Video 8.1

Vijay Kumar

Definitions

State

$$x \in \mathbb{R}^n$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

State equations

$$\dot{x} = f(x)$$

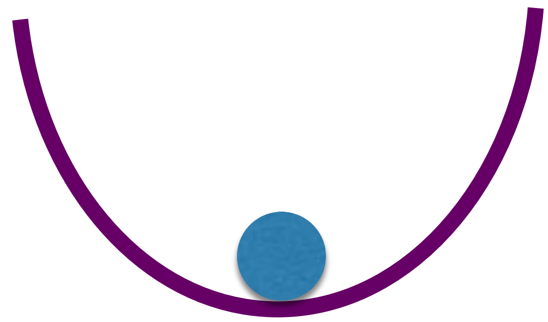
Equilibrium

$$x_e = \begin{bmatrix} q_e \\ 0 \end{bmatrix}$$

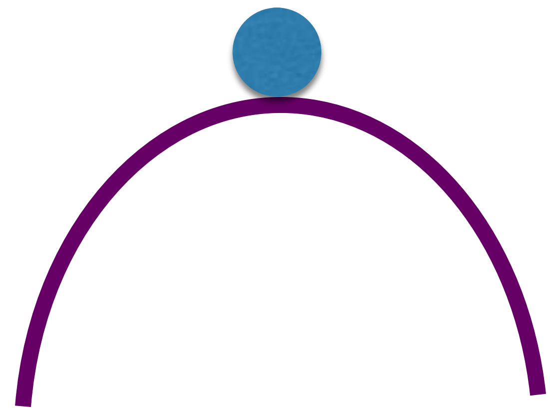
$$f(x_e) = 0$$

Stability

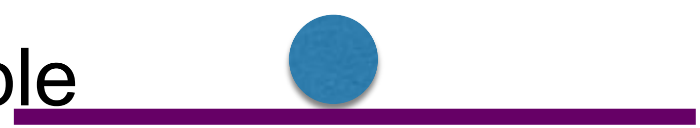
- Stable



- Unstable



- Neutrally (Critically) Stable

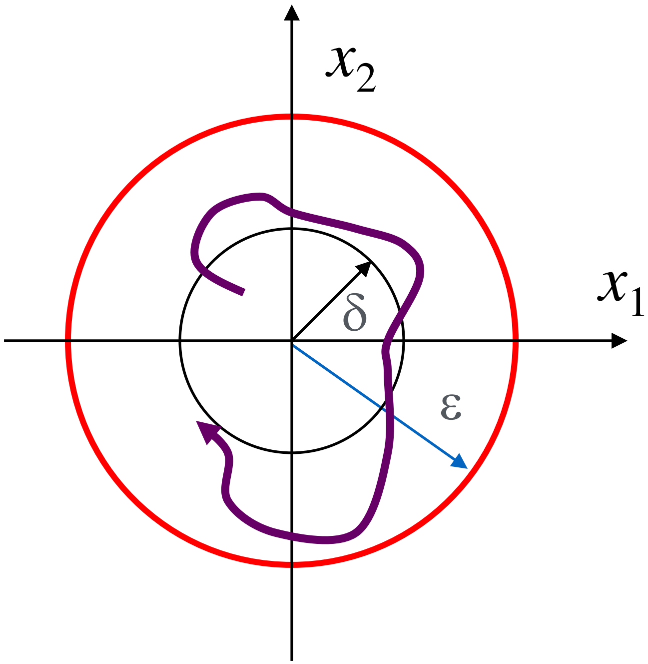


Stability

Translate the origin to x_e

$x(t)$ =0 is **stable** (Lyapunov stable) if and only if for any ε > 0, there exists a δ(ε) > 0 such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \leq \epsilon, \; \forall t > t_0$$

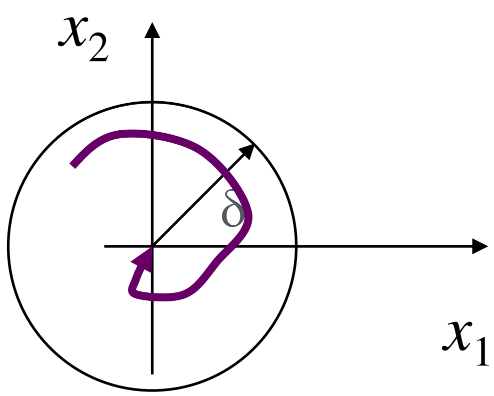


$x(t)$ =0 is **asymptotically stable** if and only if it is stable *and* there exists a δ > 0 such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \to 0;$$

$$\text{as } t \to \infty$$

Asymptotic Stability



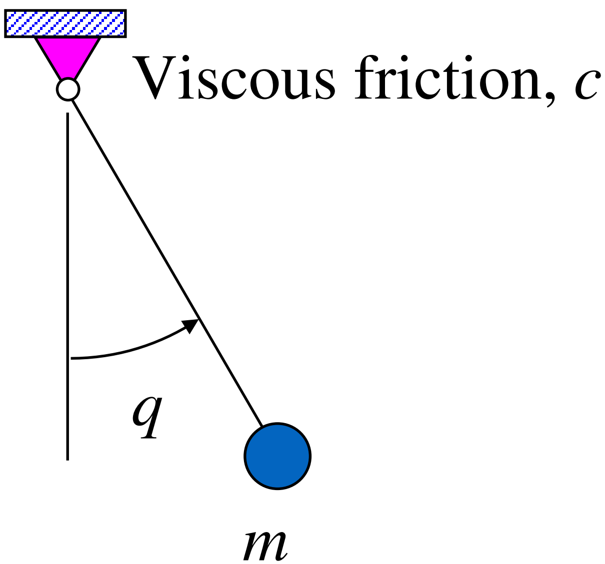
$x(t)$ =0 is **asymptotically stable** if and only if it is stable *and* there exists a δ > 0 such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \to 0;$$

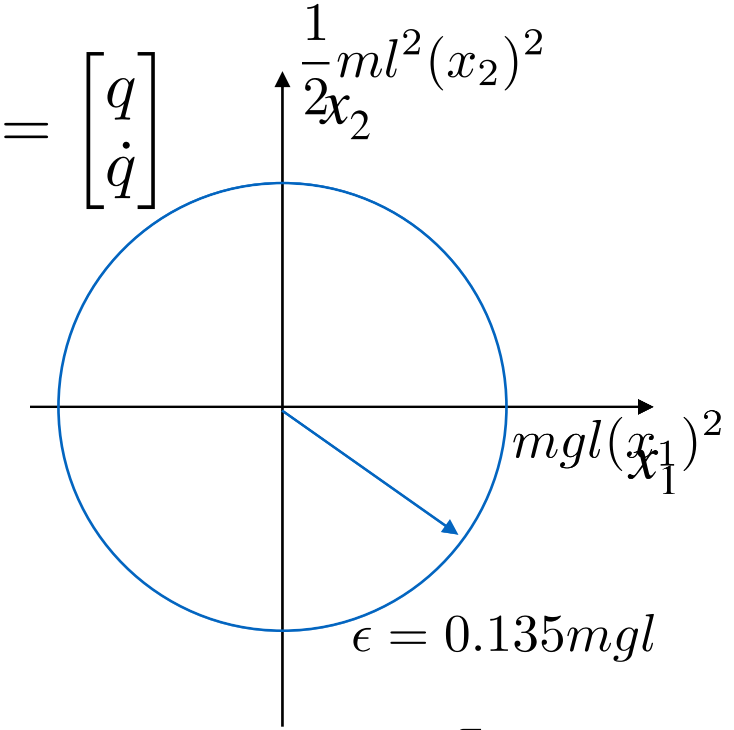
$$\text{as } t \to \infty$$

$x(t)$ =0 is **globally asymptotically stable** if and only if it is asymptotically stable *and* it is independent of $x(t_0)$

Example



$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$



$$E = T + V = \frac{1}{2}ml^2(x_2)^2 + mgl(1 - \cos x_1)$$

$E(t)$ cannot increase

$$E \leq \frac{1}{2}ml^2(x_2)^2 + mgl(x_1)^2$$

Suppose you want $x_1 < \frac{\pi}{6}$

$$E \leq \frac{1}{2}ml(0)^2 + mgl\left(1 - \cos\frac{\pi}{6}\right)$$

$$0.134mgl$$

Global Asymptotic Stability of Linear Systems

$$\dot{x} = f(x) \qquad f(x) = Ax$$

- Global Asymptotic Stability

  if and only if the real parts of all eigenvalues of A *are negative*

- Lyapunov Stability, not Global Asymptotic Stability

  if and only if the real parts of all eigenvalues are non positive, and zero eigenvalue is not repeated

- Unstable

  if and only if there is one eigenvalue of A whose real part is positive

Linear Autonomous Systems

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j \Rightarrow \dot{x} = Ax$$

Solution

eigenvectors

eigenvalues

$$x(t) = e^{(t-t_0)A} x_0 \quad \Rightarrow \quad x(t) = P e^{(t-t_0)\Lambda} P^{-1}$$

for non defective *A*

but similar story for defective A

Exponential of a matrix, *X*

$$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \ldots$$

Eigenvalues and eigenvectors for non defective *X*

$$X p_i = \lambda_i p_i$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 & p_2 & \ldots p_n \end{bmatrix}$$

$$e^X = P e^{\Lambda} P^{-1}$$

ONLINE LEARNING

Video 8.2

Vijay Kumar

Stability of "Almost Linear" Systems

$$\dot{x} = f(x) \qquad f(x) \sim Ax$$

- Global Asymptotic Stability

  if and only if the real parts of all eigenvalues of A are negative

- Lyapunov Stability, not Global Asymptotic Stability

  if and only if the real parts of all eigenvalues are non positive, and zero eigenvalue is not repeated

  Not Significant

  Significant dynamics

- Unstable

  if and only if there is one eigenvalue of A whose real part is positive

Lyapunov's theorem

- Nonlinear, autonomous systems
- Near equilibrium points

If the linearized system exhibits significant behavior, then the stability characteristics of the nonlinear system near the equilibrium point are the same as that of the linear system.

Example

- Equation of motion

$$\ddot{q} + \frac{x}{ml^2}\dot{q} + \frac{g}{l}\sin q = 0$$

- State space representation

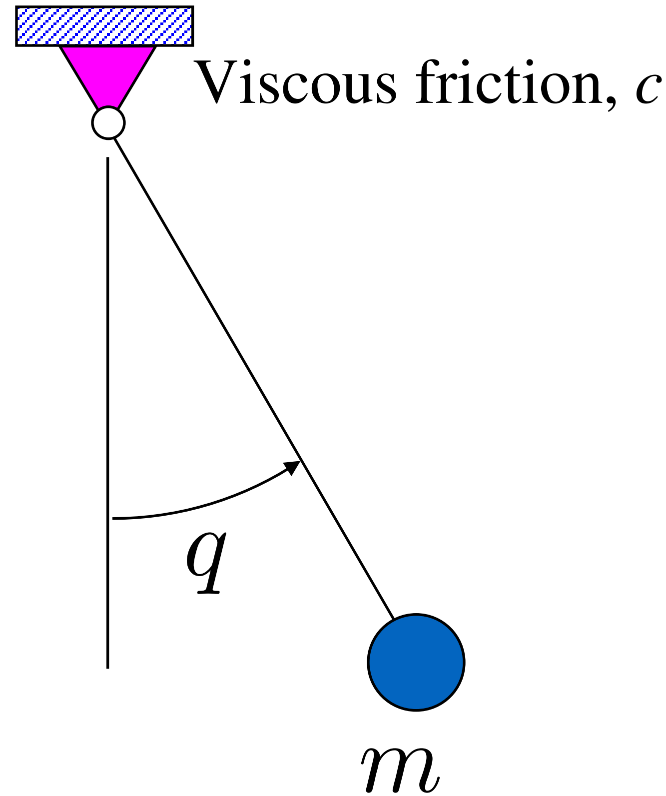
$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{c}{ml^2}x_2 \end{bmatrix}$$

- Equilibrium points

$$x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

- Change of variables

$$\tilde{x}_1 = (x - x_{e,1}), \tilde{x}_2 = (x - x_{e,2})$$



Example

- Equilibrium point number 1

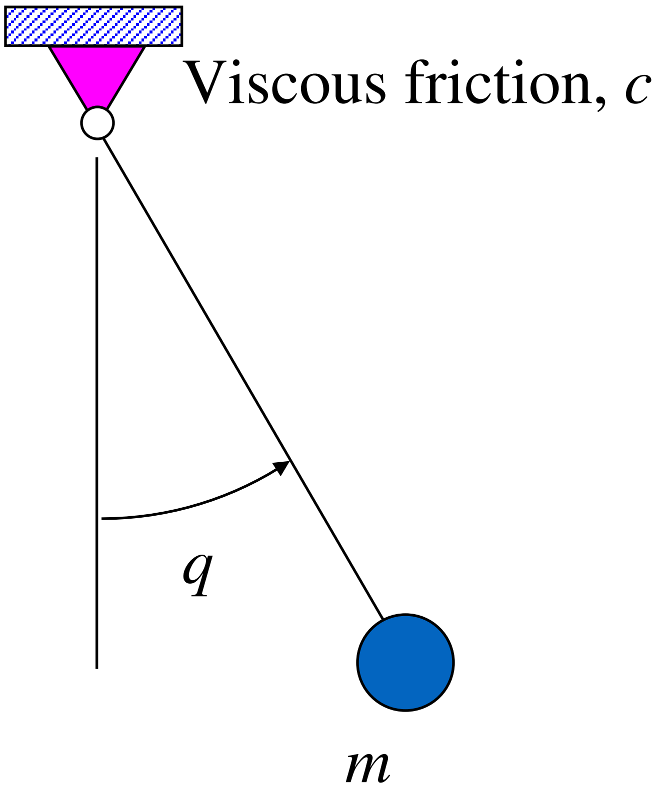
$$x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{\tilde{x}} = \begin{bmatrix} \tilde{x}_2 \\ -\frac{g}{l}\sin\tilde{x}_1 - \frac{c}{ml^2}\tilde{x}_2 \end{bmatrix}$$

- Equilibrium point number 2

$$x_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

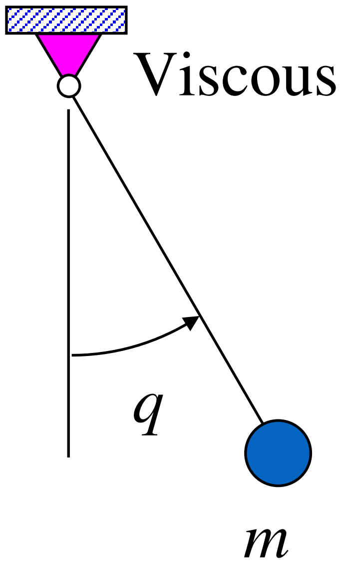
$$\dot{\tilde{x}} = \begin{bmatrix} \tilde{x}_2 \\ \frac{g}{l}\sin\tilde{x}_1 - \frac{c}{ml^2}\tilde{x}_2 \end{bmatrix}$$



Example

- Equilibrium point number 1

$$x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dot{\tilde{x}} = \begin{bmatrix} \tilde{x}_2 \\ -\frac{g}{l}\sin\tilde{x}_1 - \frac{c}{ml^2}\tilde{x}_2 \end{bmatrix}$$



- Linearization

$$f(\tilde{x}) = f(0) + \left.\frac{\partial f(\tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=0} (\tilde{x}) + O(\tilde{x}^2)$$

$$\approx Ax$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix}$$

$$\lambda^2 + \lambda\left(\frac{c}{ml^2}\right) + \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{c}{2ml^2} \pm \frac{1}{2}\sqrt{\left(\frac{c}{ml^2}\right)^2 - 4\frac{g}{l}}$$

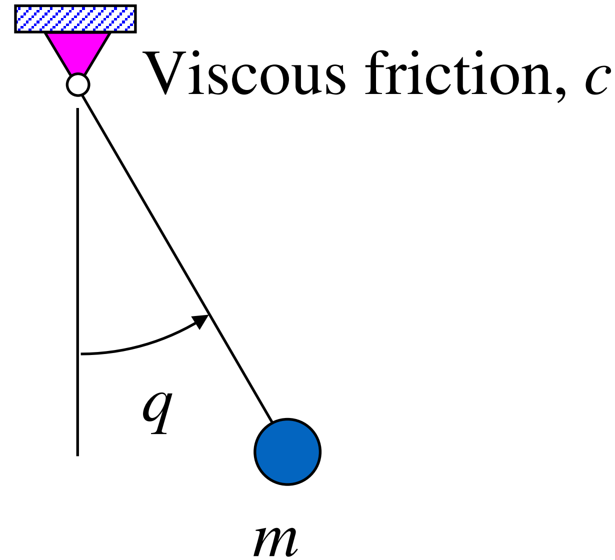
If c>0 and g>0, real parts of both eigenvalues are always negative

The system is locally asymptotically stable

Example

- Equilibrium point number 2

$$x_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \qquad \dot{\tilde{x}} = \begin{bmatrix} \tilde{x}_2 \\ \frac{g}{l}\sin\tilde{x}_1 - \frac{c}{ml^2}\tilde{x}_2 \end{bmatrix}$$



- Linearization

$$f(\tilde{x}) = f(0) + \left.\frac{\partial f(\tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=0} (\tilde{x}) + O(\tilde{x}^2)$$

$$\approx Ax$$

If c>0 and g>0, both eigenvalues are real, one is positive.

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix}$$

$$\lambda^2 + \lambda\left(\frac{c}{ml^2}\right) - \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{c}{2ml^2} \pm \frac{1}{2}\sqrt{\left(\frac{c}{ml^2}\right)^2 + 4\frac{g}{l}}$$

The system is unstable

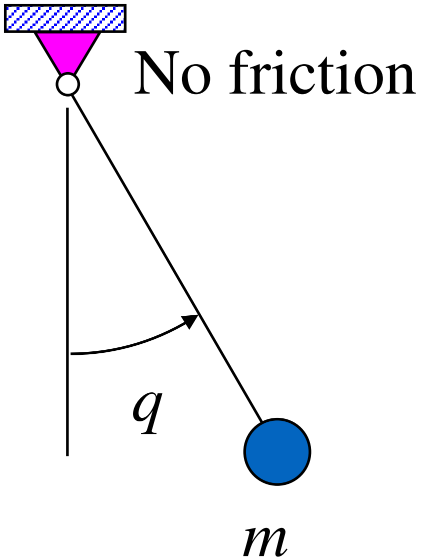
Example (c=0)

- Equilibrium point number 1

$$\dot{\tilde{x}} = \begin{bmatrix} \tilde{x}_2 \\ -\frac{g}{l}\sin\tilde{x}_1 \end{bmatrix} \qquad x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$f(\tilde{x}) = f(0) + \left.\frac{\partial f(\tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=0} (\tilde{x}) + O(\tilde{x}^2)$$

$$\approx Ax$$



- Linearization

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$$

$$\lambda^2 + \frac{g}{l} = 0$$

$$\lambda_{1,2} = \pm i\sqrt{\frac{g}{l}}$$

Real parts of both eigenvalues are non negative

No conclusive results

Summary for Nonlinear Autonomous Systems

- Write equations of motion in state space notation

$$\dot{x} = f(x)$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

- Solve $f(x)=0$

- Identify equilibrium point(s), x_e

- Linearize equations of motion to get the coefficient matrix **A**

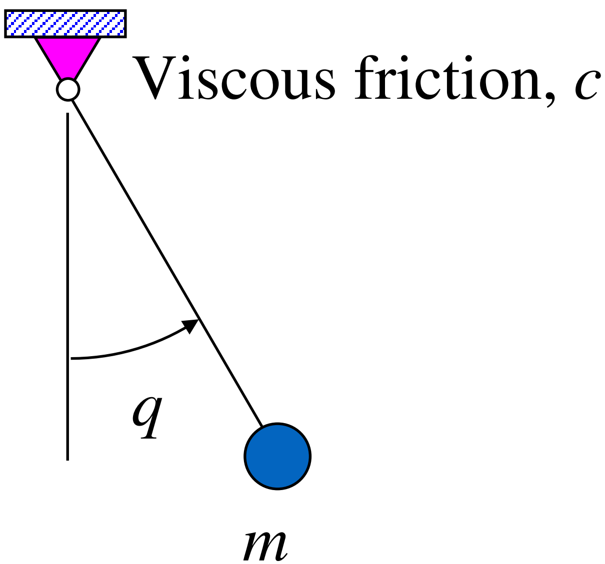
$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_e}$$

- Compute eigenvalues of **A**. Use Lyapunov's theorem. If the linearized system have significant dynamics, we can make an inference about stability.

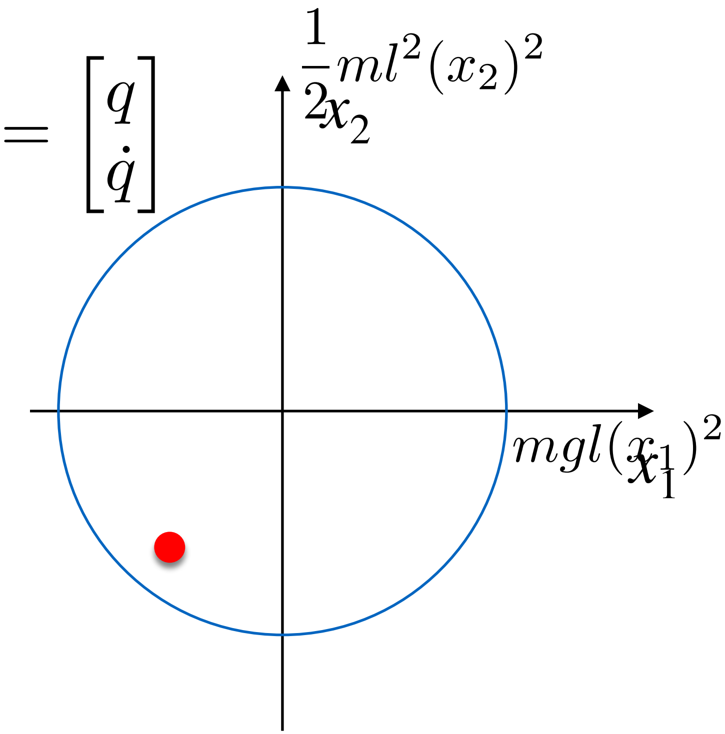
Lyapunov's Direct Method

- Avoids linearization (hence direct)

Example



$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$



$$\frac{dE}{dt} \leq 0$$

$$E = T + V = \frac{1}{2}ml^2(x_2)^2 + mgl(1 - \cos x_1)$$

$E(t)$ cannot increase

$$E \leq \frac{1}{2}ml^2(x_2)^2 + mgl(x_1)^2$$

Lyapunov's Direct Method

- $V(x)$ is a continuous function with continuous first partial derivatives

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$

- $V(x)$ is positive definite

$$V(0) = 0$$

$$V(x) > 0, \text{ for } x \neq 0$$

Such a function *V* is called a *Lyapunov Function Candidate*

V acts like a *norm*

What if you can show that *V* never increases?

Theorem

1. The (above) system is stable if there exists a Lyapunov function candidate such that the time derivative of V is negative semi-definite along all solution trajectories of the system.

$$\dot{x} = f(x)$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n$$

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$

$$V(0) = 0$$

$$V(x) > 0, \text{ for } x \neq 0$$

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) \leq 0$$

Theorem

2. The (above) system is asymptotically stable if there exists a Lyapunov function candidate such that the time derivative of V is negative definite along all solution trajectories of the system.

$$\dot{x} = f(x)$$
$$x \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n$$

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$
$$V(0) = 0$$
$$V(x) > 0, \text{ for } x \neq 0$$

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) < 0$$

Example 1

- Equation of motion

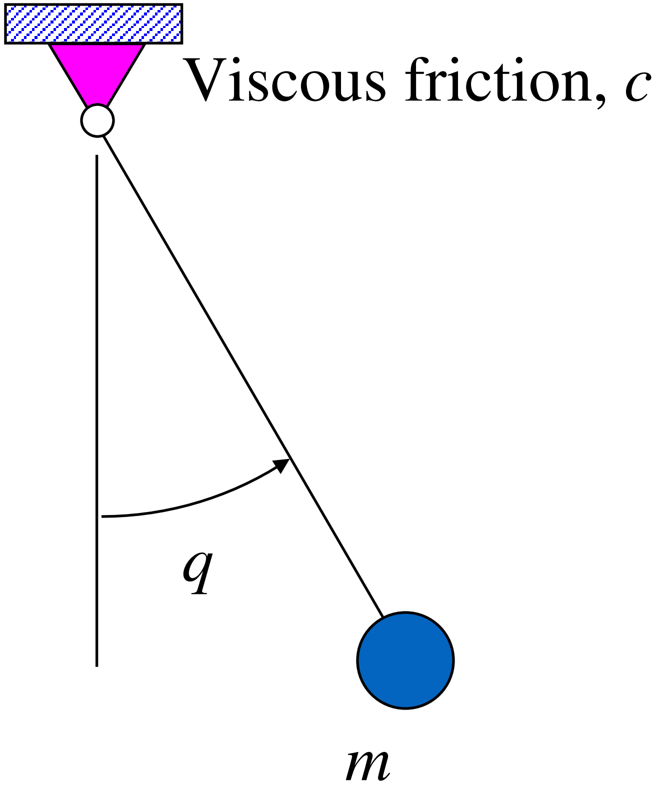
$$\ddot{q} + \frac{c}{ml^2}\dot{q} + \frac{g}{l}\sin q = 0$$

- State space representation

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{c}{ml^2}x_2 \end{bmatrix}$$

$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Equilibrium point



What is a candidate Lyapunov function?

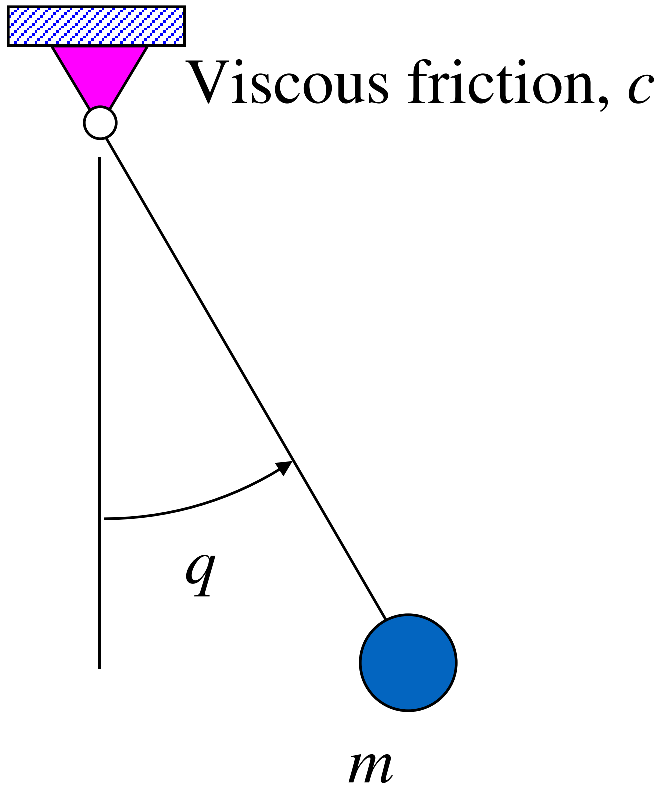
$$V(x) = \frac{1}{2}ml^2(x_2)^2 + mgl(1 - \cos x_1)$$

Example 1

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{c}{ml^2}x_2 \end{bmatrix}$$

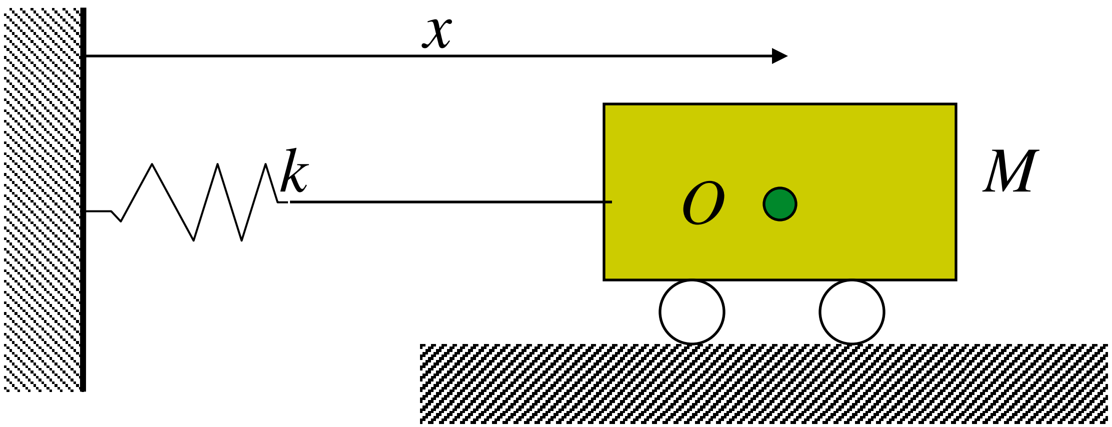
$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V(x) = \frac{1}{2}ml^2(x_2)^2 + mgl(1 - \cos x_1)$$



Example 2

•One-dimensional spring-mass-dashpot with a nonlinear spring



$$m\ddot{x} + b\dot{x} + kx^3 = 0$$

Linearized system does *not* have significant dynamics

What is a candidate Lyapunov function?

$$\frac{1}{2}m(x_2)^2 + \frac{1}{4}k(x_1)^4$$

Video 8.3
Vijay Kumar

Fully-actuated robot arm (*n* joints, *n* actuators)

Equations of Motion

$$\boxed{M(q)}\ddot{q} + \boxed{C(q,\dot{q})\dot{q}} + \boxed{N(q)} = \boxed{\tau}$$

symmetric, positive definite inertia matrix

n-dimensional vector of Coriolis and centripetal forces

n-dimensional vector of gravitational forces

n-dimensional vector of actuator forces and torques

Fully-actuated robot arm (continued)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$u = \tau \in \mathbb{R}^n$$

$$\dot{x} = \begin{bmatrix} x_2 \\ -M(x_1)^{-1}(N(x_1) + C(x_1, x_2)x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ M(x_1)^{-1} \end{bmatrix} u$$

$$y = q \in \mathbb{R}^n$$

PD Control of Robot Arms

Reference trajectory

$$q^d(t)$$

Error

$$\tilde{q} = q - q^d(t)$$

assume $\dot{q}^d = 0$

Proportional + Derivative Control

$$\tau = -K_P \tilde{q} - K_D \dot{q}$$

$$K_P = \begin{bmatrix} K_{P,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_{P,n} \end{bmatrix} \qquad K_D = \begin{bmatrix} K_{D,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_{D,n} \end{bmatrix}$$

Assume no gravitational forces

PD Control achieves Global Asymptotic Stability

Lyapunov function candidate

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P \tilde{q}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \cancelto{0}{N(q)} = \tau$$

$$\tau = -K_P\tilde{q} - K_D\dot{q}$$

Proof

$$\dot{V} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \tilde{q}^T K_P \dot{\tilde{q}}$$

$$\ddot{q} = M(q)^{-1}\left[-C(q,\dot{q})\dot{q} - K_P\tilde{q} - K_D\dot{q}\right]$$

Identity

$\dot{M}(q) - 2C(q,\dot{(q)})$ is skew symmetric

Assume no gravitational forces

PD Control achieves Global Asymptotic Stability

Lyapunov function candidate

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P \tilde{q}$$

Proof

$$\dot{V} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \tilde{q}^T K_P \tilde{q}$$

$$= -\dot{q}^T K_D \dot{q}^T \leq 0$$

decreasing as long as velocity is non zero

can it reach a state where $\dot{q} = 0,\ q \neq q^d$ *?*

Penn Engineering

Assume no gravitational forces

PD Control achieves Global Asymptotic Stability

Proof

$$\dot{V} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \tilde{q}^T K_P \tilde{q}$$
$$= \dot{q}^T K_D \dot{q}^T \leq 0$$

decreasing as long as velocity is non zero

can it reach a state where $\dot{q} = 0,\ q \neq q^d$

$$\ddot{q} = M(q)^{-1}\left[-C(q,\dot{q})\dot{q} - K_P\tilde{q} - K_D\dot{q}\right]$$

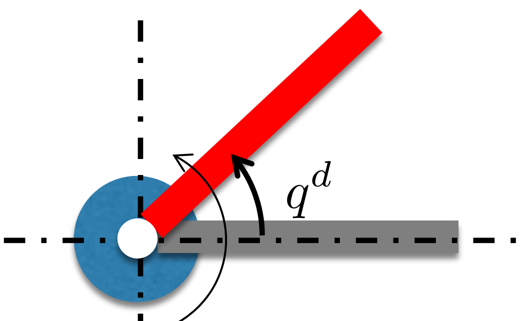
$$\dot{q} = 0,\ q \neq q^d \implies K_P\tilde{q} = 0$$

La Salle's theorem guarantees Global Asymptotic Stability

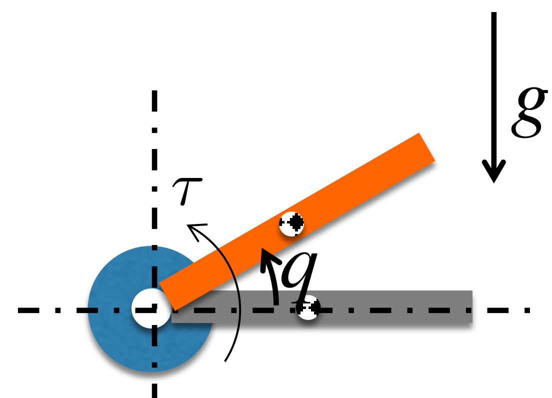
With gravitational forces

PD Control achieves Global Asymptotic Stability but with a new equilibrium point

$$K_P(q - q^d) = N(q)$$



$$\tau = -K_P\tilde{q} - K_D\dot{q}$$



$$\tau = mg\frac{l}{2}cosq$$

PD control with gravity compensation

$$\tau = N(q) - K_P\tilde{q} - K_D\dot{q}$$

Global Asymptotic Stability with the correct equilibrium configuration

Use the same Lyapunov function candidate:

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P\tilde{q}$$

Computed Torque Control

Reference trajectory

$$q^d(t),\ \dot{q}^d(t),\ \ddot{q}^d(t)$$

Compensate for gravity and inertial forces

$$\tau = C(q,\dot{q})\dot{q} + N(q) + M(q)\left(\ddot{q}^d + -K_P\tilde{q} - K_D\dot{\tilde{q}}\right)$$

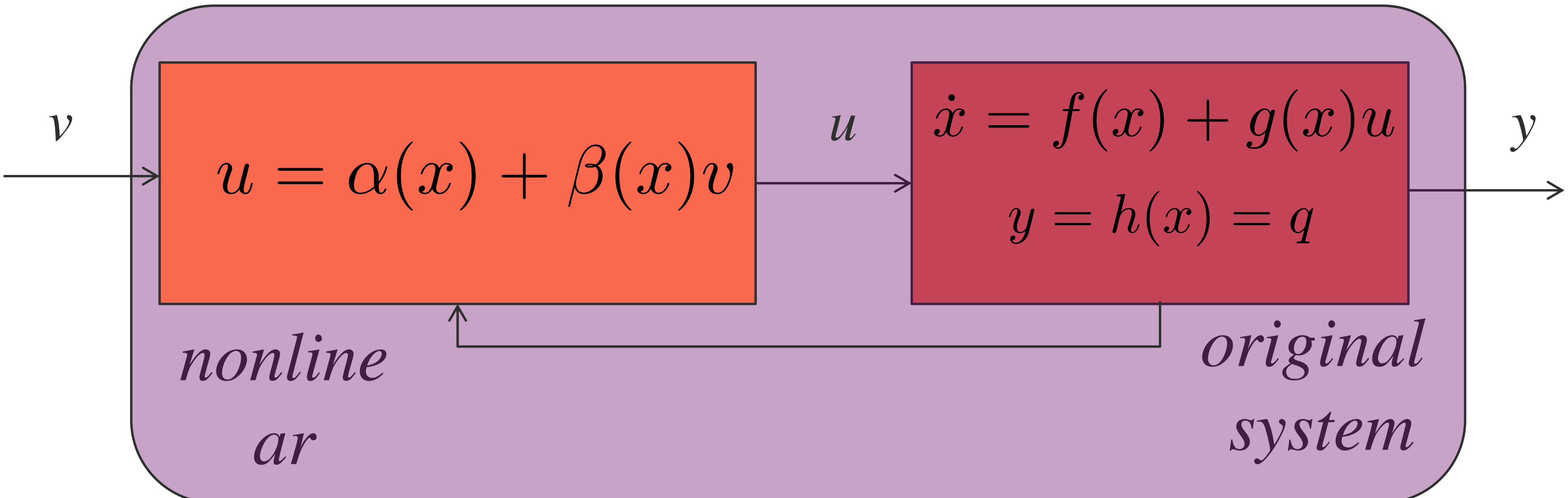
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

Global Asymptotic Stability

$$\ddot{\tilde{q}} + K_D\dot{\tilde{q}} + K_P\tilde{q} = 0$$

Computed Torque Control and Feedback Linearization

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$



new system $\ddot{y} = \ddot{q} = v$

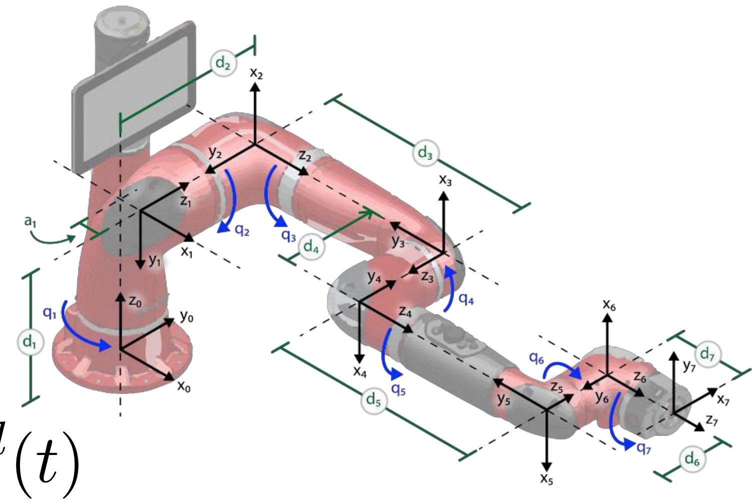
Nonlinear feedback transforms the original nonlinear system to a new linear system

Linearization is exact (distinct from linear approximations to nonlinear systems)

Joint Space versus Task Space Control

Task coordinates

$$X = \begin{bmatrix} \mathbf{p} \\ \mathbf{\Theta} \end{bmatrix}$$



Reference trajectory

$$X^d(t),\ \dot{X}^d(t),\ \ddot{X}^d(t)$$

Task space control

$$\ddot{\tilde{X}} + K_D \dot{\tilde{X}} + K_P \tilde{X} = 0$$

Kinematics

$$\dot{X} = J\dot{q}$$

$$\ddot{X} = J\ddot{q} + \dot{J}\dot{q}$$

Task Space Control

Task coordinates

$$X = \begin{bmatrix} \mathbf{p} \\ \mathbf{\Theta} \end{bmatrix}$$

Task space control

$$\ddot{\tilde{X}} + K_D \dot{\tilde{X}} + K_P \tilde{X} = 0$$

Kinematics

$$\dot{X} = J\dot{q}$$

$$\ddot{X} = J\ddot{q} + \dot{J}\dot{q}$$

Commanded joint accelerations

$$\ddot{q} = J^{-1} \left(-\dot{J}\dot{q} + \ddot{X}^d + K_D \dot{\tilde{X}} + K_P \tilde{X} \right)$$

Computed torque control

$$\tau = C(q, \dot{q})\dot{q} + N(q) + M(q) \left(J^{-1} \left(-\dot{J}\dot{q} + \ddot{X}^d + K_D \dot{\tilde{X}} + K_P \tilde{X} \right) \right)$$