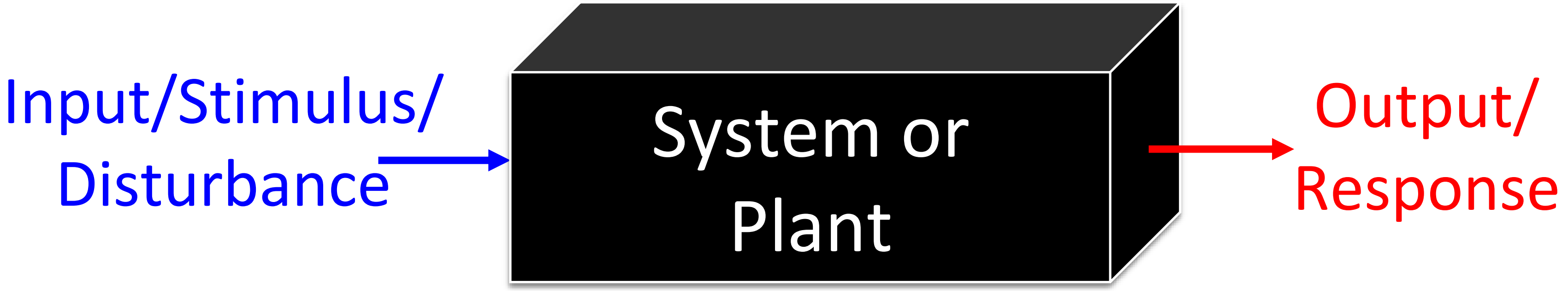
Video 5.1

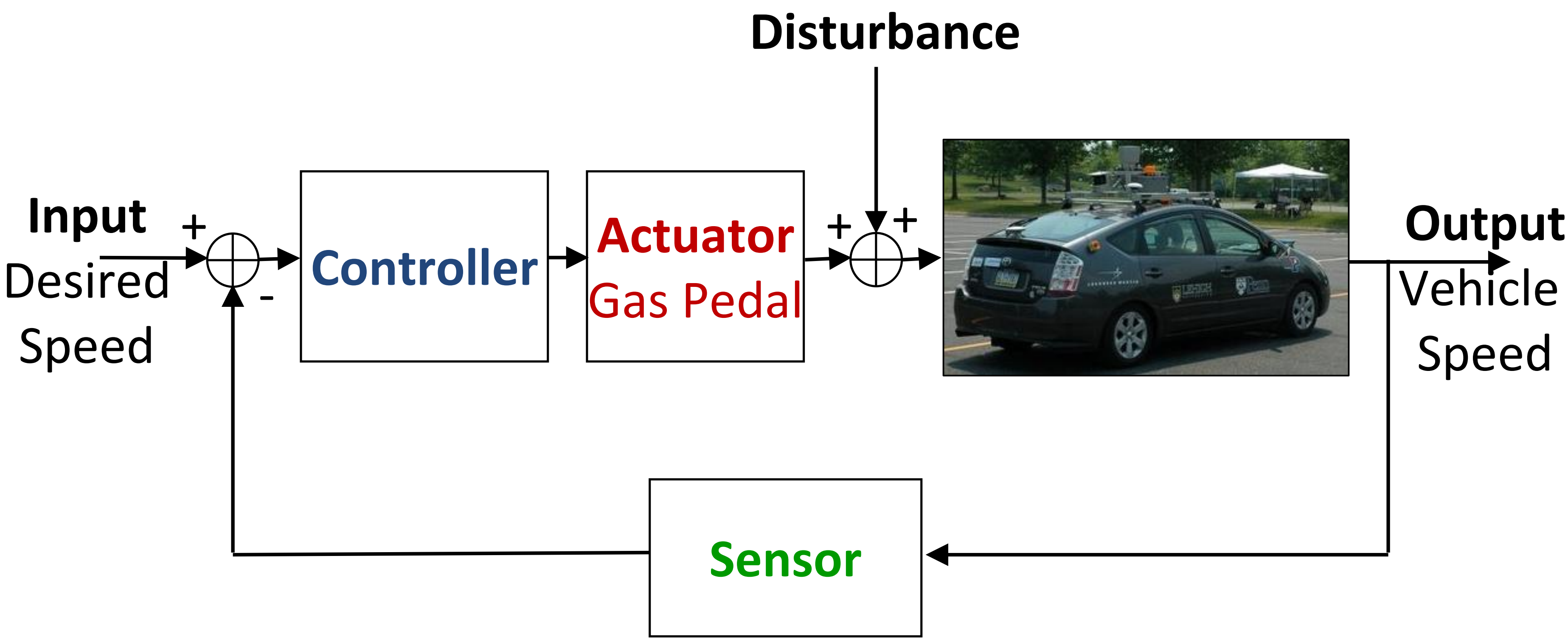
Vijay Kumar and Ani Hsieh

The Purpose of Control



- Understand the "Black Box"
- Evaluate the Performance
- Change the Behavior

Anatomy of a Feedback Control System



Twin Objectives of Control

$$m\ddot{q} = -b\dot{q} + u_{engine} + u_{hill}$$

$$u_{engine} = K(v_{des} - v)$$

- Performance
- Disturbance Rejection

Learning Objectives for this Week

- Linear Control
  - Modeling in the frequency domain
  - Transfer Functions
  - Feedback and Feedforward Control

Frequency Domain Modeling

$$a_m \frac{d^m}{dt^m} q(t) + a_{m-1} \frac{d^{m-1}}{dt^{m-1}} q(t) + \ldots + a_1 \frac{d}{dt} q(t) + a_0 q(t) = \boxed{\tau(t)}$$

$$\boxed{b_k \frac{d^k}{dt^k} \tau(t) + b_k \frac{d^{k-1}}{dt^{k-1}} \tau(t) + \ldots + b_0 \tau(t)}$$

- Algebraic vs Differential Equations
- Laplace Transforms
- Diagrams

Laplace Transforms

Integral Transform that maps functions from the *time* domain to the *frequency* domain

$$\mathbb{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

with $s = \sigma + j\omega$

Example

Let $f(t) = 1$, compute $\mathbb{L}[f(t)]$

$$\mathbb{L}[f(t)] = \int_{0^-}^{\infty} e^{-st} dt$$

$$= -\frac{e^{-st}}{s} \Big|_{0^-}^{\infty}$$

$$= 0 - (-\frac{1}{s}) = \frac{1}{s}$$

Inverse Laplace Transforms

Integral Transform that maps functions from the *frequency* domain to the *time* domain

$$\mathbb{L}^{-1}[F(s)] = \int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds$$

Example

Let $F(s) = \dfrac{1}{s+a}$, compute $\mathbb{L}^{-1}[F(s)]$

$$\mathbb{L}^{-1}[F(s)] = \int_{\sigma - j\omega}^{\sigma + j\omega} \frac{e^{st}}{s+a} ds$$

Laplace Transform Tables

$f(t)$	$F(s)$
$\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$	1
$\mathcal{U}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$sin(\phi t)$	$\frac{k}{s^2+k^2}$
$cos(\phi t)$	$\frac{s}{s^2+k^2}$
$e^{at} sin(\phi t)$	$\frac{k}{(s-a)^2+k^2}$
$e^{at} cos(\phi t)$	$\frac{s-a}{(s-a)^2+k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$

http://integral-table.com/downloads/LaplaceTable.pdf

Video 5.2

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Generalizing

Given $$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

How do we obtain $f(t)$?

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} \Rightarrow F(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Partial Fraction Expansion

$$F(s) = \frac{N(s)}{D(s)}$$

Case 1: Roots of D(s) are Real & Distinct

Case 2: Roots of D(s) are Real & Repeated

Case 3: Roots of D(s) are Complex or Imaginary

Case 1: Roots of D(s) are Real & Distinct

Compute the Inverse Laplace of

$$F(s) = \frac{1}{s^2 + 3s + 2}$$

Case 2: Roots of D(s) are Real & Repeated

Compute the Inverse Laplace of

$$F(s) = \frac{s+2}{(s+1)(s^2+6s+9)}$$

Case 3: Roots of D(s) are Complex

Compute the Inverse Laplace of

$$F(s) = \frac{3}{s(s^2 + 2s + 5)}$$

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Video 5.3

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Using Laplace Transforms

Given

$$M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = \tau(t)$$

- Solving for x(t)

1. Convert to frequency domain
2. Solve algebraic equation
3. Convert back to time domain

Properties of Laplace Transforms

Property	Name
Linearity	$\mathbb{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)$
1^{st} Derivative	$\mathbb{L}[\frac{d}{dt}f(t)] = sF(s) - f(0^-)$
2^{nd} Derivative	$\mathbb{L}[\frac{d^2}{dt^2}f(t)] = s^2F(s) - sf(0^-) - \frac{df}{dt}(0^-)$
n^{th} Derivative	$\mathbb{L}[\frac{d^n}{dt^n}f(t)] = s^nF(s) - \sum_{i=1}^{n} s^{(n-i)}f^{(i-1)}(0^-)$
Integration	$\mathbb{L}[\int_0^t f(\lambda)d\lambda] = \frac{1}{s}F(s)$
Multiplication by time	$\mathbb{L}[tf(t)] = -\frac{dF(s)}{ds}$
Time Shift	$\mathbb{L}[f(t-a)\mathcal{U}(t-a)] = e^{-as}F(s)$
Complex Shift	$\mathbb{L}[f(t)e^{-at}] = F(s+a)$
Time Scaling	$\mathbb{L}[f(\frac{t}{a})] = aF(as)$
Convolution $(*)$	$\mathbb{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$
Initial Value Thm	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
Final Value Thm	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Summary

Laplace Transforms

- time domain <-> frequency domain
- differential equation <-> algebraic equation
- Partial Fraction Expansion factorizes "complicated" $F(s)$ expressions to simplify computation of inverse Laplace Transforms

Example: Solving an ODE (1)

Given $\ddot{x}(t) - 10x(t) + 9x(t) = \tau(t)$ with

$x(0) = 0$, $\dot{x}(0) = 0$ and $\tau(t) = 5t$.

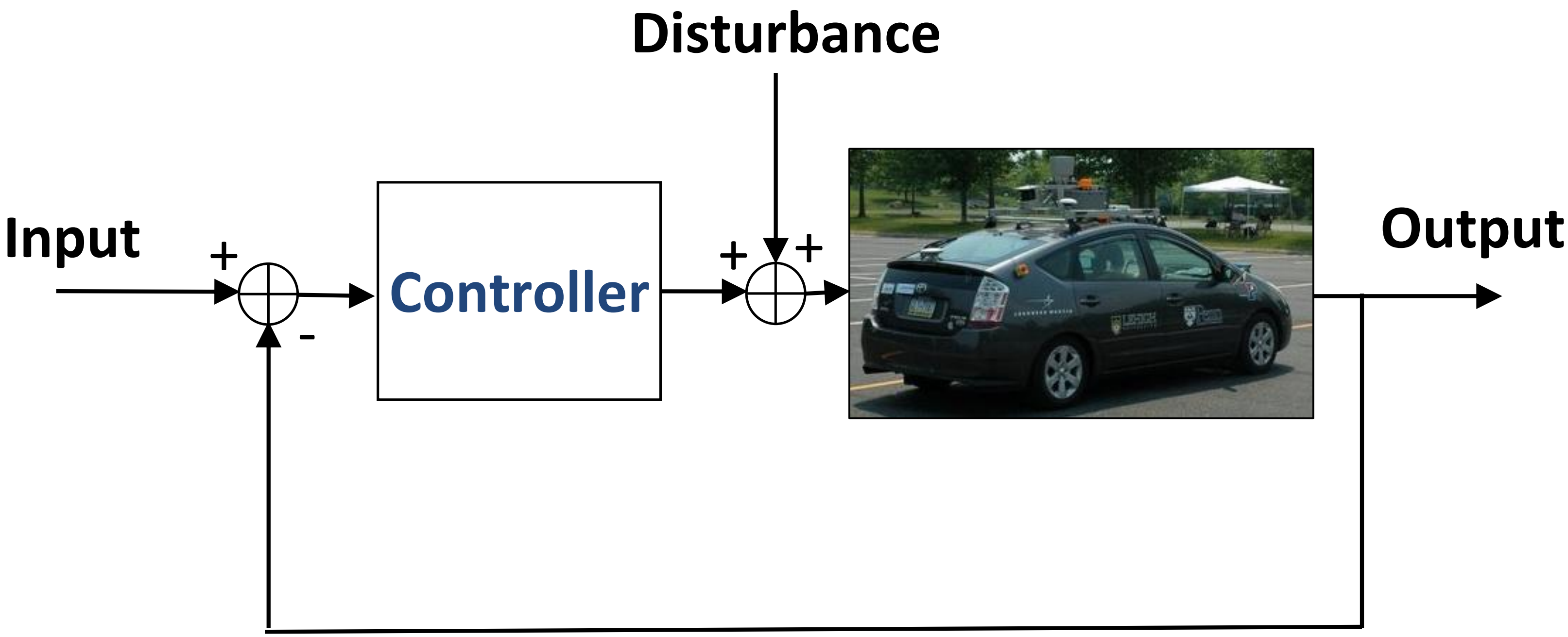
Solve for $x(t)$.

Example: Solving an ODE (2)

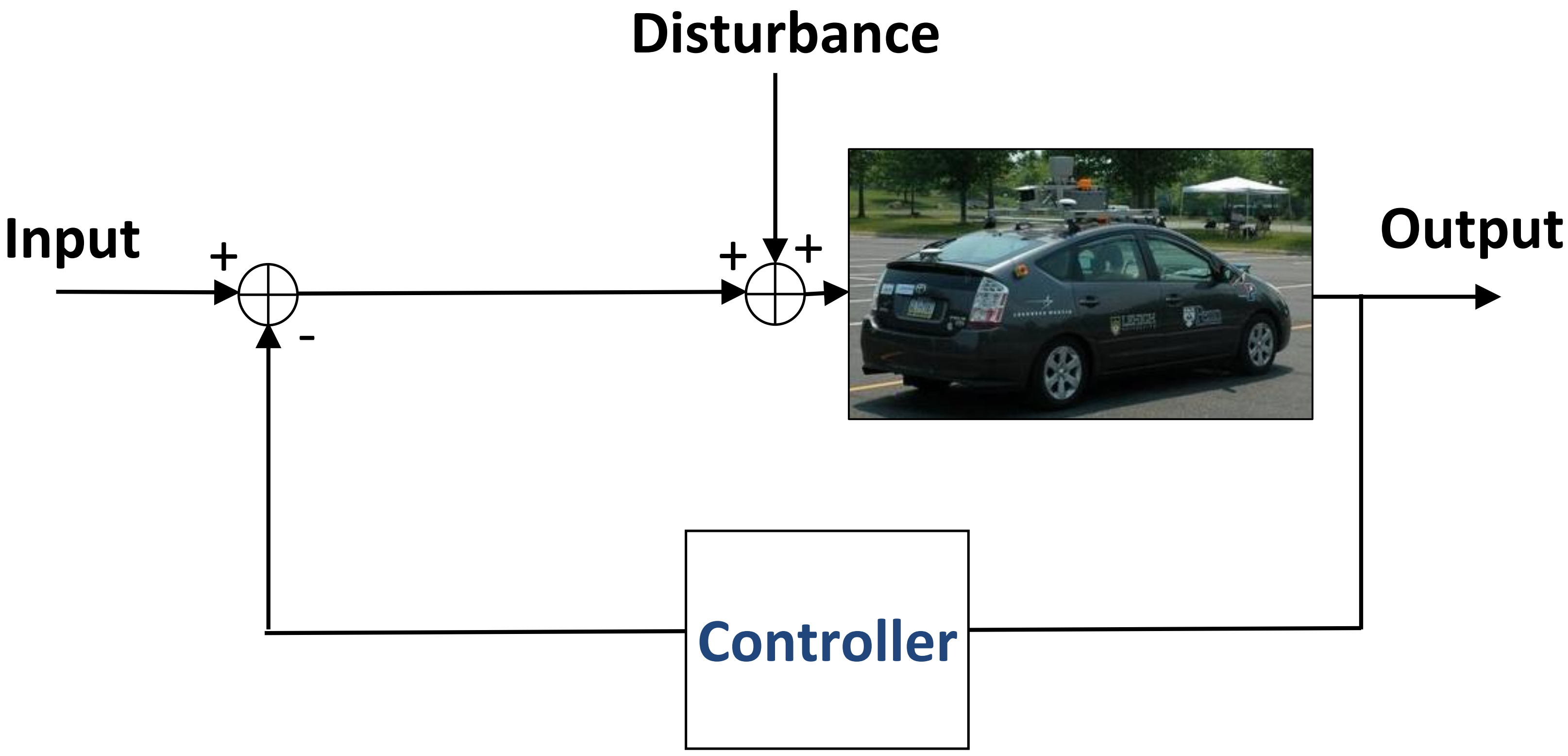
Video 5.4

Vijay Kumar and Ani Hsieh

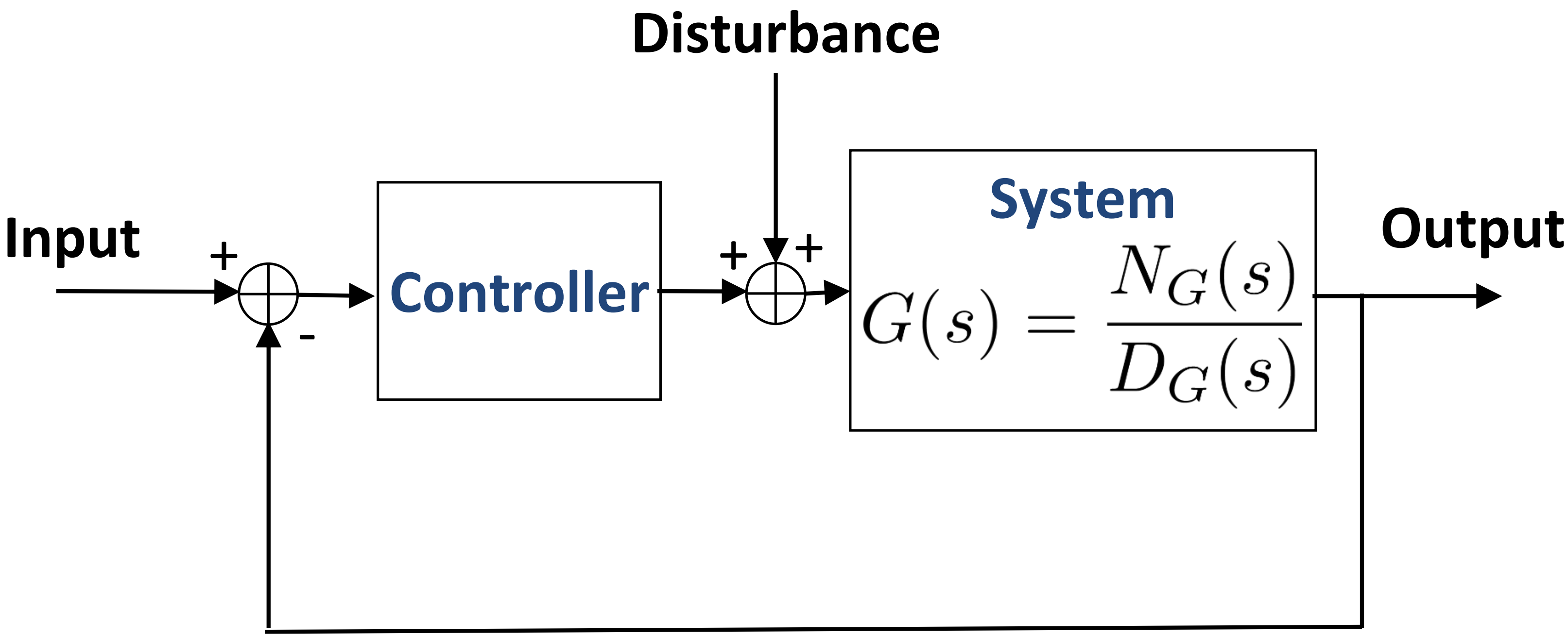
Controller Design



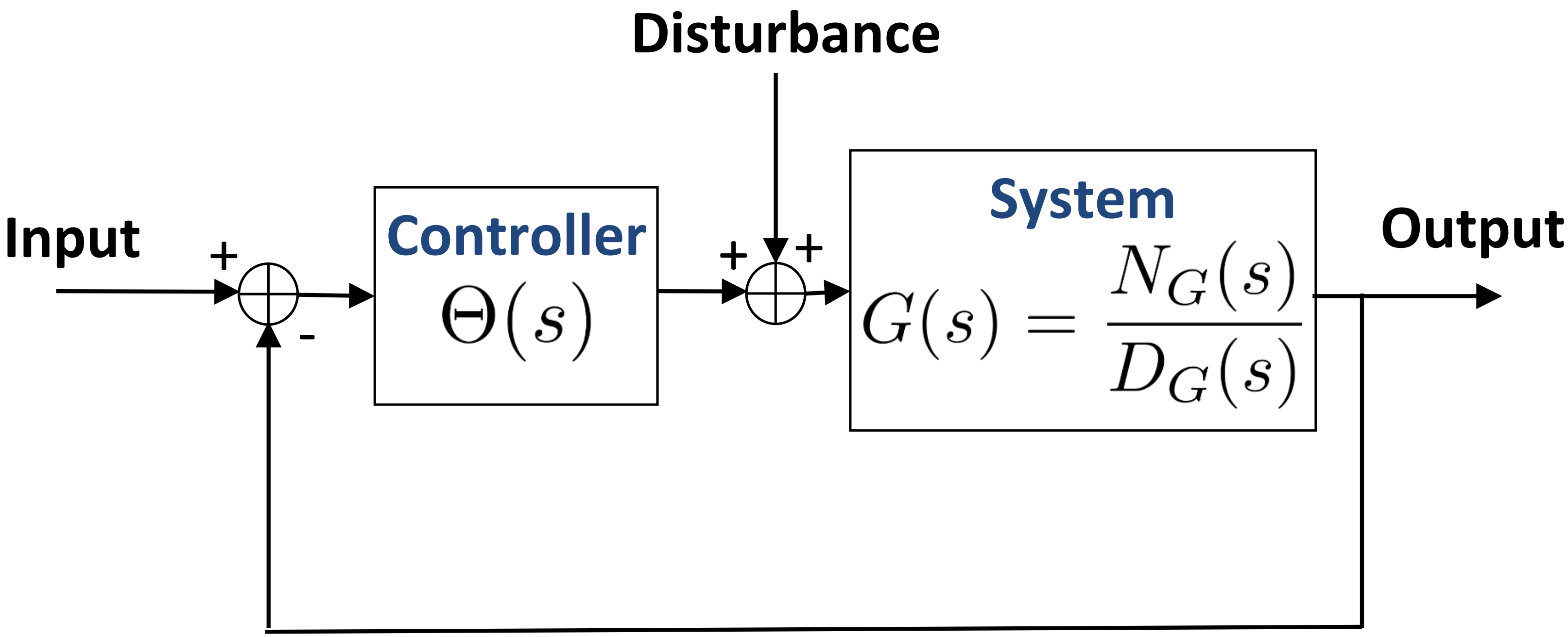
Controller Design



Controller Design



Controller Design



Transfer Function

Relate a system's output to its input

1. Easy separation of INPUT, OUTPUT, SYSTEM (PLANT)
2. Algebraic relationships (vs. differential)
3. Easy interconnection of subsystems in a MATHEMATICAL framework

In General

A General N-Order Linear, Time Invariant ODE

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \ldots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \ldots + b_0 r(t)$$

G(s) =Transfer Function = output/input

Furthermore, if we know G(s), then

output = G(s)*input

Solution given by

$$\mathcal{L}^{-1}\left[G(s) * \text{input}\right]$$

General Procedure

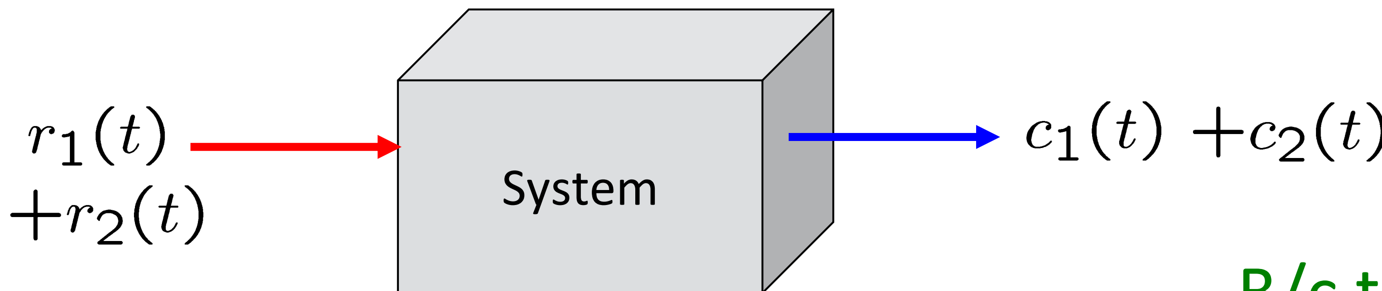
Given $f(q(t), \dot{q}(t), \ddot{q}(t), \ldots, \frac{d^m q(t)}{dt}, t)$ and desired performance criteria

1. Convert $f(\cdot)$ $\Rightarrow$ $F(s) = \mathbb{L}[f(\cdot)]$
2. Analyze $F(s)$
3. Design using $F(s)$
4. Validate using $f(\cdot)$
5. Iterate

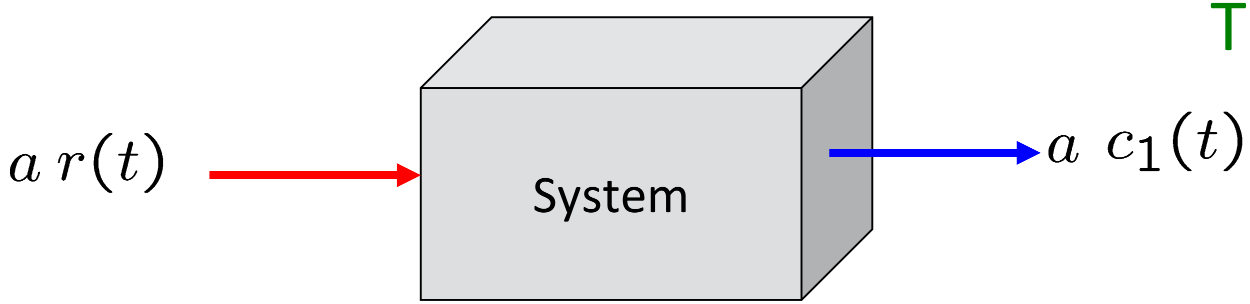
Underlying Assumptions

Linearity

1. Superposition $f(x_1 + x_2) = f(x_1) + f(x_2)$



2. Homogeneity $f(ax) = af(x)$



B/c the Laplace Transform is Linear!

Video 5.5
Vijay Kumar and Ani Hsieh

Characterizing System Response

Given $G(s) = \dfrac{Y(s)}{U(s)}$

How do we characterize the performance of a system?

- Special Case 1: 1st Order Systems
- Special Case 2: 2nd Order Systems

Poles and Zeros

Given $G(s) = \dfrac{N(s)}{D(s)}$

Poles $\{s \mid G(s) = \infty \text{ and } D(s) = 0 \text{ s.t. } N(s) = 0\}$

Zeros $\{s \mid G(s) = 0 \text{ and } N(s) = 0 \text{ s.t. } D(s) = 0\}$

Example: $G(s) = \dfrac{s+2}{s(s+5)}$

First Order Systems

In general $G(s) = \dfrac{s+a}{s+b}$

Let U(s) = 1/s, then $Y(s) = \dfrac{s+b}{s(s+a)} = \dfrac{A}{s} + \dfrac{B}{s+a}$

As such,

$$
\begin{aligned}
c(t) &= A + Be^{-at} \\
A &= \frac{b}{a} \\
B &= 1 - \frac{b}{a}
\end{aligned}
$$

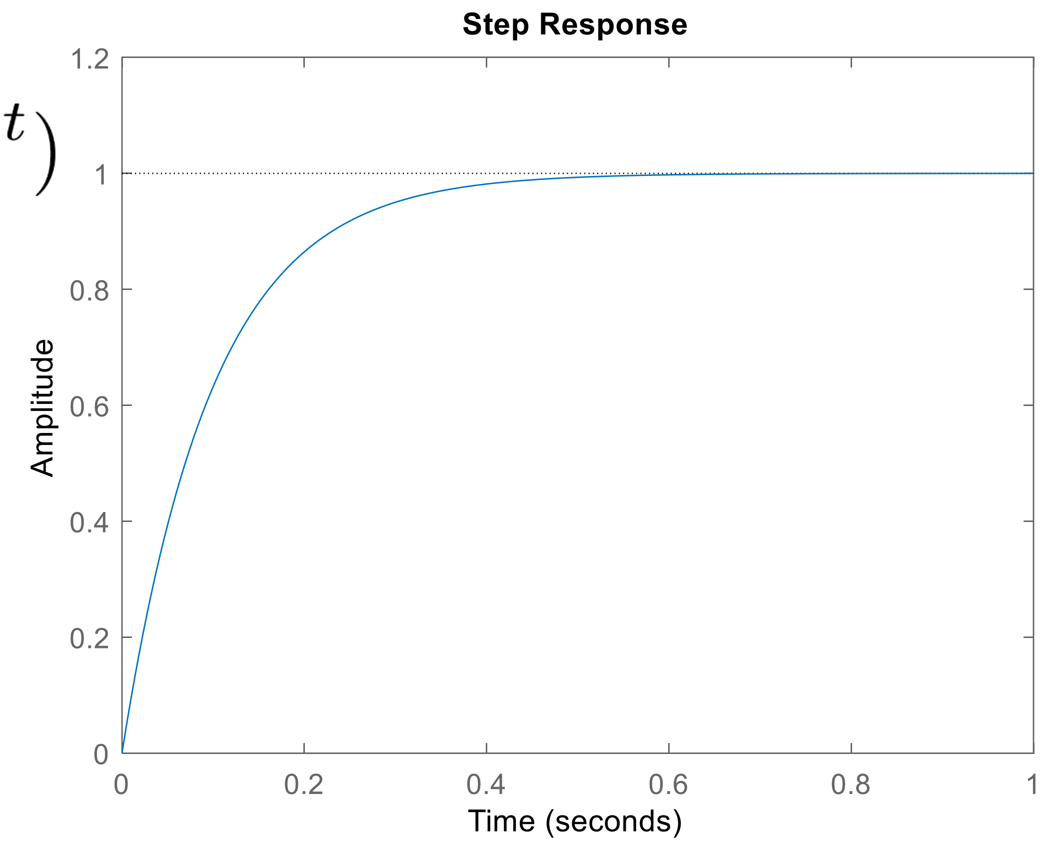
$$y(t) = \frac{b}{a} + (1 - \frac{b}{a})e^{-at}$$

Therefore,

Characterizing First Order Systems

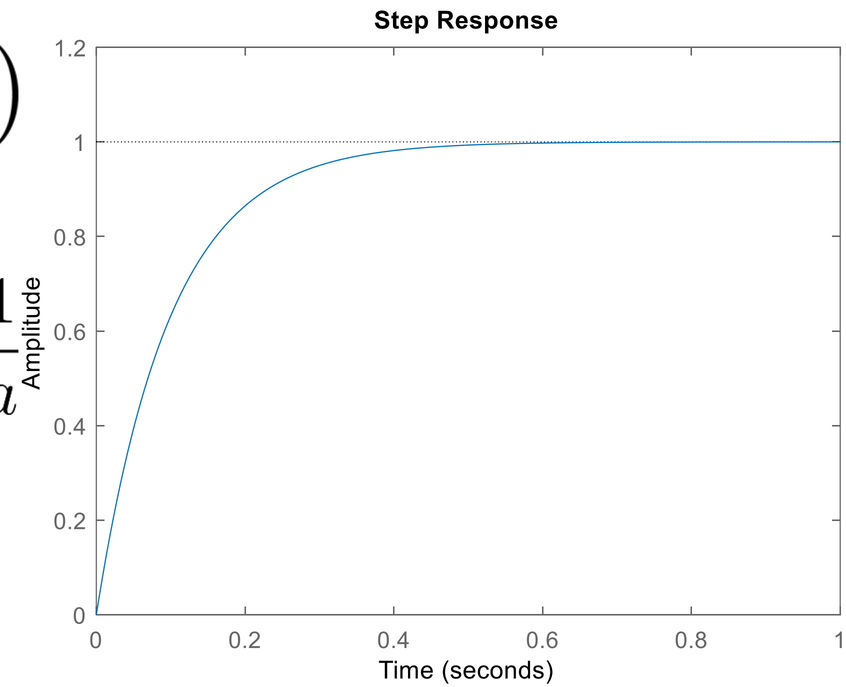
Given $G(s) = \dfrac{a}{s+a}$ with U(s) = 1/s

$$y(t) = a(1 - e^{-at})$$



Characterizing First Order Systems

$$y(t) = a(1 - e^{-at})$$



Time Constant – $T_c = \frac{1}{a}$

Rise Time – T_r

Settling Time – $T_s = \frac{4}{a}$

Second Order Systems

Given, $G(s) = \dfrac{c}{s^2 + bs + c}$ and U(s) = 1/s

$$Y(s) = \frac{1}{s(s^2 + bs + c)} = \frac{A}{s} + \frac{B}{s + r_1} + \frac{C}{s + r_2}$$

Possible Cases

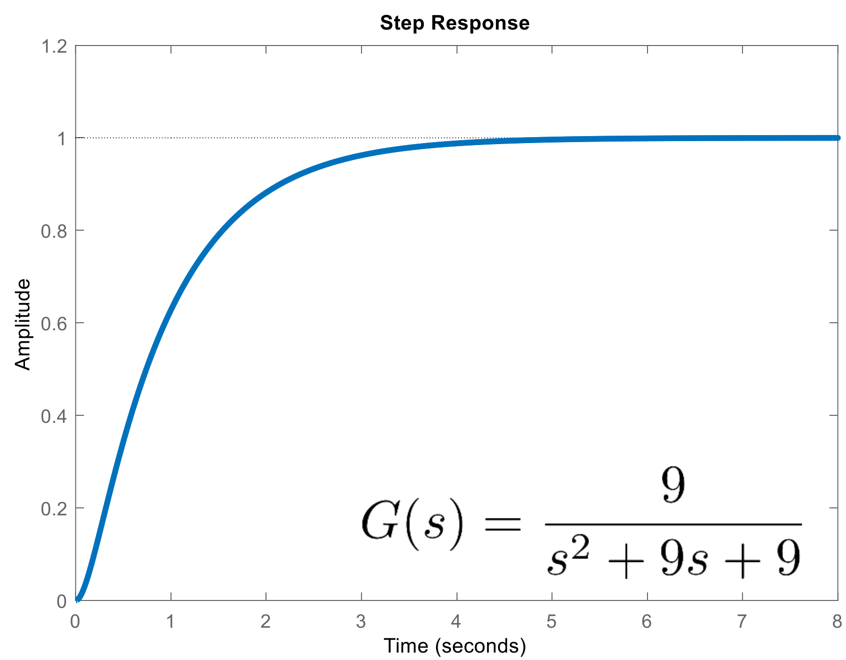
1. r_1 & r_2 are real & distinct
2. r_1 & r_2 are real & repeated
3. r_1 & r_2 are both imaginary
4. r_1 & r_2 are complex conjugates

Case 1: Real & Distinct Roots

$$Y(s) = \frac{c}{s(s^2 + bs + c)} = \frac{A}{s} + \frac{B}{s + r_1} + \frac{C}{s + r_2}$$

$$y(t) = K_1 + K_2 e^{-r_1 t} + K_3 e^{-r_2 t}$$

Overdamped response



Video 5.6

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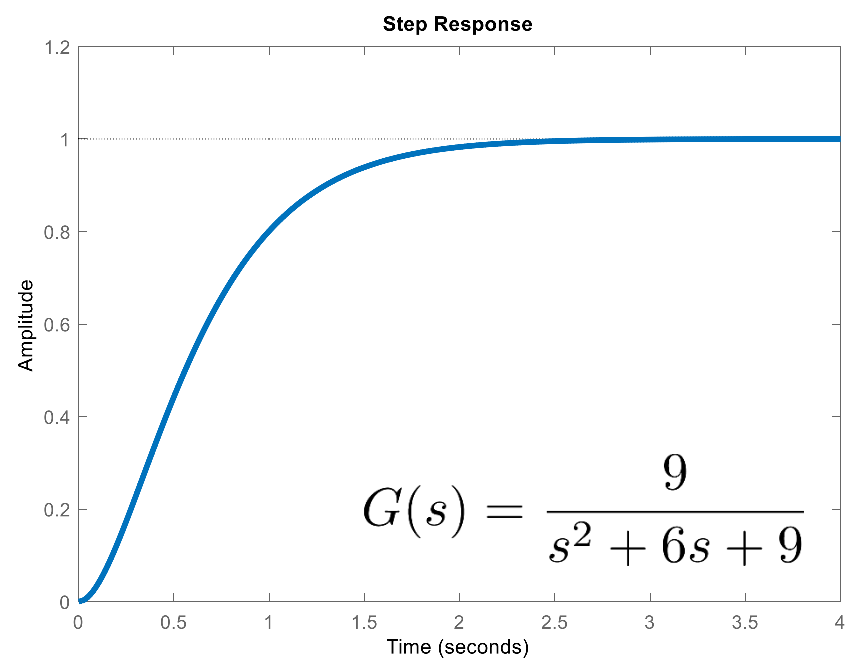
Case 2: Real & Repeated Roots

$$Y(s) = \frac{c}{s(s^2 + bs + c)} = \frac{A}{s} + \frac{B}{s + r_1} + \frac{C}{(s + r_1)^2}$$

$$y(t) = K_1 + K_2 e^{-r_1 t} + K_3 t e^{-r_1 t}$$

Critically damped response



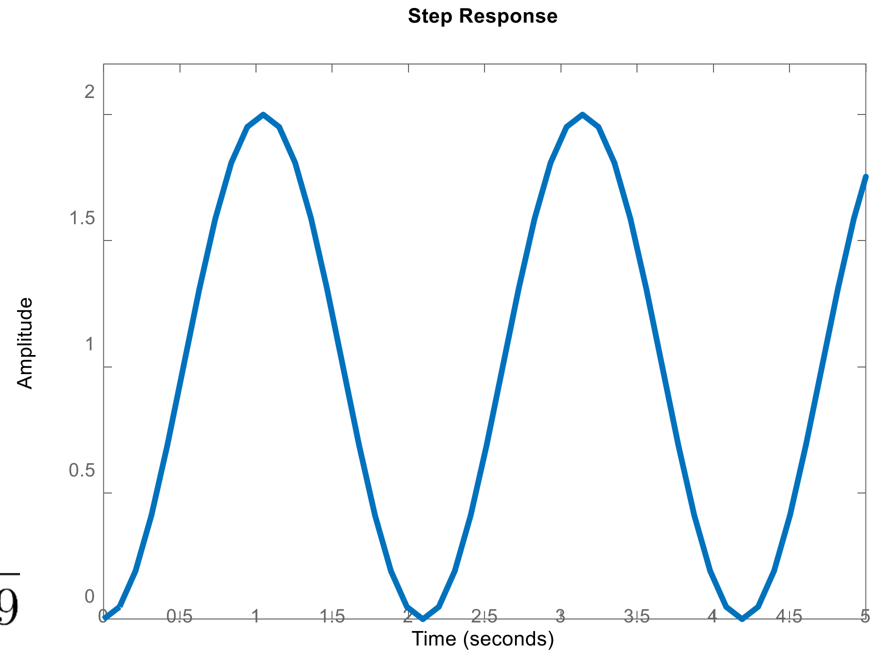
Case 3: All Imaginary Roots

$$Y(s) = \frac{\omega^2}{s(s^2+\omega^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\omega^2}$$

$$y(t) = K_1 + K_2 \cos(\omega t - \phi)$$

Undamped response

$$G(s) = \frac{9}{s^2+9}$$

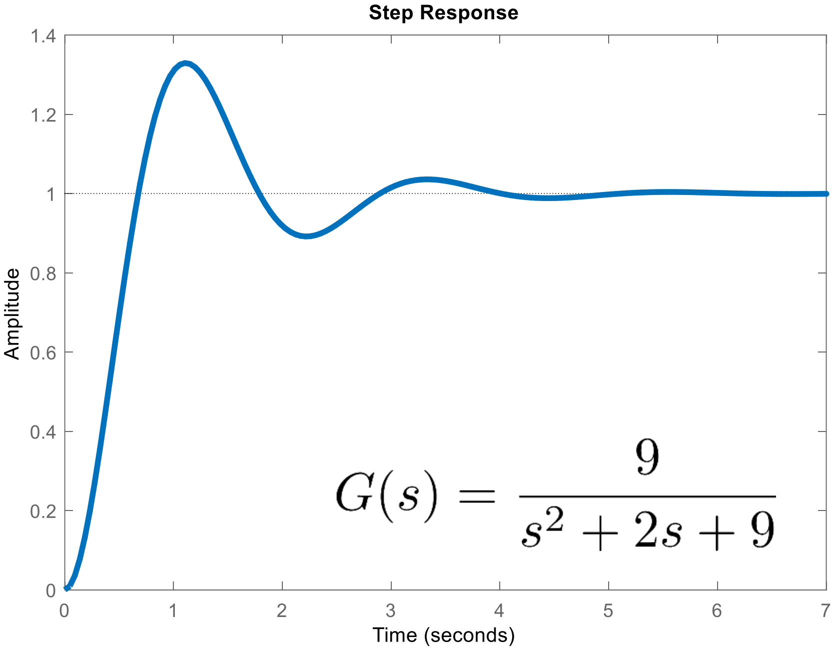


Case 4: Roots Are Complex

$$Y(s) = \frac{c}{s(s^2+bs+c)} = \frac{A}{s} + \frac{Bs+C}{as^2+bs+c} = \frac{A}{s} + \frac{D(s+\sigma)}{(s+\sigma)^2+\omega^2}$$

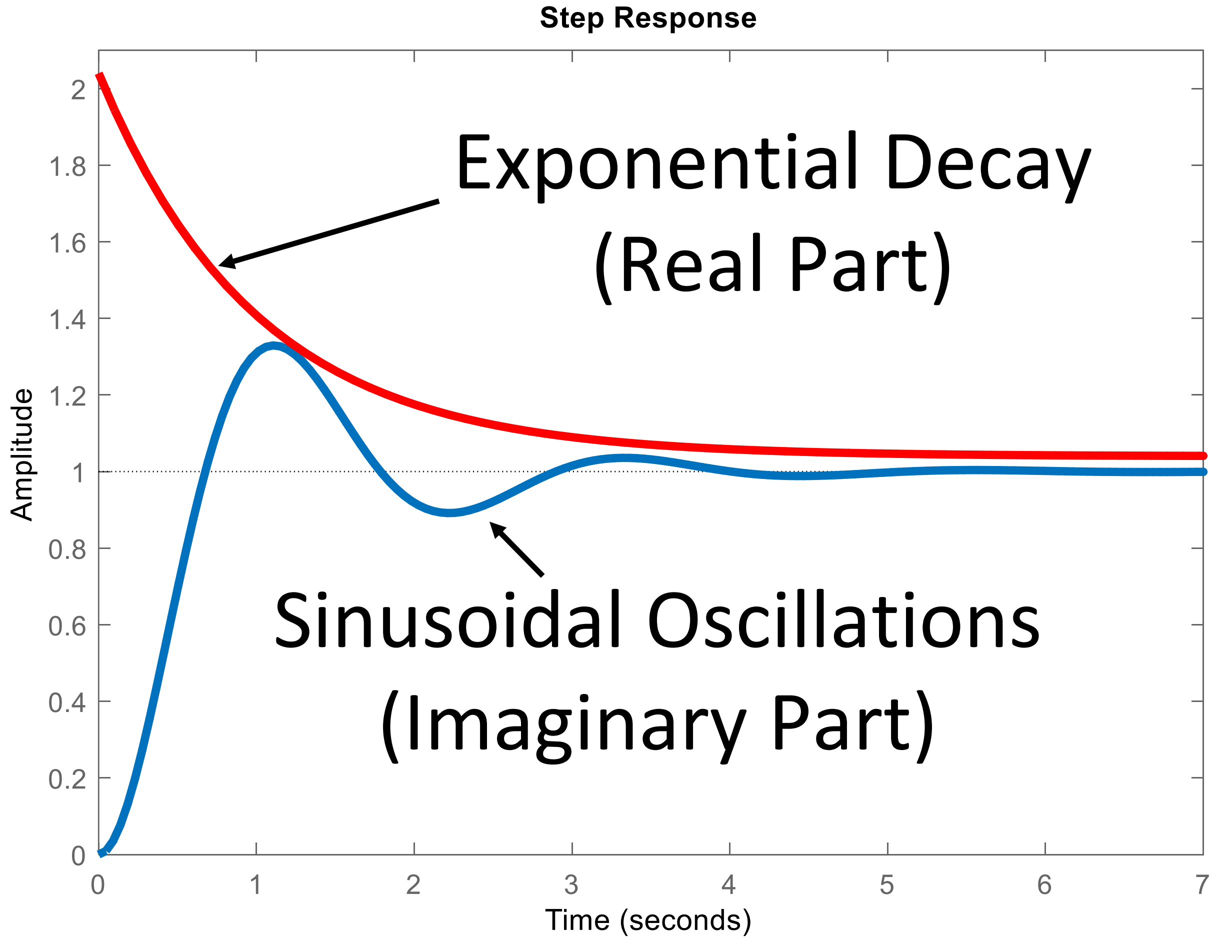
$$y(t) = K_1 + K_2 e^{-\sigma t}\cos(\omega t - \phi)$$

Underdamped response



A Closer Look at Case 4

$$y(t) = K_1 + K_2 e^{-\sigma t} \cos(\omega t - \phi)$$



Summary of 2nd Order Systems

Given, $G(s) = \dfrac{c}{s^2 + bs + c}$ and U(s) = 1/s

Solution is one of the following:

1. **Overdamped**: r_1 & r_2 are real & distinct
2. **Critically Damped**: r_1 & r_2 are real & repeated
3. **Undamped**: r_1 & r_2 are both imaginary
4. **Underdamped**: r_1 & r_2 are complex conjugates

2^{nd} Order System Parameters

Given $G(s) = \dfrac{c}{s^2 + bs + c}$ and U(s) = 1/s

- Natural Frequency – ω_n

  System's frequency of oscillation with no damping

- Damping Ratio – ζ

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/sec)}} = \frac{1}{2\pi}\frac{\text{Natural period (sec)}}{\text{Exponential time constant}}$$

General 2[nd] Order System

Given $G(s) = \frac{c}{s^2 + bs + c}$ and U(s) = 1/s

- When b = 0 $G(s) = \frac{c}{s^2 + c}$

$$s = \pm j\sqrt{c} \Rightarrow \omega_n = \sqrt{c} \Rightarrow c = \omega_n^2$$

- For an underdamped system

$$s = -\sigma \pm j\omega_n \quad \text{w/} \quad \sigma = -\frac{b}{2}$$

$$\zeta = \frac{|\sigma|}{\omega_n} = \frac{b/2}{\omega_m} \Rightarrow b = 2\zeta\omega_n$$

General 2nd Order Systems

Second-order system transfer functions have the form

$$G(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

with poles of the form $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Example: For $G(s) = \dfrac{36}{(s^2 + 4.2s + 36)}$

Compute ζ, ω_n, and $s_{1,2}$?

Video 5.7

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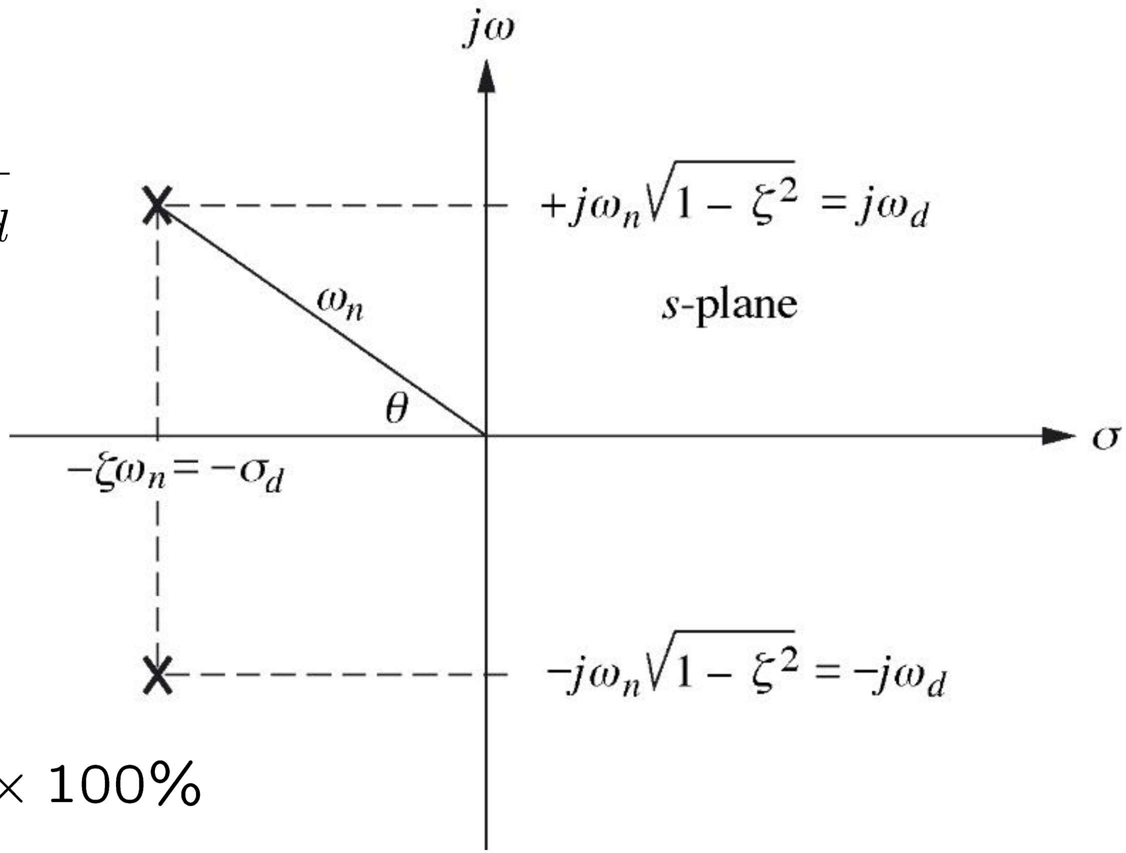
Characterizing Underdamped Systems

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

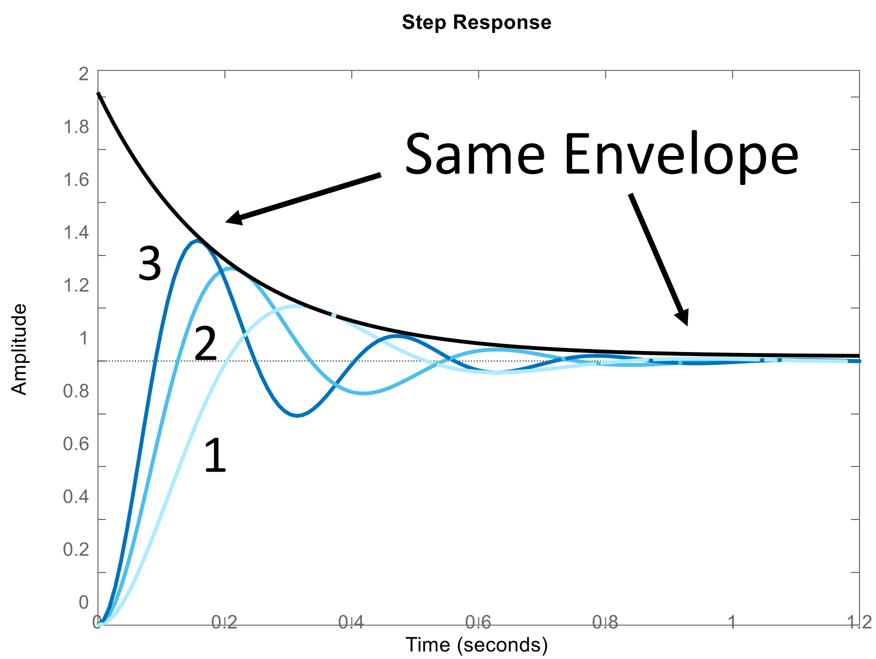
$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

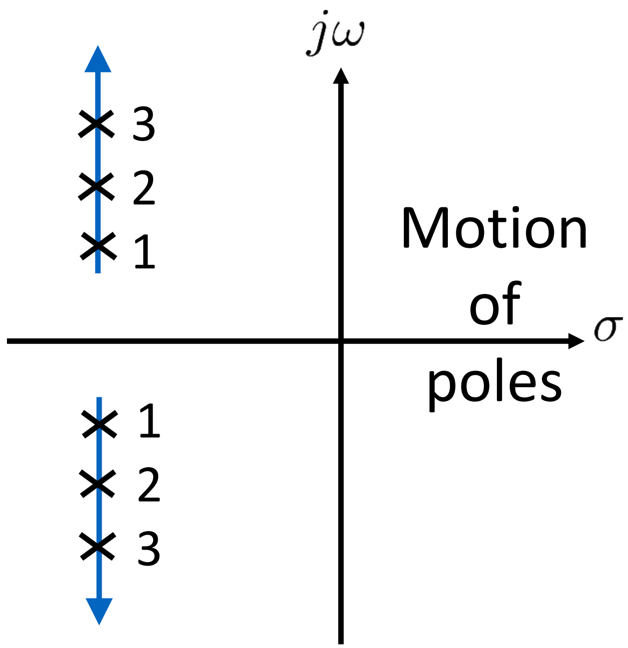


Peak Time

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} \\ &= -\sigma_d \pm j\omega_d \end{aligned}$$

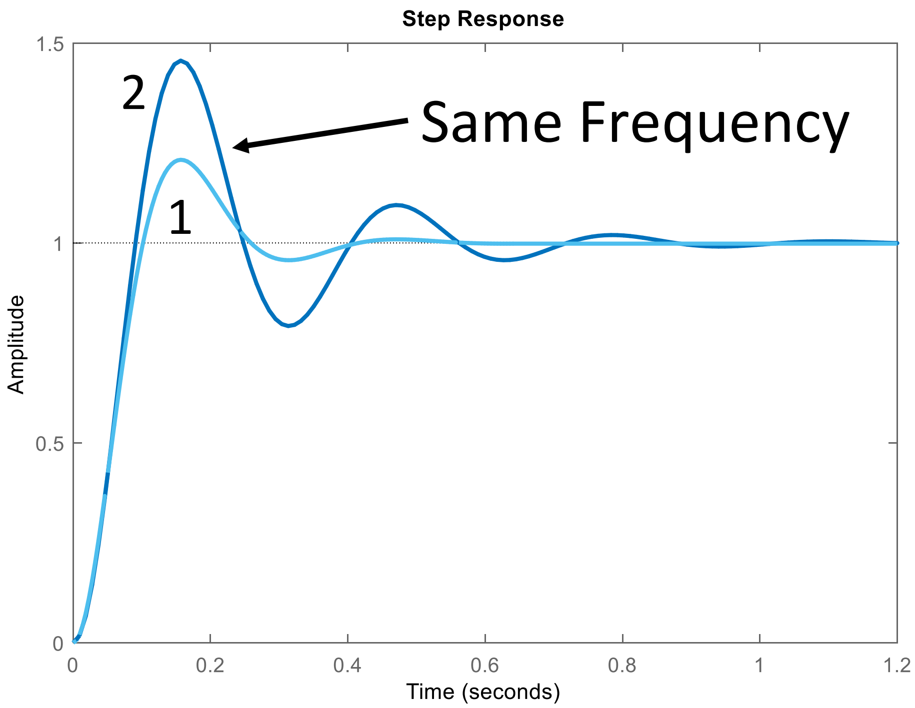


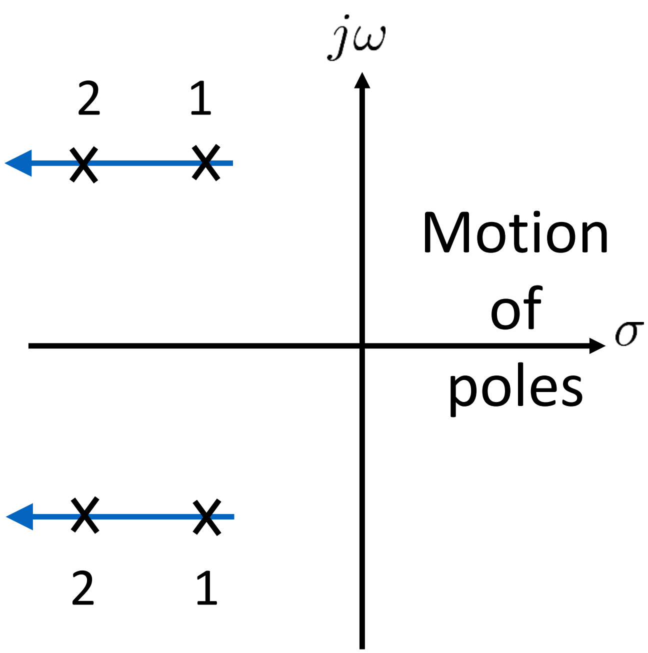


Settling Time

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\sigma_d \pm j\omega_d \end{aligned}$$

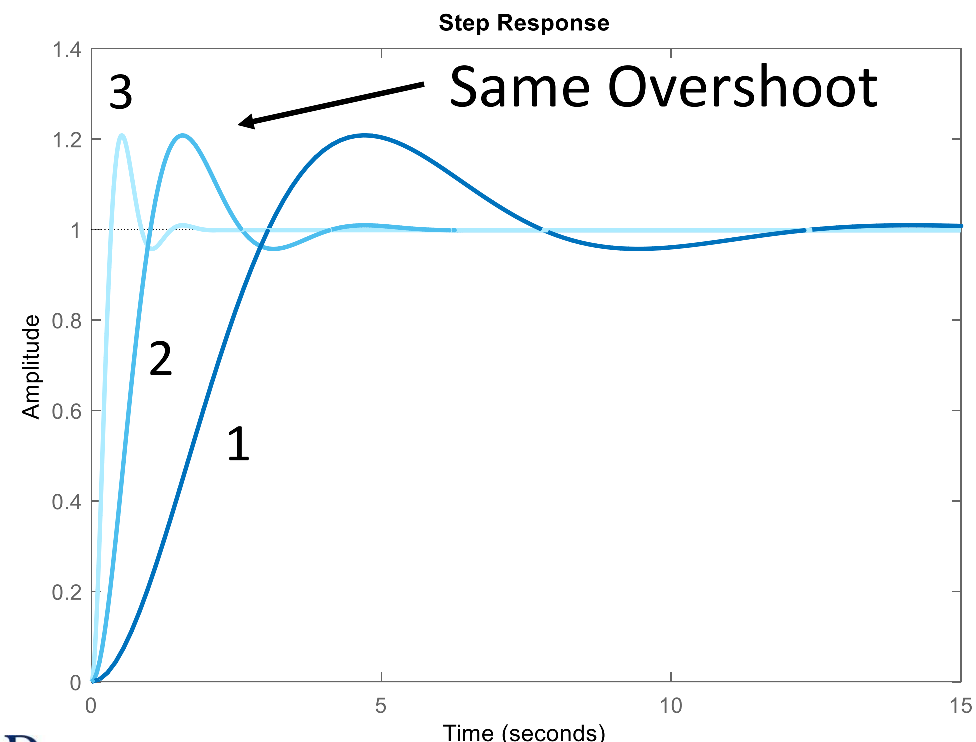


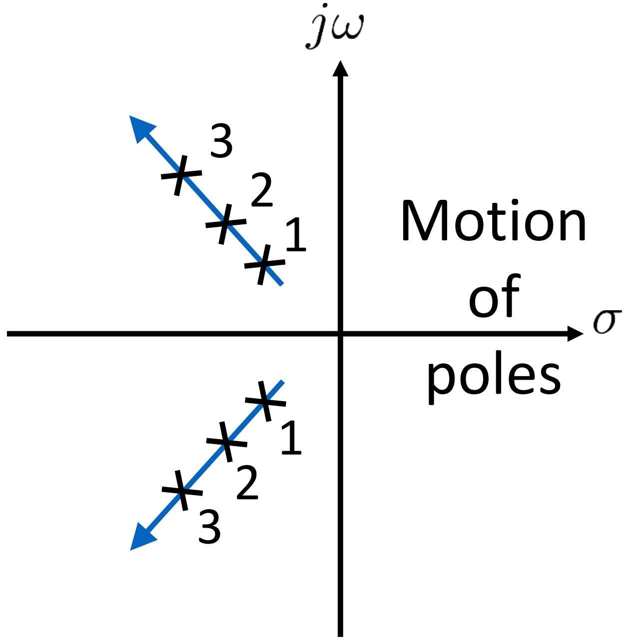


Overshoot

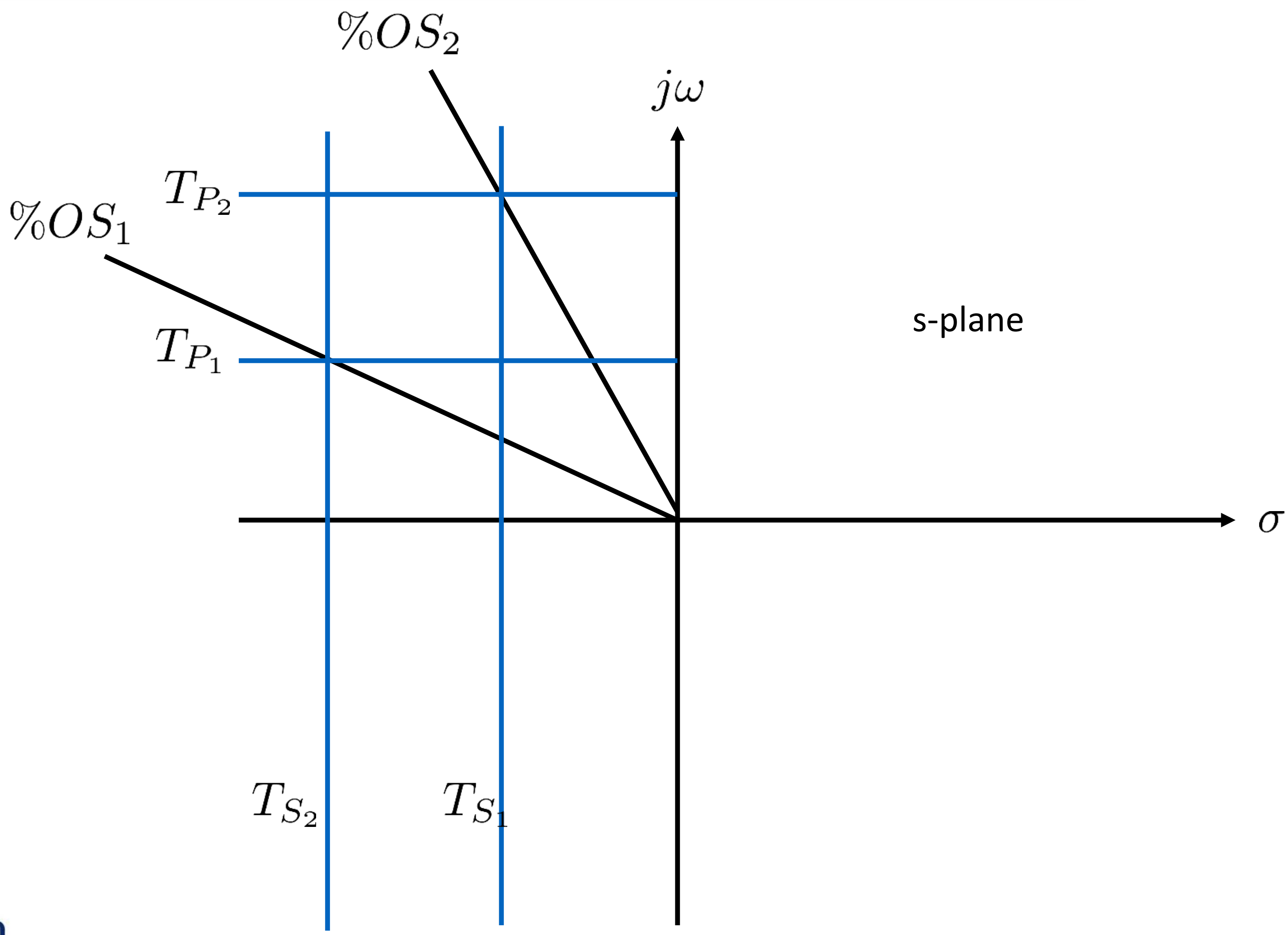
$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\sigma_d \pm j\omega_d \end{aligned}$$





In Summary



Video 5.8

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Independent Joint Control

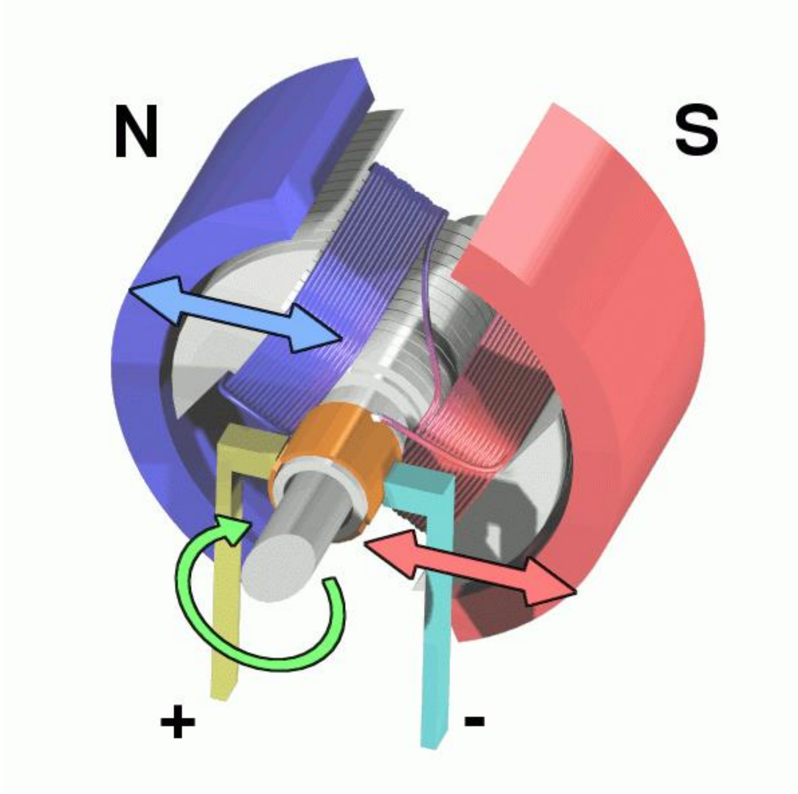
In general,

n-Link Robot Arm generally has $\geq n$ actuators

Single Input Single Output (SISO)

Single joint control

Permanent Magnet DC Motor

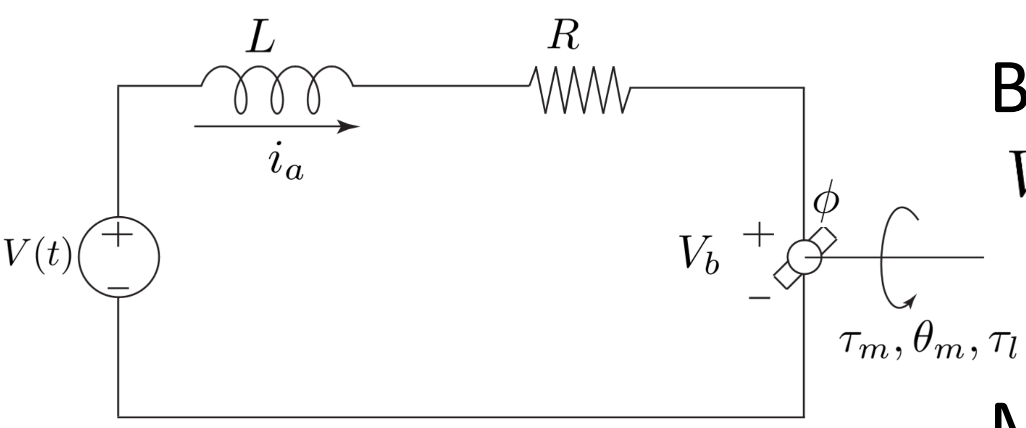


Source: Wikimedia Commons

Basic Principle

$$\mathbf{F} = \mathbf{i} \times \phi$$

Electrical Part



Armature Current

$$L\frac{di_a}{dt} + Ri_a = V - V_b$$

Back EMF

$$\begin{aligned} V_b &= K_2\phi\omega_m = K_m\omega_m \\ &= K_m\frac{d\theta_m}{dt} \end{aligned}$$

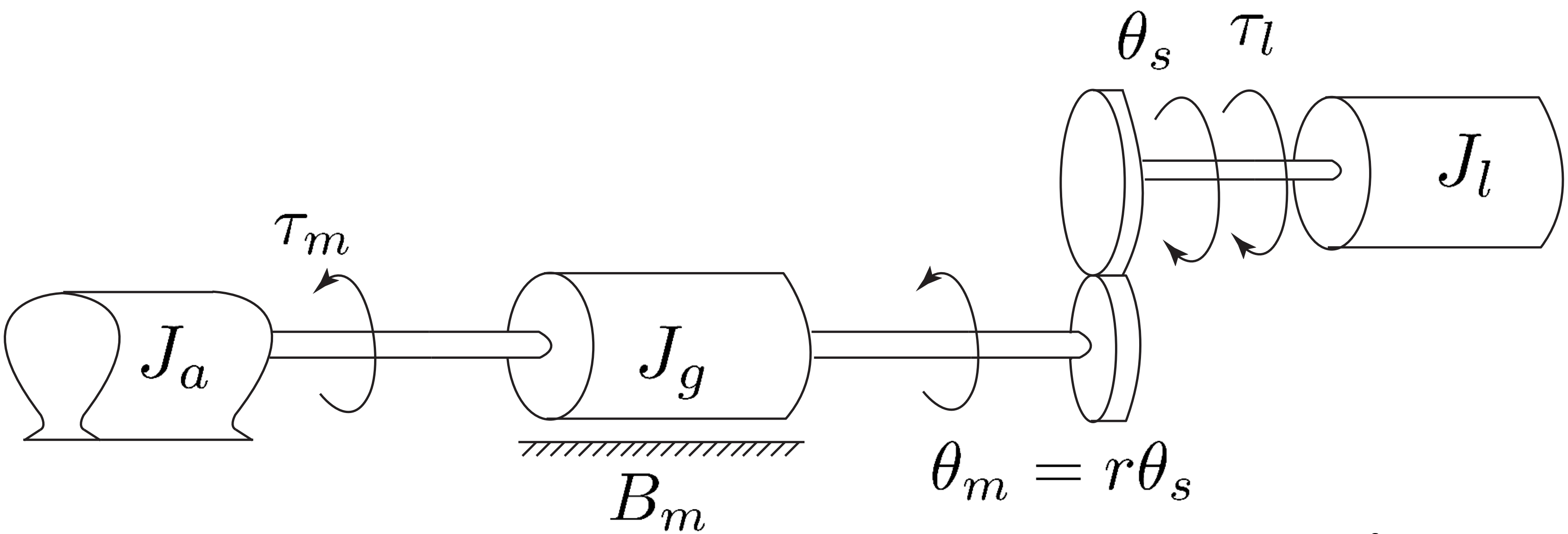
Motor Torque

$$\tau_m = K_1\phi_{i_a} = K_m i_a$$

Torque Constant

$$K_m = \frac{R\tau_0}{V_r}$$

Mechanical Part



$$\theta_m = r\theta_s$$

Actuator Dynamics

Gear ratio $r:1$

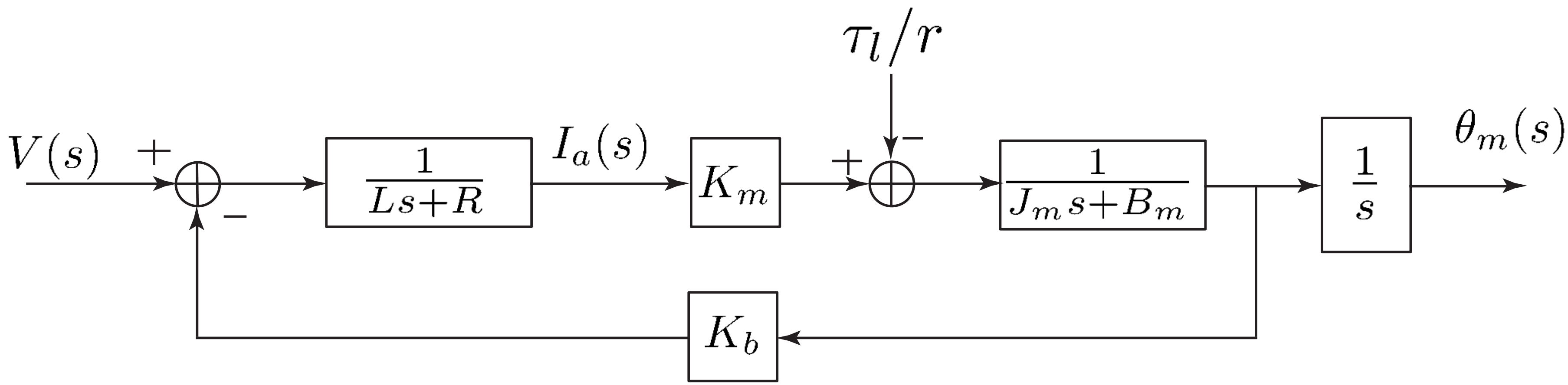
$$J_m = J_a + J_g$$

$$\begin{aligned} J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} &= \tau_m - \frac{\tau_l}{r} \\ &= K_m i_a - \frac{\tau_l}{r} \end{aligned}$$

Combining the Two

$$(Ls + R)I_a(s) = V(s) - K_b s\Theta_m(s)$$

$$(J_m s^2 + B_m s)\Theta_m(s) = K_m I_a(s) - \frac{T_l(s)}{r}$$



Correction: the K_b terms should be K_m

Two SISO Outcomes

Input Voltage – Motor Shaft Position

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s\left[(Ls+R)(J_m s + B_m) + K_b K_m\right]}$$

Load Torque – Motor Shaft Position

$$\frac{\Theta_m(s)}{T(s)} = \frac{-(Ls+R)/r}{s\left[(Ls+R)(J_m s + B_m) + K_b K_m\right]}$$

Video 5.9

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Two SISO Outcomes

Input Voltage – Motor Shaft Position

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s\left[(Ls+R)(J_m s+B_m)+K_b K_m\right]}$$

Load Torque – Motor Shaft Position

$$\frac{\Theta_m(s)}{T(s)} = \frac{-(Ls+R)/r}{s\left[(Ls+R)(J_m s+B_m)+K_b K_m\right]}$$

Assumption: $L/R \ll J_m/B_m$

2nd Order Approximation

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s\left[(Ls+R)(J_m s + B_m) + K_b K_m\right]}$$

Divide by R and set L/R = 0

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m/R}{s(J_m s + B_m + K_b K_m/R)}$$

$$\frac{\Theta_m(s)}{T(s)} = \frac{-1/r}{s(J_m s + B_m + K_b K_m/R)}$$

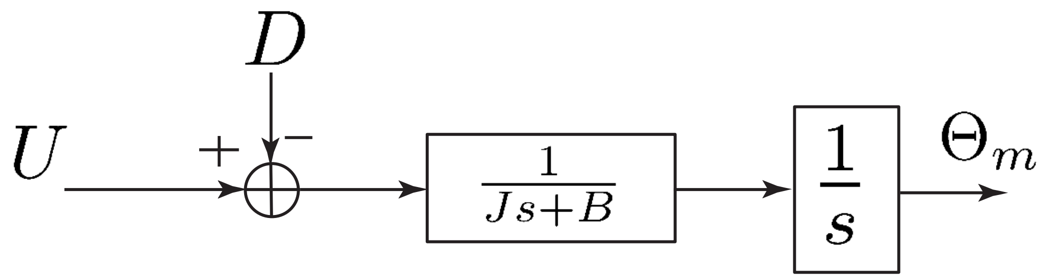
In the time domain

$$J_m \ddot{\theta}_m(t) + (B_m + K_b K_m/R)\dot{\theta}_m(t) = (K_m/R)V(t) - \tau_l(t)/r$$

Open-Loop System

Actuator Dynamics

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) - d(t)$$



- Set-point tracking (feedback)
- Trajectory tracking (feedforward)

Our Control Objectives

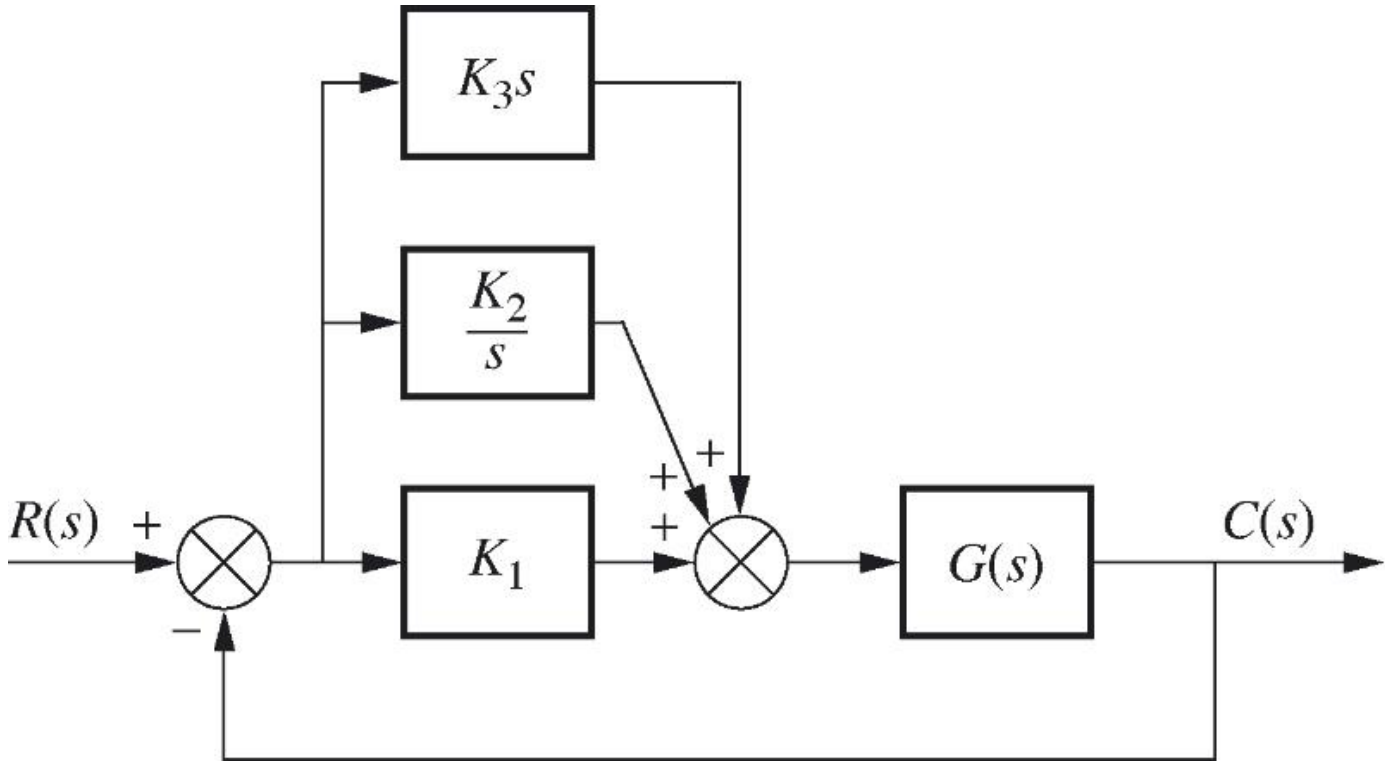
- Motion → sequence of end-effector positions and orientations (EE poses)
- EE poses → Joint Angles → Motor Commands
- Transfer function

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m/R}{s(J_m s + B_m + K_b K_m/R)}$$

- Three primary linear controller designs:
  - *P (proportional)*
  - *PD (proportional-derivative)*
  - *PID (proportional-integral-derivative)*

Set-Point Tracking

The Basic PID Controller



Proportional (P) Control

- Control input *proportional* to error

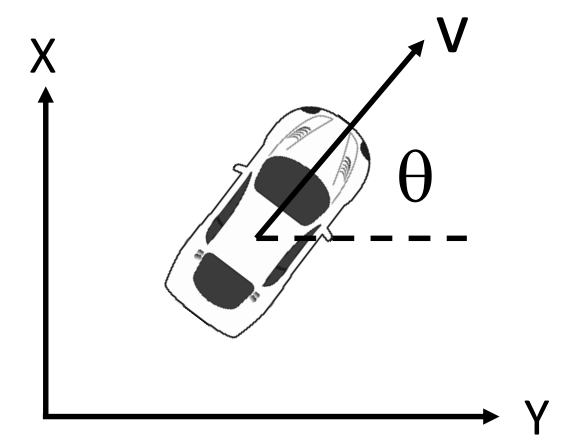
$$u(t) = K_P(\theta^d(t) - \theta(t))$$

$$U(s) = K_P(\Theta^d(s) - \Theta(s))$$

- K_P – controller gain
- Error is amplified by K_P to obtain the desired output signal

P Control of Vehicle Speed

Example: Cruise Control



Desired linear speed

$$\dot{\Theta}^d(s) = \Omega^d(s) = 0$$

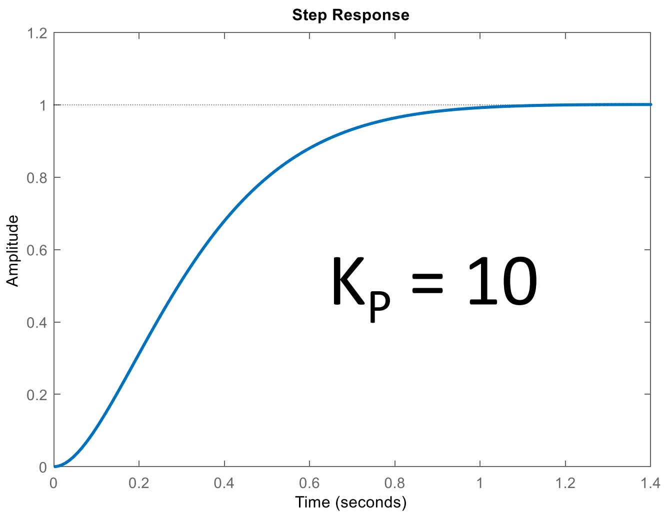
$$\Rightarrow \xi_L = \xi_R = \boxed{\xi}$$

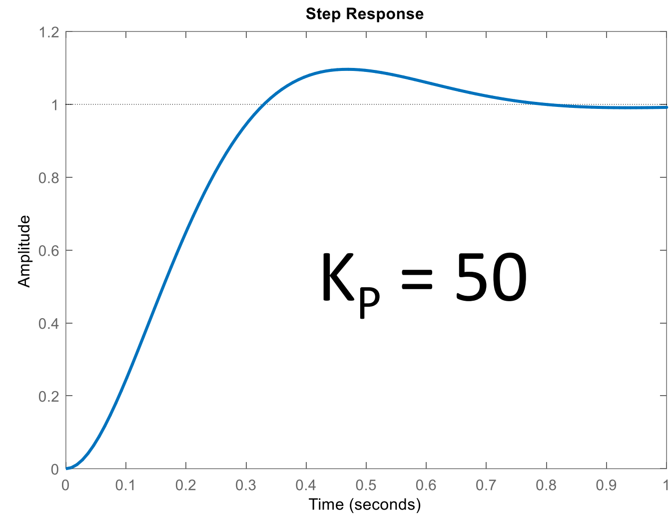
vehicle wheel speed

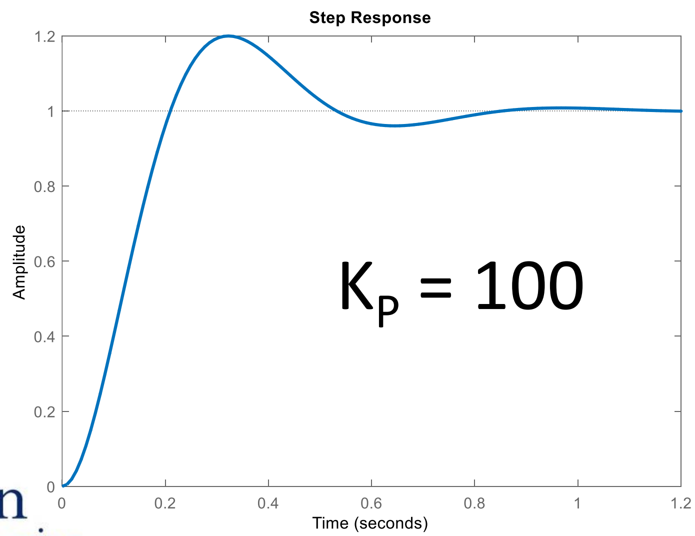
Control input proportional to error

$$U(s) = K_P(\xi^d - \xi)$$

Performance of P Controller







- Increases the controller gain decreases rise time
- Excessive gain can result in overshoot

Video 5.10

Vijay Kumar and Ani Hsieh

Proportional-Derivative (PD) Control

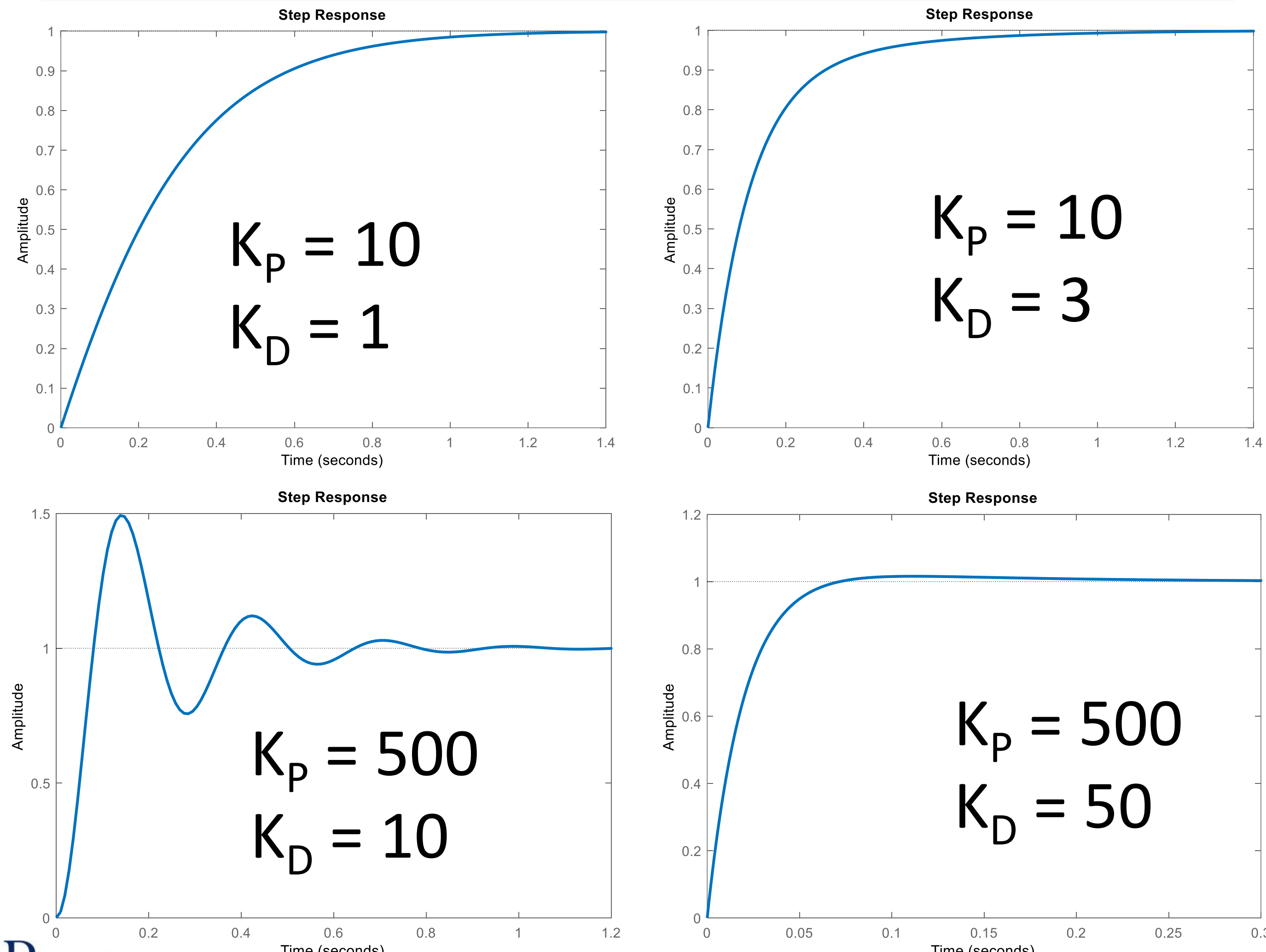
- Control input *proportional* to error AND 1st derivative of error

$$u(t) = K_P(\theta^d(t) - \theta(t)) + K_D \frac{d}{dt}(\theta^d(t) - \theta(t))$$

$$U(s) = K_P(\Theta^d(s) - \Theta(s)) - K_D s\Theta(s)$$

- Including rate of change of error helps mitigates oscillations

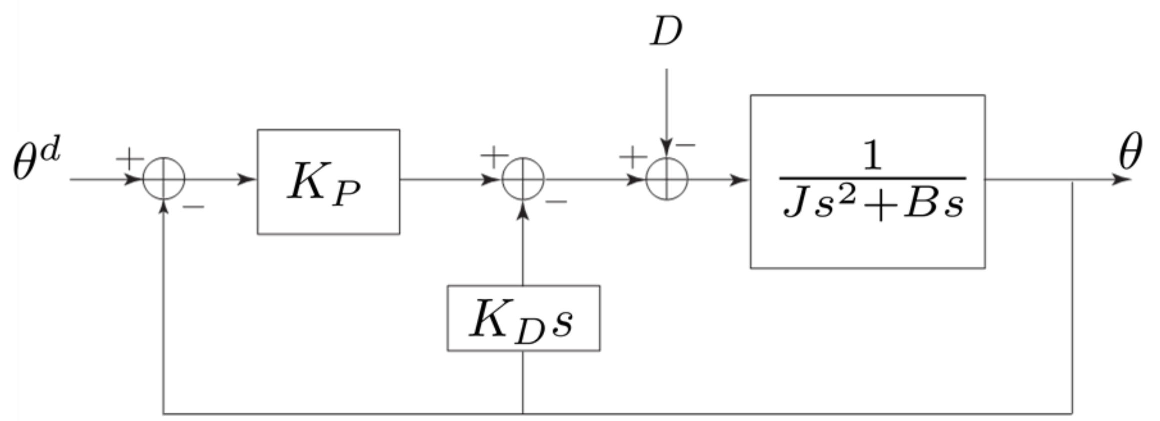
Performance of PD Controller



Decreases rise time

Reduces overshoot & Settling Time

PD Control of a Joint



$$U(s) = K_P(\Theta^d(s) - \Theta(s)) - K_D s\Theta(s)$$

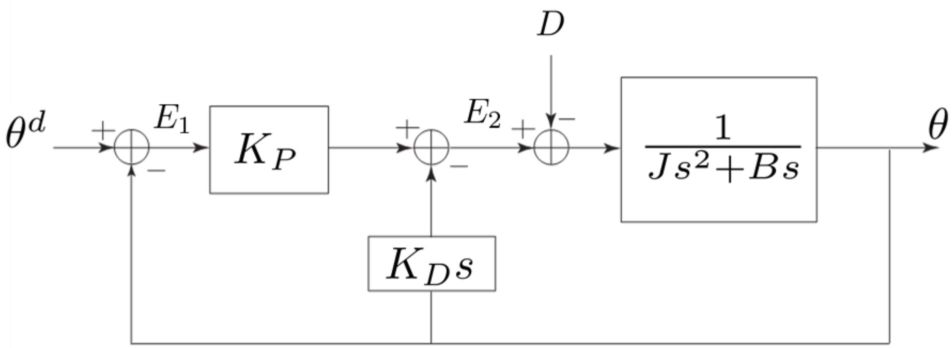
Closed loop system given by

$$\Theta(s) = \frac{K_P}{\Delta(s)}\Theta^d(s) - \frac{1}{\Delta(s)}D(s)$$

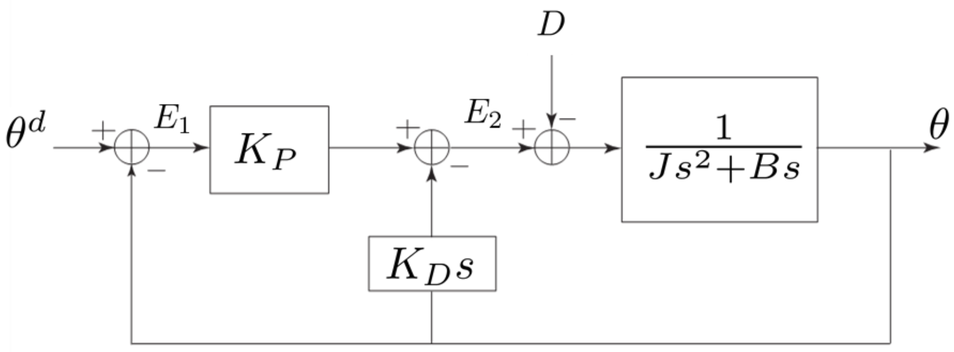
w/

$$\Delta(s) = Js^2 + (B + K_D)s + K_P$$

PD Compensated Closed Loop Response (1)



PD Compensated Closed Loop Response (2)



Picking K_P and K_D

Closed loop system $\Theta(s) = \frac{K_P}{\Delta(s)}\Theta^d(s) - \frac{1}{\Delta(s)}D(s)$

w/ $\Delta(s) = Js^2 + (B + K_D)s + K_P$

$$\Delta(s) = s^2 + \frac{(B + K_D)}{J}s + \frac{K_P}{J} = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Design Guidelines

- Critically damped w/ $\zeta = 1$
- Pick $K_P = \omega_n^2 J$ and
  $K_D = 2\zeta\omega_n J - B$

Performance of the PD Controller

Assuming $\Theta^d(s) = \frac{\Omega^d}{s}$ and $D(s) = \frac{D}{s}$

Tracking error is given by

$$\begin{aligned} E(s) &= \Theta^d(s) - \Theta(s) \\ &= \frac{Js^2 + (B + K_D)s}{\Delta(s)}\Theta^d(s) + \frac{1}{\Delta(s)}D(s) \end{aligned}$$

At steady-state $e_{ss} = \lim_{s \to 0} sE(s) = -\frac{D}{K_P}$

Video 5.11

Vijay Kumar and Ani Hsieh

Proportional-Integral-Derivative (PID) Controller

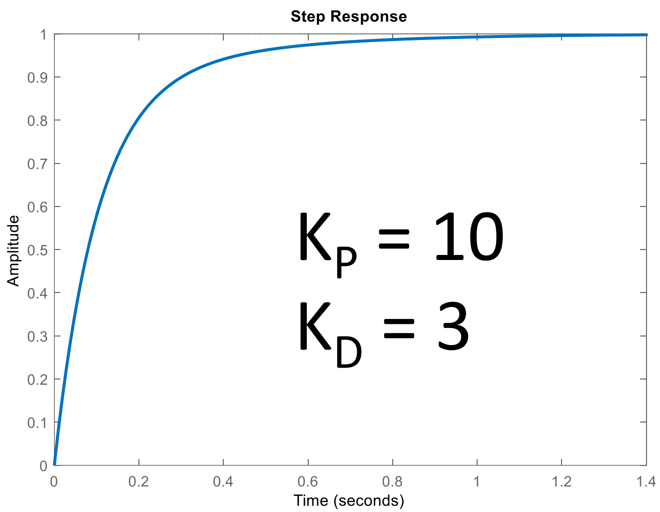
- Control input *proportional* to error, 1st derivative AND an integral of the error

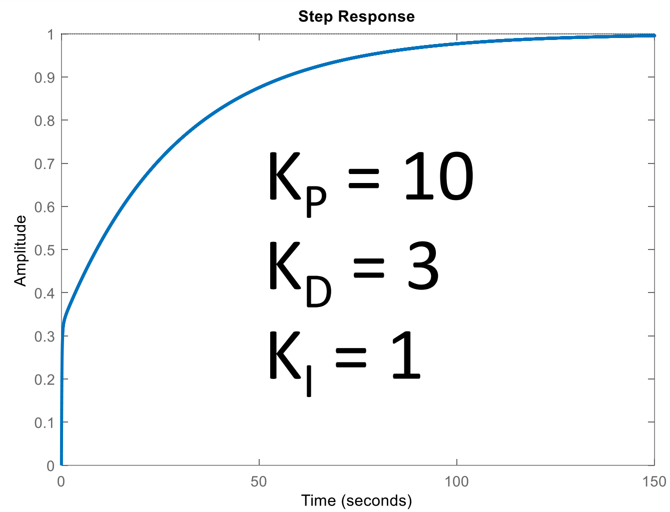
$$u(t) = K_P(\theta^d(t) - \theta(t)) + K_D \frac{d}{dt}(\theta^d(t) - \theta(t)) + K_I \int_0^t (\theta^d(\tau) - \theta(\tau))d\tau$$

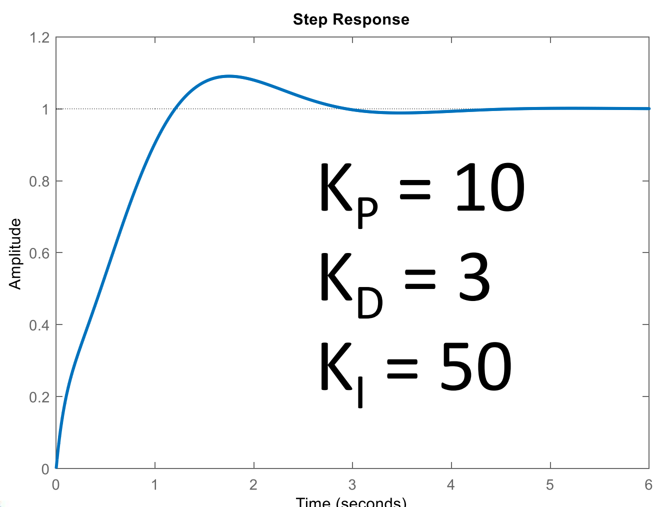
$$U(s) = K_P(\Theta^d(s) - \Theta(s)) - K_D s\Theta(s) + \frac{K_I}{s}(\Theta^d(s) - \Theta(s))$$

- The integral term offsets any steady-state errors in the system

Performance of PID Controller

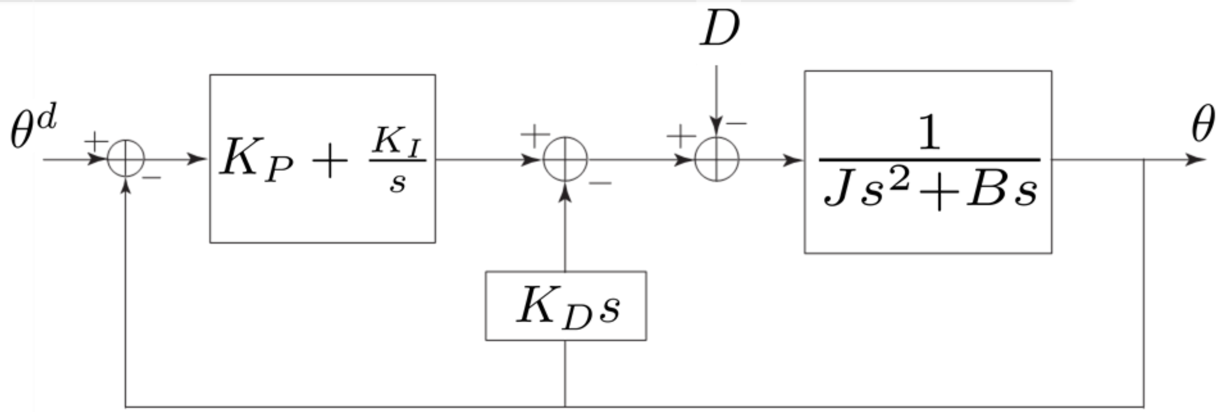






- Eliminates SS-Error
- Increases overshoot & settling time

PID Control of a Joint

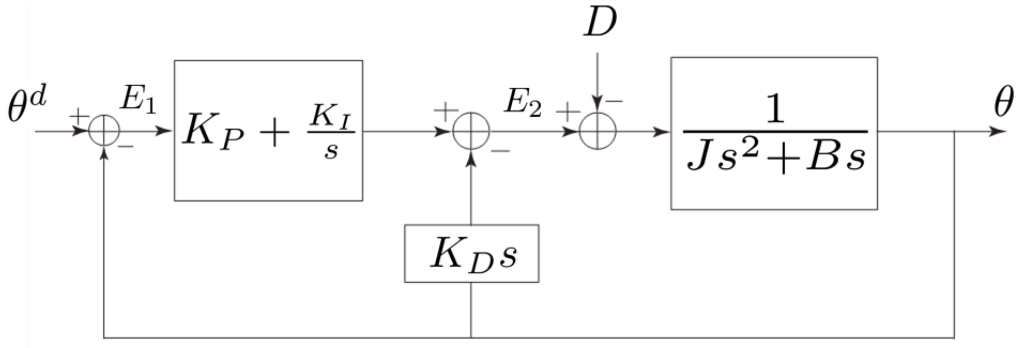


Closed-loop system is given by

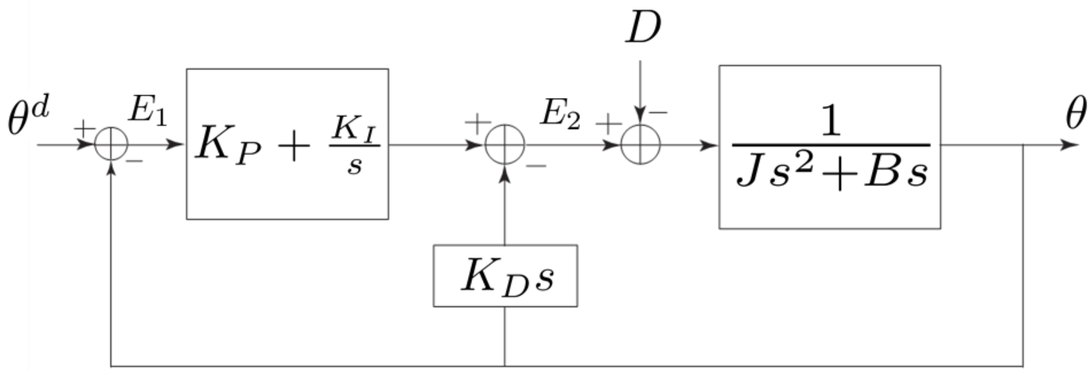
$$\Theta(s) = \frac{(K_P s + K_I)}{\Delta_2(s)}\Theta^d(s) - \frac{s}{\Delta_2(s)}D(s)$$

w/ $\Delta_2(s) = Js^3 + (B + K_D)s^2 + K_P s + K_I$

PID Compensated Closed Loop Response (1)



PID Compensated Closed Loop Response (2)



Picking K_P, K_D, and K_I

Closed loop system $\Theta(s) = \frac{(K_P s + K_I)}{\Delta_2(s)}\Theta^d(s) - \frac{s}{\Delta_2(s)}D(s)$

w/ $\Delta_2(s) = Js^3 + (B + K_D)s^2 + K_P s + K_I$

Design Guidelines

- System stable if K_P, K_D, and K_I >0
- $K_I < \frac{(B + K_D)K_P}{J}$ ★
- Set K_I = 0 and pick K_P, K_D, then go back to pick K_I w/ ★ in mind

Summary of PID Characteristics

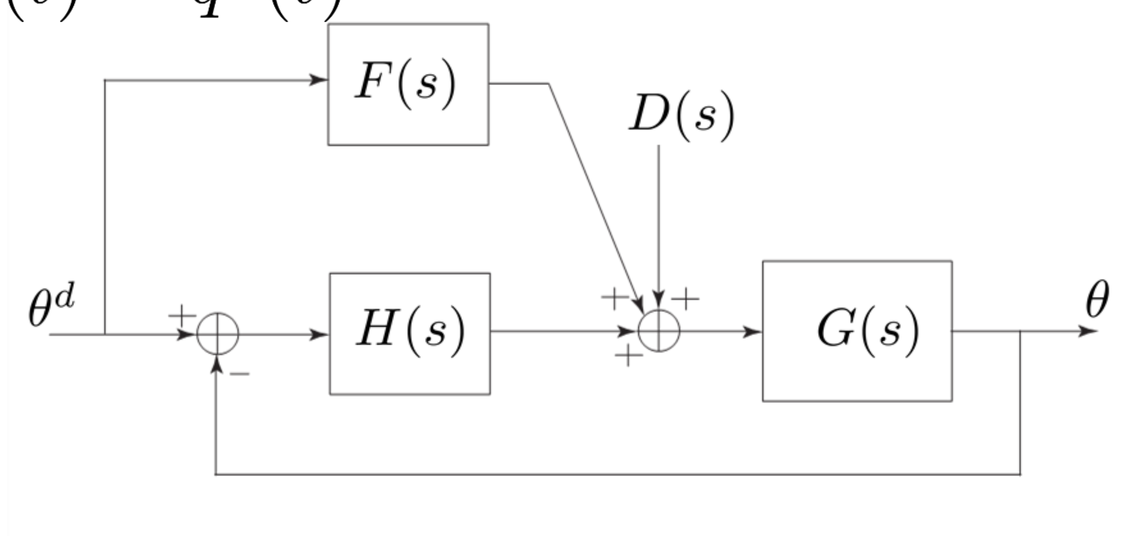
CL Response	Rise Time	% Overshoot	Settling Time	S-S Error
K_P	Decrease	Increase	Small Change	Decrease
K_D	Small Change	Decrease	Decrease	Small Change
K_I	Decrease	Increase	Increase	Eliminate

Tuning Gains

- Appropriate gain selection is crucial for optimal controller performance
  - Analytically (R-Locus, Frequency Design, Ziegler Nichols, etc)
  - Empirically
- The case for experimental validation
  - Model fidelity
  - Optimize for specific hardware
  - Saturation and flexibility

Feedforward Control

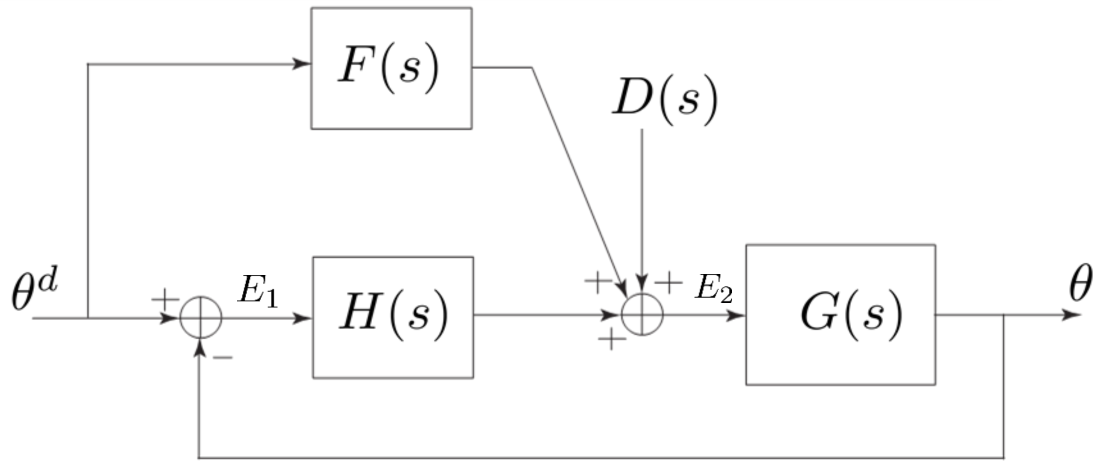
- Motion → sequence of end-effector positions and orientations (EE poses)
- What if instead of $\theta^d(t) = q^d$ we want $\theta^d(t) = q^d(t)$?



Video 5.12

Vijay Kumar and Ani Hsieh

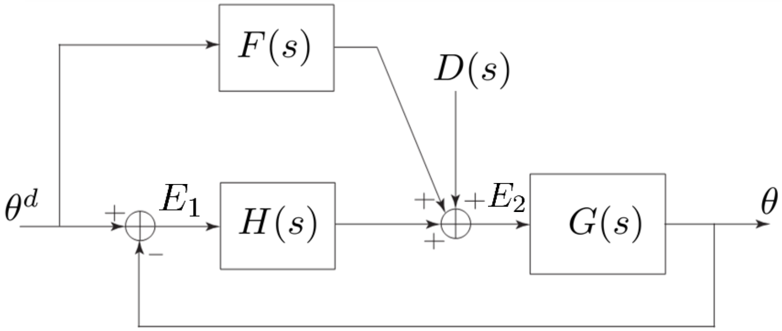
Closed Loop Transfer Function (1)



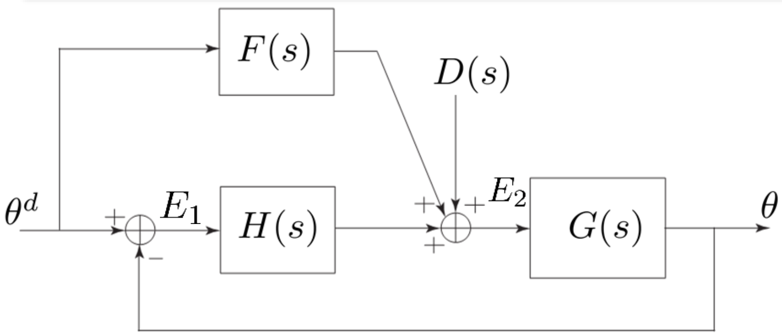
$$\Theta = G(s)E_2, \;\; E_1 = \Theta^d - \Theta, E_2 = F(s)\Theta^d + H(s)E_1$$

$$G(s) = \frac{q(s)}{p(s)}, H(s) = \frac{c(s)}{d(s)}, F(s) = \frac{a(s)}{b(s)}$$

Closed Loop Transfer Function (2)



Closed Loop Transfer Function (3)



Picking F(s)

Closed loop transfer function given by

$$T(s) = \frac{q(s)\left(c(s)b(s) + a(s)d(s)\right)}{b(s)\left(p(s)d(s) + q(s)c(s)\right)}$$

Behavior of closed loop response, given by roots of

$$b(s)\left(p(s)d(s) + q(s)c(s)\right)$$

H(s) and F(s) be chosen so that

$$Re\left(roots\left(p(s)d(s) + q(s)c(s)\right)\right) < 0$$

Will This Work?

Let F(s) = 1/G(s), *i.e.*, a(s) = p(s) and b(s) = q(s), then

$$T(s) = \frac{q(s)\left(c(s)q(s) + p(s)d(s)\right)}{q(s)\left(p(s)d(s) + q(s)c(s)\right)}$$

$$\frac{\Theta}{\Theta^d} = \frac{q(pd+qc)}{q(pd+qc)} \Rightarrow q(pd+qc)(\Theta^d - \Theta) = 0$$

$$q(pd+qc)E(s) = 0$$

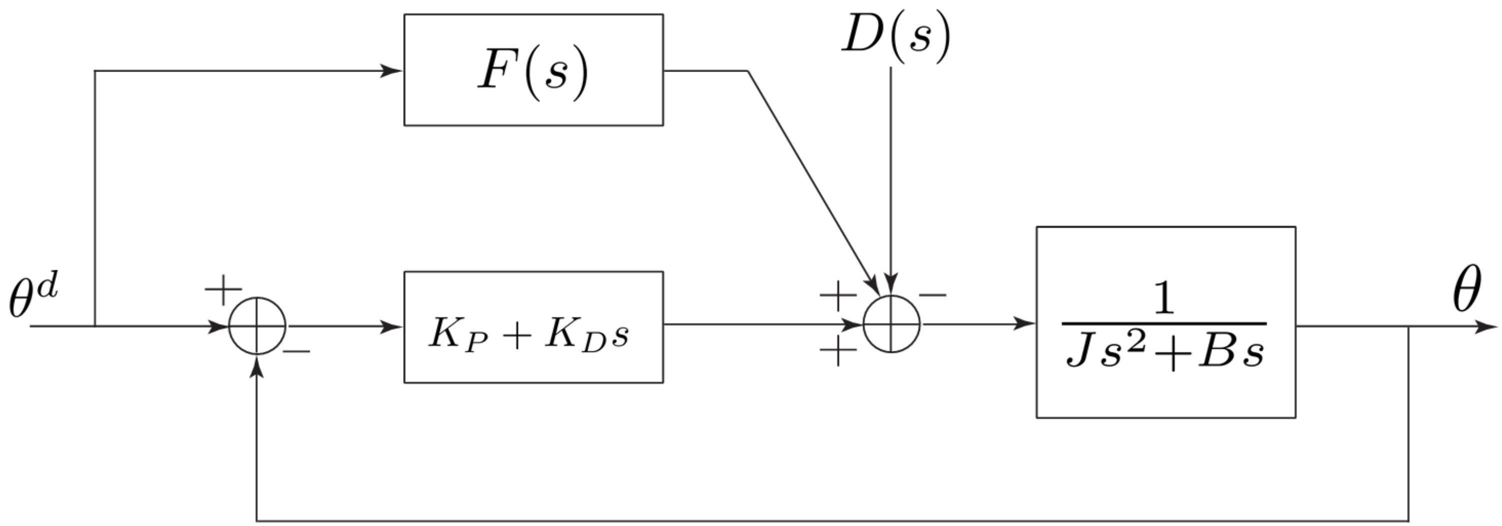
System will track any reference trajectory!

Caveats – Minimum Phase Systems

Picking F(s) = 1/G(s), leads to $q(pd + qc)E(s) = 0$

- Assume system w/o FF loop is stable
- By picking F(s) = 1/G(s), we require numerator of G(s) to be *Hurwitz* (or $Re\left(roots(q(s))\right) < 0$)
- Systems whose numerators have roots with negative real parts are called *Minimum Phase*

Feedforward Control w/ Disturbance

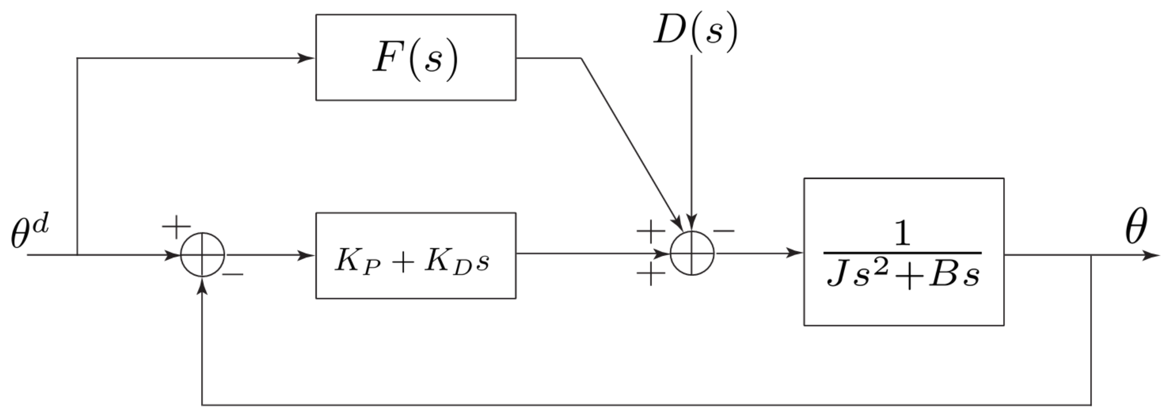


Assume: D(s) = constant w/ $G(s) = \frac{1}{Js^2 + Bs}$

Pick F(s) = 1/G(s) = $Js^2 + Bs$

Note the following:

Tracking Error



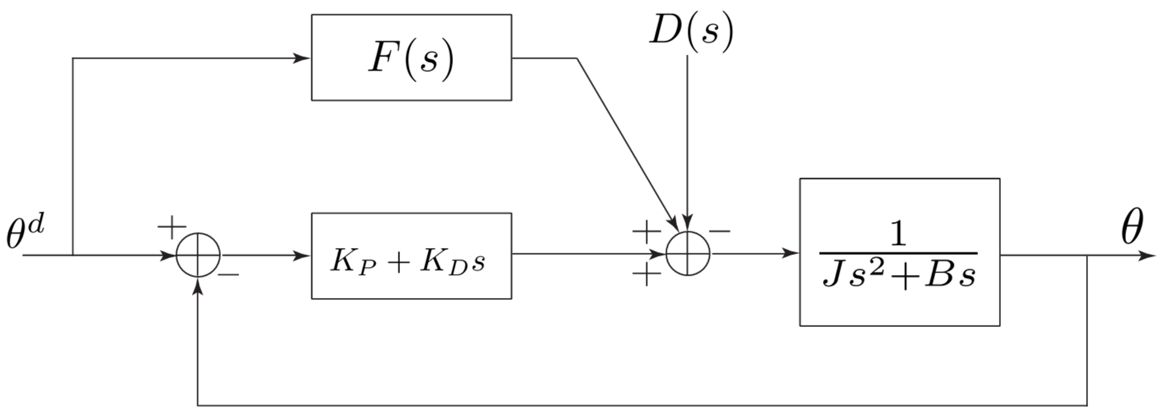
Control law in time domain

$$\begin{aligned} u(t) &= J\ddot{\theta}^d + B\dot{\theta}^d + K_D(\dot{\theta}^d - \dot{\theta}) + K_P(\theta^d - \theta) \\ &= f(t) + K_D\dot{e}(t) + K_P e(t) \end{aligned}$$

System dynamics w/ control + disturbance

$$J\ddot{\theta} + B\dot{\theta} = V(t) - rd(t)$$

Overall Performance



$$J\ddot{\theta} + B\dot{\theta} = f(t) + K_D\dot{e}(t) + K_P e(t) - rd(t)$$

$$J(\ddot{\theta}^d - \ddot{\theta}) + B(\dot{\theta}^d - \dot{\theta}) + K_D\dot{e}(t) + K_P e(t) = rd(t)$$

$$J\ddot{e} + (B + K_D)\dot{e} + K_P e = rd(t)$$