Tensor transformation

Transformation rules (addition, multiplication, quotient)

Tensor Transformation:

$\alpha' = \alpha$ zero-order tensor

$a'_i = Q_{mi} a_m$ first-order tensor (vector)

$T'_{ij} = Q_{mi} Q_{nj} T_{mn}$ second-order tensor (tensor)

$S'_{ijk} = Q_{mi} Q_{nj} Q_{rk} S_{mnr}$ third-order tensor

$C'_{ijkl} = Q_{mi} Q_{nj} Q_{pk} Q_{ql} C_{mnpq}$ fourth-order tensor

$\vdots$

Transformation Rules:

The addition rule:

$$W'_{ijk} = Q_{mi} Q_{nj} Q_{rk} W_{mnr}$$

$$T'_{ijk} + S'_{ijk} = Q_{mi} Q_{nj} Q_{rk} T_{mnr} + Q_{mi} Q_{nj} Q_{rk} S_{mnr}$$

$$= Q_{mi} Q_{nj} Q_{rk} \underbrace{[T_{mnr} + S_{mnr}]}_{W_{mnr}} = Q_{mi} Q_{nj} Q_{rk} W_{mnr}$$

The multiplication rule:

a_i is a vector
T_{ij} is a tensor

} it can be proved that the product result order is equal to the number of FREE indices.

examples: $a_i a_j = T_{ij}$, $T_{ij} T_{kl} = C_{ijkl}$

$$T_{im} T_{mk} = S_{ik}$$

For a second-order tensor: $T_{ij} = a_i a_j$

If $T_{ij} = a_i a_j$, then it is true for all coordinate systems.

$$T'_{ij} = a'_i a'_j \quad ③$$

$a'_i = Q_{mi} a_m$ ①

$a'_j = Q_{nj} a_n$ ②

Plug-in ① and ② in ③ gives

$$T'_{ij} = Q_{mi} a_m Q_{nj} a_n$$

$$T'_{ij} = Q_{mi} Q_{nj} \underbrace{a_m a_n}_{T_{mn}}$$

$$T'_{ij} = Q_{mi} Q_{nj} T_{mn}$$

For a third-order tensor: $M_{ijkl} = T_{ij} T_{kl}$

If $M_{ijkl} = T_{ij} T_{kl}$, then it is true for all coordinate systems.

$$M'_{ijkl} = T'_{ij} T'_{kl} \quad ④$$

$$T'_{ij} = Q_{mi} Q_{nj} T_{mn} \quad (5)$$

$$T'_{kl} = Q_{rk} Q_{pl} T_{rp} \quad (6)$$

Substitute (5) and (6) in (4) gives

$$M'_{ijkl} = Q_{mi} Q_{nj} T_{mn} Q_{rk} Q_{pl} T_{rp}$$

$$= Q_{mi} Q_{nj} Q_{rk} Q_{pl} \underbrace{T_{mn} T_{rp}}_{M_{mnrp}}$$

$$M'_{ijkl} = Q_{mi} Q_{nj} Q_{rk} Q_{pl} M_{mnrp}$$

The Quotient Rule:

$$\left.\begin{array}{l} a_i \text{ is a vector} \\ T_{ij} \text{ is a tensor} \end{array}\right\} \quad a_i = T_{ij} b_j \quad \therefore b_i \text{ is a vector}$$

Proof:

$$a_i = T_{ij} b_j \quad (*)$$

if $a_i = T_{ij} b_j$, then it is true in all coordinate system.

$$a'_i = T'_{ij} b'_j \quad (\blacksquare)$$

$$a'_i = Q_{mi} a_m \longrightarrow a_i = Q_{im} a'_m \quad (1)$$

$$T'_{ij} = Q_{mi} Q_{nj} T_{mn} \longrightarrow T_{ij} = Q_{im} Q_{jn} T'_{mn} \quad (2)$$

plug-in (1) and (2) in $(*)$ gives

$$Q_{im} a'_m = Q_{im} Q_{jn} T'_{mn} b_j$$

from $(\blacksquare)$, $a'_m = T'_{mn} b'_n$ ③

$$\xrightarrow{③} Q_{im} T'_{mn} b'_n = Q_{im} Q_{jn} T'_{mn} b_j$$

$Q_{mk} = Q_{mi} Q_{ik}$

$$\xrightarrow{\times Q_{ik}} \underbrace{Q_{ik} Q_{im}}_{\blacktriangle\ \delta_{mk}} T'_{mn} b'_n = \underbrace{Q_{ik} Q_{im}}_{\blacktriangle\ \delta_{mk}} Q_{jn} T'_{mn} b_j$$

$(\blacktriangle)$ $\delta_{mk} = Q_{im} Q_{ik}$

$$\to T'_{mn} b'_n = Q_{jn} T'_{mn} b_j$$

$$\to T'_{mn} b'_n - Q_{jn} T'_{mn} b_j = 0$$

$$\to T'_{mn} \left(b'_n - Q_{jn} b_j \right) = 0$$

$$b'_n - Q_{jn} b_j = 0 \Rightarrow b'_n = Q_{jn} b_j$$

Which implies b_i is a vector.

Another example: $(*)$ $T_{ij} = C_{ijkl} E_{kl}$ (generalized Hooke's law)

if $T_{ij} = C_{ijkl} E_{kl}$, then it is true for all coordinate systems.

$$T'_{ij} = C'_{ijkl} E'_{kl} \quad (\blacksquare)$$

$T'_{ij} = Q_{mi} Q_{nj} T_{mn} \to T_{ij} = Q_{im} Q_{jn} T'_{mn}$ ①

$C'_{ijkl} = Q_{mi} Q_{nj} Q_{rk} Q_{sl} C_{mnrs} \to C_{ijkl} = Q_{im} Q_{jn} Q_{kr} Q_{ls} C'_{mnrs}$ ②

Plugging in ① and ② in $(*)$ gives

$$T_{ij} = C_{ijkl} E_{kl}$$

$$Q_{im} Q_{jn} T'_{mn} = Q_{im} Q_{jn} Q_{kr} Q_{ls} C'_{mnrs} E_{kl} \quad (**)$$

T'_{mn} can be rewritten as (from ([redacted]))

$$T'_{mn} = C'_{mnrs} E'_{rs} \quad ③$$

Plug-in ③ in (**)

$$Q_{im} Q_{jn} C'_{mnrs} E'_{rs} = Q_{im} Q_{jn} Q_{kr} Q_{ls} C'_{mnrs} E_{kl}$$

$$\xrightarrow[\delta_{im} = Q_{im} Q_{if}]{Q_{mf} = Q_{mi} Q_{if}} \xrightarrow{\times Q_{if}} \underbrace{Q_{if} Q_{im}}_{\delta_{fm}} Q_{jn} C'_{mnrs} E'_{rs} = \underbrace{Q_{if} Q_{im}}_{\delta_{fm}} Q_{jn} Q_{kr} Q_{ls} C'_{mnrs} E_{kl}$$

$$Q_{jn} C'_{mnrs} E'_{rs} = Q_{jn} Q_{kr} Q_{ls} C'_{mnrs} E_{kl}$$

$$\xrightarrow[\delta_{nz} = Q_{jn} Q_{jz}]{Q_{nz} = Q_{nj} Q_{jz}} \xrightarrow{\times Q_{jz}} \underbrace{Q_{jz} Q_{jn}}_{\delta_{nz}} C'_{mnrs} E'_{rs} = \underbrace{Q_{jz} Q_{jn}}_{\delta_{nz}} Q_{kr} Q_{ls} C'_{mnrs} E_{kl}$$

$$C'_{mnrs} E'_{rs} = Q_{kr} Q_{ls} C'_{mnrs} E_{kl}$$

$$C'_{mnrs} E'_{rs} - Q_{kr} Q_{ls} C'_{mnrs} E_{kl} = 0$$

$$C'_{mnrs} \left[E'_{rs} - Q_{kr} Q_{ls} E_{kl} \right] = 0$$

$$E'_{rs} - Q_{kr} Q_{ls} E_{kl} = 0 \Rightarrow E'_{rs} = Q_{kr} Q_{ls} E_{kl}$$

$\therefore$ E_{kl} are the components of a 2nd-order tensor.