Symmetric and Antisymmetric Tensors

The Dual Vector (of an Antisymmetri Tensor)

Eigenvalues / Eigenvectors of a Tensor

Symmetric Tensor: $T = T^T$

$$T_{ij} = T_{ji} \quad ①$$

for example : $T_{12} = T_{21}$ $\quad T_{11} = T_{11}$ $\quad T_{32} = T_{23}$

Antisymmetric Tensor: $T = -T^T$

$$T_{ij} = -T_{ji} \quad ②$$

for example: $T_{12} = -T_{21}$ $\quad T_{32} = -T_{23}$ $\quad T_{11} = T_{22} = T_{33} = 0$

* Any tensor T can be written as the sum of its symmetric and antisymmetric part.

$$T = T^S + T^A$$

where $T^S = \frac{T + T^T}{2}$ and $T^A = \frac{T - T^T}{2}$

Some Important Properties:

1. $tr(T^S T^A) = 0$

proof: $tr(T^S T^A) = tr(\underbrace{T_{ij}}_{sym} \underbrace{T_{ji}}_{asym}) \overset{①}{\underset{②}{=}} tr(T_{ji} (-T_{ij}))$

$$= -tr(T_{ji}T_{ij}) = -tr(T_{jj})$$

the left-hand side can be written as $tr(T_{ii})$

$$tr(T_{ii}) = -tr(T_{jj}) \Rightarrow tr(T_{ii}) = -tr(T_{ii})$$

$$2tr(T_{ii}) = 0 \Rightarrow tr(T_{ii}) = 0$$

2. $tr(AB) = tr(A^S B^S) + tr(A^A B^A)$

3. $(T^A)^T = -T^A$

The Dual vector of an Antisymmetic Tensor:

In an antisymmetric tensor: $T = -T^T$, $T_{ij} = -T_{ji}$

$$T^A = \begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{bmatrix}$$

Definition of a dual vector: $Ta = t^A \times a$ ③

$$T_{12} = e_1 \cdot Te_2 \overset{③}{=} e_1 \cdot (t^A \times e_2)$$

from Module 1.1.2 : $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$ (*)

$$T_{12} = e_1 \cdot (t^A \times e_2) \overset{(*)}{=} t^A \cdot (e_2 \times e_1) = -t^A e_3$$

$$T_{31} = e_3 \cdot Te_1 = e_3 \cdot (t^A \times e_1) \overset{(*)}{=} t^A \cdot (e_1 \times e_3) = -t^A e_2$$

$$T_{23} = e_2 \cdot Te_3 = e_2 \cdot (t^A \times e_3) \overset{(*)}{=} t^A \cdot (e_3 \times e_2) = -t^A e_1$$

$$t^A = -\left(T_{23}\hat{e}_1 + T_{31}\hat{e}_2 + T_{12}\hat{e}_3\right)$$

$$= \left(T_{32}\hat{e}_1 + T_{13}\hat{e}_2 + T_{21}\hat{e}_3\right)$$

$$t^A = -\frac{1}{2}\epsilon_{ijk}T_{jk}e_i \Rightarrow 2t^A = -\epsilon_{ijk}T_{jk}e_i$$

- the axis of rotation of a rotation matrix is parallel to the dual vector of the rotation tensor.

$$\underset{\text{deformation gradient}}{\nabla u} = \underset{\text{infinitesimal strain tensor}}{\overset{\text{sym}}{E}} + \underset{\text{infinitesimal rotation tensor}}{\overset{\text{asym}}{\Omega}}$$

$$\Omega\, \overset{\text{direction vector}}{dX} = \underset{\text{dual vector of the infinitesimal tensor } (\Omega)}{t^A} \underset{\text{cross product}}{\times} dX$$

Proof : (Example 2.21.2)

R : a rotation tensor

m : a univector in the direction of axis of rotation.

we would like to prove q (dual vector of R) is parallel to m

in other words $q \times m = 0$

$|q||m| \sin\theta = 0$

$$(1)\quad Rm = m \xrightarrow{\times R^T} \underbrace{R^TR}_{R \text{ is a rotation tensor } \therefore\ R^TR = RR^T = I} m = R^Tm \Rightarrow Im = R^Tm \Rightarrow m = R^Tm \quad (2)$$

from (1) and (2) :

$$\begin{cases} Rm = m \\ R^Tm = m \end{cases} \xrightarrow[\text{both sides}]{\text{substract}} Rm - R^Tm = 0$$

$$m(R - R^T) = 0$$

$$R^A = \left(\frac{R-R^T}{2}\right) \Rightarrow 2R^A m = 0 \quad \text{and} \quad R^A = q$$

$$2R^A m = 2q \times m = 0 \Rightarrow q \times m = 0$$

↗ dual vector of R^A

Eigenvalues and Eigenvectors of a Tensor:

$$Ta = \lambda a \leftarrow \text{eigenvectors}$$

eigenvalues ↗

$$T(\alpha a) = \alpha Ta = \alpha \lambda a = \lambda(\alpha a)$$

scalar ↗

For the Identity tensor (I): $Ia = a$

in other words: $Ia = (1)\, a$

$I\alpha a = (1)(\alpha a)$

How to find Eigenvalues?

$a \cdot a = 1$ $\quad$ $Ta = \lambda a$ $\quad$ assume $\vec{a}$ is a unitvector

replace a on the right-hand side with Ia

$$Ta = \lambda(Ia) \Rightarrow Ta - \lambda(Ia) \Rightarrow a(T - \lambda I) = 0$$

For nontrivial solution $|T - \lambda I| = 0$

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0$$

the determinate gives us an equation which is called the "Characteristic Equation".

How to find Eigenvectors?

$$Ta = \lambda a$$

$$\xrightarrow[\text{a with } Ia]{\text{replace}} \quad Ta - \lambda Ia = 0 \Rightarrow a(T - \lambda I) = 0 \quad , (T - \lambda I)a = 0$$

indicial notation : $(T_{ij} - \lambda \delta_{ij})\ a = 0$

let $a = \alpha_i e_i$ $\quad : (T_{ij} - \lambda \delta_{ij})\ \alpha_i = 0$

Wait!
Is this correct?

$$(T_{11} - \lambda)\alpha_1 + T_{21}\alpha_2 + T_{31}\alpha_3 = 0$$

But this in incorrect because we agreed to fill in the tensors by column.

$\therefore$ indicial notation: $(T_{ij} - \lambda \delta_{ij})\alpha_j = 0$
(CORRECT)

$$\begin{cases} (T_{11} - \lambda)\alpha_1 + (T_{12})\alpha_2 + (T_{13})\alpha_3 = 0 \\ (T_{21})\alpha_1 + (T_{22} - \lambda)\alpha_2 + (T_{23})\alpha_3 = 0 \\ (T_{31})\alpha_1 + (T_{32})\alpha_2 + (T_{33} - \lambda)\alpha_3 = 0 \end{cases}$$

Since $\vec{a}$ is a unitvector $\therefore$ $\quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$