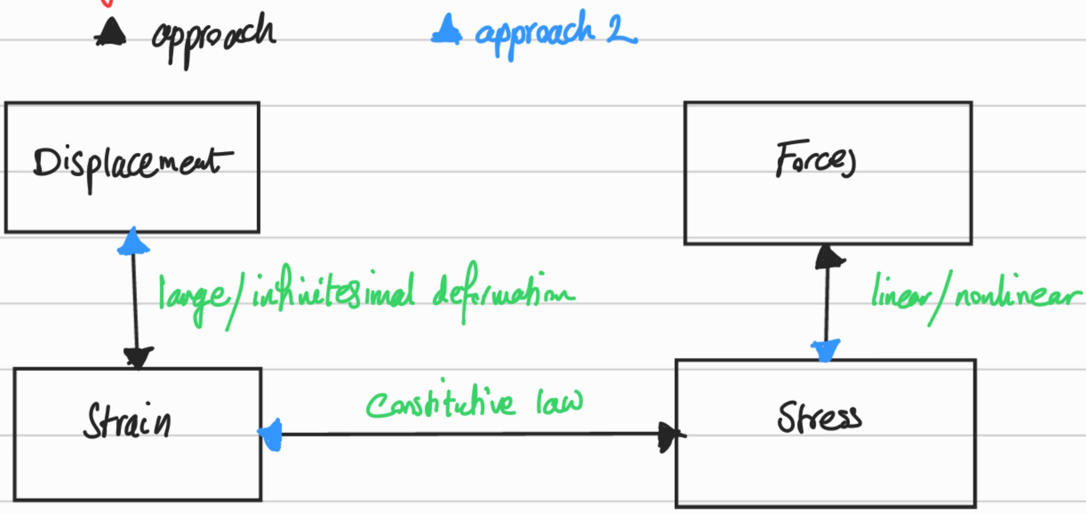
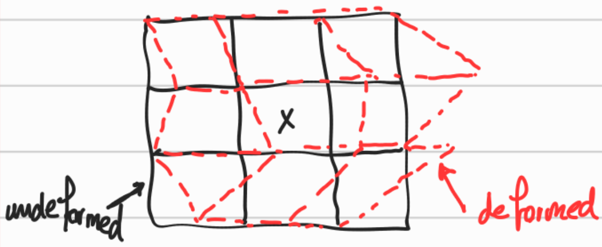
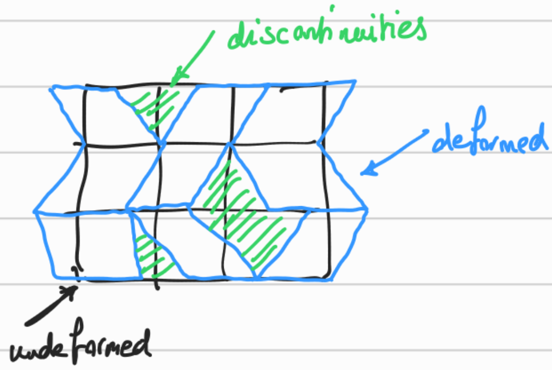
Compatibility Equations for Infinitesimal strain:





approach 1



possible answer from approach 2

Why Compatibility Equations? In elasticity we make an important assumption. This assumption is a continuous domain/continuum. Also our displacement fields should be single-valued. From infinitesimal strain tensor we had

$$E = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right] = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right]$$

From above equation we get 6 equations ($E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}$).

However, we have 3 unknowns (displacements) which are u_1, u_2, u_3

Therefore to get continuous single-valued displacements we need to satisfy compatibility equations (Saint Venant).

* It should be noted that the strain compatibility equations are satisfied automatically when the strains are derived from a displacement field (approach 1)

* Thus, one need to verify the compatibility equations only when the strains are derived from stress-strain constitutive law.

Deriving Compatibility Equations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\xrightarrow{\partial/\partial x_k \partial x_l} \quad \boxed{\varepsilon_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right)}$$

$i \quad j \,,\, k \quad l$	
$k \quad l \,,\, i \quad j$	$\varepsilon_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right)$
$j \quad l \,,\, i \quad k$	$\varepsilon_{jl,ik} = \frac{1}{2}\left(u_{j,lik} + u_{l,jik}\right)$
$i \quad k \,,\, j \quad l$	$\varepsilon_{ik,jl} = \frac{1}{2}\left(u_{i,kjl} + u_{k,ijl}\right)$

$$\boxed{\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{il,ik} = 0}$$

Saint Venant Compatibility Equation

$3^4 = 81$ equations $\longrightarrow$ we get 6 equations (3.16.7 – 3.16.12)

Cartesian Coordinate System

$\nabla \times (\nabla \times \varepsilon)$

Continuous displacement intuition:

Suppose we have infinitesimal strain below from a stress-strain relation.

$$E_{11} = k X_2^2 \;, \quad E_{22} = E_{33} = E_{12} = E_{13} = E_{23} = 0$$

We would like to check if such strains yield a continuous single-valued displacement field.

$$E_{11} = k X_2^2 = \frac{\partial u_1}{\partial X_1} \Rightarrow \int \partial u_1 = \int k X_2^2 \, \partial X_1 \Rightarrow u_1 = k X_2^2 X_1 + f_1(X_2, X_3)$$

$$E_{22} = 0 = \frac{\partial u_2}{\partial X_2} \Rightarrow \int \partial u_2 = \int 0 \, \partial X_2 \Rightarrow u_2 = f_2(X_1, X_3)$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = 0 = 2k X_1 X_2 + \frac{\partial f_1(X_2, X_3)}{\partial X_2} + \frac{\partial f_2(X_1, X_3)}{\partial X_1}$$

a function of (X_1, X_2) $\qquad$ $f_1(X_2, X_3)$ $\qquad$ $f_2(X_1, X_3)$

(impossible!)

$\therefore$ the above strain tensor is incompatible.

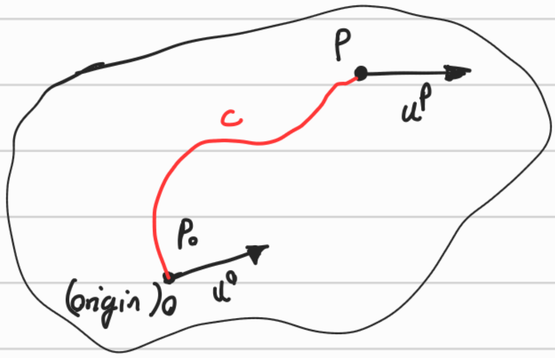
Simply connected vs. Multiply-connected:

Simply-connected

Multiply-connected

* Incompatibility equation is the necessary and sufficient condition for simply-connected domains.

Proof:



We place the origin (o) in P_0.

$$u_i^P = u_i^0 + \int_c du_i = u_i^0 + \int_c \frac{\partial u_i}{\partial X_j} dX_j = u_i^0 + \int_c \nabla u \, dX_j$$

$$= u_i^0 + \int_c (E_{ij} + W_{ij}) \, dX_j = u_i^0 + \int_c E_{ij} dX_j + \int_c W_{ij} dX_j \quad (1)$$

Let us get rid of $\int_c W_{ij} dX_j$

Using integration by parts, $d(W_{ij} X_j) = dW_{ij} X_j + W_{ij} dX_j$

$$\Rightarrow W_{ij} dX_j = d(W_{ij} X_j) - dW_{ij} X_j$$

$$\xrightarrow{\int} \int_0^P W_{ij} dX_j = (W_{ij} X_j)\Big|_0^P - \int_0^P X_j \, dW_{ij}$$

$$= (W_{ij} X_j) - \int_c X_j W_{ij,k} dX_k \quad (2)$$

Next, $\left.\begin{array}{l} \dfrac{\partial W_{ij}}{\partial X_k} = W_{ij,k} = ? \\ W_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \end{array}\right\} \quad W_{ij,k} = \frac{1}{2}(u_{i,jk} - u_{j,ik})$

$$W_{ij,k} = \frac{1}{2}(u_{i,jk} - u_{j,ik}) = \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial X_j \partial X_k} - \frac{\partial^2 u_j}{\partial X_i \partial X_k}\right)$$

$$+\frac{1}{2}\frac{\partial^2 u_k}{\partial X_j \partial X_i} - \frac{1}{2}\frac{\partial^2 u_k}{\partial X_j \partial X_i}$$

$$= \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial X_j \partial X_k} + \frac{\partial^2 u_k}{\partial X_j \partial X_i}\right)$$

$$-\frac{1}{2}\left(\frac{\partial^2 u_j}{\partial X_i \partial X_k} + \frac{\partial^2 u_k}{\partial X_j \partial X_i}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial X_j}\left(u_{i,k} + u_{k,i}\right) - \frac{1}{2}\frac{\partial}{\partial X_i}\left(u_{j,k} + u_{k,j}\right)$$

$$= \frac{\partial}{\partial X_j}\left(E_{ik}\right) - \frac{\partial}{\partial X_i}\left(E_{jk}\right) = E_{ik,j} - E_{jk,i} \qquad (2)$$

We plug (3) in (2) and in (1)

$$u_i^P = u_i^0 + \int_C E_{ij}\, dX_j + W_{ij}^P X_j^P - \int_C X_j\left(E_{ik,j} - E_{jk,i}\right) dX_k$$

$$= u_i^0 + W_{ij}^P X_j^P + \int_C E_{ik}\, dX_k - \int_C X_j\left(E_{ik,j} - E_{jk,i}\right) dX_k$$

$$= u_i^0 + W_{ij}^P X_j^P + \int_C E_{ik} - X_j\left(E_{ik,j} - E_{jk,i}\right) dX_k$$

To get a unique displacement field, the above integral on path "C" should be independent of the path we take. In mathematics, we call this integral "exact differential" and using "Stokes Theorem" (only for a simply-connected domain), the integral is path-independent.

$$U_{ik} = E_{ik} - X_j\left(E_{ik,j} - E_{jk,i}\right)$$

$$U_{ik,l} = U_{il,k}$$

$$E_{ik,l} - \underbrace{X_{j,l}}_{\delta_{jl}}\left(E_{ik,j} - E_{jk,i}\right) - X_j\left(E_{ik,jl} - E_{jk,il}\right)$$

$$= E_{il,k} - \underbrace{X_{j,k}}_{\delta_{jk}}\left(E_{il,j} - E_{jl,i}\right) - X_j\left(E_{il,jk} - E_{jl,ik}\right)$$

$$\cancel{E_{ik,l}} - \left(\cancel{E_{ik,l}} - \cancel{E_{lk,i}}\right) - X_j\left(E_{ik,jl} - E_{jk,il}\right)$$

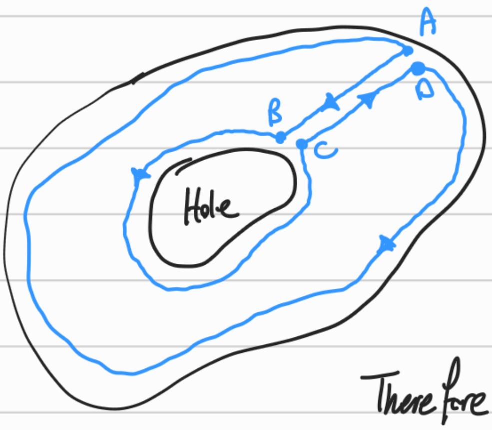
$$= \cancel{E_{il,k}} - \left(\cancel{E_{il,k}} - \cancel{E_{kl,i}}\right) - X_j\left(E_{il,jk} - E_{jl,ik}\right)$$

$$-X_j\left(E_{ik,jl} - E_{jk,il}\right) + X_j\left(E_{il,jk} - E_{jl,ik}\right) = 0$$

$$X_j\left(E_{il,jk} - E_{jl,ik} - E_{ik,jl} + E_{jk,il}\right) = 0$$

$$E_{il,jk} - E_{jl,ik} - E_{ik,jl} + E_{jk,il} = 0$$

For Multiply-connected domain,

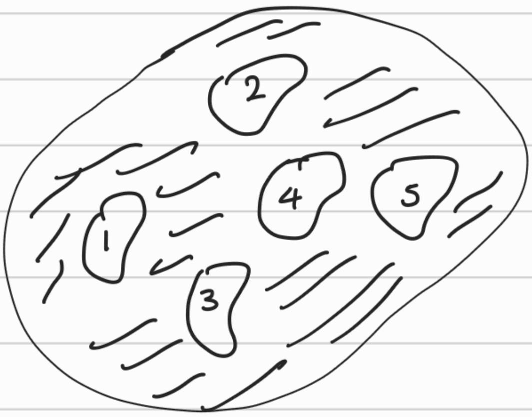


$$\int_A^B = -\int_C^D \Rightarrow \int_A^B + \int_C^D = 0$$

$$\int_B^C = -\int_D^A$$

Therefore, for a multiply-connected domain,

$$\int_{\text{hole}} du_i = 0 \quad \text{and} \quad \int_{\text{hole}} d\omega_i = 0$$



$$\left.\begin{array}{l} \displaystyle\int_{C_n} du_i = 0 \\ C_n,\ n=1,2,3,\ldots \\ \\ \displaystyle\int_{C_n} d\omega_i = 0 \\ C_n,\ n=1,2,3,\ldots \end{array}\right\} \text{Cesaro Integral}$$

* For multiply-connected domains, the incompatibility equation is only necessary but not sufficients. Also Cesaro Integrals should be zero on discontinuities.