edX Robo4 Mini MS – Locomotion Engineering

Block 1 – Week 2 – Unit 1

A Linear Time Invariant Mechanical System

Video 2.1

Segment 1.2.1.1
Prismatic 1 DoF Physics

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with
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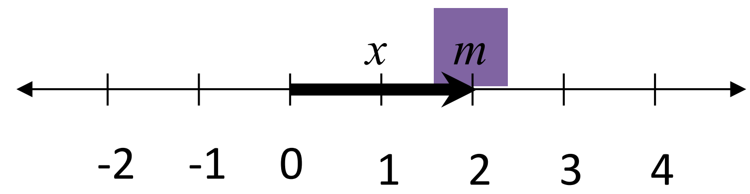
June, 2017

1 DoF (Prismatic) Kinematics



- Moving body with "mass" m
- Frame of Reference
  - coordinates
  - extension, x
- 1 DoF Kinematics
  - moving mass has velocity

$$v(t) := \dot{x}(t) = \frac{d}{dt} x(t)$$

(symbol := means "defined as")

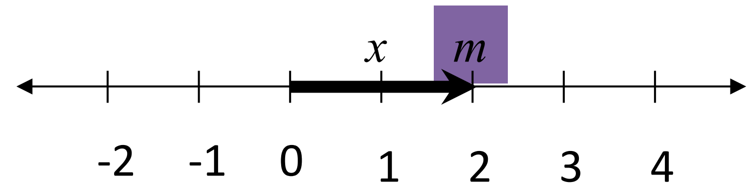
  - acceleration: changes in velocity

$$a(t) := \dot{v}(t) = \ddot{x}(t)$$

("overdots" denote time derivatives)

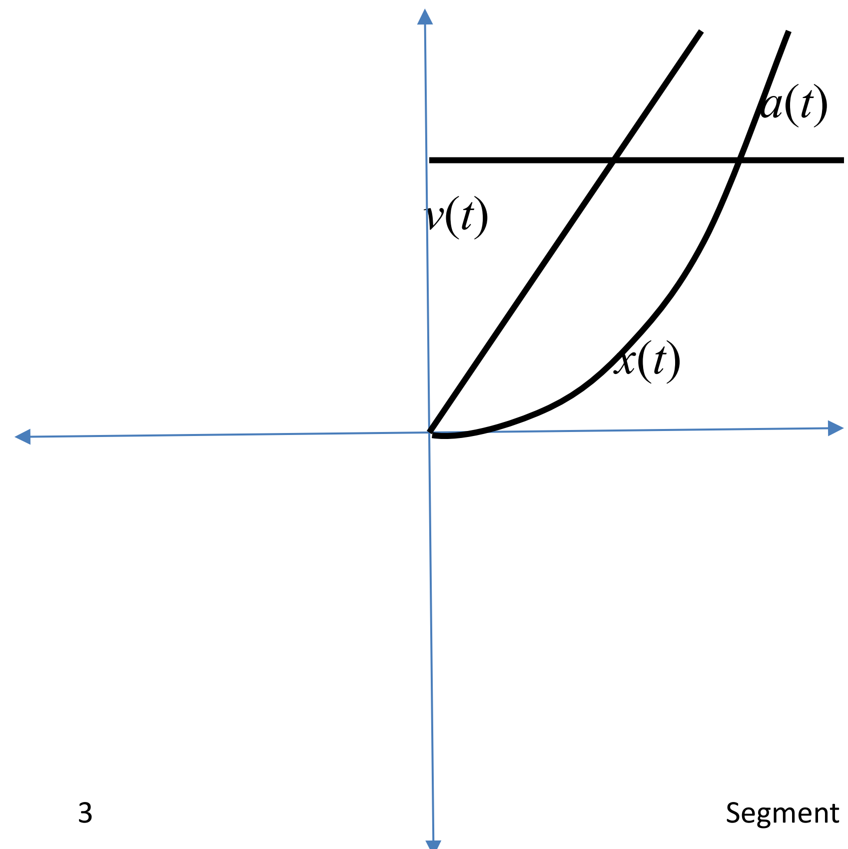
Infinitesimal 1 DoF Kinematics

- Relationships between extension, velocity, acceleration
  - completely specified by mathematics
  - as matter of "pure" geometry
  - needing no further physical insight



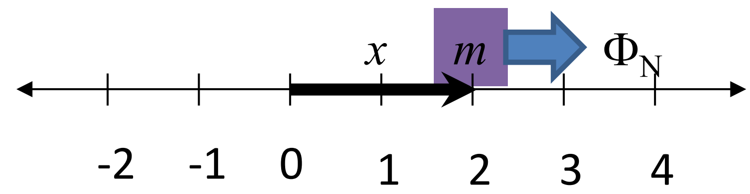
$$v(t) := \dot{x}(t) = \frac{d}{dt}x(t)$$

$$a(t) := \dot{v}(t) = \ddot{x}(t)$$



Newton's Law

- The mechanics of work
  - forces "act" on masses
  - Galileo: change velocities
  - Newton: change momentum
    - special case: position invariance
    - implies constant of proportionality
    - we call "mass"



$$\boxed{m\ddot{x} = \Phi_N}$$

ODEs from State-dependent forces

- forces often depend on motion
  - e.g. viscous damping
  - proportional (& opposed) to v

$$\Phi_{\mathrm{D}}(\dot{x}) := -b\dot{x}$$

- mechanics now gives
- an Ordinary Differential Equation (ODE)

$$m\ddot{x} = \Phi_{\mathrm{D}}(\dot{x}) = -b\dot{x}$$

From ODE to Vector Field

- Aim: view ODEs geometrically
  - rethink velocity as a "state"
  - to get a "first order" ODE (i.e., state variables isolated on LHS)
  - defined by a "vector field" (VF)

$$m\ddot{x} = -b\dot{x} =: \Phi_{\mathrm{D}}(\dot{x})$$

$$v := \dot{x}$$

$$\dot{v} = f_{\mathrm{D}}(v)$$

$$f_{\mathrm{D}}(v) := \lambda v \qquad (1)$$

$$\lambda := -\frac{b}{m}$$

Solution of the Initial Condition Problem

$$\mathrm{e}^{\lambda t} := \sum_{k=0}^{\infty} (\lambda t)^k / k!$$

$$\frac{d}{dt}\mathrm{e}^{\lambda t} = \frac{d}{dt}\left(1 + \sum_{k=1}^{\infty} (\lambda t)^k / k!\right)$$

$$= \lambda \sum_{k=1}^{\infty} k \, (\lambda t)^{(k-1)} / k!$$

$$= \lambda \mathrm{e}^{\lambda t} \stackrel{\checkmark}{=} f_{\mathrm{D}}(\mathrm{e}^{\lambda t})$$

Properties of the Exponential

- infinite series $e^{\lambda t}$
  - exists (converges) for all arguments
  - identity at $t = 0$
  - never vanishes
- so for each t
  - $e^{\lambda t} v_0$ is an invertible linear function
  - taking any initial velocity, v_0 to a new velocity, v_t
- and we get
  - a t-parametrized family
  - of invertible linear functions

$$f_{\mathrm{D}}^{t}\left(v_0\right) := \mathrm{e}^{\lambda t} v_0 \qquad (2)$$

The Flow Generated by a Vector Field

- "flow"
  - a t-parametrized family
  - of invertible (generally nonlinear)
  - functions of state

$$f_{\mathrm{D}}^{t}\left(v_0\right) := \mathrm{e}^{\lambda t} v_0 \qquad (2)$$

$$\dot{v} = f_{\mathrm{D}}(v) \qquad (1)$$

- "generated" by VF
- used for all "nice" VF
  - reflecting those properties
  - even when we don't know how
  - to write down the specific infinite series

Adding Compliance

- Robotics setting usually introduces compliance
  - (position dependent opposing force)
  - e.g., Hooke's law spring force

$$\Phi_{\mathrm{HS}}(x) := -kx$$

$$\varphi_{\mathrm{HS}}(x) := -\int_0^x \Phi_{\mathrm{HS}}(\chi)d\chi \qquad (3)$$
$$= \frac{1}{2}kx^2$$

  - with associated potential

Newton Gives A 2nd Order ODE

- sum applied forces to change momentum getting new ODE expressed as
  - change in momentum
  - or change in velocity

$$
\begin{aligned}
m\ddot{x} &= \Phi_{\mathrm{D}}(\dot{x}) + \Phi_{\mathrm{HS}}(x) \\
&= -b\dot{x} - kx \\
&=: \Phi_{\mathrm{DHO}}(\mathbf{x})
\end{aligned}
\qquad (4)
$$

$$
\mathbf{x} := \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}
$$

$$
\ddot{x} = \Xi_{\mathrm{DHO}}(\mathbf{x}) := \frac{1}{m}\Phi_{\mathrm{DHO}}(\mathbf{x})
$$

DHO := “damped harmonic oscillator” (expecting “under-damped” gains)

Moving Ahead

- Seeking a VF
  - given scalar 2nd order ODE
    $$\ddot{x} = \Xi_{\mathrm{DHO}}(\mathbf{x})$$
  - re-express via vector, $\mathbf{x}$
    $$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$
  - to get 1st order VF
    $$\dot{\mathbf{x}} = f_{\mathrm{DHO}}(\mathbf{x})$$

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LTI Flow

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From Scalar 2nd Order to Vector 1st Order

- Recall Newton's 2nd order ODE
- Now recast as 1st order vector ODE
  - using matrix calculus
    - d/dt operating on array
    - yields array of d/dt entries

$$\ddot{x} = \Xi_{\mathrm{DHO}}(\mathbf{x}) = -\frac{k}{m}x - \frac{b}{m}\dot{x}$$

$$\mathbf{x} := \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} v \\ \Xi_{\mathrm{DHO}}(\mathbf{x}) \end{bmatrix} \\ &= \begin{bmatrix} \dot{x} \\ -\frac{b}{m}\dot{x} - \frac{k}{m}x \end{bmatrix} =: f_{\mathrm{DHO}}(\mathbf{x}) \qquad (5) \\ &= A\mathbf{x}; \quad A := \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \end{aligned}$$

  - good time to review calculus, matrix algebra
  - soon start working with matrix calculus

Matrix Exponential via Diagonalization

- Notice VF is

  LTI

  ("linear time invariant")

$$f_{\mathrm{DHO}}(\mathbf{x}) = A\mathbf{x}$$

- Diagonalize

$$A = E\Lambda E^{-1}; \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- Exponentiate
  - exponent of diagonal matrix
  - is matrix of diagonal exponents

$$\begin{aligned} \mathrm{e}^{At} &= \textstyle\sum_{k=0}^{\infty} \frac{1}{k!} \left(E\Lambda E^{-1} t\right)^k \\ &= E\left(\textstyle\sum_{k=0}^{\infty} \frac{1}{k!} (\Lambda t)^k\right) E^{-1} \\ &= E\left(\textstyle\sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} (\lambda_1 t)^k & 0 \\ 0 & (\lambda_2 t)^k \end{bmatrix}\right) E^{-1} \\ &= E \begin{bmatrix} \mathrm{e}^{t\lambda_1} & 0 \\ 0 & \mathrm{e}^{t\lambda_2} \end{bmatrix} E^{-1} \end{aligned}$$

Closed Form Solution of LTI VF

- Time derivative
  - of matrix exponential
  - satisfies ODE

$$\frac{d}{dt}\mathrm{e}^{At} = \frac{d}{dt}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!} \left(E \Lambda E^{-1} t\right)^k \right)$$

$$= \frac{d}{dt} E \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!} (\Lambda t)^k \right) E^{-1}$$

$$= E \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} E^{-1}.$$

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} E \begin{bmatrix} (\lambda_1 t)^{(k-1)} & 0 \\ 0 & (\lambda_2 t)^{(k-1)} \end{bmatrix} E^{-1}$$

$$= A\mathrm{e}^{At}$$

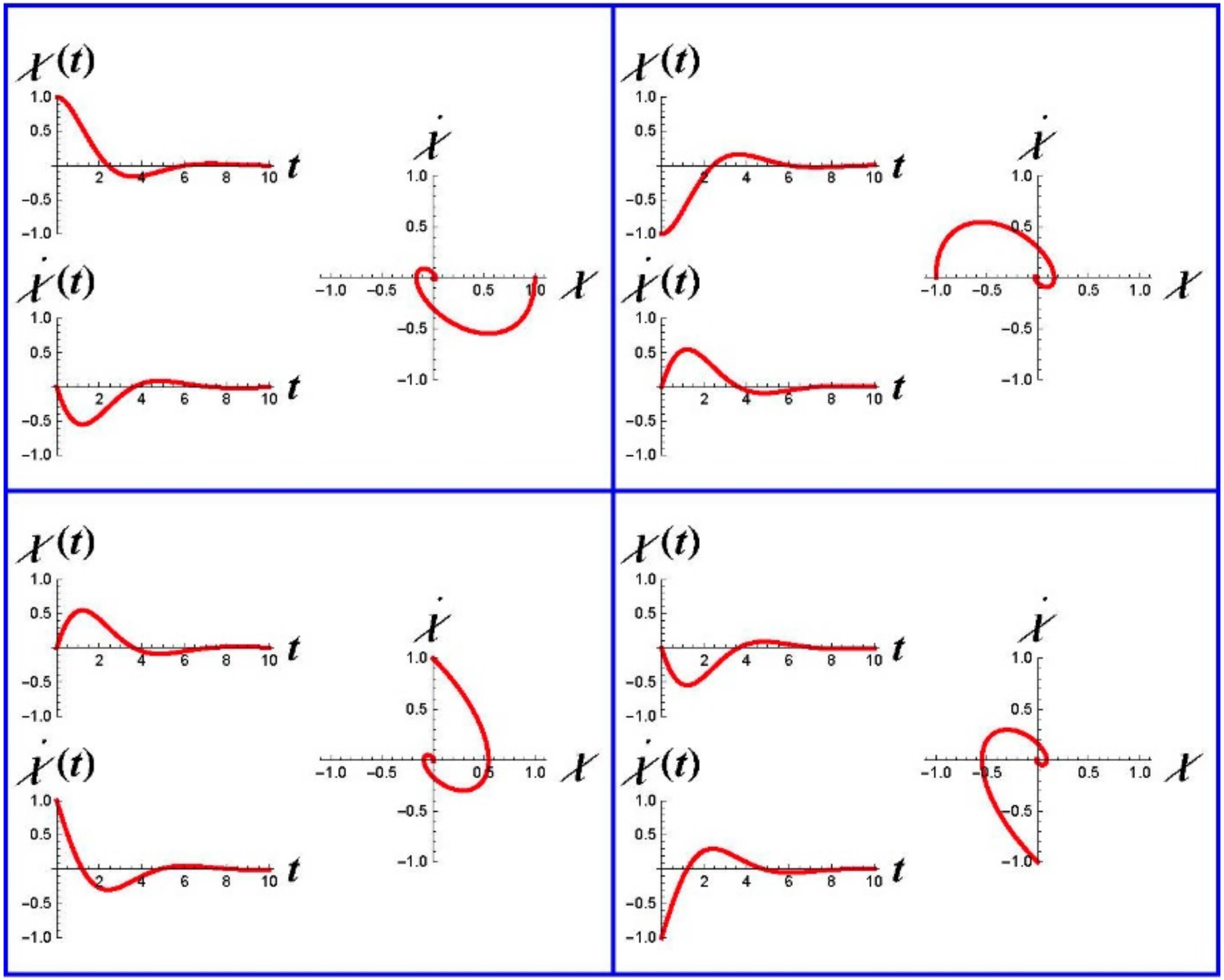
- Yields closed form solution
  - for flow $f^t_{\mathrm{DHO}}(\mathbf{x}_0) := \mathrm{e}^{At}\mathbf{x}_0 \qquad (6)$
  - reference:

[M. W. Hirsch, S. Smale, and R. L. Devaney, Differential equations, dynamical systems and an introduction to chaos, vol. 60. Access Online via Elsevier, 2004]

Compare (6) to eqn (2) in segment 1.2.1.1
- *a time-paramtrized family*
- *of invertible linear functions*

Visualizing ODE Solutions

- 2^{nd} order scalar
  - family of paired scalar-valued
  - functions of time
  - parametrized by two ICs
  - for position & velocity
- 1^{st} order vector
  - family of single vector-valued
  - functions of time
  - parametrized by vector IC

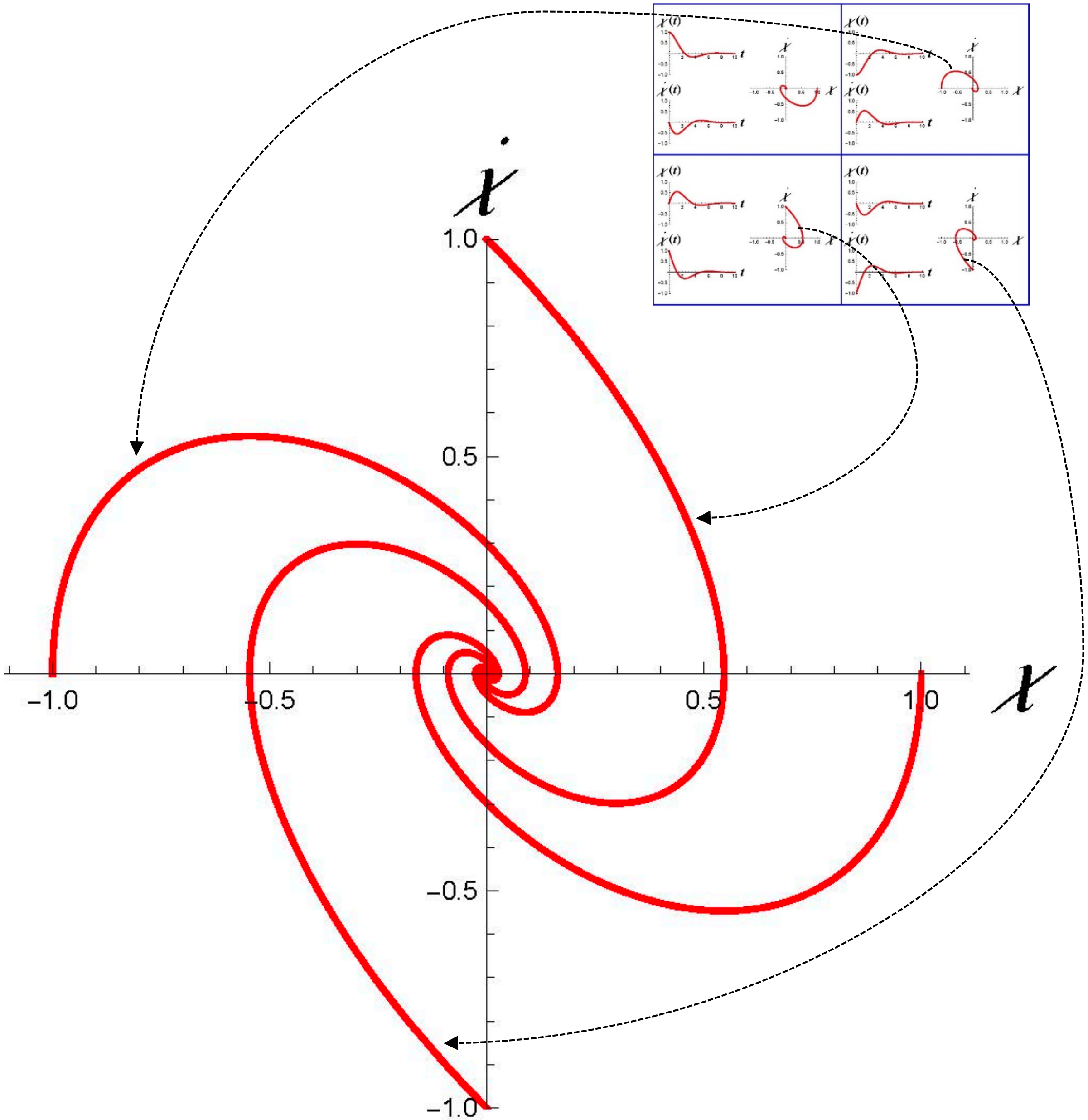


Solutions Through Four Different ICs

- 2^{nd} order scalar view: paired scalar trajectories
- 1^{st} order vector view: single vector trajectory

Introducing the Dynamical Systems View

- 2 dimensional VF
- generates 2 dimensional flow
  - family of vector-valued
  - transformations of vectors
  - parametrized by time



Robotics Virtue of Dynamical Systems View

- Family of (time-parametrized) state space transformations
- Helps in thinking about
  - Periodic Behavior
    - relates states at one phase
    - to states at any other phase
  - Asymptotic Behavior
    - never really know exact IC
    - rarely care about exact time trajectory
    - want “every” IC to “end up” in goal state(s)

Diagonalization as Change of Coordinates

$$\dot{\mathbf{x}} = f_{\mathrm{DHO}}(\mathbf{x}) = A\mathbf{x}$$

- Diagonalization: what does it "mean"?
  - it's a change of coordinates

$$\begin{aligned}\mathbf{z} := E^{-1}\mathbf{x} =: h_{\mathrm{diag}}(\mathbf{x}) \\ \Rightarrow \dot{\mathbf{z}} &= E^{-1}\dot{\mathbf{x}} \\ &= E^{-1}f_{\mathrm{DHO}}(\mathbf{x}) = E^{-1}f_{\mathrm{DHO}}(E\mathbf{z}) \\ &= E^{-1}AE\mathbf{z} = \Lambda\mathbf{z} =: f_{\mathrm{diag}}(\mathbf{z})\end{aligned} \tag{7}$$

  - yielding a new expression for the "same" VF

$$\dot{\mathbf{z}} = f_{\mathrm{diag}}(\mathbf{z}) = \Lambda\mathbf{z}$$

Change of Coordinates for Conjugacy

- Change of coordinates (CC) to get flow
  - transform from $\mathbf{x}$ to $\mathbf{z}$ coordinates
  - write out (easy) expression for flow in $\mathbf{z}$ coordinates
  - then transform expression back into $\mathbf{x}$ coordinates
- Say that the two VFs are "conjugate," $f_{\mathrm{DHO}} \overset{h_{\mathrm{diag}}}{\sim} f_{\mathrm{diag}}$

Need for Different Coordinate Systems

- Value of new $\mathbf{z}$ coordinates
  - useful: easy to write down flow
  - awkward: introduce complex numbers

$$\mathbf{x} \in \mathbb{R}^2; \quad \mathbf{z} \in \mathbb{C}^2$$

$$A \in \mathbb{R}^{2\times 2}; \quad E, \Lambda \in \mathbb{C}^{2\times 2}$$

- Consider using different coordinates
  - [Hirsch et al., 2004, Ch. 3.4]
  - substituting column sum and difference vectors
  - in place of original columns of E
  - yields "real canonical" form
    - real coordinates and coefficients
    - useful expression for flow

$$\boldsymbol{y} := S^{-1}\mathbf{x} =: h_{\mathrm{RC}}(\mathbf{x})$$

$$S := E \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Real Canonical Coordinates

$$\begin{aligned}\dot{\mathbf{y}} &= S^{-1}\dot{\mathbf{x}} \\ &= S^{-1} f_{\mathrm{DHO}}(\mathbf{x}) = S^{-1} f_{\mathrm{DHO}}(S\mathbf{y}) \\ &= S^{-1}AS\mathbf{y} = \tilde{A}\mathbf{y} =: f_{\mathrm{RC}}(\mathbf{y})\end{aligned}$$

$$\tilde{A} := (\sigma I_2 + \omega J_2) \quad I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad J_2 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma := \frac{1}{2}(\lambda_1 + \lambda_2) =: \Re(\lambda_i)$$

$$\omega := \frac{1}{2} i (\lambda_1 - \lambda_2) =: \Im(\lambda_i)$$

$$\begin{aligned}\mathrm{e}^{t\tilde{A}} &= \mathrm{e}^{t(\sigma I_2 + \omega J_2)} \\ &= \mathrm{e}^{t\sigma}\mathrm{e}^{t\omega J_2} = \mathrm{e}^{t\sigma} \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}\end{aligned}$$

Moving ahead

- need different coordinates
  - for different purposes
  - e.g. soon interpret flow as CC
- these examples were linear
  - have matrix representations
  - easy to compute in closed form
- soon need nonlinear CC
  - no matrix representation
  - get used to using functional names
    - e.g. $f_{\mathrm{DHO}}(\mathrm{x})$ rather than $A\mathrm{x}$
    - e.g. $h_{\mathrm{diag}}(\mathrm{x})$ rather than $E^{-1}\mathrm{x}$

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Energy, Power, Basins

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Total Energy as a Norm

- Recall energy components
  - kinetic energy
  - Hooke's law spring potential
  - Comprise total energy
- Now re-interpret
  - as a norm
  - on the "phase plane"
  - e.g., for $k = m = 1$

$$\kappa(\dot{x}) := \frac{1}{2} m \dot{x}^2$$

$$\varphi_{\mathrm{S}}(x) := \frac{1}{2} k x^2$$

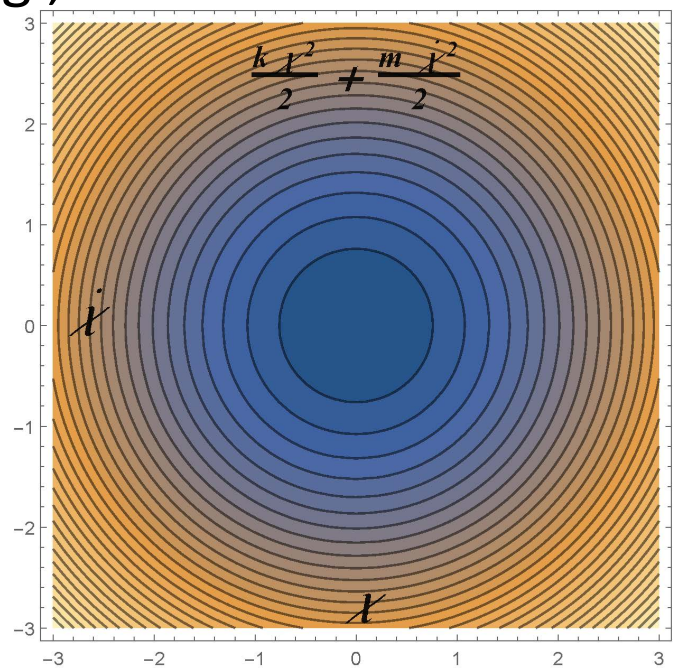
$$\eta_{\mathrm{HO}}(\mathbf{x}) := \kappa(\dot{x}) + \varphi_{\mathrm{S}}(x)$$

$$= \frac{1}{2}\left[\left(\sqrt{m}\dot{x}\right)^2 + \left(\sqrt{k}x\right)^2\right]$$

$$= \frac{1}{2}\left\| \begin{bmatrix} \sqrt{k}x \\ \sqrt{m}\dot{x} \end{bmatrix} \right\|^2$$

$$= \frac{1}{2}\left\| P^{\frac{1}{2}}\mathbf{x} \right\|^2$$

$$P := \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}$$



Total Energy Along the Flow

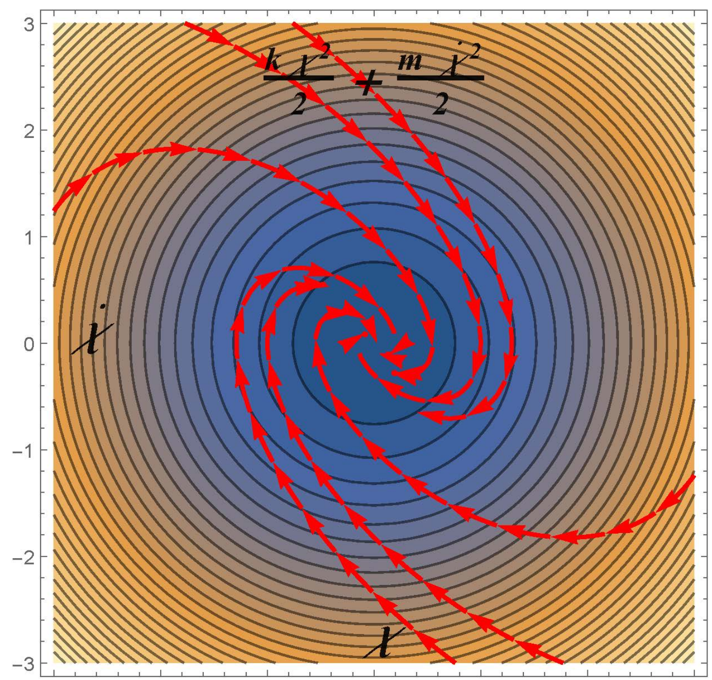
- A state trajectory
  - tracks the flow
  - from specific IC
- Visits energy levels

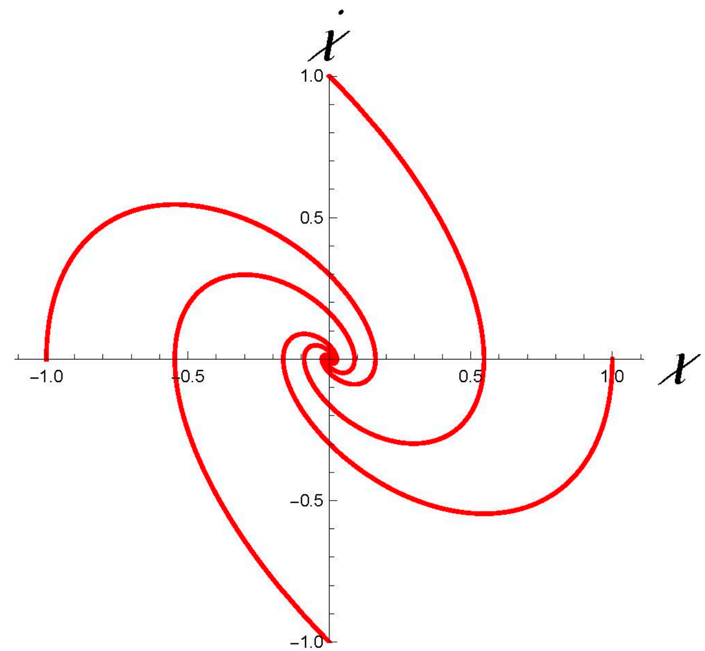
  (function composition symbol)
- Yields energy trajectory

$$f^{t}_{\mathrm{DHO}}\left(\mathbf{x}_0\right) = \mathrm{e}^{tA}\mathbf{x}_0$$

$$\eta_{\mathrm{HO}} \circ f^{t}_{\mathrm{DHO}}\left(\mathbf{x}_0\right)$$

$$= \left\| P^{\frac{1}{2}} \mathrm{e}^{tA}\mathbf{x}_0 \right\|^2$$

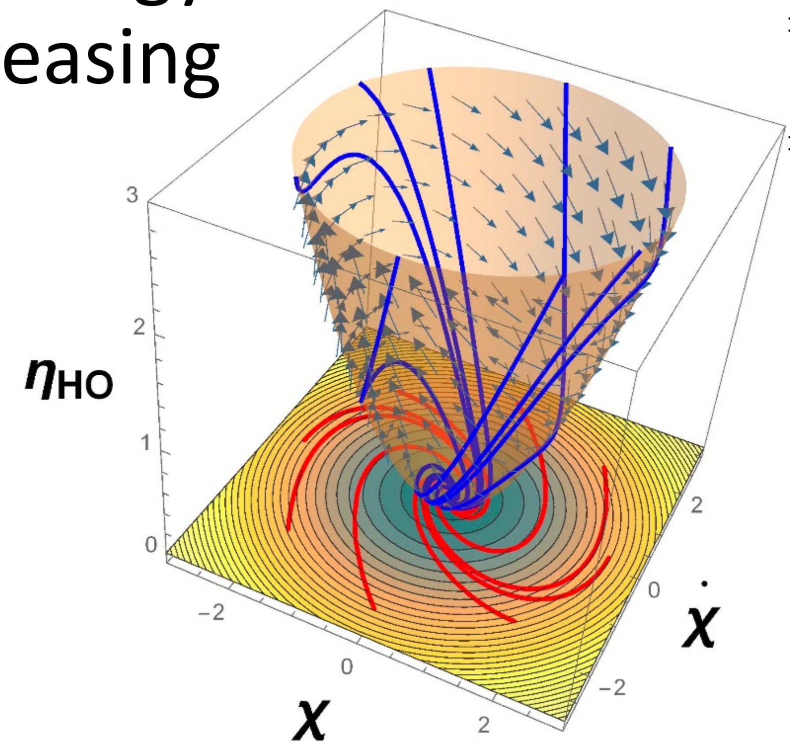


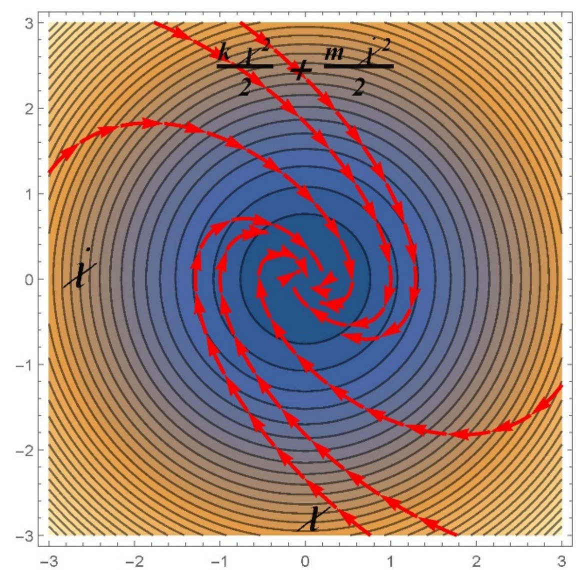


Total Energy is Non-increasing

- Energy trajectory
- Yields power trajectory
- Which is non-positive (dissipated by damping)
- So energy is non-increasing

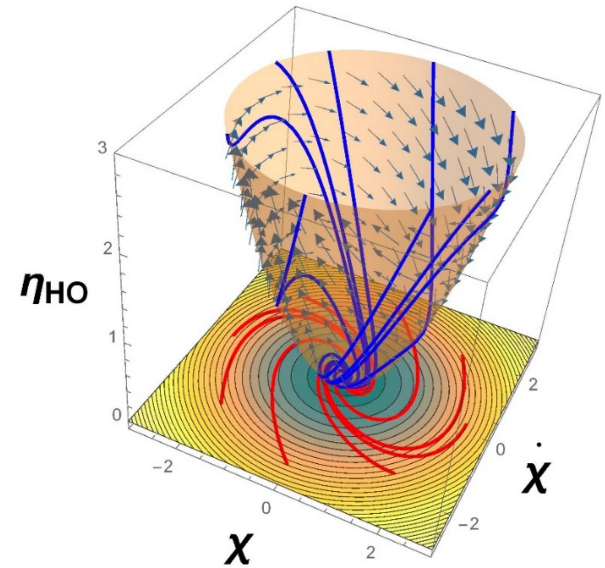
$$
\begin{aligned}
\dot{\eta}_{\mathrm{HO}} &= \frac{d}{dt}\eta_{\mathrm{HO}} \circ f^t_{\mathrm{DHO}}(\mathbf{x}_0) \\
&= D_{\mathbf{x}}\eta_{\mathrm{HO}} \cdot \frac{d}{dt} f^t_{\mathrm{DHO}}(\mathbf{x}_0) \\
&= D_{\mathbf{x}}\eta_{\mathrm{HO}} \cdot f_{\mathrm{DHO}} \circ f^t_{\mathrm{DHO}}(\mathbf{x}_0) \\
&= [\partial\varphi_{\mathrm{S}}/\partial x, \partial\kappa/\partial\dot{x}] \cdot A f^t_{\mathrm{DHO}}(\mathbf{x}_0) \\
&= [kx(t), m\dot{x}(t)] \begin{bmatrix} \dot{x}(t) \\ -\frac{b}{m}\dot{x}(t) - \frac{k}{m}x(t) \end{bmatrix} \\
&= -b\dot{x}(t)^2
\end{aligned}
$$





The Basin of Energy Decay

- Conclude [Thomson & Tait, 1888]
  - since energy is norm-like
  - and is almost always decaying
  - trajectories must settle down to lowest energy state



- Given VF, see that power
  - is a function of state
  - just like energy

$$\begin{aligned}\dot{\eta}_{\mathrm{HO}}(\mathbf{x}) &:= D_{\mathbf{x}}\eta_{\mathrm{HO}}(\mathbf{x})\cdot f_{\mathrm{DHO}}(\mathbf{x}) \\ &= -b\dot{x}^2\end{aligned} \tag{1}$$

solid mid-line dot
- *denotes array multiplication*
- *here*: (1 x 2)g(2 x 1)

- Yields concept of basin
  - ICs whose motions decay
  - to low energy state
  - as predicted by negative power