The Impact of Two Identical Bars with Different Velocities

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In this problem, the impact of two identical bars with similar density, ρ, cross-sectional area, A, and length, l but different initial velocity, v_0, is discussed. Let us name the bar on the right, s_1, and the one on the left, s_2 (see Figure. 1). Before impact, s_1 and s_2 have the initial velocity $2v_0$ and v_0, respectively. c denotes wave propagation speed. The following explains different steps of impact until two bars are separated.

2 $t = 0$

At $t = 0$, a compressive stress wave is generated at the contact area of both bars where due to Newton's third law, these generated compressive stress are equal (see Figure. 1 for the corresponding time).

From the energy point of view, before the impact, we can write

$$
\begin{aligned}
K_1 &= \frac{1}{2}m_1 v_1^2 = \frac{1}{2}(\rho A l)(2v_0)^2 = 2\rho A l v_0^2 \\
K_2 &= \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(\rho A l)(v_0)^2 = \frac{1}{2}\rho A l v_0^2 \\
U_1 &= U_2 = 0 \\
E_t &= K_1 + K_2 + U_1 + U_2 = \frac{5}{2}\rho A l v_0^2
\end{aligned}
$$

where, m is mass, K is kinetic energy, U is strain energy and E_t is total energy.

3 $0 < t < \dfrac{l}{c}$

The generated compressive wave propagates towards the free sides of both bars with the velocity c, while due to impact the particles speed changes to v' (see the corresponding time interval in Figure. 1). Since the compressive stress, $-\sigma$, at the surface of contact is the same, one can write

$$\begin{cases} \sigma_1 = \rho c(v' - 2v_0) \\ \\ \sigma_2 = \rho c(-v' - v_0) \end{cases} \xRightarrow{\sigma_1=\sigma_2} 2v_0 - v' = v' + v_0 \Rightarrow v' = \frac{v_0}{2}$$

As a results, the stresses become

$$\sigma_1 = \rho c\left(\frac{v_0}{2} - 2v_0\right) = -\frac{3}{2}\rho c v_0$$
$$\sigma_2 = \rho c\left(-\frac{v_0}{2} - v_0\right) = -\frac{3}{2}\rho c v_0$$

It is worth noting that in developing the preceding equations, it is assumed that the velocity direction follows the stress type (compressive(-) or tensile(+)).

4 $t = \frac{l}{c}$

At $t = \frac{l}{c}$, the compressive stress wave at each bar arrives at free boundary condition and affects both bars entirely. The total energy can be written as

$$K_1 = \frac{1}{2}\rho Al(\frac{v_0}{2})^2 = \frac{1}{8}\rho Alv_0^2$$
$$K_2 = \frac{1}{2}\rho Al(\frac{v_0}{2})^2 = \frac{1}{8}\rho Alv_0^2$$
$$U_1 = \left(\frac{\sigma^2}{2E}\right)V = (Al)\frac{9}{4}\frac{\rho^2c^2v_0^2}{2E} = \frac{9}{8}\rho Alv_0^2$$
$$U_2 = \left(\frac{\sigma^2}{2E}\right)V = (Al)\frac{9}{4}\frac{\rho^2c^2v_0^2}{2E} = \frac{9}{8}\rho Alv_0^2$$
$$E_t = K_1 + K_2 + U_1 + U_2 = \frac{5}{2}\rho Alv_0^2$$

where V is the bar volume.

4.1 $\frac{l}{c} < t < \frac{2l}{c}$

Due to free boundary conditions on the left and right sides of bars 1 and 2, the compressive wave reflects as a tensile wave as is illustrated in Figure. 1. At this time interval, the particles from each bar take another velocity v_1'' for bar 1 and v_2'' for bar 2 which can be determined from below knowing that the summation of both compressive and tensile stress waves is zero.

For s_1

$$-\sigma+\sigma=0 \Rightarrow \frac{3}{2}\rho c v_0=\frac{3}{2}\rho c\left(v_1''-\left(-\frac{v_0}{2}\right)\right) \Rightarrow v_0=v_1''+\frac{v_0}{2} \Rightarrow v_1''=\frac{v_0}{2}$$

For s_2

$$-\sigma+\sigma=0 \Rightarrow \frac{3}{2}\rho c v_0=\frac{3}{2}\rho c\left(v_2''-\frac{v_0}{2}\right) \Rightarrow v_0=v_2''-\frac{v_0}{2} \Rightarrow v_2''=\frac{3v_0}{2}$$

The direction of v_1'' and v_2'' is shown in Figure. 1.

5 $t=\frac{2l}{c}$

At this time, the tensile waves propagate through both bars and make them stress-free (unloading wave). At this time, the velocity for s_1 is

$$\frac{v_0}{2}-\left(-\frac{v_0}{2}\right)=v_0$$

and for s_2 is

$$-\frac{v_0}{2}+\left(-\frac{3v_0}{2}\right)=-2v_0$$

Hence, the s_1 bar rebounds in the direction of the x-axis with velocity v_0 and s_2 rebounds in the opposite direction with velocity $2v_0$ and for $t>\frac{2l}{c}$, both bars separate from each other.

The kinetic and strain energy can be written as

$$\begin{aligned}
K_1 &= \frac{1}{2}(\rho Al)(v_0)^2=\frac{1}{2}\rho Alv_0^2\\
K_2 &= \frac{1}{2}(\rho Al)(2v_0)^2=2\rho Alv_0^2\\
U_1 &= U_2=0\\
E_t &= K_1+K_2+U_1+U_2=\frac{5}{2}\rho Alv_0^2
\end{aligned}$$

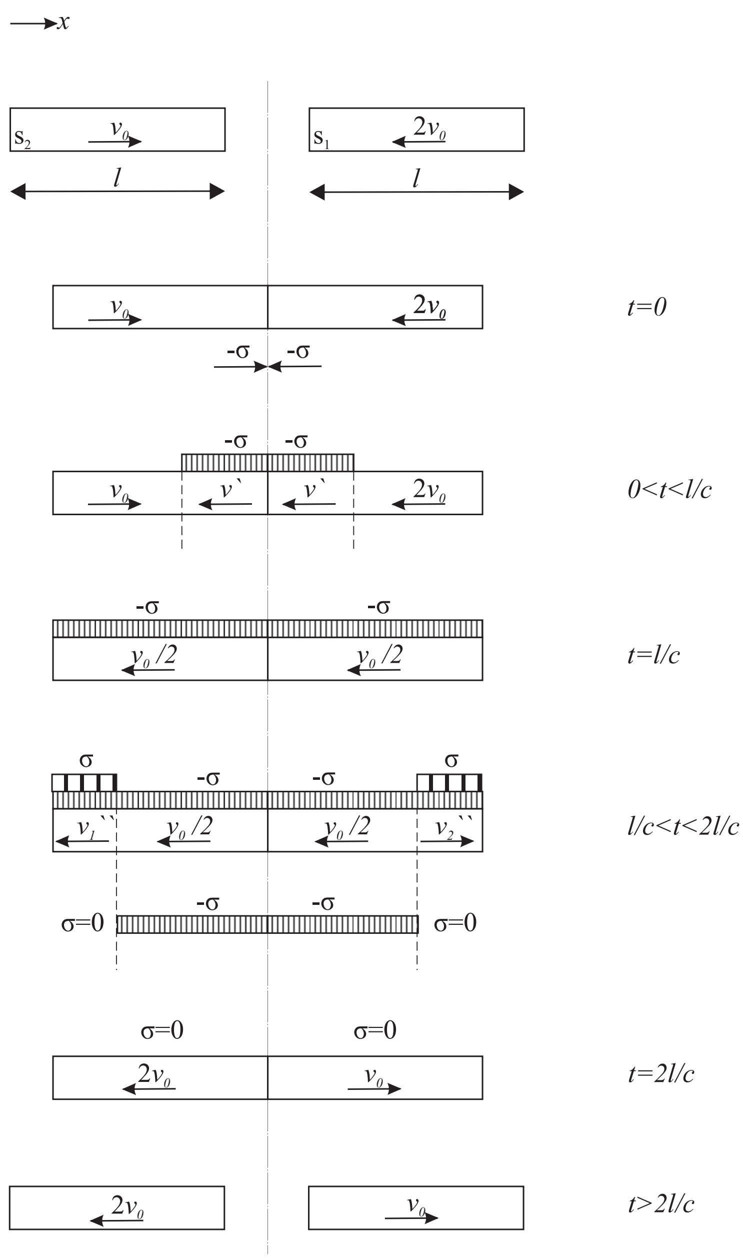

Figure 1: Different steps in the impact of two identical bars with different initial velocities.