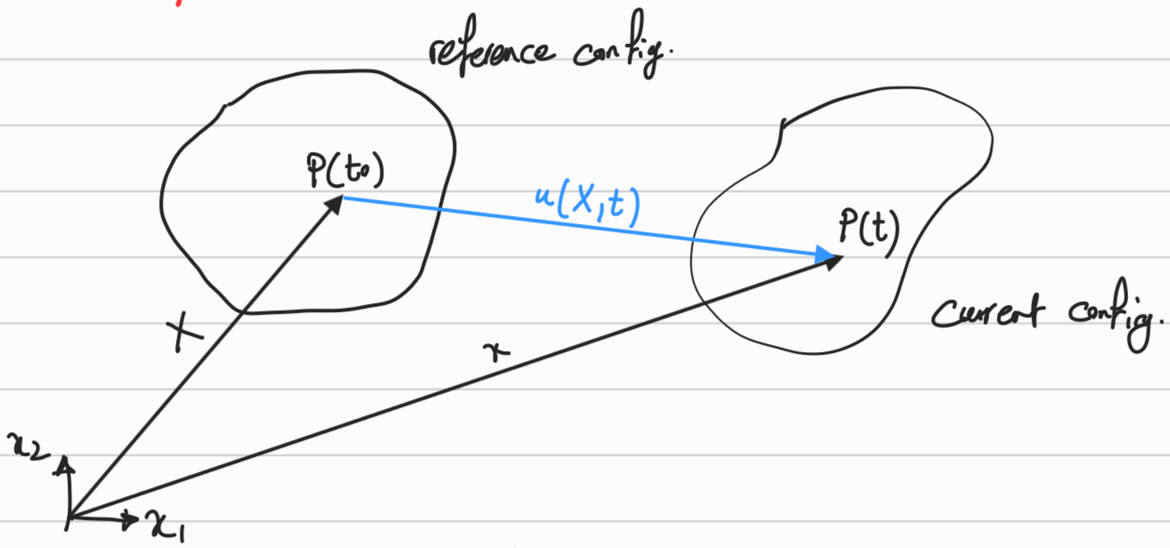
Infinitesimal deformation:



$$x(X,t) = X + u(X,t)$$

material coordinate: $u(X,t) = x(X,t) - X$

special coordinate: $u(x,t) = x - X(x,t)$

Kinematic Equation for Rigid Body Motion:

- Rigid Body Translation: $\boxed{x = X + c(t)}$ $(*)$

$$u = x - X = X + c(t) - X = c(t)$$

- Rigid Body Rotation: $x = R(t)\,X$ center of rotation is at origin

$$\boxed{x - b = R(t)\,X - b} \quad (**)$$

Example 3.6.1: Rigid Body Rotation does not change the distance between any pair of material points.

first point: $x^1 - b = R(t)\,X^1 - b$ (1)

Second point: $x^2 - b = R(t)\,X^2 - b$ (2)

$\left.\begin{matrix} (1) \\ (2) \end{matrix}\right\}$ pair

$(2)-(1)$: $x^2 - x^1 = R(t)(X^2 - X^1)$

$$\Delta x = R(t)\,\Delta X$$

let us assume the length of $\Delta x = l$ and $\Delta X = L$

$$l^2 = \Delta x \cdot \Delta x = (R(t)\,\Delta X) \cdot (R(t)\,\Delta X) = (\Delta X\, R^T(t)) \cdot (R(t)\,\Delta X)$$

$$= \Delta X \cdot I\,\Delta X = \Delta X \cdot \Delta X = L^2 \qquad \therefore\ l^2 = L^2$$

$$l = L$$

– General Rigid Body Motion

Combining $(*)$ and $(**)$ $\quad \boxed{x = R(t)(X-b) + c(t)}$

Example 3.6.2

$$x = R(t)(X-b) + c(t) \xrightarrow{\partial/\partial t} v = \dot{R}(t)(X-b) + \dot{c}(t) \quad (4)$$

$$x - c(t) = R(t)(X-b) \xrightarrow{R^T(t)} R^T(t)(x - c(t)) = (X-b) \quad (3)$$

Substitute (3) in (4) : $v = \dot{R}(t) R^T(t)\,[x - c(t)] + \dot{c}(t)$

$$\dot{R}(t) R^T(t) : \partial(R(t) R^T(t))/\partial t = \dot{R}(t) R^T(t) + R(t)\dot{R}^T(t) = 0$$

$$R(t) R^T(t) = I$$

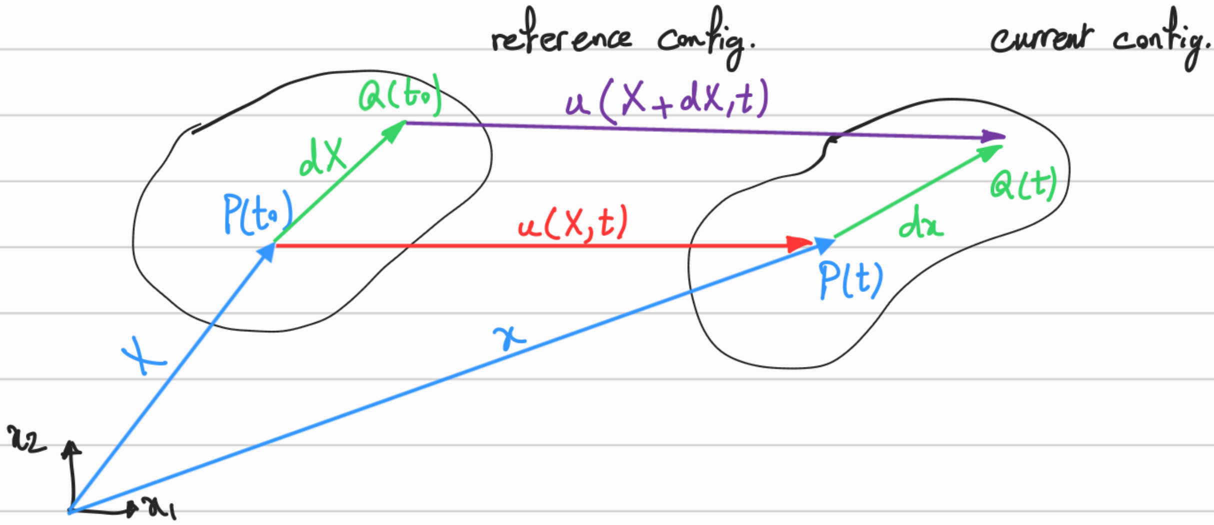
$$\Rightarrow \underbrace{\dot{R}(t) R^T(t)}_{T(t)} = -R(t)\dot{R}^T(t) = -\big(\underbrace{\dot{R}(t) R^T(t)}_{T(t)}\big)^T$$

$$T(t) = -(T(t))^T \quad \therefore \ \boxed{T(t) = \dot{R}(t) R^T(t) \text{ is antisymmetric tensor}}$$

Now that we know $\dot{R}(t)R^T(t)$ is an antisymmetric tensor, we can write it in form of a dual vector, let's assume $\dot{R}(t)R^T(t) \equiv \omega$

$$\boxed{v = \omega \times \underbrace{[x - c(t)]}_{r} + \dot{c}(t) = \omega \times r + \dot{c}(t)}$$

Infinitesimal Deformation:



$$x = X + u(X,t) \tag{5}$$

$$x + dx = X + dX + u(X+dX,t) \tag{6}$$

$$(6)-(5): \begin{cases} dx = dX + u(X+dX,t) - u(X,t) \\ \nabla u = \dfrac{u(X+dX,t) - u(X,t)}{dX} \end{cases}$$

$$\Rightarrow \boxed{dx = dX + (\nabla u)dX}$$

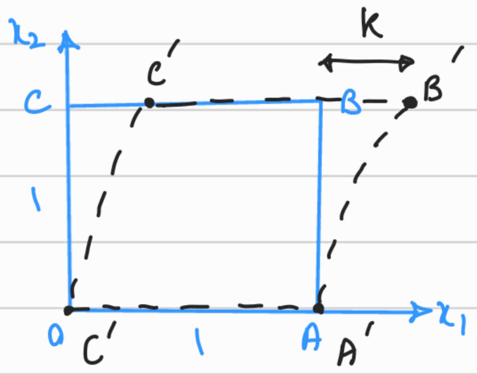
∇u : displacement gradient

$$\nabla u = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{\partial u_1}{\partial X_2} & \dfrac{\partial u_1}{\partial X_3} \\ \dfrac{\partial u_2}{\partial X_1} & \dfrac{\partial u_2}{\partial X_2} & \dfrac{\partial u_2}{\partial X_3} \\ \dfrac{\partial u_3}{\partial X_1} & \dfrac{\partial u_3}{\partial X_2} & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix}$$

in Cartesian Coordinate System

indicial notation : $\boxed{dx_i = dX_i + \frac{\partial u_i}{\partial X_j}\, dX_j}$

Example 3.7.



$$\begin{cases} u_1 = k X_2^2 \\ u_2 = 0 \\ u_3 = 0 \end{cases}, \quad k \ll 1$$

assuming $k > 0$

(a,b) $0 \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$: $dx = dX + (\nabla u)\, dX = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$\nabla u = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$B = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$: $dx = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 + 2kX_2 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1+2k \\ 1 \\ 0 \end{Bmatrix}$

(c) $\dfrac{\text{deformed length}}{\text{undeformed length}}$

$dx^1 = dX_1 \hat{e}_1$

$dx^2 = dX_2 (2k \hat{e}_1 + \hat{e}_2)$

$|dx^1| = dX_1$

$|dx^2| = dX_2 \sqrt{(4k^2+1)}$

$\dfrac{|dx^1|}{|dX^1|} = 1$

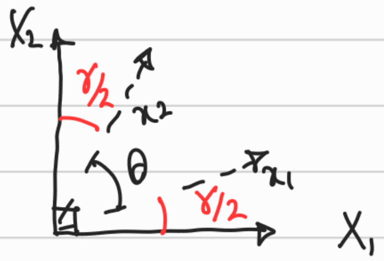
$\dfrac{|dx^2|}{|dX^2|} = \sqrt{4k^2+1}$

$$\cos\theta = \frac{dx^1 \cdot dx^2}{|dx^1| \cdot |dx^2|} = \frac{2k}{\sqrt{4k^2+1}} = \frac{2k}{1+\underbrace{2k^2+\dots}_{\text{neglect higher-order terms}}} \approx 2k$$

$$\sqrt{1+x} \approx 1 + \frac{x}{2} + \dots$$

$\cos\theta = \cos\left(\frac{\pi}{2} - \gamma\right) = \sin\gamma$ for small angle γ $\sin\gamma \simeq \gamma$

$$\gamma \simeq 2k$$



Deformation Gradient Tensor:

$$dx = dX + (\nabla u)\,dX = I\,dX + (\nabla u)\,dX = (I + \nabla u)\,dX$$

$dx = F\,dX$ where $F \equiv I + \nabla u$

deformation gradient

let as assume that the length of $|dx| = ds$ and $|dX| = dS$

$$dx \cdot dx = (F\,dX)\cdot(F\,dX) = (dX\,F^T)\cdot(F\,dX) = dX \cdot C\,dX$$

where $C \equiv F^T F$ (right Cauchy-Green deformation tensor)

$$C = F^T F = (I + \nabla u)^T (I + \nabla u) = (I + \nabla u^T)(I + \nabla u)$$

$$\boxed{C = I + \nabla u + (\nabla u)^T + (\nabla u)^T (\nabla u)}$$

$$C = I + 2E^*$$

where $E^* \equiv \frac{1}{2}\left[\nabla u + (\nabla u)^T + (\nabla u)^T(\nabla u)\right]$ Lagrange strain tensor

To get the infinitesimal strain tensor, we neglect the higher-order terms of displacement gradient in E^*

Infinitesimal strain tensor : $\boxed{E = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]}$ the symmetric part of ∇u

indicial notation: $E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right] = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right]$

C for the infinitesimal strain : $\boxed{C \simeq I + 2E}$

$\therefore$ if we have two elements, dX^1 and dX^2

$$dx^1 \cdot dx^2 = dX^1 \cdot C\, dX^2 = dX^1 \cdot (I + 2E)\, dX^2$$

$$\boxed{dx^1 \cdot dx^2 = dX^1 \cdot dX^2 + 2\, dX^1 \cdot E\, dX^2}$$

Cartesian Coordinates

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \qquad E_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\right) = \frac{\partial u_1}{\partial X_1}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right)$$

Geometrical Representation / Interpretation of $[E]$:

Diagonal elements:

$$dX^1 = dX^2 = dX = dS\,\hat{n} \qquad , \; dx^1 = dx^2 = ds\,\hat{m}$$

$$\longrightarrow \; dx \cdot dx = dX \cdot dX + 2\,dX \cdot E\,dX$$

$$ds^2 = dS^2 + 2\,dS^2 (n \cdot E\,n) \Rightarrow ds^2 - dS^2 = 2\,dS^2 (n \cdot En)$$

$$\Rightarrow \frac{(ds - dS)(ds + dS)}{2dS^2} = n \cdot En \xrightarrow[\text{strains}]{\text{infinitesimal}} ds \simeq dS$$

$$\frac{(ds - dS)\,2dS}{2dS^2} = \boxed{\frac{ds - dS}{dS} = \hat{n} \cdot E\hat{n}}$$

E_{11} is the unit elongation for an element in the x_1-direction.

E_{22} " " " " " " " " x_2 "

E_{33} " " " " " " " " x_3 "

Off-diagonal elements

$$dX^1 = dS_1\,\hat{m} \qquad \text{where } \hat{m} \cdot \hat{n} = 0$$

$$dX^2 = dS_2\,\hat{n}$$

$$\longrightarrow \; dx^1 \cdot dx^2 = dX^1 \cdot dX^2 + 2\,dX^1 \cdot E\,dX^2$$

$$ds_1\,ds_2 \cos\theta = \underbrace{\cancel{dS_1\,dS_2\,\hat{m} \cdot \hat{n}}}_{0} + 2\,dS_1\,dS_2\,\hat{m} \cdot E\hat{n}$$

$$ds_1\,ds_2 \cos\theta = 2\,dS_1\,dS_2\,\hat{m} \cdot E\hat{n}$$

Due to the assumption of infinitesmial deformation $\frac{ds_1}{dS_1} = \frac{ds_2}{dS_2} \approx 1$

And since dX^1 and dX^2 are orthogonal $\therefore \theta = \frac{\pi}{2} - \gamma$

$\therefore \cos\theta = \cos\left(\frac{\pi}{2} - \gamma\right) = \sin\gamma \equiv \gamma$

We finally get $\boxed{\gamma = 2\hat{m}\cdot E\hat{n}}$

$2E_{12}$ is the decrease in angle between two element originally in the x_1 and x_2 directions.

$2E_{13}$ " " " " " " " " " " " x_1 and x_3 " .

$2E_{23}$ " " " " " " " " " " " x_2 and x_3 " .

Example 3.8.2

$$\begin{cases} u_1 = k\left(2X_1 + X_2^2\right) \\ u_2 = k\left(X_1^2 - X_2^2\right) \\ u_3 = 0 \end{cases}, \quad k = 10^{-4}$$

(a) $dX^1 = dX_1\hat{e}_1$

$dX^2 = dX_2\hat{e}_2$

$$X = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$$

$$E = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]$$

$$\nabla u = \begin{bmatrix} 2k & 2kX_2 & 0 \\ 2kX_1 & -2kX_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\nabla u = \begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}_X$$

$$E = \frac{1}{2}\left\{\begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2k & 2k & 0 \\ -2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} =$$

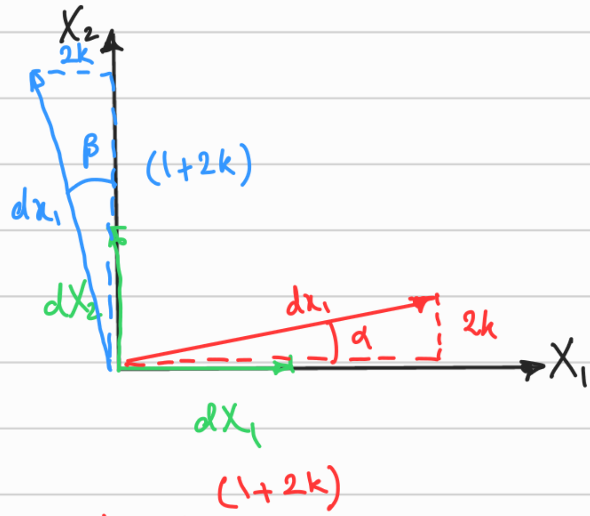
$$\begin{bmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) $$dx^1 = dX^1 + \nabla u \, dX^1 = \begin{Bmatrix} dX_1 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} dX_1 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} dX_1 + 2k\,dX_1 \\ 2k\,dX_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} (1+2k)\,dX_1 \\ 2k\,dX_1 \\ 0 \end{Bmatrix}$$

$$dx^2 = dX^2 + \nabla u \, dX^2 = \begin{Bmatrix} 0 \\ dX_2 \\ 0 \end{Bmatrix} + \begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ dX_2 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -2k\,dX_2 \\ dX_2 + 2k\,dX_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2k\,dX_2 \\ (1+2k)\,dX_2 \\ 0 \end{Bmatrix}$$

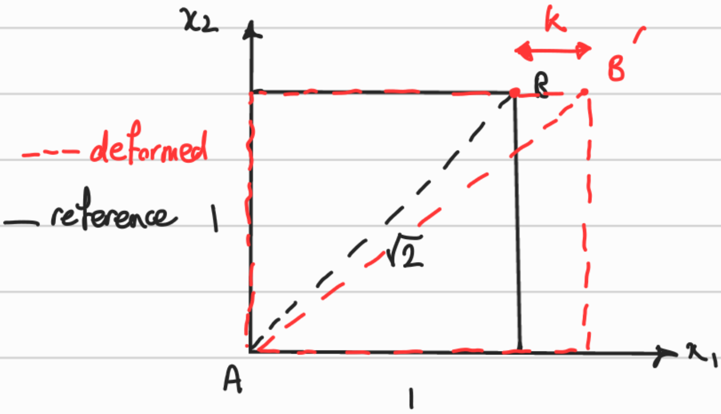


$$\tan\alpha \equiv \alpha = \frac{2k\,dX_1}{(1+2k)\,dX_1} = 2k\left(\frac{1}{1+2k}\right) = 2k(1+\dots) \simeq 2k$$

$$\beta = \frac{(2k)\,dX_2}{(1+2k)\,dX_2} \simeq 2k$$

Example 3.8.3

$$\begin{cases} u_1 = kX_1 \\ u_2 = 0 \\ u_3 = 0 \end{cases}, \quad k = 10^{-4}$$



(a) $$\nabla u = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the unit elongation in x_1-direction

$$E = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have to find the unit vector in $\overrightarrow{AB}$ direction, $\hat{n}_{AB}$

$$n_{AB} \cdot E n_{AB} = \left\{ \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0 \right\} \cdot \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{Bmatrix}$$

$$\left\{ \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0 \right\} \cdot \begin{Bmatrix} \frac{k\sqrt{2}}{2} \\ 0 \\ 0 \end{Bmatrix} = \frac{k}{2}$$ the unit elongation in $\hat{n}_{AB}$-direction

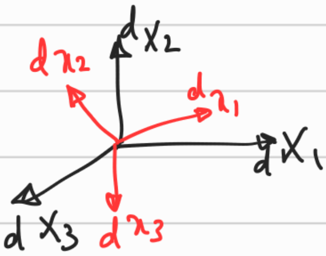
elongation in x_2-direction:

$$\{0 \quad 1 \quad 0\} \cdot \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \{0 \quad 1 \quad 0\} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = 0$$

Principal strain:

$$\underset{\text{symmetric tensor}}{[E]} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}$$

Dilatation:



$$dV_1 = dS_1\, dS_2\, dS_3$$

$$dV_2 = (dS_1 + dS_1 E_1)(dS_2 + dS_2 E_2)(dS_3 + dS_3 E_3)$$

$$= dS_1 dS_2 dS_3 (1+E_1)(1+E_2)(1+E_3)$$

$$(\text{dilation})\ e = \frac{dV_2 - dV_1}{dV_1} = \frac{dS_1 dS_2 dS_3 (1+E_1)(1+E_2)(1+E_3) - dS_1 dS_2 dS_3}{dS_1 dS_2 dS_3}$$

$$= (1+E_1)(1+E_2)(1+E_3) - 1$$

$$= E_1 + E_2 + E_3 + \text{higher-order terms}$$

Since we are dealing with infinitesimal deformation, we can neglect the higher-order terms.

$$e = E_1 + E_2 + E_3 = E_{ii} = \mathrm{div}(u) = \frac{\partial u_i}{\partial X_i}$$

Infinitesimal Rotation Tensor (Antisymmetric part of ∇u):

$$dx = dX + (\nabla u)dX = dX + (\underbrace{E}_{\text{symmetric part of } \nabla u} + \underbrace{\Omega}_{\text{Antisymmetric part of } \nabla u})dX$$

$$\Omega = \frac{1}{2}\left[\nabla u - (\nabla u)^T\right] \quad \therefore \quad \Omega\, dX = t^A \times dX$$

where t^A is the dual vector of Ω and is

$$t^A = -(\Omega_{23}\hat{e}_1 + \Omega_{31}\hat{e}_2 + \Omega_{12}\hat{e}_3)$$

$$\nabla u = E + \Omega$$

change in angle

elongation
change in angle

* Zero dilatation or $\mathrm{div}(u) = 0$ does not indicate zero deformation.