edX Robo4 Mini MS – Locomotion Engineering

Block 1 – Week 2 – Unit 3

Linearization and Lyapunov Functions

Video 5.1

Segment 1.2.3.1.a

Dynamical Systems Theory

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Penn Engineering

Agenda for this Segment

- Analytic VF
  - have "simple" Taylor approximations
  - yield "simple" closed form flow expressions
- Hope
  - "simple" closed form flow of the approximate VF
  - may approximate the "complicated" flow of the VF
- Actuality
  - for "typical" situations the hope is borne out
  - in those situations, get more than "approximation"
    - turns out that "behavior" is indistinguishable
    - up to change of coordinates (in general, nonlinear)
- Crucial issue: recognize & deal with "atypical" cases

Multivariable Taylor Series

- Taylor's Theorem
  - an "analytic" function
  - has a global Taylor expansion
    (around a specified point)

$$\text{analytic } f : \mathbb{R}^n \to \mathbb{R}^m$$
$$\Rightarrow f(x) = f(x^*) + F(x^*)\,(x - x^*) + O(\|x - x^*\|^2)$$
$$F(x^*) := D_x f\,\big|_{x^*} \in \mathbb{R}^{m\times n}$$

  - whose first order term
  - is called its linearization

Taylor Series Approximation

- Constant term $f(x) \approx f(x^*)$
  - exact at specified point
  - "close" approximation
    - in "very small"
    - neighborhood of point
- Linearization $f(x) \approx f(x^*) + F(x^*)\,(x - x^*)$
  - approximates in a "small"
  - neighborhood of the point
  - no general criterion for "small"
- … higher order terms … better …
- References
  - Taylor's Theorem

    Richard Courant. Differential and integral calculus, volume 2. John Wiley & Sons, 2011.
  - Calculus via "long polynomials"

    R. Ghrist. Funny Little Calculus Text. R. W. Ghrist, 2012.

0th Taylor Approximation of Flow

- Example: damped falling unit mass $f_{\mathrm{DG}}(v) := -\mathrm{g} - bv$

$$\Rightarrow f_{\mathrm{DG}}^{t}(v_0) = \mathrm{e}^{-tb} v_0 - \frac{\mathrm{g}}{b}\left(1 - \mathrm{e}^{-tb}\right)$$

  - Constant approximation of VF around $v_0 = 0$

$$\hat{f}_{\mathrm{DG}}(\delta) := -\mathrm{g}$$

  - Flow of constant approximation (proposed 0th order approx. of Flow)

$$\Rightarrow \hat{f}_{\mathrm{DG}}^{t}(\delta) = -\mathrm{g}t + \delta$$

  - Constant approximation of Flow around $v_0 = 0$

$$\widehat{f_{\mathrm{DG}}^{t}}(\delta) = -\frac{\mathrm{g}}{b}\left(1 - \mathrm{e}^{-tb}\right) = -\mathrm{g}t\left(\sum_{k=0}^{\infty}(-tb)^k/(k+1)!\right)$$

- Conclude: not too bad for very small t and δ

0^{th} Taylor Approximation Failure

- In general, for "typical" v_0
  - flow of the constant approximation VF
    $\hat{f}^t_{\mathrm{DG}}(v_0 + \delta) = -\mathrm{g}t + v_0 + \delta$
  - is "close" to
  - constant approximation of the flow of the actual
  - for very small t and δ
- What about "atypical" v_0 ?
  - major exception: fails badly at FP $v_0 = v_e := b/\mathrm{g}$
  - since $f^t_{\mathrm{DG}}(v_e) \equiv v_e \Rightarrow \widehat{f^t_{\mathrm{DG}}}(v_e + \delta) = v_e$
- We'll need 1^{st} Taylor Approximation for "typical" FP
- First show that "behavior is same" in typical case

Expand CC Notion

- Change of Coordinates (CC)

$$y = h(x)$$

$$x = h^{-1}(y)$$

  - continuous
  - continuously invertible

    (one-to-one & onto)

- Examples
  - Smooth (CC_D): differentiable both ways
  - Linear (CC_L): similarity (with matrix rep)

General Change of Coordinates Formulae

- maps (e.g., flows)

$$
\begin{aligned}
x(t) &= f^t(x_0) \& \text{ CC } y = h(x) \\
\Rightarrow y(t) &= h \circ x(t) = h \circ f^t(x_0) \\
&= h \circ f^t \circ h^{-1}(x_0) \\
&=: \tilde{f}^t(y_0)
\end{aligned} \tag{1}
$$

- VF

$$
\begin{aligned}
\dot{x} &= f(x) \& \text{ CC}_D \ y = h(x) \\
\Rightarrow \dot{y} &= D_x h \cdot \dot{x} = D_x h \cdot f(x) \\
&= D_x h \cdot f \circ h^{-1}(y) \\
&=: \tilde{f}(y)
\end{aligned} \tag{2}
$$

Scalar Nonlinear Change of Coordinates

- return to damped falling unit mass

$$f_{\mathrm{DG}}(v) := -(\mathrm{g} + bv)$$

- introduce proposed CC

$$u = h_{\mathrm{DG}}(v) := \frac{\mathrm{g}}{b}\ln\left(\mathrm{g} + bv\right)$$

  - to assure good $\mathrm{CC_D}$
  - assume IC "far" from FP, $v_e := b/\mathrm{g}$

- get conjugacy $\dot{u} = D_v h_{\mathrm{DG}} \cdot f_{\mathrm{DG}}(v)$
  - so behavior
  - away from FP
  - is "identical"
  - up to CC

$$= \frac{b\mathrm{g}}{b\left(\mathrm{g} + bv\right)} \cdot (-\mathrm{g} - bv)$$

$$= -\mathrm{g} =: f_{\mathrm{G}}(u)$$

Normal Form

- Previous result
  - conjugacy between VF & 0^{th} order VF approximant
  - on neighborhoods away from FP
- Leads to notion of "normal form"
  - the lowest degree polynomial approximant
  - that still admits local conjugacy to the VF
- Generalizes to analytic VF in arbitrary dimensions
  - Flowbox Theorem:
    - the normal form for a VF in the neighborhood of a nonFP
    - is the constant VF, e.g., $f_{\text{cons}}(x) := [1, 0, .., 0]^T$
    - [V. I. Arnold. Ordinary Differential Equations. MIT Press, 1973]

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Linearization and Lyapunov Functions

Video 5.2

Segment 1.2.3.1.b
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Agenda for this Segment

- Normal Form (introduced in segment just previous)
  - Flowbox theorem tells us
    - away from FP
    - VF is conjugate to constant velocity
  - now seek normal form in the neighborhood of a FP
- Taylor approximation near FP

$$f(x) = \cancel{f(x_e)}^{0} + F(x_e)\,(x - x_e) + O(\|x - x^*\|^2)$$

  - seems dominated by $F(x_e) := D_x f\,(x_e)$
  - good news when truly a normal form
  - since we understand LTI systems very well
- Now explore when this intuition holds

Conditions for FP Normal Form: Scalars

- For VF to have LTI normal form at scalar FP x_e
  - must have $F(x_e) := D_x f\,(x_e) \neq 0$
  - else

$$f(x) = f(\cancel{x_e})^{\nearrow 0} + F(\cancel{x_e})^{\nearrow 0}\,(x - x_e) + \left[D^2 f\right](x - x_e)^2 + O(\|x - x^*\|^3)$$

  - in which case linearized VF cannot be conjugate

$$\hat{f}_{\mathrm{L}}(x) := F(x_e)\,(x - x_e) = 0 \not\sim \hat{f}_{\mathrm{Q}}(x) := \left[D^2 f\right](x - x_e)^2$$

  - because, e.g., $\left[D^2 f\right] > 0 \;\&\; \eta(x) := |x - x_e|^2 /2 \Rightarrow$

$$\dot{\eta}\,(x) = D_x\eta \cdot \hat{f}_{\mathrm{Q}}(e) = \left[D^2 f\right](x - x_e)^3 = \left[D^2 f\right]\eta(x)^{3/2}$$

    - so η grows without bound along flow of quadratic field
    - whereas it is constant along flow of the linear field
  - no CC can conjugate an unbounded to a bounded fnc.

Conditions for FP Normal Form: Vectors

- But for VF to have LTI normal form at vector FP x_e
- need more than simply $F(x_e) := D_x f(x_e) \neq 0$
- e.g. $\dot{\mathbf{y}} = f_{\mathrm{NLRC}}(\mathbf{y}) := \begin{bmatrix} \|\mathbf{y}\|^2 & -1 \\ 1 & \|\mathbf{y}\|^2 \end{bmatrix}\mathbf{y}; \quad \mathbf{y}_e := 0$

  $F_{\mathrm{NLRC}}(\mathbf{y}_e) := D_{\mathbf{y}} f_{\mathrm{NLRC}}(\mathbf{y}_e) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} =: J \neq 0$

  - hence linearized VF cannot be conjugate

    $\hat{f}_{\mathrm{L}}(\mathbf{y}) := F_{\mathrm{NLRC}}(\mathbf{y}_e)\mathbf{y} \not\sim f_{\mathrm{NLRC}}(\mathbf{y})$

  - because, e.g., $\eta(\mathbf{y}) := \|\mathbf{y}\|^2/2 \Rightarrow$

    $\dot{\eta}_{\mathrm{L}}(\mathbf{y}) = D_{\mathbf{y}}\eta \cdot f_{\mathrm{L}}(\mathbf{y}) = \mathbf{y}^{\mathrm{T}} J \mathbf{y} \equiv 0$

    vs $\dot{\eta}(\mathbf{y}) = D_{\mathbf{y}}\eta \cdot f_{\mathrm{NLRC}}(\mathbf{y}) = 8\eta(\mathbf{y})^{3/2}$

    - so η grows without bound along flow of f_{NLRC}
    - whereas it is constant along flow of f_{L}

Hyperbolicity of FP

- Say that a VF is hyperbolic at a FP
- If its linearization has no purely imaginary eigenvalues
- Examples
  - scalar damped mass: $f_{\mathrm{D}}(v) = -bv \Rightarrow F_{\mathrm{D}}(0) = -b \neq 0$
  - damped pendulum:
    - FP with zero velocity
    - at $\theta_{\mathrm{e}} = n\pi$
    - again, need nonzero b

$$f_{\mathrm{DP}}(\boldsymbol{q}) := \begin{bmatrix} \dot{\theta} \\ -\frac{1}{m\ell^2} b\dot{\theta} - \frac{\mathrm{g}}{\ell} \sin\theta \end{bmatrix}$$

$$F_{\mathrm{DP}}(\boldsymbol{q}_e) := \begin{bmatrix} 0 & 1 \\ -\frac{\mathrm{g}}{\ell}\cos\theta_e & -\frac{b}{m\ell^2} \end{bmatrix}$$

Normal Form Near a FP

- If a FP of a VF $\dot{x} = f_{\text{gen}}(x); \quad f_{\text{gen}}(x_e) = 0$
- is hyperbolic

$$\Re \circ \text{evals} \circ F_{\text{gen}}(x_e) \neq 0; \quad F_{\text{gen}}(x_e) := D_x f_{\text{gen}}(x_e)$$

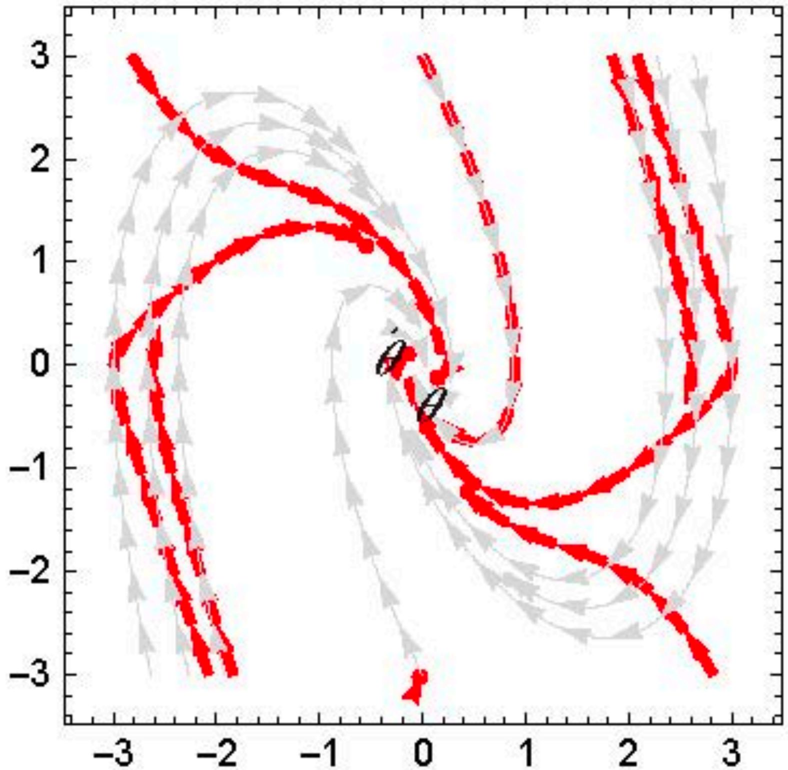
- then the linearized dynamics

$$\dot{u} = \hat{f}_{\text{gen}}(u); \quad \hat{f}_{\text{gen}}(u) := F_{\text{gen}}(x_e)u$$

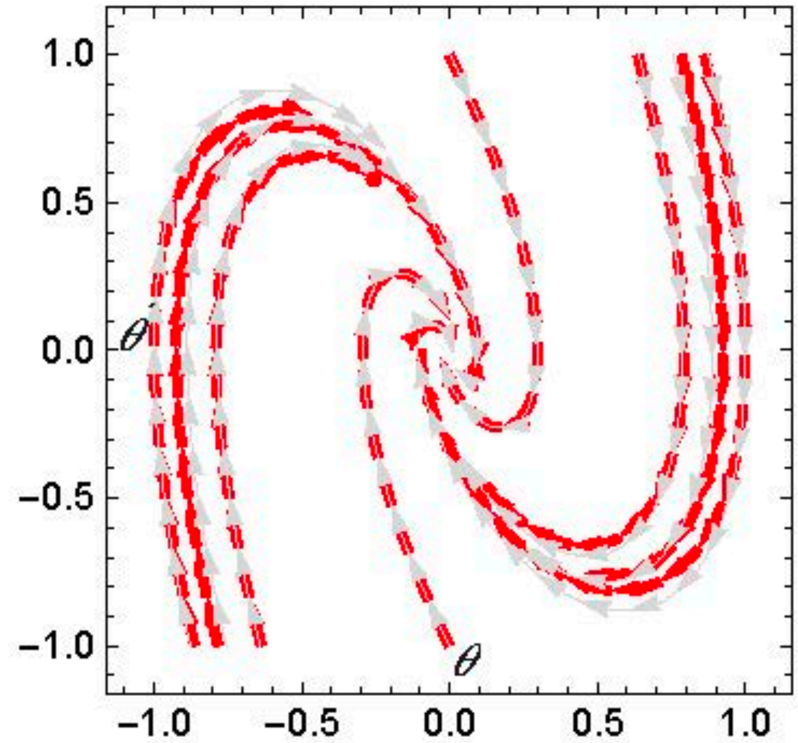
- is locally conjugate $\hat{f}_{\text{gen}} \overset{h_{\text{loc}}}{\sim} f_{\text{gen}}$
- via some CC defined in a neighborhood of the FP
- Reference:

J. Guckenheimer and P. Holmes. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, 1983.

Pendulum (red) and Linearized (gray) Orbits



Near bottom FP, $\mathbf{q}_b$



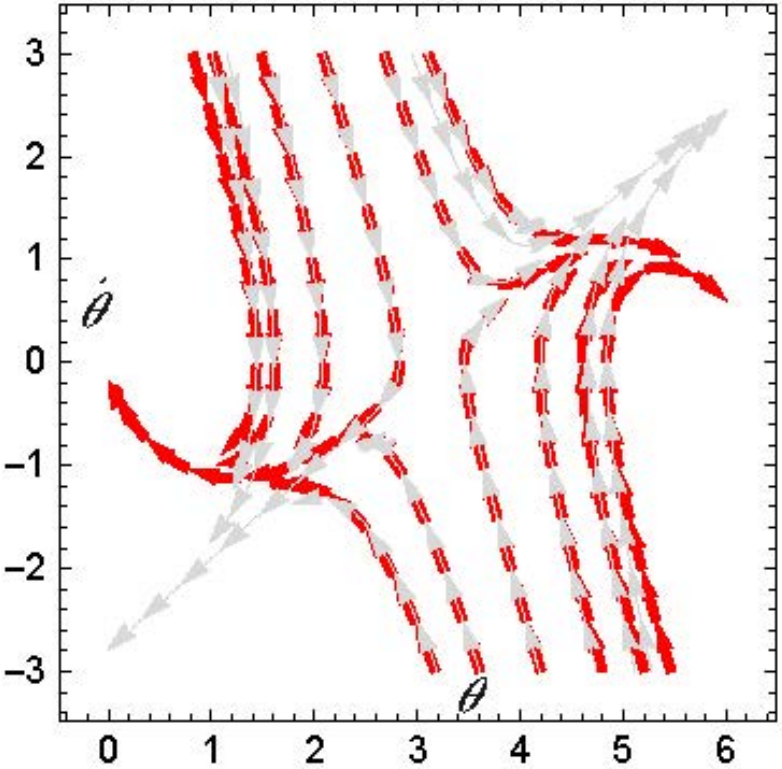
Very near bottom FP, $\mathbf{q}_b$

red: $$\dot{\boldsymbol{q}} = f_{\mathrm{DP}}(\boldsymbol{q}) := \begin{bmatrix} \dot\theta \\ -\frac{1}{m\ell^2} b\dot\theta - \frac{\mathrm{g}}{\ell}\sin\theta \end{bmatrix}$$

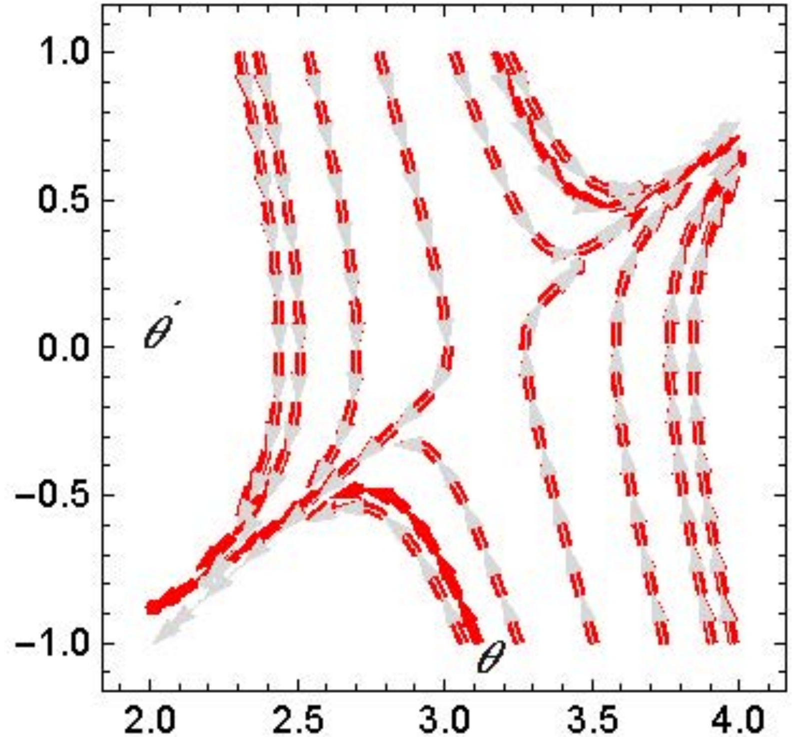
gray: $$\dot{\boldsymbol{u}} = F_{\mathrm{DP}}(\boldsymbol{q}_b)\boldsymbol{u}; \quad \boldsymbol{q}_b := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F_{\mathrm{DP}}(\boldsymbol{q}_b) := \begin{bmatrix} 0 & 1 \\ -\frac{\mathrm{g}}{\ell}\cos\theta_b & -\frac{b}{m\ell^2} \end{bmatrix}$$

Pendulum (red) and Linearized (gray) Orbits



Near top FP, $\mathbf{q}_t$



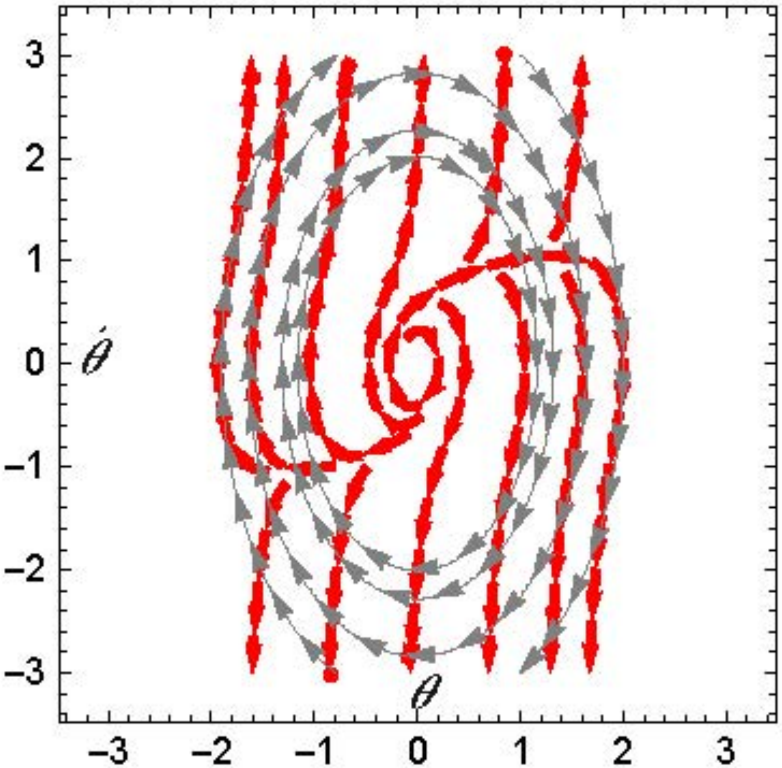
Very near top FP, $\mathbf{q}_t$

red: $$\dot{\boldsymbol{q}} = f_{\mathrm{DP}}(\boldsymbol{q}) := \begin{bmatrix} \dot{\theta} \\ -\frac{1}{m\ell^2} b\dot{\theta} - \frac{g}{\ell}\sin\theta \end{bmatrix}$$

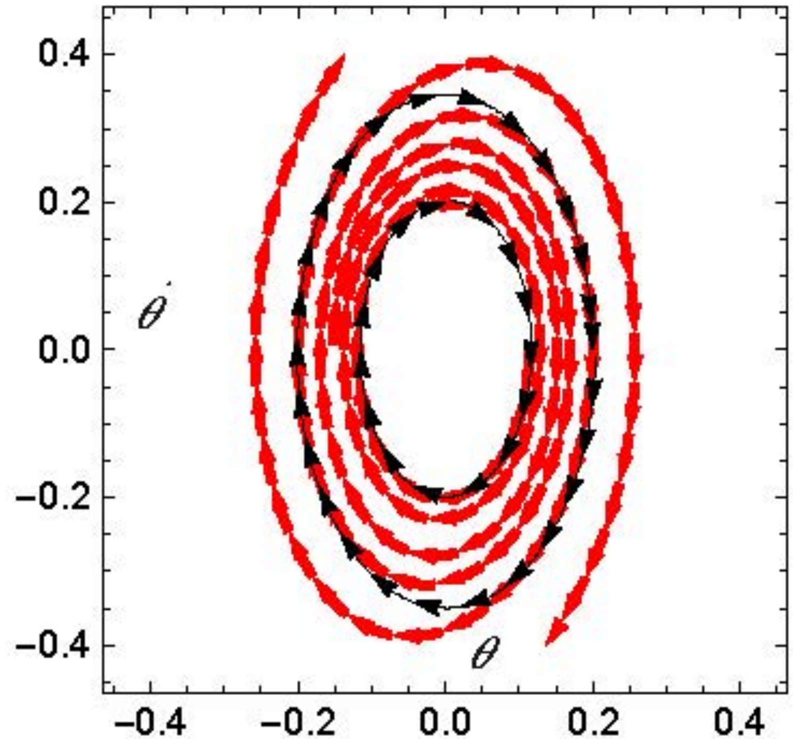
gray: $$\dot{\boldsymbol{u}} = F_{\mathrm{DP}}(\boldsymbol{q}_t)\boldsymbol{u}; \quad \boldsymbol{q}_t := \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$F_{\mathrm{DP}}(\boldsymbol{q}_t) := \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos\theta_t & -\frac{b}{m\ell^2} \end{bmatrix}$$

Unstable Pend. (red) and Lnrzd. (gray) Orbs.



Near bottom FP, $\mathbf{q}_b$



Very near bottom FP, $\mathbf{q}_b$

red: $$\dot{\boldsymbol{q}} = f_{\mathrm{UDP}}(\boldsymbol{q}) := \begin{bmatrix} \dot{\theta} \\ \frac{1}{m\ell^2} b\dot{\theta} - \frac{\mathrm{g}}{\ell} \sin\theta \end{bmatrix}$$

gray: $$\dot{\boldsymbol{u}} = F_{\mathrm{UDP}}(\boldsymbol{q}_b)\boldsymbol{u}; \quad \boldsymbol{q}_b := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F_{\mathrm{UDP}}(\boldsymbol{q}_b) := \begin{bmatrix} 0 & 1 \\ -\frac{\mathrm{g}}{\ell} \cos\theta_b & \frac{b}{m\ell^2} \end{bmatrix}$$

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Implications of FP Hyperbolicity

- Linearization of Hyperbolic FP predicts local nonlinear behavior
  - numerically: because linear term dominates Taylor expansion
  - formally: because CC preserves qualitative properties
- Most important qualitative property: stability

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Linearization and Lyapunov Functions

Video 5.3

Segment 1.2.3.2.a

Lyapunov Functions

Daniel E. Koditschek
with
Wei-Hsi Chen, T. Turner Topping and Vasileios Vasilopoulos
University of Pennsylvania
July, 2017

Agenda for this Segment

- Lyapunov functions: generalize physical energy
  - Fundamental Theorem of Dynamical Systems
    - all systems have such a generalized "energy"
    - defines "basins of attraction" around "attractors"

    (generalized steady state conditions – perhaps ***very*** complicated)
  - Our use in this course
    - infer them from hyperbolic attractors
    - find them when hyperbolicity fails
    - use them to find (conservative approximations of) basins
- Long term aim: programming work
  - attractor basins as symbols
  - energy landscapes as programs
  - seek compositional methods for designing landscapes

Positive Definite Functions

- Pin down notion of "norm-like"
  - crucial : level surfaces enclose neighborhoods
  - guaranteed at local minimum of continuous functions
  - prefer differentiability as well (want to compute power)
- A continuous, scalar valued function, V, is Positive Definite (PD) at x_e
  - if it is nonnegative on a neighborhood and vanishes only at x_e

$$V \geq 0 \;\; \& \;\; V(x) = 0 \Leftrightarrow x = x_e$$

  - often written (sloppily) as "$V > 0$"
- Say Positive Semi-Definite (PSD) when there might be other zero values
- Say Negative Definite (ND) or Negative Semi-Definite (NSD) when the sign reverses
- e.g. NSD property: $V \leq 0 \;\; \& \;\; V(x) = 0 \Leftarrow x = x_e$

Positive Definite Matrices

- A quadratic form is a scalar-valued polynomial of degree 2
  - e.g., norm-squared $\|x\|^2 = x^{\mathrm{T}}x = x_1^2 + \ldots + x_n^2$
  - e.g., total energy, $\eta_{\mathrm{HO}} = \kappa + \phi_{\mathrm{S}}$, for a Hooke's law spring potential, ϕ_{S}
  - more generally, can represent any quadratic form with a matrix,

$$V(x) = x^{\mathrm{T}} Q x; \quad Q \in \mathbb{R}^{n \times n}$$

- A quadratic form, $V = x^{\mathrm{T}} P x$, represented by the matrix P, is PD at 0
  - if and only if there is a $\mathrm{CC_L}$, $y = P^{-1/2}x$, such that

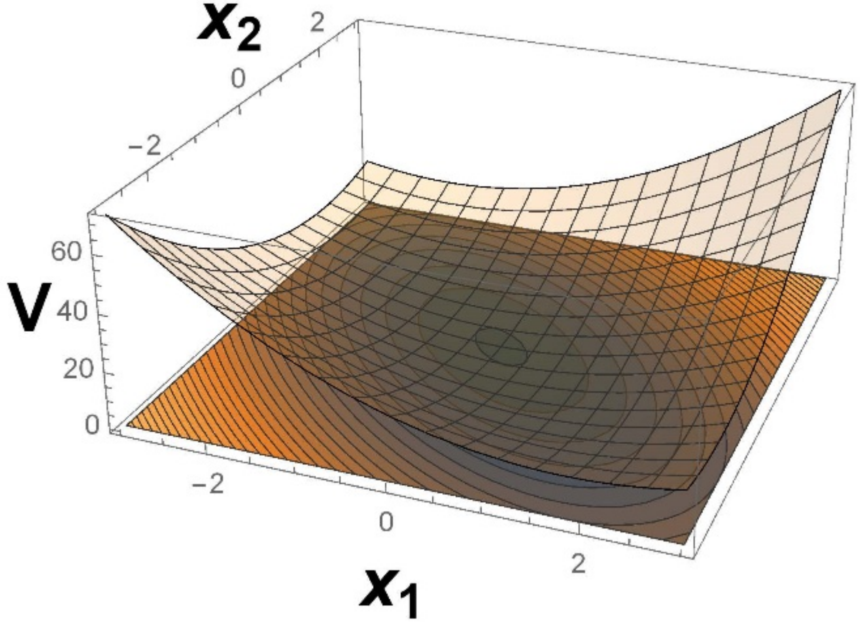
$$\tilde{V}(y) := V(P^{-\frac{1}{2}} y) = \|y\|^2$$

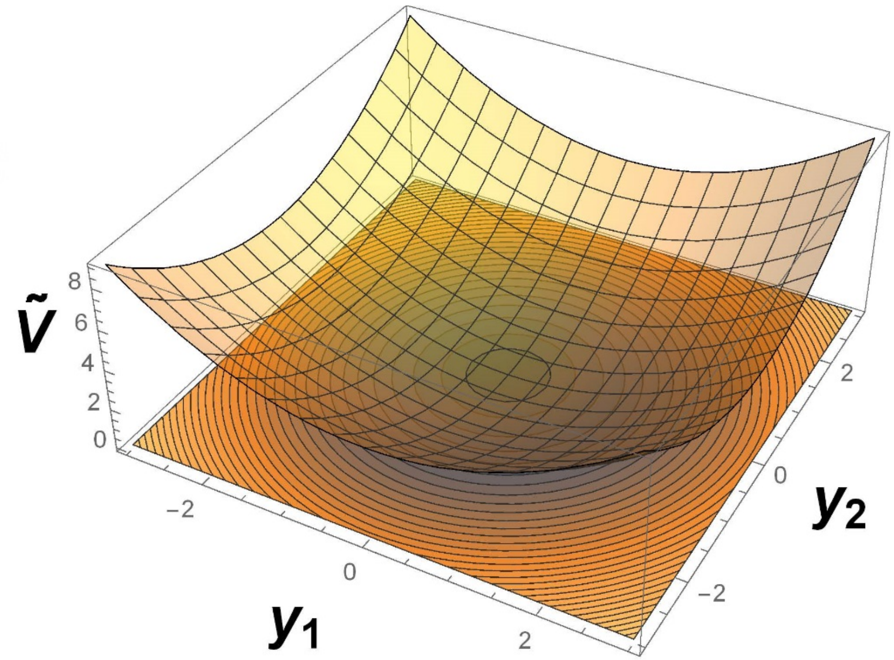
$$P^{-\frac{1}{2}} = \left[P^{\frac{1}{2}}\right]^{-1}$$

$$P^{\frac{1}{2}^{\mathrm{T}}} P^{\frac{1}{2}} = P \in \mathbb{R}^{n \times n}$$

  - in which case P is called a PD matrix written (sloppily) as "$P > 0$"

PD Quadratic Form Under CC_L





$$V(x) := x^{\mathrm{T}} P x; \quad P = R^{\mathrm{T}} \Pi R$$

$$\Pi := \begin{bmatrix} \pi_1^2 & 0 \\ 0 & \pi_2^2 \end{bmatrix}; \quad R^{\mathrm{T}} = R^{-1}$$

$$P^{\frac{1}{2}} := \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix} R$$

$$\tilde{V}(y) := V(P^{-\frac{1}{2}} y) = \|y\|^2$$

$$P^{-\frac{1}{2}} = \left[P^{\frac{1}{2}} \right]^{-1}$$

$$P^{\frac{1}{2}\,\mathrm{T}} P^{\frac{1}{2}} = P$$

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Block 1 – Week 2 – Unit 3

Linearization and Lyapunov Functions

Video 5.4

Segment 1.2.3.2.b

Lyapunov Functions

Daniel E. Koditschek
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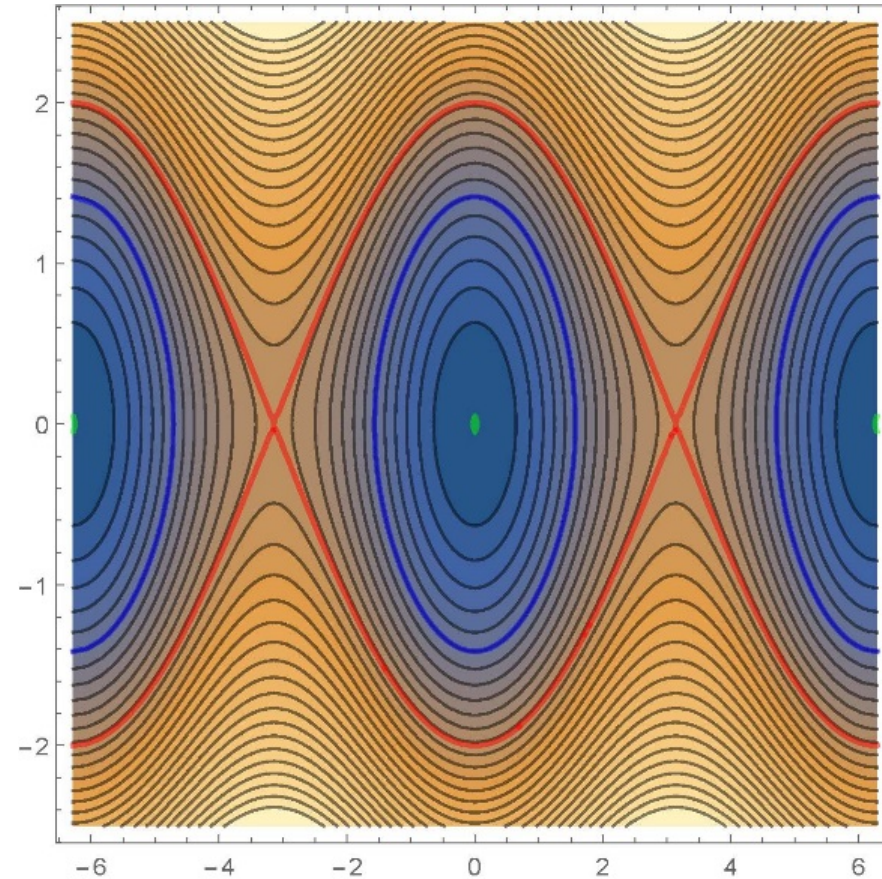
University of Pennsylvania
July, 2017

Stability

- A set is invariant
  - if the orbit through any element
  - remains within it for all time
  - it is positive invariant
    - if trajectories through its elements
    - remain within it for all future time
- A FP is stable under the flow of a VF
  - if sufficiently small neighborhoods
  - are positive invariant

Undamped Pendulum near “bottom” FP



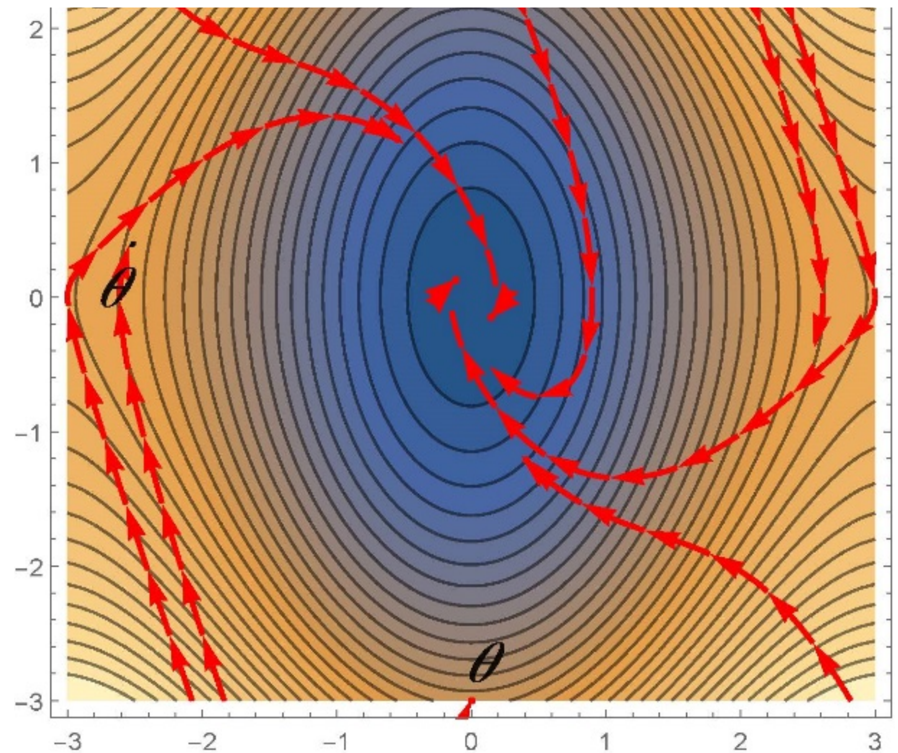
$$\dot{\boldsymbol{q}} = f_{\mathrm{R}}(\boldsymbol{q}) := \begin{bmatrix} \dot{\theta} \\ -\frac{\mathrm{g}}{\ell}\cos\theta \end{bmatrix}$$

$$\boldsymbol{q}_e = \boldsymbol{q}_b := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Asymptotic Stability

- A FP is asymptotically stable under the flow of a VF
  - if it is stable and
  - sufficiently small neighborhoods
  - approach the FP asymptotically
  - in future time
- in which case it is an attractor
- whose basin
  - is the set of ICs
  - which asymptotically approach the FP

Damped Pendulum near "bottom" FP



$$\dot{\boldsymbol{q}} = f_{\mathrm{DP}}(\boldsymbol{q}) :=$$

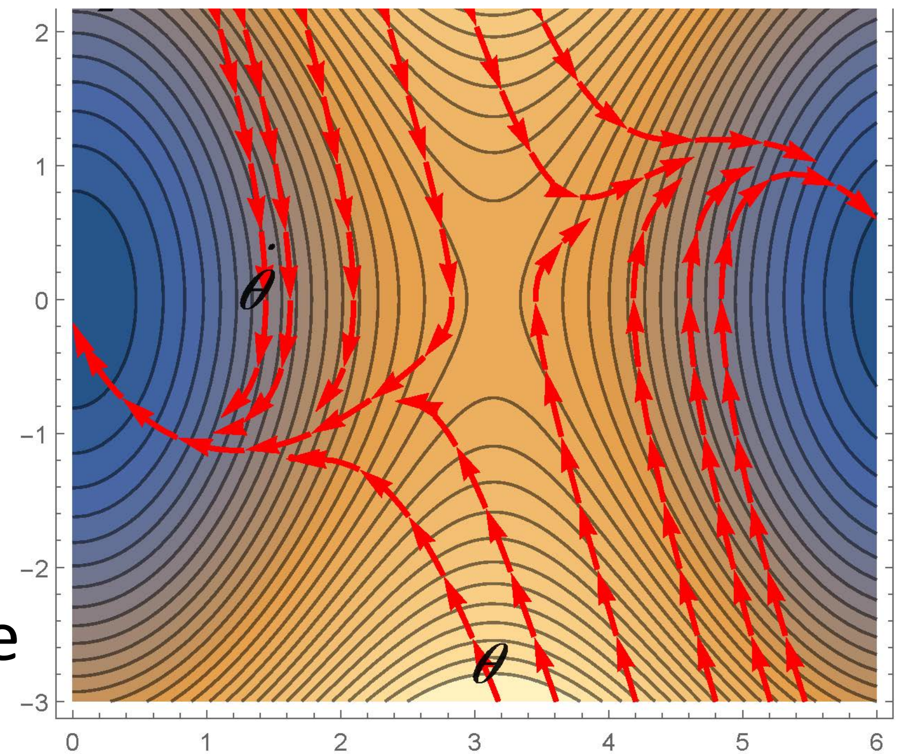
$$\begin{bmatrix} \dot{\theta} \\ -\frac{\mathrm{g}}{\ell}\cos\theta - \frac{b}{m\ell^2}\dot{\theta} \end{bmatrix}$$

$$\boldsymbol{q}_e = \boldsymbol{q}_t := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Instability

- A FP is unstable under the flow of a VF
  - if it fails to be stable
  - i.e., every neighborhood
  - has ICs whose trajectories
  - leave at some future time
- and it is a repeller
  - if it is asymptotically stable in reverse time
  - i.e., every IC in small enough neighborhoods
  - has trajectories that leave in future time

Damped Pendulum near "top" FP



$$\dot{\boldsymbol{q}} = f_{\mathrm{DP}}(\boldsymbol{q}) := \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\cos\theta - \frac{b}{m\ell^2}\dot{\theta} \end{bmatrix}$$

$$\boldsymbol{q}_e = \boldsymbol{q}_t := \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Lyapunov Theory

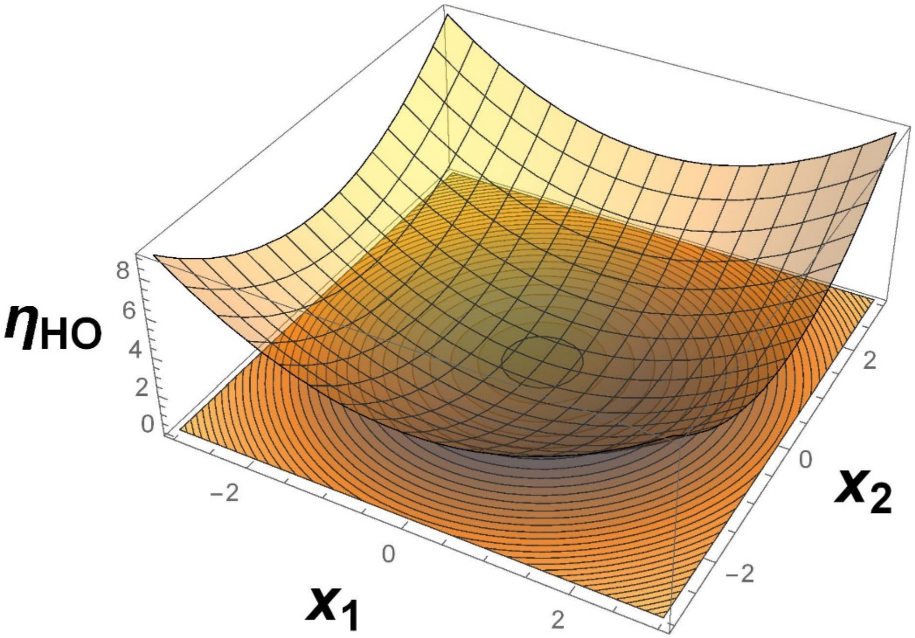
- PD V, is a Lyapunov function (LF)for a VF at a FP

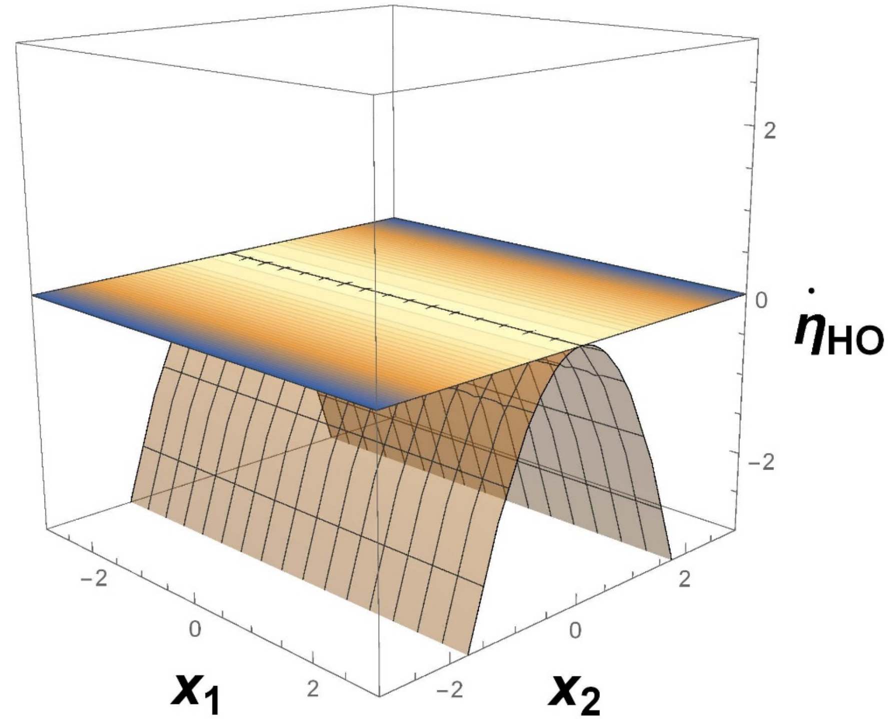
$$\dot{x} = f_{\text{gen}}(x); \quad f_{\text{gen}}(x_e) = 0$$

- if its power function is NSD: $\dot{V} = D_x V \cdot f_{\text{gen}} \leq 0$
- it is strict if the power function is ND: $\dot{V} < 0$
- Lyapunov's Theorem:
  - LF for VF at FP implies stability of FP
  - strict LF for VF at FP implies asymptotic stability of FP
- Converse Theorem:
  - if FP of VF is stable then it has a LF
  - if asymptotically stable then it has a strict LF
- Reference:

V. I. Arnold. Ordinary Differential Equations. MIT Press, 1973.

PD and NSD Quadratic Forms (Seg.1.2.1.3)





$$
\begin{aligned}
V(x) &= \eta_{\mathrm{HO}}(x) \\
&= \frac{1}{2} m x_2^2 + \frac{1}{2} k x_1^2 \\
&= \frac{1}{2} \left\| \begin{bmatrix} \sqrt{k} x_1 \\ \sqrt{m} x_2 \end{bmatrix} \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}(x) &= \dot{\eta}_{\mathrm{HO}}(x) \\
&= -b x_2^2 \\
&= -\left\| \begin{bmatrix} \sqrt{0} x_1 \\ \sqrt{b} x_2 \end{bmatrix} \right\|^2
\end{aligned}
$$

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Moving Ahead

- Local Lyapunov Theory
  - necessary and sufficient conditions for steady state stability
  - not constructive (but typically “energy-like”)
- Linearized Dynamics
  - numerically close and behaviorally exact account
  - of flow in the neighborhood of FP
  - algorithmic construction of LF
  - conservative estimate of basin; of robustness
- Global Lyapunov Theory
  - not developed in this course
  - fundamental theorem of dynamical systems
  - idea: generalized energy landscapes for programming work