi_\theta} + \frac{1}{h_z h_\theta}\frac{\partial h_z}{\partial \xi_\theta}$$

$$+ \frac{1}{h_r h_z} \frac{\partial h_r}{\partial \xi_z} + \frac{1}{h_\theta h_z} \frac{\partial h_\theta}{\partial \xi_z} + \frac{1}{h_z h_z} \frac{\partial h_z}{\partial \xi_z}$$

$$= \frac{1}{r}$$

Compare with previous solution: $\nabla \cdot \vec{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) + \frac{\partial v_z}{\partial z}$

$$v = \hat{e}_r = (1)\,\hat{e}_r + (0)\,\hat{e}_\theta + (0)\,\hat{e}_z$$

$$\nabla \cdot \hat{e}_r = \frac{1}{r}$$