

Einstein's Summation Convention

The Kronecker Delta

The Permutation Symbol

$$\alpha = a_1 x_1 + a_2 x_2 + \dots + a_n x_n = \sum_{i=1}^n a_i x_i = \sum_{j=1}^n a_j x_j = \sum_{k=1}^n a_k x_k$$

$$\alpha = \cancel{\sum_{i=1}^n} a_i x_i = a_i x_i = a_j x_j = a_k x_k = a_5 x_5$$

dummy index

$$n=3 : \alpha = a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3 \quad \left. \begin{matrix} 3=3 \\ 2=2 \end{matrix} \right\} \begin{matrix} \text{number of} \\ \text{terms} \end{matrix}$$

$$n=2 : \alpha = a_j x_j = a_1 x_1 + a_2 x_2$$

To get the number of terms (the last term in series) dummy index
dummy index

In CM, we do not exceed $n=3 \therefore 3$

1. Dummy index is an index which is repeated twice. We sum over the repeated index.
2. An index should not appear more than twice on each side of the equation.

~~$$\alpha = a_i x_i y_i$$~~

$$\alpha = \sum_{i=1}^3 a_i x_i y_i \quad \checkmark$$

$$a_{ii} = a_{11} + a_{22} + a_{33}$$

$$\alpha = \underset{\substack{i \\ \equiv \\ j}}{a_{ij}} x_i x_j = a_{1j} x_1 x_j + a_{2j} x_2 x_j + a_{3j} x_3 x_j$$

$$= a_{11} x_1 z_1 + a_{21} x_2 z_1 + a_{31} x_3 z_1$$

$$+ a_{12} x_1 z_2 + a_{22} x_2 z_2 + a_{32} x_3 z_2$$

$$+ a_{13} x_1 z_3 + a_{23} x_2 z_3 + a_{33} x_3 z_3$$

the number of the terms is $3^2 = 9$

3. If a subscript appears once on one side, it should appear once on the other side (free index)

$$\alpha_j = a_i x_i b_j \quad \begin{matrix} \text{free index} \\ \text{---} \\ \text{dummy index} \end{matrix}$$

$$\alpha_j = a_i x_i b_j = a_1 x_1 b_j + a_2 x_2 b_j + a_3 x_3 b_j$$

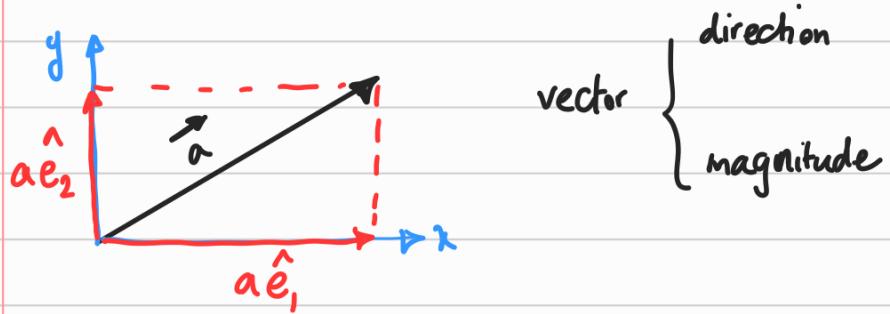
$$\begin{cases} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{cases} = \begin{cases} a_1 x_1 b_1 + a_2 x_2 b_1 + a_3 x_3 b_1 \\ a_1 x_1 b_2 + a_2 x_2 b_2 + a_3 x_3 b_2 \\ a_1 x_1 b_3 + a_2 x_2 b_3 + a_3 x_3 b_3 \end{cases} \quad \begin{matrix} j=1 \\ j=2 \\ j=3 \end{matrix}$$

\therefore the number of expressions is $3^1 = 3$

The number of expressions: 3 free index

$$T_{ij} = \underbrace{A_{im}}_{\text{---}} \underbrace{A_{jm}}_{\text{---}} \quad \begin{matrix} \text{dummy index} \\ \text{free index} \end{matrix} \quad \begin{matrix} 3^1 = 3 \\ 3^2 = 9 \end{matrix}$$

$$\begin{matrix} 9 \text{ separate expressions} \\ \vdots \\ T_{11} \\ T_{12} \\ \vdots \\ T_{33} \end{matrix} = \begin{cases} A_{11} A_{11} + A_{12} A_{12} + A_{13} A_{13} \\ \vdots \\ A_{31} A_{31} + A_{32} A_{32} + A_{33} A_{33} \end{cases}$$



$$\vec{a} = \underbrace{a \hat{e}_1}_x + \underbrace{a \hat{e}_2}_y + \underbrace{a \hat{e}_3}_z = a \hat{e}_i$$

$$a_i = a \hat{e}_i \rightarrow a_i = a e_i \quad (a_i = a e_i)$$

$$a = a_i e_i$$

transformation

$$\hat{e}'_i = Q_{mi} \hat{e}_m$$

dummy index free index

from 123 coordinate to 1'2'3' coordinate

$$\begin{cases} \hat{e}'_1 \\ \hat{e}'_2 \\ \hat{e}'_3 \end{cases} = \begin{cases} Q_{11} \hat{e}_1 + Q_{21} \hat{e}_2 + Q_{31} \hat{e}_3 \\ Q_{12} \hat{e}_1 + Q_{22} \hat{e}_2 + Q_{32} \hat{e}_3 \\ Q_{13} \hat{e}_1 + Q_{23} \hat{e}_2 + Q_{33} \hat{e}_3 \end{cases} \quad \begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix}$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\delta_{ii} \stackrel{i=j}{=} 1 = \delta_{22} = \delta_{33}$$

$i \nearrow \searrow j$

$$\delta_{13} \stackrel{i \neq j}{=} 0 = \delta_{12} = \delta_{23} = \delta_{32} = \dots$$

$i \nearrow \searrow j$

The Kronecker Delta in matrix form

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{Identity matrix} = I$$

$$\delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

$$\delta_{1m} a_m = \cancel{\delta_{11} a_1} + \cancel{\delta_{12} a_2} + \cancel{\delta_{13} a_3} = a_1 \quad (\delta_{im} a_m \text{ where } i=1)$$

$$\delta_{2m} a_m = \cancel{\delta_{21} a_1} + \cancel{\delta_{22} a_2} + \cancel{\delta_{23} a_3} = a_2 \quad i=2$$

$$\delta_{3m} a_m = \cancel{\delta_{31} a_1} + \cancel{\delta_{32} a_2} + \cancel{\delta_{33} a_3} = a_3 \quad i=3$$

$$\delta_{im} a_m = a_i$$

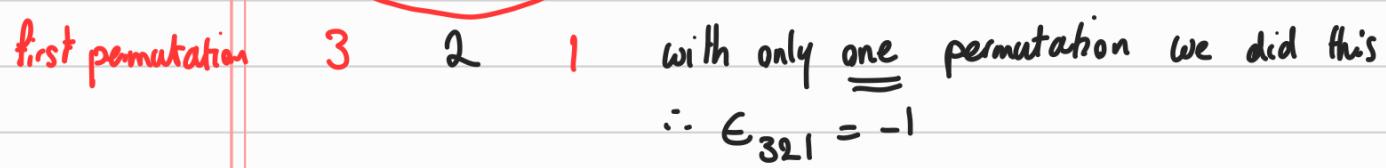
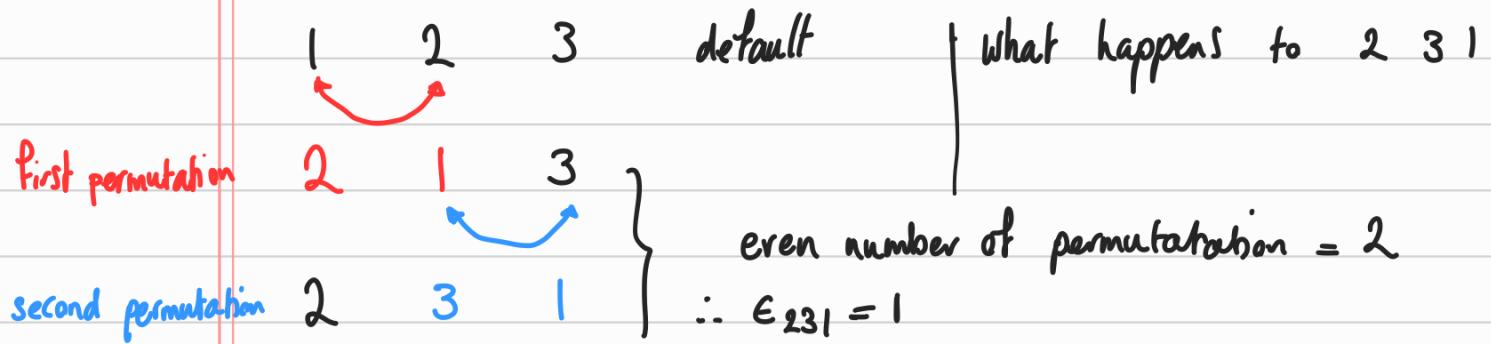
* In two-dimensional problem

$$\left\{ \begin{array}{l} \delta_{\alpha m} a_m = a_\alpha \text{ where } \alpha = 1, 2 \\ \delta_{\beta n} a_n = a_\beta \text{ where } \beta = 1, 2 \end{array} \right.$$

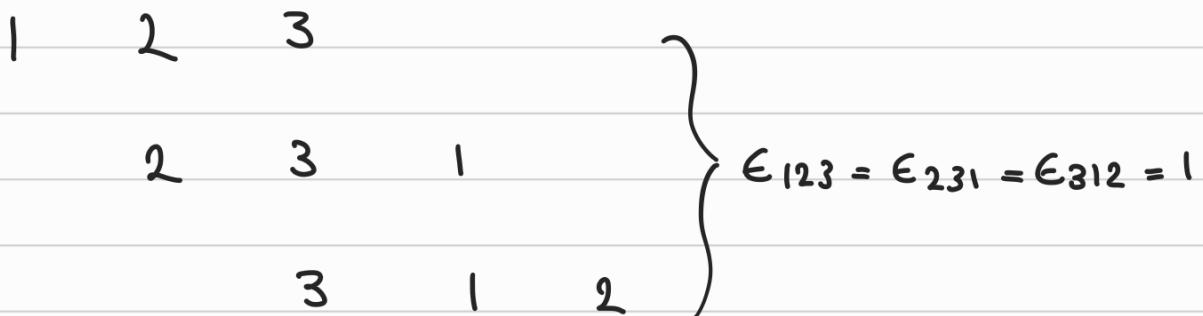
The Permutation Symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if the number of permutation of } 1, 2, 3 \text{ is even} \\ -1 & \text{if the number of permutation of } 1, 2, 3 \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

What I mean by Permutation

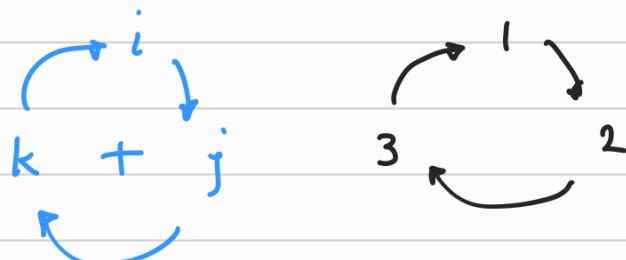


Even permutations

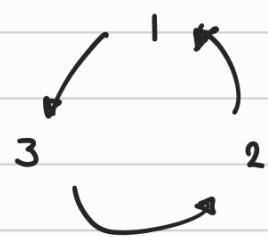
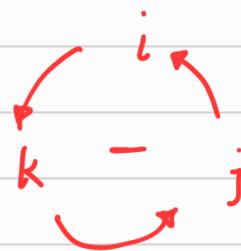


When $\epsilon_{ijk} = 0$? When you have at least two subscripts equal.

$$\epsilon_{112} = \epsilon_{211} = \epsilon_{322} = \epsilon_{212} = \dots = 0$$



$$\begin{aligned}\epsilon_{123} &= 1 \\ \epsilon_{312} &= 1 \\ \epsilon_{231} &= 1\end{aligned}$$



$$\epsilon_{321} = -1$$

$$\epsilon_{132} = -1$$

$$\epsilon_{213} = -1$$

Dot Product (.)

$$\vec{a} \cdot \vec{b} = a_i \hat{e}_i \cdot b_j \hat{e}_j = a_i b_j \hat{e}_i \cdot \hat{e}_j =$$

$$= a_1 b_1 \hat{e}_1 \cdot \hat{e}_1 + a_2 b_2 \hat{e}_2 \cdot \hat{e}_2 + a_3 b_3 \hat{e}_3 \cdot \hat{e}_3$$

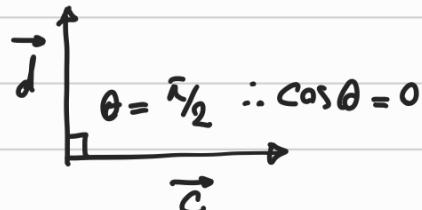
$$= a_1 b_1 \cancel{\hat{e}_1 \cdot \hat{e}_1} + a_1 b_2 \cancel{\hat{e}_1 \cdot \hat{e}_2} + a_1 b_3 \cancel{\hat{e}_1 \cdot \hat{e}_3}$$

$$+ a_2 b_1 \cancel{\hat{e}_2 \cdot \hat{e}_1} + a_2 b_2 \cancel{\hat{e}_2 \cdot \hat{e}_2} + a_2 b_3 \cancel{\hat{e}_2 \cdot \hat{e}_3}$$

$$+ a_3 b_1 \cancel{\hat{e}_3 \cdot \hat{e}_1} + a_3 b_2 \cancel{\hat{e}_3 \cdot \hat{e}_2} + a_3 b_3 \cancel{\hat{e}_3 \cdot \hat{e}_3}$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i = a_i b_i \delta_{ij}$$

$$\vec{c} \cdot \vec{d} = |c| |d| \cos \theta$$

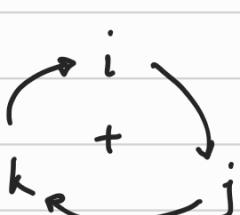


Cross Product (x)

$$a \times b = a \hat{e}_i \times b \hat{e}_j = ab \hat{e}_i \times \hat{e}_j$$

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3 = \cancel{e_{123}} \hat{e}_3$$

$$\hat{e}_i \times \hat{e}_j = \epsilon_{ijk} \hat{e}_k$$



$$\hat{e}_j \times \hat{e}_i = \epsilon_{jik} \hat{e}_k$$

$\underbrace{< 0$