The Impact Analysis of Two Identical Bars Using The Wave Equation and d'Alembert Solution

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The d'Alembert solution to wave equation in one-dimension, Eq. 1, includes two main terms: (a) $f(x+ct)$ which indicates a backward wave moving in the negative direction of x-axis (see Figure. 1 for the direction of x-axis) and (b) $f(x-ct)$ that is a forward wave moving in the positive direction of x-axis.

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] \tag{1}$$

Invoking the chain role and assuming linear elasticity, the velocity, v, and longitudinal stress, σ, can be calculated from the following equations.

$$\begin{aligned} v(x,t) &= \frac{\partial u(x,t)}{\partial t} = cf'(x+ct) - cf'(x-ct) \\ \sigma(x,t) &= E\epsilon = E\frac{\partial u(x,t)}{\partial x} = E[f'(x+ct) - f'(x-ct)] \end{aligned} \tag{2}$$

in which E is the Young's modulus and ϵ is longitudinal strain.

In the case of impact of two identical bars with similar ρ, A and l, to avoid misinterpretation as well as input conflict in Maple, we may use the following forms of wave equation where the second backward wave is replaced with letter g and subscripts 1 and 2 denotes bars $s1$ and $s2$, respectively.

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}[g_1(x+ct) + f_1(x-ct)] \\ u_2(x,t) &= \frac{1}{2}[g_2(x+ct) + f_2(x-ct)] \end{aligned} \tag{3}$$

Consequently, v and σ become

$$\begin{aligned} v_1(x,t) &= cg_1'(x+ct) - cf_1'(x-ct) \\ \sigma_1(x,t) &= E[g_1'(x+ct) - f_1'(x-ct)] \\ v_2(x,t) &= cg_2'(x+ct) - cf_2'(x-ct) \\ \sigma_2(x,t) &= E[g_2'(x+ct) - f_2'(x-ct)] \end{aligned} \tag{4}$$

The set of equations in Eq. 4 along with suitable initial and boundary conditions can be used in analyzing the impact of bars.

Suppose the problem in Figure. 1 where the bar $s1$ hits the stationary bar $s2$. The problem can be broken into different time intervals.

2 $t = 0$

At $t = 0$, both bars are stress-free and the bar $s1$ has the initial velocity v_0. As $s1$ hits $s2$, a compressive wave is generated at the contact area and propagates towards the left and right in $s1$ and $s2$, respectively which are essentially g_1 and f_2 terms in Eq. 3. At this time the following initial and boundary conditions can be written.

$$\begin{aligned}
&v_1(x,0) = v_0 \Rightarrow cg_1'(x) - cf_1'(x) = v_0 \\
&\sigma_1(x,0) = 0 \Rightarrow E[g_1'(x) - f_1'(x)] = 0 \\
&v_2(x,0) = 0 \Rightarrow cg_2'(x) - cf_2'(x) = 0 \\
&\sigma_2(x,0) = 0 \Rightarrow E[g_2'(x) - f_2'(x)] = 0
\end{aligned} \tag{5}$$

Using `dsolve` command from Maple gives

$$\begin{aligned}
g_1(x) &= \frac{v_0 x}{2c} + c_2 \\
f_1(x) &= -\frac{v_0 x}{2c} + c_4 \\
g_2(x) &= c_1 \\
f_2(x) &= c_3
\end{aligned} \tag{6}$$

c_1 to c_4 are constants and are not required to be determined as they vanish in calculating velocity and longitudinal stress. Eq. 6 shows that at $t = 0$, $s1$ is characterized by the two backward ($g_1(x + ct)$) and forward ($f_1(x - ct)$) waves. Substituting Eq. 6 in Eq. 5 gives

$$\begin{aligned}
v_1(x,0) &= v_0 \\
\sigma_1(x,0) &= 0 \\
v_2(x,0) &= 0 \\
\sigma_2(x,0) &= 0
\end{aligned} \tag{7}$$

which concludes the velocity and longitudinal stress at $t = 0$.

3 $0 < t < \dfrac{l}{c}$

After impact, the initial and boundary condition for the solution of the set of equations in Eq. 4 require updating. To find the velocity of both bars after impact we use Newton's third law.

$$\begin{cases} \sigma_1 = \rho c(v' - v_0) \\ \\ \sigma_2 = \rho c(-v' - 0) \end{cases} \xRightarrow{\sigma_1=\sigma_2} v_0 - v' = v' \Rightarrow v' = \frac{v_0}{2}$$

The initial and boundary conditions update accordingly.

$$\begin{aligned} v_1(x,t) &= \frac{v_0}{2} \Rightarrow cg_1'(x) - cf_1'(x) = \frac{v_0}{2} \\ \sigma_1(x,t) &= -\frac{\rho c v_0}{2} \Rightarrow E[g_1'(x) - f_1'(x)] = -\frac{\rho c v_0}{2} \\ v_2(x,t) &= \frac{v_0}{2} \Rightarrow cg_2'(x) - cf_2'(x) = \frac{v_0}{2} \\ \sigma_2(x,t) &= -\frac{\rho c v_0}{2} \Rightarrow E[g_2'(x) - f_2'(x)] = -\frac{\rho c v_0}{2} \end{aligned} \tag{8}$$

The following answers are obtained from Maple.

$$\begin{aligned} g_1(x) &= -\frac{\rho c v_0}{4E}x + \frac{v_0}{4c}x + c_2 \\ f_1(x) &= -\frac{\rho c v_0}{4E}x - \frac{v_0}{4c}x + c_4 \\ g_2(x) &= -\frac{\rho c v_0}{4E}x + \frac{v_0}{4c}x + c_1 \\ f_2(x) &= -\frac{\rho c v_0}{4E}x - \frac{v_0}{4c}x + c_3 \end{aligned} \tag{9}$$

Here, the interaction of the generated backward and forward waves represents the velocity and longitudinal stress. Plug-in Eq. 9 in Eq. 8, we get the velocity and longitudinal stress.

$$\begin{aligned} v_1(x,0) &= \frac{v_0}{2} \\ \sigma_1(x,0) &= -\frac{\rho c v_0}{2} \\ v_2(x,0) &= \frac{v_0}{2} \\ \sigma_2(x,0) &= -\frac{\rho c v_0}{2} \end{aligned} \tag{10}$$

which also has been determined from Newton's third law.

4 $\frac{l}{c} < t < \frac{2l}{c}$

At this time interval, the generated compressive waves reflect as tensile wave (unloading wave) due to free boundary conditions on the left and right sides of $s1$ and $s2$. The generated tensile wave make both bars stress-free and the velocities can be calculated from stress equations.

$$-\sigma_{1c} + \sigma_{1t} = 0 \Rightarrow \rho c\left(-\frac{v_0}{2} + v''\right) \Rightarrow v'' = \frac{v_0}{2}$$

The subscript c and t denote compressive and tensile waves. Using the above result, one is able to calculate the velocity of each bar.

For $s1$,

$$\frac{v_0}{2} - \frac{v_0}{2} = 0$$

For $s2$,

$$\frac{v_0}{2} + \frac{v_0}{2} = v_0$$

The initial and boundary conditions are

$$\begin{aligned}
v_1(x,0) = 0 &\Rightarrow cg_1'(x) - cf_1'(x) = 0 \\
\sigma_1(x,0) = 0 &\Rightarrow E[g_1'(x) - f_1'(x)] = 0 \\
v_2(x,0) = v_0 &\Rightarrow cg_2'(x) - cf_2'(x) = v_0 \\
\sigma_2(x,0) = 0 &\Rightarrow E[g_2'(x) - f_2'(x)] = 0
\end{aligned} \tag{11}$$

Solving the above set of differential equations results in

$$\begin{aligned}
g_1(x) &= c_2 \\
f_1(x) &= c_4 \\
g_2(x) &= \frac{v_0 x}{2c} + c_1 \\
f_2(x) &= -\frac{v_0 x}{2c} + c_3
\end{aligned} \tag{12}$$

As it can be inferred from Eq. 12 The velocity and longitudinal stress are characterized by the backward ($g_2(x+ct)$) and forward ($f_2(x-ct)$) waves of $s2$ and are

$$\begin{aligned}
v_1(x,0) &= 0 \\
\sigma_1(x,0) &= 0 \\
v_2(x,0) &= v_0 \\
\sigma_2(x,0) &= 0
\end{aligned} \tag{13}$$

Figure. 1 summarises the above explanation.

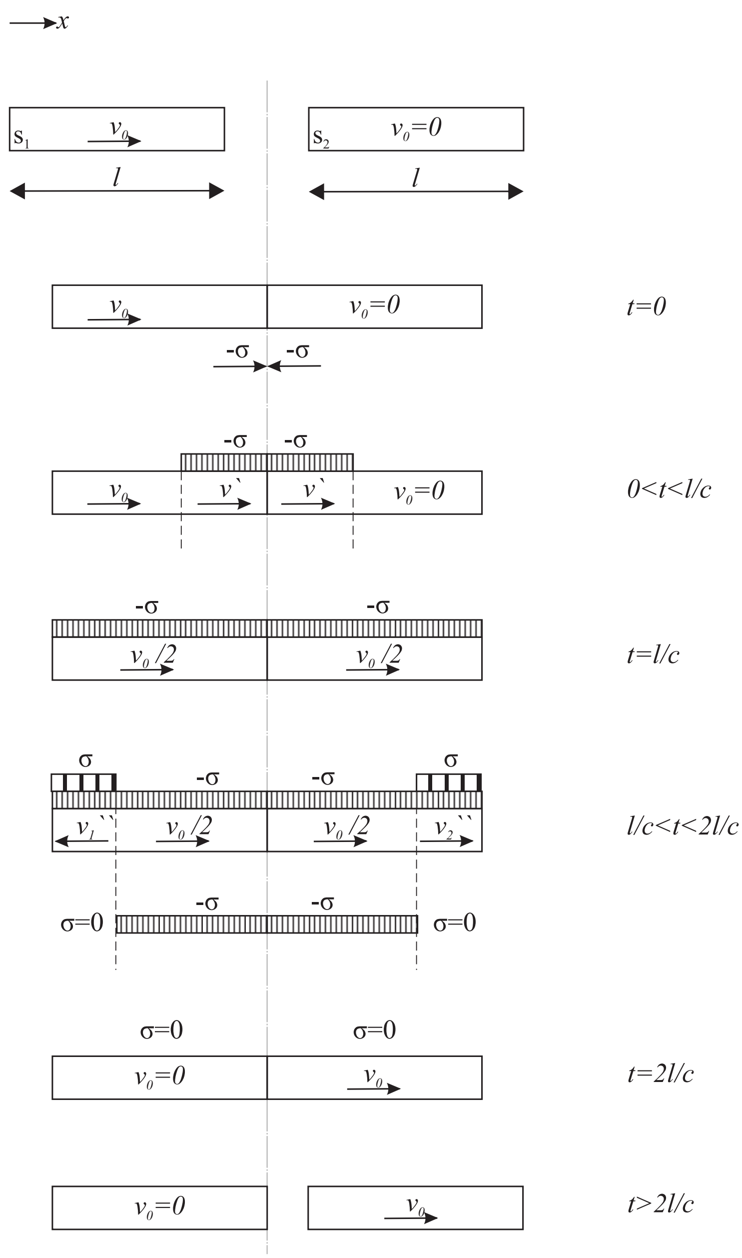

Figure 1: Different steps in the impact of two identical bars, one stationary and another with an initial velocity.