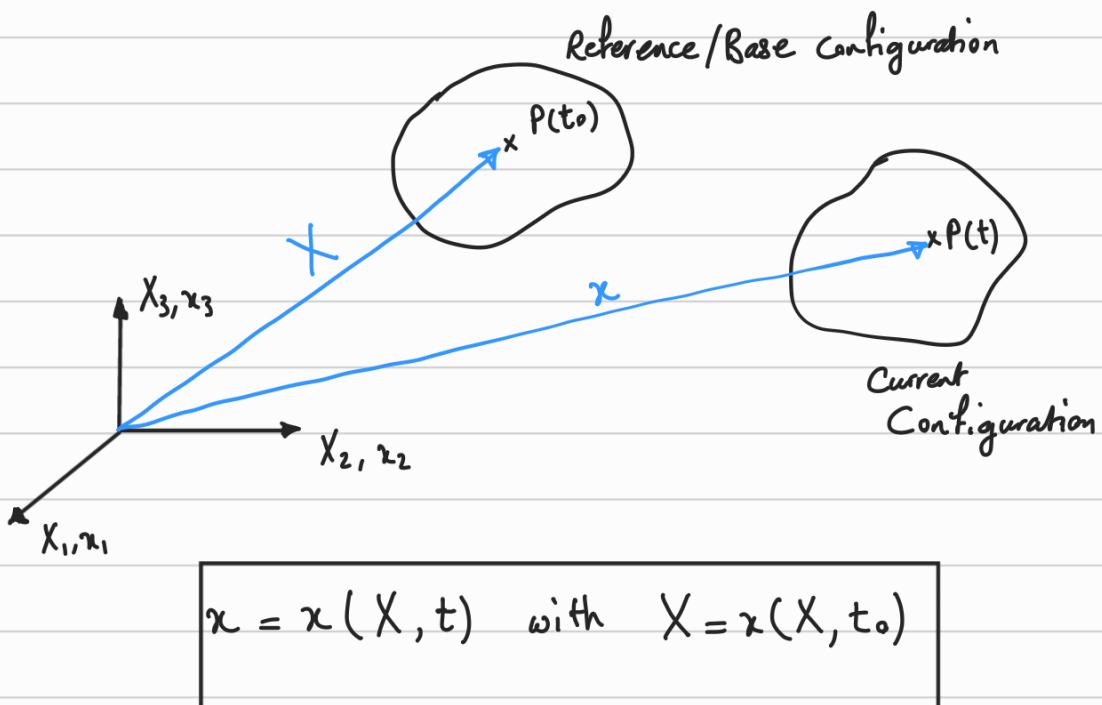


Reference and Current Configuration

Lagrangian / Eulerian Point of view

Material Derivative

Reference and Current Configuration:



$$x_1 = x_1(X_1, X_2, X_3, t), \quad X_1 = x_1(X_1, X_2, X_3, t_0)$$

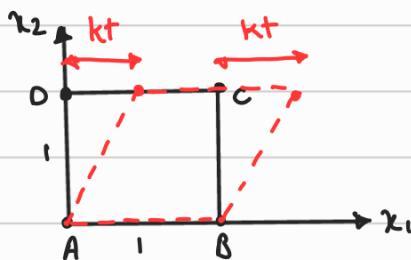
$$x_2 = x_2(X_1, X_2, X_3, t), \quad X_2 = x_2(X_1, X_2, X_3, t_0)$$

$$x_3 = x_3(X_1, X_2, X_3, t), \quad X_3 = x_3(X_1, X_2, X_3, t_0)$$

$\underbrace{\quad}_{\text{motion of continuum}}$

$\underbrace{\quad}_{\text{material coordinate}}$

Example 3.1.1



— Reference configuration

--- Current Configuration

$$\vec{x} = \vec{X} + kt \vec{X}_2 \hat{e}_1 = (X_1 \hat{e}_1 + X_2 \hat{e}_2 + X_3 \hat{e}_3) + kt X_2 \hat{e}_1 \quad (*)$$

where $\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$

$$A(0,0,0) \xrightarrow{(*)} x = 0 \hat{e}_1 + 0 \hat{e}_2 + 0 \hat{e}_3$$

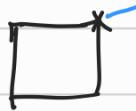
$$B(1,0,0) \xrightarrow{(*)} x = 1 \hat{e}_1 + 0 \hat{e}_2 + 0 \hat{e}_3$$

$$C(1,1,0) \xrightarrow{(*)} x = 1 \hat{e}_1 + 1 \hat{e}_2 + 0 \hat{e}_3 + kt \hat{e}_1 = (1+kt) \hat{e}_1 + 1 \hat{e}_2 + 0 \hat{e}_3$$

$$D(0,1,0) \xrightarrow{(*)} x = (0+kt) \hat{e}_1 + 1 \hat{e}_2 + 0 \hat{e}_3 = kt \hat{e}_1 + 1 \hat{e}_2 + 0 \hat{e}_3$$

Lagrangian and Eulerian Point of view:

Lagrangian (material description) you move with a particle



Reference config.

t_0



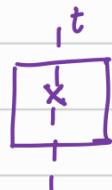
Current config

t

$$\theta = \hat{\theta}(x_1, x_2, x_3, t)$$

usually used in Solid Mechanics

Eulerian (spatial description)



$$\theta = \hat{\theta}(x_1, x_2, x_3, t)$$

usually used in Fluid Mechanics

Example 3.2.1

$$\begin{cases} x_1 = X_1 + kt X_2 \\ x_2 = (1+kt) X_2 \\ x_3 = X_3 \end{cases} \quad \theta = \alpha(x_1 + x_2) \quad \text{spatial description}$$

$$\text{a) } \theta = \alpha(x_1 + x_2) = \alpha(X_1 + kt X_2 + (1+kt) X_2) = \alpha X_1 + (1+2kt) X_2$$

material description

$$b) \quad v_i = \left(\frac{\partial x_i}{\partial t} \right)_{X_i-\text{fixed}} \quad x_i = x_i(X_j, t)$$

$$\begin{cases} v_1 = kx_2 \\ v_2 = kx_2 \\ v_3 = 0 \end{cases} \Rightarrow \vec{v} = kx_2 \hat{e}_1 + kx_2 \hat{e}_2 \quad \text{material description (**)}$$

To get \vec{v} in spatial coordinate, we find X_2 in terms of x_i and then substitute

$$\text{if with } X_2 \text{ in (**)} \quad x_2 = (1+kt) X_2 \Rightarrow X_2 = \frac{1}{1+kt} x_2$$

$$\vec{v} = \frac{k}{1+kt} x_2 \hat{e}_1 + \frac{k}{1+kt} x_2 \hat{e}_2 \quad \text{spatial description}$$

Material Derivative:

For a scalar field:

$$\theta = \theta(X_1, X_2, X_3, t)$$

$\frac{D}{Dt}$ = material derivative

$$\frac{D\theta}{Dt} = \left(\frac{\partial \theta(X_1, X_2, X_3, t)}{\partial t} \right)_{X_i-\text{fixed}} \quad \text{material description}$$

But what if we are in the spatial description!

$$\theta = \theta(x_1, x_2, x_3, t)$$

$$\frac{D\theta}{Dt} = \frac{D\theta(x_1, x_2, x_3, t)}{Dt} = \frac{\partial \theta}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial \theta}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial \theta}{\partial x_3} \frac{\partial x_3}{\partial t} + \frac{\partial \theta}{\partial t}$$

$$= \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial x_1} v_1 + \frac{\partial \theta}{\partial x_2} v_2 + \frac{\partial \theta}{\partial x_3} v_3$$

$$\boxed{\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + v_i \left(\frac{\partial \theta}{\partial x_i} \right)}$$

spatial description (indicial notation)

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \mathbf{v} \cdot (\nabla \theta)$$

spatial description (tensorial notation)

in cylindrical coordinate system: $\nabla \theta = \text{grad}(\theta) = \frac{\partial \theta}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \theta}{\partial \theta} \hat{e}_\theta + \frac{\partial \theta}{\partial z} \hat{e}_z$

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) \cdot \left\{ \frac{\partial \theta}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \theta}{\partial \theta} \hat{e}_\theta + \frac{\partial \theta}{\partial z} \hat{e}_z \right\}$$

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + v_r \frac{\partial \theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial \theta}{\partial \theta} + v_z \frac{\partial \theta}{\partial z}$$

For a vector field:

$$\mathbf{v} = \left(\frac{\partial \mathbf{x}}{\partial t} \right)_{X_i-\text{fixed}} = \frac{D\mathbf{x}}{Dt}, \quad \mathbf{a} = \left(\frac{\partial \mathbf{v}}{\partial t} \right)_{X_i-\text{fixed}} = \frac{D\mathbf{v}}{Dt} \quad \begin{matrix} \text{material} \\ \text{description} \end{matrix}$$

But what if you want to get $\frac{D\mathbf{v}}{Dt}$ in spatial coordinate.

$\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$ and also each v_i is a function of x_j and time, t ; therefore,

$$\vec{v} = v_1(x_1, x_2, x_3, t) \hat{e}_1 + v_2(x_1, x_2, x_3, t) \hat{e}_2 + v_3(x_1, x_2, x_3, t) \hat{e}_3$$

$$\begin{aligned} \mathbf{a} = \frac{D\mathbf{v}}{Dt} &= \frac{\partial v_1}{\partial x_1} \frac{\partial x_1}{\partial t} \hat{e}_1 + \frac{\partial v_1}{\partial x_2} \frac{\partial x_2}{\partial t} \hat{e}_2 + \frac{\partial v_1}{\partial x_3} \frac{\partial x_3}{\partial t} \hat{e}_3 + \frac{\partial v_1}{\partial t} \hat{e}_1 \\ &+ \frac{\partial v_2}{\partial x_1} \frac{\partial x_1}{\partial t} \hat{e}_1 + \frac{\partial v_2}{\partial x_2} \frac{\partial x_2}{\partial t} \hat{e}_2 + \frac{\partial v_2}{\partial x_3} \frac{\partial x_3}{\partial t} \hat{e}_3 + \frac{\partial v_2}{\partial t} \hat{e}_2 \\ &+ \frac{\partial v_3}{\partial x_1} \frac{\partial x_1}{\partial t} \hat{e}_1 + \frac{\partial v_3}{\partial x_2} \frac{\partial x_2}{\partial t} \hat{e}_2 + \frac{\partial v_3}{\partial x_3} \frac{\partial x_3}{\partial t} \hat{e}_3 + \frac{\partial v_3}{\partial t} \hat{e}_3 \end{aligned}$$

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

spatial description (indicial notation)

$$a = \frac{\partial v}{\partial t} + (\nabla v) v$$

spatial description (tensorial notation)

Therefore, in general (spatial description)

For a scalar field (\square) $\frac{D \square}{Dt} = \frac{\partial \square}{\partial t} + v \cdot (\nabla \square)$

For a vector field (Δ) $\frac{D \Delta}{Dt} = \frac{\partial \Delta}{\partial t} + (\nabla \Delta) v$

Example 3.4.2

$$\left\{ \begin{array}{l} v_1 = \frac{kx_1}{1+kt} \\ v_2 = \frac{kx_2}{1+kt} \\ v_3 = \frac{kx_3}{1+kt} \end{array} \right.$$

a) $\frac{D \vec{v}}{Dt} = \vec{a} = \frac{\partial}{\partial t} (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3) + (\nabla v) v$

$$\left\{ \begin{array}{l} \frac{\partial v_1}{\partial t} = \frac{-k^2}{1+kt} x_1 \\ \frac{\partial v_2}{\partial t} = \frac{-k^2}{1+kt} x_2 \\ \frac{\partial v_3}{\partial t} = \frac{-k^2}{1+kt} x_3 \end{array} \right.$$

$$\nabla v = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \frac{k}{1+kt} & 0 & 0 \\ 0 & \frac{k}{1+kt} & 0 \\ 0 & 0 & \frac{k}{1+kt} \end{bmatrix}$$

$$\begin{aligned} a &= \frac{-k^2}{1+kt} x_1 \hat{e}_1 - \frac{k^2}{1+kt} x_2 \hat{e}_2 - \frac{k^2}{1+kt} x_3 \hat{e}_3 \\ &+ \frac{k^2}{1+kt} x_1 \hat{e}_1 + \frac{k^2}{1+kt} x_2 \hat{e}_2 + \frac{k^2}{1+kt} x_3 \hat{e}_3 = 0 \hat{e}_1 + 0 \hat{e}_2 + 0 \hat{e}_3 \end{aligned}$$

$$b) x_i = x_i(x_j, t) \quad \left(\frac{Dv}{Dt} \right)_{x_i \text{-fixed}}$$

$$v_i = \left(\frac{dx_i}{dt} \right)_{x_i \text{-fixed}} = \frac{kx_i}{1+kt} \quad \begin{matrix} \text{to get} \\ \text{the displacement} \\ \text{we integrate} \end{matrix}$$

$$\frac{dx_i}{dt} = \frac{kx_i}{1+kt} \quad \Rightarrow \int \frac{dx_i}{x_i} = \int_0^t \frac{k}{1+kt} dt$$

$$\ln x_i - \ln X_i = \ln(1+kt) \Rightarrow \ln \frac{x_i}{X_i} = \ln(1+kt)$$

$$x_i = (1+kt) X_i$$

$$\text{By following similar procedures, } x_2 = (1+kt) X_2$$

$$x_3 = (1+kt) X_3$$

$$v_i = \left(\frac{\partial x_i}{\partial t} \right)_{x_i \text{-fixed}} = kX_1 \hat{e}_1 + kX_2 \hat{e}_2 + kX_3 \hat{e}_3$$

$$a_i = \left(\frac{\partial v_i}{\partial t} \right)_{x_i \text{-fixed}} = 0 \hat{e}_1 + 0 \hat{e}_2 + 0 \hat{e}_3$$