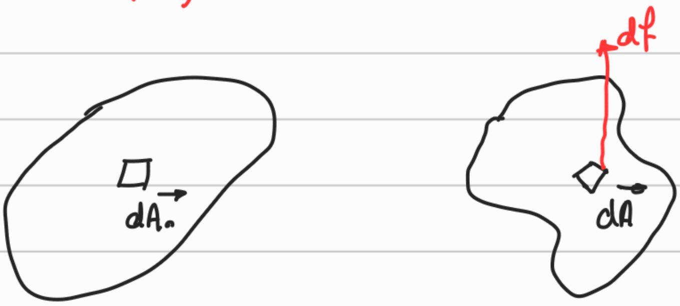
Piola-Kirchhoff Stress Tensor:

First PK Stress Tensor (T_0):



ref config. current config

(force vector/unit area) $\vec{df} = \vec{t_0}(dA_0)$ $\qquad dA_0 = \dfrac{|\vec{dA_0}|}{|n|}$

def ref

$$t_0 = T_0 n_0$$

first PK stress tensor

$$\begin{cases} df = t_0\, dA_0 \quad (*) \\ df = t\, dA \\ t_0 = T_0 n_0 \quad (**) \end{cases} \Rightarrow t\, dA = T_0 n_0\, dA_0 \Rightarrow Tn\, dA = T_0 n_0\, dA_0$$

$$T n \frac{dA}{dA_0} = T_0 n_0$$

Module 2.8 → $\begin{cases} dA\, n = dA_0\, J (F^{-1})^T n_0 \\ J = |\det F| \end{cases}$ $\Rightarrow T J (F^{-1})^T n_0 = T_0 n_0$

$$\rightarrow (T J (F^{-1})^T - T_0)\, n_0 = 0 \Rightarrow T_0 = T J (F^{-1})^T$$

1st PK stress tensor $\boxed{T_0 = T J (F^T)^{-1}}$

$$(T_0)_{ij} = J\, T_{im} (F^{-1})_{jm}$$

$$T = \frac{1}{J} T_0 \left(F^T\right)$$

$$T_{ij} = \frac{1}{J} (T_0)_{im} (F)_{jm} \quad , \quad F_{mj} = \frac{\partial x_m}{\partial X_j}$$

$$F_{jm} = \frac{\partial X_j}{\partial x_m}$$

T is symmetric : $T = T^T$

T_0 is symmetric? $T_0 = T_0^T$

$$T_0^T = \left\{ T J \left(F^T\right)^{-1} \right\}^T = J \left(\left(F^{-1}\right)^T \right)^T T^T = J F^{-1} T \neq J T \left(F^{-1}\right)^T$$

T_0 is NOT symmetric

True stress $\quad \sigma_t = \dfrac{F_{def}}{A_{def}} \quad$ Cauchy stress

Eng. stress $\quad \sigma_E = \dfrac{F_{def}}{A_{0\,ref}} \quad$ 1st PK

Second PK Stress Tensor:

ref config $\qquad\qquad$ current config

$$\left.\begin{aligned} d\tilde{f} &= \tilde{t}(dA_0) \\ \tilde{t} &= \tilde{T} n_0 \end{aligned}\right\} \quad d\tilde{f} = \tilde{T} n_0 (dA_0)$$

$$\left.\begin{aligned} df = F d\tilde{f} \Rightarrow d\tilde{f} = F^{-1} df \end{aligned}\right\} \quad F^{-1} df = \tilde{T} n_0 (dA_0)$$

from (*)

$$F^{-1} t_0 dA_0 = \tilde{T} n_0 dA_0 \xrightarrow{(\cancel{*})} F^{-1} T_0 n_0 dA_0 = \tilde{T} n_0 dA_0$$

$$F^{-1} T_0 n_0 = \tilde{T} n_0 \Rightarrow (F^{-1} T_0 - \tilde{T}) n_0 = 0 \Rightarrow \boxed{\tilde{T} = F^{-1} T_0}$$

2nd PK stress tensor

1st and 2nd PK

$$\tilde{T} = F^{-1} T_0$$

$$\tilde{T}_{ij} = (F^{-1})_{im} (T_0)_{mj}$$

2nd PK and Cauchy

$$\tilde{T} = F^{-1} T_0 = J F^{-1} T (F^{-1})^T$$

$$\tilde{T}_{ij} = J F^{-1}_{im} T_{mn} F (F^{-1})_{jn}$$

Is $\tilde{T}$ symmetric? Yes

$$(\tilde{T})^T = (J F^{-1} T (F^{-1})^T)^T = J ((F^{-1})^T)^T T^T (F^{-1})^T$$

$$= J F^{-1} T (F^{-1})^T = \tilde{T}$$

tensor		force	area	
T	(x,t)	current	current	Both dyads are in current
T_0	(X,t)	current	reference	two-point tensor (one of the dyads in current and the other in ref)
$\tilde{T}$	(X,t)	reference	reference	Both dyads are in ref

Example 4.10.3

$A(X_1, X_2, X_3)$

$$\frac{\partial}{\partial X_m} \det A = (\det A)(A^{-1})_{nj} \frac{\partial A_{jn}}{\partial X_m}$$

show $\dfrac{\partial}{\partial x_j}\left(\dfrac{F_{jm}}{J}\right) = 0$

where $F_{jm} = \dfrac{\partial x_j}{\partial X_m}$, $x_j = \hat{x}_j(X_1, X_2, X_3, t)$, $J = (\det F) > 0$

$$\frac{\partial}{\partial x_j}\left(\frac{F_{jm}}{J}\right) = \frac{1}{J}\frac{\partial F_{jm}}{\partial x_j} + F_{jm}\left(\frac{-1}{J^2}\right)\frac{\partial J}{\partial x_j}$$

$$= \frac{1}{J}\frac{\partial F_{jm}}{\partial X_n}\frac{\partial X_n}{\partial x_j} - \frac{F_{jm}}{J^2}\frac{\partial J}{\partial X_n}\frac{\partial X_n}{\partial x_j}$$

$$= \frac{1}{J}\frac{\partial F_{jm}}{\partial X_n}\frac{\partial X_n}{\partial x_j} - \frac{1}{J^2}\left(\frac{\partial x_j}{\partial X_m}\right)\frac{\partial J}{\partial X_n}\frac{\partial X_n}{\partial x_j}$$

$$\frac{\partial X_n}{\partial X_m}$$

$$= \frac{1}{J}\frac{\partial F_{jm}}{\partial X_n}\frac{\partial X_n}{\partial x_j} - \frac{1}{J^2}\underbrace{\frac{\partial X_n}{\partial X_m}\frac{\partial(\det F)}{\partial X_n}}_{\frac{\partial(\det F)}{\partial X_m}}$$

$$\frac{\partial(\det F)}{\partial X_m} = \underset{J}{\cancel{(\det F)}}\,(F^{-1})_{nj}\frac{\partial F_{jn}}{\partial X_m}$$

$$= \frac{1}{J}\frac{\partial F_{jm}}{\partial X_n}\frac{\partial X_n}{\partial x_j} - \frac{1}{J}\frac{\partial F_{jm}}{\partial X_n}\underset{\frac{\partial X_n}{\partial x_j}}{\cancel{(F^{-1})_{nj}}} = 0$$

Equation of Motion w.r.t the 2nd Piola-Kirchhoff stress Tensor:

Cauchy strain Tensor: $\operatorname{div} T + \rho B = \rho a \quad , \quad \dfrac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho a_i$

$$T_{ij} = \frac{1}{J} (T_o)_{im} (F_{mj})^T = \frac{1}{J} (T_o)_{im} F_{jm}$$

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{J}(T_o)_{im} F_{jm}\right) = \frac{\partial}{\partial x_j}\left((T_o)_{im} \frac{F_{jm}}{J}\right)$$

$$= \frac{\partial (T_o)_{im}}{\partial x_j} \frac{F_{jm}}{J} + (T_o)_{im} \cancelto{0}{\frac{\partial}{\partial x_j}\left(\frac{F_{jm}}{J}\right)}$$

Example 4.10.3

$$= \frac{\partial (T_o)_{im}}{\partial X_n} \underbrace{\frac{\partial X_n}{\partial x_j} \frac{\partial x_j}{\partial X_m}}_{\frac{\partial X_n}{\partial X_m} = \delta_{nm}} \left(\frac{1}{J}\right)$$

$$= \left(\frac{1}{J}\right) \frac{\partial (T_o)_{im}}{\partial X_n} \delta_{nm} = \frac{1}{J} \frac{\partial (T_o)_{im}}{\partial X_m} = \frac{1}{J} \frac{\partial (T_o)_{ij}}{\partial X_j}$$

Equation of Motion w.r.t 2nd PK ST : $\dfrac{1}{J} \dfrac{\partial (T_o)_{ij}}{\partial X_j} + \rho B_i = \rho a_i$

$$\frac{\partial (T_o)_{ij}}{\partial X_j} + \rho J B_i = \rho J a_i \qquad \rho J = \rho_o$$

$B_i(X_i)$ $\quad$ $a_i(X_i)$

Equation of Motion w.r.t 2nd PK ST $\quad \boxed{\dfrac{\partial (T_o)_{ij}}{\partial X_j} + \rho_o B_i = \rho_o a_i}$