Spherical Coordinate System:

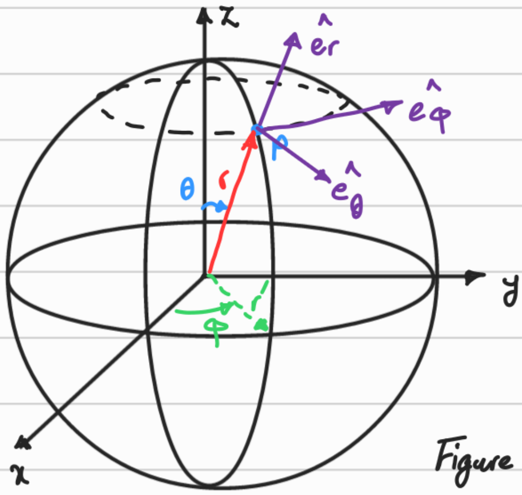


Figure 1

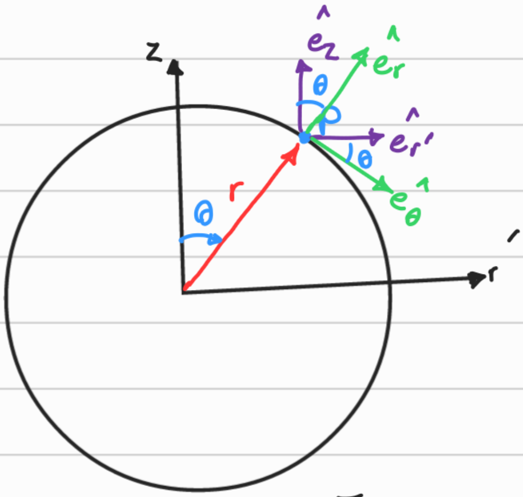


Figure 2

Position vector : $\vec{r} = r\,\hat{e}_r$

$$d\vec{r} = dr\,\hat{e}_r + r\,d\hat{e}_r$$

Figure 2:

$$\begin{array}{cc} \hat{e}_{r'}\cos\theta & + \hat{e}_{r'}\sin\theta \\ -\hat{e}_z\sin\theta & + \hat{e}_z\cos\theta \\ \hat{e}_\theta & \hat{e}_r \end{array} \Rightarrow \begin{array}{l} e_\theta = \cos\theta\,\hat{e}_{r'} - \sin\theta\,\hat{e}_z \\ e_r = \sin\theta\,\hat{e}_{r'} + \cos\theta\,\hat{e}_z \end{array}$$

$\hat{e}_\theta$

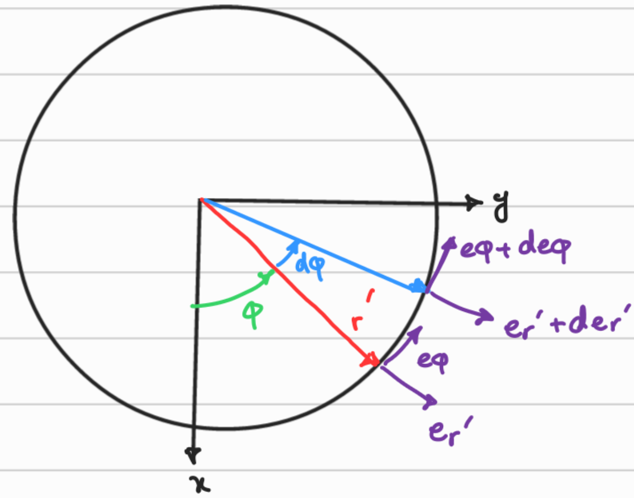
$$d\hat{e}_r = \cos\theta\,d\theta\,\hat{e}_{r'} + \sin\theta\,d\hat{e}_{r'} - \sin\theta\,d\theta\,\hat{e}_z + \cos\theta\,\underbrace{\cancel{d\hat{e}_z}}_{0}$$

$$= d\theta\,\hat{e}_\theta + \sin\theta\,d\hat{e}_{r'}$$

$-\hat{e}_r$

$$d\hat{e}_\theta = -\sin\theta\,d\theta\,\hat{e}_{r'} + \cos\theta\,d\hat{e}_{r'} - \cos\theta\,d\theta\,\hat{e}_z - \sin\theta\,\underbrace{\cancel{d\hat{e}_z}}_{0}$$

$$= -d\theta\,\hat{e}_r + \cos\theta\,d\hat{e}_{r'}$$



Recall from Polar Coordinate System:

$$d\hat{e}_r = d\theta\, \hat{e}_\theta$$

↓ let us map it into r'-φ

$$d\hat{e}_{r'} = d\varphi\, \hat{e}_\varphi \quad (*)$$

$$\boxed{d\hat{e}_r = d\theta\, \hat{e}_\theta + \sin\theta\, d\hat{e}_{r'} \overset{(*)}{=} d\theta\, \hat{e}_\theta + \sin\theta\, d\varphi\, \hat{e}_\varphi}$$

$$\boxed{d\hat{e}_\theta = -d\theta\, \hat{e}_r + \cos\theta\, d\hat{e}_{r'} \overset{(*)}{=} -d\theta\, \hat{e}_r + \cos\theta\, d\varphi\, \hat{e}_\varphi}$$

Recall from Polar Coordinate System:

$$d\hat{e}_\theta = -d\theta\, \hat{e}_r$$

↓ map it in r'-φ

$$d\hat{e}_\varphi = -d\varphi\, \hat{e}_{r'}$$

$$d\hat{e}_\varphi = -d\varphi\, \hat{e}_{r'} \xrightarrow[\hat{e}_{r'} = \cos\theta\, \hat{e}_\theta + \sin\theta\, \hat{e}_r]{\text{Figure 2}} -d\varphi\, (\cos\theta\, \hat{e}_\theta + \sin\theta\, \hat{e}_r)$$

$$\boxed{d\hat{e}_\varphi = -\sin\theta\, d\varphi\, \hat{e}_r - \cos\theta\, d\varphi\, \hat{e}_\theta}$$

$$\boxed{d\vec{r} = dr\, \hat{e}_r + r\, d\theta\, \hat{e}_\theta + r\sin\theta\, d\varphi\, \hat{e}_\varphi}$$

Gradient of a Scalar Field (∇f)

$f(r, \theta, \varphi)$

$$\frac{df}{dr} = \nabla f \Rightarrow df = \nabla f \cdot dr = \left[(\nabla f)_r\, \hat{e}_r + (\nabla f)_\theta\, \hat{e}_\theta + (\nabla f)_\varphi\, \hat{e}_\varphi \right] \cdot$$

$$\left\{ dr\, \hat{e}_r + r\, d\theta\, \hat{e}_\theta + r\sin\theta\, d\varphi\, \hat{e}_\varphi \right\}$$

$$df = (\nabla f)_r \, dr + (\nabla f)_\theta \, r d\theta + (\nabla f)_\varphi (r\sin\theta) d\varphi \quad ①$$

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \varphi} d\varphi \quad ②$$

compare ① and ② : $(\nabla f)_r = \frac{\partial f}{\partial r}$, $(\nabla f)_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta}$

$$(\nabla f)_\varphi = \frac{1}{r\sin\theta} \frac{\partial f}{\partial \varphi}$$

Gradient of a Vector Field:

$$\vec{v}(r,\theta,\varphi) = v_r(r,\theta,\varphi)\,\hat{e_r} + v_\theta(r,\theta,\varphi)\,\hat{e_\theta} + v_\varphi(r,\theta,\varphi)\,\hat{e_\varphi} \quad ⑤$$

$$\frac{d\vec{v}}{dr} = \nabla\vec{v} = T \text{ (2nd-order tensor)} \Rightarrow d\vec{v} = T dr = T\left(dr\,\hat{e_r} + r d\theta\,\hat{e_\theta} + r\sin\theta d\varphi\,\hat{e_\varphi}\right)$$

$$= dr\,T\hat{e_r} + r d\theta\,T\hat{e_\theta} + r\sin\theta d\varphi\,T\hat{e_\varphi} \quad ③$$

$$\left.\begin{aligned}
T\hat{e_r} &= T_{jr}\hat{e_j} \overset{j=r,\theta,\varphi}{=\!=} T_{rr}\hat{e_r} + T_{\theta r}\hat{e_\theta} + T_{\varphi r}\hat{e_\varphi} \\
T\hat{e_\theta} &= T_{j\theta}\hat{e_j} = T_{r\theta}\hat{e_r} + T_{\theta\theta}\hat{e_\theta} + T_{\varphi\theta}\hat{e_\varphi} \\
T\hat{e_\varphi} &= T_{j\varphi}\hat{e_j} = T_{r\varphi}\hat{e_r} + T_{\theta\varphi}\hat{e_\theta} + T_{\varphi\varphi}\hat{e_\varphi}
\end{aligned}\right\} \quad ④$$

Plug in ④ in ③ : $dv = dr\left(T_{rr}\hat{e_r} + T_{\theta r}\hat{e_\theta} + T_{\varphi r}\hat{e_\varphi}\right)$

$$+ r d\theta\left(T_{r\theta}\hat{e_r} + T_{\theta\theta}\hat{e_\theta} + T_{\varphi\theta}\hat{e_\varphi}\right)$$

$$+ r\sin\theta d\varphi\left(T_{r\varphi}\hat{e_r} + T_{\theta\varphi}\hat{e_\theta} + T_{\varphi\varphi}\hat{e_\varphi}\right)$$

$$= \left(dr\,T_{rr} + r d\theta\,T_{r\theta} + r\sin\theta\,d\varphi\,T_{r\varphi}\right)\hat{e_r}$$

$$+ \left(dr\,T_{\theta r} + r d\theta\,T_{\theta\theta} + r\sin\theta\,d\varphi\,T_{r\varphi}\right)\hat{e_\theta}$$

$$+ \left(dr\,T_{\varphi r} + r d\theta\,T_{\varphi\theta} + r\sin\theta\,d\varphi\,T_{\varphi\varphi}\right)\hat{e_\varphi} \quad ⑥$$

from ⑤ $d\mathbf{v} = dv_r\,\hat{e}_r + v_r\,d\hat{e}_r + dv_\theta\,\hat{e}_\theta + v_\theta\,d\hat{e}_\theta + dv_\varphi\,\hat{e}_\varphi + v_\varphi\,d\hat{e}_\varphi$

$$\begin{cases} dv_r = \dfrac{\partial v_r}{\partial r}dr + \dfrac{\partial v_r}{\partial \theta}d\theta + \dfrac{\partial v_r}{\partial \varphi}d\varphi \\ dv_\theta = \dfrac{\partial v_\theta}{\partial r}dr + \dfrac{\partial v_\theta}{\partial \theta}d\theta + \dfrac{\partial v_\theta}{\partial \varphi}d\varphi \\ dv_\varphi = \dfrac{\partial v_\varphi}{\partial r}dr + \dfrac{\partial v_\varphi}{\partial \theta}d\theta + \dfrac{\partial v_\varphi}{\partial \varphi}d\varphi \end{cases}$$

$$d\mathbf{v} = \left(\frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial \theta}d\theta + \frac{\partial v_r}{\partial \varphi}d\varphi\right)\hat{e}_r + v_r\left(d\theta\,\hat{e}_\theta + \sin\theta\,d\varphi\,\hat{e}_\varphi\right)$$

$$+\left(\frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial \theta}d\theta + \frac{\partial v_\theta}{\partial \varphi}d\varphi\right)\hat{e}_\theta + v_\theta\left(-d\theta\,\hat{e}_r + \cos\theta\,d\varphi\,\hat{e}_\varphi\right)$$

$$+\left(\frac{\partial v_\varphi}{\partial r}dr + \frac{\partial v_\varphi}{\partial \theta}d\theta + \frac{\partial v_\varphi}{\partial \varphi}d\varphi\right)\hat{e}_\varphi + v_\varphi\left(-\sin\theta\,d\varphi\,\hat{e}_r - \cos\theta\,d\varphi\,\hat{e}_\theta\right)$$

$$= \left(\frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial \theta}d\theta + \frac{\partial v_r}{\partial \varphi}d\varphi - v_\varphi\sin\theta\,d\varphi - v_\theta\,d\theta\right)\hat{e}_r$$

$$+\left(\frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial \theta}d\theta + \frac{\partial v_\theta}{\partial \varphi}d\varphi + v_r\,d\theta - v_\varphi\cos\theta\,d\varphi\right)\hat{e}_\theta$$

$$+\left(\frac{\partial v_\varphi}{\partial r}dr + \frac{\partial v_\varphi}{\partial \theta}d\theta + \frac{\partial v_\varphi}{\partial \varphi}d\varphi\right) + v_r\sin\theta\,d\varphi + v_\theta\cos\theta\,d\varphi\Big)\hat{e}_\varphi \quad ⑦$$

compare ⑥ and ⑦ : $T_{rr} = \dfrac{\partial v_r}{\partial r}$, $T_{r\theta} = \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right)$, $T_{r\varphi} = \dfrac{1}{r\sin\theta}\left(\dfrac{\partial v_r}{\partial \varphi} - v_\varphi\sin\theta\right)$

$T_{\theta r} = \dfrac{\partial v_\theta}{\partial r}$, $T_{\theta\theta} = \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right)$, $T_{\theta\varphi} = \dfrac{1}{r\sin\theta}\left(\dfrac{\partial v_\theta}{\partial \varphi} - v_\varphi\cos\theta\right)$

$T_{\varphi r} = \dfrac{\partial v_\varphi}{\partial r}$, $T_{\varphi\theta} = \dfrac{1}{r}\left(\dfrac{\partial v_\varphi}{\partial \theta}\right)$, $T_{\varphi\varphi} = \dfrac{1}{r\sin\theta}\left(\dfrac{\partial v_\varphi}{\partial \varphi} + v_r\sin\theta + v_\theta\cos\theta\right)$

$$\nabla\mathbf{v} = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{r\varphi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\varphi} \\ T_{\varphi r} & T_{\varphi\theta} & T_{\varphi\varphi} \end{bmatrix}$$

Divergence of a Vector Field:

$$\operatorname{div}\vec{v} = tr(\nabla v) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{1}{r\sin\theta}\left(\frac{\partial v_\varphi}{\partial \varphi} + v_r \sin\theta + v_\theta \cos\theta\right) \quad (8)$$

Curl of a Vector Field:

$$\operatorname{Curl}\vec{v} = 2\,t^{A} =$$

Divergence of A Tensor Field:

$$(\operatorname{div} T)\cdot a = \operatorname{div}(T^T a) - tr((\nabla a)T^T) \quad \forall \vec{a}$$

$$a = \hat{e_r} : (\operatorname{div} T)_r = \operatorname{div}(T^T e_r) - tr((\nabla e_r)\, T^T)$$

$$T\hat{e_r} = T_{jr}\hat{e_j} = T_{rr}\hat{e_r} + T_{\theta r}\hat{e_\theta} + T_{\varphi r}\hat{e_\varphi}$$

$$T^T e_r = \underbrace{T_{rr}}_{v_r}\hat{e_r} + \underbrace{T_{r\theta}}_{v_\theta}\hat{e_\theta} + \underbrace{T_{r\varphi}}_{v_\varphi}\hat{e_\varphi}$$

from (8)

$$\operatorname{div}(T^T e_r) = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{r\varphi}}{\partial \varphi} + 2\frac{T_{rr}}{r} + \frac{T_{r\theta}\cot\theta}{r}$$

$$\hat{e_r} = \underbrace{(1)}_{v_r}\hat{e_r} + \underbrace{(0)}_{v_\theta}\hat{e_\theta} + \underbrace{(0)}_{v_\varphi}\hat{e_\varphi}$$

$$\nabla e_r\, T^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/r \end{bmatrix}\begin{bmatrix} T_{rr} & T_{\theta r} & T_{\varphi r} \\ T_{r\theta} & T_{\theta\theta} & T_{\varphi\theta} \\ T_{r\varphi} & T_{\theta\varphi} & T_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{T_{r\theta}}{r} & \frac{T_{\theta\theta}}{r} & \frac{T_{\varphi\theta}}{r} \\ \frac{T_{r\varphi}}{r} & \frac{T_{\theta\varphi}}{r} & \frac{T_{\varphi\varphi}}{r} \end{bmatrix}$$

$$tr(\nabla e_r T^T) = \frac{T_{\theta\theta}}{r} + \frac{T_{\varphi\varphi}}{r}$$

$$(\mathrm{div}\, T)_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{r\varphi}}{\partial \varphi} + \frac{2T_{rr}}{r} + \frac{T_{r\theta}\cot\theta}{r}$$

$$- \frac{T_{\theta\theta}}{r} - \frac{T_{\varphi\varphi}}{r}$$

$$a = \hat{e}_\theta : (\mathrm{div}\, T)_\theta = \mathrm{div}(T^T e_\theta) - tr((\nabla \hat{e}_\theta) T^T)$$

$$T e_\theta = T_{j\theta}\hat{e}_j = T_{r\theta}\hat{e}_r + T_{\theta\theta}\hat{e}_\theta + T_{\varphi\theta}\hat{e}_\varphi$$

$$T^T e_\theta = \underbrace{T_{\theta r}}_{v_r}\hat{e}_r + \underbrace{T_{\theta\theta}}_{v_\theta}\hat{e}_\theta + \underbrace{T_{\theta\varphi}}_{v_\varphi}\hat{e}_\varphi$$

$$\mathrm{div}(T^T\hat{e}_\theta) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\varphi}}{\partial \varphi} + \frac{2}{r}T_{\theta r} + \frac{T_{\theta\theta}\cot\theta}{r}$$

$$\hat{e}_\theta = \underbrace{(0)}_{v_r}\hat{e}_r + \underbrace{(1)}_{v_\theta}\hat{e}_\theta + \underbrace{(0)}_{v_\varphi}\hat{e}_\varphi$$

$$\nabla\hat{e}_r T^T = \begin{bmatrix} 0 & \frac{-1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\cot\theta}{r} \end{bmatrix} \begin{bmatrix} T_{rr} & T_{\theta r} & T_{\varphi r} \\ T_{r\theta} & T_{\theta\theta} & T_{\varphi\theta} \\ T_{r\varphi} & T_{\theta\varphi} & T_{\varphi\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{T_{r\theta}}{r} & -\frac{T_{\theta\theta}}{r} & -\frac{T_{\varphi\theta}}{r} \\ 0 & 0 & 0 \\ \frac{\cot\theta}{r}T_{r\varphi} & \frac{\cot\theta}{r}T_{\theta\varphi} & \frac{\cot\theta}{r}T_{\varphi\varphi} \end{bmatrix}$$

$$tr(\nabla\hat{e}_r T^T) = -\frac{T_{r\theta}}{r} + T_{\varphi\varphi}\left(\frac{\cot\theta}{r}\right)$$

$$(\mathrm{div}\, T)_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\varphi}}{\partial \varphi} + \frac{2}{r}T_{\theta r} + \frac{T_{\theta\theta}\cot\theta}{r}$$

$$+ \frac{T_{r\theta}}{r} - T_{\varphi\varphi}\left(\frac{\cot\theta}{r}\right)$$

$$a = \hat{e}_\varphi : (\mathrm{div}\, T)_\varphi = \mathrm{div}(T^T e_\varphi) - tr((\nabla e_\varphi) T^T)$$

$$T e_\varphi = T_{j\varphi}\hat{e}_j = T_{r\varphi}\hat{e}_r + T_{\theta\varphi}\hat{e}_\theta + T_{\varphi\varphi}\hat{e}_\varphi$$

$$T^T \hat{e}_\varphi = \underbrace{T_{\varphi r}}_{v_r} \hat{e}_r + \underbrace{T_{\varphi\theta}}_{v_\theta} \hat{e}_\theta + \underbrace{T_{\varphi\varphi}}_{v_\varphi} \hat{e}_\varphi$$

$$div(T^T \hat{e}_\varphi) = \frac{\partial T_{\varphi r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\varphi\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial \varphi} + \frac{2T_{\varphi r}}{r} + \frac{T_{\varphi\theta}\cot\theta}{r}$$

$$\hat{e}_\varphi = (0)\hat{e}_r + (0)\hat{e}_\theta + (1)\hat{e}_\varphi$$

$$\nabla \hat{e}_\varphi T^T = \begin{bmatrix} 0 & 0 & -\frac{1}{r} \\ 0 & 0 & -\frac{1}{r}\cot\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{rr} & T_{\theta r} & T_{\varphi r} \\ T_{r\theta} & T_{\theta\theta} & T_{\varphi\theta} \\ T_{r\varphi} & T_{\theta\varphi} & T_{\varphi\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{T_{r\varphi}}{r} & -\frac{T_{\theta\varphi}}{r} & -\frac{T_{\varphi\varphi}}{r} \\ -\frac{1}{r}\cot\theta\, T_{r\varphi} & -\frac{1}{r}\cot\theta\, T_{\theta\varphi} & -\frac{1}{r}\cot\theta\, T_{\varphi\varphi} \\ 0 & 0 & 0 \end{bmatrix}$$

$$tr(\nabla e_\varphi T^T) = \frac{-T_{r\varphi}}{r} - \frac{\cot\theta}{r} T_{\theta\varphi}$$

$$(div\, T)_\varphi = \frac{\partial T_{\varphi r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\varphi\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial \varphi} + \frac{2T_{\varphi r}}{r} + \frac{T_{\varphi\theta}\cot\theta}{r} + \frac{T_{r\varphi}}{r} + \frac{\cot\theta}{r} T_{\theta\varphi}$$

Laplacian of a Scalar Field:

$$\nabla^2 f = \nabla \cdot (\nabla f) = div(\nabla f)$$

$$= div\left(\underbrace{\frac{\partial f}{\partial r}}_{v_r}\hat{e}_r + \underbrace{\frac{1}{r}\frac{\partial f}{\partial \theta}}_{v_\theta}\hat{e}_\theta + \underbrace{\frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}}_{v_\varphi}\hat{e}_\varphi\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\right)$$

$$+ \frac{2}{r}\left(\frac{\partial f}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)\frac{\cot\theta}{r}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 f}{\partial \varphi^2}\right) + \frac{2}{r}\frac{\partial f}{\partial r}$$

$$+ \frac{\cot\theta}{2}\frac{\partial f}{\partial \theta}$$

Laplacian of a Vector Field:

$$\nabla^2 \vec{v} = \nabla(\operatorname{div} \vec{v}) - \nabla \times (\nabla \times v)$$

$\operatorname{div} \vec{v}$ Eq (2.35.26)

$$\nabla(\operatorname{div} \vec{v}) = \nabla\Big(\underbrace{\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r}}_{f}\Big)$$

∇f Eq (2.35.36)

$$\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi$$

$$\nabla(\operatorname{div} \vec{v}) = \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)\hat{e}_r$$

$$+ \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)\hat{e}_\theta$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)\hat{e}_\varphi$$

$$= \Big(\frac{\partial^2 v_r}{\partial r^2} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v_\theta}{\partial r \partial \theta} - \frac{1}{r^2 \sin\theta}\frac{\partial v_\varphi}{\partial \varphi} +$$

$$\frac{1}{r\sin\theta}\frac{\partial^2 v_\varphi}{\partial \varphi^2} - 2\frac{v_r}{r^2} - \frac{1}{r^2} v_\theta \cot\theta + \frac{\cot\theta}{r}\frac{\partial v_\theta}{\partial r}\Big)\hat{e}_r$$

$$+ \frac{1}{r}\Big\{\frac{\partial^2 v_r}{\partial \theta \partial r} + \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{\cos\theta}{r\sin^2\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{r\sin\theta}\frac{\partial^2 v_\varphi}{\partial \theta \partial \varphi} + \frac{2}{r}\frac{\partial v_r}{\partial \theta}$$

$$+\left(\frac{\cot\theta}{r}\right)\frac{\partial v_\theta}{\partial\theta}+\frac{v_\theta}{r}\left(-1-\cot^2\theta\right)\Big\}\hat{e}_\theta$$

$$+\frac{1}{r\sin\theta}\Big\{\frac{\partial^2 v_r}{\partial\varphi\partial r}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\varphi\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\varphi}{\partial\varphi^2}+\frac{2}{r}\frac{\partial v_r}{\partial\varphi}+$$

$$\frac{\cot\theta}{r}\frac{\partial v_\theta}{\partial\varphi}\Big\}\hat{e}_\varphi \qquad (9)$$

new v_r

$$\nabla\times v=\Big\{\frac{v_\varphi\cot\theta}{r}+\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi}\Big\}\hat{e}_r$$

new v_θ

$$+\Big\{\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\varphi}-\frac{1}{r}\frac{\partial(rv_\varphi)}{\partial r}\Big\}\hat{e}_\theta$$

new v_φ

$$+\Big\{\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\Big\}\hat{e}_\varphi$$

$$\nabla\times(\nabla\times v)=\Big\{\frac{1}{r}\cot\theta\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)$$

$$-\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\varphi}-\frac{1}{r}\frac{\partial(rv_\varphi)}{\partial r}\right)\Big\}\hat{e}_r$$

$$+\Big\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{v_\varphi}{r}\cot\theta+\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi}\right)$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial(rv_\theta)}{\partial r}-\frac{\partial v_r}{\partial\theta}\right)\Big\}\hat{e}_\theta$$

$$+\Big\{\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi}-\frac{\partial(rv_\varphi)}{\partial r}\right)\right)$$

$$-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\varphi\cot\theta}{r}+\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi}\right)\Big\}\hat{e}_\varphi$$

(10)

$$\nabla^2\vec{v}=(9)-(10)$$