Principal Values and Principal Directions

Scalar Invariants of a tensor

Principal Values/Principal Directions:

for a real symmetric tensor $\longrightarrow$ there exists 3 eigenvalues (principal values)

We first prove that if there exists 3 principal directions, they are mutually perpendicular.

$$Tn_1 = \lambda_1 n_1 \quad , \quad Tn_2 = \lambda_2 n_2$$

$$\downarrow \cdot n_2 \qquad\qquad \downarrow \cdot n_1$$

$$n_2 \cdot Tn_1 = n_2 \cdot \lambda_1 n_1 \qquad n_1 \cdot Tn_2 = n_1 \cdot \lambda_2 n_2$$

$$= \lambda_1 n_2 \cdot n_1 \qquad\qquad = \lambda_2 n_1 \cdot n_2$$

for a symmetric tensor, $T = T^T$

$$\begin{cases} n_2 \cdot Tn_1 = \lambda_1 n_2 \cdot n_1 \Rightarrow n_1 \cdot T^T n_2 = \lambda_1 n_2 \cdot n_1 \xrightarrow{T=T^T} n_1 \cdot Tn_2 = \lambda_1 n_2 \cdot n_1 \quad (1) \\ n_1 \cdot Tn_2 = \lambda_2 n_1 \cdot n_2 \quad (2) \end{cases}$$

$(1)-(2)$: $\lambda_1 n_2 \cdot n_1 - \lambda_2 n_1 \cdot n_2 = 0$

$$(\lambda_1 - \lambda_2)\, n_1 \cdot n_2 = 0 \qquad \lambda_1 - \lambda_2 \neq 0 \text{ because } \lambda_1 \neq \lambda_2$$

$$\therefore n_1 \cdot n_2 = 0 \qquad n_1 \perp n_2$$

Case 1: $\lambda_1 \neq \lambda_2 \neq \lambda_3$ as already shown (above), at least one of the principal directions (eigenvectors) are perpendicular to each other.

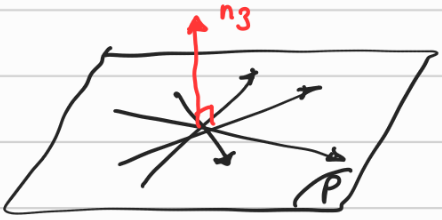
Case 2: $\lambda_1 = \lambda_2 \neq \lambda_3$

$$\lambda_1 = \lambda_2 = \lambda \qquad \begin{cases} Tn_1 = \lambda_1 n_1 = \lambda n_1 \\ Tn_2 = \lambda_2 n_2 = \lambda n_2 \end{cases}$$

For any scalar α and β

$$\begin{cases} T\alpha n_1 = \lambda \alpha n_1 \\ T\beta n_2 = \lambda \beta n_2 \end{cases}$$

$$T(\alpha n_1 + \beta n_2) = \lambda(\alpha n_1 + \beta n_2)$$



Infinite number of principal directions which for a plane (here αn_1 and βn_2) and n_3 is of course perpendicular to this plane.

principal directions: $n_1 = \pm \hat{e}_1$, $n_2 = \pm \hat{e}_2$, $n_3 = \alpha \hat{e}_1 + \beta \hat{e}_2$

Case $\lambda_1 = \lambda_2 = \lambda_3$ any vector is a principal vector (Example 2.22.1)

Matrix of a tensor w.r.t Principal directions:

spectral representation: $[T] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$T_{11} = n_1 \cdot Tn_1 = n_1 \cdot \lambda_1 n_1 = \lambda_1 n_1 \cdot n_1 = \lambda_1$$

$$T_{22} = n_2 \cdot Tn_2 = n_2 \cdot \lambda_2 n_2 = \lambda_2 n_2 \cdot n_2 = \lambda_2$$

$$T_{32} = n_3 \cdot Tn_2 = n_3 \cdot \lambda_2 n_2 = \lambda_2 n_3 \cdot n_2 = 0$$

We can show that a tensor T includes a maximum and a minimum values that the diagonal elements of any matrix T can have.

$$e_1' = \alpha n_1 + \beta n_2 + \gamma n_3$$

$$T_{11}' = e_1' \cdot T e_1' = [\alpha \quad \beta \quad \gamma] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$= [\alpha\lambda_1 \quad \beta\lambda_2 \quad \gamma\lambda_3] \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3$$

Let us assume $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$

$$\lambda_1(\alpha^2+\beta^2+\gamma^2) \geqslant \alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3$$

$$\boxed{\alpha^2+\beta^2+\gamma^2 = 1}$$

$$\therefore \quad \lambda_1 \geqslant T_{11}'$$

Using the same strategy

$$\alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3 \geqslant \lambda_3(\alpha^2+\beta^2+\gamma^2)$$

$$\therefore \quad T_{11}' \geqslant \lambda_3$$

Principal Scalar Invariants:

Characteristic equation: $|T_{ij} - \lambda\delta_{ij}| = 0$

$$\lambda^3 - \underbrace{I_1}\,\lambda^2 + \underbrace{I_2}\,\lambda - \underbrace{I_3} = 0$$

scalar invariants

$$I_1 = T_{11} + T_{22} + T_{33} = T_{ii} = tr(T)$$

$$I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix}$$

$$= \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ji} \right) = \frac{1}{2}\left[\left(tr(T)\right)^2 - tr(T^2) \right]$$

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \det[T]$$

Scalar invariants in terms of eigenvalues:

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$I_3 = \lambda_1\lambda_2\lambda_3$$