edX Robo4 Mini MS – Locomotion Engineering

Week 8 – Unit 1

Spring Loaded Inverted Pendulum

Video 9.1

Segment 8.0.1

SLIP in Biology and Robotics – Background

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Where Did Raibert's Hoppers Come From?

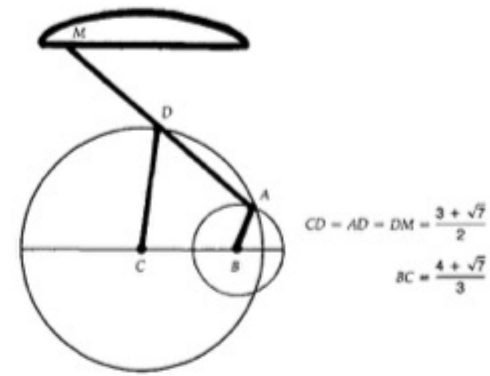
- Raibert traced engineering history back to 19th Century

  1850 – Chebyshev designs linkage used in walking mechanism

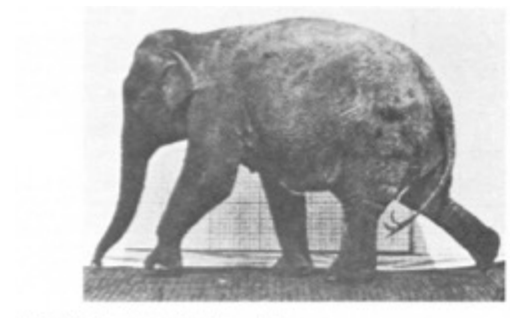
  1893 – Rygg patents human-powered mechanical horse

  1945 – Wallace patents hopping tank with stabilizing reaction wheels

- But found more compelling inspiration in biology
  - past machines were all quasi-statically stable
  - animals
    - manage their kinetic energy
    - rely on symmetry



Raibert's rendering of Lucas's 1894 design



Raibert's reproduction of Muybridge's 1872 animal studies

figures & citations from:
M. H. Raibert, "Legged Robots," *Commun. ACM*, vol. 29, no. 6, pp. 499–514, Jun. 1986.

The Virtue of Controlled Springs

- Previous engineering analysis treated the inverted pendulum
  - recall: 1 DoF IP in week 3
  - higher DoF IP important
  - in history of control theory
- Raibert's machine incorporated tunable compliance
  - biologists had already begun to realize
    - multiple inverted pendula model dynamic running
    - entailing phase exchange of kinetic and potential energy
    - as modeled by introduction of spring element

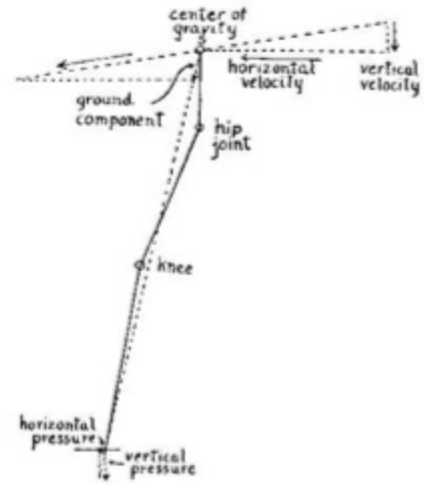


figure from:
W. O. Fenn, "WORK AGAINST GRAVITY AND WORK DUE TO VELOCITY CHANGES IN RUNNING," *American Journal of Physiology -- Legacy Content*, vol. 93, no. 2, pp. 433–462, Jun. 1930.

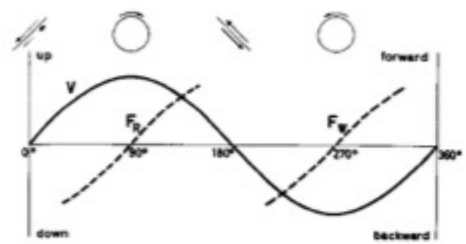


figure from:
G. A. Cavagna, F. P. Saibene, and R. Margaria, "Mechanical work in running," *Journal of Applied Physiology*, vol. 19, no. 2, pp. 249–256, Mar. 1964.

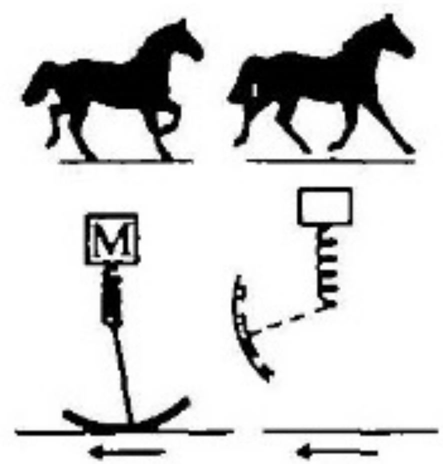


figure from:
T. A. McMahon, "The role of compliance in mammalian running gaits," *Journal of Experimental Biology*, vol. 115, no. 1, p. 263, 1985.

Animals Run Like Pogo-sticks

- Biologists discovered
  - all running animals
  - exhibit pogo-stick dynamics
- They used
  - the simplified dynamics
  - to classify
  - animal gait parameters

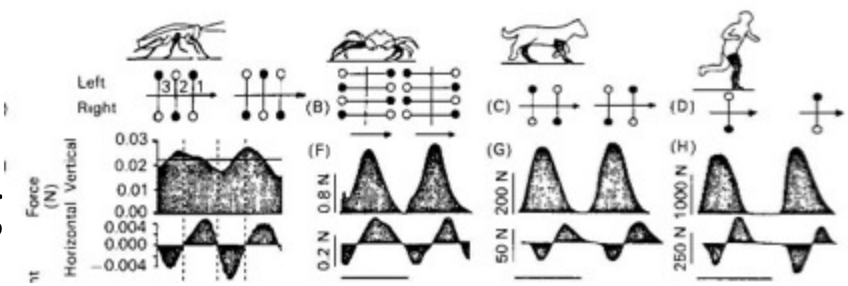


figure from:
R. J. Full, “Concepts of efficiency and economy in land locomotion.,” in *Efficiency and Economy in Animal Physiology*, R. W. Blake, Ed. Cambridge University Press, 1991, pp. 97–131.

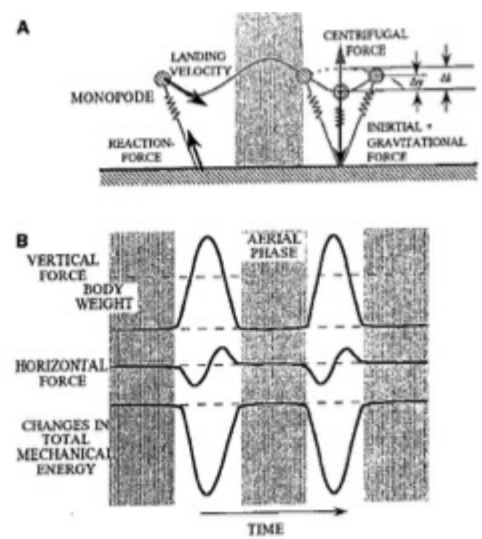


figure from:
R. Blickhan and R. J. Full, “Similarity in multilegged locomotion: Bouncing like a monopode,” *Journal of Comparative Physiology A: Sensory, Neural, and Behavioral Physiology*, vol. 173, no. 5, pp. 509–517, 1993.

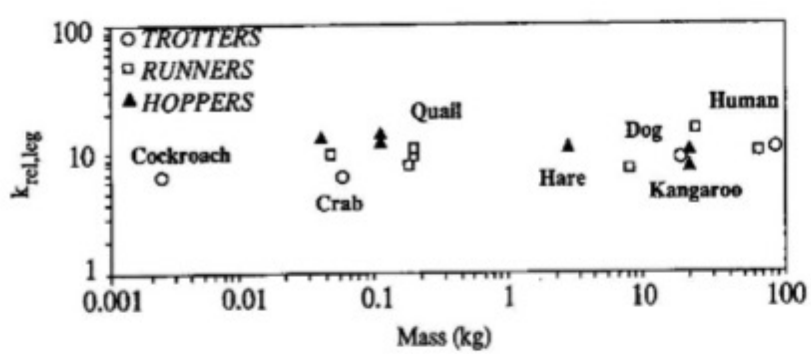
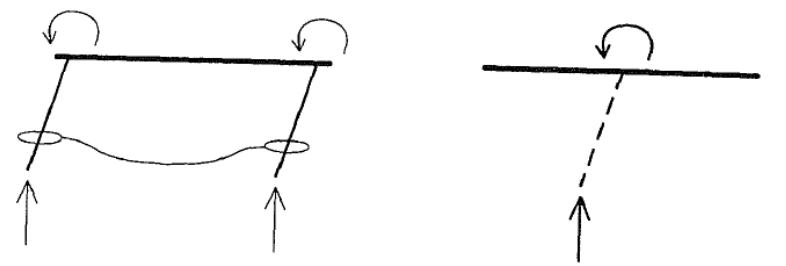


figure from:
R. J. Full and C. T. Farley, “Musculoskeletal dynamics in rhythmic systems: a comparative approach to legged locomotion,” in *Biomechanics and neural control of posture and movement*, J. M. Winters and P. Crago, Eds. Springer, 2000, pp. 192–205.

Raibert's Compositions

- Raibert showed empirically
  - 2DoF mass-spring controller
  - works for multiple legs
    - planar biped
    - spatial "virtual" biped
  - when coordinated properly
- Template/Anchor concept
  - begins to offer mathematical justification
  - still under active development



figures from:
M. H. Raibert, *Legged Robots That Balance*. Cambridge: MIT Press, 1986.

This Week

- Introduce SLIP template
  - "spring-loaded inverted pendulum"
  - 2 DoF Revolute-Prismatic chain
- Term from Schwind & Kod '95
  - to distinguish this specific "template" kinematics
  - from many other locomotion models
    - mostly higher DoF ("anchors")
    - a few alternative 2 DoF "templates"

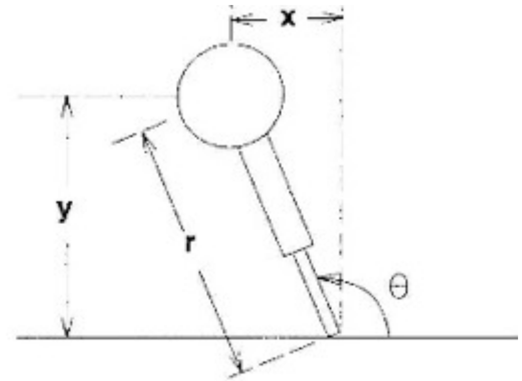


figure from:
W. J. Schwind and D. E. Koditschek, "Control of forward velocity for a simplified planar hopping robot," in *Robotics and Automation, 1995. Proceedings., 1995 IEEE International Conference on, 1995*, vol. 1, pp. 691–696.

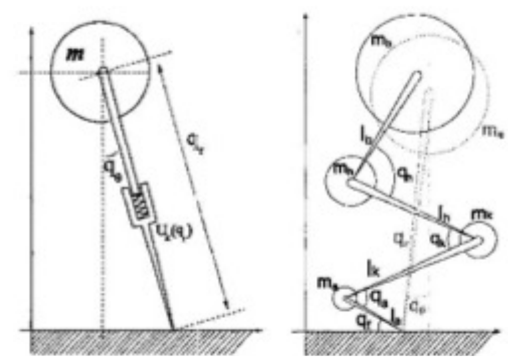


figure from:
U. Saranli, W. J. Schwind, and D. E. Koditschek, "Toward the control of a multi-jointed, monoped runner," in *Robotics and Automation, 1998. Proceedings. 1998 IEEE International Conference on*, 1998, vol. 3, pp. 2676–2682.

edX Robo4 Mini MS – Locomotion Engineering

Week 8 – Unit 1

Spring Loaded Inverted Pendulum

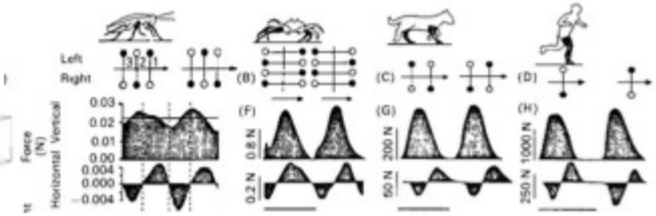
Video 9.2

Segment 8.0.2
SLIP in Biology and Robotics – Agenda

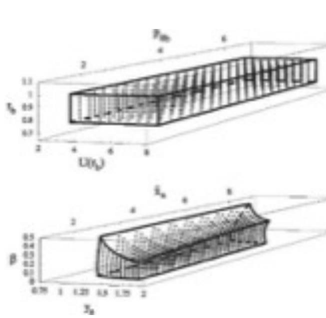
Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Where We Are Going (This Week & Next)

- This Week: SLIP running model
  - from bioinspiration
  - to modeling
- Next week: SLIP stepping
  - guards, mode flow maps, resets, p_{SLP}
  - fore-aft velocity stepping feedback control

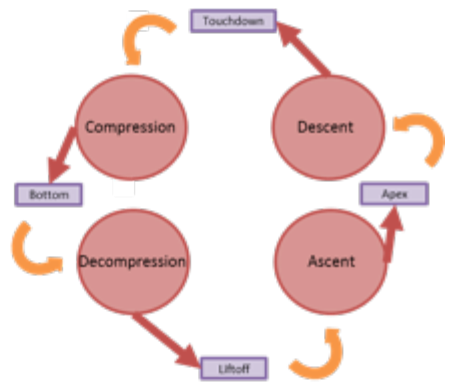


R. J. Full, Cambridge University Press, 1991, Op. Cit.

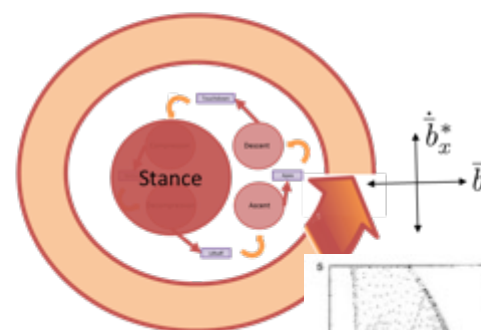


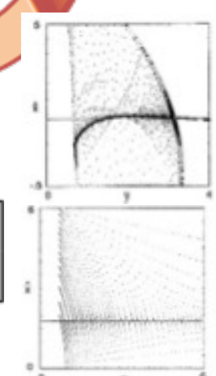
$$\begin{bmatrix} t(q_x) \\ q_\theta(q_x) \\ p_x(q_x) \\ p_\theta(q_x) \end{bmatrix} = \begin{bmatrix} t_0 + \frac{\mu q_{x0}^2}{p_{\theta 0}^2 + \mu k + q_{x0}^2 p_{x0}^2}\left[q_x p_x(q_x) - q_{x0} p_{x0}\right] \\ q_{\theta 0} + \frac{p_{\theta 0}}{\sqrt{p_{\theta 0}^2 + \mu k}} \operatorname{arccot}\left[\frac{p_{\theta 0}^2 + \mu k + q_{x0} p_{x0} q_x p_x(q_x)}{\sqrt{p_{\theta 0}^2 + \mu k}\,|q_x p_x(q_x) - q_{x0} p_{x0}|}\right] \\ \left[p_{x0}^2 + \frac{(L_0 + \mu k)(q_x^2 - q_{x0}^2)}{q_x^2 q_{x0}^2}\right]^{\frac{1}{2}} \\ p_{\theta 0} \end{bmatrix}$$

W. J. Schwind and D. E. Koditschek, "Approximating the Stance Map of a 2-DOF Monoped Runner," *Journal of Nonlinear Science*, vol. 10, no. 5, pp. 533–568, 2000



$$p_{\mathrm{SLP}}(\mathbf{x}_a) := g_{\mathrm{AP}} \circ m_{\mathrm{ASC}} \circ m_{\mathrm{Stance}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)$$



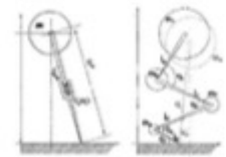


$$\begin{aligned} P_{\mathrm{F}}(\mathbf{x}^*) &:= D_{\mathbf{x}} p_{\mathrm{F}}(\mathbf{x}^*) \\ &= I_2 + ab^{\mathrm{T}} \\ a &:= \begin{bmatrix} 2k\bar{b}_x^* \sqrt{\bar{b}_y^* - \chi_\ell \sin g_{\mathrm{F}}(\mathbf{x}^*)} - \chi_\ell \cos g_{\mathrm{F}}(\mathbf{x}^*) \\ -\frac{1}{k}\sqrt{\bar{b}_y^* - \chi_\ell \sin g_{\mathrm{F}}(\mathbf{x}^*)} \end{bmatrix} \\ b^{\mathrm{T}} &:= D_{\mathbf{x}} \tilde{g}_{\mathrm{F}}(\mathbf{x}^*) \end{aligned}$$

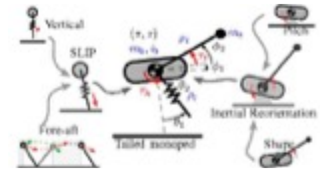
W. J. Schwind and D. E. Koditschek, IEEE, *1995 Op. Cit.*

Where We Are Going (Final Project Weeks)

- Based on a simulation study
  - from SLIP fore-aft stepping
  - to AKH 2DoF running
- Based on a physical robot
  - from SLIP/AKH running
  - to tail-energized Jerboa
- Toward real machines
  - Minitaur
  - Spatial Jerboa



U. Saranli, *et al.* , IEEE 1998, Op. Cit.



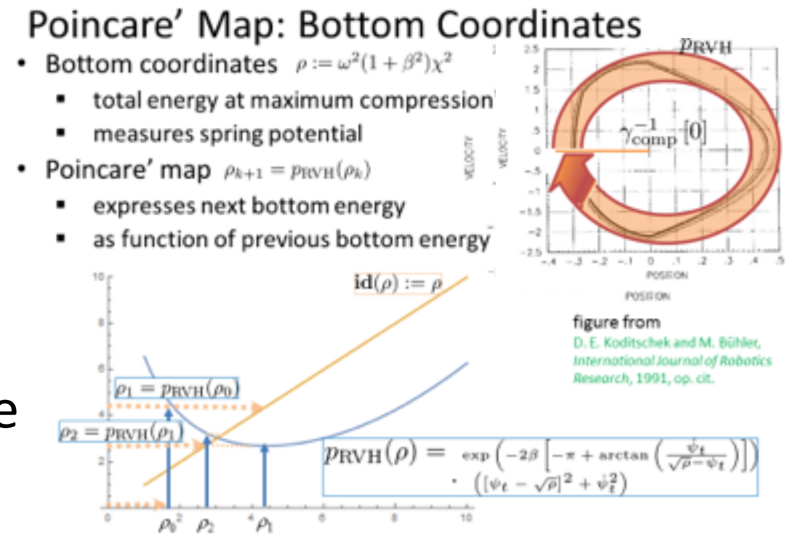
A. De and D. E. Koditschek, “Parallel composition of templates for tail-energized planar hopping,” in Robotics and Automation (ICRA), 2015 IEEE International Conference on, 2015, pp. 4562–4569.

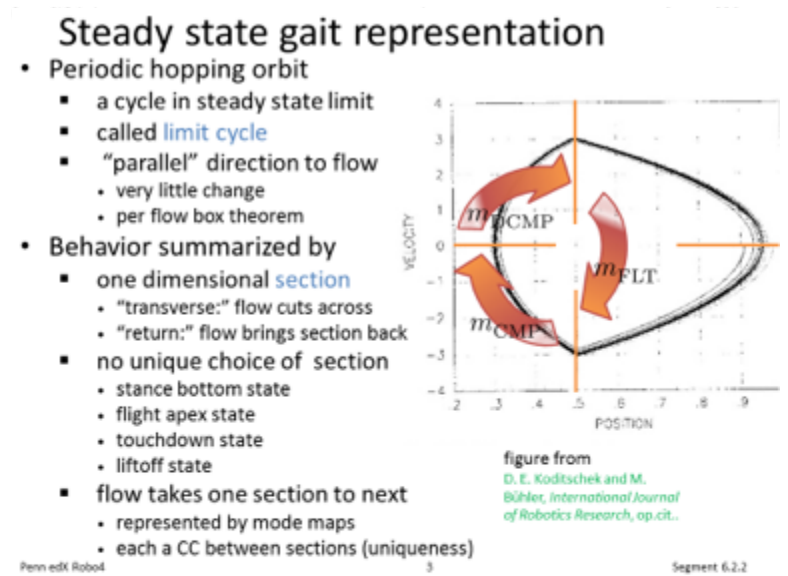
G. Kenneally, A. De, and D. E. Koditschek, “Design principles for a family of direct-drive legged robots,” IEEE Robotics and Automation Letters, vol. 1, no. 2, pp. 900–907, 2016.

A. L. Brill, A. De, A. M. Johnson, and D. E. Koditschek, “Tail-Assisted Rigid and Compliant Legged Leaping,” 2015 IEEE/RSJ International Conference on Intelligent Robots and Systems, Sep. 2015.

Where We Have Been (Vertical Hopper)

- Locomotion Model: Return Map
- Recall RVH units (Weeks 6 & 7)
  - Poincare' return map, p_{RVH}
    - models dynamics
    - of state (energy/height) at next stride
    - as function of previous
  - study stability of its FP to understand gait
- Recall how we derived p_{RVH}
    - composed the hybrid mode maps
    - depended upon closed form integration
      - of ballistic flight flow & td time
      - of stance de/compression flow & lo time

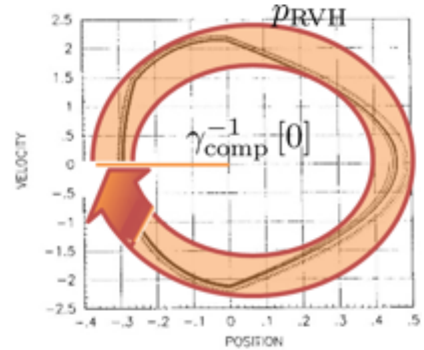




Where We Have Been (Closed Form Flows)

- Locomotion Model: Return Map

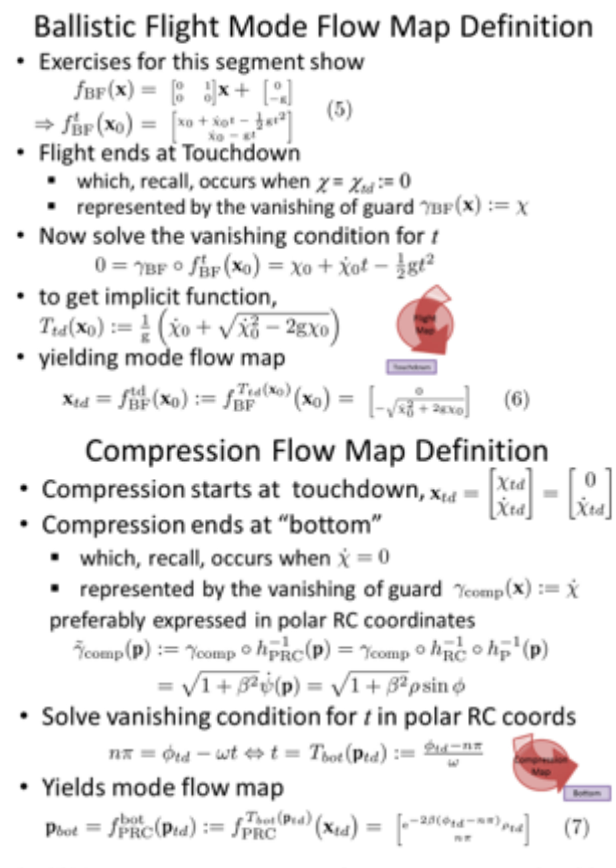
$$p_{\mathrm{RVH}}(\rho) = \exp\left(-2\beta\left[-\pi + \arctan\left(\frac{\dot{\psi}_t}{\sqrt{\rho}-\psi_t}\right)\right]\right) \cdot \left([\psi_t - \sqrt{\rho}]^2 + \dot{\psi}_t^2\right)$$



  - before: basis for stability analysis
  - now: basis for controller design
- How/Why got closed form flows?
  - superficial answer:

    both stance and flight are LTI
  - deeper answer:

    both are 1 DoF Hamiltonian



Ballistic Flight Mode Flow Map Definition

- Exercises for this segment show

$$f_{\mathrm{BF}}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

$$\Rightarrow f_{\mathrm{BF}}^t(\mathbf{x}_0) = \begin{bmatrix} \chi_0 + \dot{\chi}_0 t - \frac{1}{2}gt^2 \\ \dot{\chi}_0 - gt \end{bmatrix} \quad (5)$$

- Flight ends at Touchdown
  - which, recall, occurs when $\chi = \chi_{td} := 0$
  - represented by the vanishing of guard $\gamma_{\mathrm{BF}}(\mathbf{x}) := \chi$
- Now solve the vanishing condition for t

$$0 = \gamma_{\mathrm{BF}} \circ f_{\mathrm{BF}}^t(\mathbf{x}_0) = \chi_0 + \dot{\chi}_0 t - \frac{1}{2}gt^2$$

- to get implicit function,

$$T_{td}(\mathbf{x}_0) := \frac{1}{g}\left(\dot{\chi}_0 + \sqrt{\dot{\chi}_0^2 - 2g\chi_0}\right)$$

- yielding mode flow map

$$\mathbf{x}_{td} = f_{\mathrm{BF}}^{td}(\mathbf{x}_0) := f_{\mathrm{BF}}^{T_{td}(\mathbf{x}_0)}(\mathbf{x}_0) = \begin{bmatrix} 0 \\ -\sqrt{\dot{\chi}_0^2 + 2g\chi_0} \end{bmatrix} \quad (6)$$

Compression Flow Map Definition

- Compression starts at touchdown, $\mathbf{x}_{td} = \begin{bmatrix} \chi_{td} \\ \dot{\chi}_{td} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\chi}_{td} \end{bmatrix}$
- Compression ends at "bottom"
  - which, recall, occurs when $\dot{\chi} = 0$
  - represented by the vanishing of guard $\gamma_{\mathrm{comp}}(\mathbf{x}) := \dot{\chi}$

preferably expressed in polar RC coordinates

$$\tilde{\gamma}_{\mathrm{comp}}(\mathbf{p}) := \gamma_{\mathrm{comp}} \circ h_{\mathrm{PRC}}^{-1}(\mathbf{p}) = \gamma_{\mathrm{comp}} \circ h_{\mathrm{RC}}^{-1} \circ h_{\mathrm{P}}^{-1}(\mathbf{p})$$

$$= \sqrt{1+\beta^2}\,\dot{\psi}(\mathbf{p}) = \sqrt{1+\beta^2}\,\rho \sin\phi$$

- Solve vanishing condition for t in polar RC coords

$$n\pi = \phi_{td} - \omega t \Leftrightarrow t = T_{bot}(\mathbf{p}_{td}) := \frac{\phi_{td} - n\pi}{\omega}$$

- Yields mode flow map

$$\mathbf{p}_{bot} = f_{\mathrm{PRC}}^{\mathrm{bot}}(\mathbf{p}_{td}) := f_{\mathrm{PRC}}^{T_{bot}(\mathbf{p}_{td})}(\mathbf{x}_{td}) = \begin{bmatrix} e^{-2\beta(\phi_{td}-n\pi)}\rho_{td} \\ n\pi \end{bmatrix} \quad (7)$$

Penn edX Robo4 5 Segment 6.1.2

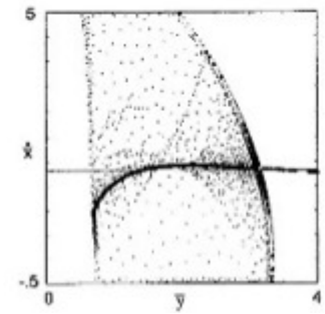
This Week: SLIP Running Model

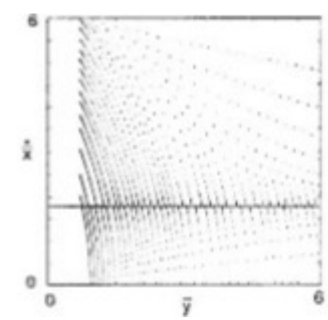
- Need Control Return Map
  - fore-aft velocity needs regulation
    - reserve the shank spring for height
    - left with stepping angle
  - seek a "plant model" $\mathbf{x}_{n+1} = p_{\mathrm{SLP}}(\mathbf{x}_n, \theta_n)$
    - use to develop feedback law $\theta_{n+1} = g_{\mathrm{F}}(\mathbf{x}_n)$
    - and study resulting closed loop dynamics

$$\mathbf{x}_{n+1} = p_{\mathrm{F}}(\mathbf{x}_n) := p_{\mathrm{SLP}}(\mathbf{x}_n, g_{\mathrm{F}}(\mathbf{x}_n))$$

- Need next two weeks to get there
  - this week develop approximate model
    - address fundamental road block
    - to closed form expressions for flow
  - next week: use it for analysis & design

$$\begin{aligned} P_{\mathrm{F}}(\mathbf{x}^*) &:= D_{\mathbf{x}} p_{\mathrm{F}}(\mathbf{x}^*) \\ &= I_2 + ab^{\mathrm{T}} \\ a &:= \begin{bmatrix} 2k\dot{\bar{b}}_x^* \sqrt{\bar{b}_y^* - \chi_\ell \sin g_{\mathrm{F}}(\mathbf{x}^*)} - \chi_\ell \cos g_{\mathrm{F}}(\mathbf{x}^*) \\ -\frac{1}{k}\sqrt{\bar{b}_y^* - \chi_\ell \sin g_{\mathrm{F}}(\mathbf{x}^*)} \end{bmatrix} \\ b^{\mathrm{T}} &:= D_{\mathbf{x}} \tilde{g}_{\mathrm{F}}(\mathbf{x}^*) \end{aligned}$$



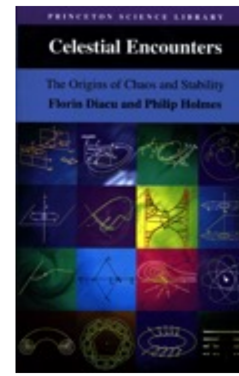


W. J. Schwind and D. E. Koditschek, IEEE, *1995 Op. Cit.*

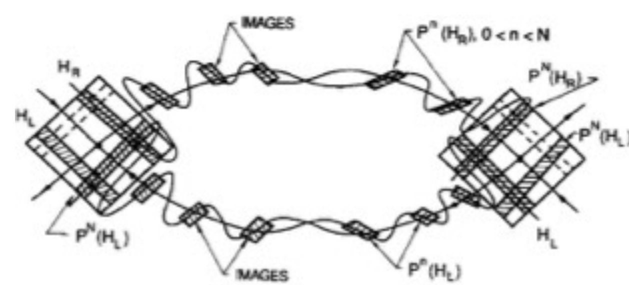
Addressing A Fundamental Road Block

- Problem: fore-aft DoF is coupled in stance
  - pinned toe becomes revolute joint
  - bioinspired running models have compliant shank
  - both polar DoFs contribute to speed
- Nonlinear 2 DoF Systems
  - Generally Non-Integrable
    - flows become “chaotic”
    - no closed form integrals can exist
  - Historical Precedent
    - find closed form approximations
    - this week: ignore stance gravity

Diacu & Holmes, P’ton U. Press, 1996.



Wikipedia



P. Holmes, “Poincaré, celestial mechanics, dynamical-systems theory and ‘chaos,’” Physics Reports, vol. 193, no. 3, pp. 137–163, Sep. 1990.

This Week

- Introduce SLIP template
  - "spring-loaded inverted pendulum"
  - 2 DoF Revolute-Prismatic chain
- Term from Schwind & Kod '95
  - to distinguish this specific "template" kinematics
  - from many other locomotion models
    - mostly higher DoF ("anchors")
    - a few alternative 2 DoF "templates"

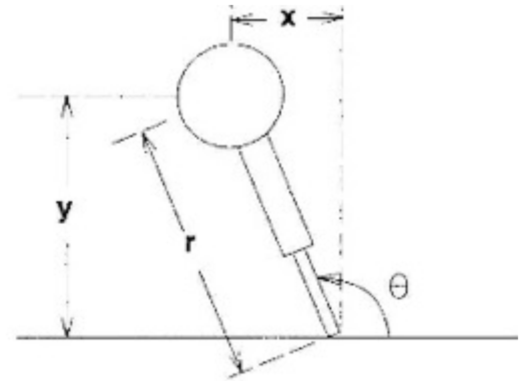


figure from:
W. J. Schwind and D. E. Koditschek, "Control of forward velocity for a simplified planar hopping robot," in *Robotics and Automation, 1995. Proceedings., 1995 IEEE International Conference on, 1995*, vol. 1, pp. 691–696.

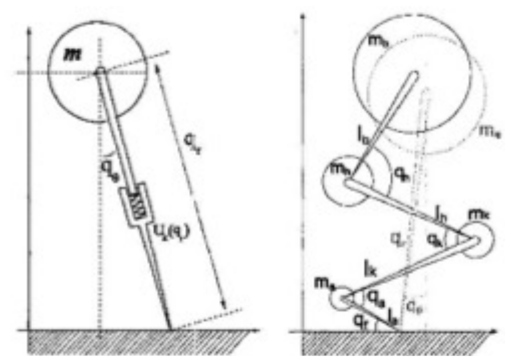


figure from:
U. Saranli, W. J. Schwind, and D. E. Koditschek, "Toward the control of a multi-jointed, monoped runner," in *Robotics and Automation, 1998. Proceedings. 1998 IEEE International Conference on*, 1998, vol. 3, pp. 2676–2682.

edX Robo4 Mini MS – Locomotion Engineering

Week 8 – Unit 1

Spring Loaded Inverted Pendulum

Video 9.3

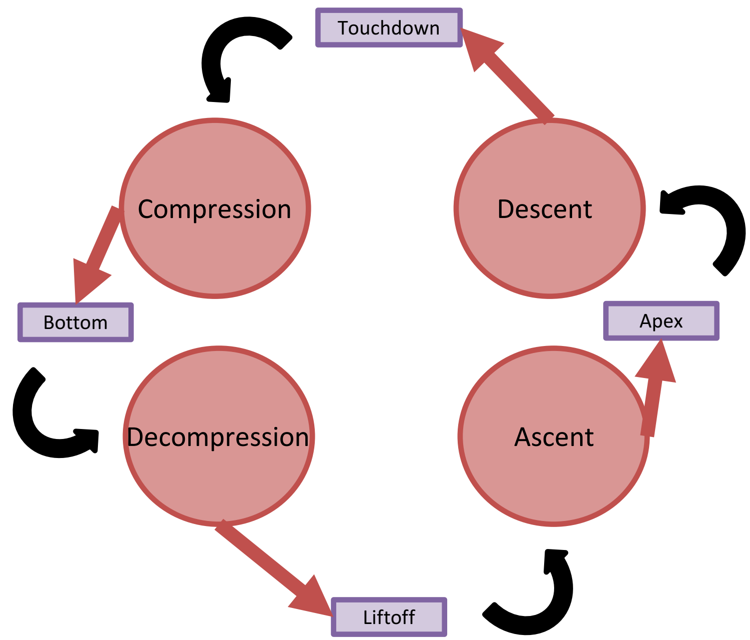
Segment 8.1.1

Continuous Time Dynamics

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Recall Approach to Vertical Hopper

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express mode map
- Model reset map
- Compose
  - mode map ◦ reset map
  - further compose each composition in turn
- End up with return map



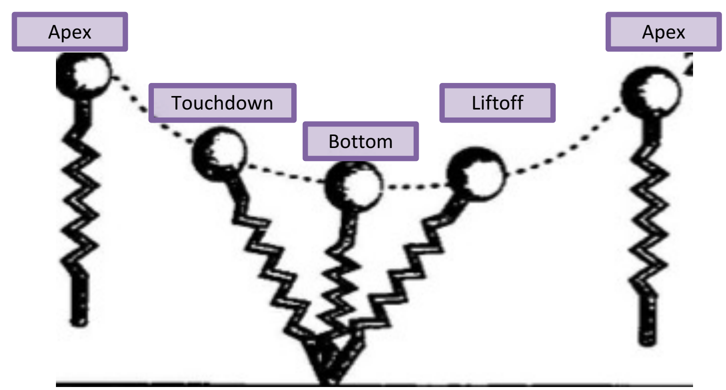
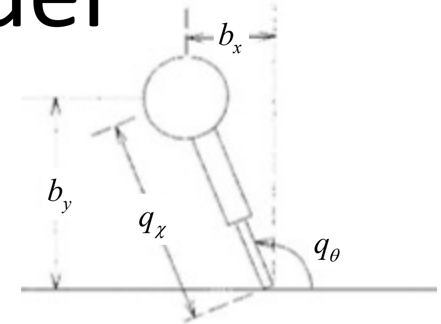


figure adapted from
W. J. Schwind, “Spring loaded inverted pendulum running: A plant model.,” University of Michigan, PhD Thesis,1998.

Planar Ballistic Flight Model

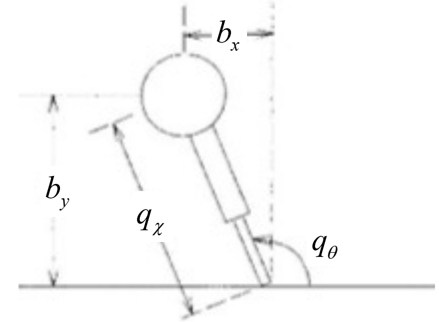


- Point mass μ body $\mathbf{b} := \begin{bmatrix} b \\ \dot{b} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \\ \begin{bmatrix} \dot{b}_x \\ \dot{b}_y \end{bmatrix} \end{bmatrix}$
- Kinetic energy $\kappa_{\mathrm{PBF}}(\mathbf{b}) = \frac{1}{2}\mu \dot{b}^{\mathrm{T}} \dot{b}$
- Gravitational potential $\varphi_{\mathrm{PBF}}(b) = \mu \mathrm{g} b_y$
- Lagrangian $\lambda_{\mathrm{PBF}}(\mathbf{b}) := \frac{1}{2}\mu \dot{b}^{\mathrm{T}} \dot{b} - \mu \mathrm{g} b_y$
- Euler-Lagrange operator

$$\begin{aligned} 0_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \left(\left[\frac{d}{dt} D_{\dot{b}} - D_b \right] \lambda_{\mathrm{PBF}} \right)^T \\ &= \left(\frac{d}{dt} D_{\dot{b}} \kappa_{\mathrm{PBF}} \right)^T - \Phi_{\mathrm{PBF}}(b); \quad \Phi_{\mathrm{PBF}} := -\left(D_b \varphi_{\mathrm{PBF}}\right)^T = -\begin{bmatrix} 0 \\ \mu \mathrm{g} \end{bmatrix} \\ &= \mu \ddot{b} - \Phi_{\mathrm{PBF}}(b) \end{aligned}$$

- Yields $\begin{bmatrix} \ddot{b}_x \\ \ddot{b}_y \end{bmatrix} = \Xi_{\mathrm{PBF}}(\mathbf{b}) := \frac{1}{\mu} \Phi_{\mathrm{PBF}}(b) = -\begin{bmatrix} 0 \\ \mathrm{g} \end{bmatrix} \qquad (1)$

RP Chain Kinematics



- Point mass μ body
  - Cartesian coordinates $b = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$
  - polar coordinates $q = \begin{bmatrix} q_\chi \\ q_\theta \end{bmatrix}$
- Related by kinematic polar CC

$$b = h_{\mathrm{RP}}^{-1}(q) := \begin{bmatrix} q_\chi \cos q_\theta \\ q_\chi \sin q_\theta \end{bmatrix}$$

  - compare to "phase" (energy-angle) CC h_{P} from Seg.6.1
  - check to verify $q = h_{\mathrm{RP}}(b) = \begin{bmatrix} \| b \| \\ \arctan b_y / b_x \end{bmatrix}$

RP Chain Infinitesimal Kinematics

- Newton: need velocities too
  - Cartesian coordinates $\mathbf{b} := \begin{bmatrix} b \\ \dot{b} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \\ \begin{bmatrix} \dot{b}_x \\ \dot{b}_y \end{bmatrix} \end{bmatrix}$
  - polar coordinates $\mathbf{q} := \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} q_\chi \\ q_\theta \end{bmatrix} \\ \begin{bmatrix} \dot{q}_x \\ \dot{q}_\theta \end{bmatrix} \end{bmatrix}$
- Related by infinitesimal kinematic polar CC

$$\mathbf{b} := \begin{bmatrix} b \\ \dot{b} \end{bmatrix} = \mathbf{h}_{\mathrm{RP}}^{-1}(\mathbf{q}) = \begin{bmatrix} h_{\mathrm{RP}}^{-1}(q) \\ Dh_{\mathrm{RP}}^{-1}(q)\ \dot{q} \end{bmatrix}$$

$$h_{\mathrm{RP}}^{-1}(q) := \begin{bmatrix} q_\chi \cos q_\theta \\ q_\chi \sin q_\theta \end{bmatrix}$$

$$\Rightarrow Dh_{\mathrm{RP}}^{-1}(q) = \begin{bmatrix} \cos q_\theta & -q_\chi \sin q_\theta \\ \sin q_\theta & q_\chi \cos q_\theta \end{bmatrix}$$

RP Chain Kinetic Energy Formulae

- Lagrangian needs kinetic energy
  - Cartesian coordinates $\kappa_{\mathrm{RPb}}(\mathbf{b}) = \frac{1}{2}\mu\|\dot{b}\|^2 = \frac{1}{2}\mu\dot{b}^{\mathrm{T}}\dot{b}$
  - polar coordinates $\kappa_{\mathrm{RP}}(\mathbf{q}) = \kappa_{\mathrm{RPb}} \circ \mathbf{h}_{\mathrm{KP}}^{-1}(\mathbf{q})$
- using infinitesimal kinematic polar CC

$$\kappa_{\mathrm{RP}}(\mathbf{q}) = \frac{1}{2}\mu\, \dot{q}^{\mathrm{T}} \left[Dh_{\mathrm{RP}}^{-1}(q) \right]^T Dh_{\mathrm{RP}}^{-1}(q)\, \dot{q}$$

$$= \frac{1}{2}\mu\, \dot{q}^{\mathrm{T}} M(q)\dot{q} = \frac{1}{2}\mu\left(\dot{q_\chi}^2 + q_\chi^2 \dot{q_\theta}^2\right)$$

$$Dh_{\mathrm{RP}}^{-1}(q) = \begin{bmatrix} \cos q_\theta & -q_\chi \sin q_\theta \\ \sin q_\theta & q_\chi \cos q_\theta \end{bmatrix} \Rightarrow M(q) := \mu \begin{bmatrix} 1 & 0 \\ 0 & q_\chi^2 \end{bmatrix}$$

RP Chain Potential Energy Formulae

- Lagrangian needs potential energy too
- Gravitational potential
  - Cartesian coordinates $\varphi_{\mathrm{PBF}}(b) = \mu g b_y$
  - polar coords $\varphi_{\mathrm{grav}}(q) = \varphi_{\mathrm{PBF}} \circ h_{\mathrm{KP}}^{-1}(q) = \mu g q_\chi \cos q_\theta$
- Introduce shank spring $\varphi_{\mathrm{S}}(q_\chi)$
  - choose specific type of spring shortly
  - purely radial "central" force (no toe torque!)
- Add up to get complete potential energy

$$\begin{aligned}\varphi_{\mathrm{RP}}(q) &= \varphi_{\mathrm{grav}}(q) + \varphi_{\mathrm{S}}(q) \\ &= \mu g q_\chi \cos q_\theta + \varphi_{\mathrm{S}}(q_\chi)\end{aligned}$$

RP Chain Lagrangian Formulae

- Lagrangian (polar cords) $\lambda_{\mathrm{RP}}(\mathbf{q}) := \kappa_{\mathrm{RP}}(\mathbf{q}) - \varphi_{\mathrm{RP}}(q)$
- Niggling Details (to be or not to be “sloppy”)
  - formally speaking, should write as

$$\lambda_{\mathrm{RP}}(\mathbf{q}) := \kappa_{\mathrm{RP}}(\mathbf{q}) - \tilde{\varphi}_{\mathrm{RP}}(\mathbf{q})$$

$$\tilde{\varphi}_{\mathrm{RP}} := \varphi_{\mathrm{RP}} \circ \Pi_q$$

$$\Pi_q(\mathbf{b}) := [I_{2\times 2}, 0_{2\times 2}] \begin{bmatrix} \begin{bmatrix} q_\chi \\ q_\theta \end{bmatrix} \\ \begin{bmatrix} \dot{q}_x \\ \dot{q}_\theta \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} q_\chi \\ q_\theta \end{bmatrix} = q$$

  - mathematics: common domain/codomain overloading
  - programming: common use of un-typed variables
- We’ll (mainly) stay sloppy

RP Chain Lagrangian Mechanics

- Lagrangian (polar cords) $\lambda_{\mathrm{RP}}(\mathbf{q}) := \kappa_{\mathrm{RP}}(\mathbf{q}) - \varphi_{\mathrm{RP}}(q)$

$$0_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left(\left[\frac{d}{dt} D_{\dot{q}} - D_q \right] \left[\kappa_{\mathrm{RP}} - \varphi_{\mathrm{RP}} \right] \right)^T$$

$$= \left(\frac{d}{dt} D_{\dot{q}} \kappa_{\mathrm{RP}} - D_q \kappa_{\mathrm{RP}} \right)^T - \Phi_{\mathrm{RP}}(q); \qquad \Phi_{\mathrm{RP}} := -\left(D_q \varphi_{\mathrm{RP}}\right)^T$$

$$= \left(\frac{d}{dt} \dot{q}^{\mathrm{T}} M(q) - \frac{1}{2} \dot{q}^{\mathrm{T}} \left[D_q M(q) \dot{q} \right] \right)^T - \Phi_{\mathrm{RP}}(q)$$

$$= M(q)\ddot{q} + \dot{M}(q)\dot{q} - \left(\dot{q}^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ \mu q_\chi \dot{q}_\theta & 0 \end{bmatrix} \right)^T - \Phi_{\mathrm{RP}}(q)$$

$$= M(q)\ddot{q} - B(q, \dot{q})\dot{q} - \Phi_{\mathrm{RP}}(q)$$

$$B(q, \dot{q}) := \mu q_\chi \begin{bmatrix} 0 & -\dot{q}_\theta \\ \dot{q}_\theta & -\dot{q}_\chi \end{bmatrix}$$

$$= -\frac{1}{2} \dot{M}(q) + \mu q_\chi \dot{q}_\theta J_2; \qquad J_2 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

RP Chain Dynamics

- Lagrangian mechanics $M(q)\ddot{q} = B(q,\dot{q})\dot{q} + \Phi_{\mathrm{RP}}(q)$
- Invertible inertia (assuming some extension)

$$M(q) := \mu \begin{bmatrix} 1 & 0 \\ 0 & q_\chi^2 \end{bmatrix} \Rightarrow M^{-1}(q) = \begin{bmatrix} 1 & 0 \\ 0 & 1/q_\chi^2 \end{bmatrix}$$

- Yields

$$\ddot{q} = M^{-1}(q)\left[B(q,\dot{q})\dot{q} + \Phi_{\mathrm{RP}}(q)\right] =: \Xi_{\mathrm{RP}}(\mathbf{q})$$

- Where

$$\begin{bmatrix} \ddot{q}_\chi \\ \ddot{q}_\theta \end{bmatrix} = \Xi_{\mathrm{RP}}(\mathbf{q}) = \begin{bmatrix} q_\chi \dot{q_\theta}^2 - g\cos q_\theta + D_{q_\chi}\varphi_{\mathrm{S}}(q_\chi) \\ -\frac{2}{q_\chi}\dot{q_\chi}\dot{q_\theta} + \frac{g}{q_\chi}\sin q_\theta \end{bmatrix}$$

edX Robo4 Mini MS – Locomotion Engineering

Week 8 – Unit 1

Spring Loaded Inverted Pendulum

Video 9.4

Segment 8.1.2

Continuous Time Models – Planar Ballistic Flight VF & Flow

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Recall Approach to Vertical Hopper

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express mode map
- Model reset map
- Compose
  - mode map ◦ reset map
  - further compose each composition in turn
- End up with return map

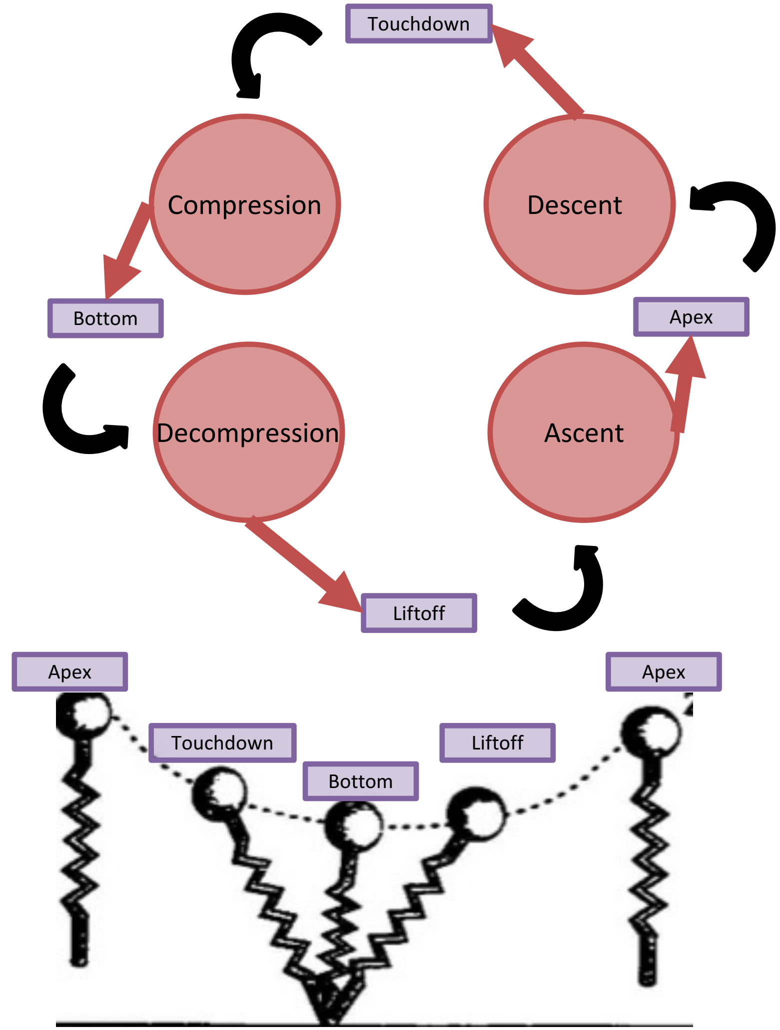


figure adapted from
W. J. Schwind, “Spring loaded inverted pendulum running: A plant model.,” University of Michigan, PhD Thesis,1998.

Planar Ballistic Flight Conjugate Momenta

- Cartesian conjugate momenta

$$p_b{}^{\mathrm{T}} := D_{\dot{b}}\kappa_{\mathrm{PBF}}(\mathbf{b}) = \frac{\mu}{2} D_{\dot{b}}\|\dot{b}\|^2 = \mu\dot{b}^{\mathrm{T}}$$

- Cartesian conjugate momentum coordinates

$$\mathbf{p}_b := \begin{bmatrix} b \\ p_b \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \\ \begin{bmatrix} p_x \\ p_y \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \\ \begin{bmatrix} \mu\dot{b}_x \\ \mu\dot{b}_y \end{bmatrix} \end{bmatrix} =: \mathbf{h}_{\mathrm{CCM}}(\mathbf{b})$$

- Hamiltonian total energy

$$\begin{aligned} \eta_{\mathrm{PBF}}(\mathbf{p}_b) &= \kappa_{\mathrm{PBF}} \circ \mathbf{h}_{\mathrm{CCM}}^{-1}(\mathbf{p}_b) + \varphi_{\mathrm{PBF}}(b) \\ &= \frac{1}{2\mu} p_b{}^{\mathrm{T}} p_b + \varphi_{\mathrm{PBF}}(b) \\ &= \frac{1}{2\mu}\left(p_x^2 + p_y^2\right) + \mu g b_y \end{aligned}$$

Planar Ballistic Flight Hamiltonian VF

- Hamiltonian reformulation of Lagrangian mechanics
  - Euler-Lagrange

$$\frac{d}{dt} D_{\dot{b}} \lambda_{\mathrm{PBF}} = D_b \lambda_{\mathrm{PBF}}$$
$$\Downarrow$$
$$\frac{d}{dt} D_{\dot{b}} \kappa_{\mathrm{PBF}} = -D_b \varphi_{\mathrm{PBF}}$$
$$\Downarrow$$
$$\frac{d}{dt} {p_b}^{\mathrm{T}} = -\left[D_b \eta_{\mathrm{PBF}}\right]^T$$

  - CC and energy def

$$\begin{aligned}\frac{d}{dt} b &= \frac{1}{\mu} p_b \\ &= \left[D_{p_b} \frac{1}{2\mu} {p_b}^{\mathrm{T}} p_b\right]^T \\ &= \left[D_{p_b} \kappa_{\mathrm{PBF}} \circ \mathbf{h}_{\mathrm{CCM}}^{-1}\right]^T \\ &= \left[D_{p_b} \eta_{\mathrm{PBF}}\right]^T\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{p}}_b = \begin{bmatrix} \dot{b} \\ \dot{p}_b \end{bmatrix} &= \begin{bmatrix} \left[D_{p_b} \eta_{\mathrm{PBF}}\right]^T \\ -\left[D_b \eta_{\mathrm{PBF}}\right]^T \end{bmatrix} (\mathbf{p}_b) \\ &= \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \left[D_b \eta_{\mathrm{PBF}}\right]^T \\ \left[D_{p_b} \eta_{\mathrm{PBF}}\right]^T \end{bmatrix} (\mathbf{p}_b) \\ &= J_4 \left[D_{\mathbf{p}_b} \eta_{\mathrm{PBF}}\right]^T (\mathbf{p}_b) \\ &= \begin{bmatrix} p_x/\mu \\ p_y/\mu \\ 0 \\ -\mu \mathrm{g} \end{bmatrix} \\ &=: f_{\mathrm{PBF}}(\mathbf{p}_b)\end{aligned}$$

Planar Ballistic Flight Conservation Laws

- Hamiltonian immediately yields energy conservation
  - since power formula

$$\begin{aligned}\dot{\eta}_{\mathrm{PBF}}\left(\mathbf{p}_b\right) &= \left(D_{\mathbf{p}_b}\eta_{\mathrm{PBF}} \cdot f_{\mathrm{PBF}}\right)\left(\mathbf{p}_b\right) \\ &= \left(D_{\mathbf{p}_b}\eta_{\mathrm{PBF}} \cdot J_4 \cdot \left[D_{\mathbf{p}_b}\eta_{\mathrm{PBF}}\right]^T\right)\left(\mathbf{p}_b\right) \\ &\equiv 0\end{aligned}$$

  - is governed by skew-symmetric $J_4 := \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}$
- Another conserved quantity (horizontal momentum)
  - revealed by cyclic ("Langrangian invariant") coordinate b_x

$$\frac{d}{dt}p_x = D_{b_x}\lambda_{\mathrm{PBF}} = D_{b_x}\left(\frac{1}{2\mu}\left(p_x^2 + p_y^2\right) - \mu g b_y\right) = 0$$

- Two conserved quantities in 2 DoF mechanical system
  - implies completely integrable dynamics
  - i.e., flow can be expressed in closed form

Planar Ballistic Flight Flow

- Restrict analysis to modes where $\dot{b}_y \neq 0$
  - shortly below introduce appropriate guards
  - yielding "ascent" to and "descent" from apex
- Allows replacement of t by b_y as independent variable
  - use conservation laws to flow conjugate momenta forward

$$p_b(b_y) = \begin{bmatrix} p_x(b_y) \\ p_y(b_y) \end{bmatrix} = \begin{bmatrix} p_{x0} \\ \pm\sqrt{p_{y0}^2 + 2\mu^2 \mathrm{g}(b_{y0} - b_y)} \end{bmatrix}$$

  - exploit simplicity of dynamics to flow t forward

$$\dot{p}_y = -\mu \mathrm{g} \Rightarrow p_y(b_y) - p_{y0} = -\mu \mathrm{g} t$$

$$\Rightarrow t(b_y) = \frac{p_y(b_y) - p_{y0}}{\mu \mathrm{g}}$$

  - and, in turn, to flow b_x as well

$$\dot{b}_x = \frac{1}{\mu} p_x \Rightarrow b_x(b_y) = b_{x0} + \frac{p_{x0}}{\mu} t(b_y)$$

edX Robo4 Mini MS – Locomotion Engineering

Week 8 – Unit 1

Spring Loaded Inverted Pendulum

Video 9.5

Segment 8.1.3

Continuous Time Models – RP Chain VF & Flow

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
August, 2017

This Segment: Stance Mode

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express mode map
- Model reset map
- Compose
  - mode map ◦ reset map
  - further compose each composition in turn
- End up with return map

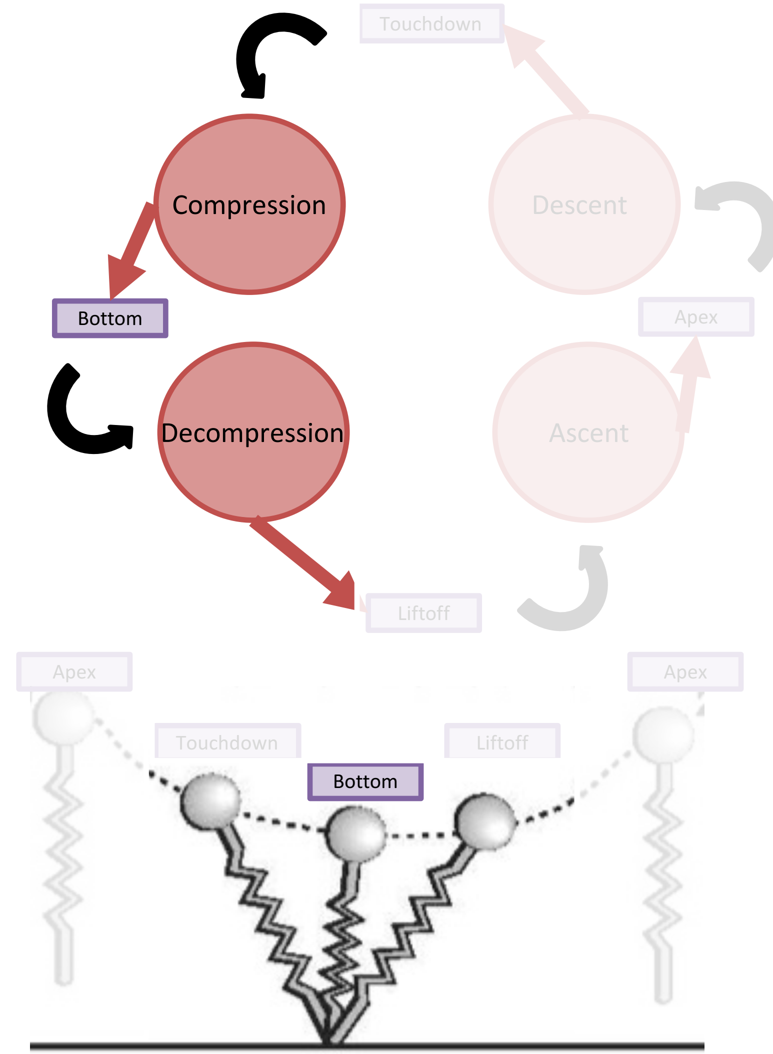


figure adapted from

W. J. Schwind and D. E. Koditschek, "Approximating the Stance Map of a 2-DOF Monoped Runner," *Journal of Nonlinear Science*, vol. 10, no. 5, pp. 533–568, 2000.

RP Chain Conjugate Momenta

- Polar conjugate momenta

$$p_q{}^{\mathrm{T}} := D_{\dot{q}}\kappa_{\mathrm{RB}}(\mathbf{q}) = \frac{1}{2}D_{\dot{q}}\left(\dot{q}^{\mathrm{T}}M(q)\dot{q}\right) = \left(M(q)\dot{q}\right)^T$$

- Polar conjugate momentum coordinates

$$\mathbf{p}_q := \begin{bmatrix} q \\ p_q \end{bmatrix} = \begin{bmatrix} q \\ M(q)\dot{q} \end{bmatrix} =: \mathbf{h}_{\mathrm{PCM}}(\mathbf{q})$$

- Hamiltonian total energy

$$\begin{aligned} \eta_{\mathrm{RP}}(\mathbf{p}_q) &= \kappa_{\mathrm{RP}} \circ \mathbf{h}_{\mathrm{PCM}}^{-1}(\mathbf{p}_q) + \varphi_{\mathrm{RP}}(q) \\ &= \frac{1}{2}p_q{}^{\mathrm{T}}M^{-1}(q)p_q + \varphi_{\mathrm{RP}}(q) \\ &= \left(p_\chi^2 + \frac{p_\theta^2}{q_\chi^2}\right) + \varphi_{\mathrm{RP}}(q) \end{aligned}$$

RP Chain Hamiltonian VF

- Hamiltonian reformulation of Lagrangian mechanics
  - developed in previous material (e.g., Seg. 3.1.2, 8.1.2)
  - same pattern as for Cartesian flight

$$\dot{\mathbf{p}}_q = \begin{bmatrix} \dot{q} \\ \dot{p}_q \end{bmatrix} = J_4 \left[D_{\mathbf{p}_q} \eta_{\mathrm{RP}} \right]^T (\mathbf{p}_q)$$

$$= \begin{bmatrix} p_\chi / \mu \\ p_\theta / \mu q_\chi^2 \\ p_\theta^2 / \mu q_\chi^3 - D_{q_\chi} \varphi_{\mathrm{S}}(q_\chi) - \mu \mathrm{g} \cos p_\theta \\ -\mu \mathrm{g}\, q_\chi \cos p_\theta \end{bmatrix}$$

$$=: f_{\mathrm{RP}}(\mathbf{p}_q)$$

- Energy conservation: $\dot{\eta}_{\mathrm{RP}} = D\eta_{\mathrm{RP}} \cdot J_4 \cdot \left[D\eta_{\mathrm{RP}} \right]^T \equiv 0$
- Cyclic coordinates? $\frac{d}{dt} p_q = D_q \varphi_{\mathrm{RP}} = ?$

Central Force RP Dynamics Approximation

- Assume spring potential dominates gravity

$$\varphi_{\mathrm{RP}}(q) = \cancelto{0}{g}\mu q_\chi \cos q_\theta + \varphi_{\mathrm{S}}(q_\chi)$$

  - getting a “central” force (varies in extension only)
  - Implies q_θ is a cyclic coordinate (integrable!)
- Introduced to SLIP by M’Closkey & Burdick ’93
  - focused on period doubling bifurcations
  - needed good (closed form) approximation to Poincare’ map
- Extended by many subsequent researchers

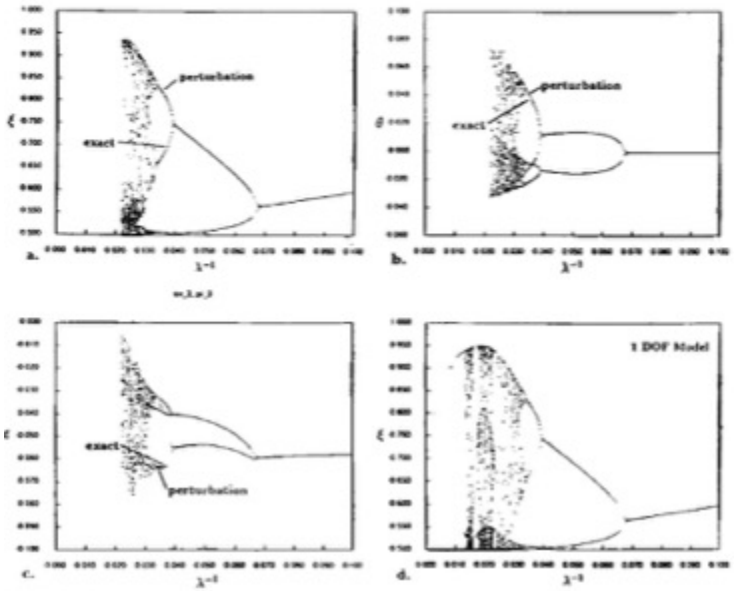
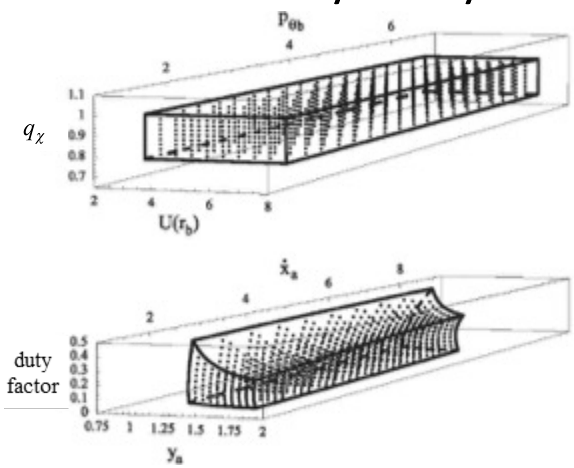
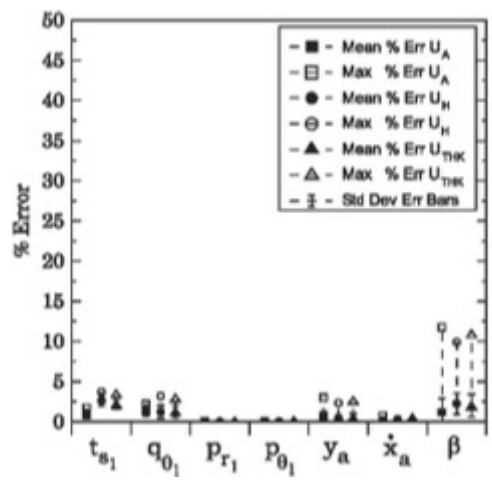


figure from

R. T. M’Closkey and J. W. Burdick, “Periodic motions of a hopping robot with vertical and forward motion,” *The International journal of robotics research*, vol. 12, no. 3, pp. 197–218, 1993.





figures from

W. J. Schwind and D. E. Koditschek, “Approximating the Stance Map of a 2-DOF Monoped Runner,” *Journal of Nonlinear Science*, vol. 10, no. 5, pp. 533–568, 2000.

.

0-Grav Approximation of RP Chain Flow

- Restrict analysis to modes where $\dot{q}_\chi \neq 0$
  - shortly below introduce appropriate guards
  - yielding "compression" to and "decompression" from bottom
- Allows replacement of t by q_χ as independent variable
  - use conservation laws to flow conjugate momenta forward

$$p_q(q_\chi) = \begin{bmatrix} p_\chi(q_\chi) \\ p_\theta(q_\chi) \end{bmatrix} = \begin{bmatrix} \left[2\mu\left[\varphi_{\mathrm{S}}(q_{\chi 0}) - \varphi_{\mathrm{S}}(q_\chi)\right] + p_{\theta 0}^2\left(\frac{1}{q_{\chi 0}^2} - \frac{1}{q_\chi^2}\right)\right]^{1/2} \\ p_{\theta 0} \end{bmatrix}$$

  - flow t forward by direct integration

$$\frac{dt}{dq_\chi} = 1/\frac{dq_\chi}{dt} \Rightarrow t(q_\chi) = t(q_{\chi 0}) + \int_{q_{\chi 0}}^{q_\chi} \frac{\mu}{p_\chi(\chi)} d\chi$$

  - and, in turn, to flow q_θ as well

$$\frac{dq_\theta}{dq_\chi} = \frac{dq_\theta}{dt}\frac{dt}{dq_\chi} = \dot{q}_\theta / \dot{q}_\chi \Rightarrow q_\theta(q_\chi) = q_\theta(q_{\chi 0}) + \int_{q_{\chi 0}}^{q_\chi} \frac{p_{\theta 0}}{\chi^2 p_\chi(\chi)} d\chi$$

Closed Form 0-Grav Approximation

- What does it mean to be "integrable"
  - Math/physics: reduction to "elliptic" integral
  - Robotics/control: need closed form analytical expression
- In general, need further approximation of integrals
- For purposes of this course, pick special spring law
  - air spring from vertical hopper unit (max. extension χ_l)

$$\varphi_{\mathrm{AS}}(q_\chi) := \frac{1}{2}k\left(\frac{1}{q_\chi^2} - \frac{1}{\chi_l^2}\right) \Rightarrow \Phi_{\mathrm{AS}}(q_\chi) := -D\varphi_{\mathrm{AS}}\left(q_\chi\right) = \frac{1}{q_\chi^3}$$

  - yields closed form expressions for elliptic integrals, hence flow: (branch selected by compression or decompression mode)

$$\begin{bmatrix} t(q_\chi) \\ q_\theta(q_\chi) \\ p_\chi(q_\chi) \\ p_\theta(q_\chi) \end{bmatrix} = \begin{bmatrix} t_0 + \frac{\mu q_{\chi 0}^2}{p_{\theta 0}^2 + \mu k + q_{\chi 0}^2 p_{\chi 0}^2}\left[q_\chi p_\chi(q_\chi) - q_{\chi 0} p_{\chi 0}\right] \\ q_{\theta 0} + \frac{p_{\theta 0}}{\sqrt{p_{\theta 0}^2 + \mu k}} \operatorname{arccot}\left[\frac{p_{\theta 0}^2 + \mu k + q_{\chi 0} p_{\chi 0} q_\chi p_\chi(q_\chi)}{\sqrt{p_{\theta 0}^2 + \mu k}\left[q_\chi p_\chi(q_\chi) - q_{\chi 0} p_{\chi 0}\right]}\right] \\ \pm\left[p_{\chi 0}^2 + \frac{(p_{\theta 0} + \mu k)(q_\chi^2 - q_{\chi 0}^2)}{q_\chi^2 q_{\chi 0}^2}\right]^{\frac{1}{2}} \\ p_{\theta 0} \end{bmatrix}$$