edX Robo4 Mini MS – Locomotion Engineering

Week 6 – Unit 2

Raibert Vertical Hopper

Video 7.1

Segment 6.2.1

Hybrid Systems Model

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Penn Engineering

Initial Approach to Vertical Hopper

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express mode map
- Model reset map
- Compose
  - mode map ◦ reset map
  - further compose each composition in turn
- End up with return map

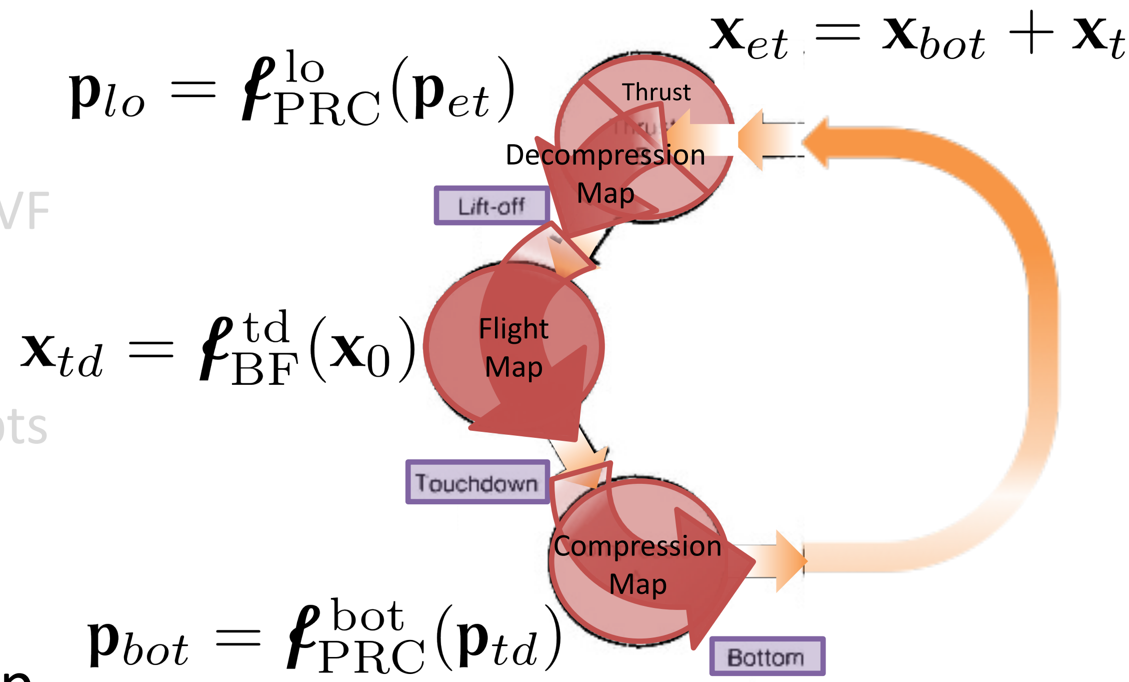
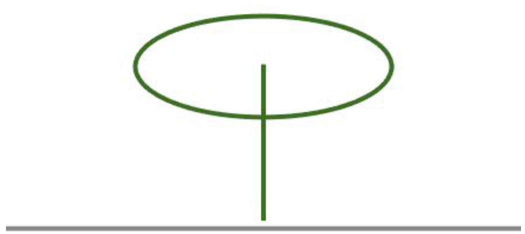


figure modified from
Raibert, *Commun. ACM*, 1986, *op. cit.*

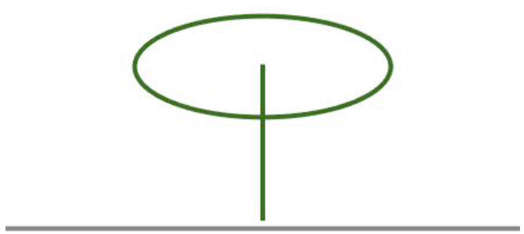
Vertical Hopper as Hybrid System

- Physical system doesn’t “know” about modes
- But users must: e.g., control of hopping height
  - Raibert: fixed duration constant thrust
  - different duration yields different behavior

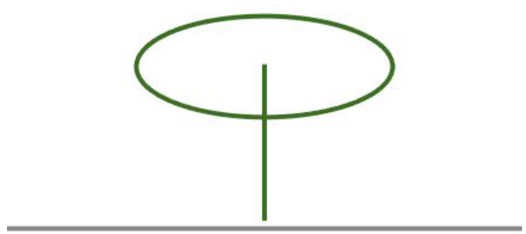
Purely passive



Radial thrust:
Duration = 20ms



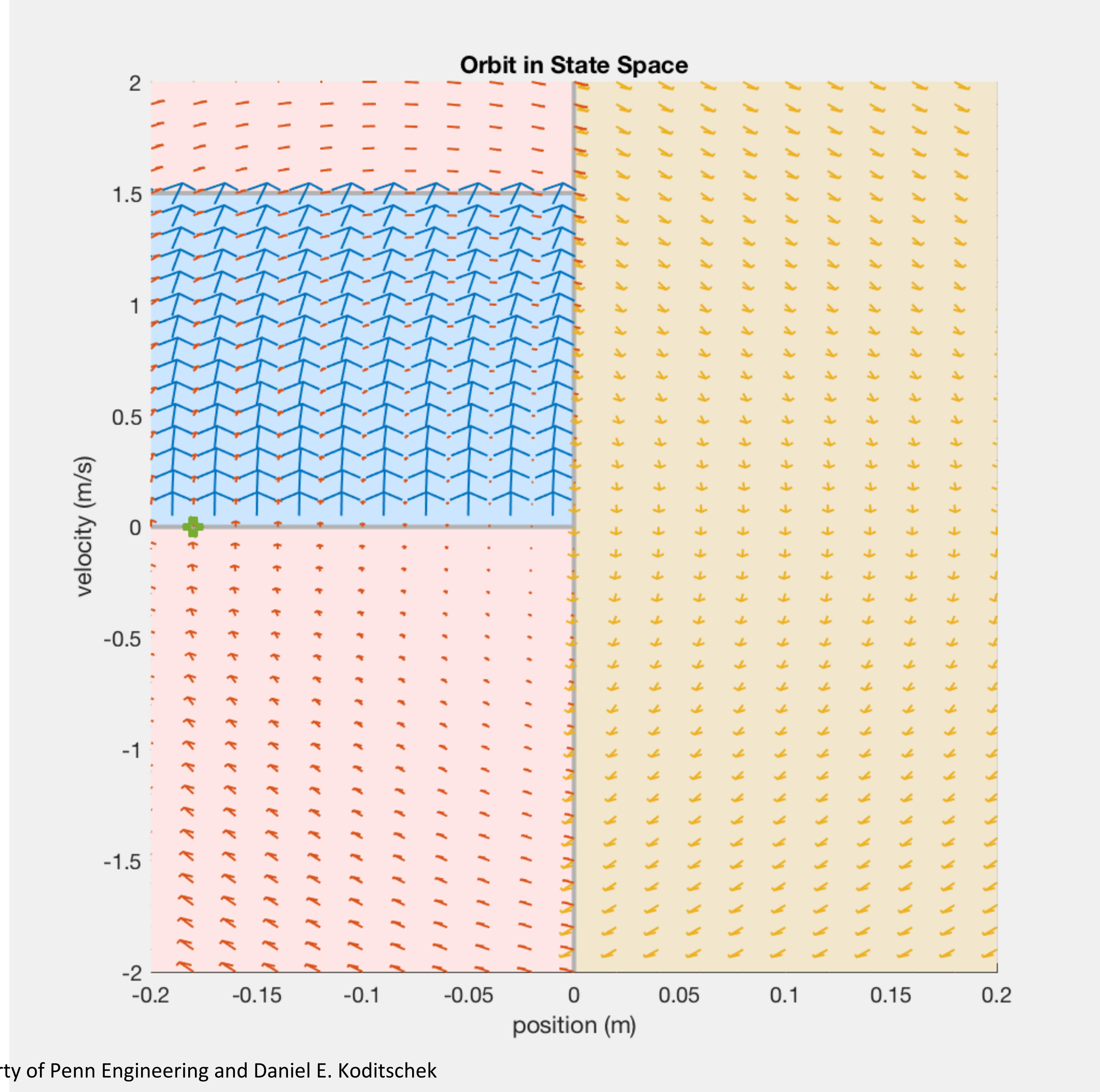
Radial thrust:
Duration = 50ms



Simulation credit: Jeffrey Duperret

Penn Engineering

Vertical Hopper as Hybrid System



Reset from Flight to Compression

- Need to "hand-off" state
  - from touchdown

$$\mathbf{x}_{td} = \boldsymbol{f}_{\mathrm{BF}}^{\mathrm{td}}(\mathbf{x}_0)$$

  - to compression

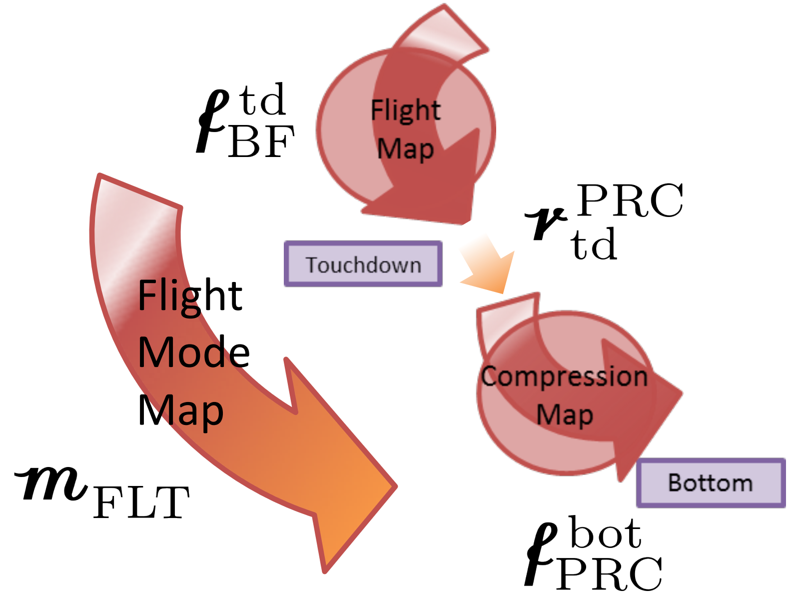
$$\mathbf{p}_{bot} = \boldsymbol{f}_{\mathrm{PRC}}^{\mathrm{bot}}(\mathbf{p}_{td})$$

- Use CC as reset

$$\mathbf{p}_{td} = h_{\mathrm{PRC}}(\mathbf{x}_{td}) =: \boldsymbol{r}_{\mathrm{td}}^{\mathrm{PRC}}(\mathbf{x}_{td})$$

- Yields mode map

$$\boldsymbol{m}_{\mathrm{FLT}} := \boldsymbol{r}_{\mathrm{td}}^{\mathrm{PRC}} \circ \boldsymbol{f}_{\mathrm{BF}}^{\mathrm{td}}$$

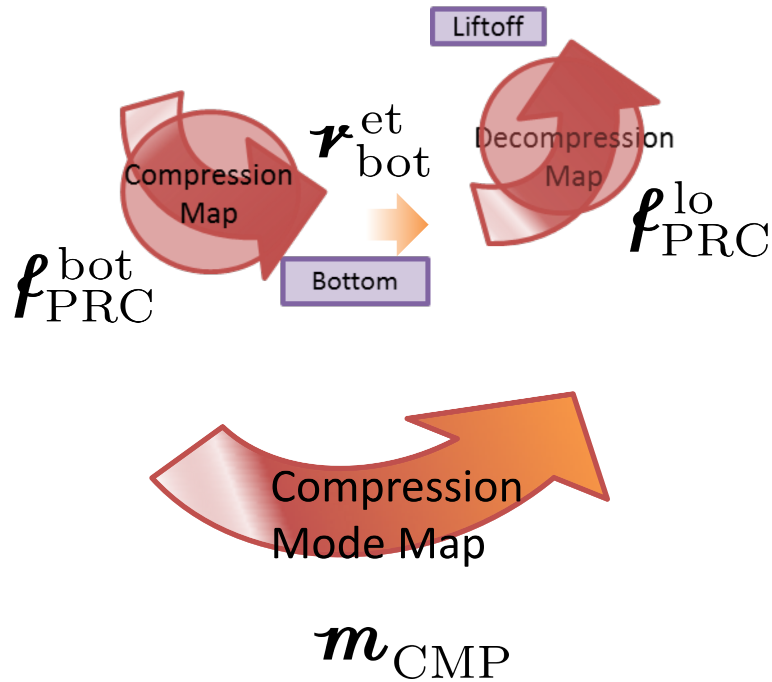


Reset from Compression to Decompression

- Thrust $\Phi_{\text{ref}}(t) \equiv \tau$
  - timed: $t \in [t_b, t_b + \Delta_\tau]$
  - not event driven
  - so plays the role of a reset
- To "hand-off" state
  - from compression
    $\mathbf{p}_{bot} = \boldsymbol{f}^{\text{bot}}_{\text{PRC}}(\mathbf{p}_{td})$
  - through thrust
    $\mathbf{x}_{et} = \mathbf{x}_{bot} + \mathbf{x}_t$
  - to decompression
    $\mathbf{p}_{lo} = \boldsymbol{f}^{\text{lo}}_{\text{PRC}}(\mathbf{p}_{et})$
- Use appropriate CC

$$\mathbf{p}_{et} = h_{\text{PRC}}(h^{-1}_{\text{PRC}}(\mathbf{p}_{bot}) + \mathbf{x}_t)$$

$$=: \boldsymbol{r}^{\text{et}}_{\text{bot}}(\mathbf{p}_{bot})$$



- Yields compression mode map $\boldsymbol{m}_{\text{CMP}} := \boldsymbol{r}^{\text{et}}_{\text{bot}} \circ \boldsymbol{f}^{\text{bot}}_{\text{PRC}}$

Reset from Decompression to Flight

- To reach flight mode map $\boldsymbol{m}_{\mathrm{FLT}}(\mathbf{x}_{lo}) := \boldsymbol{r}_{\mathrm{td}}^{\mathrm{PRC}} \circ \boldsymbol{\ell}_{\mathrm{BF}}^{\mathrm{td}}(\mathbf{x}_{lo})$
- From liftoff

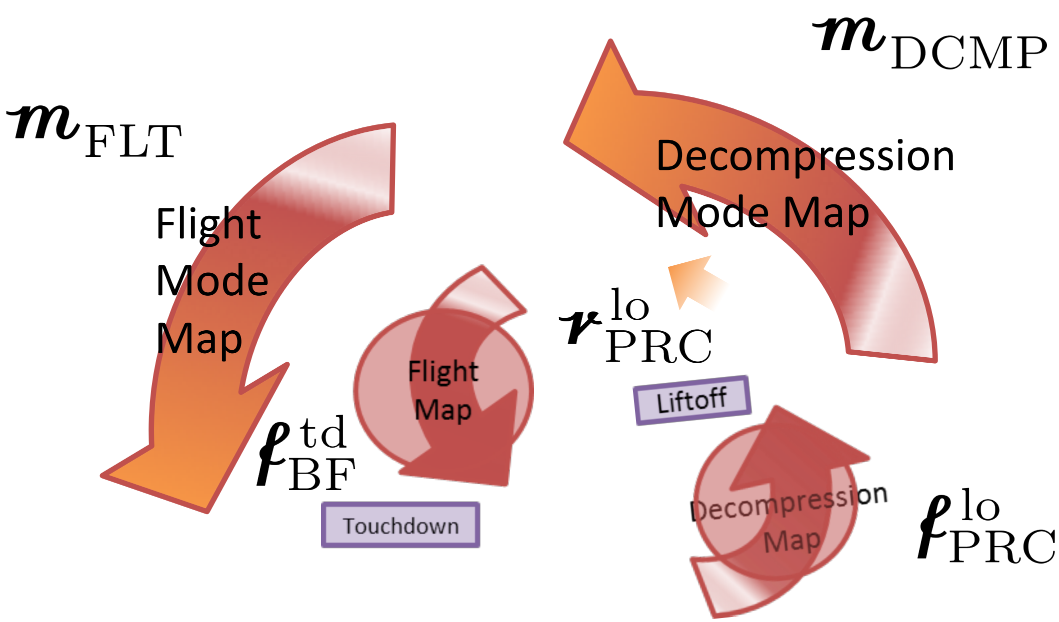
$$\mathbf{p}_{lo} = \boldsymbol{\ell}_{\mathrm{PRC}}^{\mathrm{lo}}(\mathbf{p}_{et})$$

- Use CC as reset

$$\begin{aligned} \mathbf{x}_{lo} &= h_{\mathrm{PRC}}^{-1}(\mathbf{p}_{lo}) \\ &=: \boldsymbol{r}_{\mathrm{PRC}}^{\mathrm{lo}}(\mathbf{x}_{td}) \end{aligned}$$

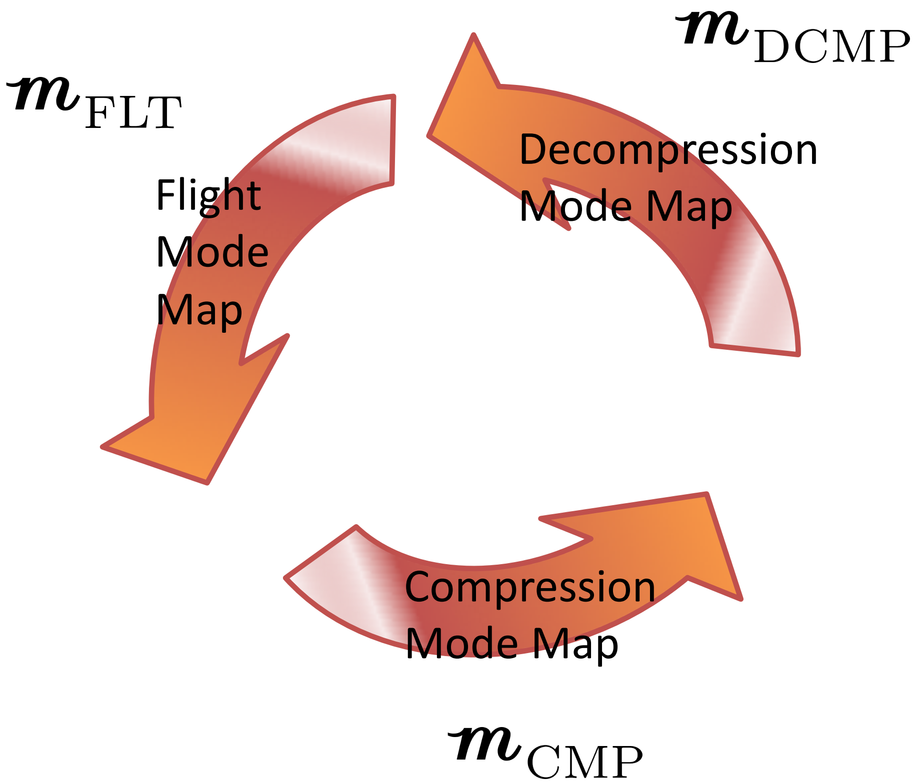
- Yields mode map

$$\boldsymbol{m}_{\mathrm{FLT}} := \boldsymbol{r}_{\mathrm{td}}^{\mathrm{PRC}} \circ \boldsymbol{\ell}_{\mathrm{BF}}^{\mathrm{td}}$$



Moving Ahead

- Finally have mode maps
  - express the physics
  - of each mode
  - and compose properly
- How to use them?
  - fear: infinite regress?
  - hope: what can they reveal?



edX Robo4 Mini MS – Locomotion Engineering

Week 6 – Unit 2

Raibert Vertical Hopper

Video 7.2

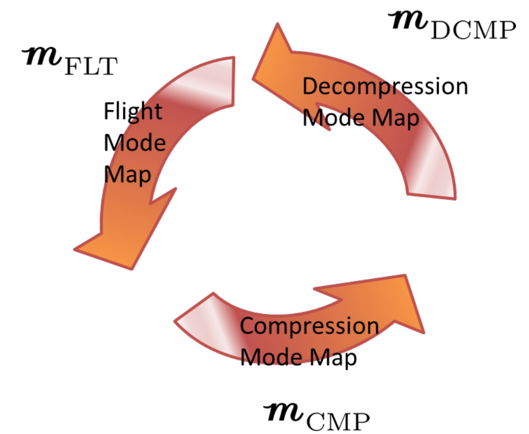
Segment 6.2.2

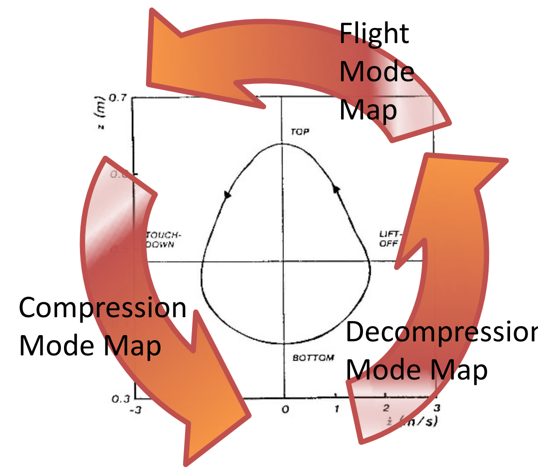
Hybrid Systems Model

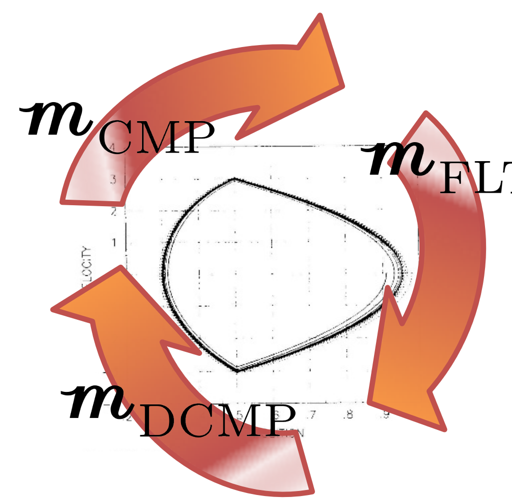
Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Where We Are Going

- Have mode maps!
- What can they reveal?
  - gait stability properties
  - parameter influence
  - gait control affordance
- How to get there
  - Insight, Data
  - Models
  - Analysis
  - Synthesis







figures from
D. E. Koditschek and M. Bühler, “Analysis of a Simplified Hopping Robot,” *The International Journal of Robotics Research*, vol. 10, no. 6, pp. 587–605, Dec. 1991.

Steady state gait representation

- Periodic hopping orbit
  - a cycle in steady state limit
  - called limit cycle
  - "parallel" direction to flow
    - very little change
    - per flow box theorem
- Behavior summarized by
  - one dimensional section
    - "transverse:" flow cuts across
    - "return:" flow brings section back
  - no unique choice of section
    - stance bottom state
    - flight apex state
    - touchdown state
    - liftoff state
  - flow takes one section to next
    - represented by mode maps
    - each a CC between sections (uniqueness)

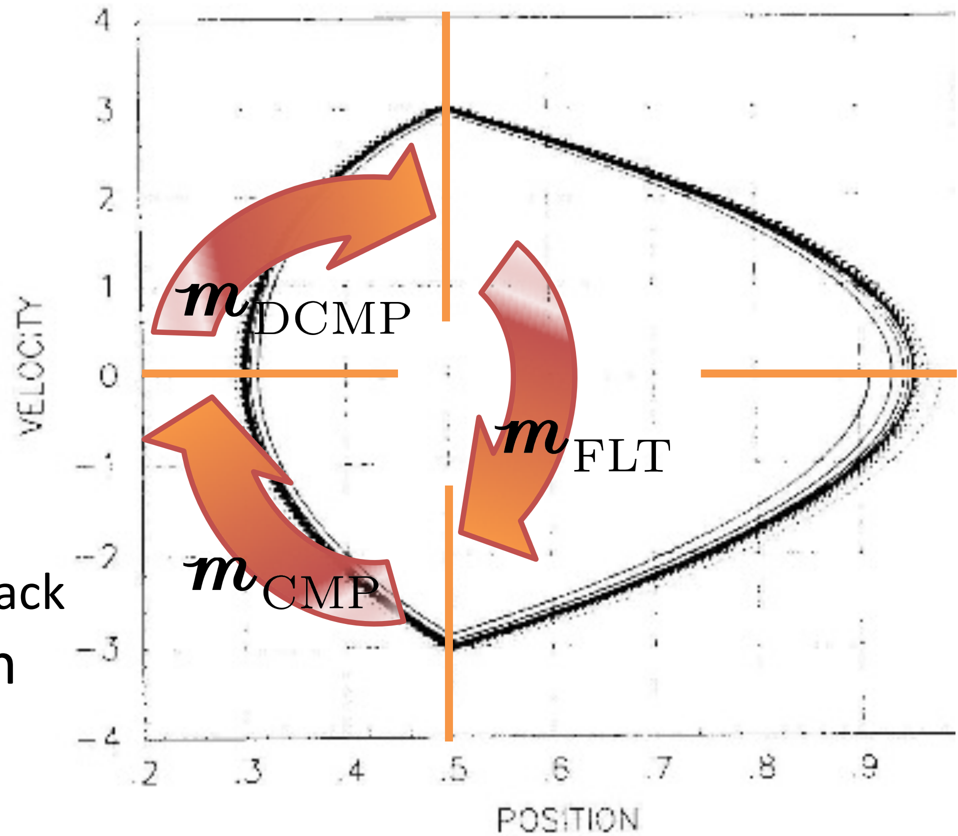
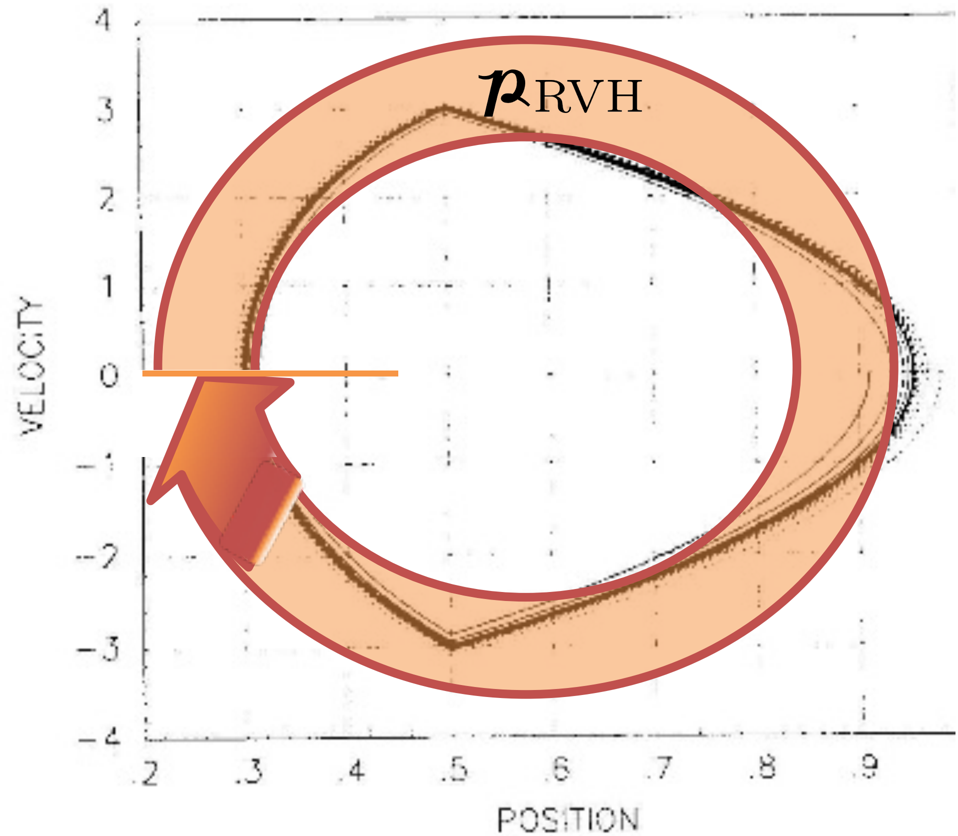


figure from
D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, op.cit..

The (Poincare') Return Map

- Strategy
  - fix a section (bottom)
  - define coordinates (energy, ρ)
  - compose mode maps
  - to get return map

$$\boldsymbol{p}_{\mathrm{RVH}} := \boldsymbol{m}_{\mathrm{DCMP}} \circ \boldsymbol{m}_{\mathrm{FLT}} \circ \boldsymbol{m}_{\mathrm{CMP}} \quad (1)$$



- Study discrete dynamics

$$\rho_{k+1} = \boldsymbol{p}_{\mathrm{RVH}}(\rho_k)$$

figure from
D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, op.cit..

Physical Meaning of Bottom Coordinates

- Polar RC coordinates include vertical total energy
  - showed in Seg.2.3

$$\rho = {e_1}^{\mathrm{T}}\mathbf{p} = {e_1}^{\mathrm{T}} h_{\mathrm{P}}(\mathbf{y}) = e_1^{\mathrm{T}} h_{\mathrm{PRC}}(\mathbf{x})$$

$$= \eta_{\mathrm{HO}}(\mathbf{x}) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

  - express same equivalence using GLH parameters

$$\rho = {e_1}^{\mathrm{T}}\mathbf{p} = {e_1}^{\mathrm{T}} h_{\mathrm{P}}(\mathbf{y}) = e_1^{\mathrm{T}} h_{\mathrm{PRC}}(\mathbf{x})$$

$$= \eta_{\mathrm{GLH}}(\mathbf{x}) = \frac{1}{2}\dot{\chi}^2 + \omega^2(1+\beta^2)\chi^2$$

- Total energy at bottom represents spring potential
  - recall bottom event guard $\gamma_{\mathrm{comp}}(\mathbf{x}) = \dot{\chi}$
  - hence bottom coordinates $\rho_b = \omega^2(1+\beta^2)\chi_b^2$
  - represent spring potential (equivalently, extension)

Decompression & Flight Mode Map Derivation

- Showed (Seg.6.1) polar RC bottom
  - since $\tilde{\gamma}_{\mathrm{comp}}(\mathbf{p}) = \gamma_{\mathrm{comp}} \circ h_{\mathrm{PRC}}^{-1}(\mathbf{p}) = \sqrt{1+\beta^2}\rho \sin\phi$
  - occurs at $\phi = n\pi$ – we'll take $n = 0$
- Show (Exrs): $\boldsymbol{y}_{et} = \boldsymbol{y}_b + \boldsymbol{y}_t = \begin{bmatrix}\psi_b \\ 0\end{bmatrix} + \begin{bmatrix}\psi_t \\ \dot{\psi}_t\end{bmatrix}$

$$\Rightarrow \mathbf{p}_{et} = \begin{bmatrix}\rho_{et} \\ \phi_{et}\end{bmatrix} = \begin{bmatrix} [\psi_t - \sqrt{\rho_b}]^2 + \dot{\psi}_t^2 \\ -\pi + \arctan\left(\frac{\dot{\psi}_t}{\sqrt{\rho_b} - \psi_t}\right) \end{bmatrix}$$

- Showed (Seg. 6.1+Exrs) reduce PRC VF to 1 dim

$$\frac{d\rho}{d\phi} = \frac{d}{dt}\rho \Big/ \frac{d}{dt}\phi = \frac{-2\beta\omega\rho}{-\omega} =: \tilde{f}_{\mathrm{PRC}}(\rho) \qquad (2)$$

$$\Rightarrow \rho_{lo} = \tilde{f}_{\mathrm{PRC}}^{\phi_{lo}-\phi_{et}}(\rho_{et}) = \mathrm{e}^{2\beta(\phi_{lo}-\phi_{et})}\rho_{et}$$

- Flight is lossless (total energy conserved): $\rho_{td} = \rho_{lo}$
- Touchdown angle – use symmetry:

$$\chi_{td} = \chi_{lo} = 0 \Rightarrow \dot{\psi}_{td} = -\dot{\psi}_{lo} \Rightarrow \phi_{td} = \phi_{lo} - \pi$$

Bottom Return Map Derivation

- Compression to next bottom
  - via reduced PRC $\rho_{b,next} = \mathrm{e}^{2\beta(\phi_{b,next}-\phi_{td})}\rho_{td}$
  - and symmetry $= \mathrm{e}^{-2\beta\phi_{lo}}\rho_{lo}$
- Compose with preceding decompression mode map

$$\begin{aligned}\rho_{b,next} &= \mathrm{e}^{-2\beta\phi_{lo}}\rho_{lo} \\ &= \mathrm{e}^{-2\beta\phi_{lo}}\mathrm{e}^{2\beta(\phi_{lo}-\phi_{et})}\rho_{et} \\ &= \mathrm{e}^{-2\beta\phi_{et}}\rho_{et}\end{aligned}$$

- Finally compose with reset from previous bottom

$$\begin{aligned}\rho_{b,next} &= \exp\left(-2\beta\left[-\pi + \arctan\left(\frac{\dot{\psi}_t}{\sqrt{\rho_b}-\psi_t}\right)\right]\right) \\ &\quad \cdot \left(\left[\psi_t - \sqrt{\rho_b}\right]^2 + \dot{\psi}_t^2\right) \\ &=: \boldsymbol{\rho}_{\mathrm{RVH}}(\rho_b)\end{aligned} \tag{3}$$

Moving Ahead

- We've now written out return map
- In bottom coordinates: $\rho_{k+1} = \boldsymbol{\mathcal{P}}_{\mathrm{RVH}}(\rho_k)$
  - total energy at "next" bottom
  - as a function of total energy at "previous" bottom
- What can it reveal?
  - gait stability properties
  - parameter influence
  - gait control affordance
- First introduce discrete dynamical systems

edX Robo4 Mini MS – Locomotion Engineering

Week 6 – Unit 3

Raibert Vertical Hopper

Video 7.3

Segment 6.3.1

Return Map Analysis - Coordinates

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Raibert's Original Hopping Analysis

- control system delivers fixed thrust each stance
  - causing bouncing to come to equilibrium
  - at hopping height for which energy injected
  - just equals energy lost (to friction and unsprung leg mass)
- because mechanical losses are monotonic with hopping height
  - a unique equilibrium hopping height
  - exists for each fixed value of thrust
  - and greater thrust results in greater height

paraphrase of Raibert's '86 MIT Press book passage from

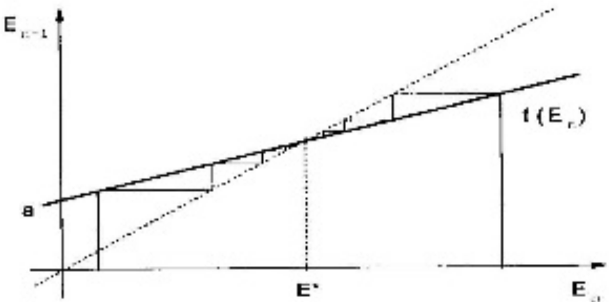
D. E. Koditschek and M. Bühler, "Analysis of a Simplified Hopping Robot," *The International Journal of Robotics Research*, vol. 10, no. 6, pp. 587–605, Dec. 1991.

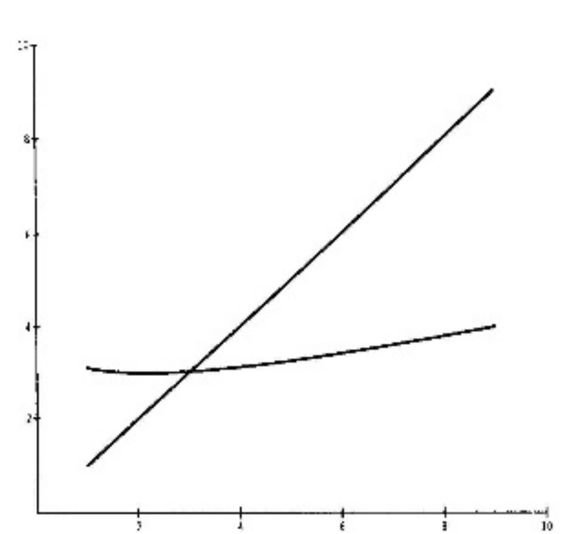
Graphical Portrayal of Raibert's Analysis

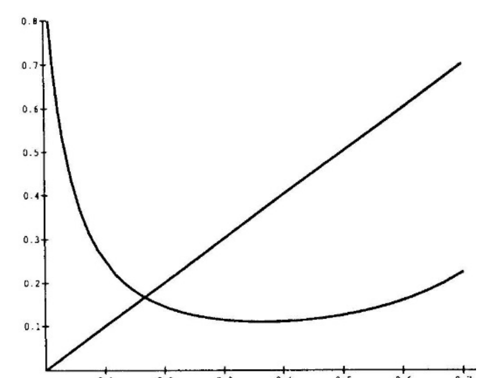
- Energy added and lost each stance

$$\boldsymbol{p}(\rho) \approx \rho + a(\rho) - l(\rho)$$
$$a(\rho) := \text{energy gained} \qquad (1)$$
$$l(\rho) := \text{energy lost}$$

- Monotonic mechanical losses
  - a unique equilibrium hopping height
  - exists for each fixed value of thrust
  - and greater thrust results in greater height
- Potential complications
  - a must fall off power limits
  - might fall off "early"

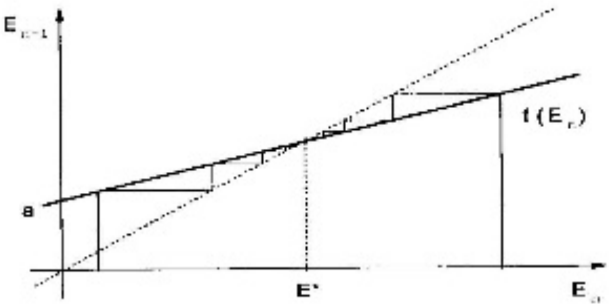






figures from
D. E. Koditschek and M. Bühler *International Journal of Roboti Research*, 1991, op. cit.

Discrete Scalar LTI System



- Exr: vertical stance hopper
  - add fixed energy, E_b, at bottom
  - but not enough to achieve liftoff

$$\rho_{k+1} = \mathrm{e}^{-2\pi\beta}\rho_k + E_b =: \boldsymbol{p}_{\mathrm{VSH}}(\rho_k) \qquad (2)$$

- Gives return map of the graphical form in eqn (1)
- Gives practice with discrete dynamical systems
  - LTI discrete theory mimics that of continuous time
  - stability of FP $\quad \rho^* = \frac{E_b}{1-\mathrm{e}^{-2\pi\beta}}$
  - from eigenvalues of magnitude less than 1

Change of Coordinates to ET Section

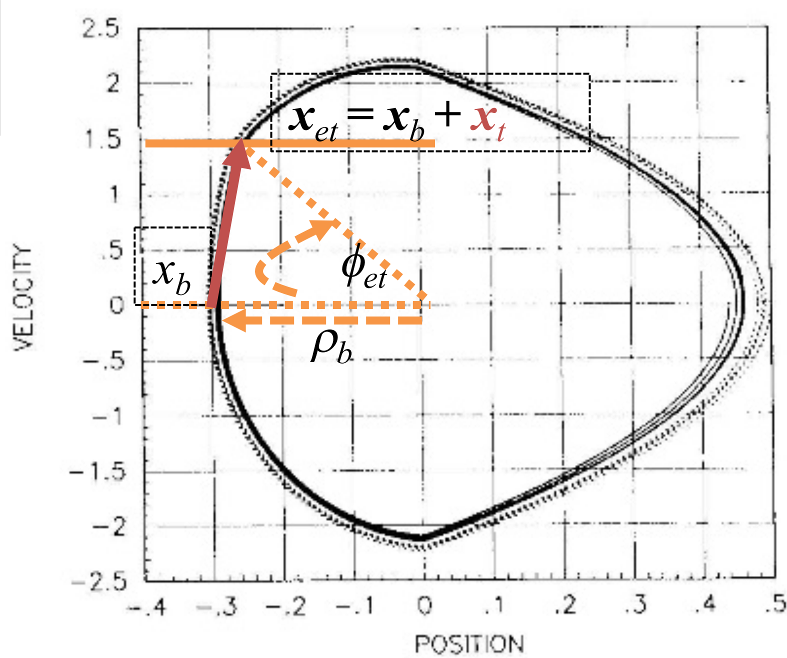
- To simplify the return map expression

$$\boldsymbol{p}_{\mathrm{RVH}}(\rho) = \exp\left(-2\beta\left[-\pi + \arctan\left(\frac{\dot{\psi}_t}{\sqrt{\rho}-\psi_t}\right)\right]\right)\left([\psi_t - \sqrt{\rho}]^2 + \dot{\psi}_t^2\right)$$

- It's convenient to rewrite in ET-coordinates

$$\phi = h_{\mathrm{bet}}(\rho) := e_2^{\mathrm{T}} \boldsymbol{r}_{\mathrm{bot}}^{\mathrm{et}}(\rho) = \arctan\left[\frac{\dot{y}_t}{\sqrt{\rho_b} - y_t}\right]$$

$$\Rightarrow h_{\mathrm{bet}}^{-1}(\phi) = \left[\frac{\dot{\psi}_t}{\tan\phi} + \psi_t\right]^2$$



- Where, recall (Seg.6.2),

$$\mathbf{p}_{et} = \begin{bmatrix} \rho_{et} \\ \phi_{et} \end{bmatrix} = \boldsymbol{r}_{\mathrm{bot}}^{\mathrm{et}}(\mathbf{p}_{bot}) := h_{\mathrm{PRC}}(h_{\mathrm{PRC}}^{-1}(\mathbf{p}_{bot})$$

$$= \begin{bmatrix} [\psi_t - \sqrt{\rho_b}]^2 + \dot{\psi}_t^2 \\ -\pi + \arctan\left(\frac{\dot{\psi}_t}{\sqrt{\rho_b}-\psi_t}\right) \end{bmatrix} \quad (4)$$

figure from
D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, 1991, op. cit.

The (Poincare') Return Map in ET Coords

- New ET coordinate return map representation

$$\tilde{\boldsymbol{p}}_{\mathrm{RVH}}(\phi) := h_{\mathrm{bet}} \circ \boldsymbol{p}_{\mathrm{RVH}} \circ h_{\mathrm{bet}}^{-1}(\phi) = \tilde{g}_{\mathrm{RVH}} \circ g_{\mathrm{RVH}}(\phi)$$

$$\tilde{g}_{\mathrm{RVH}}(u) := \arctan\left(\frac{u}{1+\alpha_t u}\right); \quad \alpha_t := \frac{\psi_t}{\dot{\psi}_t} \tag{4}$$

$$g_{\mathrm{RVH}}(\phi) := \sin\phi \cdot \mathrm{e}^{\beta(\pi-\phi)}$$

- Next: study discrete dynamics $\phi_{k+1} = \tilde{\boldsymbol{p}}_{\mathrm{RVH}}(\phi_k)$
  - by finding FP
  - and their linearized dynamics
- Simpler, but not simple

edX Robo4 Mini MS – Locomotion Engineering

Week 6 – Unit 3

Raibert Vertical Hopper

Video 7.4

Segment 6.3.2

Return Map Analysis - Stability

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Recap: One Dimensional Discrete Dynamics

- Goal: stability of limit cycle
  - conditions for convergence
  - to isolated periodic orbit
  - from all nearby ICs
- Ingredients of Analysis
  - section

$$\gamma_{\text{comp}}^{-1}[0] := \{\mathbf{x} \in \mathbb{R}^2 : \gamma_{\text{comp}}(\mathbf{x}) = 0\} \qquad (1)$$
$$\gamma_{\text{comp}}(\mathbf{x}) := \dot{\chi}$$

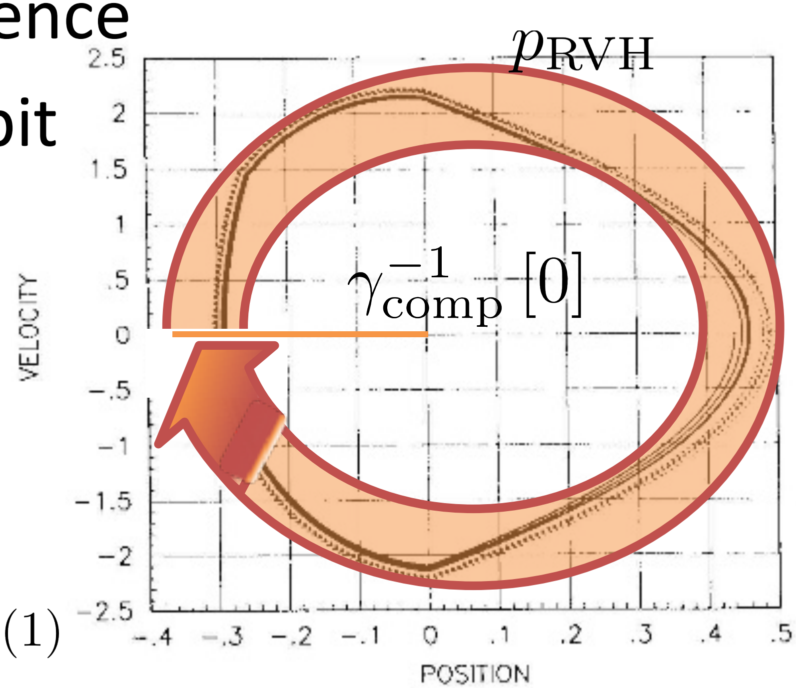


figure from
D. E. Koditschek and M. Buhler, “Analysis of a Simplified Hopping Robot,” *The International Journal of Robotics Research*, vol. 10, no. 6, pp. 587–605, Dec. 1991.

- specification using one equation
- in two variables $\mathbf{x} := \begin{bmatrix} \chi \\ \dot{\chi} \end{bmatrix}$
- yields a one dimensional set of ICs

  - Poincare’ (“return”) map

Meaning of Poincare' ("return") map

- What is it? Why is it helpful?
  - function specifying how
    - "next" 1 dim section crossing
    - depends upon "previous"
  - characterizing cyclic behavior
    - since value along any section
    - equivalently expresses qualitative properties
  - asymptotically (steady state)
- Choice of section
  - any "transverse" curve will do
    - e.g., liftoff $\gamma_{\text{decomp}}^{-1}[0] := \{\mathbf{x} \in \mathbb{R}^2 : \gamma_{\text{decomp}}(\mathbf{x}) = 0\}; \quad \gamma_{\text{decomp}}(\mathbf{x}) := \chi$
    - e.g. end-thrust points $\mathbf{p}_{et} = h_{\text{PRC}}(h_{\text{PRC}}^{-1}(\mathbf{p}_{bot}) + \mathbf{x}_t) =: r_{\text{bot}}^{\text{et}}(\mathbf{p}_{bot})$
  - all 1 dim representations
    - of the "energy" in the cycle
    - at given instant

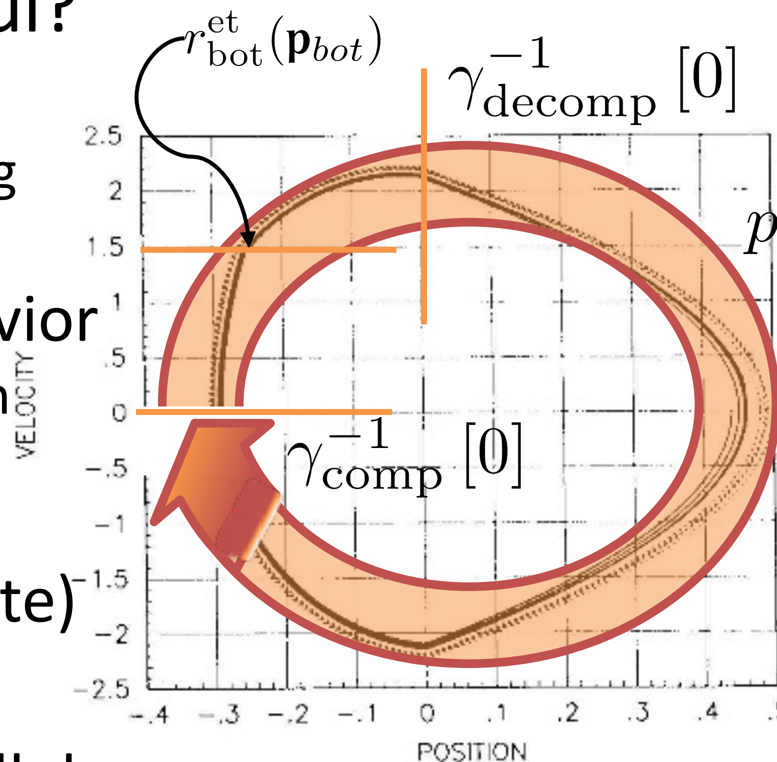


figure from D. E. Koditschek and M. Buhler, *The International Journal of Robotics Research*, 1991, op. cit.

Poincare' Map: Bottom Coordinates

- Bottom coordinates $\rho := \omega^2(1+\beta^2)\chi^2$
  - total energy at maximum compression
  - measures spring potential
- Poincare' map $\rho_{k+1} = p_{\mathrm{RVH}}(\rho_k)$
  - expresses next bottom energy
  - as function of previous bottom energy

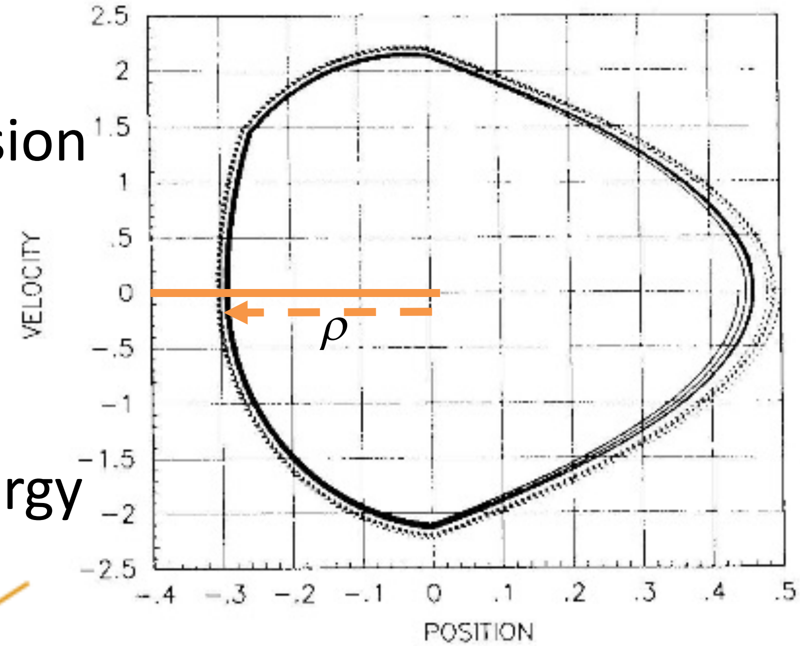
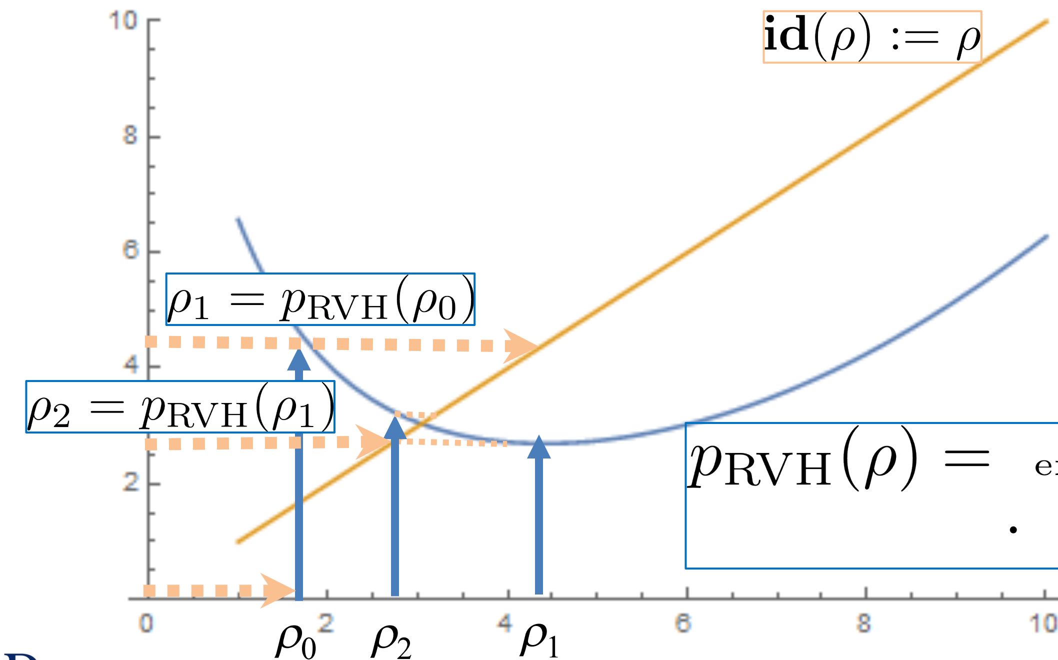


figure from
D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, 1991, op. cit.



$$p_{\mathrm{RVH}}(\rho) = \exp\left(-2\beta\left[-\pi + \arctan\left(\frac{\dot\psi_t}{\sqrt{\rho}-\psi_t}\right)\right]\right) \cdot \left([\psi_t - \sqrt{\rho}]^2 + \dot\psi_t^2\right)$$

Poincare' Map: Bot -> ET Coordinates

- Simplify map representation
  - by CC to ET-coordinates

$$\tilde{p}_{\mathrm{RVH}}(\phi) := h_{\mathrm{bet}} \circ p_{\mathrm{RVH}} \circ h_{\mathrm{bet}}^{-1}(\phi)$$

(illustrated using different parameters from previous slide)

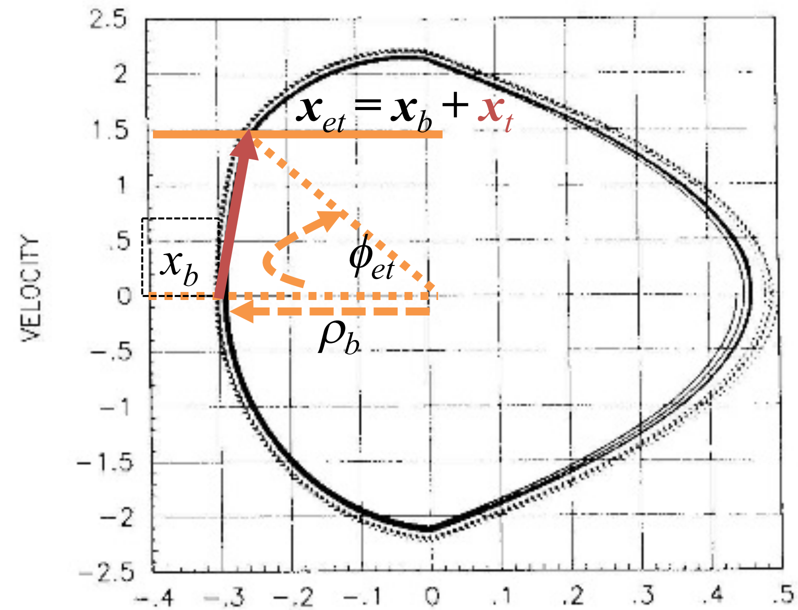
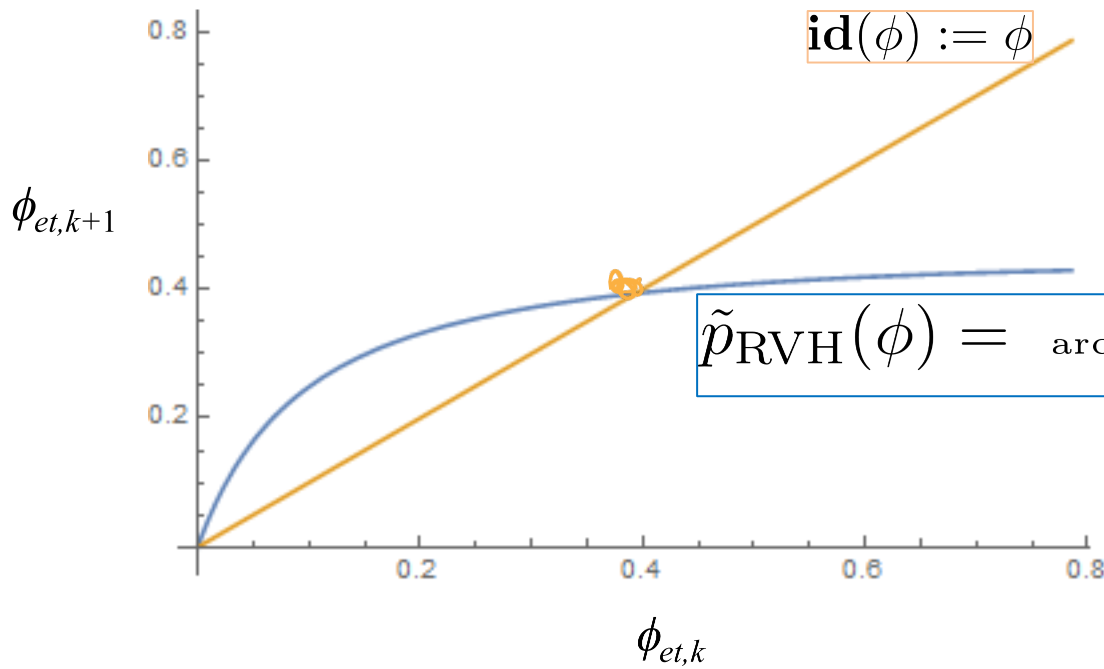


figure from
D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, 1991, op. cit.



$$\tilde{p}_{\mathrm{RVH}}(\phi) = \arctan\left(\frac{\dot{\psi}_t \sin\phi \cdot e^{\beta(\pi-\phi)}}{\dot{\psi}_t + \psi_t \sin\phi \cdot e^{\beta(\pi-\phi)}}\right)$$

First Step of Analysis: Fixed Points

- Study discrete dynamics $\phi_{k+1} = \tilde{p}_{\mathrm{RVH}}(\phi_k)$
  - by finding FP $\tilde{p}_{\mathrm{RVH}}(\phi^*) = \phi^* \Leftrightarrow \phi^* \in \{\phi_0^* := 0, \phi_1^* > 0\}$
  - and studying their linearized dynamics
- ET coordinate map can be factored

$$\tilde{p}_{\mathrm{RVH}}(\phi) = \tilde{g}_{\mathrm{RVH}} \circ g_{\mathrm{RVH}}(\phi)$$

$$\tilde{g}_{\mathrm{RVH}}(u) := \arctan\left(\frac{u}{1+\alpha_t u}\right); \quad \alpha_t := \frac{\psi_t}{\dot{\psi}_t}$$

$$g_{\mathrm{RVH}}(\phi) := \sin\phi \cdot \mathrm{e}^{\beta(\pi-\phi)}$$

- Quickly see 0 must be FP: it's FP for both factors
- More work to ascertain existence of $\phi_1^* > 0$

(consult reference: D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, 1991, op. cit.)

Second Step: Linearization at $\phi_0^* = 0$

- Calculus $\tilde{P}_{\mathrm{RVH}}(\phi) = D\tilde{p}_{\mathrm{RVH}}(\phi) = \tilde{g}'_{\mathrm{RVH}}\,|_{u=g_{\mathrm{RVH}}(\phi)} \cdot g'_{\mathrm{RVH}}(\phi)$

$$\tilde{g}'_{\mathrm{RVH}}(u) = \frac{1}{u^2+(1+\alpha_t u)^2}$$

$$g'_{\mathrm{RVH}}(\phi) = \mathrm{e}^{\beta(\pi-\phi)}\,(\cos\phi - \beta\sin\phi)$$

- Implies $\tilde{P}_{\mathrm{RVH}}(0) = \tilde{g}'_{\mathrm{RVH}}\,|_{0=g_{\mathrm{RVH}}(0)} \cdot g'_{\mathrm{RVH}}(0)$

$$= 1\cdot \mathrm{e}^{\beta\pi-0}\,(\cos 0 - \beta\sin 0)$$

$$= \mathrm{e}^{\beta\pi} > 1$$

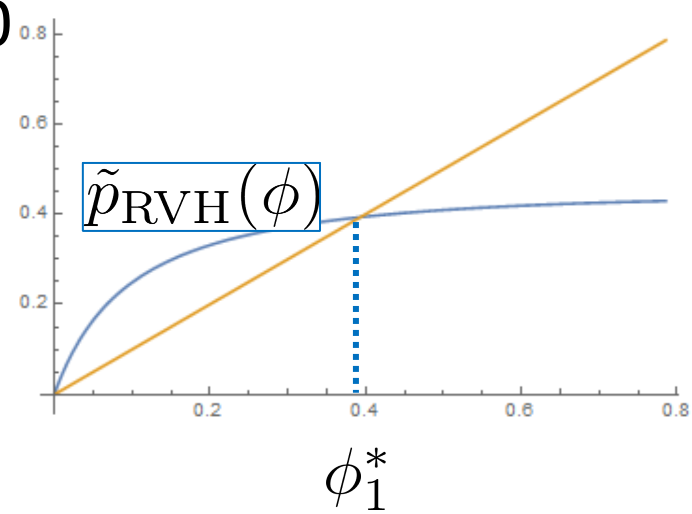
- Hence FP at 0 is unstable
  - hopping is pumped up
  - from very low energy states

Third Step: Linearization at $\phi_1^* > 0$

- Introduce quotient map on the set $\phi > 0$

$$q(\phi) := \tilde{p}_{\mathrm{RVH}}(\phi)/\phi$$

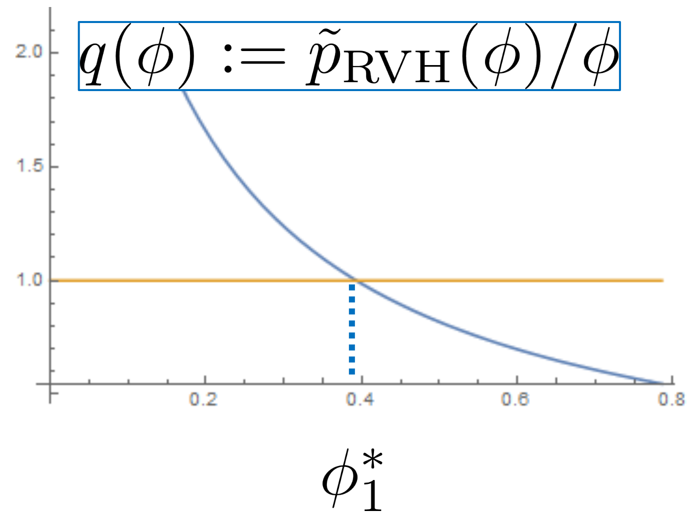
  - know $q(\phi) > 1$ for $0 < \phi < \phi_1^*$
  - since 0 is unstable FP
  - and ϕ_1^* is unique FP on $\phi > 0$



- Know $q(\phi_1^*) = 1$ so $q'(\phi_1^*) < 0$
  - now use calculus

$$0 > q'(\phi_1^*) = \frac{1}{\phi_1^*}\left[\tilde{p}'_{\mathrm{RVH}}(\phi_1^*) - 1\right]$$

$$\Rightarrow 1 > \tilde{p}'_{\mathrm{RVH}}(\phi_1^*)$$



- More arguments show $\tilde{p}'_{\mathrm{RVH}}(\phi_1^*) > -1$

(consult reference: D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, 1991, op. cit.)

- So linearization is asymptotically stable

$$1 > \tilde{P}_{\mathrm{RVH}}(\phi_1^*) := D\tilde{p}_{\mathrm{RVH}}(\phi_1^*) > -1$$

Summary and Conclusion

- ET-coordinate representation of Poincare' map
  - has two FP $\tilde{p}_{\mathrm{RVH}}(\phi^*) = \phi^* \Leftrightarrow \phi^* \in \{\phi_0^* := 0, \phi_1^* > 0\}$
  - whose linearized dynamics
    - is unstable at ϕ_0^*: $\tilde{P}_{\mathrm{RVH}}(\phi_0^*) = \mathrm{e}^{\beta\pi} > 1$
    - and asymptotically stable at ϕ_1^*: $\left|\tilde{P}_{\mathrm{RVH}}(\phi_1^*)\right| < 1$
- Conjugation preserves FP and linearized eigenvalues
- Hence stance energy map has same properties
  - two FP

$$p_{\mathrm{RVH}}(\rho^*) = \rho^* \Leftrightarrow \rho^* \in \{\rho_0^* := h_{\mathrm{bet}}^{-1}(\phi_0^*) = 0, \rho_1^* := h_{\mathrm{bet}}^{-1}(\phi_1^*) > 0\}$$

  - same stability properties $P_{\mathrm{RVH}}(\rho^*) = \tilde{P}_{\mathrm{RVH}}(\phi^*)$

edX Robo4 Mini MS – Locomotion Engineering

Week 6 – Unit 3

Raibert Vertical Hopper

Video 7.5

Segment 6.3.3
Return Map Analysis - Conclusion

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

What Has Been Shown?

- Used Hooke's law stance model of hopper
  - to show that constant thrust pumps energy
  - with a unique locally asymptotically stable limit cycle
- Additional arguments using this model give

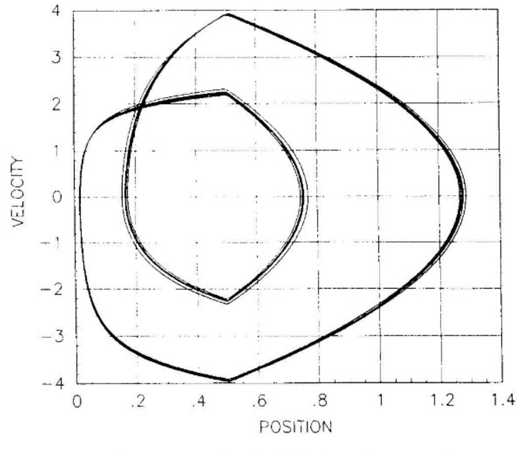
  (consult reference: D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, 1991, op. cit.)
  - essentially global basin for unique limit cycle
  - with possible "hunting" (negative slope linearization)
  - but preclude "limping" (period-two FP)
    - occurs only in physically meaningless parameter regime
    - wherein end-thrust resets directly into flight mode

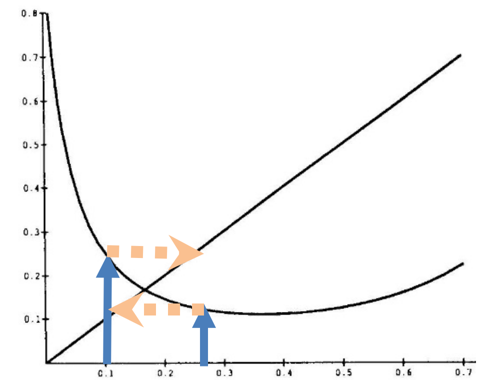
Verified Much of Raibert's Original Analysis

- control system delivers fixed thrust each stance
  - causing bouncing to come to equilibrium
  - at hopping height for which energy injected
  - just equals energy lost (to friction)
- although mechanical losses may not be monotonic
  - a unique equilibrium hopping height
  - exists for each fixed value of thrust
  - and greater thrust results in greater height
- <u>however</u> poorly chosen parameters
  - may result in "hunting" (oscillatory convergence to FP)
  - and closely related nonlinear model exhibits "limping"

Limping: When can it Happen?

- Simulations of pneumatic spring model
  - in physically plausible regime
    - where compression force from high prior apex
    - back-drives the pneumatic pressure chamber
  - exhibit robust “limping”
    - convergence to alternation between
    - identically repeated
    - high-long & short-low hops
- Poincare’ map analysis
  - reveals FP-destabilizing bifurcation
  - to asymptotically stable period-two orbit
- Raibert reported empirical “limping”
  - (personal comm.)
  - but seemed due to higher dof effects

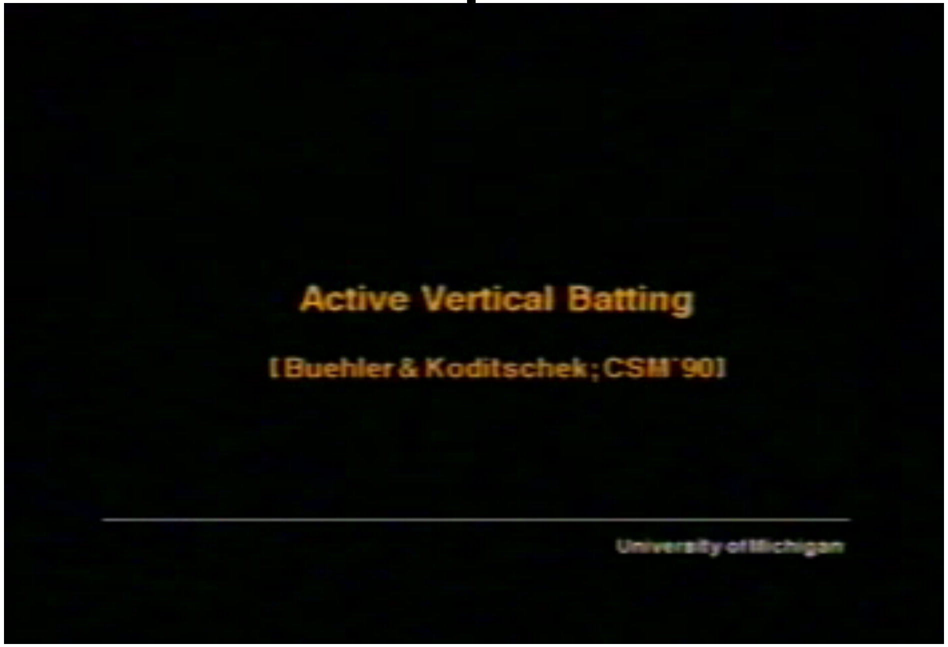


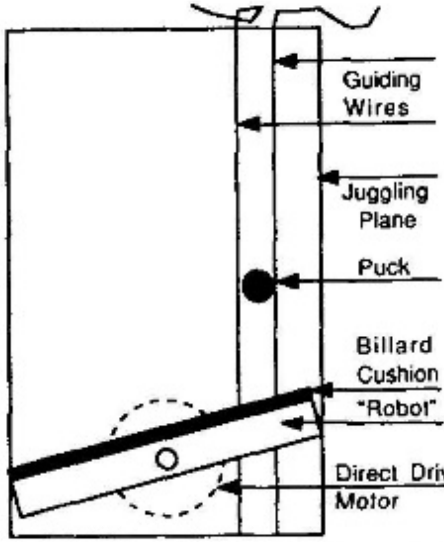


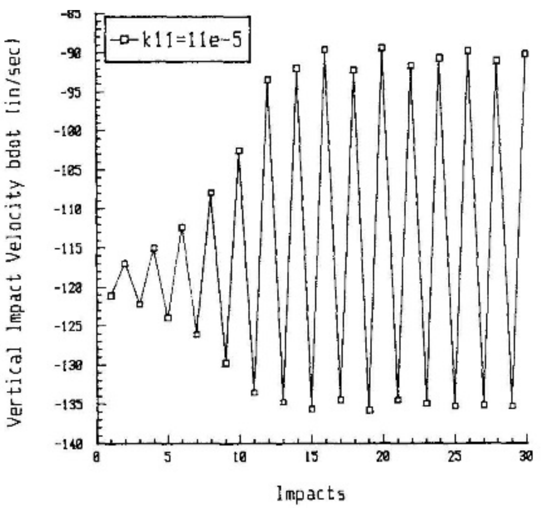
figures from
D. E. Koditschek and M. Bühler, *International Journal of Robotics Research*, 1991, op. cit.

The RVH as Dynamical Template

- Approximate RVH Poincare' Map
- Is "anchored" in Buehler's juggler
  - system settles down
  - to purely vertical orbits
  - whose return maps
  - have steady state properties
  - as predicted
- Introduce more formal notion soon







figures from:

M. Buhler and D. E. Koditschek, "From stable to chaotic juggling: Theory, simulation, and experiments," in *Robotics and Automation, 1990. Proceedings., 1990 IEEE International Conference on*, 1990, pp. 1976–1981.

Bifurcation Studies with Buehler's Juggler

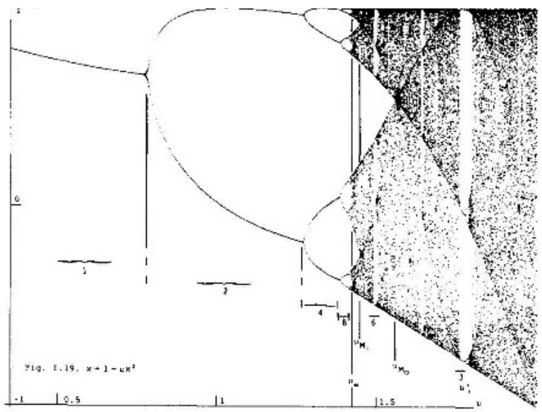
- Bifurcation
  - qualitative change
  - in attractor structure
  - due to systematic parameter adjustment
- Extensive theory available

textbook reference:

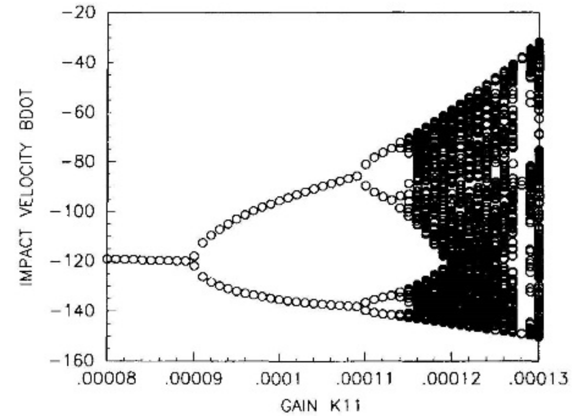
P. Collet and J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*. Springer Science & Business Media, 2009.

figures from:

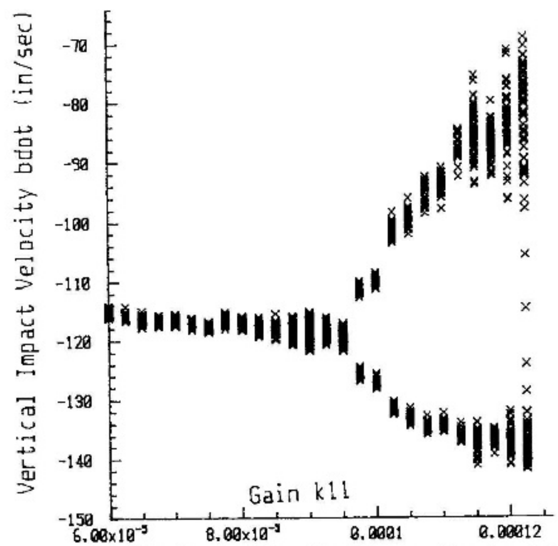
M. Buhler and D. E. Koditschek, *Int. Conf. Robotics and Automation, 1990. op.cit.*



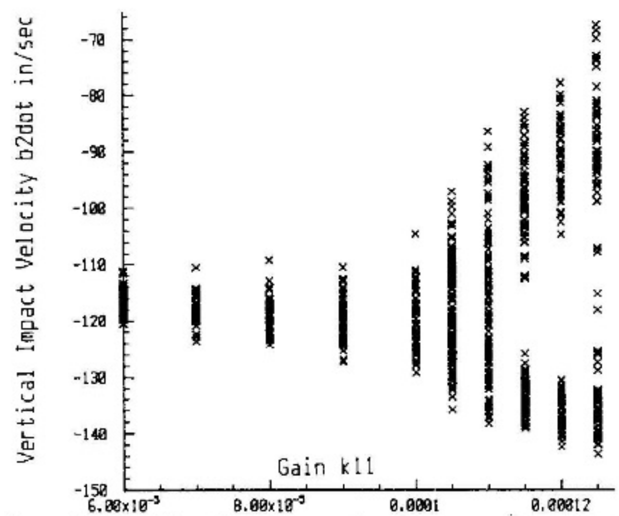
Textbook Bifurcation Diagram



Simulation Bifurcation Diagram



1 DoF Constrained Juggler



Unconstrained Planar Juggler

Dynamical Systems Thinking in Robotics

- We've encoded our tasks as dynamical attractors
- Whose basins function as abstract symbols
  - regions of state space
  - wherein the task is guaranteeably programmed
  - and indefatigably achieved
- To be "composed"
  - achieving some more complicated behavior
  - from some simpler, well understood components
- Dynamical Systems Theory
  - gives mathematically tractable
  - physically robust and achievable
  - predictive and composable symbols