Substitution, Multiplication, Factoring, Contraction

Scalar Triple Product, Vector Triple Product

Substitution

$a_i = U_{im} b_m$

$b_i = V_{im} c_m$

$\xrightarrow[b_i \text{ in } a_i]{\text{plug in}}$ $\cancel{a_i = U_{im} V_{im} c_m}$

Step 1. $b_j = V_{jn} c_n$

Step 2. $a_i = U_{im} b_m = U_{ij} b_j$

$\overset{\text{Step}}{\underset{1}{=}} U_{ij} V_{jn} c_n = a_i$

Multiplication

$p = a_m b_m$

$q = c_m d_m$

$\longrightarrow$ $\cancel{pq = a_m b_m c_m d_m}$

Step 1. $q = c_i d_i$

Step 2. $pq = a_m b_m c_i d_i$

Factoring

$$T_{ij} n_j - \underbrace{\lambda n_i} = 0$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\lambda n_i = \lambda \underbrace{\delta_{ij} n_j}_{n_i}$ ①

$$T_{ij}n_j - \lambda n_i = 0 \xrightarrow{①} T_{ij}n_j - \lambda\delta_{ij}n_j = 0$$

$$(T_{ij} - \lambda\delta_{ij})n_j = 0$$

Contraction

T_{ii} is the contraction of T_{ij}

$$T_{ij} \xrightarrow{\text{Contraction}} T_{ii} = T_{11} + T_{22} + T_{33}$$

$$T_{ij} = \lambda\Delta\delta_{ij} + 2\mu E_{ij} \xrightarrow{\text{Contraction}} T_{ii} = \lambda\Delta\underset{3}{\cancel{\delta_{ii}}} + 2\mu E_{ii}$$

$$= 3\lambda\Delta + 2\mu E_{ii}$$

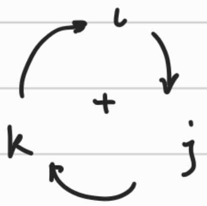
Scalar Triple Product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_i\hat{e}_i \cdot \underbrace{(b_j\hat{e}_j \times c_k\hat{e}_k)}_{②} = a_i\hat{e}_i \cdot b_jc_k\epsilon_{jkm}\hat{e}_m$$

$$= a_ib_jc_k\epsilon_{jkm}\hat{e}_i\cdot\hat{e}_m = a_ib_jc_k\epsilon_{jkm}\delta_{im}$$

$$\overset{i=m}{=} a_ib_jc_k\epsilon_{jki} = a_ib_jc_k\epsilon_{ijk}$$

② $b_j\hat{e}_j \times c_k\hat{e}_k = b_jc_k\hat{e}_j\times\hat{e}_k = b_jc_k\epsilon_{jkm}\hat{e}_m$



$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

$a \cdot (b \times c) = a_i b_j c_k \epsilon_{ijk}$

$b \cdot (c \times a) = b_j c_k a_i \epsilon_{jki}$

$c \cdot (a \times b) = c_k a_i b_j \epsilon_{kij}$

Let α be a scalar

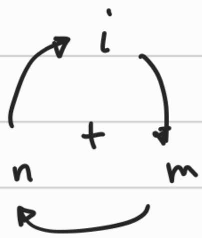
$\alpha a \cdot (b \times c) = \alpha a_i b_j c_k \epsilon_{ijk}$

$\alpha b \cdot (c \times a) = \alpha b_j c_k a_i \epsilon_{jki}$

Vector Triple Product

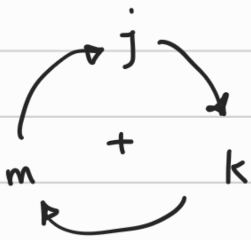
$$\vec{a} \times (\vec{b} \times \vec{c}) = a_i \hat{e}_i \times \underbrace{\left(b_j \hat{e}_j \times c_k \hat{e}_k \right)}_{3} = a_i \hat{e}_i \times b_j c_k \epsilon_{jkm} \hat{e}_m$$

$$= a_i b_j c_k \epsilon_{jkm} \hat{e}_i \times \hat{e}_m = a_i b_j c_k \epsilon_{jkm} \epsilon_{imn} \hat{e}_n \quad 4$$



$\epsilon_{imn} = \epsilon_{mni} = \epsilon_{nim}$

3 $b_j \hat{e}_j \times c_k \hat{e}_k = b_j c_k \hat{e}_j \times \hat{e}_k = b_j c_k \epsilon_{jkm} \hat{e}_m$



④ Eq(2.4.7) $\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$

$$a_i b_j c_k \underbrace{\epsilon_{jkm}\epsilon_{imn}}_{⑤} \hat{e}_n$$

⑤ $\epsilon_{jkm}\epsilon_{imn} = \epsilon_{jkm}\epsilon_{nim} = \delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn}$

④ $a_i b_j c_k(\delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn})\hat{e}_n = a_i b_j c_k \underbrace{\delta_{jn}}_{j=n}\underbrace{\delta_{ki}}_{k=i}\hat{e}_n - a_i b_j c_k \underbrace{\delta_{ji}}_{j=i}\underbrace{\delta_{kn}}_{k=n}\hat{e}_n$

$$= a_i b_j c_i \hat{e}_j - a_i b_i c_k \hat{e}_k = \underbrace{(a_i c_i)}_{\vec{a}\cdot\vec{c}}\underbrace{(b_j \hat{e}_j)}_{\vec{b}} - \underbrace{a_i b_i}_{\vec{a}\cdot\vec{b}}\underbrace{(c_k \hat{e}_k)}_{\vec{c}}$$

$$= (a\cdot c)b - (a\cdot b)c$$

$$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$$

$$b\times(c\times a) = (b\cdot a)c - (b\cdot c)a$$

$$c\times(a\times b) = (c\cdot b)a - (c\cdot a)b$$

$a\times(b\times c) = \cancel{(a\cdot c)b} - \cancel{(a\cdot b)c}$ $\quad c\cdot a = a\cdot c$

$+ \qquad +$

$b\times(c\times a) = \cancel{(b\cdot a)c} - \cancel{(b\cdot c)a}$ $\quad c\times a = -a\times c$

$+ \qquad +$

$c\times(a\times b) = \cancel{(c\cdot b)a} - \cancel{(c\cdot a)b}$

$$a\times(b\times c) + b\times(c\times a) + c\times(a\times b) = 0$$