Video 2.1a

Vijay Kumar and Ani Hsieh

Introduction to Lagrangian Mechanics

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Analytical Mechanics

- Aristotle
- Galileo
- Bernoulli
- Euler
- Lagrange
- D'Alembert

1. Principle of Virtual Work: Static equilibrium of a particle, system of *N* particles, rigid bodies, system of rigid bodies
2. D'Alembert's Principle: Incorporate inertial forces for dynamic analysis
3. Lagrange's Equations of Motion

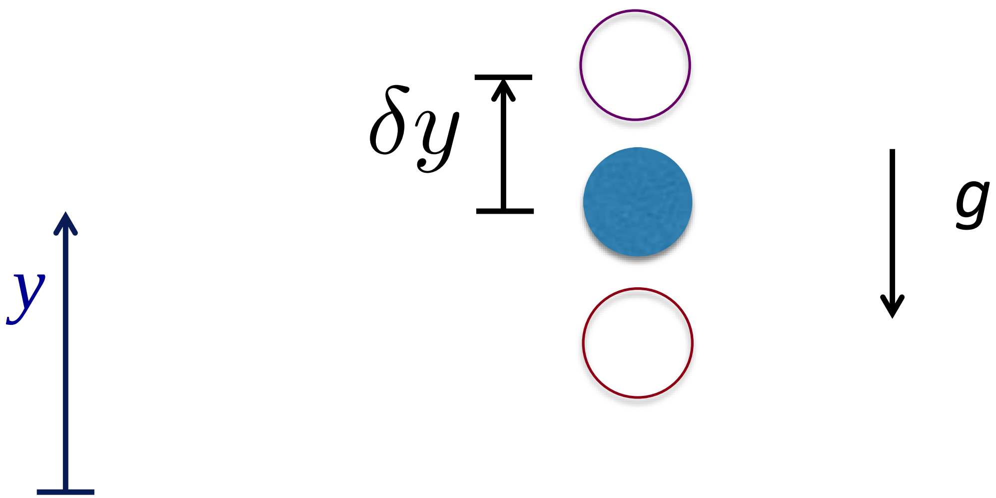
Generalized Coordinate(s)

A minimal set of coordinates required to describe the configuration of a system

No. of generalized coordinates = no. of degrees of freedom

Virtual Displacements

Virtual displacements are small displacements consistent with the constraints

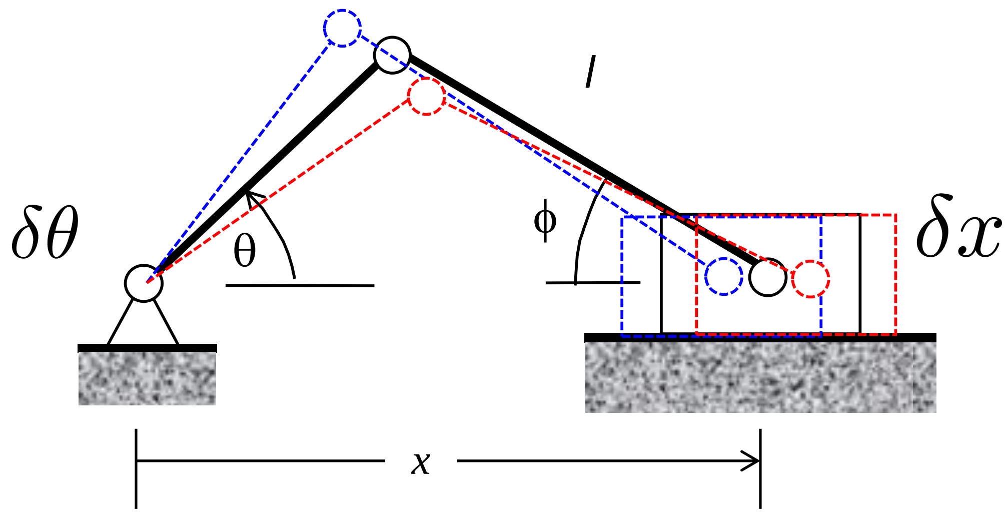


A particle of mass *m* constrained to move vertically

generalized coordinate y

Virtual Displacements

Virtual displacements are small displacements consistent with the constraints

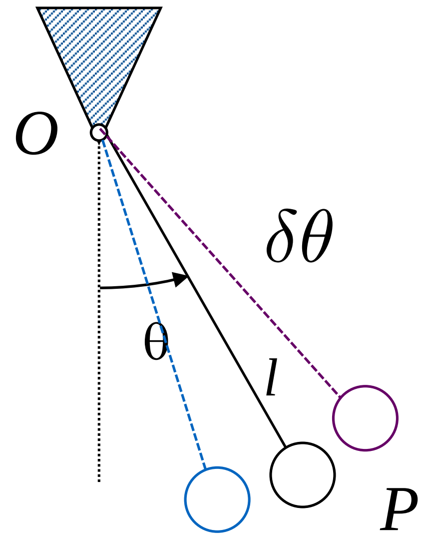


The slider crank mechanism: a single degree of freedom linkage

generalized coordinate x, θ, or ϕ

Virtual Displacements

Virtual displacements are small displacements consistent with the constraints

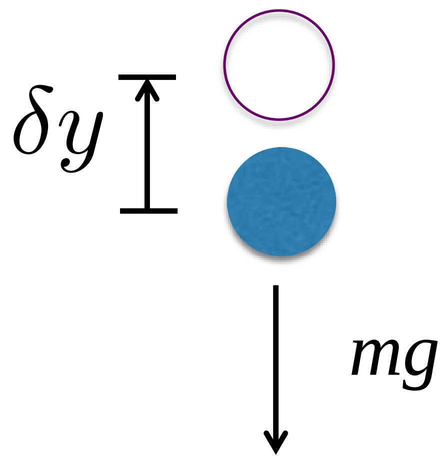


The pendulum: a single degree of freedom linkage

generalized coordinate θ

Virtual Work

The work done by applied (external) forces through the virtual displacement, δW

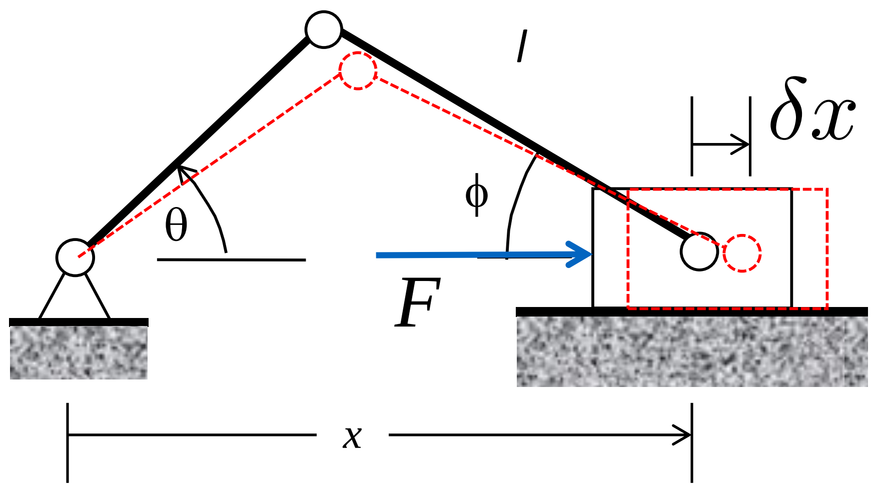


A particle of mass *m* constrained to move vertically

$$\delta W = -mg\delta y$$

Virtual Work

The work done by applied (external) forces through the virtual displacement, δW



The slider crank mechanism: a single degree of freedom linkage

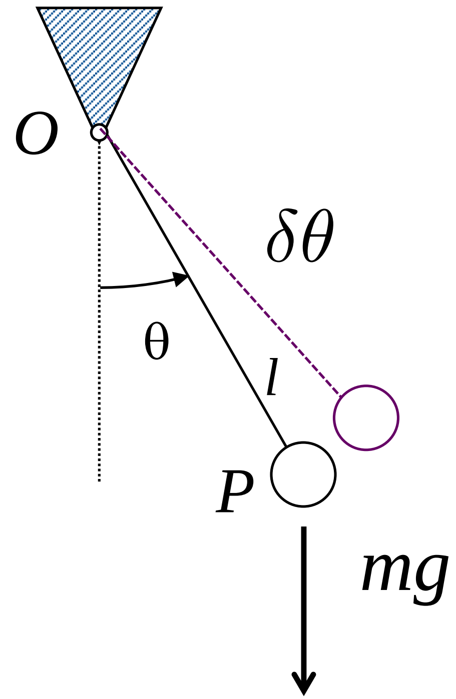
$$\delta W = F\delta x$$

Video 2.1b

Vijay Kumar and Ani Hsieh

Virtual Work

The work done by applied (external) forces through the virtual displacement, δW

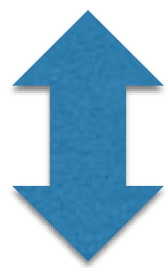


$$\delta W = -mgl\sin\theta\delta\theta$$

The pendulum: a single degree of freedom linkage

The Principle of Virtual Work

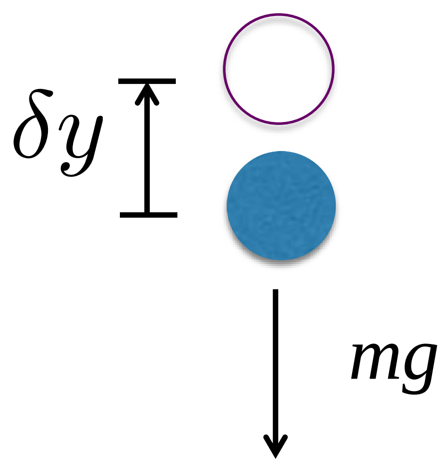
The virtual work done by all applied (external) forces through any virtual displacement is zero



The system is in equilibrium

Static Equilibrium

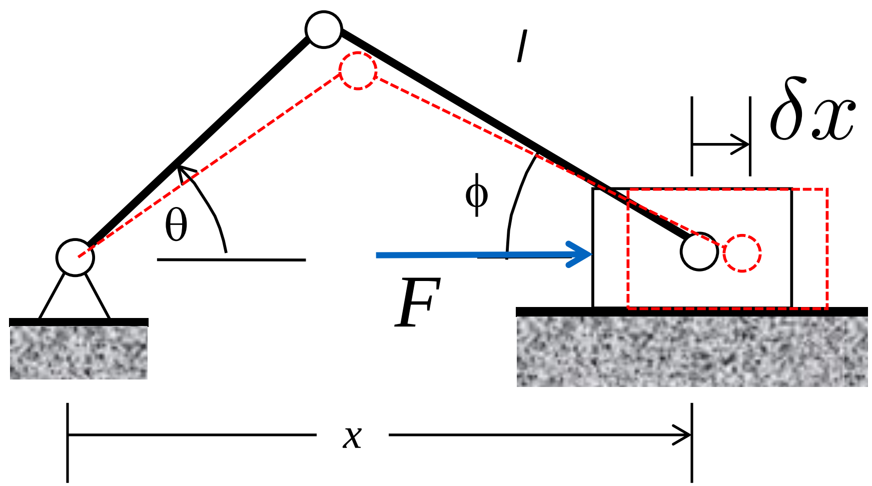
$$\delta W = 0$$



$$\delta W = -mg\delta y = 0$$

Static Equilibrium

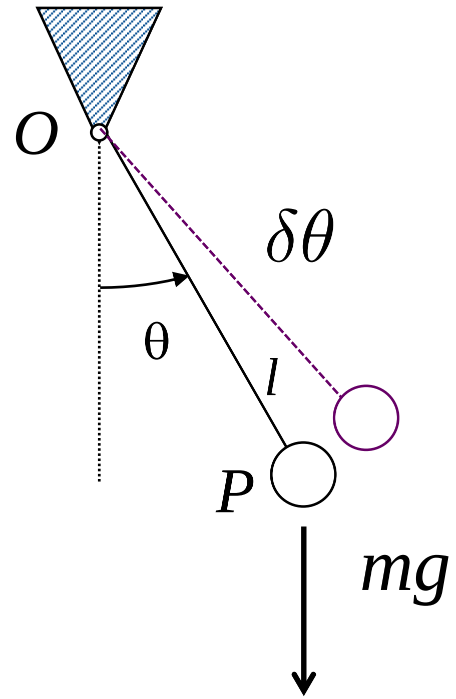
$$\delta W = 0$$



$$\delta W = F\delta x = 0$$

Static Equilibrium

$$\delta W = -mgl \sin\theta \delta\theta = 0$$



D'Alembert's Principle

The virtual work done by all applied (external) forces through any virtual displacement is zero

include inertial forces

⇕ principle of virtual work

The system is in *static* equilibrium

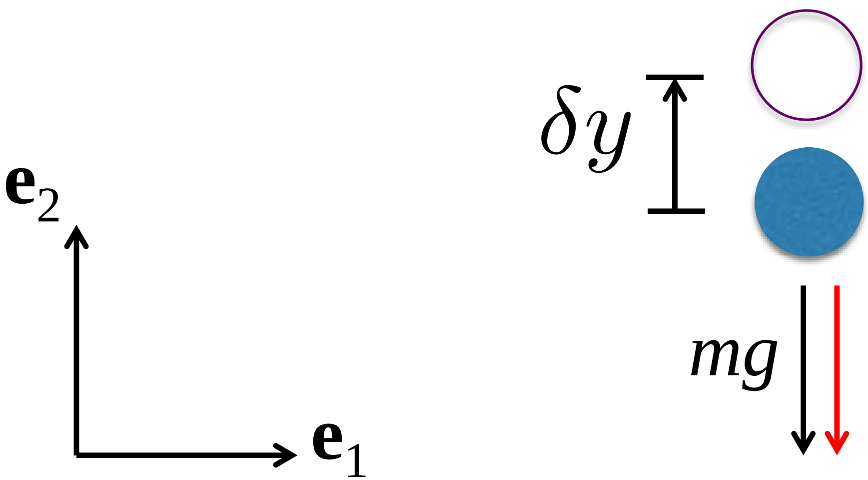
Equations of motion for the system

inertial force = - mass x acceleration

D’Alembert’s Principle

acceleration

$$\mathbf{a} = \ddot{y}\mathbf{e}_2$$

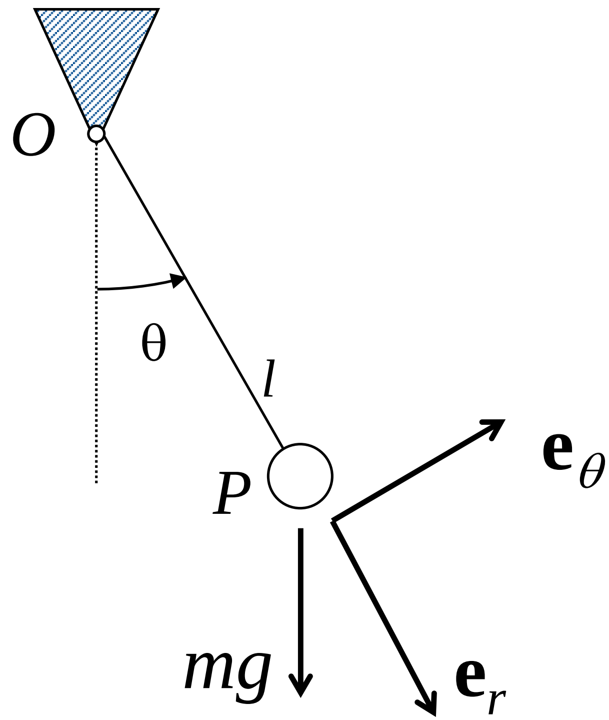


inertial force

$$-m\ddot{y}\mathbf{e}_2$$

$$\delta W = -(mg + m\ddot{y})\delta y = 0$$

D'Alembert's Principle



acceleration

$$\mathbf{a} = l(\ddot{\theta}\mathbf{e}_\theta - \dot{\theta}^2\mathbf{e}_r)$$

inertial force

$$-ml(\ddot{\theta}\mathbf{e}_\theta - \dot{\theta}^2\mathbf{e}_r)$$

equation of motion

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$mg(-\sin\theta\mathbf{e}_\theta + \cos\theta\mathbf{e}_\mathbf{r})$$

$$\delta W = (-ml^2\ddot{\theta} - mgl\sin\theta)\delta\theta = 0$$

D'Alembert's principle with generalized coordinates

generalized coordinate

$$\delta W = (Q + Q^{\star})\delta q = 0$$

contribution from external force(s)

contribution from inertial force(s)

Particle in the vertical plane

$$\delta W = -(mg + m\ddot{y})\delta y = 0$$

Simple pendulum

$$\delta W = (-ml^2\ddot{\theta} - mgl\sin\theta)\delta\theta = 0$$

The Key Idea

The contribution from the inertial forces can be expressed as a function of the kinetic energy and its derivatives

Kinetic Energy

$$Q^\star = -\left[\frac{d}{dt}\left(\frac{\partial \boxed{\mathcal{K}}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{K}}{\partial q}\right]$$

Lagrange's Equation of Motion

$$\left[\frac{d}{dt}\left(\frac{\partial \mathcal{K}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{K}}{\partial q}\right] = Q$$

Particle in the vertical plane

$$\mathcal{K} = \frac{1}{2} m \dot{y}^2 \quad Q^{\star} = -m\ddot{y} \quad Q = -mg$$

Simple pendulum

$$\mathcal{K} = \frac{1}{2} m l^2 \dot{\theta}^2 \quad Q^{\star} = -ml^2\ddot{\theta} \quad Q = -mgl\sin\theta$$

Lagrange's Equation of Motion for a Conservative System

Conservative System

There exists a scalar function such that all applied forces are given by the gradient of the potential function

Scalar Function

$$Q = -\frac{d}{dq}\boxed{\mathcal{P}}$$

Gravitational force is conservative

Scalar function is the potential energy

Standard Form of Lagrange's Equation of Motion

$$\left[\frac{d}{dt}\left(\frac{\partial \mathcal{K}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{K}}{\partial q}\right] = Q$$

$$Q = -\frac{d}{dq}\mathcal{P}$$

The Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

$$\boxed{\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q}\right] = 0}$$

Lagrange's Equation of Motion

$$\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q}\right] = 0$$

Particle in the vertical plane

$$\mathcal{L} = \frac{1}{2}m\dot{y}^2 - mgy$$

Simple pendulum

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\sin\theta$$

Video 2.2

Vijay Kumar and Ani Hsieh

Newton Euler Equations

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Analytical Mechanics

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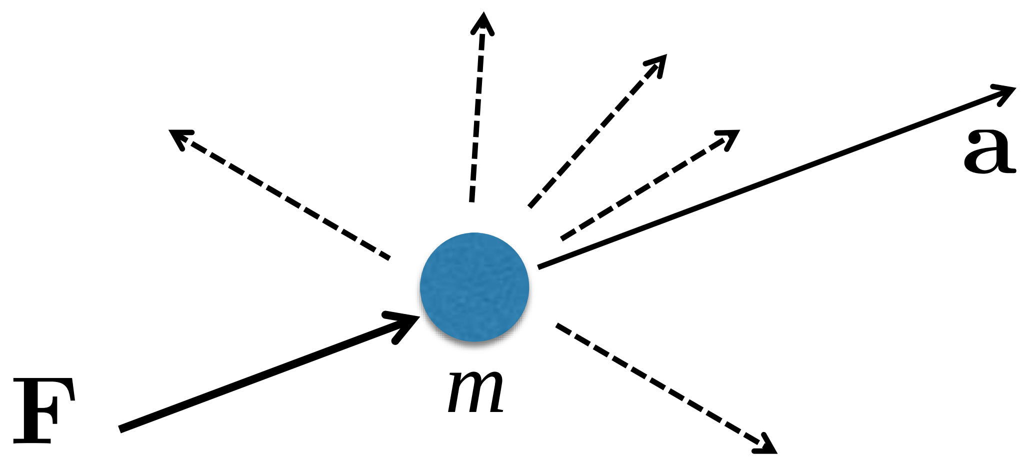
$$\left[\frac{d}{dt}\left(\frac{\partial \mathcal{K}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{K}}{\partial q}\right] = Q$$

Newton Euler Equations

Recall Newton's 2nd law of motion



$$\mathbf{F} = m\mathbf{a}$$

net external force = inertia x acceleration

Newton Euler Equations

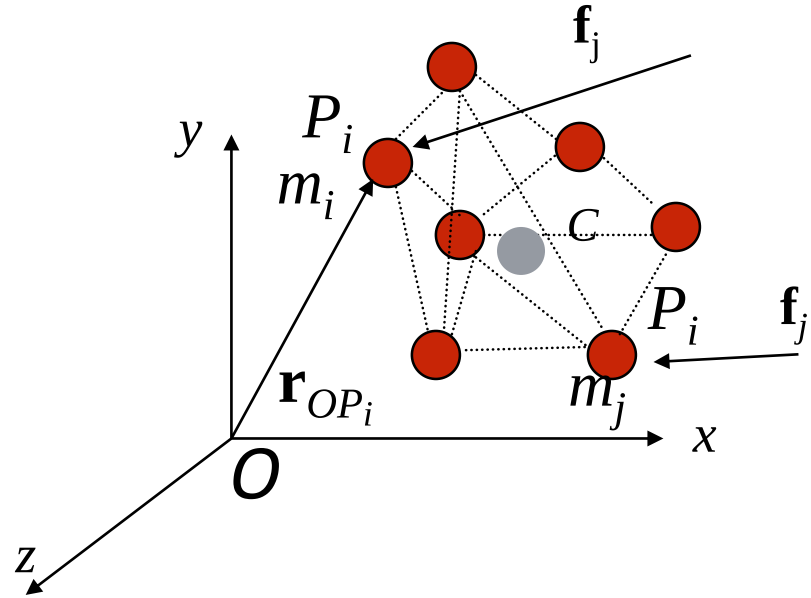
Newton's equations of motion for a translating rigid body

Euler's equations of motion for a rotating rigid body

Motion of Systems of Particles

Center of Mass

$$\mathbf{r}_C = \frac{1}{m}\sum_{i=1}^{k} m_i \mathbf{r}_{OP_i}$$



Newton's equations of motion

$$\boxed{\mathbf{F}} = \sum_{i=1}^{k} \mathbf{f}_i = \boxed{m}\boxed{\mathbf{a}_C}$$

net external force = *total mass* x *acceleration of center of mass*

Newton's Second Law for a System of Particles

The center of mass for a system of particles, *S*, accelerates in an inertial frame, *A*, as if it were a single particle with mass *m* (equal to the total mass of the system) acted upon by a force equal to the net external force.

$$\mathbf{F} = m\frac{d\mathbf{v}_C}{dt}$$

Linear momentum

$$\mathbf{L} = m\mathbf{v}_C \qquad \mathbf{F} = \frac{d\mathbf{L}}{dt}$$

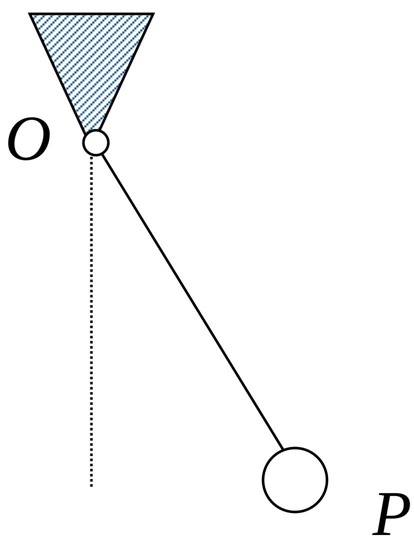
Rate of change of linear momentum in an inertial frame is equal to the net external force acting on the system.

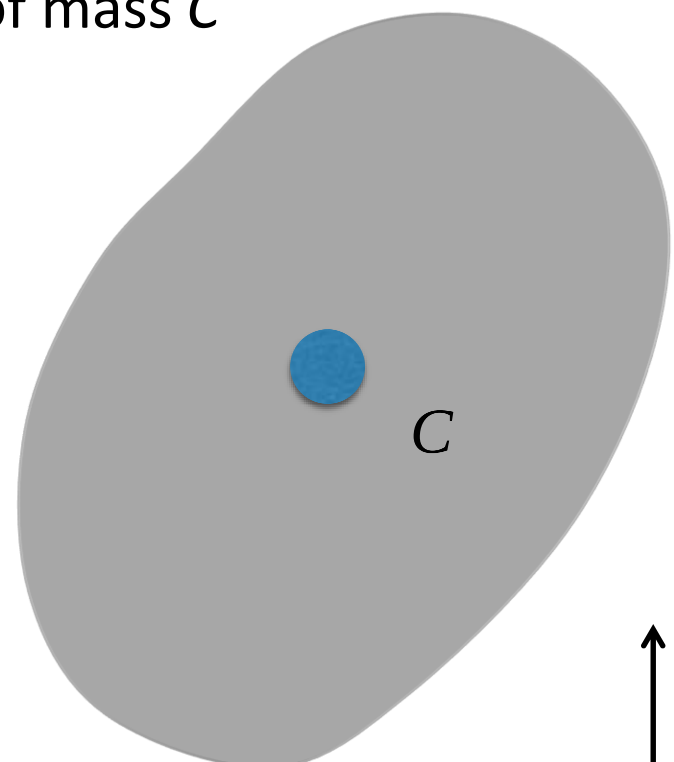
Also true for a rigid body

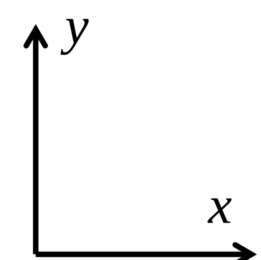
Equations of Motion for a Rotating Rigid Body

- Rigid body with a point *O* fixed in an inertial frame
- Rigid body with the center of mass *C*

Angular momentum







$$\mathbf{H}_O = I_O \omega$$

$$\mathbf{H}_C = I_C \omega$$

moment of inertia

Equations of Motion for a Rotating Rigid Body

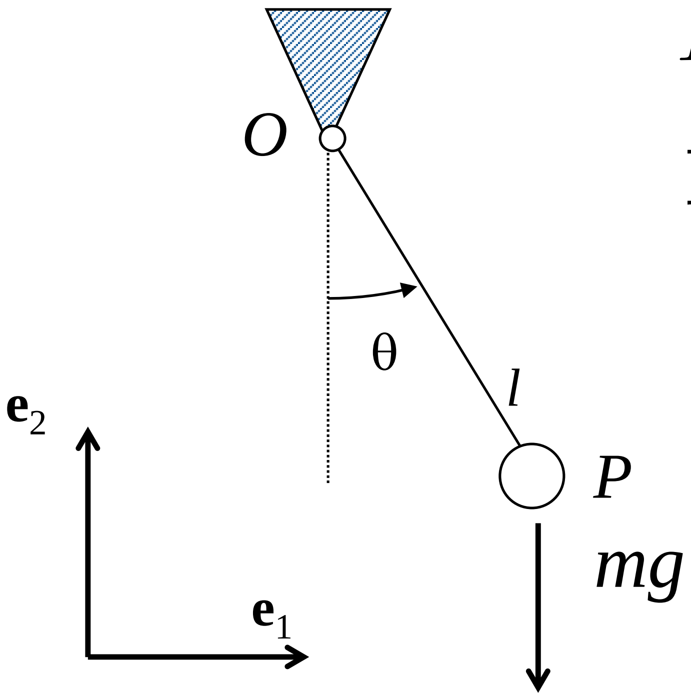
- Rigid body with a point *O* fixed in an inertial frame
- Rigid body with the center of mass *C*

The rate of change of angular momentum of a rigid body (or a system of rigidly connected particles) in an inertial frame with *O* or *C* as an origin is equal to the net external moment acting (with the same origin) on the body.

$$\frac{d\mathbf{H}_O}{dt} = \frac{d(I_O\omega)}{dt} = \mathbf{M}_O$$

$$\frac{d\mathbf{H}_C}{dt} = \frac{d(I_C\omega)}{dt} = \mathbf{M}_C$$

Example



$$I_O = ml^2 \qquad \omega = \dot{\theta}\mathbf{e}_3$$

$$\mathbf{M}_O = \mathbf{r}_{OP} \times (-mg)\mathbf{e}_2$$

$$\mathbf{r}_{OP} = l(\sin\theta\mathbf{e}_1 - \cos\theta\mathbf{e}_2)$$

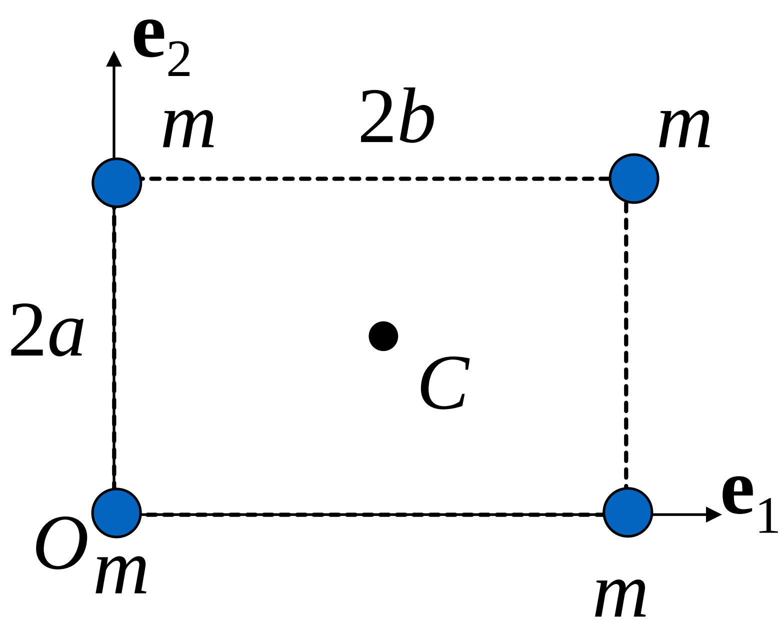
$$\mathbf{M}_O = -mgl\sin\theta\mathbf{e}_3$$

$$\frac{d\mathbf{H}_O}{dt} = \frac{d(I_O\omega)}{dt} = \mathbf{M}_O$$

equation of motion

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Moment of Inertia of Planar Objects



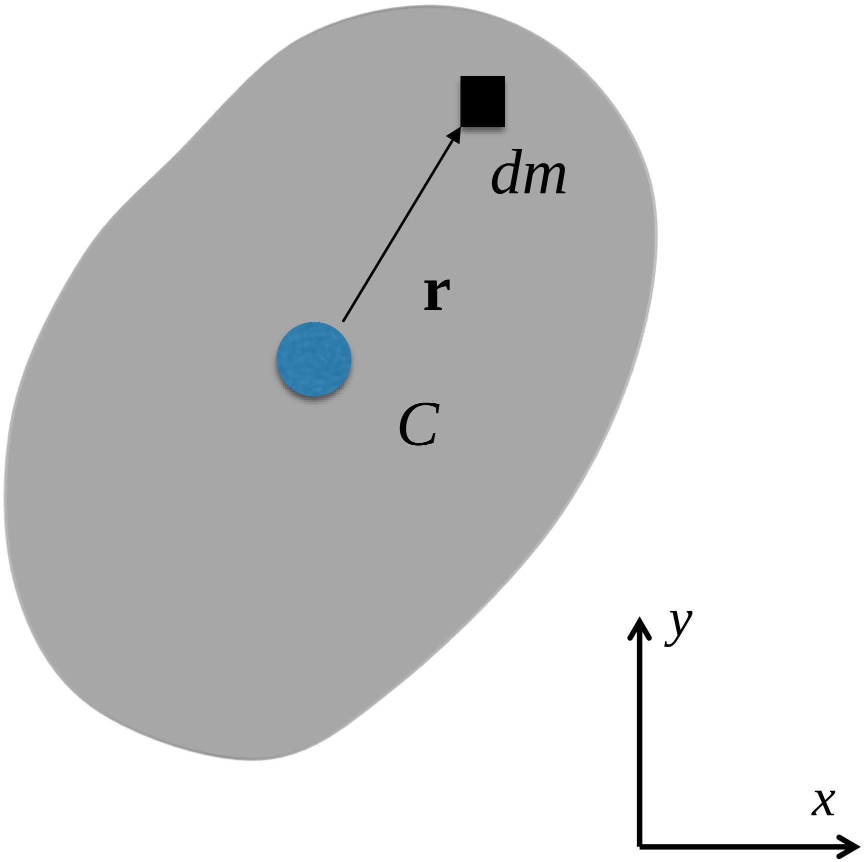
$$I_O = \Sigma_i m_i \|\mathbf{r}_{OP_i}\|^2$$

$$8m(a^2 + b^2)$$

$$I_C = \Sigma_i m_i \|\mathbf{r}_{PC_i}\|^2$$

$$4m(a^2 + b^2)$$

Moment of Inertia of Planar Objects



$$I_C = \int_M \|\mathbf{r}\|^2 dm$$

Video 2.3

Vijay Kumar and Ani Hsieh

Dynamics and Control
Acceleration Analysis: Revisited

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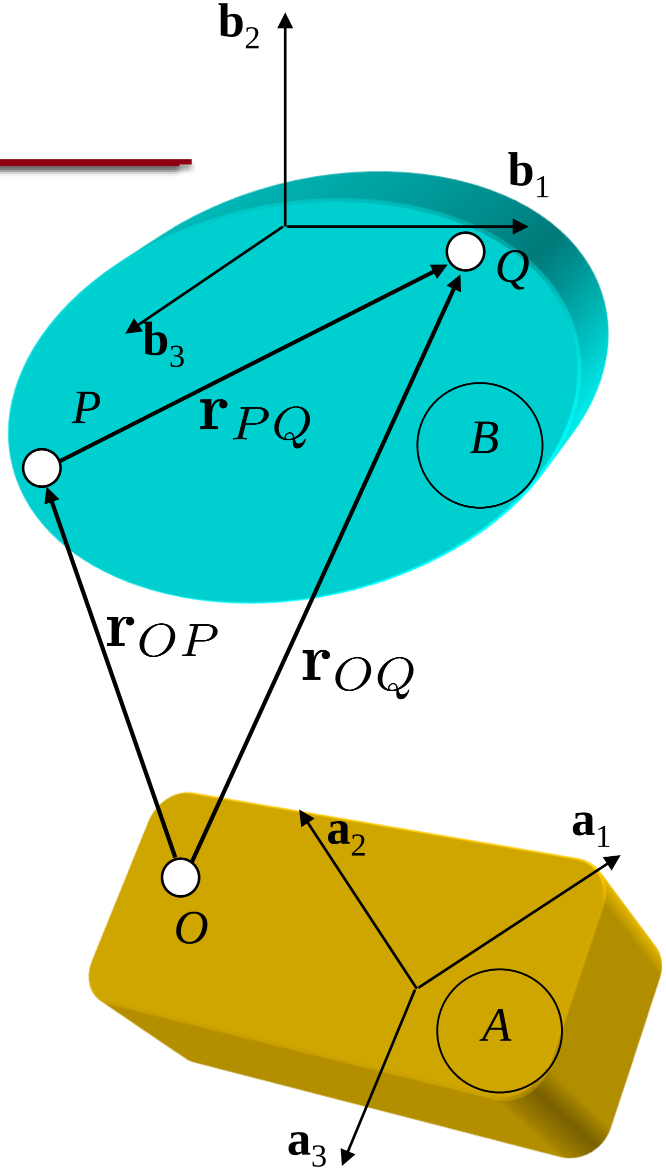
Position Vectors

- Reference frame A
  - Origin O
  - Basis vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$
- Position Vectors
  - Position vectors for P and Q in A

    $\mathbf{r}_{OP}$ $\mathbf{r}_{OQ}$
  - Position vector of Q in B

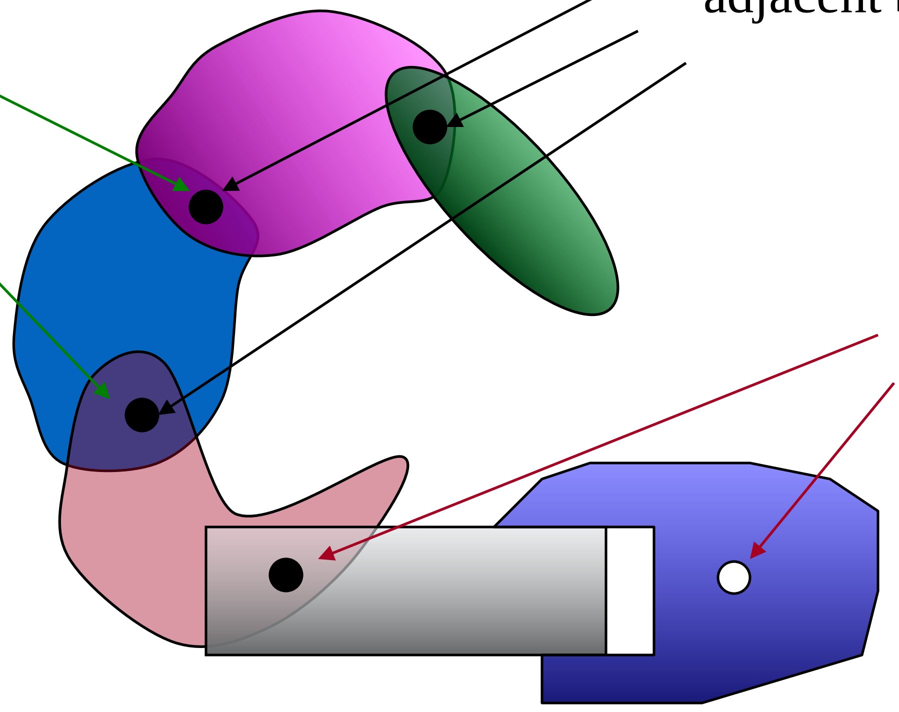
    $\mathbf{r}_{PQ}$



General Approach to Analyzing Multi-Body System

2. Pair of points fixed to the same body

1. Points common to adjacent bodies



3. Pair of points on adjacent bodies whose relative motions can be easily described

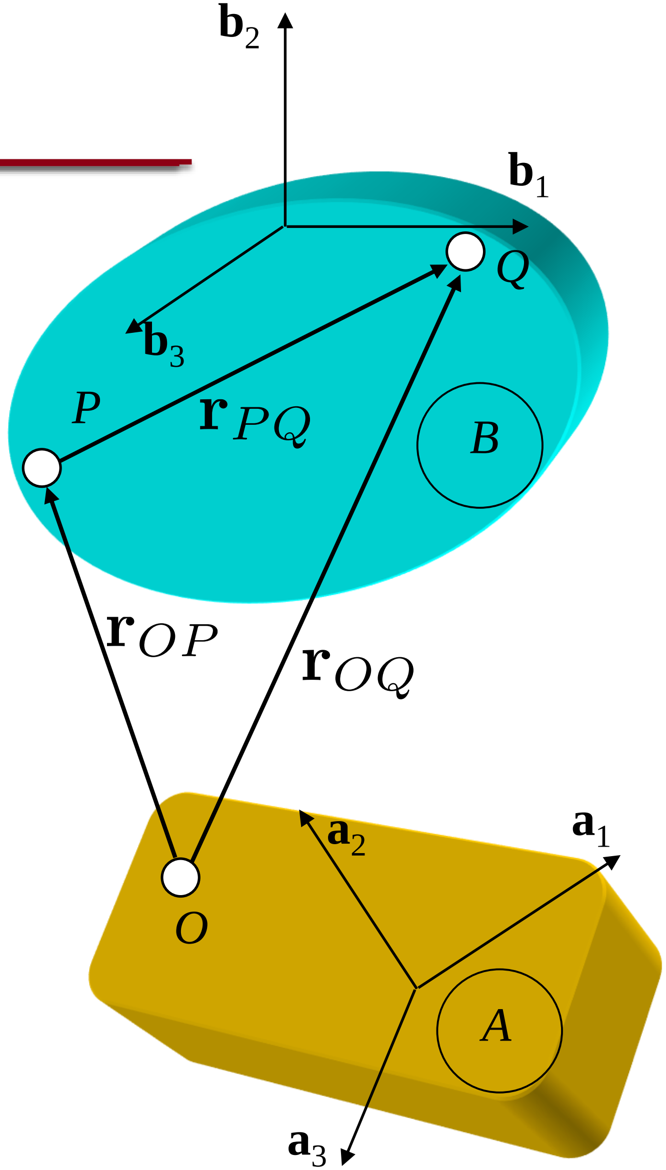
4. Write equations relating pairs of points either on <u>same</u> or <u>adjacent bodies</u>.

Position Analysis

$$\mathbf{r}_{OQ} = \mathbf{r}_{OP} + \mathbf{r}_{PQ}$$

$$q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$
$$= p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$
$$+ q_1'\mathbf{b}_1 + q_2'\mathbf{b}_2 + q_3'\mathbf{b}_3$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \mathbf{R}_{AB} \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix}$$



Velocity Analysis

$$\mathbf{r}_{OQ} = \mathbf{r}_{OP} + \mathbf{r}_{PQ}$$

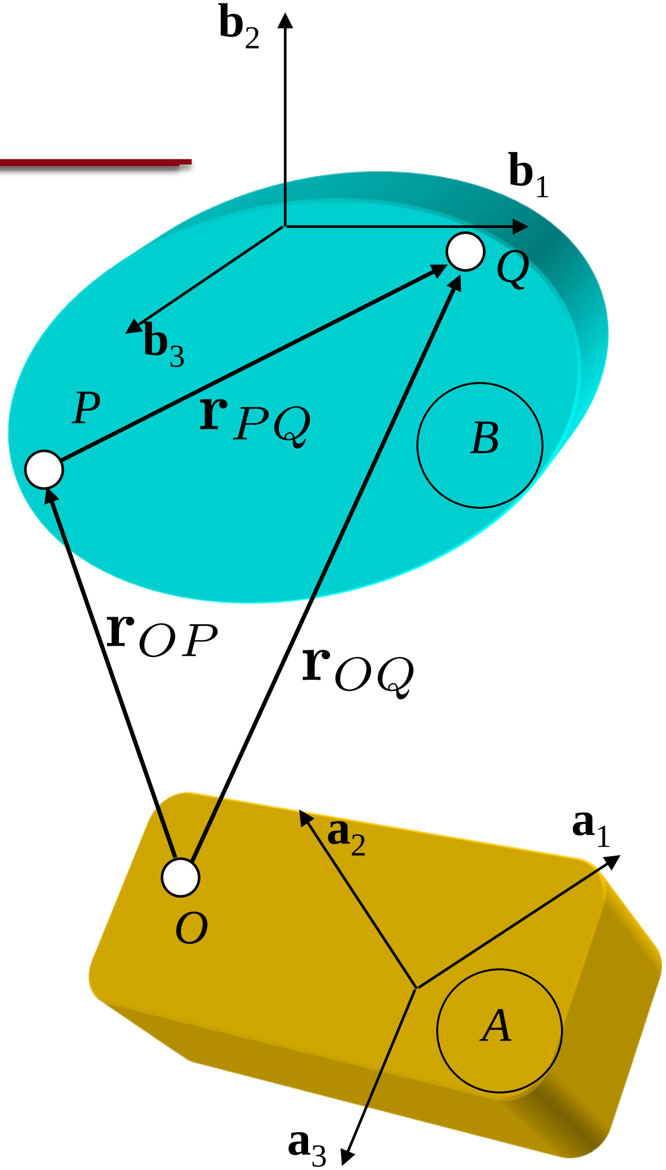
$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} + \underbrace{\boxed{\dot{\mathbf{R}}_{AB}\,\mathbf{R}_{AB}^T}}_{\hat{\omega}_{AB}} \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix}$$

3x3 skew symmetric matrix

$$\hat{\omega}_{AB} \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix} = \omega_{AB} \times \mathbf{r}_{PQ}$$

$$\boxed{\mathbf{v}_Q = \mathbf{v}_P + \omega_{AB} \times \mathbf{r}_{PQ}}$$

angular velocity of B in A



Acceleration Analysis

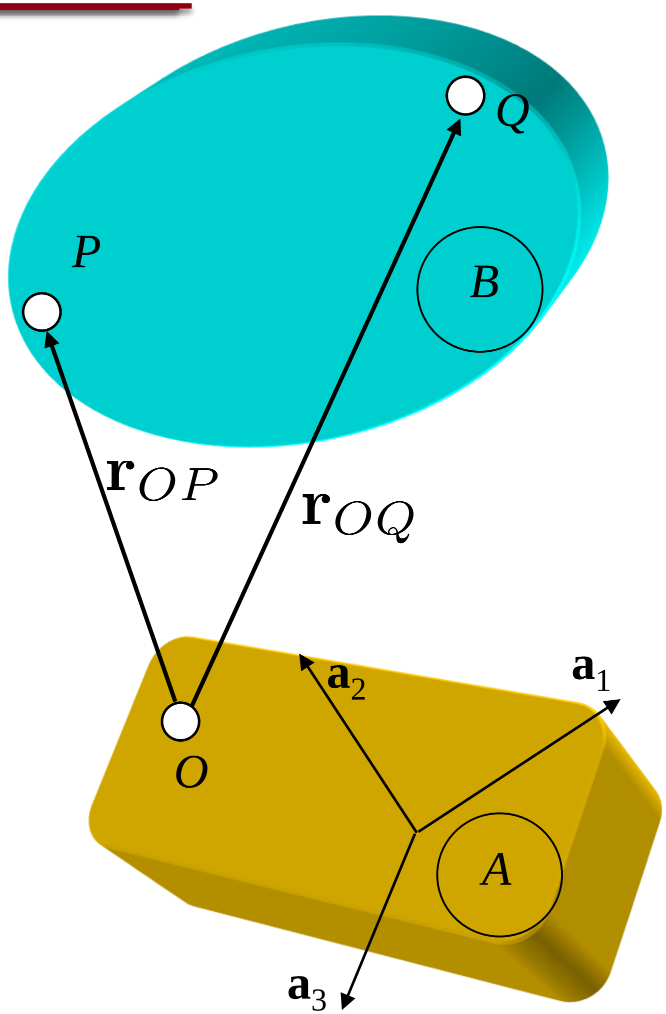
- Acceleration of P and Q in A

$$\mathbf{a}_P = \ddot{p}_1\mathbf{a}_1 + \ddot{p}_2\mathbf{a}_2 + \ddot{p}_3\mathbf{a}_3$$

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \\ \ddot{p}_3 \end{bmatrix}$$

$$\mathbf{a}_Q = \ddot{q}_1\mathbf{a}_1 + \ddot{q}_2\mathbf{a}_2 + \ddot{q}_3\mathbf{a}_3$$

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}$$



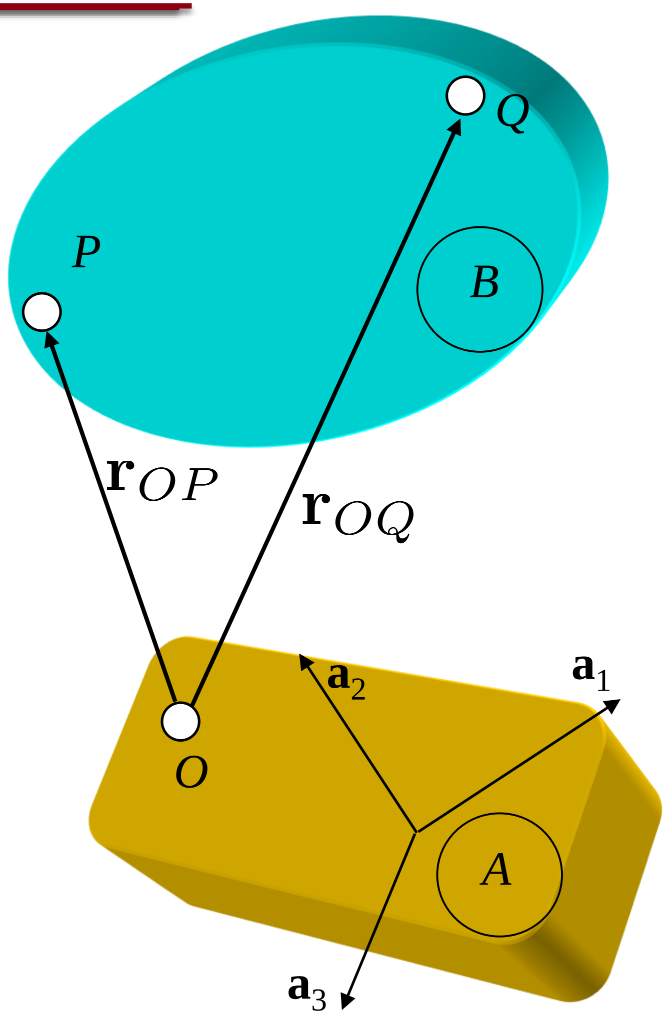
Angular Acceleration

The *angular acceleration of B in A*, is defined as the derivative of the angular velocity of B in A:

$$\alpha_{AB} = \frac{d\omega_{AB}}{dt}$$

$$\hat{\omega}_{AB} = \dot{\mathbf{R}}_{AB}\mathbf{R}^T_{AB}$$

$$\hat{\alpha}_{AB} = \ddot{\mathbf{R}}_{AB}\mathbf{R}^T_{AB} + \dot{\mathbf{R}}_{AB}\dot{\mathbf{R}}^T_{AB}$$

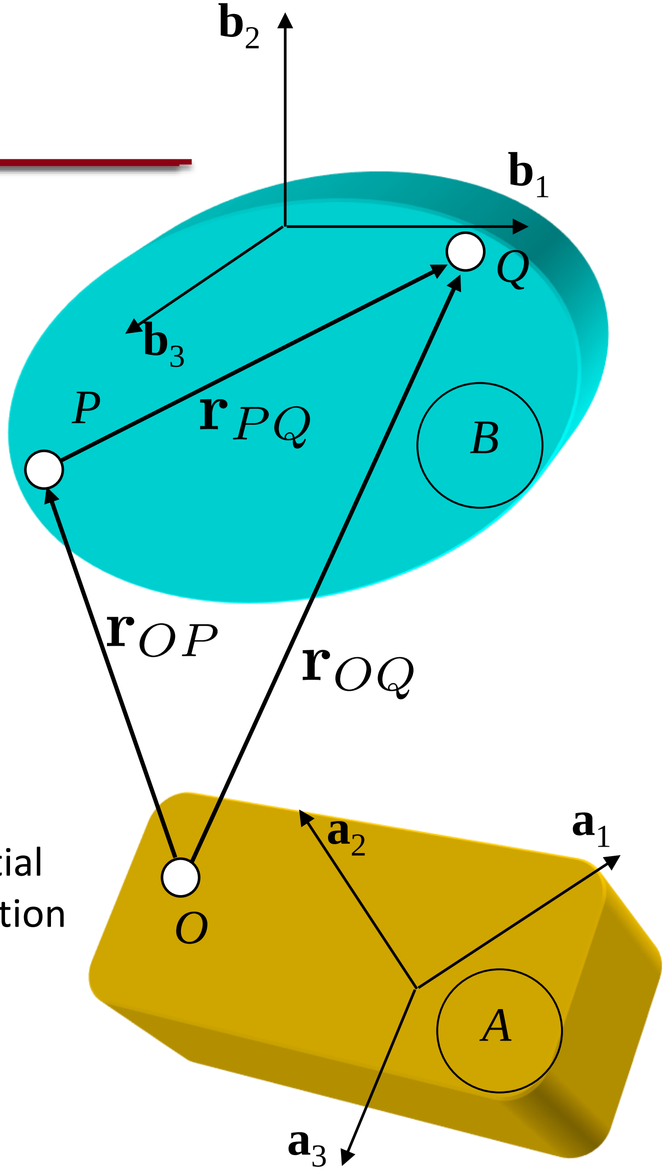


Acceleration Analysis

$$\boxed{\mathbf{v}_Q = \mathbf{v}_P + \omega_{AB} \times \mathbf{r}_{PQ}}$$

$$\frac{d}{dt}$$

$$\mathbf{a}_Q = \mathbf{a}_P + \dot{\omega}_{AB} \times \mathbf{r}_{PQ} + \omega_{AB} \times \dot{\mathbf{r}}_{PQ}$$



$$\mathbf{a}_Q = \mathbf{a}_P + \dot{\omega}_{AB} \times \mathbf{r}_{PQ} + \omega_{AB} \times (\omega_{AB} \times \mathbf{r}_{PQ})$$

tangential acceleration

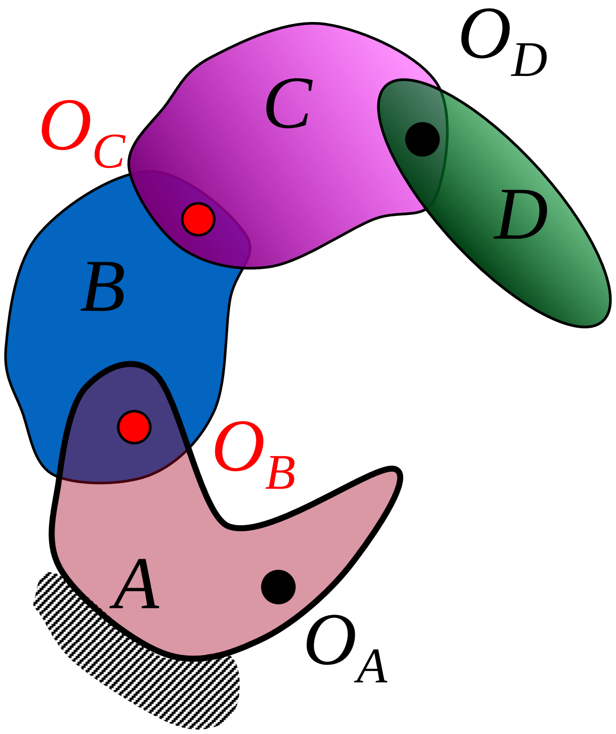
centripetal (normal) acceleration

Serial Chain of Rigid Bodies

$$\mathbf{r}_{O_A O_C} = \mathbf{r}_{O_A O_B} + \mathbf{r}_{O_B O_C}$$

$$\mathbf{v}_{O_C} = \mathbf{v}_{O_B} + \omega_{AB} \times \mathbf{r}_{O_B O_C}$$

$$\mathbf{a}_{O_C} = \mathbf{a}_{O_B} + \dot{\omega}_{AB} \times \mathbf{r}_{O_B O_C} + \omega_{AB} \times (\omega_{AB} \times \mathbf{r}_{O_B O_C})$$

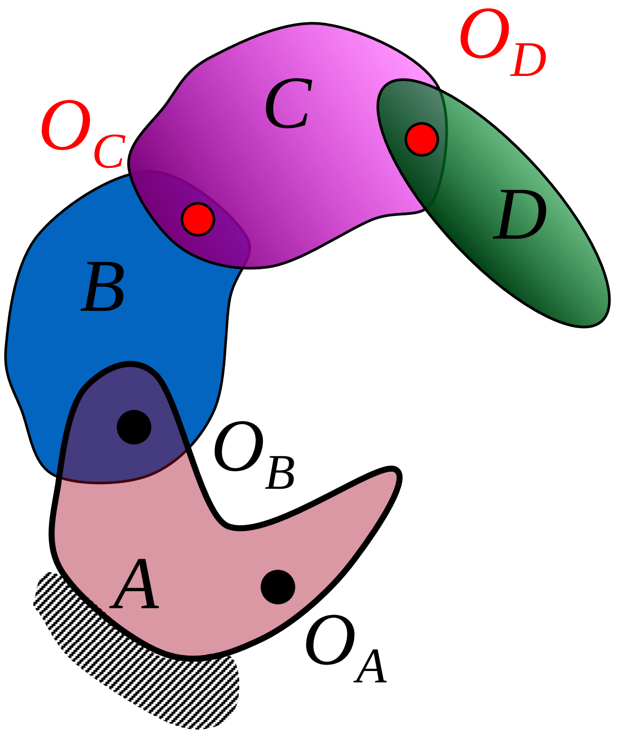


Serial Chain of Rigid Bodies

$$\mathbf{r}_{O_A O_D} = \mathbf{r}_{O_A O_C} + \mathbf{r}_{O_C O_D}$$

$$\mathbf{v}_{O_D} = \mathbf{v}_{O_C} + \omega_{AC} \times \mathbf{r}_{O_C O_D}$$

$$\mathbf{a}_{O_D} = \mathbf{a}_{O_C} + \dot{\omega}_{AC} \times \mathbf{r}_{O_C O_D} + \omega_{AC} \times (\omega_{AC} \times \mathbf{r}_{O_C O_D})$$



Video 2.4
Vijay Kumar and Ani Hsieh

Newton Euler Equations (continued)

Vijay Kumar and Ani Hsieh
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Newton Euler Equations

Newton's equations of motion

A rigid body *B* accelerates in an inertial frame *A* as if it were a single particle with the same mass *m* (equal to the total mass of the system) acted upon by a force equal to the net external force.

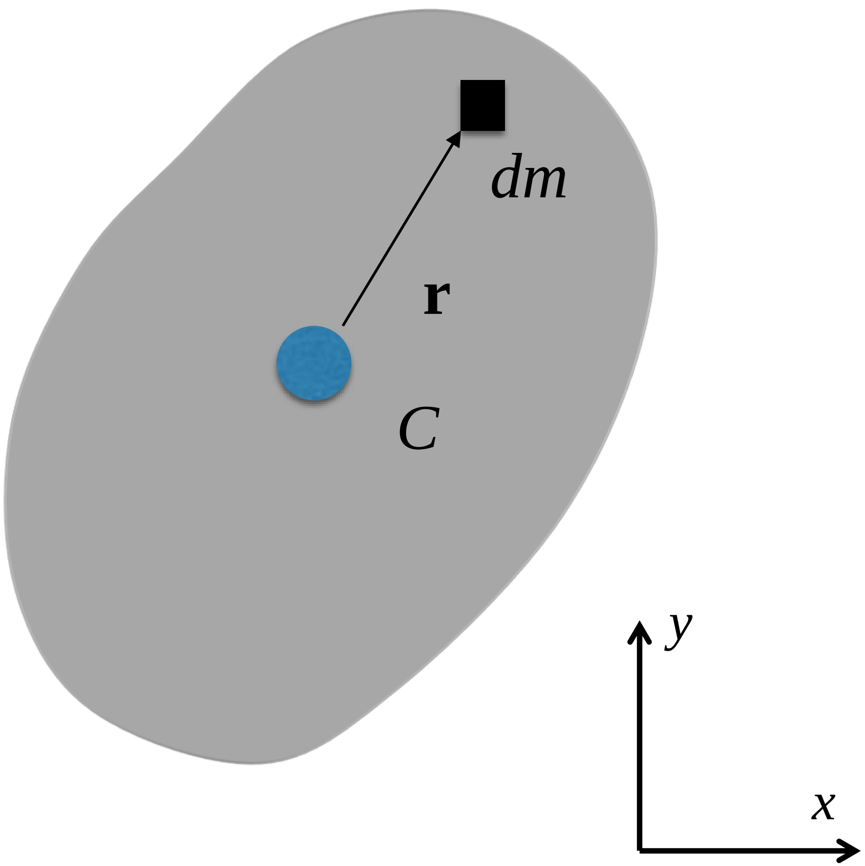
$$\mathbf{F} = m\frac{d\mathbf{v}_C}{dt}$$

Euler's equations of motion

The rate of change of angular momentum of the rigid body B with the center of mass *C* as the origin in *A* is equal to the resultant moment of all external forces acting on the body with *C* as the origin

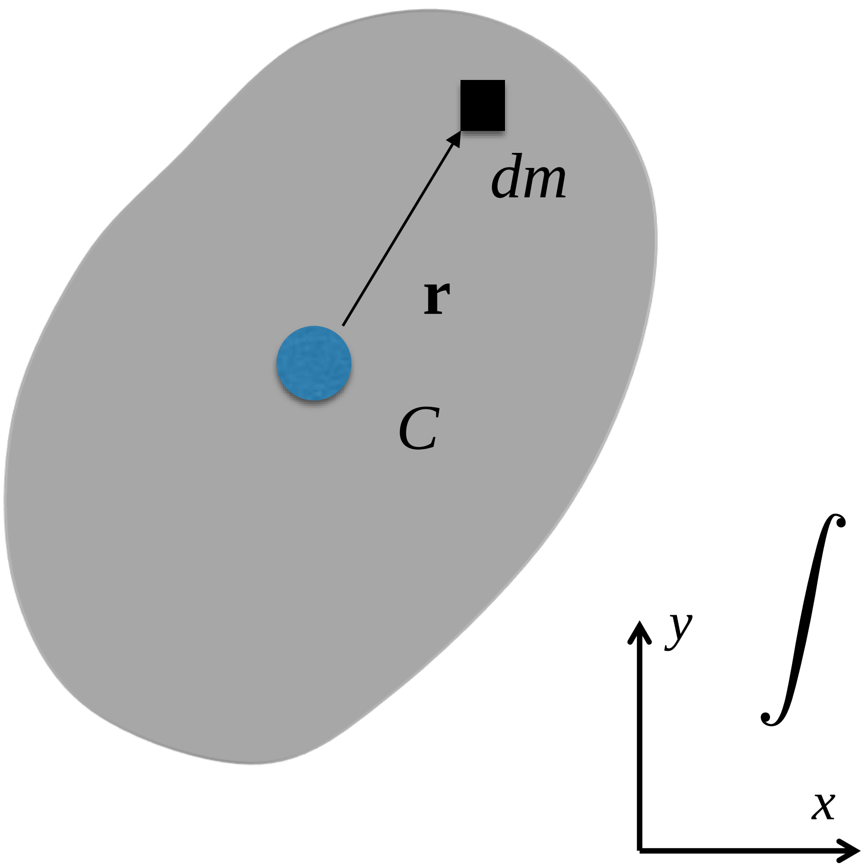
$$\mathbf{M}_C = \frac{\mathbf{I}_C\omega}{dt}$$

Moment of Inertia of 3-D Objects?



$$\cancel{I_C = \int_M \|\mathbf{r}\|^2 dm}$$

Inertia Tensor of 3-D Objects



$$\int \left(\mathbf{r}.\mathbf{r} \;\; \mathbf{I}_{3\times 3} - \mathbf{r}\mathbf{r}^T \right) dm$$

Inertia Dyadic

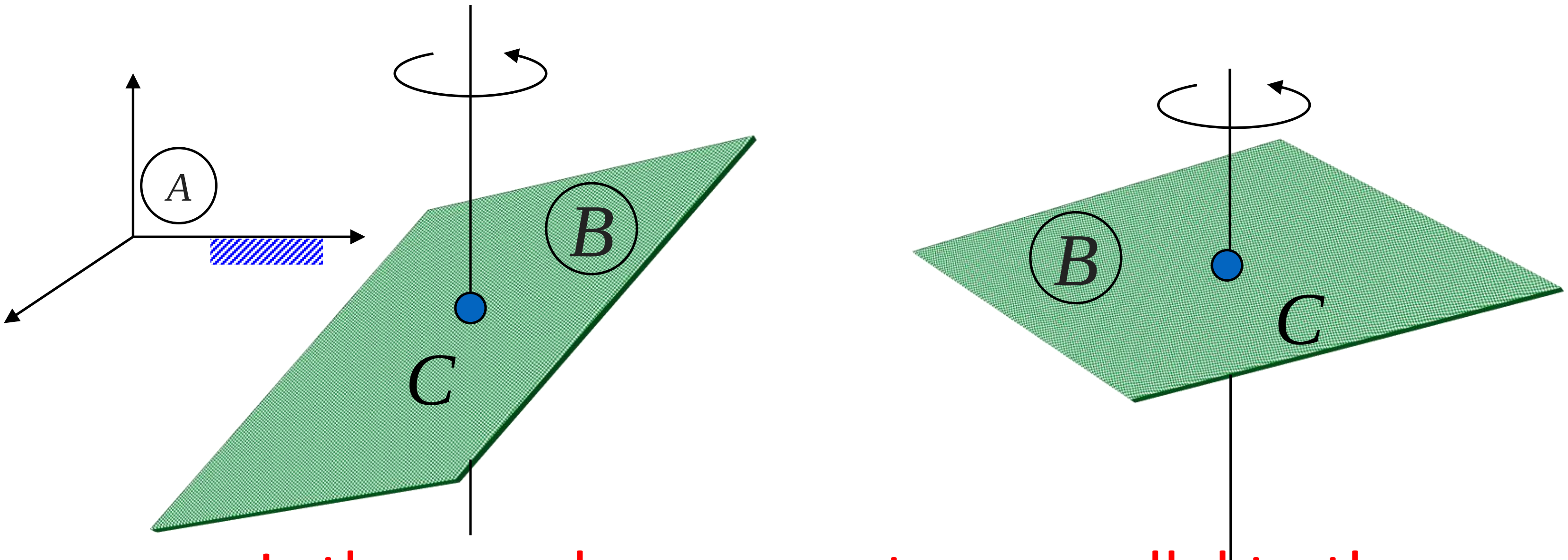
$$\begin{bmatrix} \boxed{\int (y^2 + z^2) dm} & \int -xy dm & \int -xz dm \\ \int -xy dm & \boxed{\int (x^2 + z^2) dm} & \int -yz dm \\ \int -xz dm & \int -yz dm & \boxed{\int (x^2 + y^2) dm} \end{bmatrix}$$

Products of Inertia

Principal Moments of Inertia

Example: Rectangular Plate Rotating about Axis through Center of Mass

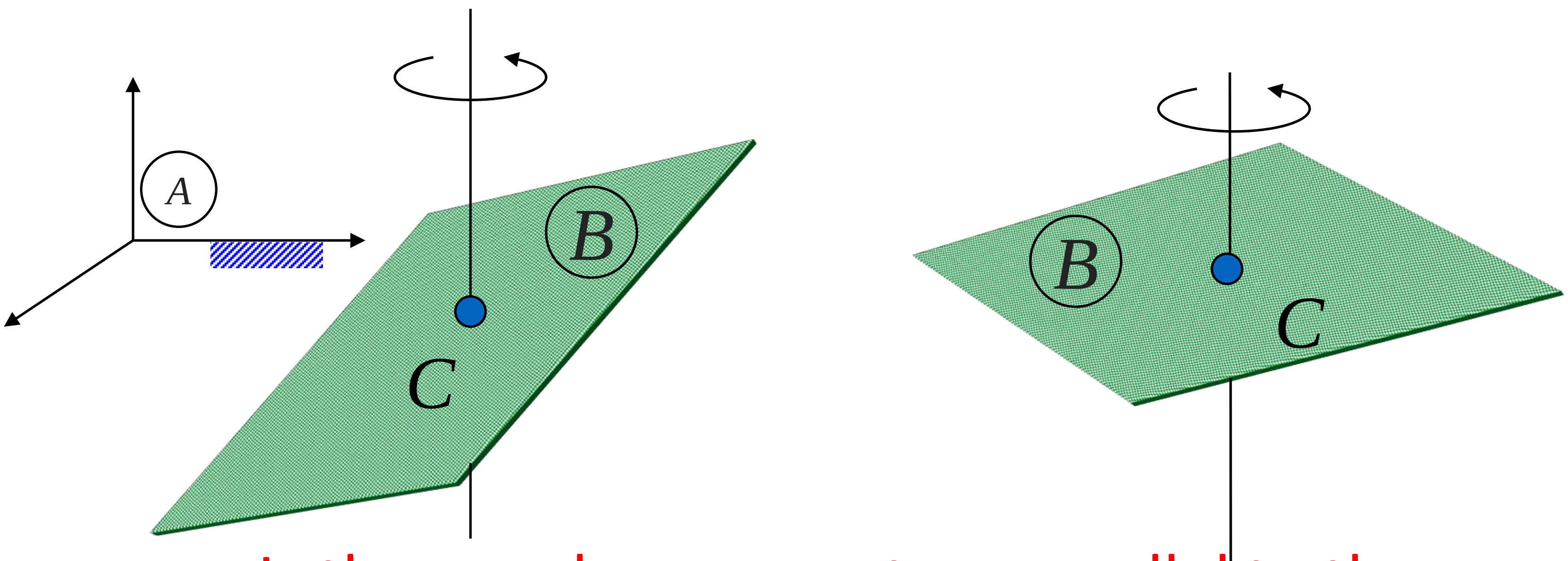
$$\begin{bmatrix} \int (y^2+z^2)dm & \int -xydm & \int -xzdm \\ \int -xydm & \int (x^2+z^2)dm & \int -yzdm \\ \int -xzdm & \int -yzdm & \int (x^2+y^2)dm \end{bmatrix}$$



Is the angular momentum parallel to the angular velocity?

Example: Rectangular Plate Rotating about Axis through Center of Mass

$$\begin{bmatrix} I_{xx} & \times & \times \\ \times & I_{yy} & \times \\ \times & \times & I_{zz} \end{bmatrix} \qquad \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$



Is the angular momentum parallel to the angular velocity?

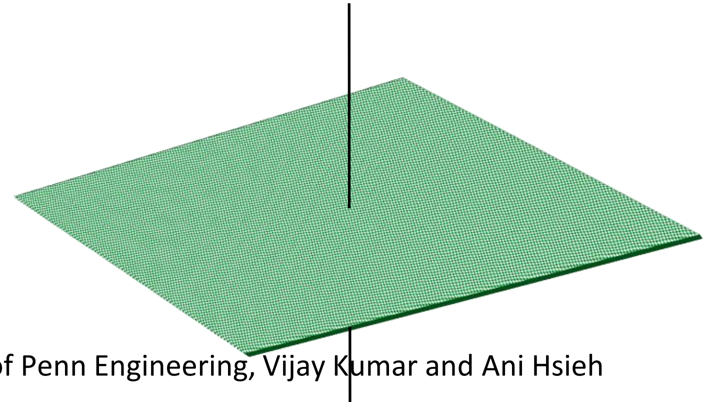
Principal Axes and Principal Moments

Principal axis

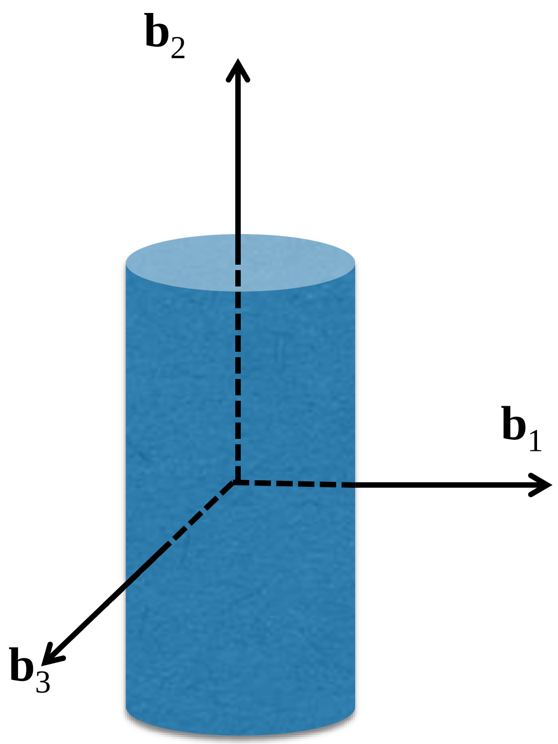
- **u** is a unit vector along a principal axis if **I u** is parallel to **u**
- There are 3 independent principal axes!

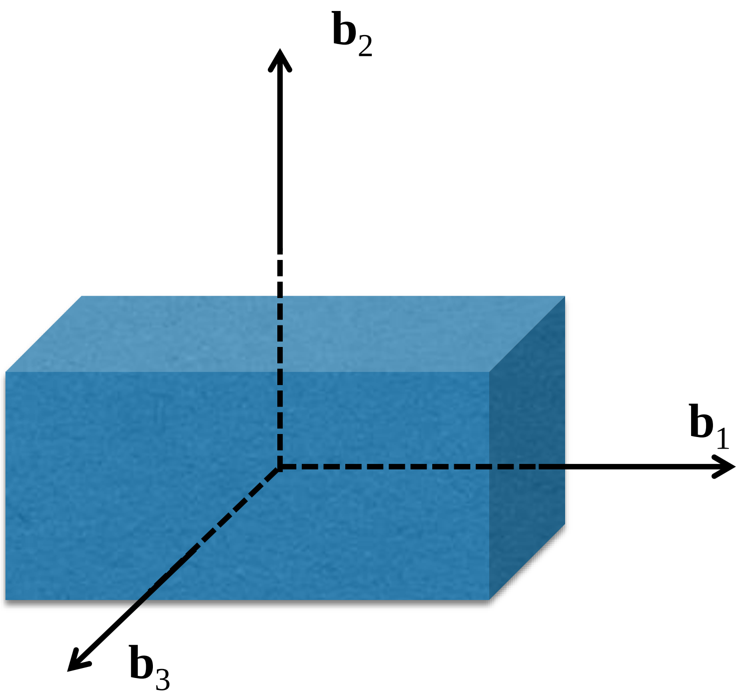
Principal moment of inertia

- The moment of inertia with respect to a principal axis, $\mathbf{u}^T \mathbf{I}\ \mathbf{u}$, is called a principal moment of inertia.



Examples





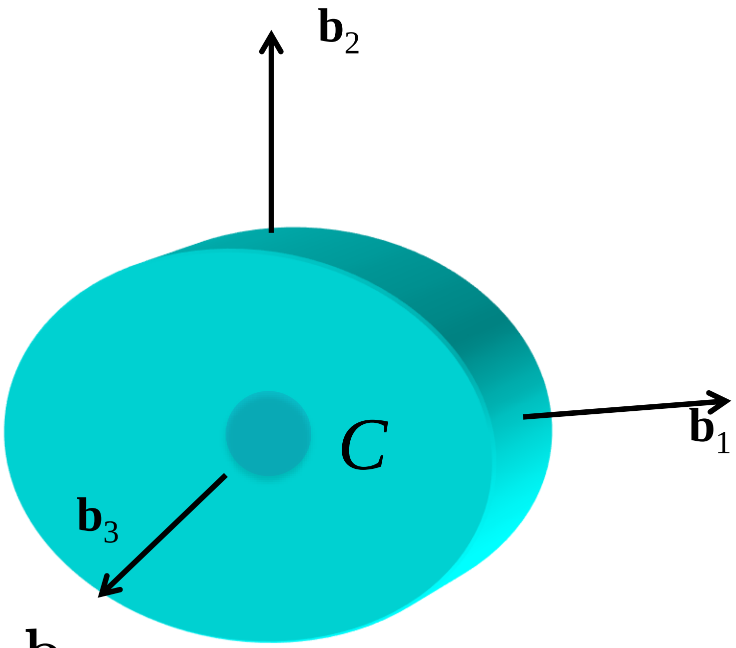
Euler's Equations of Motion

$$\mathbf{H}_C = \mathbf{I}_C \omega_{AB}$$

In frame *B*

$$\omega_{AB} = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3$$

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$



$$\mathbf{H}_C = I_{11}\omega_1 \mathbf{b}_1 + I_{22}\omega_2 \mathbf{b}_2 + I_{33}\omega_3 \mathbf{b}_3$$

$$\mathbf{H}_C = \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix}$$

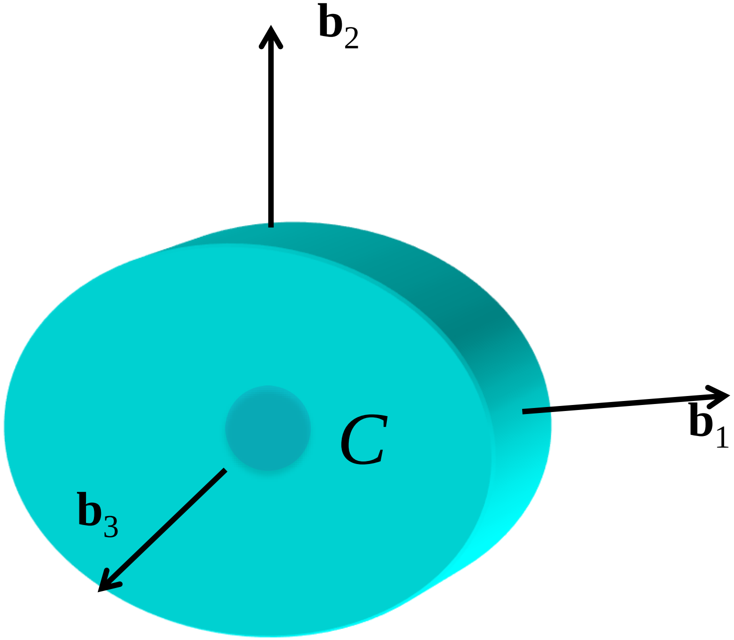
Euler's Equations of Motion

$$\mathbf{H}_C = \mathbf{I}_C \omega_{AB}$$

need to differentiate in A

In frame *A*

$$\mathbf{H}_C = \mathbf{R}_{AB} \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix}$$



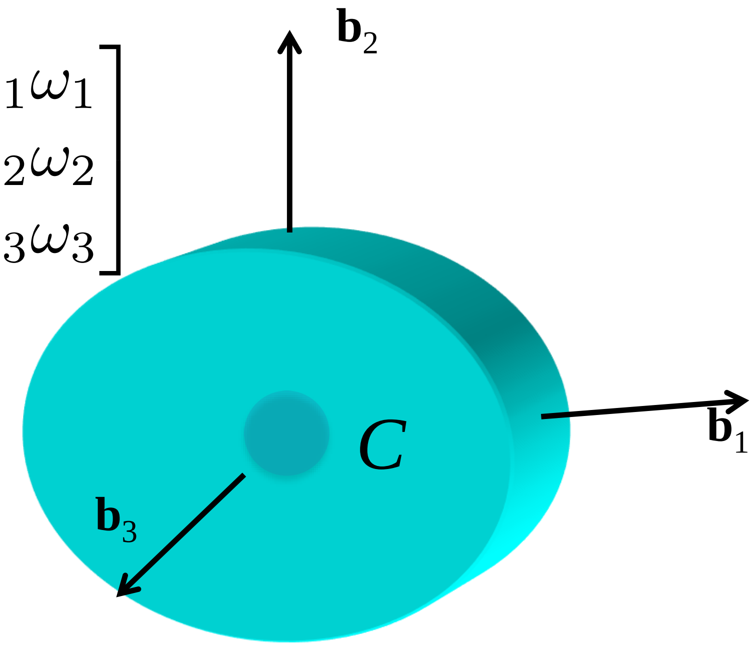
$$\dot{\mathbf{H}}_C = \mathbf{R}_{AB} \begin{bmatrix} I_{11}\dot{\omega}_1 \\ I_{22}\dot{\omega}_2 \\ I_{33}\dot{\omega}_3 \end{bmatrix} + \dot{\mathbf{R}}_{AB} \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix}$$

Euler's Equations of Motion

Transform back to frame *B*

$$\dot{\mathbf{H}}_C = \mathbf{R}_{AB}\begin{bmatrix} I_{11}\dot{\omega}_1 \\ I_{22}\dot{\omega}_2 \\ I_{33}\dot{\omega}_3 \end{bmatrix} + \dot{\mathbf{R}}_{AB}\begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix}$$

$$\mathbf{R}_{AB}^T$$



$$\hat{\omega}_{AB} = \mathbf{R}_{AB}^T \mathbf{R}_{AB}$$

angular velocity in frame *B*

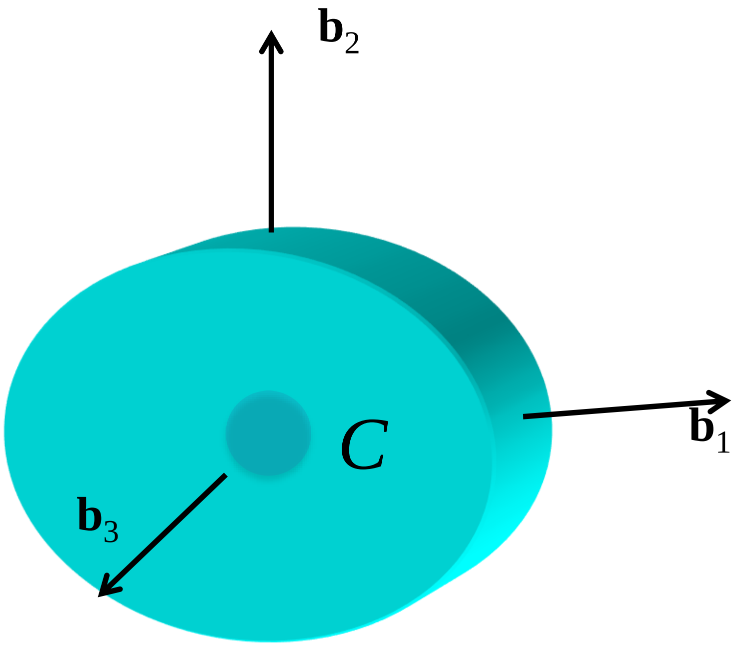
$$\dot{\mathbf{H}}_C = \begin{bmatrix} I_{11}\dot{\omega}_1 \\ I_{22}\dot{\omega}_2 \\ I_{33}\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix}$$

Euler's Equations of Motion

1. frame B ($\mathbf{b}_i$) along principal axes

2. center of mass as origin

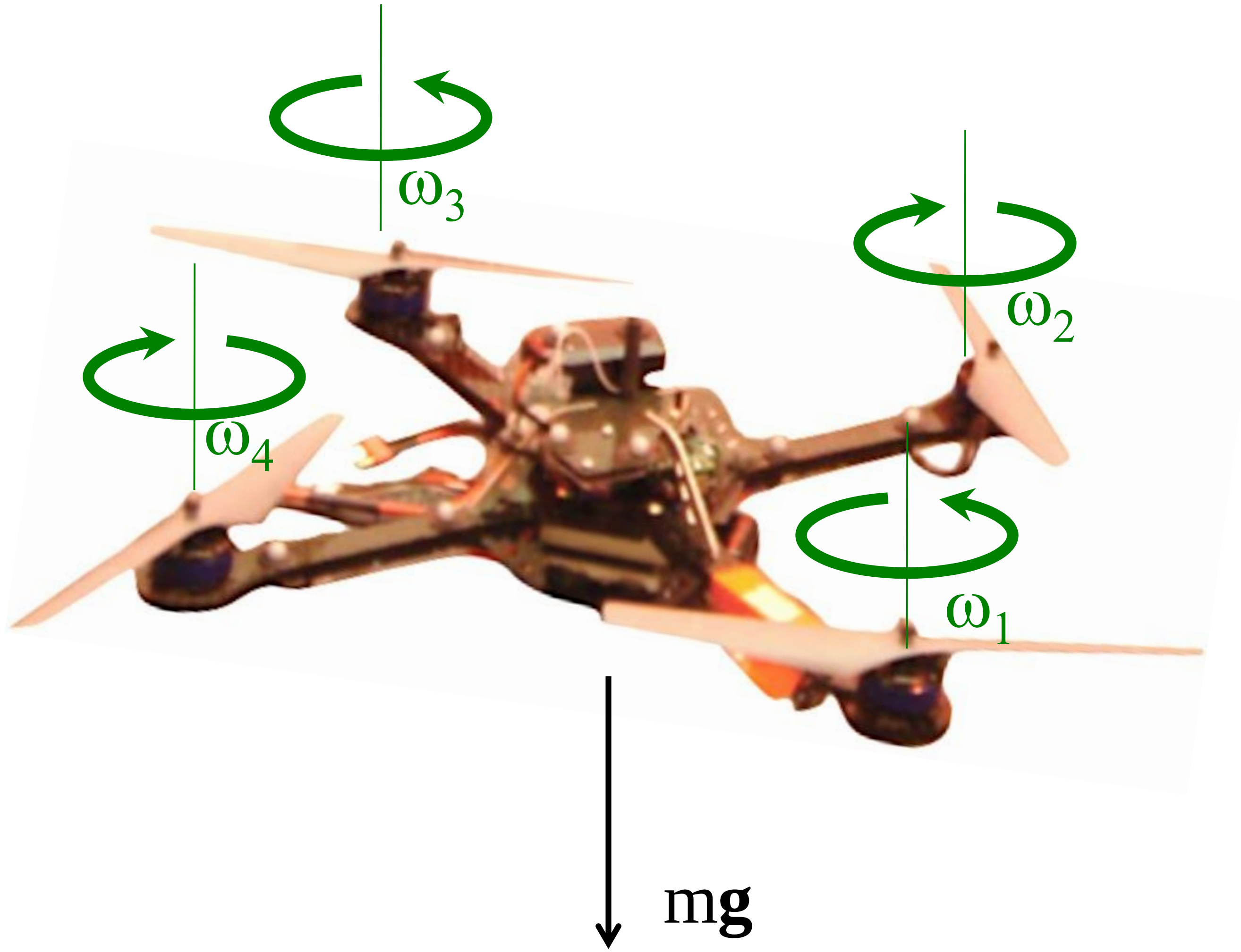
3. all components along $\mathbf{b}_i$



$$\begin{bmatrix} I_{11}\dot{\omega}_1 \\ I_{22}\dot{\omega}_2 \\ I_{33}\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix} = \mathbf{M}_C$$

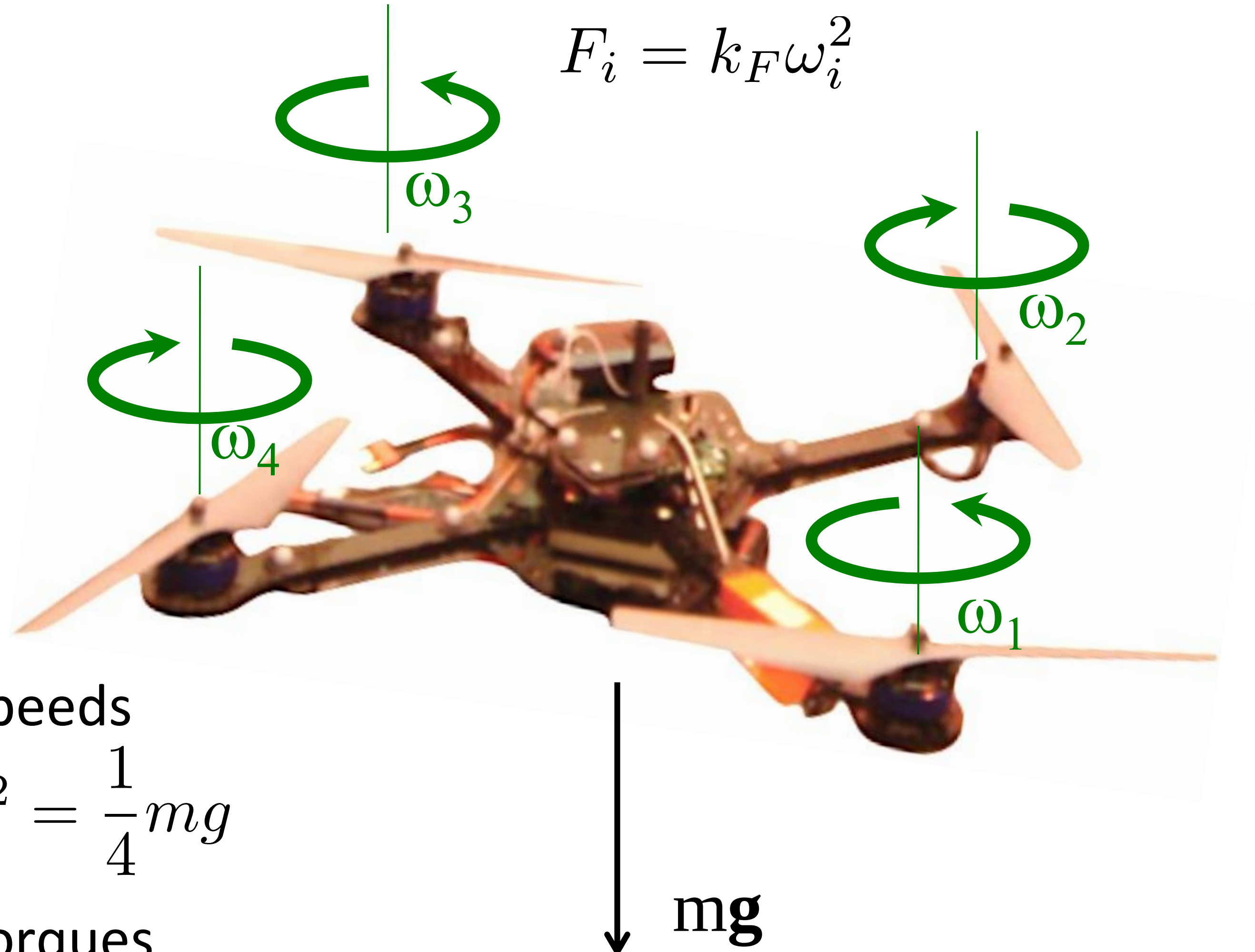
angular velocity, moments of inertia, moments in frame B

Quadrotor



Static Equilibrium (Hover)

$$F_i = k_F \omega_i^2$$



Motor Speeds

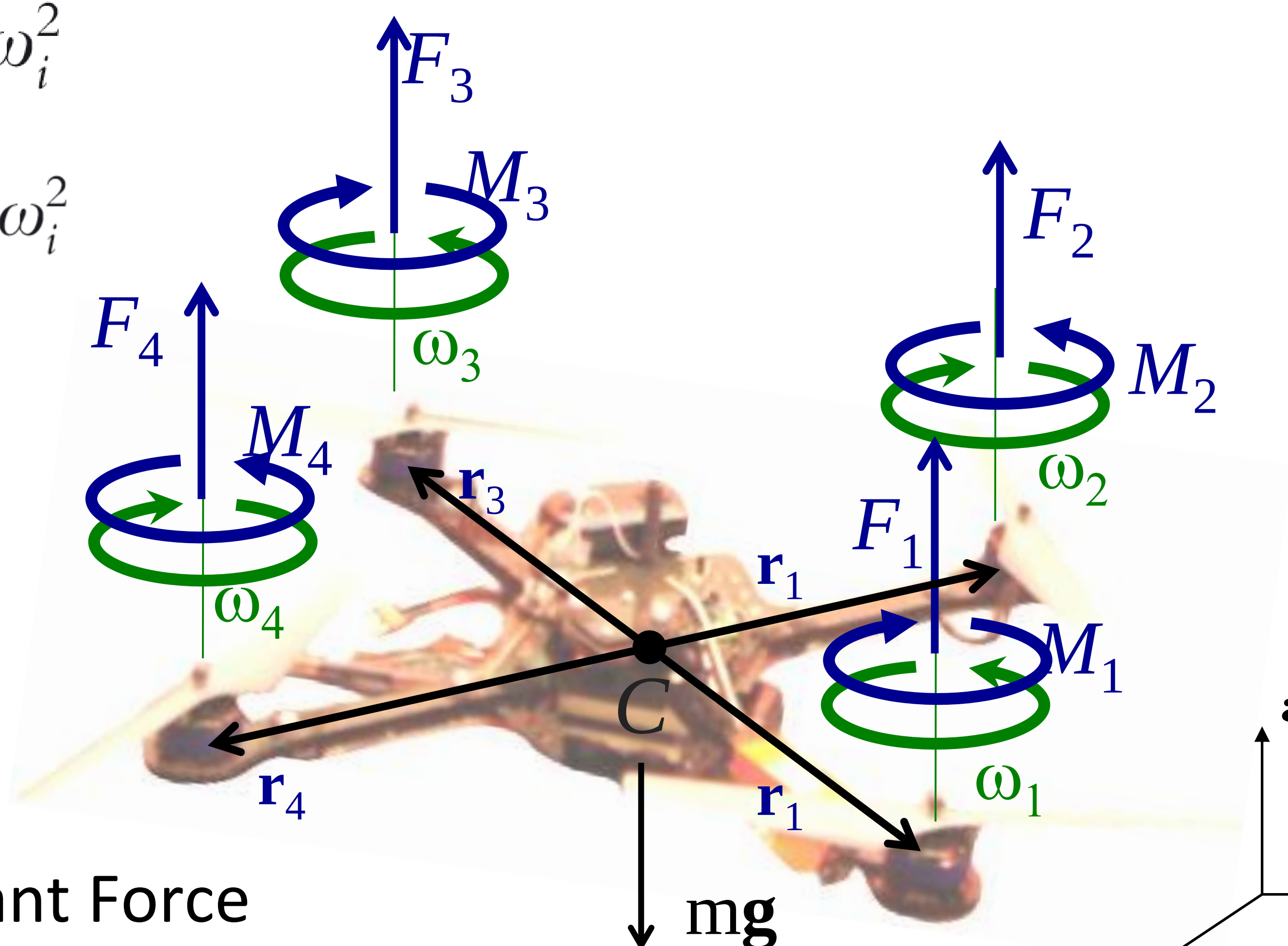
$$k_F \omega_i^2 = \frac{1}{4} mg$$

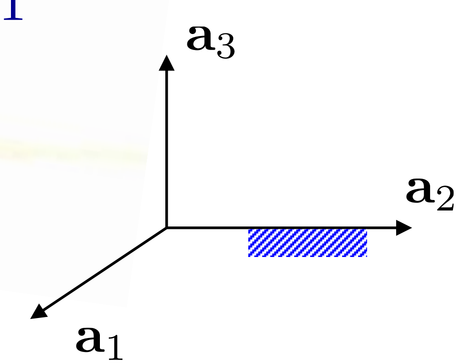
Motor Torques

$$\tau_i = k_M \omega_i^2$$

$$F_i = k_F \omega_i^2$$

$$M_i = k_M \omega_i^2$$





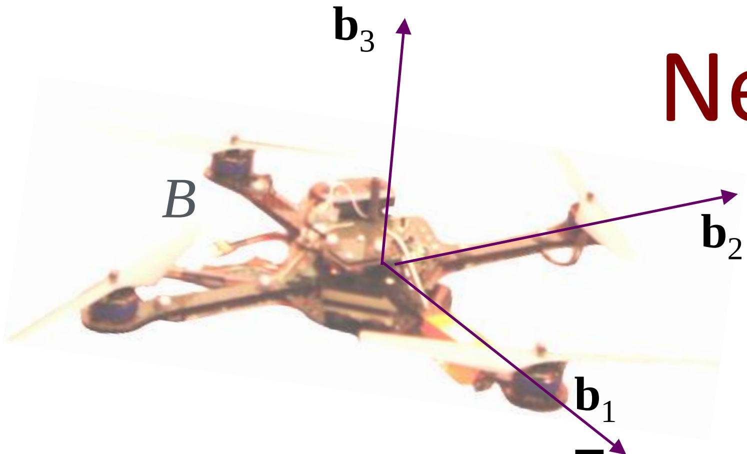
Resultant Force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 - mg\mathbf{a}_3$$

Resultant Moment about *C*

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4$$
$$+ \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4$$

Newton-Euler Equations



$$\omega_{AB} = p\mathbf{b}_1 + q\mathbf{b}_2 + r\mathbf{b}_3$$

Rotation of thrust vector from *B* to *A*

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \overset{\mathbf{R}_{AB}}{R} \begin{bmatrix} 0 \\ 0 \\ \boxed{F_1 + F_2 + F_3 + F_4} \end{bmatrix}$$

u_1

Components in the inertial frame along $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \boxed{\begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$\mathbf{u}_2$

Components in the body frame along $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, the principal axes