Rate of Change of a Material Element:

$$dx = x(X+dX,t) - x(X,t)$$

$$\xrightarrow{\frac{D}{Dt}} \quad \frac{D(dx)}{Dt} = \frac{D\,x(X+dX,t)}{Dt} - \frac{D\,x(X,t)}{Dt}$$

$$\frac{D(dx)}{Dt} = \hat{v}(X+dX,t) - \hat{v}(X,t) = \tilde{v}(x+dx,t) - \tilde{v}(x,t)$$

$$\boxed{\frac{D(dx)}{Dt} = (\nabla_X \hat{v})\,dX = (\nabla_x \tilde{v})\,dx}$$

Spatial coordinate: $\dfrac{D(dx)}{Dt} = (\nabla v)\,dx \quad (\boxtimes)$

$$\nabla v = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \quad \text{(Cartesian Coordinate)}$$

velocity gradient

$$\nabla v = D + W$$

spin tensor $W = \frac{1}{2}\left(\frac{\nabla v - (\nabla v)^T}{2}\right)$

rate of deformation tensor $D = \frac{1}{2}\left(\frac{\nabla v + (\nabla v)^T}{2}\right)$

Representation of Rate of Deformation Tensor, D:

Diagonals of D:

$$\vec{dx} = ds\,\hat{n} \qquad dx\cdot dx = ds\,n\cdot ds\,n = ds^2\,n\cdot n = ds^2$$

$$\therefore\ dx\cdot dx = ds^2$$

$$\xrightarrow{D/Dt} \quad \frac{D(dx)}{Dt}\cdot dx + dx\cdot\frac{D(dx)}{Dt} = 2\,ds\,\frac{D(ds)}{Dt}$$

$$dx\cdot\frac{D(dx)}{Dt} = ds\,\frac{D(ds)}{Dt} \qquad (*)$$

Next, $dx\cdot\dfrac{D(dx)}{Dt} \overset{(*)}{=} dx\cdot(\nabla v\, dx) = dx\cdot[(D+W)\,dx]$

$$= dx\cdot D\,dx + \cancelto{0}{dx\cdot W\,dx} = dx\cdot D\,dx$$

$$dx\cdot W\,dx \xrightarrow[\substack{\text{antisymmetric}\\ W=-W^T}]{W\text{ is}} dx\cdot W\,dx = -\,dx\cdot W^T dx$$

$$dx\cdot W\,dx + dx\cdot W^T dx = 0$$

$$dx\cdot(\cancelto{0}{W+W^T})\,dx = 0$$

$$(*)\quad ds\,\frac{D(ds)}{Dt} = dx\cdot D\,dx \xrightarrow{dx=ds\,n} \boxed{\frac{1}{ds}\,\frac{D(ds)}{Dt} = n\cdot Dn}$$

D_{11} is the rate of extension of an element in e_1-direction.

D_{22} " " " " " " " " e_2 "

D_{33} " " " " " " " " e_3 "

off-diagonals of D:

Assumption: dX^1 and dX^2 are initially orthogonal.

$$\begin{cases} dx^1 = ds_1\,\hat{n} \\ dx^2 = ds_2\,\hat{m}\end{cases} \qquad dx^1\cdot dx^2 = (ds_1\,\hat{n})\cdot(ds_2\,\hat{m})$$

$$\xrightarrow{D/Dt} \frac{D(dx^1)}{Dt}\cdot dx^2 + dx^1\cdot\frac{D(dx^2)}{Dt} = \underbrace{\frac{D}{Dt}\left(ds_1\,ds_2\,\hat{n}\cdot\hat{m}\right)}_{\text{RHS}}$$

$$\frac{D(dx)}{Dt} = (\nabla v)dx \longrightarrow (\nabla v)\, dx^1 \cdot dx^2 + dx^1 \cdot (\nabla v)\, dx^2 = RHS$$

$$dx^1 \cdot (\nabla v)^T dx^2 + dx^1 \cdot (\nabla v)\, dx^2 = RHS$$

$$dx^1 \cdot \underbrace{\left((\nabla v)^T + \nabla v\right)}_{2D} \cdot dx^2 = RHS$$

$$2\, dx^1 \cdot D\, dx^2 = \frac{D}{Dt}\left(ds_1 ds_2\, \hat{n}\cdot\hat{m}\right) \quad \leftarrow \overset{1}{\cancel{|n|}}\,\overset{1}{\cancel{|m|}}\cos\theta$$

$$2\, dx^1 \cdot D dx^2 = \frac{D}{Dt}\left(ds_1 ds_2 \cos\theta\right)$$

$$2\, dx^1 \cdot D\, dx^2 = \frac{D(ds_1)}{Dt} ds_2 \cos\theta + ds_1 \frac{D(ds_2)}{Dt}\cos\theta + ds_1 ds_2 \cancel{\frac{D(\cos\theta)}{Dt}}$$

$$-\sin\theta \frac{D\theta}{Dt}$$

$$\xrightarrow[dx^2 = ds_2 \hat{m}]{dx^1 = ds_1 \hat{n}} 2\, ds_1 ds_2 (n \cdot Dm) = \frac{D(ds_1)}{Dt} ds_2 \cos\theta + ds_1 \frac{D(ds_2)}{Dt}\cos\theta - ds_1 ds_2 \sin\theta \frac{D\theta}{Dt}$$

$$\xrightarrow{\frac{1}{ds_1 ds_2}} 2(n\cdot Dm) = \frac{1}{ds_1}\frac{D(ds_1)}{Dt}\cos\theta + \frac{1}{ds_2}\frac{D(ds_2)}{Dt}\cos\theta - \sin\theta\frac{D\theta}{Dt}$$

$$\boxed{2(n\cdot Dm) = \left\{\frac{1}{ds_1}\frac{D(ds_1)}{Dt} + \frac{1}{ds_2}\frac{D(ds_2)}{Dt}\right\}\cos\theta - \sin\theta\frac{D\theta}{Dt}}$$

Proof for $n \cdot Dm = m \cdot Dn$

$$n \cdot Dm = Dm \cdot n = m \cdot D^T n \xrightarrow[\text{is symmetric}]{D} m \cdot Dn$$

$2D_{12}$ is the rate of decrease of angle (from $\frac{\pi}{2}$) of two element in e_1 and e_2-directions.

$2D_{13}$ " " " " " " " " " " " " " e_1 "
e_3 "

$2D_{23}$ " " " " " " " " " " " " " e_2 "
e_3 "

Principal values and Directions of D:

$$D_{11} + D_{22} + D_{33} = \frac{1}{dV}\frac{D\,dV}{Dt} \quad \text{where } V \text{ is volume}$$

$$\frac{1}{dV}\frac{D\,dV}{Dt} = \frac{\partial v_i}{\partial x_i} = \operatorname{div} \vec{v} \qquad (\blacktriangle)$$

We call the diagonal elements of D as "stretching" and the off-diagonals as "shearing".

Spin Tensor, W:

W is an anti-symmetric tensor; therefore, we can write it as a dual vector, ω (omega)

$$Wa = \omega \times a$$

$$\omega = -(W_{23}\hat{e}_1 + W_{31}\hat{e}_2 + W_{12}\hat{e}_3)$$

$$\boxed{W\,dx = \omega \times dx}$$

$$\frac{D(dx)}{Dt} = (\nabla v)\,dx = (D+W)\,dx = D\,dx + W\,dx = D\,dx + \omega \times dx$$

D
- diagonal → stretching → rate of extension (extension)
- off diagonal → shearing → rate of decrease in angle (rotation)

W spin tensor (rotation)

Conservation of Mass:

The principal of conservation of mass states that for an infinitesimal volume of particle, its volume and density may change through its motion ; however, its mass remains constant.

$$dm = const \Rightarrow \frac{D(dm)}{Dt} = 0 \Rightarrow \frac{D(\rho dV)}{Dt} = 0$$

$$\frac{D\rho}{Dt} dV + \rho \frac{D(dV)}{Dt} = 0 \xrightarrow{(\blacktriangle)} \frac{D\rho}{Dt} dV + \rho\, dV (div\, \vec{v}) = 0$$

$$dV \left\{ \frac{D\rho}{Dt} + \rho (\nabla \cdot v) \right\} = 0 \Rightarrow \boxed{\frac{D\rho}{Dt} + \rho (\nabla \cdot v) = 0}$$

Conservation of mass
(Material Coordinate)

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v \cdot (\nabla \rho)$$

$$\boxed{\frac{\partial \rho}{\partial t} + v \cdot (\nabla \rho) + \rho (\nabla \cdot v) = 0}$$

Conservation of mass
(Spatial Coordinate)

$$\frac{\partial \rho}{\partial t} + v_1 \cdot \frac{\partial \rho}{\partial x_1} + v_2 \cdot \frac{\partial \rho}{\partial x_2} + v_3 \cdot \frac{\partial \rho}{\partial x_3} + \rho \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) = 0$$

in Cartesian Coordinate

incompressible material: $\boxed{div\, v = 0 \quad \text{(no change in volume)}}$