edX Robo4 Mini MS – Locomotion Engineering

Block 1 – Week 2 – Unit 2

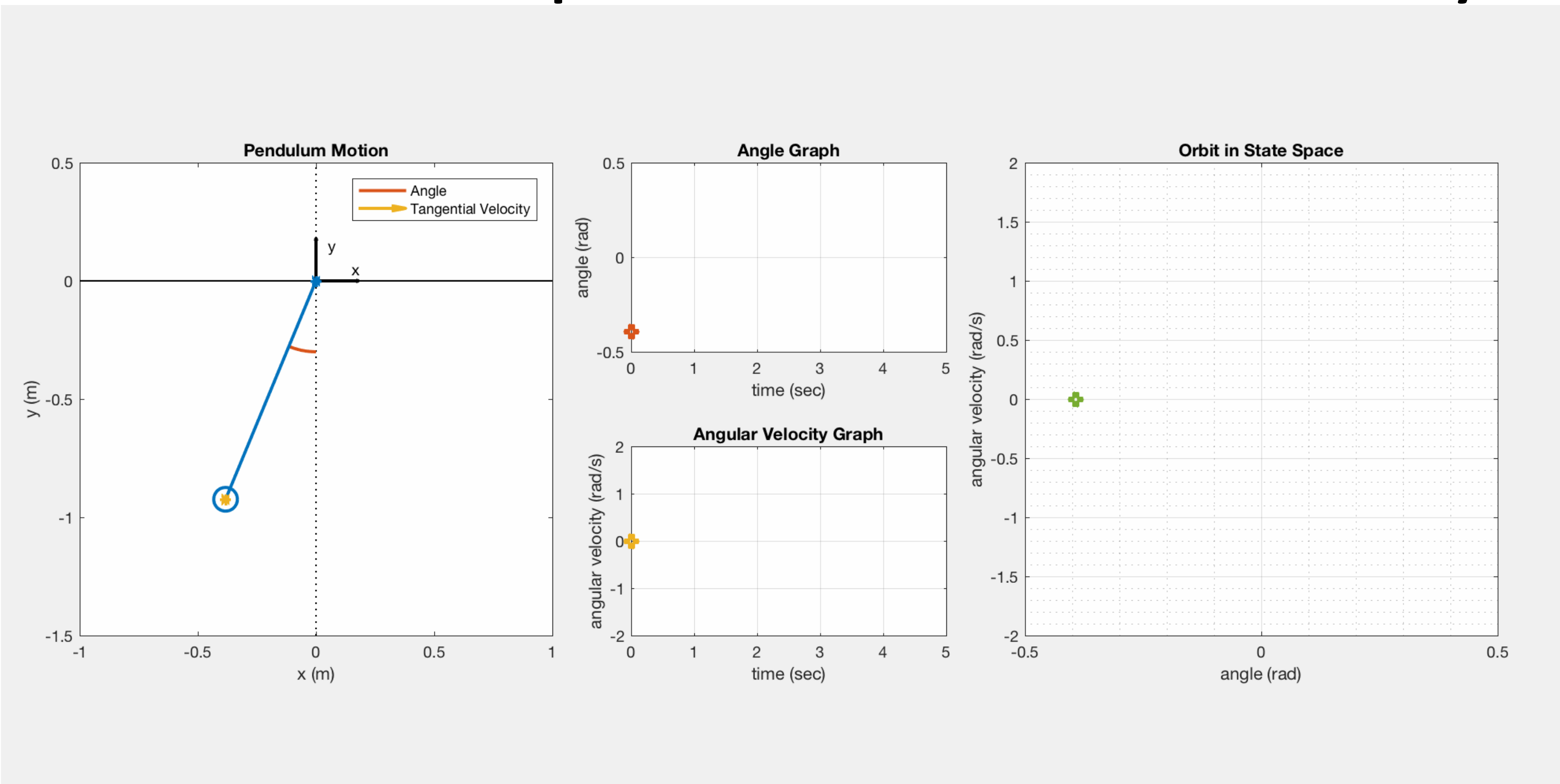
A Nonlinear Time Invariant Mechanical System

Video 3.1

Segment 1.2.2.1.a
Revolute 1 DoF Physics

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
June, 2017

The Simple Pendulum in Gravity

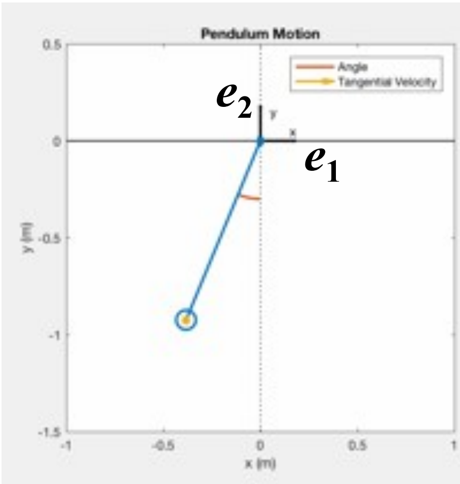


- 1 DoF revolute kinematics
  - deceptively familiar
  - archetypal nonlinear system
  - nonlinear space & dynamics

$$\theta \in \mathbb{S} := \left\{u := \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : \|u\| = 1\right\}$$

$$\boldsymbol{q} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \in T\mathbb{S} \approx \mathbb{S} \times \mathbb{R}$$

One DoF Revolute Kinematics



$$e_1 := \begin{bmatrix}1\\0\end{bmatrix}; \quad e_2 := \begin{bmatrix}0\\1\end{bmatrix}$$

$$b = \begin{bmatrix}b_x\\b_y\end{bmatrix}$$

$$= b_x e_1 + b_y e_2$$

$$b = \begin{bmatrix}\ell \cos\theta\\ \ell \sin\theta\end{bmatrix} \quad (1)$$

$$=: g_{\mathrm{R}}(\theta)$$

$$\dot{b} = D_\theta g_{\mathrm{R}} \dot{\theta}$$

$$= J g_{\mathrm{R}}(\theta)\dot{\theta}$$

$$J := \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

- Review Robo3 Week 1
  - adopt unit vector convention
  - body position entries
  - denote unit vector components
- 1 DoF revolute kinematics
  - a map from angles, θ
  - into the xy-plane b
  - lying on radius l circle
- Need infinitesimal kinematics too

One DoF Revolute Energies

- Gravitational potential energy

$$\begin{aligned}\varphi_{\mathrm{R}}(\theta) &= mgb_z \\ &= mg\ell\sin\theta\end{aligned} \qquad (2)$$

- Kinetic energy

$$\begin{aligned}\kappa_{\mathrm{R}}(\dot{\theta}) &= \frac{1}{2}m\|\dot{b}\|^2 \\ &= \frac{1}{2}m\left[Jg_{\mathrm{R}}\dot{\theta}\right]^T Jg_{\mathrm{R}}\dot{\theta} \\ &= \frac{1}{2}m\dot{\theta}^2 {g_{\mathrm{R}}}^{\mathrm{T}} J^{\mathrm{T}} J g_{\mathrm{R}} \\ &= \frac{1}{2}m\ell^2\dot{\theta}^2\end{aligned} \qquad (3)$$

- Introduce angle tangent vector

$$\boldsymbol{q} := \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

- Lagrangian

$$\begin{aligned}\lambda_{\mathrm{R}}(\boldsymbol{q}) := \kappa_{\mathrm{R}}(\dot{\theta}) \\ - \varphi_{\mathrm{R}}(\theta)\end{aligned} \qquad (4)$$

- Total energy

$$\begin{aligned}\eta_{\mathrm{R}}(\boldsymbol{q}) := \kappa_{\mathrm{R}}(\dot{\theta}) \\ + \varphi_{\mathrm{R}}(\theta)\end{aligned} \qquad (5)$$

Equations of Motion: Lossless Case

- Review Robo3 Week 2
  - Lagrangian mechanics
    - generated by Euler-Lagrange operator
    - applied to Lagrangian function
    - balanced by external forces
  - worked out for our example
    - no damping for now
    - add in shortly

$$\mathcal{D}_{\boldsymbol{q}}\lambda_{\mathrm{R}}(\boldsymbol{q}) = 0 \qquad \mathcal{D}_{\boldsymbol{q}} := D_t D_{\dot\theta} - D_\theta$$

$$\begin{aligned}
\mathcal{D}_{\boldsymbol{q}}\lambda_{\mathrm{R}}(\boldsymbol{q}) &= \left[D_t D_{\dot\theta} - \cancel{D_\theta}^{0}\right]\kappa_{\mathrm{R}} \\
&\quad - \left[D_t \cancel{D_{\dot\theta}}^{0} - D_\theta\right]\varphi_{\mathrm{R}} \\
&= D_t D_{\dot\theta}\kappa_{\mathrm{R}}(\dot\theta) + D_\theta\varphi_{\mathrm{R}}(\theta) \\
&= D_t D_{\dot\theta}\left(\frac{m\ell^2}{2}\dot\theta^2\right) \\
&\quad + D_\theta\left(mg\ell\sin\theta\right)) \\
&= m\ell^2\ddot\theta + m\ell g\cos\theta
\end{aligned}$$

$$\begin{gathered}
m\ell^2\ddot\theta = -m\ell g\cos\theta =: \Phi_{\mathrm{R}}(\boldsymbol{q}) \\
\Updownarrow \\
\ddot\theta = -\frac{g}{\ell}\cos\theta =: \Xi_{\mathrm{R}}(\boldsymbol{q})
\end{gathered} \tag{6}$$

- Reference:

Herbert Goldstein. Classical mechanics. Addison-Wesley World Student Series, Reading, Mass.: Addison-Wesley, 1950

edX Robo4 Mini MS – Locomotion Engineering

Block 1 – Week 2 – Unit 2

A Nonlnear Time Invariant Mechanical System

Video 3.2

Segment 1.2.2.1.b

Revolute 1 DoF Physics

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania

June, 2017

Lossless One DoF Revolute Vector Field

$$\ddot{\theta} = -\frac{\mathrm{g}}{\ell}\cos\theta =: \Xi_{\mathrm{R}}(\boldsymbol{q})$$

$$\Updownarrow$$

$$\begin{aligned}\dot{\boldsymbol{q}} &= \begin{bmatrix}\dot{\theta}\\ \Xi_{\mathrm{R}}(\boldsymbol{q})\end{bmatrix}\\ &= \begin{bmatrix}\dot{\theta}\\ -\frac{\mathrm{g}}{\ell}\cos\theta\end{bmatrix}\\ &=: f_{\mathrm{R}}(\boldsymbol{q})\end{aligned} \qquad (7)$$

- Once again, want VF
  - rewrite 2nd order scalar ODE
  - as 1st order vector ODE
- Nonlinear VF!
  - no matrices
  - no closed form series $f^t_{\mathrm{R}}(\boldsymbol{q}_0) = ??$
- Role of theory:
  - guarantees flow exists
  - gives rigorous methods
    - qualitative (nothing closed form) reasoning
    - formally guaranteed properties

Compare to eqns (2) & (6) in segment 1.2.1.2

- *there:*
  - *closed form (infinite series) expression of*
  - *a time-paramtrized family*
  - *of invertible nonlinear functions*
- *here:*
  - *guaranteed existence of*
  - *a time-paramtrized family*
  - *of invertible nonlinear functions*

Lossless Mechanism Conserves Energy

- Examine power
  - energy change along trajectory
    - total energy (5)
    - VF (7)
  - as function of state
- Conclude:
  - no power dissipation
  - energy is conserved
- No damping
  - we better get that result
  - … also get for DHO
    - when b:=0
    - power (Seg.1.2.1.3-eqn1) vanishes

$$
\begin{aligned}
\dot{\eta}_{\mathrm{R}} &= \frac{d}{dt}\eta_{\mathrm{R}} \circ f_{\mathrm{R}}^{t}(\boldsymbol{q}_0) \\
&= D_{\boldsymbol{q}}\eta_{\mathrm{R}} \cdot \frac{d}{dt} f_{\mathrm{R}}^{t}(\boldsymbol{q}_0) \\
&= \left[D_{\theta}\varphi_{\mathrm{R}}, D_{\dot{\theta}}\kappa_{\mathrm{R}}\right] \cdot f_{\mathrm{R}} \circ f_{\mathrm{R}}^{t}(\boldsymbol{q}_0) \\
&= \left(\left[-\Phi_{\mathrm{R}}, D_{\dot{\theta}}\kappa_{\mathrm{R}}\right] \cdot \left[\begin{smallmatrix} \dot{\theta} \\ \Xi_{\mathrm{R}} \end{smallmatrix}\right]\right)(t) \\
&= \left(\left[-\Phi_{\mathrm{R}}, m\ell^2\dot{\theta}\right] \cdot \left[\begin{smallmatrix} \dot{\theta} \\ \frac{1}{m\ell^2}\Phi_{\mathrm{R}} \end{smallmatrix}\right]\right)(t) \\
&= \left(\Phi_{\mathrm{R}}\dot{\theta} - \dot{\theta}\Phi_{\mathrm{R}}\right)(t) \equiv 0
\end{aligned}
$$

Compare with Seg.1.2.1.3 power computation

- *just used calculus & VF*
- *didn't really need closed form flow expression*

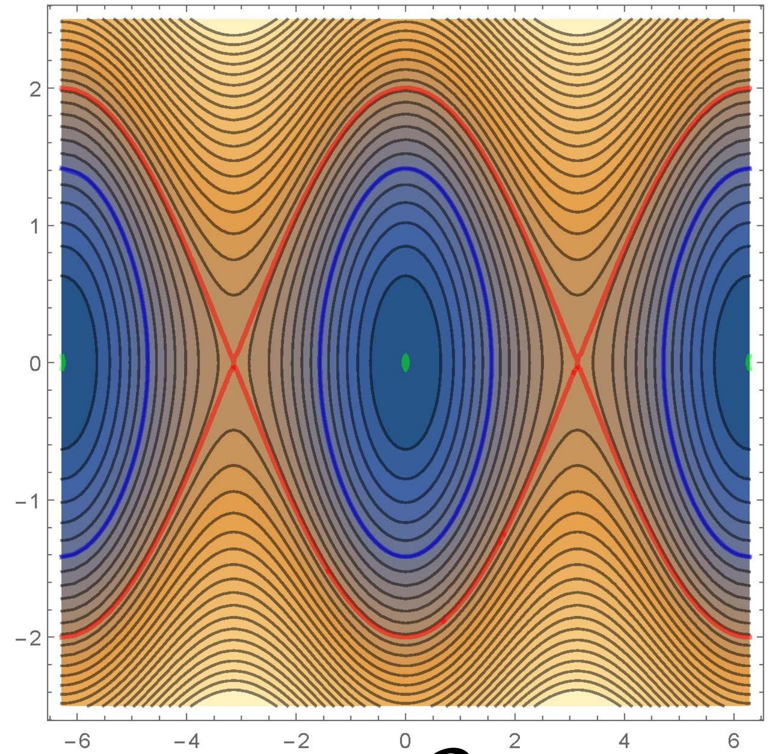
Example of Qualitative Method

- Energy is conserved
  - hence can solve constraint eqn
  - e.g. for velocity
  - as a function of angle
  - and IC energy
- Can't "get" trajectory
  - flow has no closed form expression
  - instead get "orbit"

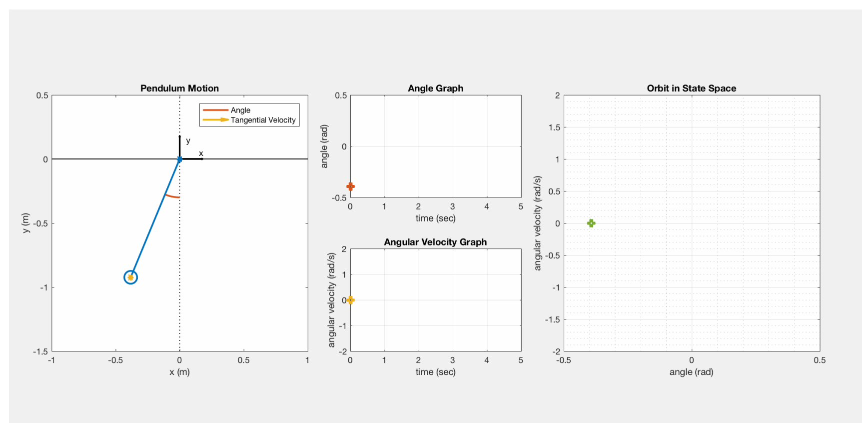
  (geometric curve on which trajectory must lie)

$$E_0 := \eta_{\mathrm{R}}(\boldsymbol{q}_0) =: $$
$$\equiv \eta_{\mathrm{R}} \circ f^t_{\mathrm{R}}(\boldsymbol{q}_0)$$
$$= \frac{m\ell^2}{2}\dot{\theta}^2(t) + mg\ell \sin\theta(t) \qquad (8)$$
$$\Downarrow$$
$$\dot{\theta} = \pm\sqrt{\frac{2}{m\ell^2}}\sqrt{E_0 - mg\ell\sin\theta}$$

$\dot{\theta}$



θ



Moving ahead

- Such geometric techniques
- Will be very important
- Although usually more complicated
- As state space dimension grows

edX Robo4 Mini MS – Locomotion Engineering

Block 1 – Week 2 – Unit 2

A Nonlnear Time Invariant Mechanical System

Video 3.3

Segment 1.2.2.2.a

Stable & Unstable Fixed Points

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Steady State Notion

- "steady state": a colloquial term
  - intuitively: a persistent operating condition
  - what is a "condition"?
  - what is "persistent"?
- this unit: formalize simplest case
  - fixed point (FP)

$$f^t(x_0) \equiv x_0$$

    - IC whose solutions are constant

VF View of Steady State Notion

- this unit: formalize simplest case
  - fixed point (FP)

$$f^t(x_0) \equiv x_0$$

    - IC whose solutions are constant
    - Implies zero of the VF

$$\begin{aligned} 0 = \frac{d}{dt} x_0 &= \frac{d}{dt} f^t(x_0) \\ &= f \circ f^t(x_0) \\ &= f(x_0) \end{aligned}$$

  - notice "condition" means "orbit"

(soon: more interesting orbits)

Stability: First Pass at “Persistence” Notion

- Persistence in state
  - if we perturb IC

$$\tilde{x}_0 := x_0 + \delta$$

  - will we remain in “same” operating condition?

$$f^t(\tilde{x}_0) \equiv ??$$

- Formalize for FP condition
  - e.g., 1 dim VF

$$f_{\mathrm{D}}(x) = \lambda x$$

  - from Seg. 1.2.1.1

Stability: First Pass “Persistence” Example

- Persistence in state
  - if we perturb IC $\tilde{x}_0 := x_0 + \delta$
  - will we remain in “same” operating condition?
- e.g., 1 dim VF from Seg.1.2.1.1 $f_{\mathrm{D}}(x) = \lambda x$
  - check flow $$f_{\mathrm{D}}^t(\tilde{x}_0) = \mathrm{e}^{t\lambda}(x_0 + \delta) = \mathrm{e}^{t\lambda}x_0 + \mathrm{e}^{t\lambda}\delta$$
  - impose FP assumption
$$0 = f_{\mathrm{D}}(x_0) = \lambda x_0$$
$$\Updownarrow$$
$$\lambda = 0 \text{ or } x_0 = 0$$
  - to realize: “same” (or near) implies λ nonpositive

$$\left|\mathrm{e}^{\lambda}\right| << 1 \Leftrightarrow \lambda \leq 0$$

Next Example: Damped Harmonic Oscillator

This image cannot currently be displayed.

- Recall from Seg.1.2.1.2
  - 2^{nd} order scalar ODE (two scalar time trajectories)
  - 1^{st} order vector ODE (planar trajectory lies on “orbit”)

Penn Engineering

Recall Damped Harmonic Oscillator VF

- DHO (Seg.1.2.1.2)
  - instead of physical coordinates
  - use real canonical cords

$$\mathbf{y} := S^{-1}\mathbf{x} =: h_{\mathrm{RC}}(\mathbf{x})$$

$$S := E \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

  - write out conjugate matrix representation
  - assume either σ or ω nonzero
  - conclude FP only at origin

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} v \\ \Xi_{\mathrm{DHO}}(\mathbf{x}) \end{bmatrix} \\ &= \begin{bmatrix} \dot{x} \\ -\frac{b}{m}\dot{x} - \frac{k}{m}x \end{bmatrix} =: f_{\mathrm{DHO}}(\mathbf{x}) \\ &= A\mathbf{x}; \quad A := \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \end{aligned}$$

$$f_{\mathrm{RC}}(\mathbf{y}) = \tilde{A}\mathbf{y}$$

$$\tilde{A} = \sigma I + \omega J$$

$$\Downarrow$$

$$\left|\tilde{A}\right| = \sigma^2 + \omega^2$$

Stability of Damped Harmonic Oscillator

- formalize for FP condition
  - perturb orbit through IC at origin

$$\tilde{\boldsymbol{y}}_0 = \delta := \begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix}$$

  - closed form perturbed trajectory

$$f^t_{\mathrm{RC}}(\delta) = \mathrm{e}^{t\sigma} R(t\omega)\delta$$

$$\Downarrow \quad R^{-1} = R^T$$

$$\| f^t_{\mathrm{RC}}(\delta) \|^2 = \mathrm{e}^{2t\sigma} \|\delta\|^2$$

  - easy to compute magnitude
  - conclude: stability implies nonpositive σ

Capturing Planar Stability via Norm

- Seg.1.2.1.3 exercises:
  - total energy is a norm

$$\tilde{\eta}_{\mathrm{HO}}(\mathbf{y}) = \eta_{\mathrm{DHO}} \circ h_{\mathrm{nrml}}^{-1}(\mathbf{y})$$
$$= \frac{1}{2}\|\mathbf{y}\|^2$$

  - in real canonical coordinates

- now compute power

$$\dot{\tilde{\eta}}_{\mathrm{HO}}(\mathbf{y}) = \frac{1}{2}\frac{d}{dt}\mathbf{y}^{\mathrm{T}}\mathbf{y}$$
$$= \mathbf{y}^{\mathrm{T}} f_{\mathrm{RC}}(\mathbf{y})$$
$$= \mathbf{y}^{\mathrm{T}}(\sigma I + \omega J)\,\mathbf{y}$$
$$= \sigma \mathbf{y}^{\mathrm{T}}\mathbf{y}$$

  - also a norm

A Scalar View of Vector Flow Stability

- power also a norm

$$\dot{\tilde{\eta}}_{\mathrm{HO}}\left(\mathbf{y}\right) = \frac{1}{2}\frac{d}{dt}\mathbf{y}^{\mathrm{T}}\mathbf{y}$$
$$= \mathbf{y}^{\mathrm{T}} f_{\mathrm{RC}}(\mathbf{y})$$
$$= \mathbf{y}^{\mathrm{T}}\left(\sigma I + \omega J\right)\mathbf{y}$$
$$= \sigma \mathbf{y}^{\mathrm{T}}\mathbf{y}$$

$$\tilde{\eta}_{\mathrm{HO}}(\mathbf{y}) = \frac{1}{2}\|\mathbf{y}\|^2$$
$$\Downarrow$$
$$\dot{\tilde{\eta}}_{\mathrm{HO}} = 2\sigma\tilde{\eta}_{\mathrm{HO}}$$

- so energy satisfies scalar ODE
  - stability from scalar LTI ODE
  - inherited by vector flow
    - neighborhoods of 0 in R
    - define neighborhoods of 0 in R^2
  - again conclude: stability implies σ nonpositive

edX Robo4 Mini MS – Locomotion Engineering

Block 1 – Week 2 – Unit 2

A Nonlnear Time Invariant Mechanical System

Video 3.4

Segment 1.2.2.2.b

Stable & Unstable Fixed Points

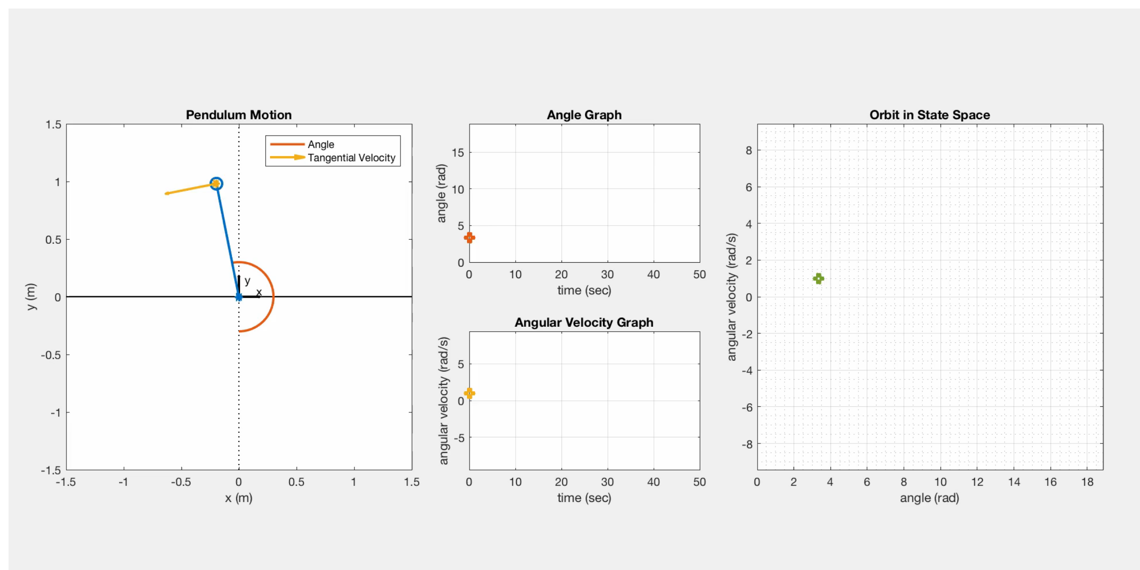
Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania

July, 2017

Damped Pendulum: Stable & Unstable FP

This image cannot currently be displayed.

- Look at Pendulum again
  - this time with damping
  - previously
    - “top” IC
    - unstable FP
  - damping
    - stabilizes bottom
    - “top” FP still unstable



Multiple FP: Different Stability Properties

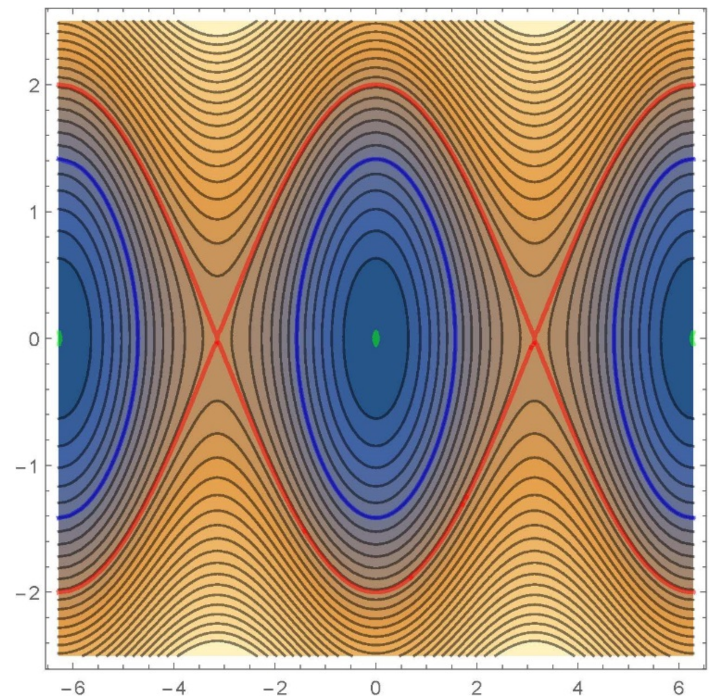
- DP VF (Seg.1.2.2.1)

$$f_{\mathrm{DP}}(\boldsymbol{q}) := \begin{bmatrix} \dot\theta \\ -\frac{\mathrm{g}}{\ell}\cos\theta - \frac{b}{m\ell^2}\dot\theta \end{bmatrix} \qquad (1)$$

- DP Power (Exr.1.2.2.1)

$$\dot\eta_{\mathrm{DP}}(\boldsymbol{q}) := D_{\boldsymbol{q}}\eta_{\mathrm{R}} \cdot f_{\mathrm{DP}}(\boldsymbol{q}) \\ = -b\dot\theta^2 \qquad (2)$$

- NL systems
  - can have multiple FP
  - pendulum:
    - "bottom" & "top"



Predict Different Stability Properties ?

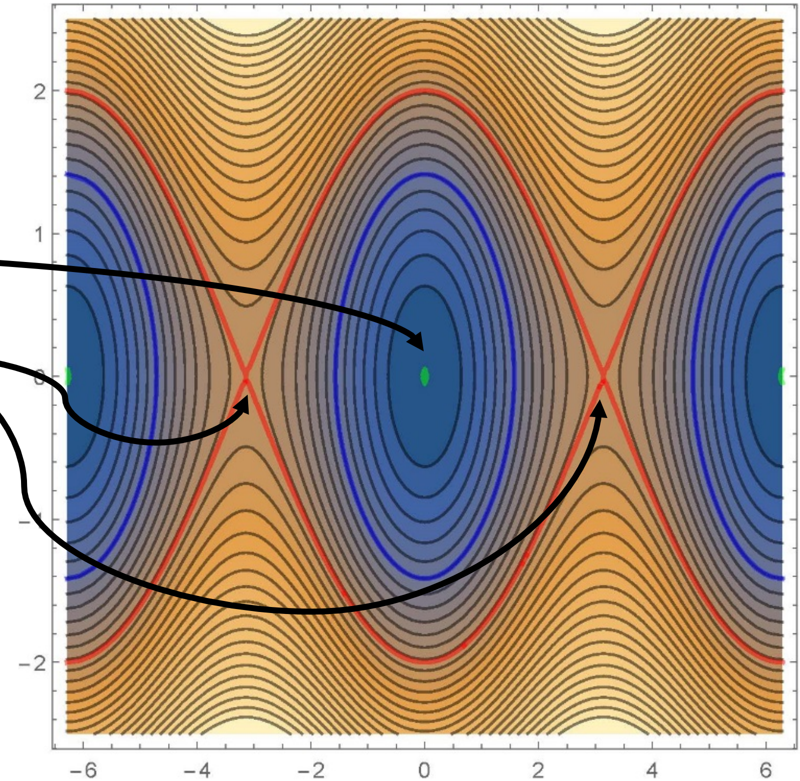
- Showed power is nonpositive

$$\dot{\eta}_{\mathrm{DP}}\left(\boldsymbol{q}\right) = -b\dot{\theta}^2$$

- But FP may or may not be stable
- Total energy
  - norm-like at bottom

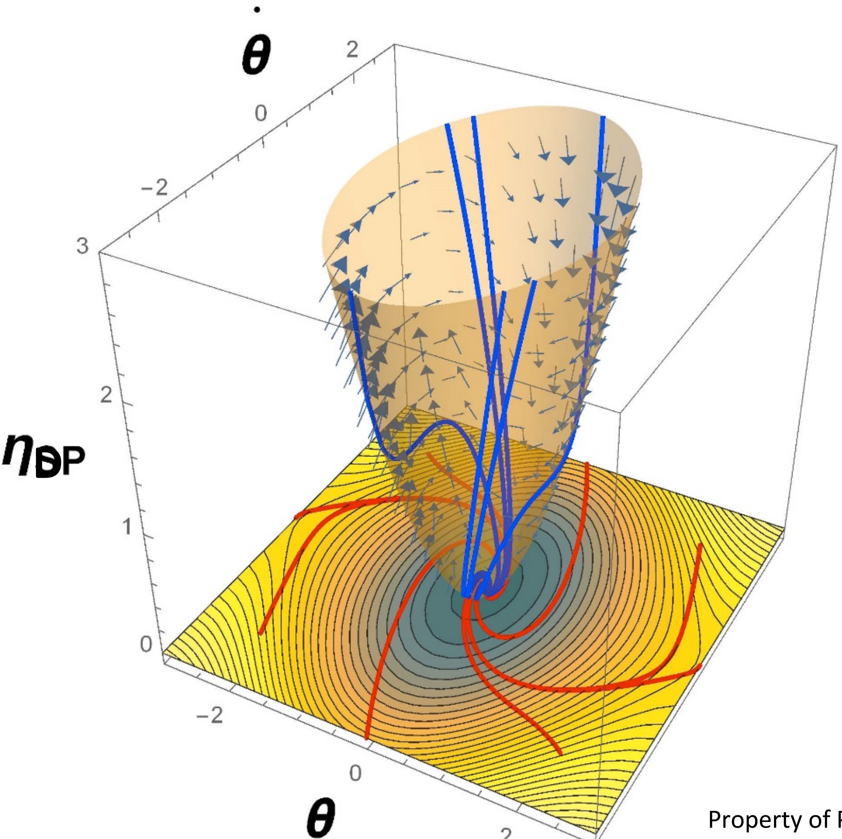
$$\boldsymbol{q}_b := 0 \;\; \& \;\; \boldsymbol{q}_t := \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

  - “saddle” at top

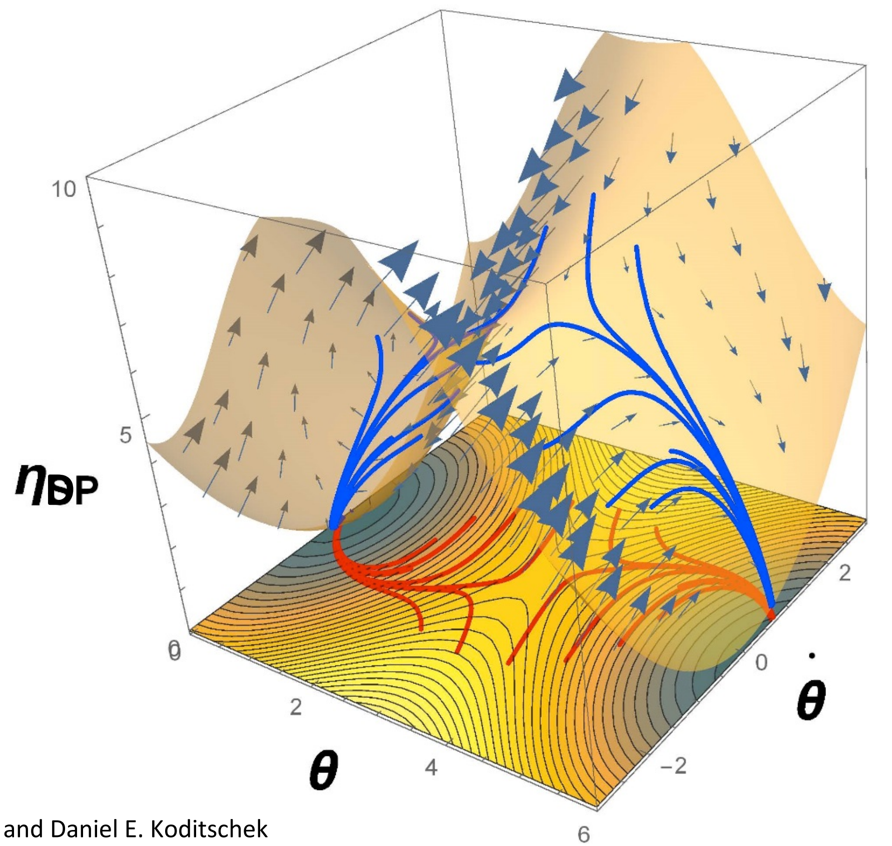


Simulations Suggest What’s Happening

- total energy at “bottom”
  - level curves enclose neighborhoods (norm-like)
  - stable FP
  - with basin of attraction



- total energy at “top”
  - level curves run off
  - unstable FP
  - no basin



Need a Qualitative Theory

- Damped Harmonic Oscillator
  - LTI: closed form solutions
  - total energy: norm-like
    - an explicit norm
    - in the "right" coordinates
    - yields scalar LTI energy ODE
- Damped Pendulum
  - NLTI: no closed form solutions
  - total energy: sometimes norm-like
    - what is the "norm-like" property?
    - how to get rigorous conclusions without a scalar ODE?

$$\tilde{\eta}_{\mathrm{HO}}(\mathbf{y}) = \frac{1}{2}\|\mathbf{y}\|^2$$

$$\dot{\tilde{\eta}}_{\mathrm{HO}} = \sigma\tilde{\eta}_{\mathrm{HO}}$$

$$\eta_{\mathrm{DP}}(\boldsymbol{q}) = \tfrac{m\ell^2}{2}\dot{\theta}^2 + mg\ell\sin\theta$$

$$\dot{\eta}_{\mathrm{DP}}(\boldsymbol{q}) = -b\dot{\theta}^2 \overset{?}{\neq} \mathrm{func}(\eta_{\mathrm{DP}})$$

Moving Ahead

- Linearization
  - locally (e.g. near FP)
  - often can use Taylor approximation of NL VF
  - to get Taylor approximation of NL flow
- Lyapunov Functions
  - locally (e.g. near FP)
  - stability properties formally characterized
  - by generalized "total energy"
- Beyond this course
  - notion of "global" Lyapunov function
  - gives "fundamental theorem of dynamical systems"