Eulerian Strain Tensor

Change of Area/Volume due to Deformation

Eulerian Strain Tensor (e^*):

$$dx = F\,dX \quad \longleftrightarrow \quad \boxed{F^{-1} dx = dX} \quad \left(F^{-1}_{ij} dx_j = dX_i \right)$$

$$F^{-1}_{ij} = \frac{\partial X_i}{\partial x_j}$$

$$F^{-1} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}$$

assumption: $dX^1 = dS_1 \hat{n}$ $\quad dx^1 = ds_1 \hat{e}_1$

$dX^2 = dS_2 \hat{m}$ $\quad dx^2 = ds_2 \hat{e}_2$ $\Big\}$ elements in current config. are orthogonal

$$dX^1 . dX^2 = F^{-1} dx^1 . F^{-1} dx^2 = dx^1 . (F^{-1})^T F^{-1} dx^2 = dx^1 . (F^T)^{-1} F^{-1} dx^2$$

$$= dx^1 . (F F^T)^{-1} dx^2 = dx^1 . B^{-1} dx^2$$

$$\frac{dx^1 . dx^2}{\times(-)} \longrightarrow \quad dx^1 . dx^2 - dX^1 . dX^2 = dx^1 . I dx^2 - dx^1 . B^{-1} dx^2$$

$$dx^1 . dx^2 - dX^1 . dX^2 = dx^1 . \underbrace{(I - B^{-1})}_{2e^*} dx^2$$

We call $(I - B^{-1}) = 2e^*$ where e^* is Eulerian strain tensor

$$\boxed{dx^1 . dx^2 - dX^1 . dX^2 = 2 dx^1 . e^* dx^2} \quad (*)$$

Diagonals: $dx^1 = ds_1 \hat{e}_1$ $\qquad dX^1 = dS_1 \hat{n}$

$dx^2 = dx^1$ $\qquad dX^2 = dX^1$

(*) $ds_1^2 - dS_1^2 = 2\, ds_1^2\, e_1 \cdot e^* e_1 \Rightarrow e^*_{11} = \dfrac{ds_1^2 - dS_1^2}{2 ds_1^2}$

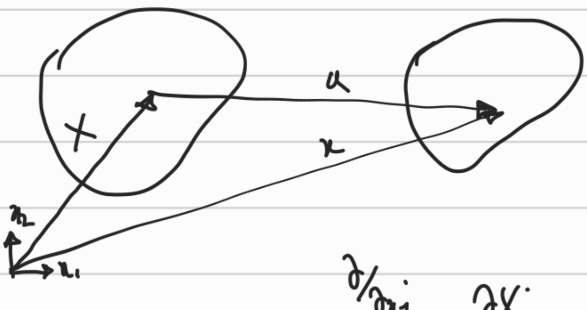
Off-diagonals: $dx^1 = ds_1 \hat{e}_1$ $\qquad dX^1 = dS_1 \hat{n}$

$dx^2 = ds_2 \hat{e}_2$ $\qquad dX^2 = dS_2 \hat{m}$

(*) $\quad - dS_1 dS_2 \cos\beta = 2 ds_1 ds_2\, e^*_{12}$

$e^*_{12} = \dfrac{-dS_1 dS_2}{2 ds_1 ds_2} \cos\beta$ where $\cos\beta$ is the angle between $\hat{n}$ and $\hat{m}$ in the reference configuration.

Current Config. $\qquad$ Reference config.



$x = X + u \Rightarrow X = x - u$

$X = x - u(x_1, x_2, x_3, t)$

$\xrightarrow{\partial/\partial x_j} \dfrac{\partial X_i}{\partial x_j} = \underbrace{\dfrac{\partial x_i}{\partial x_j}}_{\delta_{ij}} - \dfrac{\partial u_i}{\partial x_j}$

$F^{-1} = I - \nabla_x u$

$B^{-1} = (FF^T)^{-1} = (F^T)^{-1} F^{-1} = (F^{-1})^T F^{-1} = (I - \nabla_x u)^T (I - \nabla_x u)$

$= I - \nabla_x u - (\nabla_x u)^T + (\nabla_x u)^T (\nabla_x u)$

$$2e^* = I - B^{-1} = \nabla_x u + (\nabla_x u)^T - (\nabla_x u)^T (\nabla_x u)$$

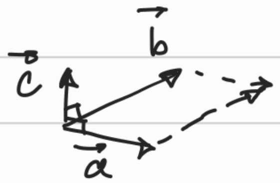
$$\boxed{\begin{aligned} e^* &= \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^T\right) - \frac{1}{2}\underbrace{(\nabla_x u)^T (\nabla_x u)}_{T_{ij}} \\ e^*_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{2}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} \end{aligned}}$$

$$T_{ij} = (\nabla_x u)^T_{im} (\nabla_x u)_{mj} = (\nabla_x u)_{mi} (\nabla_x u)_{mj} = \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j}$$

For infinitesimal deformation : $\frac{\partial u_i}{\partial x_j} \simeq \frac{\partial u_i}{\partial X_j}$, $\frac{\partial u_j}{\partial x_i} \simeq \frac{\partial u_j}{\partial X_i}$

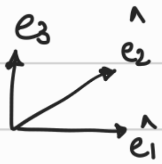
$\therefore$ " " " : $e^* \simeq E$

Change of Area due to Deformation:



$\vec{a} \times \vec{b}$ = area of parellogram

In the Reference config. $\quad \vec{dA_0} = dX^1 \times dX^2 = dS_1 \hat{e}_1 \times dS_2 \hat{e}_2$



$$= \underbrace{dS_1 \, dS_2}_{dA_0} \hat{e}_3$$

$$\vec{dA_0} = dA_0 \hat{e}_3$$

In the Current Config. $\quad \vec{dA} = dx^1 \times dx^2 = F dX^1 \times F dX^2$

$$= dS_1 F\hat{e}_1 \times dS_2 F\hat{e}_2 = dS_1 dS_2 \, F\hat{e}_1 \times F\hat{e}_2$$

$$\vec{dA} = dA_0 \, F\hat{e}_1 \times F\hat{e}_2 \qquad (1)$$

We know $\vec{dA} = dA \, \hat{n} \qquad (2)$

(2) in (1) : $dA\,\hat{n} = dA_0\, F\hat{e}_1 \times F\hat{e}_2 \rightarrow \hat{n} = \dfrac{dA_0}{dA}\, F\hat{e}_1 \times F\hat{e}_2$ (3)

$F\hat{e}_1 \cdot (F\hat{e}_1 \times F\hat{e}_2) = F\hat{e}_2 \cdot (F\hat{e}_1 \times F\hat{e}_2) = 0$

$\therefore F\hat{e}_1 \cdot n = F\hat{e}_2 \cdot n = 0$

$\therefore e_1 \cdot F^T n = e_2 \cdot F^T n = 0$

$\therefore e_1 \perp F^T n$ and $e_2 \perp F^T n$

(3) $F\hat{e}_3 \cdot \hat{n} = \dfrac{dA_0}{dA}\, \underbrace{F\hat{e}_3 \cdot (F\hat{e}_1 \times F\hat{e}_2)}_{\det F}$

$Ta \cdot (Tb \times Tc) = Ta_i\hat{e}_i \cdot (Tb_j\hat{e}_j \times Tc_k\hat{e}_k)$

$= a_i b_j c_k\, T\hat{e}_i \cdot (T\hat{e}_j \times T\hat{e}_k)$

$T\hat{e}_i = T_{ri}\hat{e}_r \longrightarrow = a_i b_j c_k\, T_{ri}\hat{e}_r \cdot (T_{mj}\hat{e}_m \times T_{nk}\hat{e}_n)$

$= a_i b_j c_k\, T_{ri} T_{mj} T_{nk}\, \hat{e}_r \cdot \underbrace{(\hat{e}_m \times \hat{e}_n)}_{\epsilon_{mnz}\hat{e}_z}$

$= a_i b_j c_k\, T_{ri} T_{mj} T_{nk}\, \epsilon_{mnz}\, \underbrace{\hat{e}_r \cdot \hat{e}_z}_{\delta_{rz}}$

$= a_i b_j c_k\, T_{ri} T_{mj} T_{nk}\, \epsilon_{mnr}$

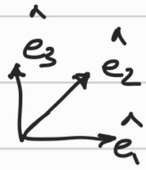
$\epsilon_{rmn} = \epsilon_{mnr} = \epsilon_{nrm} \longrightarrow = a_i b_j c_k\, \underbrace{T_{ri} T_{mj} T_{nk}\, \epsilon_{rmn}}_{(\det T)}$

(3) $F\hat{e}_3 \cdot \hat{n} = \dfrac{dA_0}{dA}(\det F)$

$$\hat{e}_3 \cdot F^T n = \frac{dA_0}{dA}(\det F)$$

$$\underset{(F^T)^{-1}}{F^T} n = \frac{dA_0}{dA}(\det F)\,\underset{(F^T)^{-1}}{\hat{e}_3} \Rightarrow n = \frac{dA_0}{dA}(\det F)(F^{-1})^T \hat{e}_3$$

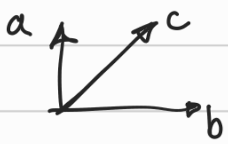
$$\boxed{\begin{aligned} dA\, n &= dA_0 (\det F)(F^{-1})^T \hat{e}_3 \\ dA &= dA_0 (\det F)\left|(F^{-1})^T \hat{e}_3\right| \end{aligned}}$$



In general,

$$\boxed{\begin{aligned} dA\, n &= dA_0 (\det F)(F^{-1})^T n_0 \\ dA &= dA_0 (\det F)\left|(F^{-1})^T n_0\right| \end{aligned}}$$

Change of Volume due to Deformation:



$a\cdot(b\times c)$ = volume of parallelpiped

In reference config $\quad dV_0 = dX^1 \cdot (dX^2 \times dX^3) = dS_1\hat{e}_1 \cdot (dS_2\hat{e}_2 \times dS_3\hat{e}_3)$

$$= dS_1\, dS_2\, dS_3\, \hat{e}_1 \cdot \cancel{(\hat{e}_2 \times \hat{e}_3)}^{\,1} = dS_1\, dS_2\, dS_3$$

In Current Config. $\quad dV = \left|dx^1 \cdot (dx^2 \times dx^3)\right| = \left|F dX^1 \cdot (F dX^2 \times F dX^3)\right|$

$$= \underbrace{dS_1\, dS_2\, dS_3}_{dV_0}\, \left|\underbrace{F\hat{e}_1 \cdot (F\hat{e}_2 \times F\hat{e}_3)}_{(\det F)}\right|$$

$$\therefore \quad dV = dV_0 (\det F) = J\, dV_0 \quad , \quad J = \det F$$

$C = F^T F$ and $B = F F^T$

$\det C = \det B = (\det F)^2$

$$dV = \sqrt{\det C}\, dV_0 = \sqrt{\det B}\, dV_0$$

For an incompressible material, $dV = dV_0$

From conservation of mass, $m = m_0$

$$\rho\, dV = \rho_0\, dV_0$$

$$\rho J \cancel{dV_0} = \rho_0 \cancel{dV_0}$$

$$\rho = \frac{\rho_0}{J} \quad , \quad J = \det F = \sqrt{\det C} = \sqrt{\det B}$$

* if $|\det F| = 0$ then $dV = 0$ that is in contrast with the principal of conservation of mass.

If $|\det F| < 0$ or $|\det F| > 0$ in the reference config., it should also be the same in current config.

Ref conf.		current config.
$\lvert\det F\rvert > 0$	$\longrightarrow$	$\lvert\det F\rvert > 0$
$\lvert\det F\rvert < 0$	$\longrightarrow$	$\lvert\det F\rvert < 0$

Because if $|\det F|$ changes sign, it shows that $|\det F| = 0$ somewhere which is in contrast to the principal of conservation of mass (see above statement)

Dilatation for infinitesimal deformation : $e = tr(E) = \operatorname{div} u$

" " large " : $\frac{dV}{dV_0} = |\det F| = J$