edX Robo4 Mini MS – Locomotion Engineering

Week 8 – Unit 2

Spring Loaded Inverted Pendulum

Video 10.1

Segment 8.2.1

Hybrid Systems Model – Guards, Resets, Mode Maps

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
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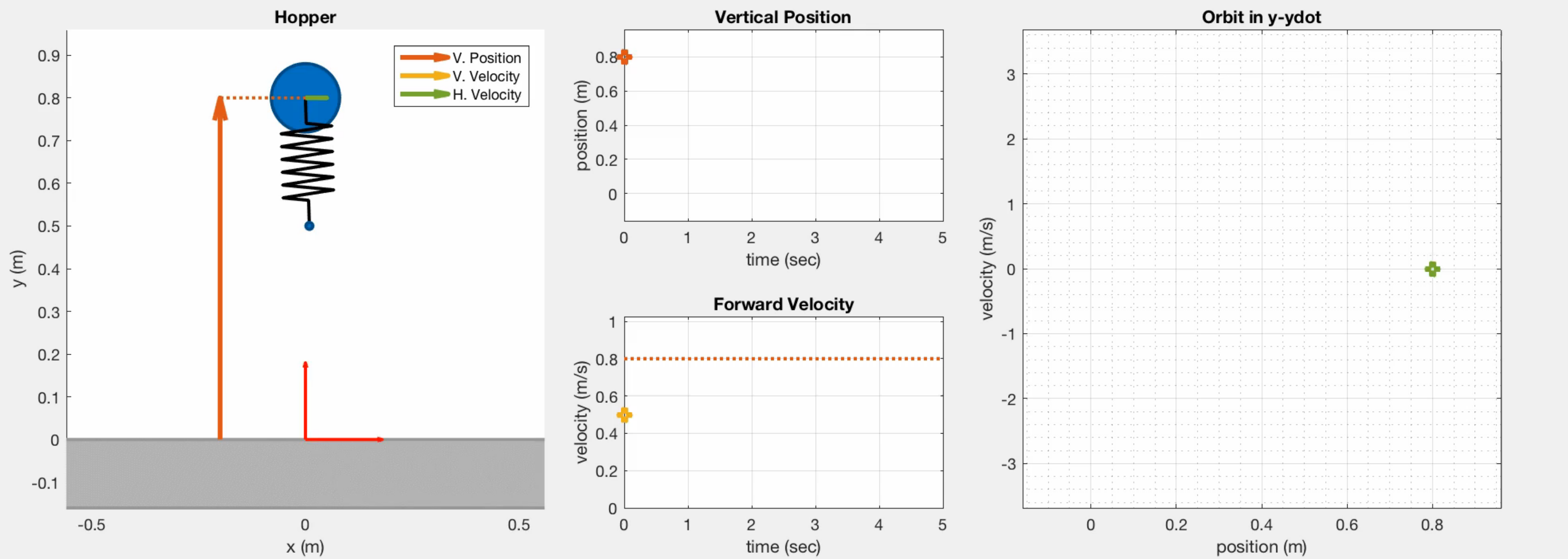
August, 2017

Course: Heading Toward Real Locomotion



Simulation credit: Vassilis Vasilopoulos

This segment:

define and compose the various modes to get Poincare' Map

Recall Approach to Vertical Hopper

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express mode map
- Model reset map
- Compose
  - mode map ∘ reset map
  - further compose each composition in turn
- End up with return map

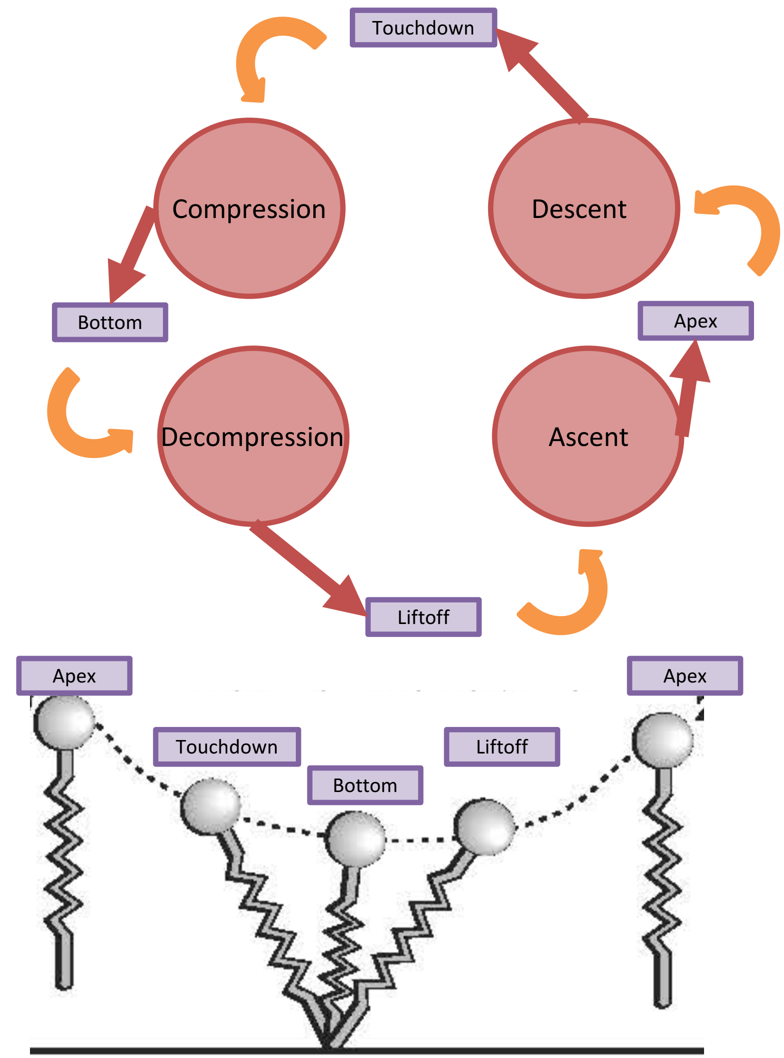
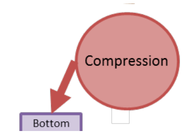


figure adapted from
W. J. Schwind and D. E. Koditschek, "Approximating the Stance Map of a 2-DOF Monoped Runner," *Journal of Nonlinear Science*, vol. 10, no. 5, pp. 533–568, 2000.

Hybrid Ingredients: Guards

- Bottom condition (polar cords)
  - leg extension velocity sign change
  - via compression guard

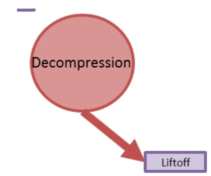
$$\gamma_{\text{comp}}(\mathbf{p}_q) := p_\chi$$



$$\mathbf{p}_{q_b} \in \gamma_{\text{comp}}^{-1}[0]$$

- Liftoff condition (polar coords)
  - leg extension at spring rest length χ_l
  - via decompression guard

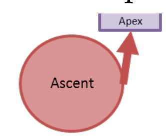
$$\gamma_{\text{dcmp}}(\mathbf{p}_q) := q_\chi - \chi_\ell$$



$$\mathbf{p}_{q_l} \in \gamma_{\text{dcmp}}^{-1}[0]$$

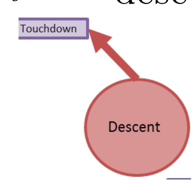
- Apex condition (Cartesian coords)
  - vertical velocity sign change
  - via apex guard $\gamma_{\text{apex}}(\mathbf{p}_b) := p_y$

$$\mathbf{p}_{b_a} \in \gamma_{\text{apex}}^{-1}[0]$$



- Touchdown condition (Cartesian coords)
  - leg extension at spring rest length χ_l
  - this via descent guard $\gamma_{\text{desc}}(\mathbf{p}_b) := b_y - \chi_\ell \cos q_{\theta_t}$

$$\mathbf{p}_{b_t} \in \gamma_{\text{desc}}^{-1}[0]$$

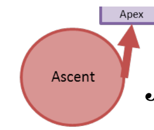


Hybrid Ingredients: Flight Event Maps

- Apex event (vertical height flow)
  - use $p_y(b_y)$ (Seg. 8.1.2) & solve for b_y

$$0 = \gamma_{\text{apex}} \circ \tilde{f}^{b_y}_{\text{PBF}}(\mathbf{p}_{b_l}) = \sqrt{p^2_{y_l} + 2\mu^2 \mathrm{g}(b_{y_l} - b_y)}$$

$\mathbf{p}_{b_a} \in \gamma^{-1}_{\text{apex}}[0]$



$f^{\text{AP}}_{\text{LO}}$

$\mathbf{p}_{b_l} \in \gamma^{-1}_{\text{dcmp}}[0]$

  - get apex height and time

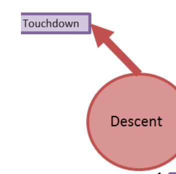
$$b_{y_a} = b_{y_l} + \frac{p^2_{y_l}}{2\mu^2 \mathrm{g}}; \qquad T_a(\mathbf{p}_{q_l}) = \frac{p_{y_l}}{\mu \mathrm{g}}$$

  - to complete ascent flow map

$$\mathbf{p}_{b_a} = f^{\text{AP}}_{\text{LO}}(\mathbf{p}_{q_l}) := f^{T_a(\mathbf{p}_{q_l})}_{\text{PBF}}(\mathbf{p}_{q_l})$$

- TD event (vertical height flow)

$\mathbf{p}_{b_t} \in \gamma^{-1}_{\text{desc}}[0]$



$f^{\text{TD}}_{\text{AP}}$

$\mathbf{p}_{b_a} \in \gamma^{-1}_{\text{apex}}[0]$

  - guard zero height $b_{y_t} = \chi_\ell \cos q_{\theta_t}$
  - enters $T(b_y)$ (Seg. 8.1.2) with , choice

$$T_t(\mathbf{p}_{b_a}, q_{\theta_t}) := \sqrt{\frac{2\left(b_{y_a} - q_{\theta_t} \cos q_{\theta_t}\right)}{\mathrm{g}}}$$

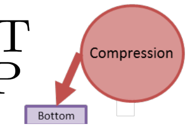
  - yields descent flow map $f^{\text{TD}}_{\text{AP}}(\mathbf{p}_{b_a}, q_{\theta_t}) := f^{T_t(\mathbf{p}_{b_a}, q_{\theta_t})}_{\text{PBF}}(\mathbf{p}_{b_a})$

Hybrid Ingredients: Stance Event Maps

- Bottom event (leg extension flow)

$$\mathbf{p}_{q_t} \in \gamma_{\text{desc}}^{-1}[0]$$

$$f_{\text{RP}}^{\text{BT}}$$



$$\mathbf{p}_{q_b} \in \gamma_{\text{comp}}^{-1}[0]$$

  - use $p_\chi(q_\chi)$ (Seg. 8.1.3) & solve for q_χ

$$0 = \gamma_{\text{comp}} \circ \tilde{f}_{\text{RP}}^{q_\chi}(\mathbf{p}_{q_t}) = \left[p_{\chi_t}^2 + \frac{(p_{\theta_t} + \mu k)(q_\chi^2 - q_{\theta_t}^2)}{q_\chi^2 q_{\theta_t}^2} \right]^{\frac{1}{2}}$$

  - get bottom extension and time

$$q_{\chi_b} = \sqrt{\frac{q_{\theta_t}^2 (p_{\theta_t}^2 + \mu k)}{q_{\theta_t}^2 p_{\chi_t}^2 + (p_{\theta_t}^2 + \mu k)}}; \qquad T_b(\mathbf{p}_{q_t}) := \frac{-\mu q_{\theta_t}^3 p_{\chi_t}}{p_{\theta_t}^2 + \mu k + q_{\theta_t}^2 p_{\chi_t}^2}$$

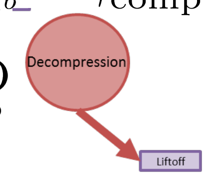
  - yields compression mode flow map

$$\mathbf{p}_b = f_{\text{RP}}^{\text{BT}}(\mathbf{p}_{q_t}) := f_{\text{RP}}^{T_b(\mathbf{p}_{q_t})}(\mathbf{p}_{q_t})$$

- Liftoff event (extension flow)

$$\mathbf{p}_{q_b} \in \gamma_{\text{comp}}^{-1}[0]$$

$$f_{\text{RP}}^{\text{LO}}$$



$$\mathbf{p}_{q_l} \in \gamma_{\text{dcmp}}^{-1}[0]$$

  - guard shaft rest-length $q_{\chi_b} = \chi_\ell$
  - enters $T(q_\chi)$ (Seg. 8.1.3)

$$T_l(\mathbf{p}_b) := \mu q_{\chi_b} \sqrt{\frac{\chi_\ell^2 - q_{\chi_b}}{p_{\theta_b}^2 + \mu k}}$$

  - yields decompression mode flow map $f_{\text{RP}}^{\text{LO}}(\mathbf{p}_b) := f_{\text{RP}}^{T_l(\mathbf{p}_b)}(\mathbf{p}_b)$

Hybrid Ingredients: Resets

- Liftoff event

  polar -> Cartesian

$$\mathbf{p}_{b_l} = \mathbf{h}_{\mathrm{RP}}^{-1}(\mathbf{p}_{q_l})$$
$$=: r_{\mathrm{LO}}^{\mathrm{FL}}(\mathbf{p}_{q_l})$$

- Touchdown event
  - Cartesian-> polar

$$\mathbf{p}_q = \mathbf{h}_{\mathrm{RP}}(\mathbf{p}_{b_t})$$
$$=: r_{\mathrm{TD}}^{\mathrm{RP}}(\mathbf{p}_{b_t})$$

- No reset (identity map) needed
  - Bottom

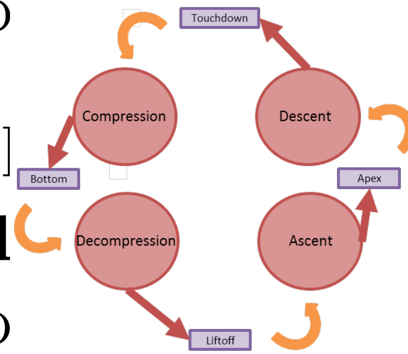
  (polar -> polar)
  - Apex

  (Cartesian -> Cartesian)

$\mathbf{p}_{q_t} \in \gamma_{\mathrm{desc}}^{-1}[0]$ $\quad r_{\mathrm{TD}}^{\mathrm{RP}}$ $\quad \mathbf{p}_{b_t} \in \gamma_{\mathrm{desc}}^{-1}[0]$

$f_{\mathrm{RP}}^{\mathrm{TD}}$ $\quad f_{\mathrm{AP}}^{\mathrm{TD}}$

$\mathbf{p}_{q_b} \in \gamma_{\mathrm{comp}}^{-1}[0]$ $\quad \mathbf{id}$

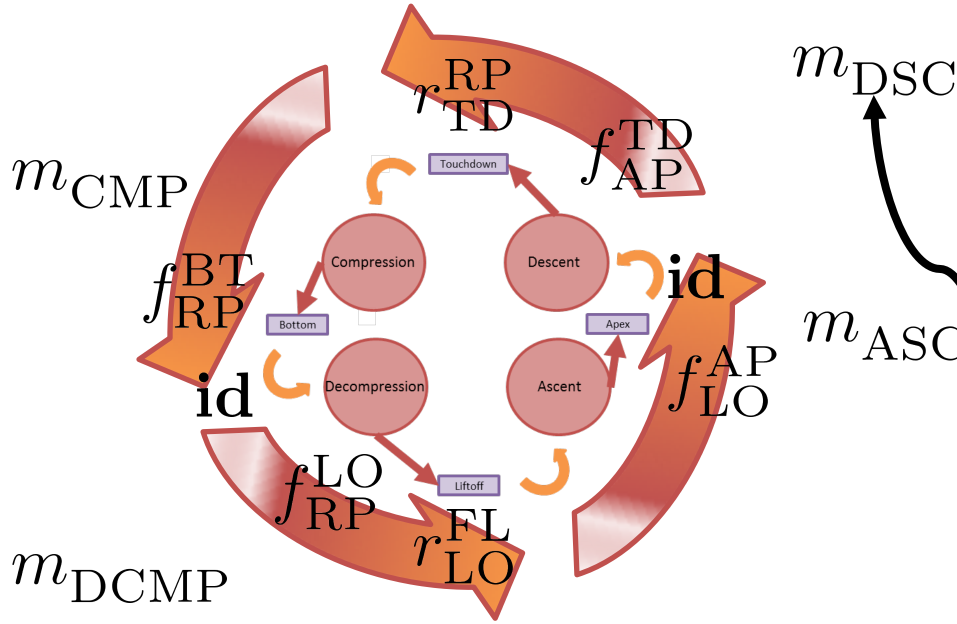


$\mathbf{id}$ $\quad \mathbf{p}_{b_a} \in \gamma_{\mathrm{apex}}^{-1}[0]$

$f_{\mathrm{RP}}^{\mathrm{LO}}$ $\quad f_{\mathrm{LO}}^{\mathrm{AP}}$

$\mathbf{p}_{q_l} \in \gamma_{\mathrm{dcmp}}^{-1}[0]$ $\quad r_{\mathrm{LO}}^{\mathrm{FL}}$ $\quad \mathbf{p}_{b_l} \in \gamma_{\mathrm{dcmp}}^{-1}[0]$

Hybrid Model: Mode Maps



- Ascent mode

$$m_{\mathrm{ASC}} := \mathbf{id} \circ f_{\mathrm{LO}}^{\mathrm{AP}}$$

- Descent mode

$$m_{\mathrm{DSC}} := r_{\mathrm{TD}}^{\mathrm{RP}} \circ f_{\mathrm{AP}}^{\mathrm{TD}}$$

- Compression mode

$$m_{\mathrm{CMP}} = \mathbf{id} \circ f_{\mathrm{RP}}^{\mathrm{TD}}$$

- Decompression mode

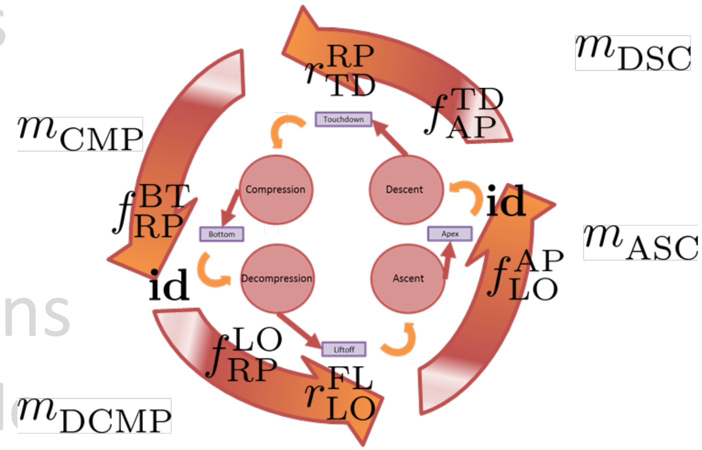
$$m_{\mathrm{DCMP}} := r_{\mathrm{LO}}^{\mathrm{FL}} \circ f_{\mathrm{RP}}^{\mathrm{LO}}$$

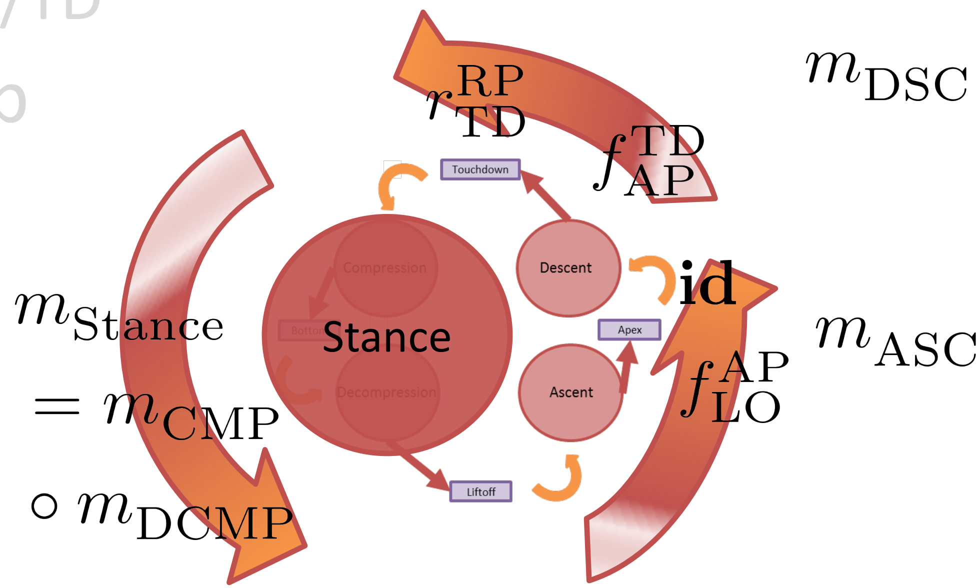
Descent Control Parameter

$$\begin{cases} m_{\mathrm{DSC}}(\mathbf{p}_{b_a}, q_{\theta_t}) \\ := r_{\mathrm{TD}}^{\mathrm{RP}} \circ f_{\mathrm{AP}}^{\mathrm{TD}}(\mathbf{p}_{b_a}, q_{\theta_t}) \\ = r_{\mathrm{TD}}^{\mathrm{RP}} \circ f_{\mathrm{PBF}}^{T_t(\mathbf{p}_{b_a}, q_{\theta_t})}(\mathbf{p}_{b_a}) \end{cases}$$

Next: Depart Vertical Hopper (Stance Mode)

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express mode map
- Model reset map
- Compose
  - mode map ∘ reset map
  - further compose
  - **impose symmetry**
- **End up with Fore-Aft control return map**





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Hybrid Systems Model – Guards, Resets, Mode Maps

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Hybrid model: SLIP F-A Relative to RVH

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express mode map
- Model reset map
- Compose
  - mode map ◦ reset map
  - further compose each composition in turn
- End up with return map

RVH:
Control
asserted
during stance

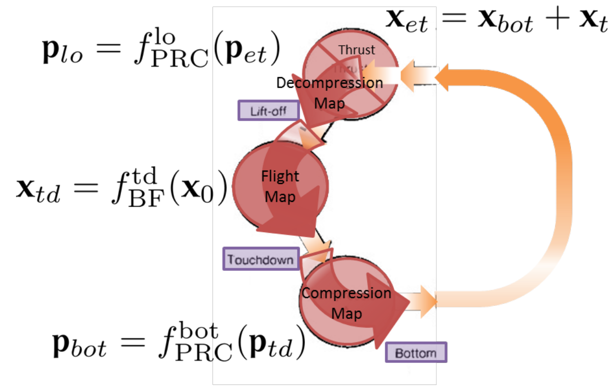

figure modified from
Raibert, *Commun. ACM*, 1986, *op. cit.*

SLIP F-A:
Control
asserted
during flight

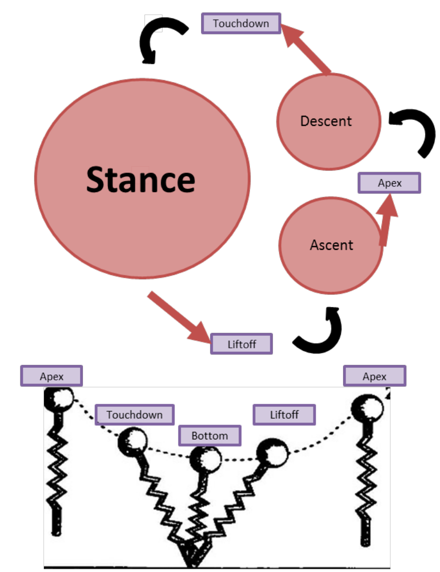

figure adapted from
W. J. Schwind, "Spring loaded inverted pendulum running: A plant model," University of Michigan, PhD Thesis,1998.

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Week 8 – Unit 2

Spring Loaded Inverted Pendulum

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Segment 8.2.2

Hybrid Systems Model – Poincare'

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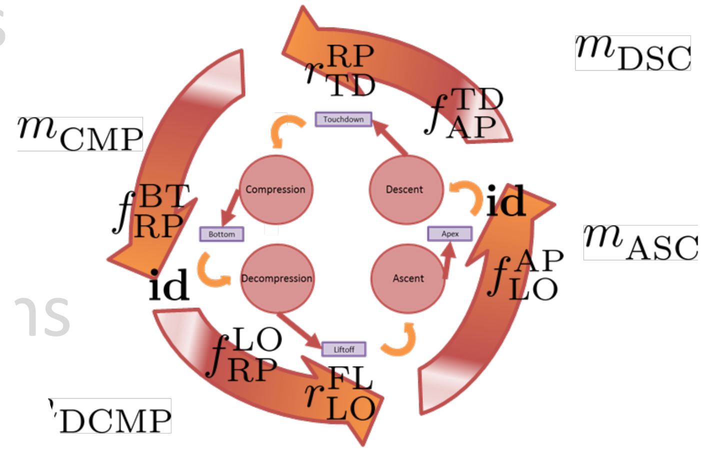
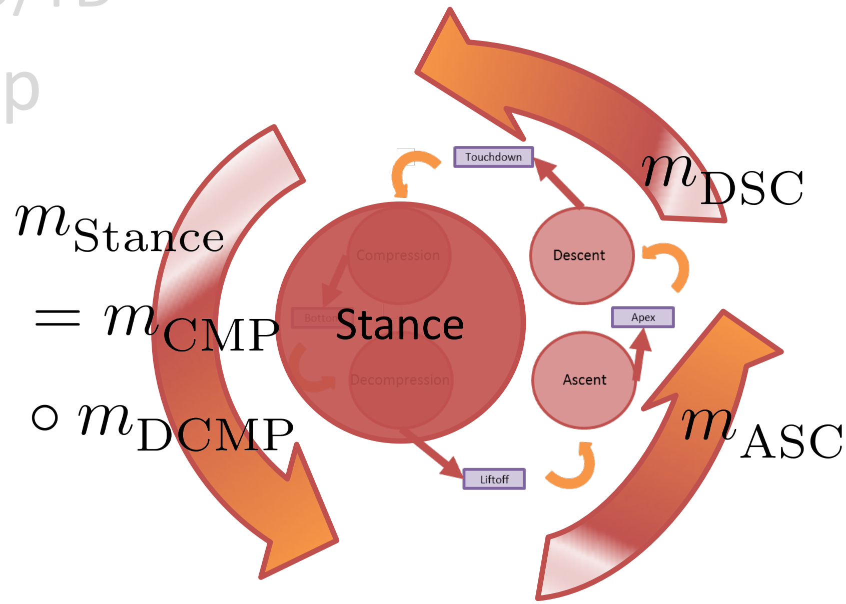
August, 2017

Fore-Aft Velocity Control Poincare' Map

- Model Continuous tim... ws
  - each mode ...
  - ge...
  - ...ns
  - .../ID
- S... mode map
- M... reset map
- Compose
  - mode map ∘ reset map
  - further compose
  - impose symmetry
- End up with Fore-Aft control Poincare' ("return") map

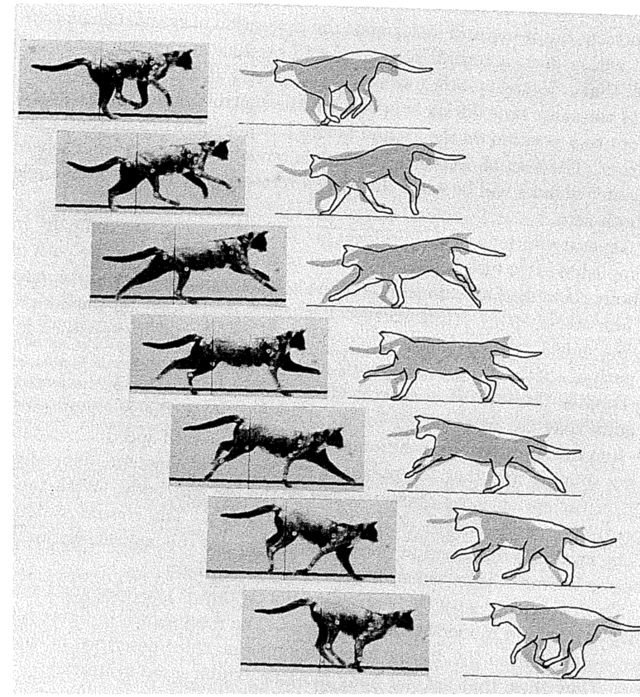
This segment:
- focus on fore-aft speed
- assume well regulated height
- merge de/compression
- yields single stance mode

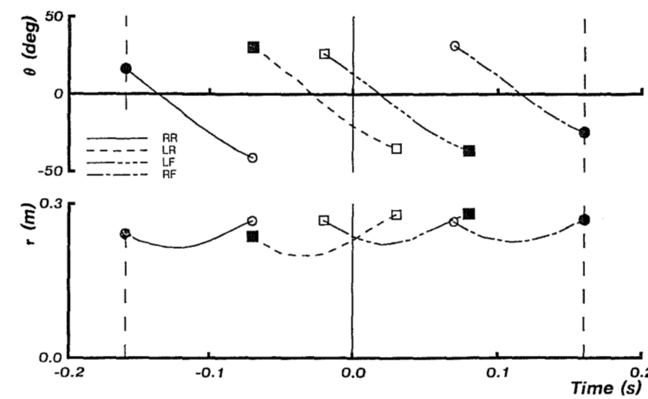




Introducing Symmetry

- Raibert identified the importance of symmetry
  - apparent forward-reverse time similarity
  - in animals and machines
- Time reversal symmetry
  - deep, important idea in Physics
    - lossless systems are reversible
    - "time's arrow" due to dissipation
  - growing insights for locomotion
    - SLIP model: "piecewise Hamiltonian"
    - generally: non-conservative effects play a "merely perturbative" role





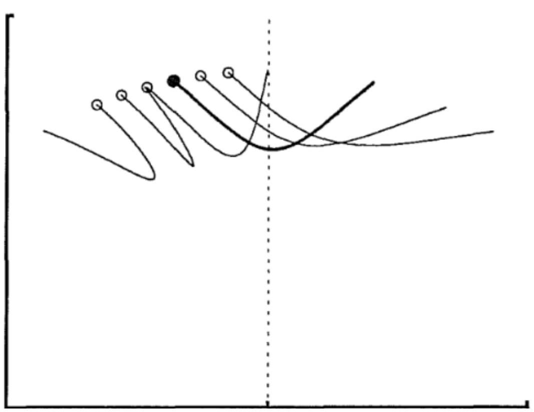
figures from
M. H. Raibert, Legged Robots That Balance. Cambridge: MIT Press, 1986.

Formalizing SLIP Symmetry: Neutral Angle

- Introduce "linear" CC, S, for polar (RP) coordinates :

$$\begin{bmatrix} \begin{bmatrix} q_\chi \\ -q_\theta \end{bmatrix} \\ \begin{bmatrix} -p_\chi \\ p_\theta \end{bmatrix} \end{bmatrix} = S \begin{bmatrix} \begin{bmatrix} q_\chi \\ q_\theta \end{bmatrix} \\ \begin{bmatrix} p_\chi \\ p_\theta \end{bmatrix} \end{bmatrix}; \quad S := \begin{bmatrix} L & 0 \\ 0 & -L \end{bmatrix}; \quad L := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  - reverses extension time & vertical toe angle
  - acts as an involution $S^{-1} = S$
  - self-conjugates stance VF f_{RP} i.e., $S f_{\mathrm{RP}}(\mathbf{p}_q) = f_{\mathrm{RP}}(S\mathbf{p}_q)$
- Formalizes Raibert's "neutral" angle
  - unique td toe angle, q_{θ_t} ,
  - such that fore-aft td velocity, p_{x_t} ,
  - achieves unaccelerated lo, $p_{x_l} = p_{x_t}$



M. H. Raibert, 1986, op. cit.

Interpreting SLIP Symmetry

- Bottom symmetry interpretation of neutral
  - neutral angles lie on orbits symmetric about $\gamma_{\text{bottom}}^{-1}[0]$
  - or, equivalently, orbits passing through FP of S,

$$\text{Fix}_S := \{\mathbf{p}_q \equiv S\mathbf{p}_q\} = \{\mathbf{p}_q : q_\theta = 0 \ \& \ p_\chi = 0\} \subset \gamma_{\text{bottom}}^{-1}[0]$$

  - corresponding to trajectories through bottom IC
    - with odd-in-time q_θ–components
    - with even-in-time q_χ–components
  - or, equivalently $\mathbf{p}_{q_b} \in \text{Fix}_S \Rightarrow f_{\text{RP}}^{-t}(\mathbf{p}_{q_b}) = f_{\text{RP}}^{t}(S\mathbf{p}_{q_b})$
- Further simplification from neglecting stance gravity
  - all orbits are "neutral" wrt their bottom state

$$\mathbf{p}_{q_b} \in \gamma_{\text{bottom}}^{-1}[0] \Rightarrow \hat{f}_{\text{RP}}^{-t}(\mathbf{p}_{q_b}) = \hat{f}_{\text{RP}}^{t}(S\mathbf{p}_{q_b})$$

  - due to cyclic nature of q_θ of in 0-grav RP VF $\hat{f}_{\text{RP}}$

Exploiting 0-Grav SLIP Symmetry

- Now focus on desired apex fore-aft velocity $\dot{\bar{b}}_x^*$
  - maintain constant energy throughout stance
  - ignore height – live with whatever apex value $\bar{b}_y$ we get
- Greatly simplifies composition of stance mode maps
  - symmetry relates td & bot. and bot. & lo via S
  - yields stance mode map

$$\mathbf{p}_{q_l} = m_{\text{Stance}}(\mathbf{p}_{q_t}) := m_{\text{DCMP}} \circ m_{\text{CMP}}(\mathbf{p}_{q_t}) = \begin{bmatrix} q_{\chi_t} \\ 2\theta_{\text{S}}(\mathbf{p}_{q_t}) + q_{\theta_t} \\ -p_{\chi_t} \\ p_{\theta_t} \end{bmatrix}$$

  - with closed form entries

$$\theta_{\text{S}}(\mathbf{p}_{q_t}) := \frac{p_{\theta_t}}{\sqrt{p_{\theta_t}^2 + \mu k}} \operatorname{arccot}\left[\frac{p_{\theta_t}^2 + \mu k + q_{\chi_t} p_{\chi_t} q_{\chi_b} p_{\chi}(q_{\chi_b})}{\sqrt{p_{\theta_t}^2 + \mu k}\left[q_{\chi_b} p_{\chi}(q_{\chi_b}) - q_{\chi_t} p_{\chi_t}\right]}\right]$$

  - where $p_{\chi}(q_{\chi_b})$ solves $0 = p_{\chi}(q_{\chi_b}) = \pm\left[p_{\chi_t}^2 + \frac{(p_{\theta_t} + \mu k)(q_{\chi_b}^2 - q_{\chi_t}^2)}{q_{\chi_b}^2\, q_{\chi_t}^2}\right]^{\frac{1}{2}}$

Introducing Apex Coordinates

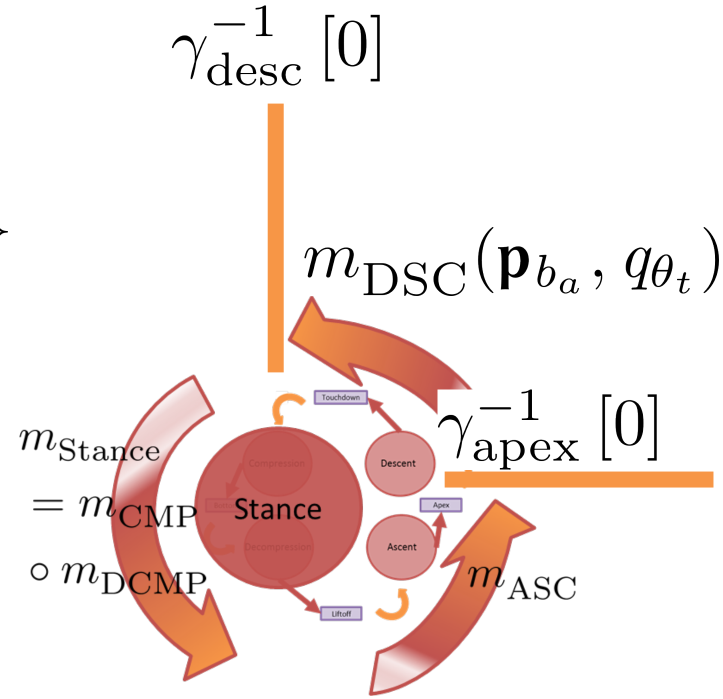
- Apex coordinates

$$\gamma_{\mathrm{apex}}^{-1}[0] = \{\mathbf{p}_{b_a} = \begin{bmatrix} \begin{bmatrix} \star \\ \bar{b}_y \end{bmatrix} \\ \begin{bmatrix} \mu \dot{\bar{b}}_x \\ 0 \end{bmatrix} \end{bmatrix} : \bar{b}_y, \dot{\bar{b}}_x \in \mathbb{R}^2\}$$

  - horizontal speed at maximum height
  - where * denotes ignored horizontal position
  - projection isolates apex state

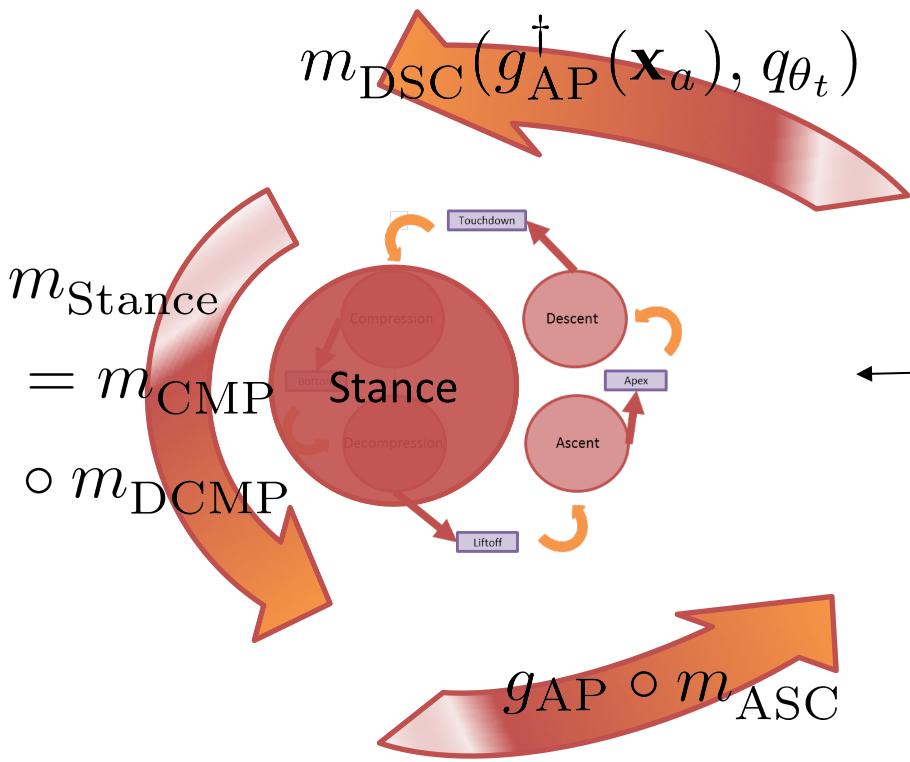
$$\mathbf{x}_a := \begin{bmatrix} \bar{b}_y \\ \dot{\bar{b}}_x \end{bmatrix} =: g_{\mathrm{AP}}(\mathbf{p}_{b_a})$$

  - pseudo-inverse $\mathbf{p}_{b_a} = \begin{bmatrix} \begin{bmatrix} 0 \\ \bar{b}_y \end{bmatrix} \\ \begin{bmatrix} \mu \dot{\bar{b}}_x \\ 0 \end{bmatrix} \end{bmatrix} =: g_{\mathrm{AP}}^{\dagger}(\mathbf{x}_a)$
    - re-parametrizes mode map input
    - promotes conjugacy to 2Dim

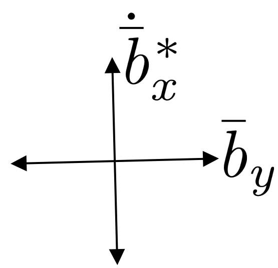


- Apex view promotes touchdown angle control analysis

Using Apex Coordinates



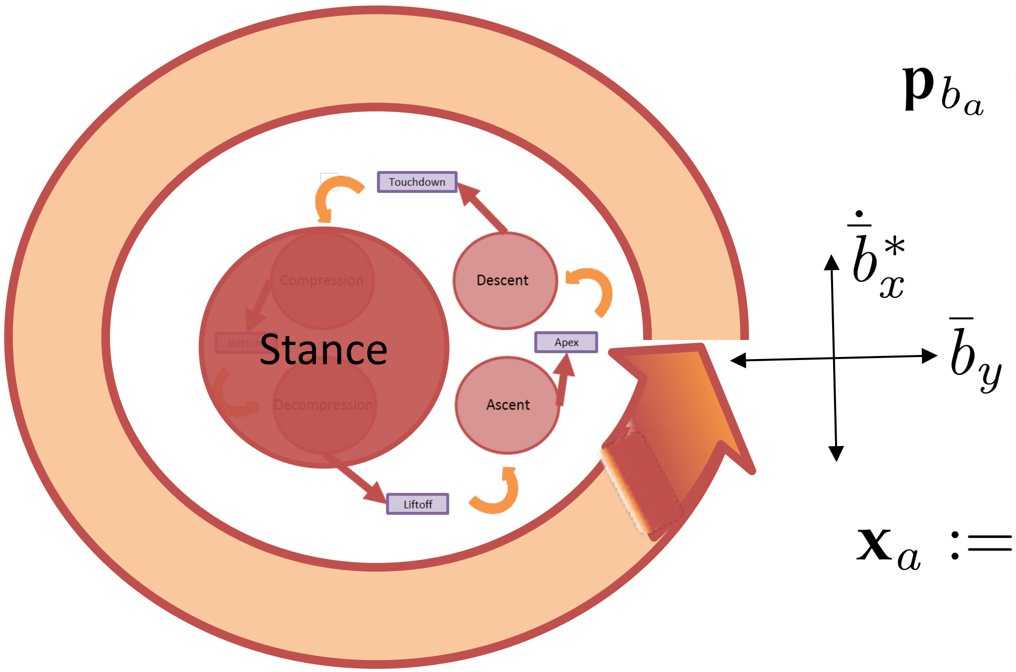
$$\mathbf{p}_{b_a} = \begin{bmatrix} \begin{bmatrix} 0 \\ \bar{b}_y \end{bmatrix} \\ \begin{bmatrix} \mu \dot{\bar{b}}_x \\ 0 \end{bmatrix} \end{bmatrix} =: g_{\mathrm{AP}}^{\dagger}(\mathbf{x}_a)$$



$$\mathbf{x}_a := \begin{bmatrix} \bar{b}_y \\ \dot{\bar{b}}_x \end{bmatrix} =: g_{\mathrm{AP}}(\mathbf{p}_{b_a})$$

Apex Coord. Return Map Representation

$$p_{\mathrm{SLP}}(\mathbf{x}_a) := g_{\mathrm{AP}} \circ m_{\mathrm{ASC}} \circ m_{\mathrm{Stance}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)$$



$$\mathbf{p}_{b_a} = \begin{bmatrix} \begin{bmatrix} 0 \\ \bar{b}_y \end{bmatrix} \\ \begin{bmatrix} \mu \dot{\bar{b}}_x \\ 0 \end{bmatrix} \end{bmatrix} =: g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a)$$

$$\mathbf{x}_a := \begin{bmatrix} \bar{b}_y \\ \dot{\bar{b}}_x \end{bmatrix} =: g_{\mathrm{AP}}(\mathbf{p}_{b_a})$$

Closed Form Fore-Aft SLIP Return Map

- Foregoing analysis now yields

$$p_{\mathrm{SLP}}(\mathbf{x}_a, \theta_t) = g_{\mathrm{AP}} \circ m_{\mathrm{ASC}} \circ m_{\mathrm{Stance}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)$$

$$= \begin{bmatrix} \frac{1}{2\mathrm{g}} \left(\dot{\bar{b}}_x \sin\Theta + \sqrt{2\mathrm{g}(\bar{b}_y - \chi_\ell \sin\theta_t)} \cos\Theta \right)^2 + \chi_\ell \sin(\Theta - \theta_t) \\ -\dot{\bar{b}}_x \cos\Theta + \sqrt{2\mathrm{g}(\bar{b}_y - \chi_\ell \sin\theta_t)} \sin\Theta \end{bmatrix}$$

$$\Theta := 2\left[\theta_t + \theta_{\mathrm{S}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)\right]$$

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Week 8 – Unit 3

Raibert Vertical Hopper

Video 10.3

Segment 8.3.1

Return Map Analysis - Factorization

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August, 2017

Where Have We Been?

- Week 6: purely vertical "hopping" gait
- SLIP has a naturally decoupled 1 DoF invariant subspace
  - hopping in place with no fore-aft energy
  - maintains pure vertical behavior (until/unless disturbed)
- Raibert's height controller regulated shank energy
  - as if the leg were "locked" in purely vertical configuration
  - analysis showed why this works if "locked"
  - recent analysis (beyond this course) begins to shed light on "free" case

Where Does This Go in the Long Run?

- This week: purely fore-aft stepping control
  - no analogously decoupled 1DoF invariant subspace
  - introduction of simplifying assumptions yields insight
    - will show Raibert's proportional control around the "neutral" angle
    - succeeds in regulating fore-aft speed – but generally perturbs height
- Recent analysis (beyond this course) sheds light on full SLIP
  - this course: explore that case (and more complex) numerically
  - pursue further studies to apply analysis to those complex cases

Where Were Just Before?

- Recall closed form fore-aft SLIP return map

$$p_{\mathrm{SLP}}(\mathbf{x}_a, \theta_t) = g_{\mathrm{AP}} \circ m_{\mathrm{ASC}} \circ m_{\mathrm{Stance}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)$$

$$= \begin{bmatrix} \frac{1}{2\mathrm{g}}\left(\dot{\bar{b}}_x \sin\Theta + \sqrt{2\mathrm{g}(\bar{b}_y - \chi_\ell \sin\theta_t)}\cos\Theta\right)^2 + \chi_\ell \sin(\Theta - \theta_t) \\ -\dot{\bar{b}}_x \cos\Theta + \sqrt{2\mathrm{g}(\bar{b}_y - \chi_\ell \sin\theta_t)}\sin\Theta \end{bmatrix}$$

$$\Theta := 2\left[\theta_t + \theta_{\mathrm{S}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)\right]$$

- Obtained from simplifying assumptions
  - ignore stance mode gravitational potential
  - conserved angular momentum yields second integral

Where Are We Going Just now?

- Recall closed form fore-aft SLIP return map

$$p_{\mathrm{SLP}}(\mathbf{x}_a, \theta_t) = g_{\mathrm{AP}} \circ m_{\mathrm{ASC}} \circ m_{\mathrm{Stance}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)$$

$$= \begin{bmatrix} \frac{1}{2\mathrm{g}} \left(\dot{\bar{b}}_x \sin\Theta + \sqrt{2\mathrm{g}(\bar{b}_y - \chi_\ell \sin\theta_t)} \cos\Theta \right)^2 + \chi_\ell \sin(\Theta - \theta_t) \\ -\dot{\bar{b}}_x \cos\Theta + \sqrt{2\mathrm{g}(\bar{b}_y - \chi_\ell \sin\theta_t)} \sin\Theta \end{bmatrix}$$

$$\Theta := 2\left[\theta_t + \theta_{\mathrm{S}} \circ m_{\mathrm{DSC}}(g^{\dagger}_{\mathrm{AP}}(\mathbf{x}_a), \theta_t)\right]$$

- This unit: study touchdown angle feedback control

$$\theta_t(n+1) = g_{\mathrm{F}}(\mathbf{x}_n) \Rightarrow p_{\mathrm{F}}(\mathbf{x}) := p_{\mathrm{SLP}}(\mathbf{x}, g_{\mathrm{F}}(\mathbf{x}))$$

- This segment: factor return map
  - reveals inherent simplicity of isolated fore-aft system
  - as horizontal translation punctuated by toe rotation

Return Map Factorization

- Schwind & Kod ICRA 1995, op. cit. showed

$$p_{\mathrm{SLP}} = \boldsymbol{t}_{\Theta - \theta_{\mathrm{t}}} \circ \mathbf{s} \circ \mathbf{r}_{\Theta} \circ \mathbf{s}^{-1} \circ \boldsymbol{t}_{-\theta_{\mathrm{t}}}$$

  - where $\boldsymbol{t}_{\alpha}(\mathbf{x}) := \mathbf{x} + \begin{bmatrix} \chi_{\ell} \sin \alpha \\ 0 \end{bmatrix} \Leftrightarrow \boldsymbol{t}_{\alpha}^{-1}(\mathbf{x}) = \boldsymbol{t}_{\pi - \alpha}(\mathbf{x})$
  - and $\mathbf{s}(\boldsymbol{u}) := \begin{bmatrix} (k u_2)^2 \\ u_1 \end{bmatrix} \Leftrightarrow \mathbf{s}^{-1}(\boldsymbol{v}) = \begin{bmatrix} v_2 \\ \frac{1}{k}\sqrt{v_1} \end{bmatrix}$
  - with $\mathbf{r}_{\alpha}(\mathbf{x}) := - \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \mathbf{x} \Leftrightarrow \mathbf{r}_{\alpha}^{-1}(\mathbf{x}) = \mathbf{r}_{-\alpha}(\mathbf{x})$
  - and where the control variable, θ_t, enters as

$$\Theta = \tilde{g}_{\mathrm{F}}(\mathbf{x}, \theta_t) := 2 \left[\theta_t + \theta_{\mathrm{S}} \circ m_{\mathrm{DSC}}(g_{\mathrm{AP}}^{\dagger}(\mathbf{x}), \theta_t) \right]$$

- Next segment: investigate two control laws
  - Raibert stepping: “proportional control”
  - inverse dynamics: “exact linearization”

edX Robo4 Mini MS – Locomotion Engineering

Week 8 – Unit 3

Spring Loaded Inverted Pendulum

Video 10.4

Segment 8.3.2

Return Map Analysis - Stability

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August, 2017

Fore-Aft SLIP Return Map Equilibria

- Given return map $p_{\mathrm{SLP}}(\mathbf{x}, \theta_t) = \boldsymbol{t}_{\Theta-\theta_{\mathrm{t}}} \circ \mathbf{s} \circ \mathbf{r}_{\Theta} \circ \mathbf{s}^{-1} \circ \boldsymbol{t}_{-\theta_{\mathrm{t}}}$
- Closed loop from touchdown angle feedback policy

$$\theta_t(n+1) = g_{\mathrm{F}}(\mathbf{x}_n) \Rightarrow p_{\mathrm{F}}(\mathbf{x}) := p_{\mathrm{SLP}}(\mathbf{x}, g_{\mathrm{F}}(\mathbf{x}))$$

- Yields equilibria (FP), i.e., $p_{\mathrm{F}}(\mathbf{x}^*) = \mathbf{x}^* = \begin{bmatrix} \bar{b}_y^* \\ \dot{\bar{b}}_x^* \end{bmatrix}$
  - when $\Theta = \tilde{g}_{\mathrm{F}}(\mathbf{x}^*, \theta_t^*) := 2\left[\theta_t^* + \theta_{\mathrm{S}} \circ m_{\mathrm{DSC}}(g_{\mathrm{AP}}^{\dagger}(\mathbf{x}^*), \theta_t^*)\right] = \pi$
  - since $\boldsymbol{t}_{\pi-\alpha} = \boldsymbol{t}_{\alpha} = \boldsymbol{t}_{-\alpha}^{-1}$ implies

$$\begin{aligned} p_{\mathrm{F}}(\mathbf{x}^*) &= \boldsymbol{t}_{\pi-\mathrm{g_F}} \circ \mathbf{s} \circ \mathbf{r}_{\pi} \circ \mathbf{s}^{-1} \circ \boldsymbol{t}_{-\mathrm{g_F}}(\mathbf{x}^*) \\ &= (\boldsymbol{t}_{\mathrm{g_F}} \circ \mathbf{s}) \circ \mathbf{id} \circ (\boldsymbol{t}_{\mathrm{g_F}} \circ \mathbf{s})^{-1}(\mathbf{x}^*) \\ &= \mathbf{x}^* \end{aligned}$$

  - turns out to be a necessary condition as well

Meaning and Consequences of FP Condition

- Physical Meaning
  - Recall that θ_S is the toe angle between td & bottom
  - so $\pi/2 = \Theta/2 = q_{\theta_b}$ implies vertical bottom
  - hence perfectly symmetric (non-accelerating) stance
- Mathematical Consequences
  - FP condition $\tilde{g}_{\mathrm{F}}(\mathbf{x}^*, \theta_t^*) = \pi$ (1 equation in 3 unknowns)
    - gives rise to a smooth, 2D surface $\tilde{g}_{\mathrm{F}}^{-1}(\pi)$
    - of apex-td-angle pairs $(\mathbf{x}^*, \theta_t^*)$ that parametrize FP set
  - Formally, these FP must all be "degenerate"
    - linearized dynamics can only be marginally stable
    - i.e., nearby equilibria don't attract each other

Conditions for FP Stability

- Compute linearized dynamics at FP

$$
\begin{aligned}
P_{\mathrm{F}}(\mathbf{x}^*) &:= D_{\mathbf{x}} p_{\mathrm{F}}(\mathbf{x}^*) \\
&= I_2 + ab^{\mathrm{T}} \\
a &:= \begin{bmatrix} 2k\dot{\bar{b}}_x^* \sqrt{\bar{b}_y^* - \chi_\ell \sin g_{\mathrm{F}}(\mathbf{x}^*)} - \chi_\ell \cos g_{\mathrm{F}}(\mathbf{x}^*) \\ -\frac{1}{k}\sqrt{\bar{b}_y^* - \chi_\ell \sin g_{\mathrm{F}}(\mathbf{x}^*)} \end{bmatrix} \\
b^{\mathrm{T}} &:= D_{\mathbf{x}} \tilde{g}_{\mathrm{F}}(\mathbf{x}^*)
\end{aligned}
$$

- So eigenvalues of $P_{\mathrm{F}}(\mathbf{x}^*)$ are $\lambda_1 = 1$, $\lambda_2 = 1 + a^{\mathrm{T}}b$
  - degeneracy: was inevitable
    - physically: total energy conserved; each energy IC has own FP
    - mathematically: unity eigenvector is tangent to $\tilde{g}_{\mathrm{F}}^{-1}(\pi)$
  - use Center Manifold Thm.
    - to show $0 > a^{\mathrm{T}}b > -2 \Rightarrow |\lambda_2| < 1$
    - implies local attraction to FP curve of apex states in $\tilde{g}_{\mathrm{F}}^{-1}(\pi)$

Raibert's Stepping Controller

- Proportional error feedback: $x_f = \frac{\dot{\bar{b}}_x T_s}{2} + k_{\dot{x}} \left(\dot{\bar{b}}_x - \dot{\bar{b}}_x^* \right)$
  - place foot at previous "half-stance"
  - with proportional speed correction around set point
- Recast as touchdown angle control law

$$\begin{aligned}\theta_t = g_{\mathrm{RF}}(\mathbf{x}) &:= \arccos\left(\frac{\chi_\ell \cos\theta_t^* + k_{\dot{x}} \left(\dot{\bar{b}}_x - \dot{\bar{b}}_x^* \right)}{\chi_\ell} \right) \\ &\approx \arccos\left(\frac{\frac{1}{2}\dot{\bar{b}}_x T_s + k_{\dot{x}} \left(\dot{\bar{b}}_x - \dot{\bar{b}}_x^* \right)}{\chi_\ell} \right) \\ &= \arccos\left(x_f / \chi_\ell \right)\end{aligned}$$

  - where $(\mathbf{x}^*, \theta_t^*) \in \tilde{g}_{\mathrm{F}}^{-1}(\pi)$ satisfy FP condition & $|\lambda_2| < 1$

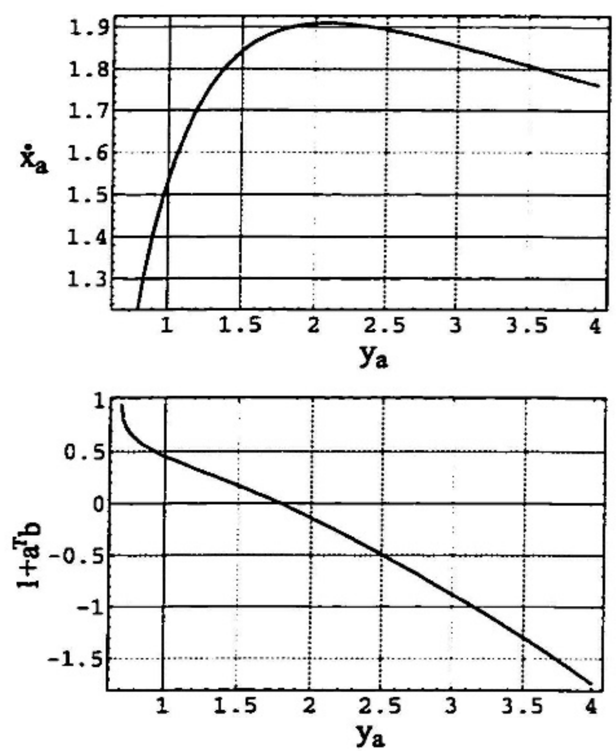
Implementing Raibert Stepping

- Two ways of thinking about implementation
  - given desired apex fore-aft speed, $\dot{\bar{b}}_x^*$
    - find stability region, i.e., $1 > |\lambda_2| = \left|1 + \left(a^{\mathrm{T}} b\right)(\mathbf{x}^*, \theta_t^*)\right|$
    - on solution curve relating $\bar{b}_y^*, \theta_t^*$
    - in FP surface $\tilde{g}_{\mathrm{F}}^{-1}(\pi)$
  - or, fix θ_t^*
    - find stability region, i.e.,

$$1 > |\lambda_2| = \left|1 + \left(a^{\mathrm{T}} b\right)(\mathbf{x}^*, \theta_t^*)\right|$$

    - on solution curve relating $\bar{b}_y^*, \dot{\bar{b}}_x^*$
    - in FP surface $\tilde{g}_{\mathrm{F}}^{-1}(\pi)$

FP and λ_2 Curves due to g_{RF}



figures from
W. J. Schwind, "Spring loaded inverted pendulum running: A plant model.," PhD. Thesis, University of Michigan, 1998.

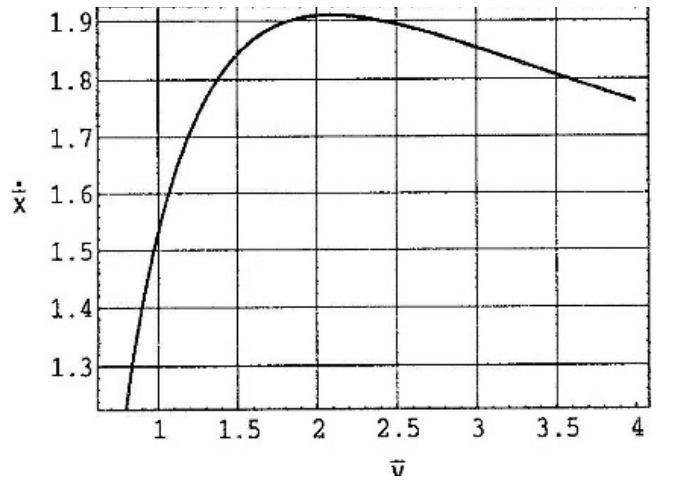
Simulation Study of Raibert Stepping

- Implementation: Fix θ_t^*
  - find stability region, i.e.,

$$1 > |\lambda_2| = \left|1 + \left(a^{\mathrm{T}} b\right)\left(\mathbf{x}^*, \theta_t^*\right)\right|$$

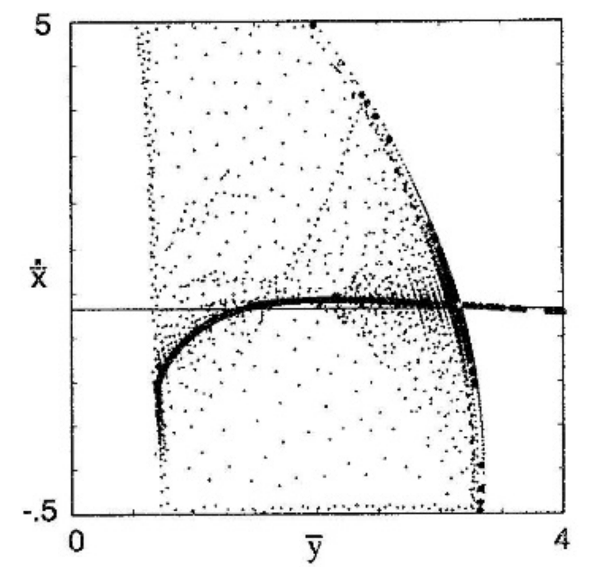
  - on solution curve relating $\bar{b}_y^*, \dot{\bar{b}}_x^*$
  - in FP surface $\tilde{g}_{\mathrm{F}}^{-1}(\pi)$

FP Curve due to g_{RF}



- Numerical results
  - shaded region shows basin of attraction to equilibrium curve
  - good volume; inexact target
  - uncontrolled height

FP Curve & Basin due to g_{RF}



figures from

W. J. Schwind and D. E. Koditschek, “Control of forward velocity for a simplified planar hopping robot,” in Robotics and Automation, 1995. Proceedings., 1995 IEEE International Conference on, 1995,, pp. 691–696.

Inverse Dynamics: Exact Linearization

- A more abstract version inverse dynamics
  - Seg. 6.1.1 : "inverse statics" for series elastic actuator
    - used the computed inverse force function, Φ_{GHS}^{-1}
    - to insert desired force profile Φ_{ref}
  - Now, given apex state, $\boldsymbol{x}$, take $\theta_t = \tilde{g}_{\mathrm{F}}^{-1}(\pi, \mathbf{x})$
    - solving implicit equation $\tilde{g}_{\mathrm{F}}(\mathbf{x}, \tilde{g}_{\mathrm{F}}^{-1}(\pi, \mathbf{x})) = \pi$
    - yields return map FP $\Rightarrow p_{\mathrm{SLP}}(\mathbf{x}, \tilde{g}_{\mathrm{F}}^{-1}(\pi, \mathbf{x})) = \mathbf{x}$
  - Holds true for any state in $\tilde{g}_{\mathrm{F}}^{-1}(\pi)$ so pick desirable $\boldsymbol{x}_{\mathrm{d}}$
- Two alternative inverse dynamics schemes:
  - "deadbeat" strategy $\boldsymbol{x}_{\mathrm{d}} = \boldsymbol{x}^*$ (target FP)
  - linear regulation strategy: given desired $\dot{\bar{b}}_x^*$
    - aim for linear regulator

$$\mathbf{x}_d = p_{\mathrm{IDF}}(\mathbf{x}) := \begin{bmatrix} \bar{b}_y \\ \dot{\bar{b}}_x^* + k(\dot{\bar{b}}_x^* - \dot{\bar{b}}_x) \end{bmatrix}$$

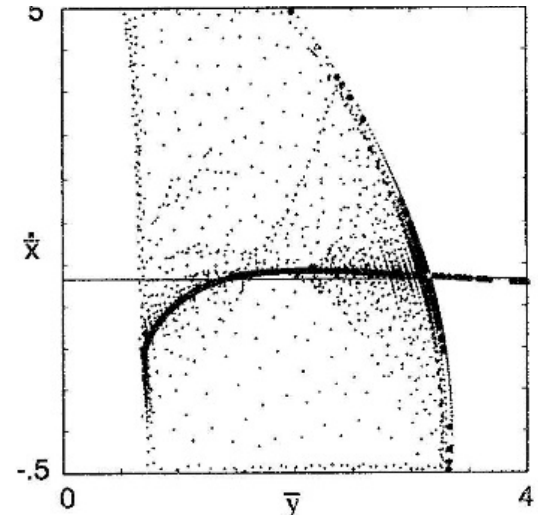
    - via partial inverse

$$g_{\mathrm{IDF}}(\mathbf{x}) := \tilde{g}_{\mathrm{F}}^{-1}(\pi, p_{\mathrm{IDF}}(\mathbf{x}))$$

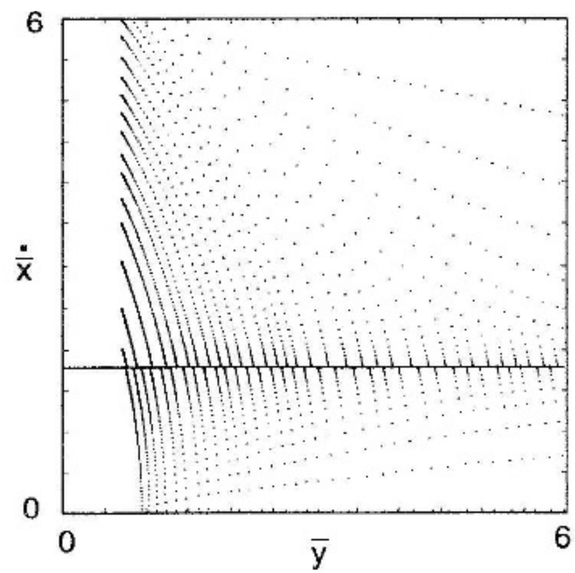
Simulation Study of Exact Linearization

- Numerical results compare
  - Raibert stepping (g_{RF})
  - Inverse dynamics (g_{ARF})
- Inverse dynamics
  - better basin volume
  - better regulation
- Raibert stepping
  - more robust against modeling errors

FP Curve & Basin due to g_{RF}



FP Curve & Basin due to g_{IDF}



figures from
W. J. Schwind and D. E. Koditschek, 1995. Op. cit.