Einstein's Summation Convention

The Kronecker Delta

The Permutation Symbol

$$\alpha = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{i=1}^{n} a_i x_i = \sum_{j=1}^{n} a_j x_j = \sum_{k=1}^{n} a_k x_k$$

dummy index

$$\alpha = \cancel{\sum_{i=1}^{n}} a_i x_i = a_i x_i = a_j x_j = a_k x_k = a_s x_s$$

$n = 3 : \alpha = a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3$ $\quad 3' = 3$

$n = 2 : \alpha = a_j x_j = a_1 x_1 + a_2 x_2$ $\quad 2' = 2$

$\left.\begin{matrix} 3' = 3 \\ 2' = 2 \end{matrix}\right\}$ number of terms

To get the number of terms (the last term in series)$^{\text{dummy index}}$

In CM, we do not exceed $n = 3 \therefore 3^{\text{dummy index}}$

1. Dummy index is an index which is repeated twice. We sum over the repeated index.
2. An index should not appear more than twice on each side of the equation.

$$\cancel{\alpha = a_i x_i y_i} \qquad \alpha = \sum_{i=1}^{3} a_i x_i y_i \checkmark$$

$$a_{ii} = a_{11} + a_{22} + a_{33}$$

$$\alpha = a_{ij} x_i x_j = a_{1j} x_1 x_j + a_{2j} x_2 x_j + a_{3j} x_3 x_j$$

$$= a_{11}x_1x_1 + a_{21}x_2x_1 + a_{31}x_3x_1$$

$$+ a_{12}x_1x_2 + a_{22}x_2x_2 + a_{32}x_3x_2$$

$$+ a_{13}x_1x_3 + a_{23}x_2x_3 + a_{33}x_3x_3$$

the number of the terms is $3^2 = 9$

3. If a subscript appears once on one side, it should appear once on the other side (free index)

$\alpha_j = a_i x_i b_j$ free index

dummy index

$$\alpha_j = a_i x_i b_j = a_1x_1b_j + a_2x_2b_j + a_3x_3b_j$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{Bmatrix} a_1x_1b_1 + a_2x_2b_1 + a_3x_3b_1 \\ a_1x_1b_2 + a_2x_2b_2 + a_3x_3b_3 \\ a_1x_1b_3 + a_2x_2b_3 + a_3x_3b_3 \end{Bmatrix} \begin{array}{l} j=1 \\ j=2 \\ j=3 \end{array}$$

$\therefore$ the number of expressions is $3^1 = 3$

The number of expressions : $3^{\text{free index}}$

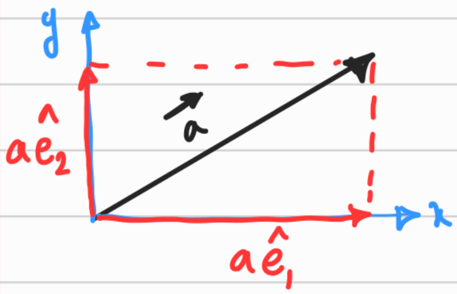
$T_{ij} = A_{im} A_{jm}$

dummy index $3^1 = 3$

free index $3^2 = 9$

9 separate expressions

$$\begin{Bmatrix} T_{11} \\ T_{12} \\ \vdots \\ T_{33} \end{Bmatrix} = \begin{Bmatrix} A_{11}A_{11} + A_{12}A_{12} + A_{13}A_{13} \\ \vdots \\ A_{31}A_{31} + A_{32}A_{32} + A_{33}A_{33} \end{Bmatrix}$$



vector $\begin{cases} \text{direction} \\ \text{magnitude} \end{cases}$

$$\vec a = \underbrace{a\hat e_1}_{x} + \underbrace{a\hat e_2}_{y} + \underbrace{a\hat e_3}_{z} = a\hat e_i$$

$$a_i = a\hat e_i \longrightarrow a_i = a e_i \quad \left(a_i = a e_i \right)$$

$$a = a_i \hat e_i$$

$$\underbrace{e'_i}_{} = \underbrace{Q_{mi}}_{} \underbrace{e_m}_{}$$

dummy index

free index

transformation from 123 coordinate to 1'2'3' coordinate

$$\begin{Bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{Bmatrix} = \begin{Bmatrix} Q_{11}\hat e_1 + Q_{21}\hat e_2 + Q_{31}\hat e_3 \\ Q_{12}\hat e_1 + Q_{22}\hat e_2 + Q_{32}\hat e_3 \\ Q_{13}\hat e_1 + Q_{23}\hat e_2 + Q_{33}\hat e_3 \end{Bmatrix} \quad \begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix}$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

$$\delta_{11} \overset{i=j}{=\!=} 1 = \delta_{22} = \delta_{33}$$

$i \quad j$

$$\delta_{13} \overset{i\neq j}{=\!=} 0 = \delta_{12} = \delta_{23} = \delta_{32} = \dots$$

$i \quad j$

The Kronecker Delta in matrix form

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{Identity matrix} = \mathbf{I}$$

$$\delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

$$\delta_{1m} a_m = \underbrace{\cancel{\delta_{11}}}_{1} a_1 + \underbrace{\cancel{\delta_{12}}}_{0} a_2 + \underbrace{\cancel{\delta_{13}}}_{0} a_3 = a_1 \quad (\delta_{im} a_m \text{ where } i=1)$$

$$\delta_{2m} a_m = \underbrace{\cancel{\delta_{21}}}_{0} a_1 + \underbrace{\cancel{\delta_{22}}}_{1} a_2 + \underbrace{\cancel{\delta_{23}}}_{0} a_3 = a_2 \quad i = 2$$

$$\delta_{3m} a_m = \underbrace{\cancel{\delta_{31}}}_{0} a_1 + \underbrace{\cancel{\delta_{32}}}_{0} a_2 + \underbrace{\cancel{\delta_{33}}}_{1} a_3 = a_3 \quad i = 3$$

$$\delta_{im} a_m = a_i$$

* In two-dimensional problem

$$\begin{cases} \delta_{\alpha m} a_m = a_\alpha & \text{where } \alpha = 1,2 \\ \delta_{\beta n} a_n = a_\beta & \text{where } \beta = 1,2 \end{cases}$$

The Permutation Symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if the number of permutation of } 1,2,3 \text{ is even} \\ -1 & \text{if the number of permutation of } 1,2,3 \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

What I mean by Permutation

1 2 3 default

2 1 3 first permutation

2 3 1 second permutation

What happens to 2 3 1

even number of permutataion = 2

$\therefore \epsilon_{231} = 1$

1 2 3 $\longrightarrow$ 3 2 1

3 2 1 first permutation

with only one permutation we did this

$\therefore \epsilon_{321} = -1$

Even permutations

1 2 3

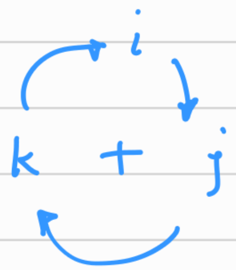
2 3 1

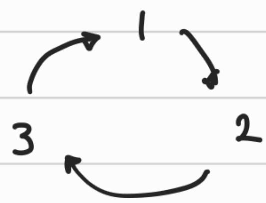
3 1 2

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

When $\epsilon_{ijk} = 0$? when you have at least two subscripts equal.

$\epsilon_{112} = \epsilon_{211} = \epsilon_{322} = \epsilon_{212} = \dots = 0$

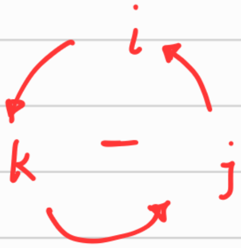


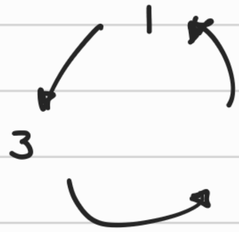


$\epsilon_{123} = 1$

$\epsilon_{312} = 1$

$\epsilon_{231} = 1$





$\epsilon_{321} = -1$

$\epsilon_{132} = -1$

$\epsilon_{213} = -1$

Dot Product (.)

$$\vec{a}\cdot\vec{b} = a\hat{e}_i \cdot b\hat{e}_j = ab\,\hat{e}_i\cdot\hat{e}_j =$$

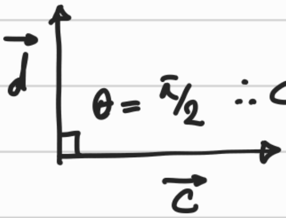
$$= ab\{\hat{e}_1\cdot\hat{e}_j + \hat{e}_2\cdot\hat{e}_j + \hat{e}_3\cdot\hat{e}_j\}$$

$$= ab\{\underbrace{\cancel{\hat{e}_1\cdot\hat{e}_1}}_{1} + \underbrace{\cancel{\hat{e}_2\cdot\hat{e}_1}}_{0} + \underbrace{\cancel{\hat{e}_3\cdot\hat{e}_1}}_{0}$$

$$+ \underbrace{\cancel{\hat{e}_1\cdot\hat{e}_2}}_{0} + \underbrace{\cancel{\hat{e}_2\cdot\hat{e}_2}}_{1} + \underbrace{\cancel{\hat{e}_3\cdot\hat{e}_2}}_{0}$$

$$+ \underbrace{\cancel{\hat{e}_1\cdot\hat{e}_3}}_{0} + \underbrace{\cancel{\hat{e}_2\cdot\hat{e}_3}}_{0} + \underbrace{\cancel{\hat{e}_3\cdot\hat{e}_3}}_{1}\} = ab$$

$$\vec{c}\cdot\vec{d} = |c||d|\cos\theta$$



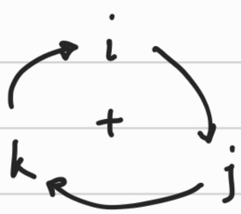
$\theta = \pi/2 \quad \therefore \cos\theta = 0$

Cross Product (x)

$$a \times b = a\hat{e}_i \times b\hat{e}_j = ab\,\hat{e}_i \times \hat{e}_j$$

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3 = \underbrace{\cancel{\epsilon_{123}}}_{1}\,\hat{e}_3$$

$$\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\,\hat{e}_k$$



$$\hat{e}_j \times \hat{e}_i = \underbrace{\epsilon_{jik}\,\hat{e}_k}_{<0}$$