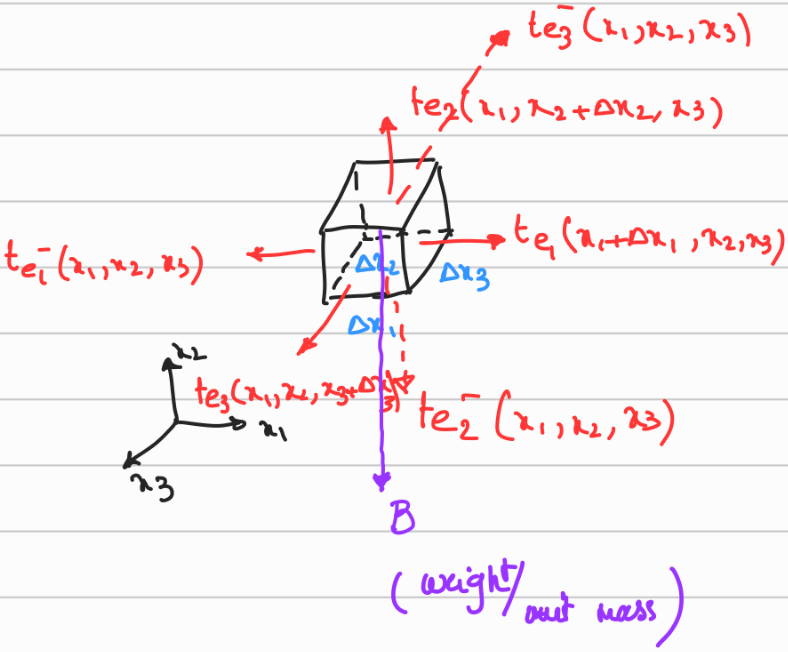
Stress Power (Strain Energy Rate):



$$dW = \vec{F}.d\vec{r}$$

$$\text{Power} = \frac{dW}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \vec{F}.\vec{v}$$

surface + Body force

$t_{e_i}\,dA$ $\quad$ $B_i(\rho dV)$

(traction/unit area)

surface force : $(t_{e_1}\cdot dx_2 dx_3)\cdot v_{\{x_1+dx_1,x_2,x_3\}} + (t_{e_1}^-\cdot dx_2 dx_3)\cdot v_{\{x_1,x_2,x_3\}}$

$$= \left\{ t_{e_1}\cdot v_{\{x_1+dx_1,x_2,x_3\}} - t_{e_1}\cdot v_{\{x_1,x_2,x_3\}} \right\} dx_2 dx_3$$

$$= \frac{\partial(t_{e_1}\cdot v)}{\partial x_1} dx_1\, dx_2\, dx_3$$

$$t_{e_1}\cdot v = (Te_1)\cdot v = e_1\cdot T^T_{1j} v_j e_1 = e_1\cdot T_{j1} v_j e_1 = T_{j1} v_j \underbrace{e_1\cdot e_1}_{\delta_{11}}$$

$$t_{e_1}\cdot v = T_{j1} v_j$$

$$t_{e_2}\cdot v = T_{j2} v_j \quad , \quad t_{e_3}\cdot v = T_{j3} v_j$$

$$P = \left[\frac{\partial (T_{ij} v_i)}{\partial x_j} dV + \rho B_i \, dV \, v_i \right]$$

$$= \left[\frac{\partial (T_{ij} v_i)}{\partial x_j} + \rho B_i v_i \right] dV$$

$$= \left[\frac{\partial T_{ij}}{\partial x_j} v_i + T_{ij} \frac{\partial v_i}{\partial x_j} + \rho B_i v_i \right] dV$$

$$\left(\frac{\partial T_{ij}}{\partial x_j} + \rho B_i \right) v_i = \rho a_i v_i = \rho \frac{D v_i}{Dt} v_i$$

$$P = \left[\rho \frac{D v_i}{Dt} v_i + T_{ij} \frac{\partial v_i}{\partial x_j} \right] dV = \frac{D(KE)}{Dt} + \underbrace{tr(TD)}_{P_s}$$

P_s (stress power)

$$\rho \frac{D v_i}{Dt} v_i \, dV$$ dm : conservation of mass : $\frac{D}{Dt}(\rho \, dV) = 0$

$$\frac{D(v_i v_i / 2)}{Dt} = \frac{1}{2} \frac{D v_i}{Dt} v_i + \frac{1}{2} v_i \frac{D v_i}{Dt} = v_i \frac{D v_i}{Dt}$$

$$\rho \frac{D v_i}{Dt} v_i \, dV = \rho \, dV \frac{D(v_i v_i / 2)}{Dt} = \frac{D(\rho \, dV \, v_i v_i / 2)}{Dt} = \frac{D(KE)}{Dt}$$

$$T_{ij} \frac{\partial v_i}{\partial x_j} \overset{T = T^T}{=\!=} T^T_{ij} \frac{\partial v_i}{\partial x_j} = T_{ji} \frac{\partial v_i}{\partial x_j} = tr(T^T \nabla v)$$

$$T_{ij} \frac{\partial v_i}{\partial x_j} \xrightarrow[i,j]{\text{swap}} T_{ji} \frac{\partial v_j}{\partial x_i} = \frac{1}{2} T_{ji} \frac{\partial v_j}{\partial x_i} + \underbrace{\frac{1}{2} T_{ji} \frac{\partial v_j}{\partial x_i}}_{\text{swap } i,j}$$

$$= \frac{1}{2} T_{ji} \frac{\partial v_j}{\partial x_i} + \frac{1}{2} \cancel{T_{ij}}^{T_{ji}} \frac{\partial v_i}{\partial x_j} = T_{ji} \underbrace{\left\{ \frac{1}{2} \left((\nabla v)^T + (\nabla v) \right) \right\}}_{D} = T_{ji} D_{ij}$$

$$= tr(TD)$$

P_s in terms of 1st PKST:

$$\frac{D}{Dt} dx = (\nabla v)\, dx \quad (2) \qquad \text{also} \quad F = \frac{dx}{dX} \Rightarrow dx = F dX \quad (1)$$

spatial coordinates

(1) in (2): $\frac{D}{Dt}(F dX) = (\nabla v)\, F dX$

$$\frac{DF}{Dt} dX + F \cancel{\frac{D(dX)}{Dt}}^{\,0} = \nabla v\, F dX \Rightarrow \frac{DF}{Dt} = \nabla v\, F$$

$$\frac{DF}{Dt} F^{-1} = \nabla v$$

$$P_s = tr(T^T \nabla v) \quad \left\{ \begin{array}{l} \nabla v = \frac{DF}{Dt} F^{-1} \\ T = \frac{1}{J} T_0 F^T \Rightarrow T^T = \frac{1}{J} F T_0^T \end{array} \right. \Rightarrow$$

$$P_s = \frac{1}{J} tr\left(F T_0^T \frac{DF}{Dt} F^{-1}\right) \overset{(3)}{=} \frac{1}{J} tr\left(T_0^T \frac{DF}{Dt}\right)$$

$$tr\left(F T_0^T \frac{DF}{Dt} F^{-1}\right)$$

$$= tr\left(T_0^T \frac{DF}{Dt} F^{-1} F\right) \quad (3)$$

$$= tr\left(\frac{DF}{Dt} F^{-1} F T_0^T\right)$$

$$= tr\left(F^{-1} F T_0^T \frac{DF}{Dt}\right)$$

$$\boxed{P_s = \frac{1}{J} tr\left(T_0^T \frac{DF}{Dt}\right)}$$

$$\rho_0 = \rho J \qquad \frac{1}{J} = \frac{\rho}{\rho_0}$$

$$P_s = \frac{\rho}{\rho_0} tr\left(T_0^T \frac{DF}{Dt}\right)$$

P_s in terms of 2nd PKST:

$$P_s = tr(T D) = tr(T^T \nabla v) \quad \left. \begin{array}{l} \\ T = \frac{1}{J} F \tilde{T} F^T \end{array} \right\} \quad P_s = tr\left(\frac{1}{J} F \tilde{T} F^T D\right)$$

$$= \frac{1}{J} tr\big(\tilde{T} \underbrace{F^T D F}_{?}\big)$$

What is $F^T D F$?

Proof 1: $$dx^1 \cdot dx^2 - dX^1 \cdot dX^2 = 2\, dX^1 \cdot E^* dX^2$$

$$dx^1 = dx^2 = ds\, \hat{n} \quad , \quad dX^1 = dX^2 = dS\, \hat{m} = dX$$

$$ds^2 - dS^2 = 2\, dX \cdot E^* dX \xrightarrow{\frac{D}{Dt}}$$

$$\frac{D}{Dt}(ds^2) = 2\, dX \cdot \frac{DE^*}{Dt} dX$$

$$ds \cdot \frac{D(ds)}{Dt} = dX \cdot \frac{DE^*}{Dt} dX$$

$$ds \cdot \frac{D(ds)}{Dt} = dx \cdot D\, dx \quad \Rightarrow \quad dx \cdot D\, dx = dX \cdot \frac{DE^*}{Dt} dX$$

$$\xrightarrow{dx = F dX} \quad F dX \cdot D\, F dX = dX \cdot F^T D F\, dX = dX \cdot \frac{DE^*}{Dt} dX$$

$$\Rightarrow \frac{DE^*}{Dt} = F^T D F$$

Proof 2: $$E^* = \frac{1}{2}(C - I)$$

$$E^* = \frac{1}{2}(F^T F - I) \xrightarrow{D/Dt} \frac{DE^*}{Dt} = \frac{1}{2}\left(\frac{DF^T}{Dt} F + F^T \frac{DF}{Dt}\right)$$

$$\frac{DF}{Dt} = \frac{D\left(\frac{dx}{dX}\right)}{Dt} = \frac{D\, dx}{Dt} \cdot \frac{1}{dX} \overset{(2)}{=} (\nabla v) \frac{dx}{dX} = \nabla v\, F$$

$$\left(\frac{DF}{Dt}\right)^T = (\nabla v\ F)^T = F^T (\nabla v)^T$$

$$\frac{DE^*}{Dt} = \frac{1}{2}\left(F^T (\nabla v)^T F + F^T (\nabla v) F\right) = F^T \underbrace{\left(\frac{1}{2}\left((\nabla v)^T + (\nabla v)\right)\right)}_{D} F$$

$$\frac{DE^*}{Dt} = F^T D F$$

$$P_s = \frac{1}{J}\left(\tilde{T}\,\frac{DE^*}{Dt}\right) \longrightarrow \boxed{P_s = \frac{\rho}{\rho_0}\left(\tilde{T}\,\frac{DE^*}{Dt}\right)}$$

Conjugate pairs		Stress power	
T	D	T	D
T_0	F	T_0	$\dot{F}$
$\tilde{T}$	E^*	$\tilde{T}$	$\dot{E}^*$