The Longitudinal Impact of a Rigid Mass Against an Elastic Bar

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1 Problem Statement

Figure. 1 illustrates the longitudinal impact of a rigid mass against an elastic bar. The rigid body has a mass m_2 and an initial velocity v_0 whereas the elastic bar is fixed at $x = l$ and is free at the origin and has a mass m_1, density ρ, length l with a cross-sectional area A and Young's modulus E. In the following, c is the wave speed.

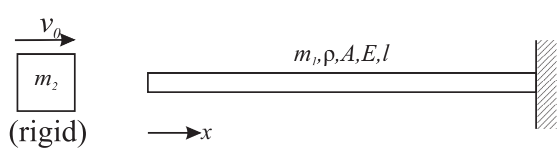

Figure 1: The problem geometry and coordinate system.

2 $0 < ct < 2l$ at $x = 0$

At time interval $0 < ct < 2l$, assuming that $t = 0$ is the time when the rigid mass hits the bar, a compressive forward wave $f(ct - x)$ is generated at $x = 0$ which takes $t = \dfrac{2l}{c}$ for the generated wave f_0 to return to $x = 0$. The subscript zero denotes the number given to the generated wave. The displacement, u, and strain, ϵ, for this time interval are given here for convenience as it is used later in the subsequent sections.

$$\begin{aligned} u_0(x,t) = f_0 &= \frac{lv_0}{Mc}\left(1 - e^{-\frac{M}{l}ct}\right) \\ \epsilon &= -\frac{v_0}{c}e^{-\frac{M}{l}ct} \end{aligned} \tag{1}$$

where

$$M = \frac{\rho A l}{m_2} = \frac{EAl}{m_2 c^2} \tag{2}$$

3 $l < ct < 3l$ at $x = l$

As soon as the compressive backward wave $f_0(ct-x)$ arrives at the fixed boundary ($x = l$), it reflects as a compressive backward wave $g_1(ct+x)$. g_1 has a latency of $\frac{l}{c}$; therefore, this backward wave is written as $g_!(ct+x-2l)$. Writing and simplifying the relevant equations results in

$$g_1 = \frac{v_0 l}{cM}\left(e^{-\frac{M}{l}(ct+x-2l)} - 1\right) \tag{3}$$

The subscript one represents the generated wave number.

4 $2l < ct < 4l$ at $x = 0$

At this time interval, both the forward and backward wave contribute to the displacement and it can be written as

$$\begin{aligned} u_1(x,t) &= f_1(ct-x-2l) + g_1(ct+x-2l) \\ &= f_1(\xi_1) + g_1(\xi_2) \end{aligned} \tag{4}$$

in which

$$\begin{cases} \xi_1 = ct - x - 2l \\ \xi_2 = ct + x - 2l \end{cases}$$

In Eq. 4, g_1 is substituted from Eq. 3 and f_1 is unknown.

4.1 Determination of f_1

One can write Newton's second law for Figure. 1 for $2l < ct < 4l$ at $x = 0$

$$\begin{aligned} -m_2\frac{\partial u^2(x,t)}{\partial t^2} &= -EA\frac{\partial u(x,t)}{\partial x} \\ m_2c^2(f_1''(\xi_1) + g_1''(\xi_2)) &= EA(g_1'(\xi_2) - f_1'(\xi_1)) \end{aligned}$$

where the chain rule is utilized in calculating the derivatives, *i.e.*, $\frac{\partial f_1}{\partial \xi_1}\frac{\partial \xi_1}{\partial t}$.

Because all the above calculations are done at the free end of the bar ($x = 0$), ξ_1 and ξ_2 can be replaced by $\xi = ct - 2l$.

$$\begin{aligned} m_2c^2(f_1''(\xi) + g_1''(\xi)) &= EA(g_1'(\xi) - f_1'(\xi)) \\ f_1''(\xi) + g_1''(\xi) &= \frac{M}{l}(g_1'(\xi) - f_1'(\xi)) \end{aligned} \tag{5}$$

M can be found from Eq. 2.

g_1 and its derivatives in Eq. 5 can be calculated from Eq. 3.

$$\begin{aligned}
g_1'(ct+x-2l) &= -\frac{v_0}{c}e^{-\frac{M}{l}(ct+x-2l)} \\
g_1'(\xi)_{\{x=0\}} &= -\frac{v_0}{c}e^{-\frac{M}{l}\xi} \\
g_1''(ct+x-2l) &= \frac{Mv_0}{lc}e^{-\frac{M}{l}(ct+x-2l)} \\
g_1''(\xi)_{\{x=0\}} &= \frac{Mv_0}{lc}e^{-\frac{M}{l}\xi}
\end{aligned}$$

One should note that the above derivatives are taking w.r.t x.

Replacing $g_1'(\xi)$ and $g_1''(\xi)$ from above set of equations and using the `dsolve` command from Maple, the solution to Eq. 5 is

$$f_1(\xi) = \frac{(2Mv_0\xi - Bcl + 2lv_0)}{Mc}e^{-\frac{M}{l}\xi} + D \tag{6}$$

where B and D are constants and can be determined from initial conditions. In order to find B and D, we use the continuity principle of displacements and velocity where both the initial displacement and velocity in Eq. 4 should have the same value of u and v at $ct = 2l$ in Eq. 1, that is

For the displacement:

$$\begin{aligned}
u_{1\{x=0,ct=2l\}} &= u_{0\{x=0,ct=2l\}} \\
f_{1\{x=0,ct=2l\}} + g_{1\{x=0,ct=2l\}} &= f_{0\{x=0,ct=2l\}}
\end{aligned} \tag{7}$$

For the velocity:

$$\begin{aligned}
v_{1\{x=0,ct=2l\}} &= v_{0\{x=0,ct=2l\}} \\
\frac{\partial}{\partial t}f_{1\{x=0,ct=2l\}} + \frac{\partial}{\partial t}g_{1\{x=0,ct=2l\}} &= \frac{\partial}{\partial t}f_{0\{x=0,ct=2l\}}
\end{aligned} \tag{8}$$

Solving Eq. 7 and Eq. 8 simultaneously

$$B = \frac{v_0}{c}(e^{-2M}+1) \quad , \quad D = 0$$

As a result

$$f_1(\xi) = \frac{v_0}{Mc}(2M\xi - le^{-2M} + l)e^{-\frac{M}{l}\xi} \tag{9}$$

5 Displacement and Strain Plots

The displacement for $2l < ct < 4l$ can be found by plugging-in Eq. 3 and Eq. 9 into Eq. 4.

$$u_1(x,t) = \frac{v_0}{Mc}\left\{\left[2M\left(ct-2l-x\right)-l\left(e^{-2M}+1\right)+2l\right]e^{-\frac{M}{l}(ct-2l-x)}+l\left(e^{-\frac{M}{l}(ct-2l+x)}-1\right)\right\} \tag{10}$$

Figure. 2 shows the piecewise combination of Eq. 1 and Eq. 10 for $0 < ct < 4l$ for $c = 1$, $l = 1$, $M = 1$ at the free side of the elastic bar ($x = 0$). As expected, the displacement plot is continuous.

The strain can be calculated by taking the derivative of displacements from Eq. 1 and Eq. 10 w.r.t x.

$$\begin{cases} \epsilon_0 = \frac{v_0}{c}e^{-\frac{M}{l}ct} & ,0 < ct < 2l \\ \epsilon_1 = \frac{v_0}{c}\left\{\left(\frac{2M(ct-2l-x)-l(e^{-2M}+1)+2l}{l}-2\right)e^{-\frac{M}{l}(ct-2l-x)}-e^{-\frac{M}{l}(ct-2l+x)}\right\} & ,2l < ct < 4l \end{cases} \tag{11}$$

Eq. 11 is plotted in Figure. 3. As it is seen, the discontinuity in the strain plot occurs at $t = 2l$ where the strain jumps to higher but negative value. It is also evident that for $0 < ct < 2l$, the strain does not become zero as it appears in form of an exponential function (see Eq. 11_1), while the strain and subsequently the stress (due to linear elastic assumption between stress and strain) vanishes at around $3l$. To find the corresponding time, we may solve Eq. 11_2 for t.

$$t = \frac{1}{2}\frac{4Ml + 2Mx + le^{-2M} + l + le^{-\frac{2M}{l}x}}{Mc}$$

If we remove x from above equation

$$t = \frac{1}{2}\frac{4Ml + l\,e^{-2M} + 2l}{Mc} \tag{12}$$

and for $c = 1$, $l = 1$, $M = 1$, the time at which stress vanishes is 3.068s.

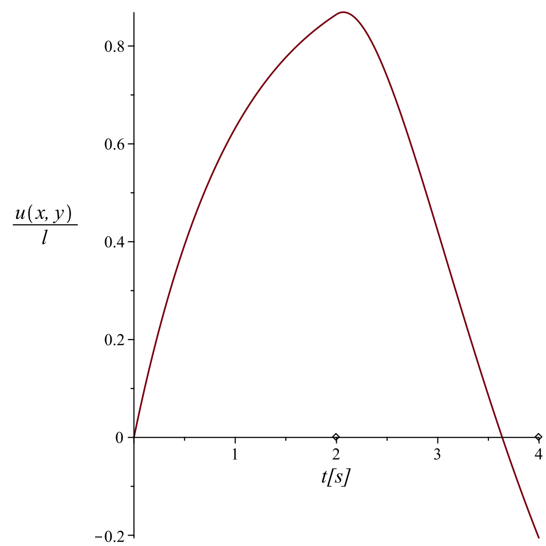

Figure 2: Normalized displacement against time for $c = 1$, $l = 1$, $M = 1$ at the free side of the elastic bar ($x = 0$).

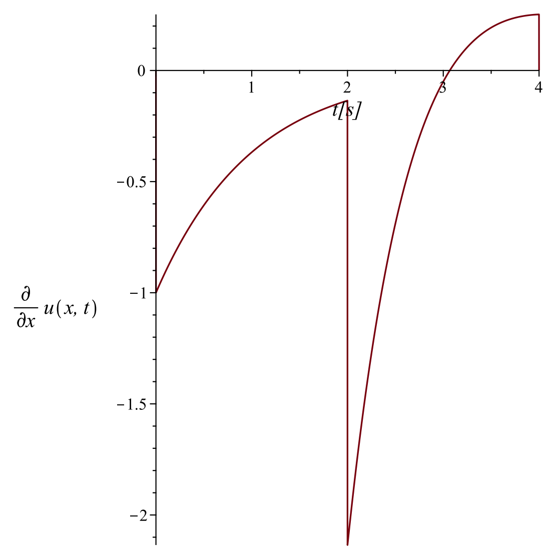

Figure 3: Strain against time for $c = 1$, $l = 1$, $M = 1$ at the free side of the elastic bar ($x = 0$).