Right/Left Cauchy-Green Deformation Tensors

Lagrange Strain Tensor

Right Cauchy-Green Deformation Tensor (C):

$$C = F^T F = U^2$$

Diagonals:

$$dx^1 = ds_1 \hat{n} \qquad dX^1 = dS_1 \hat{e}_1$$

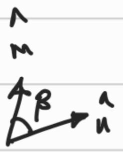
$$\left.\begin{aligned} dx^2 &= dx^1 \\ dX^1 &= dX^2 \end{aligned}\right\} \quad dx^1 \cdot dx^2 = F dX^1 \cdot F dX^2 = dX^1 \cdot F^T F dX^2 = dX^1 \cdot C dX^2$$

$$ds_1^2 = dS_1^2 \underbrace{\hat{e}_1 \cdot C \hat{e}_1} \Rightarrow C_{11} = \left(\frac{ds_1}{dS_1}\right)^2$$

$$C_{22} = \left(\frac{ds_2}{dS_2}\right)^2 , \quad C_{33} = \left(\frac{ds_3}{dS_3}\right)^2$$

off-diagonals:

$$dx^1 = ds_1 \hat{n} \qquad dX^1 = dS_1 \hat{e}_1$$



$$dx^2 = ds_2 \hat{m} \qquad dX^2 = dS_2 \hat{e}_2$$

$$dx_1 \cdot dx_2 = F dX_1 \cdot F dX_2 = dX_1 \cdot F^T F dX_2 = dX_1 \cdot C dX_2$$

$$ds_1 \hat{n} \cdot ds_2 \hat{m} = ds_1 ds_2 \, \hat{n} \cdot \hat{m} = ds_1 ds_2 \underbrace{\cancel{|n|}}_{1} \underbrace{\cancel{|m|}}_{1} \cos\beta = ds_1 ds_2 \cos\beta \quad (1)$$

$$dX_1 \cdot C dX_2 = dS_1 \hat{e}_1 \cdot C \, dS_2 \hat{e}_2 = dS_1 dS_2 \, \hat{e}_1 \cdot C \hat{e}_2 \quad (2)$$

$$(1) = (2) \Rightarrow C_{12} = \frac{ds_1 ds_2}{dS_1 dS_2} \cos\beta \quad \text{for } dX^1 = dS_1 \hat{e}_1 , \; dX^2 = dS_2 \hat{e}_2 \quad (3)$$

$$C_{13} = \frac{ds_1 ds_3}{dS_1 dS_3} \cos\alpha \quad \text{for } dX^1 = dS_1 \hat{e}_1 \,,\; dX^2 = dS_3 \hat{e}_3$$

$$C_{23} = \frac{ds_2 ds_3}{dS_2 dS_3} \cos\gamma \quad \text{for } dX^2 = dS_2 \hat{e}_2 \,,\; dX^3 = dS_3 \hat{e}_3$$

(3) $$C_{12} = \left(\frac{ds_1}{dS_1}\right)\left(\frac{ds_2}{dS_2}\right)\cos\beta = \sqrt{C_{11}}\sqrt{C_{22}}\cos\beta$$

$$\Rightarrow \cos\beta = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}}$$

Example 3.23.1

$$\begin{cases} x_1 = X_1 + 2X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

(a) $[C] = ?$

$$F = \nabla x = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = F^T F = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\lambda_1 = 5.8 \qquad n_1 = 0.4\hat{e}_1 + 0.9\hat{e}_2$

$\lambda_2 = 0.2 \qquad n_2 = 0.9\hat{e}_1 - 0.4\hat{e}_2$

$\lambda_3 = 1 \qquad n_3 = \hat{e}_3$

(c) $$C = \begin{bmatrix} 5.8 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \,,\quad C = U^2$$

$$\left.\begin{array}{l} U_{(c)} = \begin{bmatrix} \sqrt{5.8} & 0 & 0 \\ 0 & \sqrt{0.2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \\ U_{(c)}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{5.8}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{0.2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}\right\} \text{ w.r.t principal directions}$$

(d) $U_{\{e_i\}}$ $U^{-1}_{\{e_i\}}$

$\nearrow$ reference config.

$$\boxed{\begin{array}{c} U_{current} = Q^T U_{ref} Q \\ \\ Q U_{current} Q^T = U_{ref} \end{array}}$$

$$Q = \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.9 & 0 \\ 0.9 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$U_{\{e_i\}} = Q U_{(c)} Q^T = \begin{bmatrix} 0.4 & 0.9 & 0 \\ 0.9 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5.8} & 0 & 0 \\ 0 & \sqrt{0.2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0.9 & 0 \\ 0.9 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U^{-1}_{\{e_i\}} = Q U^{-1}_{(c)} Q^T = \begin{bmatrix} 0.4 & 0.9 & 0 \\ 0.9 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5.8}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{0.2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0.9 & 0 \\ 0.9 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) $R_{\{e_i\}} = ?$ $\quad F = RU \Rightarrow F U^{-1} = R$

$$R_{\{e_i\}} = F U^{-1}_{\{e_i\}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.1 & -0.7 & 0 \\ -0.7 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3.23.2

$$\begin{cases} x_1 = X_1 + kX_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

(a) stretch in direction of $\hat{e}_1$

$$C = F^T F \quad , \quad F = \nabla x = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ k & k^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_{11} = \{1 \ 0 \ 0\} \cdot \begin{bmatrix} 1 & k & 0 \\ k & k^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \{1 \ 0 \ 0\} \cdot \begin{Bmatrix} 1 \\ k \\ 0 \end{Bmatrix} = 1$$

$$C_{11} = \left(\frac{ds_1}{dS_1}\right)^2 = 1$$

(c) stretch in direction $\hat{e}_1 + \hat{e}_2$ $\left(\frac{1}{\sqrt{2}}\hat{e}_1 + \frac{1}{\sqrt{2}}\hat{e}_2\right)$

$$\frac{1}{2}\{1 \ 1 \ 0\} \cdot \begin{bmatrix} 1 & k & 0 \\ k & k^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \frac{1}{2}\{1 \ 1 \ 0\} \cdot \begin{Bmatrix} 1+k \\ k+k^2+1 \\ 0 \end{Bmatrix}$$

$$= \frac{1}{2}\underbrace{(1+k+k+k^2+1)}^{k^2+2k+2}$$

$$= \frac{k^2}{2} + k + 1 = \left(\frac{ds}{dS}\right)^2$$

$$\frac{ds}{dS} = \sqrt{\frac{k^2}{2} + k + 1}$$

(d) $\cos\beta = \dfrac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \dfrac{k}{\sqrt{k^2+1}}$

Example 3.23.3

(a) eigenvectors and eigenvalues of [C]

$$Un = \lambda n \quad , \quad C = U^2 \Rightarrow U^2 n = \lambda \underbrace{Un}_{\lambda n} \Rightarrow Cn = \lambda^2 n$$

$\therefore$ eigenvalues of C are λ^2 if eigenvalues of U are λ

eigenvectors of C are n if eigenvectors of U are n

(b) if n is the principal direction of C, in the deformed state, an element is in the direction of Rn

reference config.: $dX = dS\hat{n}$

deformed ": $dx = F dX$

$$dx = F dS\hat{n} = dS F\hat{n} \overset{F=RU}{=\!=} dS RU\hat{n} \overset{Un=\lambda n}{=\!=} dS \underbrace{\lambda}_{\text{eigenvalue}} \underbrace{(Rn)}_{\text{eigenvector}}$$

Lagrangian Strain Tensor (E^*):

$$E^* = \frac{1}{2}(C - I) \Rightarrow C = 2E^* + I \quad \blacktriangle$$

Diagonals:

$$dx^1 = ds_1 \hat{n} \qquad dX^1 = dS_1 \hat{e}_1$$

$$dx^2 = dx^1 \overset{(*)}{\Rightarrow} dx^1 \cdot dx^2 = dX^1 \cdot C\, dX^2 \overset{\blacktriangle}{=} dX^1 \cdot (2E^* + I)\, dX^2$$

$$dx^1\cdot dx^2 = 2dX^1\cdot E^* dX^2 + dX^1\cdot dX^2$$

$$dx^1\cdot dx^2 - dX^1\cdot dX^2 = 2\, dX^1\cdot E^* dX^2$$

$$ds_1^2 - dS_1^2 = 2\, dS_1^2\, \underbrace{e_1\cdot E^* e_1}_{E^*_{11}} \Rightarrow E^*_{11} = \frac{ds_1^2 - dS_1^2}{2dS_1^2}$$

for $dX_1 = dS_1\hat{e}_1$
$dx_1 = ds_1\hat{n}$

$$E^*_{22} = \frac{ds_2^2 - dS_2^2}{2dS_2^2} \quad \text{for } dX_2 = dS_2\hat{e}_2,\ dx_2 = ds_2\hat{m}$$

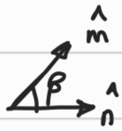
$\vdots$

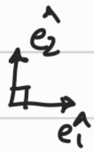
off-diagonals:

$$dx_1\cdot dx_2 - \cancelto{0}{dX^1\cdot dX^2} = 2\, dX^1\cdot E^* dX^2, \quad dx_1 = ds_1\hat{n}, \quad dX_1 = dS_1\hat{e}_1$$

$$dx_2 = ds_2\hat{m}, \quad dX_2 = dS_2\hat{e}_2$$

$$ds_1 ds_2\cos\beta = 2\, dS_1 dS_2\, \underbrace{\hat{e}_1\cdot E^*\hat{e}_2}_{E^*_{12}}$$





$$2E^*_{12} = \frac{ds_1 ds_2}{dS_1 dS_2}\cos\beta \quad \text{for } dX_1 = dS_1\hat{e}_1 \quad dx_1 = ds_1\hat{n}$$

$$dX_2 = dS_2\hat{e}_2 \quad dx_2 = ds_2\hat{m}$$

$$E^* = \frac{1}{2}(C - I) = \frac{1}{2}(F^T F - I), \quad F = (\nabla u + I)$$

$$E^* = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right] + \frac{1}{2}\underbrace{(\nabla u)^T(\nabla u)}_{T_{ij}}$$

$$E^*_{ij} = \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]}_{\text{linear}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_m}{\partial X_i}\right)\left(\frac{\partial u_m}{\partial X_j}\right)}_{\text{nonlinear}}$$

$$T_{ij} = (\nabla u)^T_{im} (\nabla u)_{mj} = (\nabla u)_{mi} (\nabla u)_{mj} = \frac{\partial u_m}{\partial X_i} \frac{\partial u_m}{\partial X_j}$$

Left Cauchy-Green Deformation Tensor (B):

$$F = VR \quad , \quad B = V^2 \quad , \quad B = FF^T$$

[B] is also known as "Finger deformation tensor"

correlation between [B] and [C]; $\quad B = FF^T = (RU)(RU)^T = RUU^TR^T$

$$= RU^2R^T = \boxed{RCR^T = B} \quad (5)$$

$$C = F^TF = (VR)^T(VR) = R^TV^TVR = R^TV^2R = \boxed{R^TBR = C} \quad (4)$$

Principal directions and values of [B]:

$$Un = \lambda n \longrightarrow Cn = \lambda^2 n \xrightarrow{(4)} R^TBRn = \lambda^2 n$$

$$B(Rn) = \lambda^2 (Rn)$$

$\therefore$ eigenvalues of B are λ^2 if eigenvalues of U are λ

eigenvectors of B are Rn if eigenvectors of U are n

Diagonals:

$$dX = dS\,n \quad , n = R^T\hat{e}_1$$

$$dx^2 = dx^1 : (\;) \quad dx^1 \cdot dx^2 = dX^1 \cdot C\, dX^2$$

$$\xrightarrow[dX^1 = dX^2]{dx^1 = ds_1\hat{n}} \qquad ds_1^2 = dS_1 (R^T\hat{e}_1) \cdot C\,(dS_1 (R^T\hat{e}_1))$$

$$ds_1^2 = dS_1^2 \underbrace{(R^T\hat{e}_1)\cdot C(R^T\hat{e}_1)}_{\hat{e}_1\cdot \underbrace{RCR^T}_{B}\hat{e}_1}$$

$$\overset{(5)}{=} dS_1^2 \underbrace{\hat{e}_1\cdot B\hat{e}_1}_{B_{11}}$$

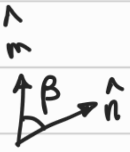
$$\Rightarrow B_{11} = \left(\frac{ds_1}{dS_1}\right)^2 \quad \text{for } dX = dS_1(R^T e_1)$$

$$B_{22} = \left(\frac{ds_2}{dS_2}\right)^2 \quad \text{for } dX = dS_2(R^T e_2)$$

$$\vdots$$

off-diagonals:

$$dx_1 = ds_1\hat{n} \qquad dX^1 = dS_1\hat{n} = dS_1(R^T\hat{e}_1)$$



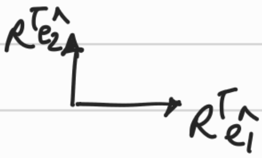
$$dx_2 = ds_2\hat{m} \qquad dX^2 = dS_2\hat{m} = dS_2(R^T\hat{e}_2)$$

$$dx_1\cdot dx_2 = dX^1\cdot C\,dX^2$$

$$ds_1 ds_2\cos\beta = dS_1 dS_2 \underbrace{\hat{e}_1\cdot B\hat{e}_2}_{B_{12}} \Rightarrow B_{12} = \frac{ds_1 ds_2}{dS_1 dS_2}\cos\beta$$

for $dX^1 = dS_1(R^T\hat{e}_1)$ and $dX^2 = dS_2(R^T\hat{e}_2)$

Initially $R^T\hat{e}_1$ and $R^T\hat{e}_2$ are orthogonal.



$$B = FF^T = (I+\nabla u)(I+(\nabla u)^T) = I + (\nabla u)^T + \nabla u + \underbrace{(\nabla u)(\nabla u)^T}_{T_{ij}}$$

$$B_{ij} = \delta_{ij} + \left(\frac{\partial u_j}{\partial X_i}\right) + \left(\frac{\partial u_i}{\partial X_j}\right) + \frac{\partial u_i}{\partial X_m}\frac{\partial u_j}{\partial X_m}$$

$$T_{ij} = (\nabla u)(\nabla u)^T = (\nabla u)_{im}(\nabla u)^T_{mj} = (\nabla u)_{im}(\nabla u)_{jm}$$

$$= \frac{\partial u_i}{\partial X_m}\frac{\partial u_j}{\partial X_m}$$

for infinitesimal deformation: $B = 2E + I$

$$C = 2E + I$$