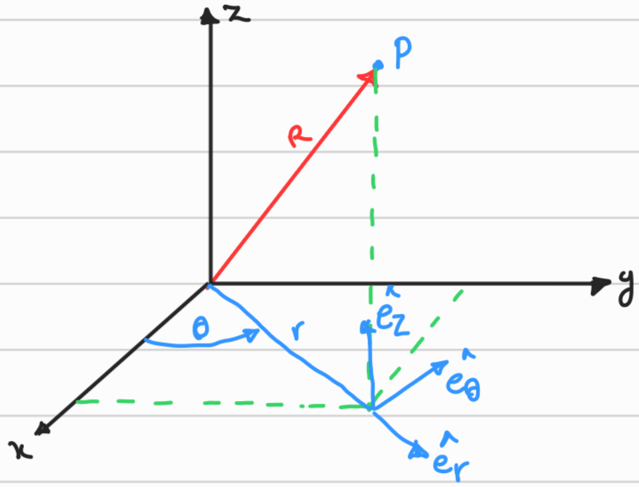
Cylindrical Coordinate:

Introduction:

$$R = r\hat{e_r} + z\hat{e_z}$$



$$dR = dr\,\hat{e_r} + r\underbrace{d\hat{e_r}}_{d\theta\hat{e_\theta}} + dz\,\hat{e_z} + \underbrace{\cancel{z\,d\hat{e_z}}}_{0} = dr\hat{e_r} + rd\theta\hat{e_\theta} + dz\,\hat{e_z} \quad (*)$$

Gradient of a Scalar Field:

Let f be a scalar field, $f(r,\theta,z)$

$$\frac{df}{dr} = \nabla f \Rightarrow df = \nabla f . dr \overset{(*)}{=} \left(a_r\hat{e_r} + a_\theta\hat{e_\theta} + a_z\hat{e_z}\right).\left(dr\hat{e_r} + rd\theta\hat{e_\theta} + dz\,\hat{e_z}\right)$$

$$= a_r dr + a_\theta r d\theta + a_z dz \qquad (1)$$

$$f(r,\theta,z): \; df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial z}dz \qquad (2)$$

compare (1) and (2) $\quad a_r = \frac{\partial f}{\partial r}, \quad a_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}, \quad a_z = \frac{\partial f}{\partial z}$

$$\nabla f = \frac{\partial f}{\partial r}\hat{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e_\theta} + \frac{\partial f}{\partial z}\hat{e_z}$$

Gradient of a Vector Field:

$$v(r,\theta,z) = v_r(r,\theta,z)\hat{e_r} + v_\theta(r,\theta,z)\hat{e_\theta} + v_z(r,\theta,z)\hat{e_z}$$

$$\frac{dv}{dr} = \nabla v \Rightarrow dv = \underbrace{\nabla v}_{\text{2nd-order tensor}}\, dr \overset{\nabla v = T}{=} T\,dr$$

$$dv = T\,dr \overset{(*)}{=} T\left(dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z\right) = dr\,(T\hat{e}_r) + r\,d\theta\,(T\hat{e}_\theta) + dz\,(T\hat{e}_z)$$

$$\begin{cases} T\hat{e}_r = T_{jr}\,\hat{e}_j \overset{j=r,\theta,z}{=} T_{rr}\,\hat{e}_r + T_{\theta r}\,\hat{e}_\theta + T_{zr}\,\hat{e}_z \\ T\hat{e}_\theta = T_{j\theta}\,\hat{e}_j = T_{r\theta}\,\hat{e}_r + T_{\theta\theta}\,\hat{e}_\theta + T_{z\theta}\,\hat{e}_z \\ T\hat{e}_z = T_{jz}\,\hat{e}_j = T_{rz}\,\hat{e}_r + T_{\theta z}\,\hat{e}_\theta + T_{zz}\,\hat{e}_z \end{cases}$$

$$dv = dr\,T_{rr}\,\hat{e}_r + dr\,T_{\theta r}\,\hat{e}_\theta + dz\,T_{zr}\,\hat{e}_z$$

$$+ r\,d\theta\,T_{r\theta}\,\hat{e}_r + r\,d\theta\,T_{\theta\theta}\,\hat{e}_\theta + r\,d\theta\,T_{z\theta}\,\hat{e}_z$$

$$+ dz\,T_{rz}\,\hat{e}_r + dz\,T_{\theta z}\,\hat{e}_\theta + dz\,T_{zz}\,\hat{e}_z$$

$$= \left(dr\,T_{rr} + r\,d\theta\,T_{r\theta} + dz\,T_{rz}\right)\hat{e}_r + \left(dr\,T_{\theta r} + r\,d\theta\,T_{\theta\theta} + dz\,T_{\theta z}\right)\hat{e}_\theta$$

$$+ \left(dr\,T_{zr} + dz\,T_{z\theta} + dz\,T_{zz}\right)\hat{e}_z \quad (3)$$

Next, $v(r,\theta,z) = v_r(r,\theta,z)\,\hat{e}_r + v_\theta(r,\theta,z)\,\hat{e}_\theta + v_z(r,\theta,z)\,\hat{e}_z$

$$dv = dv_r\,\hat{e}_r + v_r\,\underbrace{d\hat{e}_r}_{d\theta\,\hat{e}_\theta} + dv_\theta\,\hat{e}_\theta + v_\theta\,\underbrace{d\hat{e}_\theta}_{-d\theta\,\hat{e}_r} + dv_z\,\hat{e}_z + v_z\,\cancelto{0}{d\hat{e}_z}$$

$$= dv_r\,\hat{e}_r + (v_r\,d\theta)\,\hat{e}_\theta + dv_\theta\,\hat{e}_\theta - (v_\theta\,d\theta)\,\hat{e}_r + dv_z\,\hat{e}_z$$

$$\begin{cases} dv_r = \dfrac{\partial v_r}{\partial r}dr + \dfrac{\partial v_r}{\partial \theta}d\theta + \dfrac{\partial v_r}{\partial z}dz \\ dv_\theta = \dfrac{\partial v_\theta}{\partial r}dr + \dfrac{\partial v_\theta}{\partial \theta}d\theta + \dfrac{\partial v_\theta}{\partial z}dz \\ dv_z = \dfrac{\partial v_z}{\partial r}dr + \dfrac{\partial v_z}{\partial \theta}d\theta + \dfrac{\partial v_z}{\partial z}dz \end{cases}$$

Substitute above equations in dv gives

$$dv = \left(\frac{\partial v_r}{\partial r} dr + \frac{\partial v_r}{\partial \theta} d\theta + \frac{\partial v_r}{\partial z} dz \right) \hat{e_r} + (v_r d\theta)\, \hat{e_\theta}$$

$$+ \left(\frac{\partial v_\theta}{\partial r} dr + \frac{\partial v_\theta}{\partial \theta} d\theta + \frac{\partial v_\theta}{\partial z} dz \right) \hat{e_\theta} - (v_\theta d\theta)\, \hat{e_r}$$

$$+ \left(\frac{\partial v_z}{\partial r} dr + \frac{\partial v_z}{\partial \theta} d\theta + \frac{\partial v_z}{\partial z} dz \right) \hat{e_z}$$

$$= \left(\frac{\partial v_r}{\partial r} dr + \left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) d\theta + \frac{\partial v_r}{\partial z} dz \right) \hat{e_r}$$

$$+ \left(\frac{\partial v_\theta}{\partial r} dr + \left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) d\theta + \frac{\partial v_\theta}{\partial z} dz \right) \hat{e_\theta}$$

$$+ \left(\frac{\partial v_z}{\partial r} dr + \frac{\partial v_z}{\partial \theta} d\theta + \frac{\partial v_z}{\partial z} dz \right) \hat{e_z} \qquad (4)$$

compare (3) and (4)

$$\begin{bmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & T_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence of a vector Field:

$$\operatorname{div} \vec{v} = \operatorname{tr}(\nabla \vec{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) + \frac{\partial v_z}{\partial z}$$

Curl of a Vector Field:

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = 2t^{A} \qquad [\nabla v]^{A} = \frac{[\nabla v] - [\nabla v]^{T}}{2}$$

$$\frac{1}{2}\left\{\begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}-v_\theta\right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\left(\frac{\partial v_z}{\partial \theta}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix} - \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} & \partial v_z \\ \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}-v_\theta\right) & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right) & \frac{1}{r}\frac{\partial v_z}{\partial \theta} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_\theta}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}-v_\theta\right)-\frac{\partial v_\theta}{\partial r} & \frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r} \\ \frac{\partial v_\theta}{\partial r}-\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}-v_\theta\right) & 0 & \frac{\partial v_\theta}{\partial z}-\frac{1}{r}\frac{\partial v_z}{\partial \theta} \\ \frac{\partial v_z}{\partial r}-\frac{\partial v_r}{\partial z} & \frac{1}{r}\frac{\partial v_\theta}{\partial z}-\frac{\partial v_\theta}{\partial z} & 0 \end{bmatrix}$$

$$2t^A = -2\left(T_{23}\hat{e}_1 + T_{31}\hat{e}_2 + T_{12}\hat{e}_3\right) = -2\left(T_{\theta z}\hat{e}_r + T_{zr}\hat{e}_\theta + T_{r\theta}\hat{e}_z\right)$$

$$= -\left[\left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)\hat{e}_r + \left(\frac{\partial v_z}{\partial r} - \frac{\partial v_z}{\partial z}\right)\hat{e}_\theta + \left(\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) - \frac{\partial v_\theta}{\partial r}\right)\hat{e}_z\right]$$

$$= \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right)\hat{e}_z$$

Divergence of a Tensor Field:

$$(\text{div}\, T)\cdot a = \text{div}\left(T^T a\right) - \text{tr}\left((\nabla a)\, T^T\right) \quad \forall a$$

$$a = \hat{e}_r \; : \; (\text{div})^T_r = \text{div}\left(T^T \hat{e}_r\right) - \text{tr}\left((\nabla \hat{e}_r)\, T^T\right)$$

$$T\hat{e}_r = T_{jr}\hat{e}_j \overset{j=r,\theta,z}{=} T_{rr}\hat{e}_r + T_{\theta r}\hat{e}_\theta + T_{zr}\hat{e}_z$$

$$T^T\hat{e}_r = \underbrace{T_{rr}}_{v_r}\hat{e}_r + \underbrace{T_{r\theta}}_{v_\theta}\hat{e}_\theta + \underbrace{T_{rz}}_{v_z}\hat{e}_z$$

$$\text{Eq}(2.34.6) : \text{div}\left(T^T e_r\right) = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{r\theta}}{\partial \theta} + T_{rr}\right) + \frac{\partial T_{rz}}{\partial z}$$

$$\hat{e_r} = \underbrace{(1)}_{v_r} \hat{e_r} + \underbrace{(0)}_{v_\theta} \hat{e_\theta} + \underbrace{(0)}_{v_z} \hat{e_z}$$

Eq (2.34.5) : $\nabla \hat{e_r}\, T^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r}\hat{e_r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{rr} & T_{\theta r} & T_{zr} \\ T_{r\theta} & T_{\theta\theta} & T_{z\theta} \\ T_{rz} & T_{\theta z} & T_{zz} \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{r} T_{r\theta}\hat{e_r} & \frac{1}{r} T_{\theta\theta}\hat{e_r} & \frac{1}{r} T_{z\theta}\hat{e_r} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow tr((\nabla \hat{e_r})\, T^T) = \frac{1}{r} T_{\theta\theta} \hat{e_r}$$

$$(div T)_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{r\theta}}{\partial \theta} + T_{rr}\right) + \frac{\partial T_{rz}}{\partial z} - \frac{1}{r} T_{\theta\theta}$$

$a = \hat{e_\theta}$: $(div T)_\theta = div(T^T \hat{e_\theta}) - tr((\nabla \hat{e_\theta})\, T^T)$

$$T\hat{e_\theta} = T_{j\theta}\hat{e_j} = T_{r\theta}\hat{e_r} + T_{\theta\theta}\hat{e_\theta} + T_{z\theta}\hat{e_z}$$

$$T^T\hat{e_\theta} = T_{\theta j}\hat{e_j} = \underbrace{T_{\theta r}}_{v_r}\hat{e_r} + \underbrace{T_{\theta\theta}}_{v_\theta} e_\theta + \underbrace{T_{\theta z}}_{v_z} \hat{e_z}$$

Eq (2.34.6) : $div(T^T e_\theta) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial \theta} + T_{\theta r}\right) + \frac{\partial T_{\theta z}}{\partial z}$

$$\hat{e_\theta} = \underbrace{(0)}_{v_r} \hat{e_r} + \underbrace{(1)}_{v_\theta} \hat{e_\theta} + \underbrace{(0)}_{v_z} \hat{e_z}$$

Eq (2.34.5) $(\nabla \hat{e_\theta}\; T^T) = \begin{bmatrix} 0 & \frac{1}{r}(-\hat{e_\theta}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{rr} & T_{\theta r} & T_{zr} \\ T_{r\theta} & T_{\theta\theta} & T_{z\theta} \\ T_{rz} & T_{\theta z} & T_{zz} \end{bmatrix}$

$$= \begin{bmatrix} -\frac{1}{r} T_{r\theta}\hat{e_\theta} & -\frac{1}{r} T_{\theta\theta}\hat{e_\theta} & -\frac{1}{r} T_{z\theta}\hat{e_\theta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$tr\left((\nabla e_r)T^T\right) = -\frac{1}{r} T_{r\theta}\hat{e}_\theta$$

$$(\mathrm{div}\, T)_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial \theta} + T_{\theta r}\right) + \frac{\partial T_{\theta z}}{\partial z} + \frac{1}{r} T_{r\theta}$$

$$a = \hat{e}_z \quad : \quad (\mathrm{div}\, T)_z = \mathrm{div}(T^T e_z) - tr((\nabla e_z)T^T)$$

$$T\hat{e}_z = T_{jz}\hat{e}_j = T_{rz}\hat{e}_r + T_{\theta z}\hat{e}_\theta + T_{zz}\hat{e}_z$$

$$T^T\hat{e}_z = T_{zj}\hat{e}_j = \underbrace{T_{zr}}_{v_r}\hat{e}_r + \underbrace{T_{z\theta}}_{v_\theta}\hat{e}_\theta + \underbrace{T_{zz}}_{v_z}\hat{e}_z$$

$$\text{Eq } (2.34.6) : \mathrm{div}(T^T e_r) = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{z\theta}}{\partial \theta} + T_{zr}\right) + \frac{\partial T_{zz}}{\partial z}$$

$$\hat{e}_z = (0)\hat{e}_r + (0)\hat{e}_\theta + (1)\hat{e}_z$$

$$\text{Eq } (2.34.5) \quad (\nabla\hat{e}_z T^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}[T^T] = [0]$$

$$(\mathrm{div}\, T)_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{zr}}{\partial \theta} + T_{zr}\right) + \frac{\partial T_{zz}}{\partial z}$$

Laplacian of a Scalar Field:

$$\nabla f = \underbrace{\frac{\partial f}{\partial r}}_{v_r}\hat{e}_r + \underbrace{\frac{1}{r}\frac{\partial f}{\partial \theta}}_{v_\theta}\hat{e}_\theta + \underbrace{\frac{\partial f}{\partial z}}_{v_z}\hat{e}_z$$

$$\nabla^2 f = \nabla\cdot(\nabla f) = \mathrm{div}(\underbrace{\nabla f}_{\text{vector}}) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial f}{\partial r}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian of a vector Field:

$$\nabla^2 \vec{v} = \nabla(\mathrm{div}\,\vec{v}) - \nabla \times (\nabla \times \vec{v})$$

$$\mathrm{div}\,\vec{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}$$

$$\nabla(\mathrm{div}\,\vec{v}) = \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}\right)\hat{e_r}$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}\right)\hat{e_\theta}$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}\right)\hat{e_z}$$

$$= \left(\frac{\partial^2 v_r}{\partial r^2} - \frac{1}{r^2}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{1}{r}\left(\frac{\partial^2 v_\theta}{\partial r \partial \theta} + \frac{\partial v_r}{\partial r}\right) + \frac{\partial^2 v_z}{\partial r \partial z}\right)\hat{e_r}$$

$$+\left(\frac{1}{r}\frac{\partial^2 v_r}{\partial \theta \partial r} + \frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r}\frac{\partial^2 v_z}{\partial \theta \partial z}\right)\hat{e_\theta}$$

$$+\left(\frac{\partial^2 v_r}{\partial z \partial r} + \frac{1}{r}\left(\frac{\partial^2 v_\theta}{\partial z \partial \theta} + \frac{\partial v_r}{\partial z}\right) + \frac{\partial^2 v_z}{\partial z^2}\right)\hat{e_z} \qquad (5)$$

$$\nabla \times V = \underbrace{\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)}_{v_r}\hat{e_r} + \underbrace{\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)}_{v_\theta}\hat{e_\theta} + \underbrace{\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)}_{v_z}\hat{e_z}$$

$$\nabla \times (\nabla \times v) = \left(\frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right\} - \frac{\partial}{\partial z}\left\{\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right\}\right)\hat{e_r}$$

$$+\left(\frac{\partial}{\partial z}\left\{\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right\} - \frac{\partial}{\partial r}\left\{\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right\}\right)\hat{e_\theta}$$

$$+\left(\frac{\partial}{\partial r}\left\{\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right\} + \frac{1}{r}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\right\}\hat{e_z}$$

$$= \left\{\frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta \partial r} + \frac{1}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{\partial^2 v_r}{\partial z^2} + \frac{\partial^2 v_z}{\partial z \partial r}\right\}\hat{e_r}$$

$$+\left\{\frac{1}{r}\frac{\partial^2 v_z}{\partial z\partial\theta}-\frac{\partial^2 v_\theta}{\partial z^2}-\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}v_\theta-\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right.$$

$$\left.+\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta}\right\}\hat{e_\theta}$$

$$+\left\{\frac{\partial^2 v_r}{\partial r\partial z}-\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial z}-\frac{1}{r}\frac{\partial v_z}{\partial r}-\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta\partial z}\right\}\hat{e_z}$$

(6)

(5) - (6) gives

$$(\nabla^2 v)_r=\frac{\partial^2 v_r}{\partial r^2}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}v_r+\cancel{\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\cancel{\frac{\partial^2 v_z}{\partial r\partial z}}$$

$$-\cancel{\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial r}}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\cancel{\frac{\partial^2 v_z}{\partial z\partial r}}$$

$$=\frac{\partial^2 v_r}{\partial r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}v_r+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}$$

$$(\nabla^2 v)_\theta=\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}$$

$$-\frac{1}{r}\frac{\partial^2 v_z}{\partial z\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{\partial^2 v_\theta}{\partial r^2}-\frac{1}{r^2}v_\theta+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}$$

$$=\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{\partial^2 v_\theta}{\partial r^2}-\frac{1}{r^2}v_\theta+\frac{1}{r}\frac{\partial v_\theta}{\partial r}$$

$$(\nabla^2 v)_z=\frac{\partial^2 v_r}{\partial z\partial r}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial z\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial z}+\frac{\partial^2 v_z}{\partial z^2}$$

$$-\frac{\partial^2 v_r}{\partial v\partial z}+\frac{\partial^2 v_z}{\partial r^2}-\frac{1}{r}\frac{\partial v_r}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}-\frac{1}{r}\frac{\partial v_\theta}{\partial\theta\partial z}$$

$$=\frac{\partial^2 v_z}{\partial z^2}+\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}$$