Polar decomposition / Introduction to Right and Left Cauchy-Green Tensors

Objective and Nonobjective tensors

Polar Decomposition Theorem:

Polar decomposition theorem: $F = RU$, $F = VR$

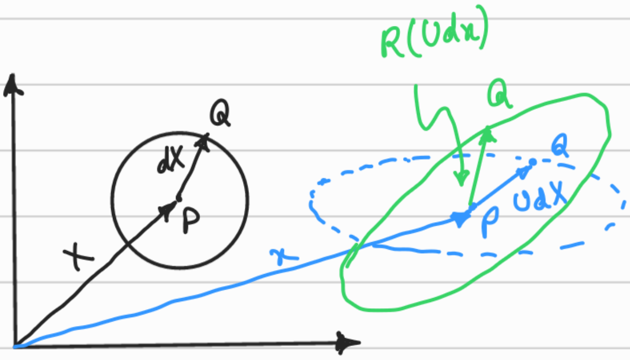
left stretch tensor

orthogonal tensor

right stretch tensor (symmetric positive definite)

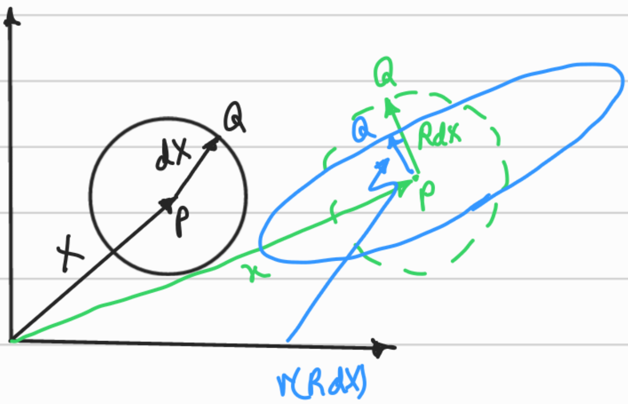
$F = RU$, $dx = F\,dX$

$dx = R(U\,dx)$ first stretching then rotation



$F = VR$, $dx = F\,dX$

$dx = V(R\,dX)$ first rotation then stretching



Correlation between U and V,

$$RU = VR \xrightarrow{\times R^T} \underbrace{R^T R}_{I} U = R^T V R \Rightarrow U = R^T V R$$

$$VR = RU \xrightarrow{\times R^T} V \underbrace{R R^T}_{I} = R U R^T \Rightarrow V = R U R^T$$

Introduction to the Right (C) and Left (B) Cauchy-Green Tensors:

Right Cauchy-Green tensor: $C = F^T F = (RU)^T (RU) = U^T \underbrace{R^T R}_{I} U = U^T U$

$U^T U \xrightarrow[\text{symmetric}]{U \text{ is}} UU \Rightarrow C = U^2 \quad (C \equiv U^2)$

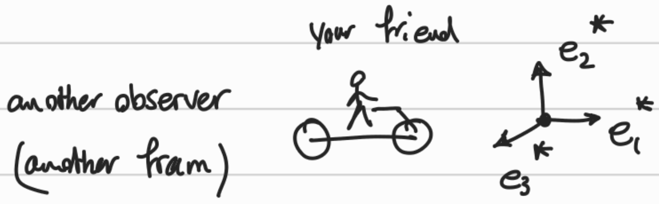
Left Cauchy-Green tensor: $B = F F^T = (VR)(VR)^T = V \underbrace{R R^T}_{I} V^T$

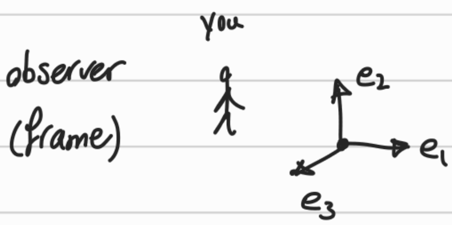
$= V V^T \xrightarrow[\text{symmetric}]{V \text{ is}} VV \Rightarrow B = V^2 \quad (B \equiv V^2)$

Objective vs. Nonobjective Tensors:

Part C of Chapter 5

Motivation: In continuum mechanics, the position of observer should not change the quantity of interest.

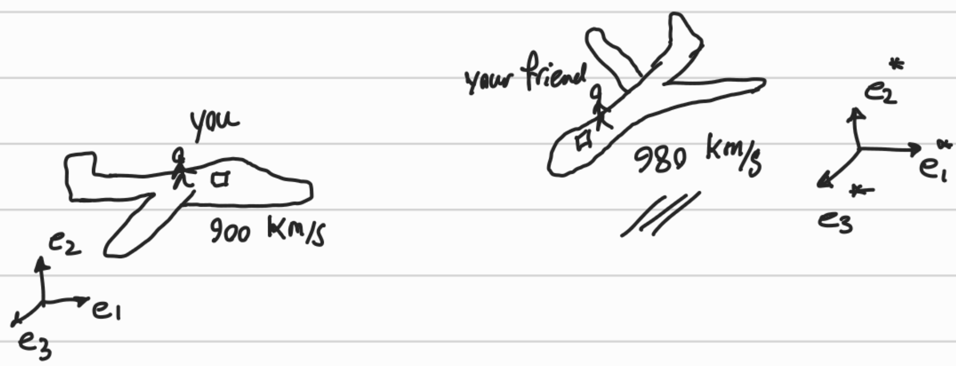




Both coordinates in both frames rotate and translate w.r.t each other.

An objective quantity is frame-independent and (there is one more condition)
e.g, distance, relative velocity

A nonobjective quantity is frame-dependent, e.g, position vector, velocity vector, speed



Change of frame is different from change of coordinate.

You can perform any number of coordinate transformation within your frame.

Rigid body motion should not affect the stress or strain tensor or mechanical properties of a material.

(there is one more condition) ?

observer : rigid body + clock

$$\{x, t\} \xrightarrow{\text{transformation}} \{x^*, t^*\}$$

from rigid body motion: $x = c(t) + R(t)(X - b)$

relative displacement

since we assume a right-hand basis vector, we can call this $Q(t)$

proper orthogonal tensor

$$\left.\begin{aligned} x_1^* &= c(t) + Q(t)(x_1 - x_0) \\ x_2^* &= c(t) + Q(t)(x_2 - x_0) \end{aligned}\right\} \underbrace{x_2^* - x_1^*}_{b^*} = Q(t)\underbrace{(x_2 - x_1)}_{b}$$

$$\boxed{b^* = Q(t)\, b}$$ objective vector

Now for a tensor, T

$c = Tb$ (3) and c, b are objective vectors $\left\{\begin{aligned} c^* &= Q(t)c \quad (1) \\ b^* &= Q(t)b \end{aligned}\right.$

$Q^T(t)b^* = b$ (2)

$c^* = T^* b^*$

$$c^* \overset{(1)}{=} Q(t)c \overset{(3)}{=} Q(t)Tb \overset{(2)}{=} Q(t)TQ^T(t)b^*$$

$$\boxed{c^* = T^* b^* \quad \text{where} \quad T = Q(t)\, T\, Q^T(t)}$$ objective tensor

(there is one more condition) = follow transformation law (for tensor)

> An objective quantity is frame-independent and follows transformation laws for tensors.

Example 5.56.1

dx is an objective vector

$$x^* = c(t) + Q(t)(x - x_0) \qquad (4)$$

$$x^* + dx^* = c(t) + Q(t)(x + dx - x_0) \qquad (5)$$

(5)-(4) : $\boxed{dx^* = Q(t)\, dx}$ $\therefore$ dx is an objective vector

$ds = |dx|$ is an objective scalar

$$(ds^*)^2 = dx^* \cdot dx^* = \underbrace{Q(t)\, dx}_{dx\, Q^T(t)} \cdot Q(t)\, dx = dx \cdot \underbrace{Q^T(t)\, Q(t)}_{I}\, dx = dx \cdot dx = (ds)^2$$

$\Rightarrow \boxed{ds^* = ds}$ $\therefore$ ds is an objective scalar.

Example 5.56.2 :

$$v^* = Q(t)\, v + \dot{Q}(t)(x - x_0) + \dot{c}(t)$$

$$x^* = c(t) + Q(t)(x - x_0) \xrightarrow{d/dt} \frac{dx^*}{dt} = \dot{c}(t) + \dot{Q}(t)(x - x_0) + Q(t)\frac{dx}{dt}$$

$\boxed{v^* = \dot{c}(t) + \dot{Q}(t)(x - x_0) + Q(t)\, v}$ $\therefore$ velocity vector is nonobjective

$$\nabla^* v^* = Q(t)(\nabla v)\, Q^T(t) + \dot{Q}\, Q^T$$

$$\nabla^* v^* = \frac{v^*(x^* + dx^*, t) - v^*(x^*, t)}{dx^*}$$

$$v^*(x^* + dx^*, t) = \dot{c}(t) + \dot{Q}(t)(x + dx - x_0) + Q(t)\, v(x + dx, t) \quad (6)$$

$$v^*(x^*, t) = \dot{c}(t) + \dot{Q}(t)(x - x_0) + Q(t)\, v(x, t) \quad (7)$$

$$\nabla^* v^* dx^* = (6) - (7) = \dot{Q}(t)\, dx + Q(t)\big[\underbrace{v(x + dx, t) - v(x, t)}_{(\nabla v) dx}\big]$$

$$\nabla^* v^* dx^* = \dot{Q}(t)\, dx + Q(t)\, (\nabla v) dx$$

$$\xrightarrow{dx^* = Q(t) dx} \nabla^* v^* Q(t)\, dx = \dot{Q}(t)\, dx + Q(t) \nabla v\, dx)$$

$$[\nabla^* v^* Q(t) - \dot{Q}(t) - Q(t) \nabla v]\, dx = 0$$

$$\nabla^* v^* Q(t) - \dot{Q}(t) - Q(t) \nabla v = 0 \xrightarrow{Q^T(t)} \nabla^* v^* - \dot{Q}(t) Q^T(t) - Q(t) \nabla v\, Q^T(t) = 0$$

$$\boxed{\nabla^* v^* = \dot{Q}(t) Q^T(t) + Q(t) \nabla v\, Q^T(t)}$$

$\therefore$ velocity gradient is nonobjective

Example 5.56.3

$$F^* = Q(t) F$$

$$dx = F\, dX$$

$$\left.\begin{array}{l} dx^* = F^* dX^* \\ dx^* = Q(t)\, dx \end{array}\right\} F^* dX^* = Q(t)\, dx$$

$$\Rightarrow F^* dX^* = Q(t) F\, dX \Rightarrow F^* = Q(t) F$$

Example 5.56.4

Transformation law for the Right Cauchy-Green tensor

$$C = F^T F \quad , \quad F^* = Q(t) F \quad (8)$$

$$C^* = F^{*T} F^*$$

$$C^* = F^{*T} F^* \overset{(8)}{=} (QF)^T (QF) = F^T \underbrace{Q^T Q}_{I} F = F^T F = C$$

$\boxed{C^* = C}$ $\therefore$ C is nonobjective

Transformation law for the Left Cauchy-Green tensor

$B = FF^T$, $F^* = Q(t) F$

$$B^* = F^* F^{*T}$$

$$B^* = F^* F^{*T} \overset{(8)}{=} (QF)(QF)^T = Q \underbrace{F F^T}_{B} Q^T = QBQ^T$$

$\boxed{B^* = QBQ^T}$ $\therefore$ B is objective