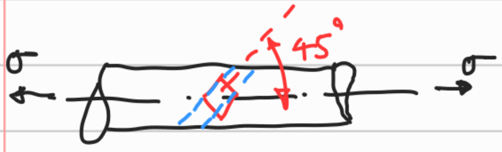
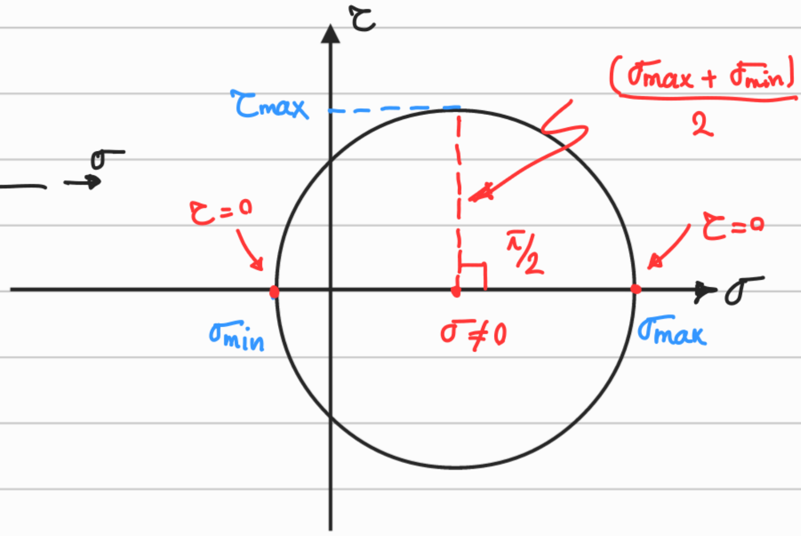
Principal Stress / Maximum Shear Stress

Motivation:





Principal Stresses:

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

T is a real symmetric tensor $\Rightarrow$ real values for principal values (T_1, T_2, T_3)
mutually perpendicular principal directions (n_1, n_2, n_3)

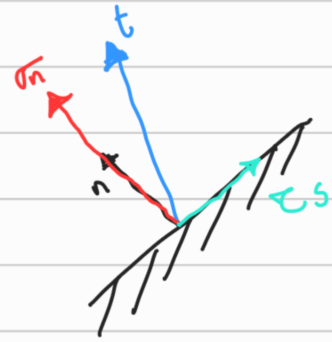
$$T = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix}$$

Maximum Shear Stresses:

Assume that T_1, T_2, T_3 are principal values
$\hat{e}_1, \hat{e}_2, \hat{e}_3$ are principal directions

We have an arbitrary $\vec{n} = n_1\hat{e}_1 + n_2\hat{e}_2 + n_3\hat{e}_3$ where the maximum shear stresses act on.

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$



$$\boxed{\vec{t} = T_1 n_1 \hat{e}_1 + T_2 n_2 \hat{e}_2 + T_3 n_3 \hat{e}_3}$$

$$t = \sigma_n + \tau_s$$
$$\tau_s = t - \sigma_n$$
$$|\tau_s|^2 = |t|^2 - \sigma_n^2$$

$$\sigma_n = t \cdot n = (T_1 n_1 \hat{e}_1 + T_2 n_2 \hat{e}_2 + T_3 n_3 \hat{e}_3) \cdot$$
$$(n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3)$$

$$\boxed{\sigma_n = T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2}$$

$$|\tau_s|^2 = |t|^2 - \sigma_n^2$$

$$\Rightarrow \boxed{\tau_s^2 = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - (T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2)^2}$$

$$(*) \quad \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} \pm 1 \\ 0 \\ 0 \end{Bmatrix} \quad \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \pm 1 \\ 0 \end{Bmatrix} \quad \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \pm 1 \end{Bmatrix} \rightarrow \tau_s = 0$$

minimum shear stress

Since we have n_1^2, n_2^2, n_3^2 and also T_1, T_2, T_3 are real values; therefore,

we get a positive answer and also a negative that hold true.

Now, let us find the maximum shear stress,

$$\tau_s^2 = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - (T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2)^2$$

unit vector

Since n_1, n_2, n_3 are unit vectors, $n_1^2 + n_2^2 + n_3^2 = 1$

$$\boxed{\begin{aligned}&\max \quad \tau_s^2 = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - (T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2)^2 \\ &\text{s.t} \quad n_1^2 + n_2^2 + n_3^2 - 1 = 0\end{aligned}}$$

$$\tau_s^2 = f(n_1, n_2, n_3)$$

$$d\tau_s^2 = \frac{\partial \tau_s^2}{\partial n_1} dn_1 + \frac{\partial \tau_s^2}{\partial n_2} dn_2 + \frac{\partial \tau_s^2}{\partial n_3} dn_3 = 0 \qquad (1)$$

$$n_1^2 + n_2^2 + n_3^2 - 1 = 0 \xrightarrow{\text{derivative}} \cancel{2} n_1 dn_1 + \cancel{2} n_2 dn_2 + \cancel{2} n_3 dn_3 = 0 \qquad (2)$$

Comparing (1) and (2) :

$$\frac{\partial \tau_s^2}{\partial n_1} = \lambda n_1 , \quad \frac{\partial \tau_s^2}{\partial n_2} = \lambda n_2 , \quad \frac{\partial \tau_s^2}{\partial n_3} = \lambda n_3 \qquad (3)$$

where λ is known as Lagrange Multiplier

$$n_1^2 + n_2^2 + n_3^2 - 1 = 0 \qquad (4)$$

Equations (3) and (4) form 4 equation from which we can find n_1, n_2, n_3, λ

(3.1) $$\frac{\partial}{\partial n_1}\left[T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - (T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2)^2\right] = \lambda n_1$$

$$2 n_1 T_1^2 - 4\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)\left(n_1 T_1\right) = \lambda n_1$$

(3.2) $$\frac{\partial}{\partial n_2}\left[T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - (T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2)^2\right] = \lambda n_2$$

$$2 n_2 T_2^2 - 4\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)\left(n_2 T_2\right) = \lambda n_2$$

(3.3) $$\frac{\partial}{\partial n_3}\left[T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - (T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2)^2\right] = \lambda n_3$$

$$2 n_3 T_3^2 - 4\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)\left(n_3 T_3\right) = \lambda n_3$$

Solving (3.1) – (3.3) and (4) simultaneously, we get 12 answers and with 6 answers from (#) we get 18 overall answers for $\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$ where by removing repeated answers we get 9 direction. These directions are given in Table 4.1.

$$\tau_s = \max\left(\frac{|T_1 - T_2|}{2}, \frac{|T_1 - T_3|}{2}, \frac{|T_2 - T_3|}{2}\right)$$

$$\tau_{s_{max}} = \frac{(T_i)_{max} - (T_i)_{min}}{2} \qquad \text{corresponding } \sigma_n = \frac{(T_i)_{max} + (T_i)_{min}}{2}$$

Section 4.6 First paragraph on page 166

Example 4.6.3

$$T_1 = T_2 \neq T_3$$

$$100 = 100 \neq 500$$

$$e_1 \quad e_2 \quad e_3$$

$$\left\{\begin{array}{lll} \hat{e}_1 - \hat{e}_3 : & \dfrac{e_1 \pm e_3}{\sqrt{2}} & e_1^2 + e_3^2 = 1 \\ \hat{e}_2 - e_3 : & \dfrac{e_2 \pm e_3}{\sqrt{2}} & \hat{e}_2 + \hat{e}_3 = 1 \\ \alpha \hat{e}_1 + \beta \hat{e}_2 + \dfrac{1}{\sqrt{2}} \hat{e}_3 & & \alpha^2 + \beta^2 + \dfrac{1}{2} = 1 \end{array}\right.$$

Module 1.2.6

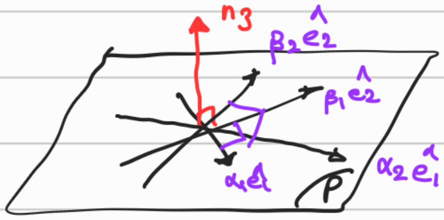
Case 1: $\lambda_1 \neq \lambda_2 \neq \lambda_3$ as already shown (above), at least one of the principal directions (eigenvectors) are perpendicular to each other.

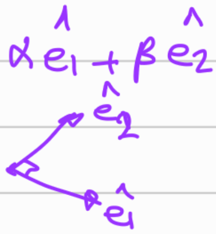
Case 2: $\lambda_1 = \lambda_2 \neq \lambda_3$

$$\lambda_1 = \lambda_2 = \lambda \qquad \begin{cases} T n_1 = \lambda_1 n_1 = \lambda n_1 \\ T n_2 = \lambda_2 n_2 = \lambda n_2 \end{cases}$$

For any scalar α and β
$$\begin{cases} T \alpha n_1 = \lambda \alpha n_1 \\ T \beta n_2 = \lambda \beta n_2 \end{cases}$$

$$T(\alpha n_1 + \beta n_2) = \lambda(\alpha n_1 + \beta n_2)$$





Infinite number of principal directions which for a plane (here αn_1 and βn_2) and n_3 is of course perpendicular to this plane.

principal directions: $\quad n_1 = \pm \hat{e}_1 \quad , \quad n_2 = \pm \hat{e}_2 \quad , \quad n_3 = \alpha \hat{e}_1 + \beta \hat{e}_2$

Case 3: $\lambda_1 = \lambda_2 = \lambda_3$ any vector is a principal vector (Example 2.22.1)