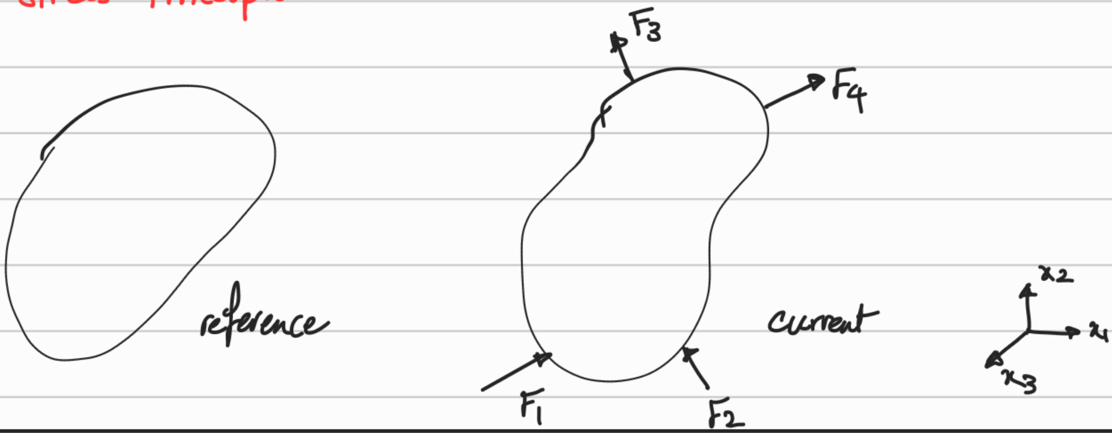
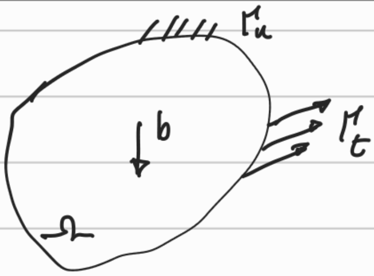
Cauchy Stress Principle $\begin{cases} \text{Cauchy stress vector} \\ \text{Cauchy stress tensor} \end{cases}$

Cauchy Stress Principle:

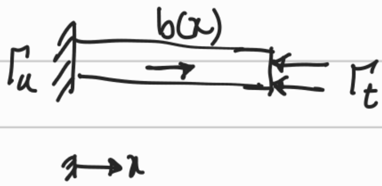


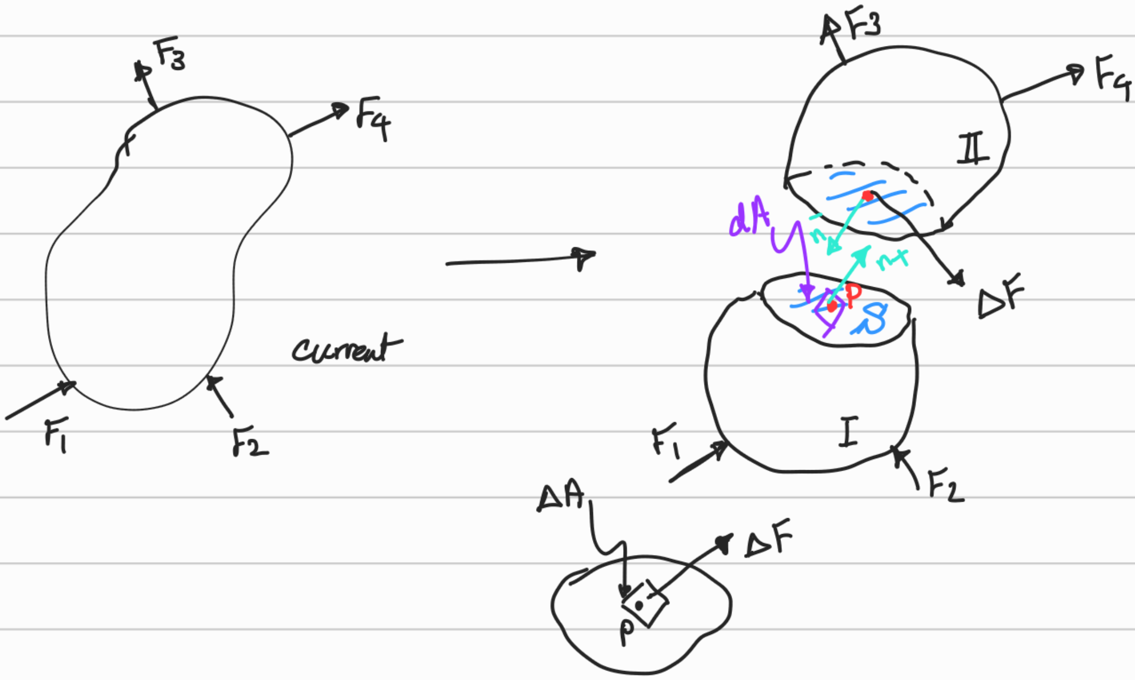


b: body force

Γ_t: surface force (Natural B.C)

Γ_u: B.C (displacement) (Essential/Geometrical B.C)





Cauchy Stress Principle (Assumption)

Body ~~forces~~ 0 + surface forces

↳ can be characterized via the concept of stress vector

Part I : $t_{n^+} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$

Part II : $t_{n^-} = \lim_{\Delta A \to 0} \frac{-\Delta F}{\Delta A}$

Newton's 3rd law $\Rightarrow t_{n^+} = -t_{n^-}$

$$\boxed{\text{Cauchy stress vector: } t = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}}$$

Cauchy stress Principle:

stress vector ($\boldsymbol{t}$) (traction) $\begin{cases} x \text{ (position)} \\ \text{time} \end{cases}$ has same values where

1. tangent plane to point P (normal $\vec{n}$)
2. on one side of plane

from Cauchy stress Principle we conclude that stress (t) vector is a function of :

$$\boldsymbol{t} = \boldsymbol{t}(x, t, n) \quad \text{(deformed config.)}$$

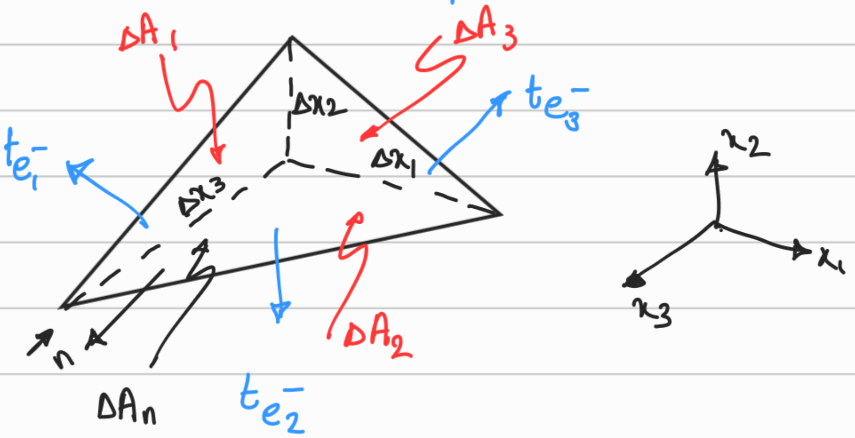
$$\boldsymbol{t} = T(x,t)\,\boldsymbol{n}$$

stress vector, linear transformation, normal vector

T is a second-order tensor which is a function of position and time.

It is called Cauchy Stress tensor. $T_{ij} = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

T is a linear transformation: (proof)



$$\sum F = ma \Rightarrow t_{e_1^-}(\Delta x_2 \Delta x_3)/2 + t_{e_2^-}(\Delta x_3 \Delta x_1)/2$$

$$+ t_{e_3^-}(\Delta x_2 \Delta x_1)/2 + t_{e_n} \Delta A_n$$

$$= \rho\left(\frac{1}{6} \Delta x_1 \Delta x_2 \cancel{\Delta x_3}\right) a = 0$$

$$\left.\begin{matrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{matrix}\right\} \to 0 \quad \therefore \frac{1}{6} \Delta x_1 \Delta x_2 \Delta x_3 \longrightarrow 0$$

$$t_{e_1^-} \Delta A_1 + t_{e_2^-} \Delta A_2 + t_{e_3^-} \Delta A_3 + t_{e_n} \Delta A_n = 0 \quad (1)$$

$$\vec{n} = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3$$

$$n_1 = \frac{\Delta A_1}{\Delta A_n}, \quad n_2 = \frac{\Delta A_2}{\Delta A_n} \quad n_3 = \frac{\Delta A_3}{\Delta A_n} \quad (2)$$

(2) in (1) $t_{e_1^-}(n_1 \Delta A_n) + t_{e_2^-}(n_2 \Delta A_n) + t_{e_3^-}(n_3 \Delta A_n) + t_{e_n} \Delta A_n = 0$

$$t_{e_1^-} n_1 + t_{e_2^-} n_2 + t_{e_3^-} n_3 + t_{e_n} = 0$$

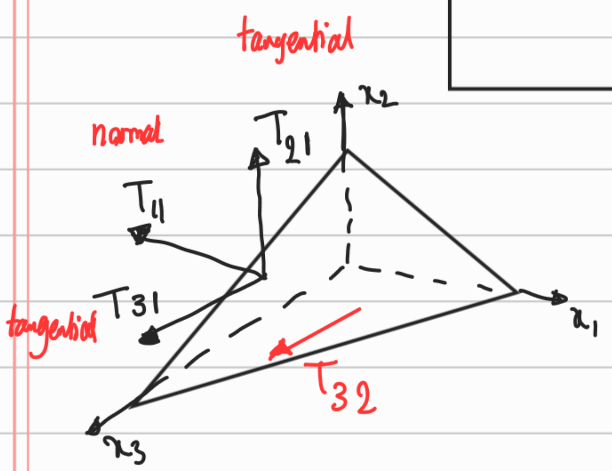
$$\xrightarrow[\vdots]{t_{e_i^-} = -t_{e_1}} \quad -t_{e_1} n_1 - t_{e_2} n_2 - t_{e_3} n_3 + t_{e_n} = 0$$

$$\begin{cases} t_{e_n} = t_{e_1} n_1 + t_{e_2} n_2 + t_{e_3} n_3 & (3) \\ T n = n_1 T e_1 + n_2 T \hat{e}_2 + n_3 T \hat{e}_3 & (4) \end{cases}$$

comparing (3) and (4)

$$\begin{cases} t_n = T n & \therefore T \text{ is linear transformation} \\ t_{e_1} = T e_1 = T_{j1} e_j = T_{11} e_1 + T_{21} e_2 + T_{31} e_3 \\ t_{e_2} = T e_2 = T_{j2} e_j = T_{12} e_1 + T_{22} e_2 + T_{23} e_3 \\ t_{e_3} = T e_3 = T_{j3} e_j = T_{13} e_1 + T_{23} e_2 + T_{33} e_3 \end{cases}$$

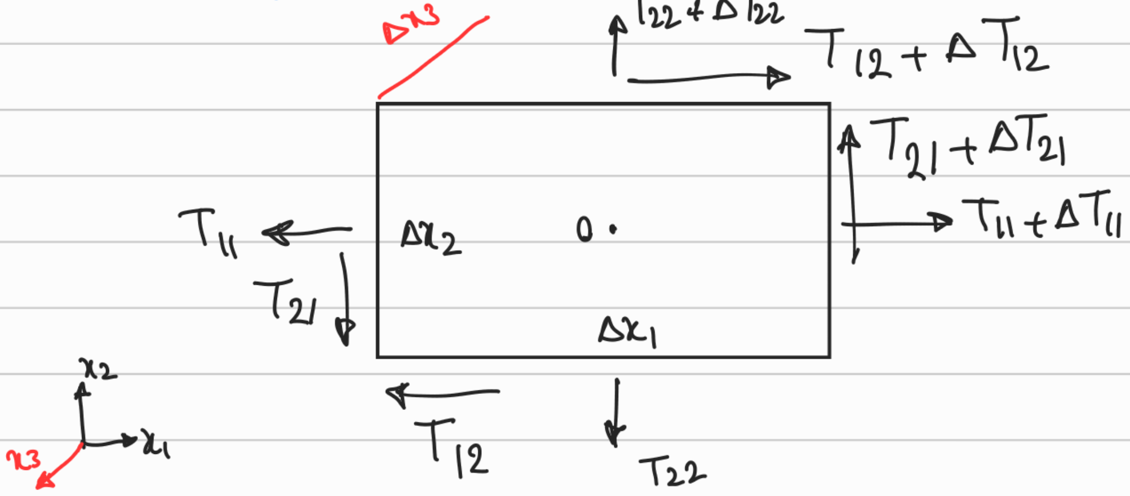
$t = T n$	tensorial notation	
$t_i = T_{ij} n_j$	indicial	"
$\{t\} = [T]\{n\}$	matrix	"



T_{31} ← normal

direction

Stress tensor is symmetric: (proof)



$$\sum (M_0)_3 = (T_{21}+\Delta T_{21})(\Delta x_3 \Delta x_2)\left(\frac{\Delta x_1}{2}\right) + T_{21}(\Delta x_2 \Delta x_3)\left(\frac{\Delta x_1}{2}\right)$$

$$- (T_{12}+\Delta T_{12})(\Delta x_3 \Delta x_1)\left(\frac{\Delta x_2}{2}\right) - T_{12}(\Delta x_3 \Delta x_1)\left(\frac{\Delta x_2}{2}\right)$$

$$= \cancel{I_3}^{\to 0}\, \alpha = 0$$

where $I_3 = \frac{1}{12} m (\Delta x_1^2 + \Delta x_2^2) = \frac{1}{12} \rho (\Delta x_1 \Delta x_2 \Delta x_3)(\Delta x_1^2 + \Delta x_2^2)$

$$\left.\begin{matrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{matrix}\right\} \to 0 \quad \therefore \quad I \simeq 0$$

let us neglect higher order terms like $\Delta T_{12} \Delta x_1 \Delta x_2 \Delta x_3 \cong 0$

$$(T_{21} - T_{12})(\Delta x_1 \Delta x_2 \Delta x_3) = 0 \Rightarrow T_{21} = T_{12}$$

Following the same strategy; $T_{13} = T_{31}$, $T_{23} = T_{32}$

therefore; T is a symmetric tensor.

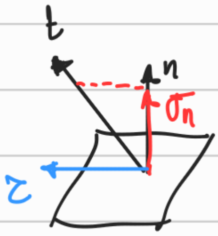
Example 4.4.2:

$$T = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix} \text{MPa}$$

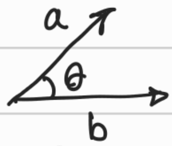
plane : $x_1 + 2x_2 + 2x_3 - 6 = 0$

$$\vec{n} = \nabla f = \frac{1}{3}(1, 2, 2)$$

$$t = \frac{1}{3}\begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3}\begin{Bmatrix} 16 \\ 4 \\ 1 \end{Bmatrix}$$



$\sigma_n = t \cdot n$



$a \cdot b = |a||b|\cos\theta$

$$\sigma_n = t \cdot n = n \cdot t = n \cdot Tn$$

$$\sigma_n = \frac{1}{9}\begin{Bmatrix} 16 \\ 4 \\ 1 \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{9}(16 + 8 + 2) = 2.89 \text{ MPa}$$

$$\tau = t - \sigma_n n = \frac{1}{3}\begin{Bmatrix} 16 \\ 4 \\ 1 \end{Bmatrix} - \frac{2.89}{3}\begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}$$