Cartesian Coordinates (X, Y, Z) for Reference Configuration

Cylindrical Coordinates (r, θ, z) for Current Configuration

$$dx = F\,dX \quad , \quad dx = dr\, e_r + r\,d\theta\, e_\theta + dz\, e_z$$

$$dX = dX\, e_X + dY\, e_Y + dZ\, e_Z$$

$$x_i = x_i(X_j, t) \qquad \begin{cases} r = r(X, Y, Z, t) \\ \theta = \theta(X, Y, Z, t) \\ z = z(X, Y, Z, t) \end{cases}$$

$[F] = ?$

current, reference

$$dx = F\,dX \Rightarrow \begin{Bmatrix} dr \\ r\,d\theta \\ dz \end{Bmatrix} = \begin{bmatrix} F_{rX} & F_{rY} & F_{rZ} \\ F_{\theta X} & F_{\theta Y} & F_{\theta Z} \\ F_{zX} & F_{zY} & F_{zZ} \end{bmatrix} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$\{e_r, e_\theta, e_z\}\{e_X, e_Y, e_Z\}$

$$\begin{cases} dr = F_{rX}\, dX + F_{rY}\, dY + F_{rZ}\, dZ \\ dr = \dfrac{\partial r}{\partial X} dX + \dfrac{\partial r}{\partial Y} dY + \dfrac{\partial r}{\partial Z} dZ \end{cases}$$

$e_r \cdot F e_X = \dfrac{\partial r}{\partial X}$	$e_r \cdot F e_Y = \dfrac{\partial r}{\partial Y}$	$e_r \cdot F e_Z = \dfrac{\partial r}{\partial Z}$
$e_\theta \cdot F e_X = \dfrac{r\,\partial \theta}{\partial X}$	$e_\theta \cdot F e_Y = \dfrac{r\,\partial \theta}{\partial Y}$	$e_\theta \cdot F e_Z = \dfrac{r\,\partial \theta}{\partial Z}$
$e_z \cdot F e_X = \dfrac{\partial z}{\partial X}$	$e_z \cdot F e_Y = \dfrac{\partial z}{\partial Y}$	$e_z \cdot F e_X = \dfrac{\partial z}{\partial Z}$

$$\begin{cases} d\theta = \frac{1}{r} F_{\theta X}\, dX + \frac{1}{r} F_{\theta Y}\, dY + \frac{1}{r} F_{\theta Z}\, dZ \\ d\theta = \dfrac{\partial \theta}{\partial X} dX + \dfrac{\partial \theta}{\partial Y} dY + \dfrac{\partial \theta}{\partial Z} dZ \end{cases}$$

$$\begin{cases} dz = F_{zx}\, dX + F_{zy}\, dy + F_{zZ}\, dZ \\ dz = \dfrac{\partial z}{\partial X} dX + \dfrac{\partial z}{\partial y} dy + \dfrac{\partial z}{\partial Z} dZ \end{cases}$$

$$Fe_X = F_{jX}\, e_j \overset{j=r,\theta,z}{=} F_{rX}\, e_r + F_{\theta X}\, e_\theta + F_{zX}\, e_z$$

$$\boxed{Fe_X = \frac{\partial r}{\partial X} e_r + \frac{r\partial\theta}{\partial X} e_\theta + \frac{\partial z}{\partial X} e_z}$$

$$Fe_Y = F_{jY}\, e_j = F_{rY}\, e_r + F_{\theta Y}\, e_\theta + F_{zY}\, e_z$$

$$\boxed{Fe_Y = \frac{\partial r}{\partial Y} e_r + \frac{r\partial\theta}{\partial Y} e_\theta + \frac{\partial z}{\partial Y} e_z}$$

$$Fe_Z = F_{jZ}\, e_j = F_{rZ}\, e_r + F_{\theta Z}\, e_\theta + F_{zZ}\, e_z$$

$$\boxed{Fe_Z = \frac{\partial r}{\partial Z} e_r + \frac{r\partial\theta}{\partial Z} e_\theta + \frac{\partial z}{\partial Z} e_z}$$

$$[F]^T = ?$$

$$e_r \cdot Fe_X = Fe_X \cdot e_r = e_X \cdot F^T e_r = \frac{\partial r}{\partial X}$$

$$e_r \cdot Fe_Y = Fe_Y \cdot e_r = e_Y \cdot F^T e_r = \frac{\partial r}{\partial Y}$$

$$e_r \cdot Fe_Z = Fe_Z \cdot e_r = e_Z \cdot F^T e_r = \frac{\partial r}{\partial Z}$$

$$F^T e_r = F^T_{jr}\, e_j \overset{j=X,Y,Z}{=} F^T_{Xr}\, e_X + F^T_{Yr}\, e_Y + F^T_{Zr}\, e_Z$$

$$\boxed{F^T e_r = \frac{\partial r}{\partial X} e_X + \frac{\partial r}{\partial Y} e_Y + \frac{\partial r}{\partial Z} e_Z}$$

$$e_\theta \cdot Fe_X = Fe_X \cdot e_\theta = e_X \cdot F^T e_\theta = \frac{r\partial\theta}{\partial X}$$

$$e_\theta \cdot F e_y = F e_y \cdot e_\theta = e_y \cdot F^T e_\theta = \frac{r\,\partial\theta}{\partial y}$$

$$e_\theta \cdot F e_z = F e_z \cdot e_\theta = e_z \cdot F^T e_\theta = \frac{r\,\partial\theta}{\partial z}$$

$$F^T e_\theta = F^T_{j\theta}\, e_j = F^T_{x\theta}\, e_x + F^T_{y\theta}\, e_y + F^T_{z\theta}\, e_z$$

$$\boxed{F^T e_\theta = \frac{r\,\partial\theta}{\partial x} e_x + \frac{r\,\partial\theta}{\partial y} e_y + \frac{r\,\partial\theta}{\partial z} e_z}$$

$$e_z \cdot F e_x = F e_x \cdot e_z = e_x \cdot F^T e_z = \frac{\partial z}{\partial x}$$

$$e_z \cdot F e_y = F e_y \cdot e_z = e_y \cdot F^T e_z = \frac{\partial z}{\partial y}$$

$$e_z \cdot F e_x = F e_x \cdot e_z = e_x \cdot F^T e_z = \frac{\partial z}{\partial z}$$

$$F^T e_z = F^T_{jz}\, e_j = F^T_{xz}\, e_x + F^T_{yz}\, e_y + F^T_{zz}\, e_z$$

$$\boxed{F^T e_z = \frac{\partial z}{\partial x} e_x + \frac{\partial z}{\partial y} e_y + \frac{\partial z}{\partial z} e_z}$$

$[B] = ?$ w.r.t current configuration

$$B_{\theta z} = e_\theta \cdot B e_z = e_\theta \cdot (F F^T) e_z = e_\theta \cdot F F^T e_z$$

$$= e_\theta \cdot F \left(\frac{\partial z}{\partial x} e_x + \frac{\partial z}{\partial y} e_y + \frac{\partial z}{\partial z} e_z \right)$$

$$= \frac{\partial z}{\partial x}\, e_\theta \cdot F e_x + \frac{\partial z}{\partial y}\, e_\theta \cdot F e_y + \frac{\partial z}{\partial z}\, e_\theta \cdot F e_z$$

$$= \left(\frac{\partial z}{\partial x}\right)\left(\frac{r\,\partial\theta}{\partial x}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{r\,\partial\theta}{\partial y}\right) + \left(\frac{\partial z}{\partial z}\right)\left(\frac{r\,\partial\theta}{\partial z}\right)$$

$[C] = ?$ w.r.t reference configuration

$$C_{xy} = e_x \cdot C e_y = e_x \cdot F^T F e_y$$

$$= e_X \cdot F^T\left(\frac{\partial r}{\partial y} e_r + \frac{r\partial\theta}{\partial y} e_\theta + \frac{\partial z}{\partial y} e_z\right)$$

$$= \frac{\partial r}{\partial y} e_X \cdot F^T e_r + \frac{r\partial\theta}{\partial y} e_X \cdot F^T e_\theta + \frac{\partial z}{\partial y} e_X \cdot F^T e_z$$

$$= \left(\frac{\partial r}{\partial y}\right)\left(\frac{\partial r}{\partial X}\right) + \left(\frac{r\partial\theta}{\partial y}\right)\left(\frac{r\partial\theta}{\partial X}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial X}\right)$$

$[B^{-1}] = ?$ w.r.t current configuration

$$B^{-1} = (FF^T)^{-1} = (F^{-1})^T F^{-1}$$

$$dX = F^{-1} dx$$

$$\begin{cases} X = X(r,\theta,z,t) \\ Y = Y(r,\theta,z,t) \\ Z = Z(r,\theta,z,t) \end{cases}$$

$$dX = F^{-1} dx \Rightarrow \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} = \begin{bmatrix} F^{-1}_{Xr} & F^{-1}_{X\theta} & F^{-1}_{Xz} \\ F^{-1}_{Yr} & F^{-1}_{Y\theta} & F^{-1}_{Yz} \\ F^{-1}_{Zr} & F^{-1}_{Z\theta} & F^{-1}_{Zz} \end{bmatrix} \begin{Bmatrix} dr \\ r d\theta \\ dz \end{Bmatrix}$$

$$\begin{cases} dX = F^{-1}_{Xr} dr + F^{-1}_{X\theta}(r d\theta) + F^{-1}_{Xz} dz \\ dX = \dfrac{\partial X}{\partial r} dr + \dfrac{\partial X}{\partial \theta} d\theta + \dfrac{\partial X}{\partial z} dz \end{cases}$$

$e_X \cdot F^{-1} e_r = \frac{\partial X}{\partial r}$	$e_X \cdot F^{-1} e_\theta = \frac{1}{r}\frac{\partial X}{\partial \theta}$	$e_X \cdot F^{-1} e_z = \frac{\partial X}{\partial z}$
$e_Y \cdot F^{-1} e_r = \frac{\partial Y}{\partial r}$	$e_Y \cdot F^{-1} e_\theta = \frac{1}{r}\frac{\partial Y}{\partial \theta}$	$e_Y \cdot F^{-1} e_z = \frac{\partial Y}{\partial z}$
$e_Z \cdot F^{-1} e_r = \frac{\partial Z}{\partial r}$	$e_Z \cdot F^{-1} e_\theta = \frac{1}{r}\frac{\partial Z}{\partial \theta}$	$e_Z \cdot F^{-1} e_z = \frac{\partial Z}{\partial z}$

$$\begin{cases} dY = F^{-1}_{Yr}\, dr + F^{-1}_{Y\theta}\,(r d\theta) + F^{-1}_{Yz}\, dz \\ dY = \dfrac{\partial Y}{\partial r} dr + \dfrac{\partial Y}{\partial \theta} d\theta + \dfrac{\partial Y}{\partial z} dz \end{cases}$$

$$\begin{cases} dZ = F^{-1}_{Zr}\, dr + F^{-1}_{Z\theta}\,(r d\theta) + F^{-1}_{Zz}\, dz \\ dZ = \dfrac{\partial Z}{\partial r} dr + \dfrac{\partial Z}{\partial \theta} d\theta + \dfrac{\partial Z}{\partial z} dz \end{cases}$$

$[F^{-1}]^T = ?$

$$e_X \cdot F^{-1} e_r = F^{-1} e_r \cdot e_X = e_r \cdot (F^{-1})^T e_X = \frac{\partial X}{\partial r}$$

$$e_X \cdot F^{-1} e_\theta = F^{-1} e_\theta \cdot e_X = e_\theta \cdot (F^{-1})^T e_X = \frac{1}{r} \frac{\partial X}{\partial \theta}$$

$$e_X \cdot F^{-1} e_z = F^{-1} e_z \cdot e_X = e_z \cdot (F^{-1})^T e_X = \frac{\partial X}{\partial z}$$

$$(F^{-1})^T e_X = (F^{-1}_{jX})^T e_j \overset{j=r,\theta,z}{=} (F^{-1}_{rX})^T e_r + (F^{-1}_{\theta X})^T e_\theta + (F^{-1}_{zX})^T e_z$$

$$\boxed{(F^{-1})^T e_X = \frac{\partial X}{\partial r} e_r + \frac{1}{r} \frac{\partial X}{\partial \theta} e_\theta + \frac{\partial X}{\partial z} e_z}$$

$$e_Y \cdot F^{-1} e_r = F^{-1} e_r \cdot e_Y = e_r \cdot F^{-1} e_Y = \frac{\partial Y}{\partial r}$$

$$e_Y \cdot F^{-1} e_\theta = F^{-1} e_\theta \cdot e_Y = e_\theta \cdot F^{-1} e_Y = \frac{1}{r} \frac{\partial Y}{\partial \theta}$$

$$e_Y \cdot F^{-1} e_z = F^{-1} e_z \cdot e_Y = e_z \cdot F^{-1} e_Y = \frac{\partial Y}{\partial z}$$

$$(F^{-1})^T e_Y = (F^{-1}_{jY})^T e_j \overset{j=r,\theta,z}{=} (F^{-1}_{rY})^T e_r + (F^{-1}_{\theta Y})^T e_\theta + (F^{-1}_{zY})^T e_z$$

$$\boxed{(F^{-1})^T e_Y = \frac{\partial Y}{\partial r} e_r + \frac{1}{r} \frac{\partial Y}{\partial \theta} e_\theta + \frac{\partial Y}{\partial z} e_z}$$

$$e_z \cdot F^{-1} e_r = F^{-1} e_r \cdot e_z = e_r \cdot F^{-1} e_z = \frac{\partial z}{\partial r}$$

$$e_z \cdot F^{-1} e_\theta = F^{-1} e_\theta \cdot e_z = e_\theta \cdot F^{-1} e_z = \frac{1}{r}\frac{\partial z}{\partial \theta}$$

$$e_z \cdot F^{-1} e_z = F^{-1} e_z \cdot e_z = e_z \cdot F^{-1} e_z = \frac{\partial z}{\partial z}$$

$$(F^{-1})^T e_z = (F^{-1}_{jz})^T e_j \overset{j=r,\theta,z}{=} (F^{-1}_{rz})^T e_r + (F^{-1}_{\theta z})^T e_\theta + (F^{-1}_{zz})^T e_z$$

$$\boxed{(F^{-1})^T e_z = \frac{\partial z}{\partial r} e_r + \frac{1}{r}\frac{\partial z}{\partial \theta} e_\theta + \frac{\partial z}{\partial z} e_z}$$

$$F^{-1} e_r = F^{-1}_{jr} e_j \overset{j=x,y,z}{=} F^{-1}_{xr} e_x + F^{-1}_{yr} e_y + F^{-1}_{zr} e_z$$

$$\boxed{F^{-1} e_r = \frac{\partial x}{\partial r} e_x + \frac{\partial y}{\partial r} e_y + \frac{\partial z}{\partial r} e_z}$$

$$F^{-1} e_\theta = F^{-1}_{j\theta} e_j = F^{-1}_{x\theta} e_x + F^{-1}_{y\theta} e_y + F^{-1}_{z\theta} e_z$$

$$\boxed{F^{-1} e_\theta = \frac{1}{r}\frac{\partial x}{\partial \theta} e_x + \frac{1}{r}\frac{\partial y}{\partial \theta} e_y + \frac{1}{r}\frac{\partial z}{\partial \theta} e_z}$$

$$F^{-1} e_z = F^{-1}_{jz} e_j = F^{-1}_{xz} e_x + F^{-1}_{yz} e_y + F^{-1}_{zz} e_z$$

$$\boxed{F^{-1} e_z = \frac{\partial x}{\partial z} e_x + \frac{\partial y}{\partial z} e_y + \frac{\partial z}{\partial z} e_z}$$

$$B^{-1}_{r\theta} = e_r \cdot B^{-1} e_\theta = e_r \cdot (FF^T)^{-1} e_\theta = e_r \cdot (F^{-1})^T (F^{-1}) e_\theta$$

$$= e_r \cdot (F^{-1})^T \left(\frac{1}{r}\frac{\partial x}{\partial \theta} e_x + \frac{1}{r}\frac{\partial y}{\partial \theta} e_y + \frac{1}{r}\frac{\partial z}{\partial \theta} e_z \right)$$

$$= \frac{1}{r}\frac{\partial x}{\partial \theta} e_r \cdot (F^{-1})^T e_x + \frac{1}{r}\frac{\partial y}{\partial \theta} e_r \cdot (F^{-1})^T e_y + \frac{1}{r}\frac{\partial z}{\partial \theta} e_r \cdot (F^{-1})^T e_z$$

$$= \left(\frac{1}{r}\frac{\partial x}{\partial \theta}\right)\left(\frac{\partial x}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right)\left(\frac{\partial y}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial z}{\partial \theta}\right)\left(\frac{\partial z}{\partial r}\right)$$

$[C]^{-1} = ?$ w.r.t reference configuration

$$C^{-1}_{xy} = e_x \cdot C^{-1} e_y = e_x \cdot (F^T F)^{-1} e_y = e_x \cdot (F)^{-1} (F^{-1})^T e_y$$

$$= e_x \cdot F^{-1}\left(\frac{\partial y}{\partial r} e_r + \frac{1}{r}\frac{\partial y}{\partial \theta} e_\theta + \frac{\partial y}{\partial z} e_z\right)$$

$$= \frac{\partial y}{\partial r} e_x \cdot F^{-1} e_r + \frac{1}{r}\frac{\partial y}{\partial \theta} e_x \cdot F^{-1} e_\theta + \frac{\partial y}{\partial z} e_x \cdot F^{-1} e_z$$

$$= \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial x}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial x}{\partial z}\right)$$