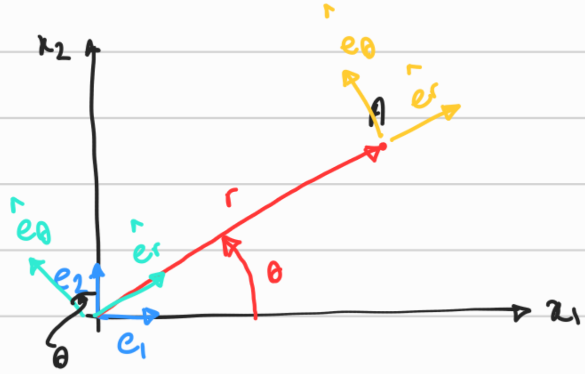
Introduction to Polar Coordinate System:



$$\hat{e}_r = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2 \xrightarrow{d} d\hat{e}_r = -\sin\theta\,d\theta\,\hat{e}_1 + \cos\theta\,d\theta\,\hat{e}_2$$

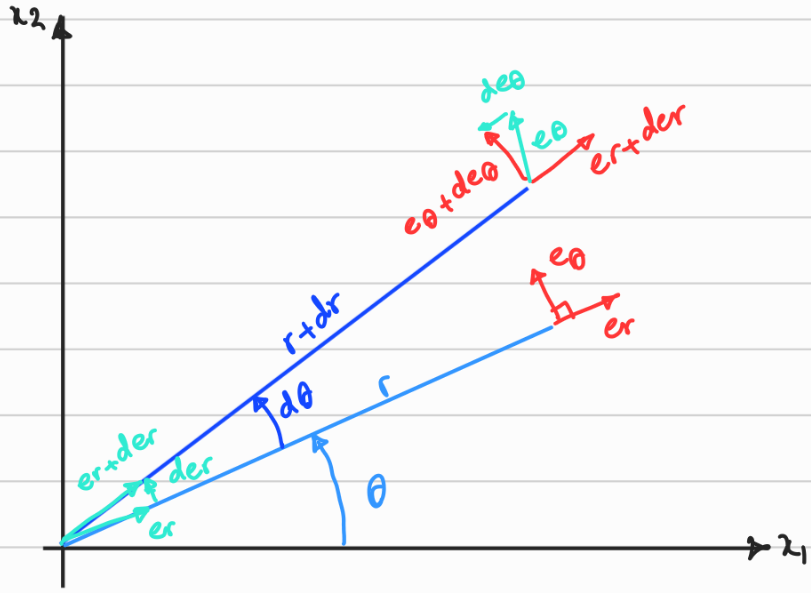
$$\hat{e}_\theta = -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2 \xrightarrow{d} d\hat{e}_\theta = -\cos\theta\,d\theta\,\hat{e}_1 - \sin\theta\,d\theta\,\hat{e}_2$$

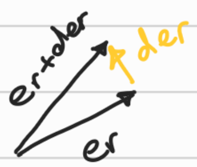
$$d\hat{e}_r = \underbrace{\left(-\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2\right)}_{e_\theta} d\theta = d\theta\,\hat{e}_\theta$$

$$(*)\begin{cases} d\hat{e}_r = d\theta\,\hat{e}_\theta \\ d\hat{e}_\theta = -d\theta\,\hat{e}_r \end{cases}$$

$$d\hat{e}_\theta = \underbrace{\left(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2\right)}_{\hat{e}_r}(-d\theta) = -d\theta\,\hat{e}_r$$

Geometrical Interpretation of $d\hat{e}_r$ and $d\hat{e}_\theta$:





Position Vector in Polar Coordinate System:

$$\vec{r} = r\,\hat{e}_r$$

$$d\vec{r} = dr\,\hat{e}_r + r\,\underbrace{d\hat{e}_r}_{d\theta\,\hat{e}_\theta} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta \quad (1)$$

Gradient of a Scalar Field (∇f):

Let $f(r,\theta)$ be a scalar field.

$$\frac{df}{dr} = \nabla f \Rightarrow df = \nabla f . dr \overset{(1)}{=} \underbrace{(a_r\,\hat{e}_r + a_\theta\,\hat{e}_\theta)}_{\text{assume}} \cdot (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta)$$

$$df = a_r\,dr + a_\theta\,r\,d\theta \quad (2)$$

$$f(r,\theta) \overset{d}{\Rightarrow} df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta \quad (3)$$

Comparing (2) and (3)
Based on the fundamental theorem in Algebra:

$$\begin{cases} a_r = \dfrac{\partial f}{\partial r} \\ a_\theta\,r = \dfrac{\partial f}{\partial \theta} \end{cases} \Rightarrow \begin{cases} a_r = \dfrac{\partial f}{\partial r} \\ a_\theta = \dfrac{1}{r}\dfrac{\partial f}{\partial \theta} \end{cases}$$

$$\therefore \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta$$

Gradient of a Vector Field:

Let $v(r,\theta) = v_r(r,\theta)\,\hat{e}_r + v_\theta(r,\theta)\,\hat{e}_\theta$

$$\frac{dv}{dr} = \nabla v \Rightarrow dv = \underbrace{\nabla v}_{\text{2nd-order tensor}}\,dr \longrightarrow dv = T\,dr \overset{(1)}{=} T(dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta)$$

$$= dr\,(T\hat{e}_r) + r\,d\theta\,(T\hat{e}_\theta) \quad (6)$$

$$T\hat{e}_i = T_{ji}\,\hat{e}_j \quad , T\hat{e}_r = T_{jr}\,\hat{e}_j \xrightarrow{j=r,\theta} T\hat{e}_r = T_{rr}\,\hat{e}_r + T_{\theta r}\,\hat{e}_\theta \quad (4)$$

$$T\hat{e}_\theta = T_{j\theta}\,\hat{e}_j \xrightarrow{j=r,\theta} T\hat{e}_\theta = T_{r\theta}\,\hat{e}_r + T_{\theta\theta}\,\hat{e}_\theta \quad (5)$$

The above expression can also be achieved using

$$\{ T\hat{e_r} \quad T\hat{e_\theta} \} = \{ \hat{e_r} \quad \hat{e_\theta} \} \begin{bmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{bmatrix} = \{ T_{rr}\hat{e_r} + T_{\theta r}\hat{e_\theta} \quad T_{r\theta}\hat{e_r} + T_{\theta\theta}\hat{e_\theta} \}$$

④ and ⑤ in ⑥ : $dv = dr(T_{rr}\hat{e_r} + T_{\theta r}\hat{e_\theta}) + rd\theta(T_{r\theta}\hat{e_r} + T_{\theta\theta}\hat{e_\theta})$

$$= dr T_{rr}\hat{e_r} + dr T_{\theta r}\hat{e_\theta} + rd\theta T_{r\theta}\hat{e_r} + rd\theta T_{\theta\theta}\hat{e_\theta}$$

$$dv = (dr T_{rr} + rd\theta T_{r\theta})\hat{e_r} + (dr T_{\theta r} + rd\theta T_{\theta\theta})\hat{e_\theta} \quad ⑨$$

Also $v(r,\theta) = v_r(r,\theta)\hat{e_r} + v_\theta(r,\theta)\hat{e_\theta}$

$$dv(r,\theta) = dv_r(r,\theta)\hat{e_r} + v_r(r,\theta)d\hat{e_r} + dv_\theta(r,\theta)\hat{e_\theta} + v_\theta(r,\theta)d\hat{e_\theta}$$

$$dv = dv_r\hat{e_r} + v_r d\hat{e_r} + dv_\theta\hat{e_\theta} + v_\theta d\hat{e_\theta}$$

$$\overset{(*)}{=} dv_r\hat{e_r} + v_r d\theta\hat{e_\theta} + dv_\theta\hat{e_\theta} + v_\theta(-d\theta\hat{e_r})$$

$$dv = (dv_r - v_\theta d\theta)\hat{e_r} + (v_r d\theta + dv_\theta)\hat{e_\theta} \quad ⑧$$

$$\left\{ \begin{array}{l} v_r(r,\theta) \xrightarrow{d} dv_r = \frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial \theta}d\theta \\ v_\theta(r,\theta) \xrightarrow{d} dv_\theta = \frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial \theta}d\theta \end{array} \right. \quad ⑦$$

⑦ in ⑧ : $dv = \left(\frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial \theta}d\theta - v_\theta d\theta\right)\hat{e_r}$

$$+ \left(v_r d\theta + \frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial \theta}d\theta\right)\hat{e_\theta} \quad ⑩$$

$$dv = (dr T_{rr} + rd\theta T_{r\theta})\hat{e_r} + (dr T_{\theta r} + rd\theta T_{\theta\theta})\hat{e_\theta} \quad ⑨$$

compare (9) and (10) : $T_{rr} = \frac{\partial v_r}{\partial r}$

$$r\,T_{r\theta} = \frac{\partial v_r}{\partial \theta} - v_\theta \rightarrow T_{r\theta} = \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)$$

$$T_{\theta r} = \frac{\partial v_\theta}{\partial r}$$

$$r\,T_{\theta\theta} = v_r + \frac{\partial v_\theta}{\partial \theta} \Rightarrow T_{\theta\theta} = \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right)$$

$$\nabla v = \begin{bmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) \end{bmatrix}$$

Divergence of a Vector Field:

$$\operatorname{div} \vec{v} = \operatorname{tr}(\nabla v)$$

$$\operatorname{div} \vec{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right)$$

Curl of a Vector Field:

$$\operatorname{curl} \vec{v} = 2t^A$$

$$T^A = \frac{T - T^T}{2} \quad : \quad [\nabla v]^A = \frac{[\nabla v] - [\nabla v]^T}{2} \Rightarrow$$

$$\frac{1}{2}\left\{\begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) \end{bmatrix} - \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} \\ \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) - \frac{\partial v_\theta}{\partial r} \\ \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & 0 \end{bmatrix}$$

$$\operatorname{curl} \vec{v} = 2t^A = -2\left(\underbrace{\cancel{T_{23}}}_{0}\,\hat{e}_1 + \underbrace{\cancel{T_{31}}}_{0}\,\hat{e}_2 + T_{12}\,\hat{e}_3\right)$$

$$= -T_{r\theta}\,\hat{e}_3 = -\left\{\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) - \frac{\partial v_\theta}{\partial r}\right\}\hat{e}_3$$

$$\operatorname{curl}\vec{v} = \left\{\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right\}\hat{e}_3$$

Divergence of a Tensor Field:

$$(\operatorname{div} T)\cdot a = \operatorname{div}(T^T a) - \operatorname{tr}((\nabla a)T^T) \quad \forall \vec{a}$$

$(\operatorname{div} T)_r$: Let's assume $\vec{a} = \hat{e}_r$

$$(\operatorname{div} T)_r = \operatorname{div}(T^T \hat{e}_r) - \operatorname{tr}((\nabla \hat{e}_r)\, T^T)$$

$$T\hat{e}_r = T_{jr}\,\hat{e}_j = T_{rr}\,\hat{e}_r + T_{\theta r}\,\hat{e}_\theta$$

$$T^T\hat{e}_r = T_{rj}\,\hat{e}_j = T_{rr}\,\hat{e}_r + T_{r\theta}\,\hat{e}_\theta$$

$$\operatorname{div}(T^T\hat{e}_r) = \operatorname{div}(\underbrace{T_{rr}}_{v_r}\hat{e}_r + \underbrace{T_{r\theta}}_{v_\theta}\hat{e}_\theta)$$

$$\operatorname{div}\vec{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right)$$

$$\operatorname{div}(T^T\hat{e}_r) = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(T_{rr} + \frac{\partial T_{r\theta}}{\partial \theta}\right)$$

$\nabla\hat{e}_r$: $\hat{e}_r = \underbrace{(1)}_{v_r}\hat{e}_r + \underbrace{(0)}_{v_\theta}\hat{e}_\theta$

$$\nabla\vec{v} = \begin{bmatrix} \cancel{\frac{\partial v_r}{\partial r}}\,0 & \frac{1}{r}\left(\cancel{\frac{\partial v_r}{\partial \theta}}^{0} - \cancel{v_\theta}^{0}\right) \\ \cancel{\frac{\partial v_\theta}{\partial r}}\,0 & \frac{1}{r}\left(v_r + \cancel{\frac{\partial v_\theta}{\partial \theta}}_{0}\right) \end{bmatrix}$$

$$\nabla\vec{e}_r = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix}$$

$$\nabla\hat{e}_r\, T^T = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \begin{bmatrix} T_{rr} & T_{\theta r} \\ T_{r\theta} & T_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{r} T_{r\theta} & \frac{1}{r} T_{\theta\theta} \end{bmatrix}$$

$$tr\left(\nabla\hat{e}_r T^T\right) = \frac{1}{r} T_{\theta\theta}$$

$$\left(\operatorname{div} T\right)_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(T_{rr} + \frac{\partial T_{r\theta}}{\partial\theta}\right) - \frac{1}{r} T_{\theta\theta}$$

$$\left(\operatorname{div} T\right)_\theta = \operatorname{div}\left(T^T \hat{e}_\theta\right) - tr\left(\left(\nabla\hat{e}_\theta\right) T^T\right)$$

$$T\hat{e}_\theta = T_{j\theta}\, \hat{e}_j$$

$$T^T e_\theta = T_{\theta j}\, \hat{e}_j = \underbrace{T_{\theta r}}_{v_r} \hat{e}_r + \underbrace{T_{\theta\theta}}_{v_\theta} \hat{e}_\theta$$

$$\operatorname{div}\left(T^T e_\theta\right) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial\theta} + T_{\theta r}\right)$$

$$\hat{e}_\theta = \underbrace{0}_{v_r}\, \hat{e}_r + \underbrace{(1)}_{v_\theta}\, \hat{e}_\theta$$

$$\nabla\hat{e}_\theta = \begin{bmatrix} 0 & -\frac{1}{r} \\ 0 & 0 \end{bmatrix}$$

$$\nabla\hat{e}_\theta\, T^T = \begin{bmatrix} 0 & -\frac{1}{r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{rr} & T_{\theta r} \\ T_{r\theta} & T_{\theta\theta} \end{bmatrix} = \begin{bmatrix} -\frac{1}{r} T_{r\theta} & -\frac{1}{r} T_{\theta\theta} \\ 0 & 0 \end{bmatrix}$$

$$tr\left(\nabla\hat{e}_\theta T^T\right) = -\frac{1}{r} T_{r\theta}$$

$$\left(\operatorname{div} T\right)_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial\theta} + T_{\theta r}\right) + \frac{1}{r} T_{r\theta} \qquad Eq\ (2.33.33)$$

Laplacian of a Scalar Field:

$$\nabla^2 f = \nabla\cdot(\nabla f) = \operatorname{div}(\nabla f) \quad , \; f(r,\theta)$$

$$\nabla f = \underbrace{\frac{\partial f}{\partial r}}_{v_r}\hat{e}_r + \underbrace{\frac{1}{r}\frac{\partial f}{\partial \theta}}_{v_\theta}\hat{e}_\theta$$

$$\operatorname{div}\vec{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

$$\operatorname{div}(\nabla f) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{1}{r}\frac{\partial f}{\partial r}$$

$$\operatorname{div}(\nabla f) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}$$

Laplacian of a Vector Field:

$$\nabla^2 v = \nabla(\nabla\cdot v) - \nabla\times(\nabla\times v)$$

$$\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = \underbrace{\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}}_{f}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta \quad \Longrightarrow$$

$$\nabla(\nabla\cdot v) = \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\hat{e}_r$$
$$+ \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\hat{e}_\theta$$
$$= \left(\frac{\partial^2 v_r}{\partial r^2} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial \theta} - \frac{v_r}{r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r}\right)\hat{e}_r$$
$$+ \frac{1}{r}\left(\frac{\partial^2 v_r}{\partial \theta\partial r} + \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{e}_\theta$$

$$\nabla\times v = \underbrace{\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)}_{v_z}\hat{e}_{\cancel{\theta}\,z}$$

15

$\nabla\times(\nabla\times v)$ Since $\hat{e}_3$ is now involved in calculating $\nabla\times(\nabla\times v)$, we borrow Eq (2.34.7) from the Cylindrical Coordinate System.

$$\nabla\times(v) = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \cancelto{0}{\frac{\partial v_\theta}{\partial z}}\right)\hat{e}_r + \left(\cancelto{0}{\frac{\partial v_r}{\partial z}} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\theta$$
$$+ \left(\cancelto{0}{\frac{\partial v_\theta}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}} - \frac{1}{r}\cancelto{0}{\frac{\partial v_r}{\partial\theta}}\right)\hat{e}_z \qquad (16)$$

Substituting (15) in (16)

$$\nabla\times(\nabla\times v) = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\hat{e}_r$$
$$- \frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\hat{e}_\theta$$
$$= \frac{1}{r}\left(\frac{\partial^2 v_\theta}{\partial\theta\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} - \frac{1}{r}\frac{\partial^2 v_r}{\partial\theta^2}\right)\hat{e}_r$$
$$+\left(-\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}v_\theta - \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{1}{r^2}\frac{\partial v_r}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}\right)\hat{e}_\theta$$

$(\nabla v)_r$:

$$\left(\frac{\partial^2 v_r}{\partial r^2} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta} + \cancel{\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}} - \frac{v_r}{r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r}\right)$$
$$\cancel{-\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial r}} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\Bigg)\hat{e}_r$$

$$(\nabla v)_r = \frac{\partial^2 v_r}{\partial r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta} \qquad \text{Eq}(2.33.40)$$

N.B. In Eq (2.33.40), $\frac{\partial^2 v_r}{\partial z^2}$ should be removed.

$$(\nabla v)_\theta = \left(\cancel{\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right.$$

$$+ \frac{\partial^2 v_\theta}{\partial r^2} - \frac{1}{r^2} v_\theta + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial v_r}{\partial \theta} - \cancel{\frac{1}{r} \frac{\partial^2 v_r}{\partial r \partial \theta}} \Big) \hat{e}_\theta$$

$$(\nabla v)_\theta = \frac{2}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial r^2} - \frac{1}{r^2} v_\theta + \frac{1}{r} \frac{\partial v_\theta}{\partial r}$$

Eq(2.33.41)