What is a tensor? Tensor vs. Matrix

Tensor Transformation

Vector Transformation

What is a tensor?

A tensor is a mathematical object/structure which explains a physical property that follows certain transformation. A tensor has a dimension and a rank.

$$\sigma = \begin{array}{c c} \begin{array}{ccc} x & y & z \end{array} & \\ \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

dimension 3 rank 2

$$\text{plane stress, } \sigma = \begin{array}{c c} \begin{array}{cc} x & y \end{array} & \\ \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} & \begin{array}{c} x \\ y \end{array} \end{array}$$

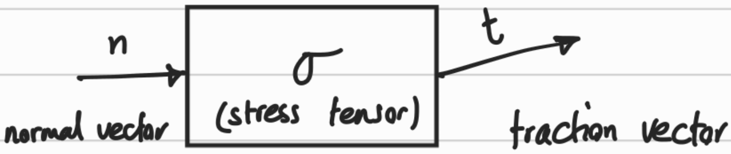
dimension 2 rank 2

Why certain transformation?

tensor blackbox

A tensor is defined in terms of its actions on a vector.

$\sigma n = t$:



Linear Transformation:
$$\begin{cases} T(a+b) = Ta + Tb \\ T(\alpha b) = \alpha Tb \end{cases}$$

T: 2nd-order tensor

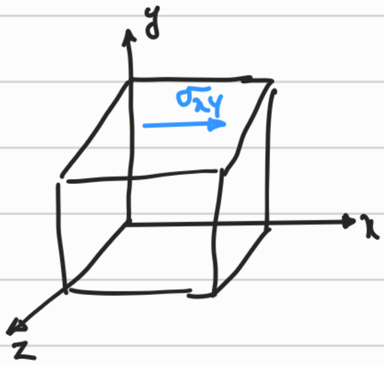
a, b: 1st-order // vector

α: zero-order // scalar

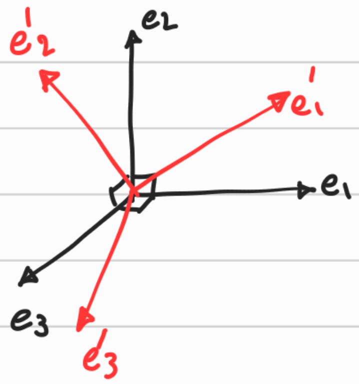
1st-order tensor

$\vec{v}$ $\longrightarrow$ $\quad v = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$

2nd-order tensor



Tensor Transformation:

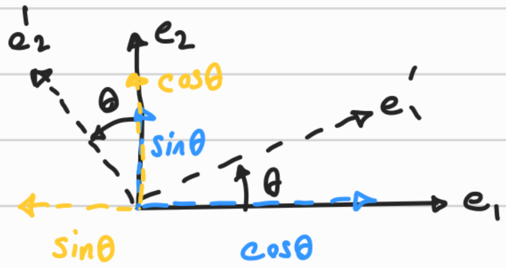


$$[e'_1 \quad e'_2 \quad e'_3] = [e_1 \quad e_2 \quad e_3] \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

Eq (2.7.1)
$$\begin{cases} Te_1 = e'_1 = T_{11}e_1 + T_{21}e_2 + T_{31}e_3 \\ Te_2 = e'_2 = T_{12}e_1 + T_{22}e_2 + T_{32}e_3 \\ Te_3 = e'_3 = T_{13}e_1 + T_{23}e_2 + T_{33}e_3 \end{cases}$$

$$e'_i = T_{ji}\, e_j = Te_i$$

Example 2.7.3



$$e_1' = \cos\theta\, e_1 + \sin\theta\, e_2 + 0\, e_3$$

$$e_2' = -\sin\theta\, e_1 + \cos\theta\, e_2 + 0\, e_3$$

$$e_3' = 0\, e_1 + 0\, e_2 + 1\, e_3$$

$$[e_1' \quad e_2' \quad e_3'] = [e_1 \quad e_2 \quad e_3] \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{11} = e_1 \cdot Te_1 \qquad T_{12} = e_1 \cdot Te_2 \qquad T_{32} = e_3 \cdot Te_2$$

$$T_{ij} = e_i \cdot Te_j \qquad T_{ij}' = e_i' \cdot Te_j'$$

Vector Transformation:

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3$$

$$\mathbf{b} = \mathbf{T}\mathbf{a} = T(a_1 e_1 + a_2 e_2 + a_3 e_3)$$

$$b_1 = \mathbf{b} \cdot e_1 = (b_1 e_1 + b_2 e_2 + b_3 e_3) \cdot e_1 = e_1 \cdot \underbrace{(b_1 e_1 + b_2 e_2 + b_3 e_3)}_{\mathbf{b}}$$

$$= e_1 \cdot T(a_1 e_1 + a_2 e_2 + a_3 e_3) = a_1 (e_1 \cdot Te_1) + a_2 (e_1 \cdot Te_2) + a_3 (e_1 \cdot Te_3)$$

$$= a_1 T_{11} + a_2 T_{12} + a_3 T_{13}$$

$$b_2 = a_1 T_{21} + a_2 T_{22} + a_3 T_{23}$$

$$b_3 = a_1 T_{31} + a_2 T_{32} + a_3 T_{33}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$b_m = \mathbf{b} \cdot e_m = (\mathbf{T}\mathbf{a}) \cdot e_m = \mathbf{T}(a_i e_i) \cdot e_m = a_i \underbrace{\mathbf{T} e_i}_{T_{ji} e_j} \cdot e_m$$

$$= a_i T_{ji} \underbrace{e_j \cdot e_m}_{\delta_{jm}} = a_i T_{mi} = b_m$$

$$[b] = [T][a]$$

Let us rewrite the above expression in another way by substituting

$$T e_i = T_{ij} e_j$$

$$a_i T e_i \cdot e_m = a_i T_{ij} \underbrace{e_j \cdot e_m}_{\delta_{jm}} = a_i T_{im} = \underbrace{T_{im}} a_i = b_m$$

T_{im} is the transpose of T_{mi}

Thus in matrix form : $[b] = [T]^T [a]$

The reason we chose the book notation $(T e_i = e_i' = T_{ji} e_j)$ is to be consistent in all chapter and simply write $[b] = [T][a]$.