Extensions of L1 Trend Filtering: Seasonality

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Abstract

This paper extends and further elucidates ideas from Kim, Koh & Boyd for L1 regularized trend filtering and acts a documentation to a repository of a python implementation using the cvxopt library which can be found at https://github.com/dave31415/myl1tf

1 The primary and dual quadratic programming problems

We will start off at Section 5.2 of KKB where they state the quadratic programming problem for L1TF and also state the dual problem.

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\ ||y-x||_2^2 + \lambda ||z||_1 && (1)\\ \text{subject to} \quad & z = Dx && (2) \end{aligned}$$

The first is an l_2 norm and the second an l_1 norm. The Lagrangian, with a dual variable $\nu \in \mathbf{R}^{n-2}$, is

$$L(x,z,\nu) = ||y-x||_2^2 + \lambda ||z||_1 + \nu^T(Dx - z)$$

The dual function is

$$\inf_{x,z}\ L(x,z,\nu) = \begin{cases} -\frac{1}{2}\ \nu^T DD^T\nu + y^T D^T \nu & -\lambda\mathbf{1} < \nu < \lambda\mathbf{1} \\ -\infty & \text{otherwise.} \end{cases}$$

and so the dual problem is

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\ \nu^T DD^T\nu - y^T D^T \nu && (3)\\ \text{subject to} \quad & -\lambda\mathbf{1} < \nu < \lambda\mathbf{1} && (4) \end{aligned}$$

From the solution ν^* of the dual problem, we can compute the L1TF solution,

$$x^* = y - D\nu^*$$

2 Seasonality

As suggested by KKB we can adapt this to add a seasonal component.

$$\begin{aligned}
\text{minimize} \quad & \tfrac{1}{2}\ ||y-x-s||_2^2 + \lambda ||z||_1 && (5)\\
\text{subject to} \quad & z = Dx && (6)\\
\textit{and} \quad & \textstyle\sum_i^P s_i = 0 && (7)\\
\textit{and} \quad & s_{i+P} = s_i && (8)
\end{aligned}$$

KKB does not go into detail on how to proceed with this and so we will begin here by deriving the dual problem. We will do this by putting the constraints on s directly into the equation to be minimized.

To do this, we define p to be the vector of independent variables defining the periodic components. This vector has dimension $(P-1)$ as the P-th, dependent value is $-\sum_{i=1}^{P-1} p_i$ which enforces the constraint that they sum to zero. This constraint is required if there is to exists a unique solution as otherwise one could add any constant to x and subtract it from s without changing the model.

We can define $\tilde{p} \in \mathbf{R}^P$ as $\tilde{p} = \left(p, -\sum_{i=1}^{P-1} p_i\right) = Tp$. T will be a $P \times (P-1)$ matrix with a $(P-1)$ identity matrix at the top and an extra row consisting of all -1. The vector s is now just a periodic re-cycling of $\tilde{p}$ which we can represent as s matrix B which is formed by row-wise stacking some number (ceil(N/P)) of $P \times P$ identity matrices and truncating rows to dimension N. So finally we can write $s = BTp \equiv Qp$. By doing this we can rewrite the optimization problem as

$$\begin{aligned}
\text{minimize} \quad & \tfrac{1}{2}\ ||y-x-Qp||_2^2 + \lambda ||z||_1 && (9)\\
\text{subject to} \quad & z = Dx && (10)
\end{aligned}$$

with the $p \in \mathbf{R}^{(P-1)}$ now unconstrained. To improve stabilization and allow more control over p, we will add a l_2 regularization constraint and write our problem.

$$\begin{aligned}
\text{minimize} \quad & \tfrac{1}{2}\ ||y-x-Qp||_2^2 + \lambda ||z||_1 + \eta\ \tfrac{1}{2} p^T p && (11)\\
\text{subject to} \quad & z = Dx && (12)
\end{aligned}$$

We will now proceed as before and derive the dual problem. To do this we first write down the Lagrangian

$$L(x, z, p, \nu, \eta, \lambda) = ||y-x-Qp||_2^2 + \lambda ||z||_1 + \nu^T (Dx - z) + \eta\ \frac{1}{2} p^T p$$

and calculate $\inf_{x,z,p} L(x, z, p, \nu, \eta, \lambda)$ by setting gradients w.r.t. x and p to zero. The more subtle minimization w.r.t. z will result in the same constraint in the dual problem as before. The reader should ensure that they understand how the terms $\lambda ||z||_1 - \nu^T z$ result in the constraint $-\lambda \mathbf{1} < \nu < \lambda \mathbf{1}$. One can show that outside of this range, the inf_z of this term is $-\infty$ (at either $z = \pm\infty$) and within is 0 (at $z = 0$) and so feasible solutions must lie within.

Setting $\nabla_x L$ =0 yields the equation.

$$y - x - Qp = D^T \nu$$

or

$$x = y - Qp - D^T \nu$$

and $\nabla_p L$ =0 yields

$$p = \eta^{-1} Q^T D^T \nu$$

and we can use this last equation for p to solve for x and we can write these solutions as

$$\begin{aligned} p^* &= \eta^{-1} Q^T D^T \nu \qquad (13) \\ &\mathit{and} \\ x^* &= y - D^T \nu - \eta^{-1} Q Q^T D^T \nu \qquad (14) \end{aligned}$$

We then construct the dual problem by plugging these solutions into the Lagrangian and multiplying it by -1 (minimize rather than maximize) and we arrive at

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\ \nu^T A \nu - y^T D^T \nu \qquad (15) \\ \text{subject to} \quad & -\lambda \mathbf{1} < \nu < \lambda \mathbf{1} \qquad (16) \end{aligned}$$

with

$$A = DD^T + \eta^{-1} D Q Q^T D^T$$

We solve this quadratic programming problem as before for ν and then use the equations above to calculate x and p from ν. It is apparent that as $\nu \to \infty$ (seasonality is suppressed), $p \to 0$ and we recover the same solution as before for x.

3 Seasonality using l_1 regularization

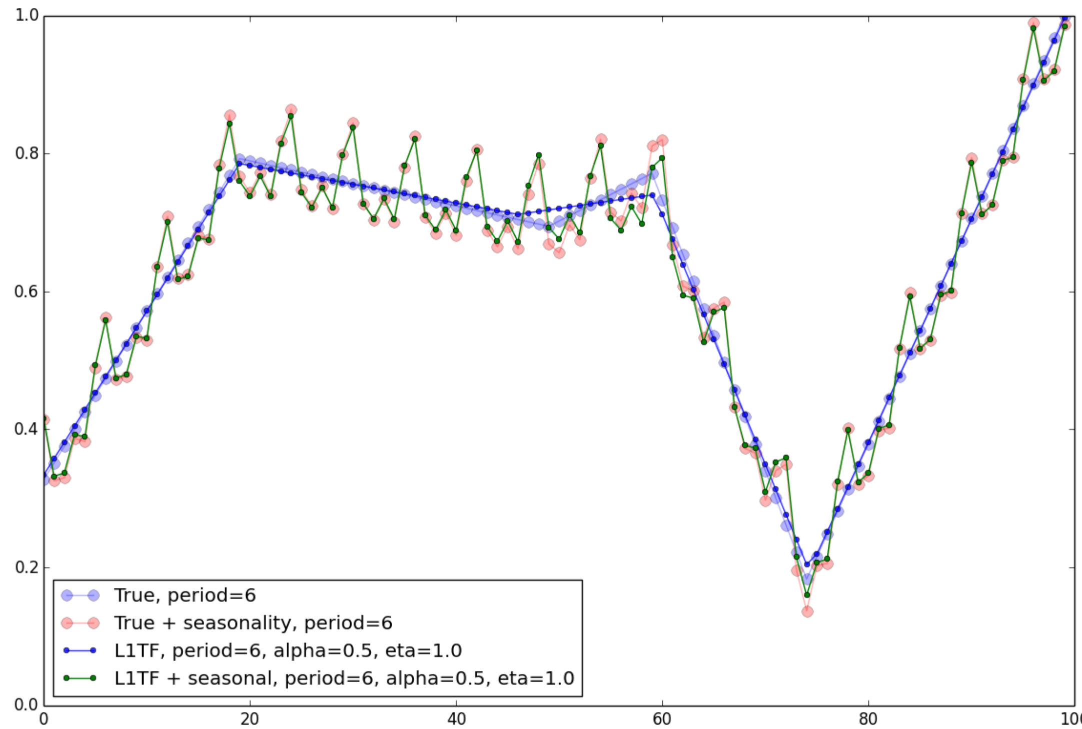
Finish

4 Modeling outliers for more robust fits

Finish

5 Implementation

The Githib repository https://github.com/dave31415/myl1tf contains an implementation for this L1TF modeling with seasonality. This is a fork of the



repository https://github.com/elsonidoq/py-l1tf by Pablo Zivic which implements the simpler version without seasonality. Both versions are in python and use the python cvxopt library to solve the quadratic programming problems. Our version contains some test programs. For example, the following command,

```
test_myl1tf.test_l1tf_on_mock_with_period(period=6, eta=1.0,alpha=0.5)
```

creates a mock data-set, fits the model and displays the following plot. (Note *eta* $= \eta$ and *alpha* $= \lambda$ as *lambda* is a reserved word in python).